

Energies with values in networks and geometric quasiconvexity

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Abstract

We study the asymptotic behaviour, as $\varepsilon \rightarrow 0^+$, of the p -Dirichlet energy of maps $u: \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$ constrained to take values in the rescaled periodic network $\varepsilon \mathcal{Z}$, where $\mathcal{Z}$ is a standard square network in $\mathbb{R}^2$. The oscillations are carried by the constraint in the target, not by the energy density, and their effect depends on p . For $1 < p < 2$ the constraint disappears in the limit: the Γ -limit is an integral functional on the whole of $W^{1,p}(\Omega, \mathbb{R}^2)$, whose density is given by an asymptotic cell formula on tilted cells. This density is equal to $(|a_1| + |a_2|)^p$ on rank-one matrices $a \otimes \nu$, $|\nu| = 1$, and exceeds its convex envelope by a quantity comparable to $|\det A| |A|^{p-2}$, with constants of order $(2-p)^{-1}$. Consequently, although the original energy density is convex, the limit density is quasiconvex but neither convex nor polyconvex: quasiconvexity is generated by the geometry of the constraint. For $p \geq 2$ the constraint survives in the limit as $\det \nabla u = 0$, and the Γ -limit is the degenerate energy $\int_{\Omega} (|\nabla u_1| + |\nabla u_2|)^p dx$, that is, the p -energy of u considered as a map into the normed plane $(\mathbb{R}^2, |\cdot|_1)$, the metric limit of the networks.

Dedicated to David Kinderlehrer on the occasion of his 85th birthday

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1 Introduction and main results

In this paper, we study the asymptotic behavior of a prototypical family of variational integrals defined on vector-valued Sobolev functions, and whose values are constrained to lie on a periodic network, which is seen at a scale ε that is small compared to the scale of observation. Given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ and an exponent $p > 1$, we consider

$$F_\varepsilon^p(u) = \begin{cases} \int_\Omega |\nabla u(x)|^p dx & \text{if } u(x) \in \varepsilon\mathcal{Z} \text{ for almost every } x \in \Omega \\ +\infty & \text{otherwise,} \end{cases} \quad (1)$$

where $\mathcal{Z}$ is a periodic network in $\mathbb{R}^2$ and $|\cdot|$ denotes the Frobenius norm on $\mathbb{R}^{2 \times 2}$. The prototypical example, that is our focus here, is the union of the coordinate lines

$$\mathcal{Z} = \left\{ (a, b) \in \mathbb{R}^2 : a \in \frac{1}{2} + \mathbb{Z} \text{ or } b \in \frac{1}{2} + \mathbb{Z} \right\} = \bigcup_{i \in \mathbb{Z}^2} (i + \partial Q), \quad Q = (-\frac{1}{2}, \frac{1}{2})^2. \quad (2)$$

The effective domain of F_ε^p is the class of constrained Sobolev maps $W^{1,p}(\Omega, \varepsilon\mathcal{Z}) = \{u \in W^{1,p}(\Omega, \mathbb{R}^2) : u(x) \in \varepsilon\mathcal{Z} \text{ for almost all } x \in \Omega\}$. Constraints of this type arise naturally in models of media with a fine internal architecture, where admissible deformations, order parameters or internal variables are forced onto a network of preferred states: lattice or network-like microstructures, strongly anisotropic media, and discrete systems in which the order parameter is quantized. Moreover, this can be regarded as a prototype for energies with values on an oscillating manifold, with the heuristic motivation that low-energy functions will be located approximately on a network with small oscillations within the manifold (which is the case for curves [18]). In all such models, one is led to ask what effective continuum theory survives the limit $\varepsilon \rightarrow 0^+$.

We will refer to this limit as a homogenization process, although it is not a usual one. Two features distinguish (1) from the energies usually encountered in homogenization

theory. First, the integrand is the p -Dirichlet integrand: all oscillations are carried by the *constraint*, not by the energy density. Second, and more importantly, the oscillations take place in the *target* space rather than in the spatial x -variable. The classical homogenization results deal with integrands $f(x/\varepsilon, \nabla u)$ oscillating in x , and the effective energy is obtained through a cell formula on the domain (see Braides [15], Müller [55] and the monograph [19]); here, instead, the cell problem lives in the target, and the relevant cell is the pre-image $Q_A = A^{-1}(Q)$ of the target cell under the affine datum with gradient A . This is why, unlike in the classical setting, the homogenization formula depends on the rank of the matrix A . More importantly, although the original integrand is convex, the homogenized density is not: it is quasiconvex but neither convex nor polyconvex, and the distance from its convex envelope is controlled by the determinant. This lack of convexity is produced by the geometry of the constraint alone and it is an important aspect of the present work.

The main goal of the paper is to characterize the Γ -limit of F_ε^p as $\varepsilon \rightarrow 0^+$ and to understand how the interplay between p and the geometry of $\mathcal{Z}$ affects the limit model. A dichotomy appears at $p = 2$: for $p < 2$ the constraint disappears in the limit, but it leaves its trace in a genuinely quasiconvex energy density; for $p > 2$ it survives as the differential constraint $\det \nabla u = 0$. For $p < 2$ the result is the following.

Theorem A (subcritical regime: geometric quasiconvexity). *Let $1 < p < 2$. Then F_ε^p Γ -converges, with respect to the strong L^p -topology, to*

$$F_{\text{hom}}^p(u) = \int_{\Omega} f_{\text{hom}}^p(\nabla u(x)) \, dx, \quad u \in W^{1,p}(\Omega, \mathbb{R}^2),$$

where $f_{\text{hom}}^p: \mathbb{R}^{2 \times 2} \rightarrow [0, +\infty)$ is a continuous, p -homogeneous, quasiconvex integrand with the following properties:

(i) (asymptotic homogenization formula) *on matrices of rank two*

$$f_{\text{hom}}^p(A) = \liminf_{T \rightarrow +\infty} \frac{|\det A|}{T^2} \inf \left\{ \int_{TQ_A} |\nabla u|^p \, dx : u \in W^{1,p}(TQ_A, \mathcal{Z}), \right. \\ \left. u = u_A \text{ in a neighbourhood of } \partial TQ_A \right\}, \quad (3)$$

where u_A is the projection of the linear function Ax on $\mathcal{Z}$ (see Definition 2.3);

(ii) (qualified two-sided estimates) *there exist $\bar{c}, \bar{C} > 0$ independent of p and explicitly computable, such that*

$$N(A)^p + \bar{c}(p-1)8^{\frac{1}{2(1-p)}} \frac{|\det A|}{(2-p)|A|^{2-p}} \leq f_{\text{hom}}^p(A) \leq N(A)^p + \bar{C} \frac{|\det A|}{(2-p)|A|^{2-p}} \quad (4)$$

for every $A \neq 0$, where, if A_1 and A_2 denote the rows of A ,

$$N(A) := |A_1| + |A_2|. \quad (5)$$

In particular, $f_{\text{hom}}^p(A) = N(A)^p$ on rank-one matrices, and the function f_{hom}^p is neither convex nor polyconvex.

The density f_{hom}^p is given by an asymptotic cell formula on the tilted cells $TQ_A = TA^{-1}(Q)$, with the projected linear map $u_A = P_0(A \cdot)$ as boundary datum (Definitions 2.3 and 2.7); by (4) it has standard p -growth, $|A|^p \leq f_{\text{hom}}^p(A) \leq C|A|^p$. The restriction $p < 2$ is exactly what makes the radial projection onto the boundary of a cell an admissible operation in $W^{1,p}$, the singularity $|x|^{-p}$ integrable in the plane precisely for $p < 2$ (Lemma 2.2): every Sobolev map can be projected onto the network at a uniformly controlled energy cost, which is why the limit domain is the whole of $W^{1,p}(\Omega, \mathbb{R}^2)$.

The lower bound in (4) shows that f_{hom}^p is not convex. Indeed, writing $I = \frac{1}{2}(2e_1 \otimes e_1) + \frac{1}{2}(2e_2 \otimes e_2)$, by (i) and (ii) we have

$$\frac{1}{2}f_{\text{hom}}^p(2e_1 \otimes e_1) + \frac{1}{2}f_{\text{hom}}^p(2e_2 \otimes e_2) = 2^p < 2^p + \bar{c} \frac{p-1}{2-p} 8^{\frac{1}{2(1-p)}} 2^{\frac{p-2}{2}} \leq f_{\text{hom}}^p(I). \quad (6)$$

As a consequence, f_{hom}^p is not polyconvex either: a polyconvex function with growth of order $p < 2$ must be convex, since an affine function of $(A, \det A)$ lying below a function with subquadratic growth cannot depend on $\det A$.

Geometric quasiconvexity. We consider f_{hom}^p as a quasiconvex function generated by the geometry of the constraint, from which the title of this paper comes. It is worth commenting on its form. The two terms in (4) have a transparent meaning. The first is the convex envelope of f_{hom}^p . Indeed, N^p is convex and lies below f_{hom}^p ; conversely, if $A_1, A_2 \neq 0$ and $\lambda = |A_1|/N(A)$, then

$$A = \lambda \left(\frac{1}{\lambda} e_1 \otimes A_1 \right) + (1 - \lambda) \left(\frac{1}{1 - \lambda} e_2 \otimes A_2 \right)$$

is a convex combination of two matrices of rank one at which, by (i), f_{hom}^p takes the value $N(A)^p$. Hence $(f_{\text{hom}}^p)^{**} = N^p$, and by (i) and (4) the function f_{hom}^p coincides with its convex envelope exactly on the matrices of rank at most one: it is convex at each of them, in the sense that it has an affine minorant touching it there, and at no matrix of rank two. Note that N is a norm on $\mathbb{R}^{2 \times 2}$ with $|A| \leq N(A) \leq \sqrt{2}|A|$ and $N(a \otimes \nu) = |a_1| + |a_2|$ for every $a \in \mathbb{R}^2$ and every unit vector $\nu \in \mathbb{R}^2$. The value $N(A)^p$ is also identified in Section 2.7 as the minimal p -th moment of a probability measure with barycentre A supported on

$$K := (e_1 \otimes \mathbb{R}^2) \cup (e_2 \otimes \mathbb{R}^2) \subset \mathbb{R}^{2 \times 2}, \quad (7)$$

the set of matrices with at most one non-zero row, which contains the gradients of all maps with values in $\varepsilon\mathcal{Z}$ (Lemma 2.18). In other words, $N(A)^p$ is the best lower bound that can be extracted from the pointwise constraint on the gradients alone. For matrices of rank one, this bound is sharp, the optimal measure being a laminate that we can generate with maps with values in $\varepsilon\mathcal{Z}$. However, for $\det A \neq 0$, instead, the unique optimal measure on K

is supported on two matrices that are not rank-one connected, so it is not a homogeneous $W^{1,p}$ -gradient Young measure [11]. The fact that the generating sequence must be gradients for maps valued in $\varepsilon\mathcal{Z}$ is responsible for the nonconvexity of f_{hom}^p . Indeed, the second term in (4) is the energy of the topological defects forced by the constraint. Maps with values in $\varepsilon\mathcal{Z}$ satisfy $\det \nabla u_\varepsilon = 0$ almost everywhere, while the affine datum has Jacobian $\det A \neq 0$; the difference is carried by *vortices*, that is, by points around which u_ε winds once around a cell of the network. For projected linear maps, this can be computed explicitly: the distributional Jacobian of $\varepsilon u_A(x/\varepsilon)$ is

$$\text{sign}(\det A) \varepsilon^2 \sum_{j \in \mathbb{Z}^2} \delta_{\varepsilon A^{-1}j},$$

with a Dirac mass, weighted by the area ε^2 of a cell, at each point of a lattice of density $|\det A| \varepsilon^{-2}$. A vortex has finite p -energy only if $p < 2$, and its cost at the scale $\varepsilon/|A|$, below which it dominates the affine part of the gradient, is of the order $\varepsilon^2 |A|^{p-2}/(2-p)$; multiplied by the density of vortices, this accounts heuristically for the second term in (4), including the order $(2-p)^{-1}$ of the constants.

The proof of the lower bound makes these heuristics rigorous without any description of the vortices: the Jacobian of the boundary datum must be paid for by large gradients. Precisely, a Lipschitz truncation [1] at level $\lambda \geq 4|A|$, combined with the null-Lagrangian property of the determinant, shows that for every competitor u in the cell problem the maximal function of $|\nabla u - A|$ exceeds λ on a set whose measure, relative to that of the cell, is at least of order $|\det A| \lambda^{-2}$. Integrating in λ gives a lower bound of order $|\det A| |A|^{p-2}/(2-p)$ for the p -th moment of the large values of $|\nabla u|$ (Lemma 3.3), which is then added to $N(A)^p$ by a Jensen inequality that keeps track of the tail (Lemma 3.1). The upper bound follows from the rank-one convexity of f_{hom}^p , by laminating a rank-one matrix with a multiple of a rotation, whose cost is computed through the projection P_0 . It is possible to give a more precise lower bound for the density f_{hom}^p using recent results [3]. First, we note that $N(A) \geq \|A\|$, where $\|\cdot\|$ denotes the spectral norm, with equality for $A \in K$. Using that the Burkholder integrands defined for exponents $p \in (1, 2)$ as

$$B_p^\pm(X) = \left(\frac{2}{p} - 1\right) \|X\|^p \pm \|X\|^{p-2} \det X, \quad X \in \mathbb{R}^{2 \times 2},$$

are quasiconvex one may show that

$$f_{\text{hom}}^p(X) \geq \|X\|^p \left(1 + \frac{p}{2-p} \frac{|\det X|}{\|X\|^2}\right) \quad (8)$$

holds for all $X \in \mathbb{R}^{2 \times 2}$. The lower bound in (8) equals $\max\{B_p^+(X), B_p^-(X)\}/(\frac{2}{p} - 1)$ and as such is quasiconvex.

We now describe the result for $p > 2$. By (4), the densities f_{hom}^p converge pointwise, as $p \rightarrow 2^-$, to a function equal to $N(A)^2$ if $\det A = 0$ and to $+\infty$ otherwise, which is the

limit density for $p = 2$ given by Theorem B below: the vortex energy becomes infinite and turns into a constraint.

Theorem B (supercritical regime: rigidity). *Let $p > 2$. Then F_ε^p Γ -converges, with respect to the strong L^p -topology, to*

$$F_0^p(u) = \begin{cases} \int_{\Omega} (|\nabla u_1(x)| + |\nabla u_2(x)|)^p dx & \text{if } u \in W^{1,p}(\Omega, \mathbb{R}^2) \text{ and } \det \nabla u = 0 \text{ a.e.} \\ +\infty & \text{otherwise.} \end{cases}$$

In the supercritical regime, the constraint produces a rigidity phenomenon that survives in the limit: limit deformations have gradients of rank at most one, and the effective energy is the degenerate energy $N(\nabla u)^p = (|\nabla u_1| + |\nabla u_2|)^p$ induced by the network metric. The limit density is a convex function of $(A, \det A)$; that is, it is polyconvex in the sense of Ball [10], with extended values. Crossing $p = 2$, the limit density thus passes from a finite, genuinely quasiconvex function to a polyconvex, constrained one; the two regimes agree on matrices of rank one, where both densities equal $(|a_1| + |a_2|)^p$ for $A = a \otimes \nu$, $|\nu| = 1$. For functions u with an image a curve, the proof of Theorem B (Section 4) is short: the lower bound follows from the identity $|\nabla u_\varepsilon| = N(\nabla u_\varepsilon)$, valid almost everywhere for maps with values in $\varepsilon\mathcal{Z}$, from the convexity of N^p and from the weak continuity of the determinant, while recovery sequences are obtained by composing u with maps that replace segments in its image by staircases in $\varepsilon\mathcal{Z}$.

We want to remark how the two ways in which the exponent 2 appears are the two sides of the same coin. From below, $p < 2$ is the integrability condition that makes the projection of the radial projection onto ∂Q bounded in $W^{1,p}$ (Lemma 2.2). Averaging over the projection centers goes back to Federer and Fleming [31], and was introduced for Sobolev maps with values in manifolds by Hardt, Kinderlehrer, and Lin [40]; here the network plays the role of the one-dimensional skeleton of the cubical decomposition of the target, and projecting cell by cell allows us to destroy the constraint in the limit.

From above, $p \geq 2$ is what guarantees that the pointwise constraint $\nabla u_\varepsilon \in K$ passes to the weak limit as the differential constraint $\det \nabla u = 0$. The determinant vanishes on K and is a null Lagrangian, so it is affine along rank-one segments, and $\{\det = 0\}$ is the lamination convex hull of K . For $p \geq 2$ the pointwise Jacobian of a $W^{1,p}$ map coincides with its distributional Jacobian, which is weakly continuous [10], so that $\det \nabla u_\varepsilon = 0$ passes to the limit. Below $p = 2$ the distributional Jacobian, although still weakly continuous for $p > 4/3$, may differ from the pointwise one by concentrations at vortex points, and maps with values in $\varepsilon\mathcal{Z}$ can carry such vortices: the Jacobians of the maps $\varepsilon u_A(x/\varepsilon)$ computed above converge weakly* to $\det A$, although $\det \nabla(\varepsilon u_A(x/\varepsilon)) = 0$ almost everywhere. This is the phenomenon exhibited, e.g. by Ball and Murat [12], with the map $x \mapsto x/|x|$, which shows that in dimension n the determinant is not $W^{1,p}$ -quasiconvex for $p < n$. In our setting, it means that for $p < 2$ the inclusion $\nabla u \in K$ is compatible with every affine

boundary datum: u_A satisfies $\nabla u_A \in K$ almost everywhere and $u_A(x) = Ax$ on $\partial(TQ_A)$ for every odd integer T , while for $p \geq 2$ such a map exists only if $\det A = 0$.

The dichotomy also has a topological reading. For $p > 2$ every $u \in W^{1,p}(\Omega, \varepsilon\mathcal{Z})$ is continuous and, if Ω is simply connected, it lifts to the universal covering of $\varepsilon\mathcal{Z}$, which is a tree, so no vortex can occur. For $p < 2$, instead, the maps that wind around a cell, such as $x \mapsto \varepsilon x/(2|x|_\infty)$ near the origin, have finite p -energy, and are precisely the singularities created by the projection of Lemma 2.2 at the points where u meets the projection centers. The borderline case $p = 2$ is the analog, for graph-valued maps, of the lifting problem for circle-valued maps [21, 14], which we do not pursue here.

For $p \geq 2$ Theorem B can also be interpreted as a study of the limits of some metric spaces. We refer to Section 5 for a more detailed discussion of this aspect.

Relation to the literature

Homogenization under constraints on the target. The homogenization of integral functionals for maps constrained to a fixed smooth manifold has been studied by Babadjian and Millot [8], who introduced a notion of tangential homogenization modeled on the tangential quasiconvexity of Dacorogna, Fonseca, Malý and Trivisa [27]. In those works the target is fixed and smooth and the oscillation is in the space variable, so that our setting is in a precise sense complementary.

Oscillating targets and effective metrics. The closest antecedent is the analysis of Braides, Cancedda and Chiadò Piat [18], who consider the Dirichlet integral of *curves* taking values in fast-oscillating manifolds collapsing onto a linear subspace, and show that the limit is described by a Finsler metric obtained by optimizing the contribution of the oscillations on level sets, with formulas mixing homogenization and convexification. Their result is the one-dimensional domain counterpart of our supercritical regime, in which the oscillating target degenerates into a metric. Related constructions of effective metrics by Γ -convergence appear in Braides, Buttazzo, and Fragalà [17].

Quasiconvexity produced by homogenization. Quasiconvexity in the sense of Morrey [54] is the natural convexity condition for vectorial integral functionals, and polyconvexity in the sense of Ball [10] is its most widely used sufficient condition; see [3, 24, 26, 39] for the relations between the various notions. Explicit quasiconvex functions that are not polyconvex are plentiful, even in the 2×2 case, but explicit examples are fewer. The early examples include [4, 65], and homogenization is a natural source of them: Braides [16] showed that polyconvexity is not preserved by periodic homogenization. The mechanism we exhibit is of a different nature: the energy density is convex and fixed, and it is the non-convex constraint in the target that generates a quasiconvex, non-convex limit density.

Homogeneous quasiconvex functions of low degree. On $\mathbb{R}^{2 \times 2}$, functions positively homogeneous of degree $p \geq 2$ that are quasiconvex but not convex are easy to produce: $|\det A|^{p/2}$ is polyconvex and not convex. For $p < 2$ the picture changes since, as observed after Theorem A, polyconvex functions with growth of order p are convex. That quasiconvex functions with subquadratic growth need not be convex was first observed by Šverák [64]. Positively

homogeneous functions of degree one were studied by Müller [56] and by Dacorogna and Haeberly [28]; in this class, Kirchheim and Kristensen [48] proved that rank-one convexity forces convexity at every matrix of rank at most one, while the examples of [64, 56, 28] show that convexity may fail at matrices of rank two and even quasiconvexity may fail at rank two matrices in higher dimensions [24]. For every $p \in (1, 2)$, the density f_{hom}^p is a quasiconvex function homogeneous of degree p that exhibits this picture in its sharpest form, and it arises naturally from a homogenization problem: it is convex at every matrix of rank at most one and at no matrix of rank two, and its distance from its convex envelope is comparable to $|\det A| |A|^{p-2}$, with constants of order $(2 - p)^{-1}$.

Vortices. The energetics of point singularities is classical for maps into the circle: the Dirichlet energy of a vortex diverges logarithmically, which is at the origin of the Ginzburg–Landau asymptotics [13], while for $p < 2$ the p -energy of a vortex is of order $(2 - p)^{-1}$, and Hardt and Lin [41] showed that p -energy minimizing maps from planar domains into the circle, with boundary data of degree d , develop exactly $|d|$ singularities, whose behaviour as $p \rightarrow 2^-$ they described. The constants in (4) display the same order $(2 - p)^{-1}$, the Jacobian, rather than the degree, measuring the number of vortices.

Differential inclusions and gradient Young measures. The set K in (7) is the union of two two-dimensional subspaces of $\mathbb{R}^{2 \times 2}$; the rank-one segments joining them lie in $\{\det = 0\}$, which is both the lamination convex hull and the quasiconvex hull of K . In the moment problem of Section 2.7, whose value is $N(A)^p$, the optimal measure is a laminate when A has rank one and, by Ball and James [11], it is not a gradient Young measure when $\det A \neq 0$. Which measures supported on K are generated by gradients of maps with values in $\varepsilon\mathcal{Z}$, a constrained version of the characterization of gradient Young measures in Sobolev spaces by Kinderlehrer and Pedregal [45], is a question of independent interest. For the general theory of differential inclusions and of hulls of sets of matrices we refer to [37, 57, 47, 66].

Maps into singular spaces and normed planes. At fixed ε the target is a one-dimensional metric graph, so that the variational theory of maps into singular targets applies: Gromov and Schoen [38], Korevaar and Schoen [49], Eells and Fuglede [30], Daskalopoulos and Mese [29]. Its estimates, however, are tied to the curvature scale of the target. The shortest embedded circles of $\varepsilon\mathcal{Z}$ are the boundaries of the cells, of length 4ε , so that $(\varepsilon\mathcal{Z}, d_\varepsilon)$ is $\text{CAT}(\kappa)$ exactly for $\kappa \geq \pi^2/(4\varepsilon^2)$ [22, Part II]; the regularity theory for maps into targets with curvature bounded above by $\kappa > 0$ requires the image to lie in a ball of radius smaller than $\pi/(2\sqrt{\kappa})$, which for the optimal value $\kappa = \pi^2/(4\varepsilon^2)$ equals ε (see [42] and, for singular spaces, [62, 35, 34]), and therefore gives no estimate uniform in ε . In the limit, the convergence $d_\varepsilon \rightarrow |\cdot|_1$ is an instance of the stable norm of a periodic metric space [59, 23], which would replace $|\cdot|_1$ for a general connected periodic network. Finally, for $p = 2$ the limit functional of Theorem B is the energy of a map into a normed plane, a setting in which energy and area interact subtly: several notions of area coexist, and Lytchak and Wenger [51, 52] showed that energy minimization solves the Plateau problem

once the notion of area is chosen appropriately, quasiconvexity of the area being a key ingredient. Comparing the quasiconvexity of these notions of area with that of our limit densities seems an interesting question, which we do not pursue here.

Plan of the paper: Section 2 is devoted to the subcritical case $p < 2$, containing the proof of Theorem A(i), although the construction of staircase paths in Section 2.3 is also used in the regime $p \geq 2$. In the subcritical regime, a fundamental issue is the estimate of projections on the networks (Section 2.1), together with the adaptation of nonlinear-homogenization techniques to functionals constrained in the target space; in particular, the modification of boundary values in Section 2.2. In Section 2.4 we separately define the homogenized energy densities on matrices of rank two and of rank one, respectively. In the first case, through an asymptotic homogenization formula, in the second case, by optimizing over paths. This is suggested by the necessity of two different variants of the blow-up method applied in Section 2.5. Moreover, since the lower bound gives two different formulas, we prove that the resulting function satisfies the continuity properties necessary to prove the Γ -limsup inequality through a density argument in Section 2.6. Section 2.7 develops the Young-measure counterpart of this analysis, identifying $N(A)^p$ as the value of a moment problem on K and constructing, for rank-one A , the optimal laminate using maps with values in $\varepsilon\mathcal{Z}$. Section 3 contains the proof of Theorem A(ii); that is, of the two-sided estimate (4). Section 4 treats the case $p \geq 2$ (Theorem B). In this case, the functions in the limit domain satisfy $\det \nabla u = 0$. The necessary notions from Topology and Graph Theory are presented and used to overcome the difficulties due to the rigidity of this constraint, which forbids the usual arguments by density. Section 5 concludes the paper with some remarks on the connections with convergence issues of metric spaces.

2 The subcritical case $p < 2$

As noted in the Introduction the behaviour of functionals (1) is different in the two cases $p < 2$ and $p \geq 2$. We start with the case $p < 2$.

We set $Q = (-1/2, 1/2)^2$, so that the network $\mathcal{Z}$ in (2) can be written as

$$\mathcal{Z} = \bigcup_{i \in \mathbb{Z}^2} (i + \partial Q).$$

For $a \in \mathbb{R}$ we let $\lfloor a \rfloor$ denote the largest number in $\frac{1}{2} + \mathbb{Z}$ less than or equal to a .

The asymptotic part of Theorem A in the Introduction is the following result.

Theorem 2.1 (subcritical homogenization theorem). *Let $1 < p < 2$ and let $\Omega \subset \mathbb{R}^2$ be a Lipschitz bounded open set. Let $\mathcal{Z} \subset \mathbb{R}^2$ be given by (2), and let F_ε^p be the functionals defined in (2), with domain $W^{1,p}(\Omega, \varepsilon\mathcal{Z})$. Then the Γ -limit as $\varepsilon \rightarrow 0$ with respect to the strong $L^p(\Omega)$ -convergence and weak $W^{1,p}(\Omega, \mathbb{R}^2)$ -convergence of F_ε^p exists and it is given by*

$$F_{\text{hom}}^p(u) = \int_{\Omega} f_{\text{hom}}^p(\nabla u(x)) dx \tag{9}$$

with domain $W^{1,p}(\Omega, \mathbb{R}^2)$. The quasiconvex function $f_{\text{hom}}^p: \mathbb{R}^{2 \times 2} \rightarrow [0, +\infty)$ is characterized on matrices A with full rank by the asymptotic homogenization formula (3).

This section will be devoted to the proof of Theorem 2.1. To that end, we will first study the properties of projections on $\mathcal{Z}$, that ensure that the Γ -limit is finite on the whole $W^{1,p}(\Omega, \mathbb{R}^2)$, and allow us to state the boundary conditions in the homogenization formula. A crucial step is the characterization of f_{hom}^p on the subset of matrices of rank at most one, showing that $f_{\text{hom}}^p(a \otimes \nu) = (|a_1| + |a_2|)^p$ for all $a, \nu \in \mathbb{R}^2$ with $|\nu| = 1$, or in other words that this formula coincides with the extension by continuity of the formula for full-rank matrices. This is one of the new technical parts of the arguments used. The liminf inequality is then proved by a blow-up argument, using the two different characterizations, and the limsup inequality by using the correctors given by the homogenization formula.

Further properties of f_{hom}^p , which imply that f_{hom}^p is not convex, will be proved in Section 3, completing the proof of Theorem A.

2.1 Projections

In the case $p < 2$ every function $u \in W^{1,p}(\Omega, \mathbb{R}^2)$ can be “projected” to a $W^{1,p}$ function with values in $\varepsilon\mathcal{Z}$. This is done by locally projecting on the boundary of an ε -cube from a suitable point in the interior. In this section, we introduce the necessary notation.

For all $c \in \frac{1}{2}Q = (-1/4, 1/4)^2$ and $i \in \mathbb{Z}^2$ we set

$$Q_i^\varepsilon = \varepsilon i + \varepsilon Q, \quad x_c^{\varepsilon, i} = \varepsilon i + \varepsilon c,$$

and consider the projection from $x_c^{\varepsilon, i}$ to ∂Q_i^ε : $P_c^{\varepsilon, i}: Q_i^\varepsilon \setminus \{x_c^{\varepsilon, i}\} \rightarrow \partial Q_i^\varepsilon$ defined by

$$P_c^{\varepsilon, i}(x) = y \text{ such that } \{y\} = \{x_c^{\varepsilon, i} + t(x - x_c^{\varepsilon, i}) : t > 0\} \cap \partial Q_i^\varepsilon.$$

This projection is continuous in $Q_i^\varepsilon \setminus \{x_c^{\varepsilon, i}\}$ and smooth outside the four segments that join $x_c^{\varepsilon, i}$ to the vertices of Q_i^ε . Setting $P_c := P_c^{1,0}$, for fixed c we have

$$P_c^{\varepsilon, i}(x) = P_c^{\varepsilon, 0}(x - \varepsilon i) + \varepsilon i = \varepsilon P_c^{1, i}\left(\frac{x}{\varepsilon}\right) = \varepsilon i + \varepsilon P_c\left(\frac{x}{\varepsilon} - i\right),$$

so that, for x outside these segments,

$$|\nabla P_c^{\varepsilon, i}(x)| = \left| \nabla P_c\left(\frac{x}{\varepsilon} - i\right) \right| \leq \frac{5}{2\left|\frac{x}{\varepsilon} - i - c\right|} = \frac{5\varepsilon}{2|x - x_c^{\varepsilon, i}|},$$

where we used $|\nabla P_c(x)| \leq \frac{5}{2|x-c|}$. To prove this estimate, one can divide the square Q in the four triangles given by joining c with the four vertices of Q . In the interior of each triangle, its definition can be found explicitly; for example, in the right triangle

$P_c(x) = (\frac{1}{2}, c_2 + \frac{x_2 - c_2}{x_1 - c_1}(\frac{1}{2} - c_1))$, hence the gradient is immediate. By reversing roles and signs for the four triangles, it holds the (sharp) bound

$$|\nabla P_c(x)| \leq \frac{K(c)}{|x - c|},$$

$$K(c) := \max \left\{ \frac{(\frac{1}{2} + |c_2|)^2}{\frac{1}{2} - |c_1|} + \frac{1}{2} - |c_1|, \frac{(\frac{1}{2} + |c_1|)^2}{\frac{1}{2} - |c_2|} + \frac{1}{2} - |c_2| \right\} \leq \frac{5}{2},$$

which follows by choosing $|c_1| = |c_2| = \frac{1}{4}$, corresponding to the four corners of $\frac{1}{2}Q$.

Alternatively, for $c \in \frac{1}{2}Q$ and x out of the four segments joining c to the corners, let ν be the outer normal to the side of Q met by the ray from c through x , and let $h := \frac{1}{2} - c \cdot \nu \geq \frac{1}{4}$ be the distance of c from that side. Then $P_c(x) = c + \frac{h}{(x-c) \cdot \nu} (x - c)$, and a direct computation gives

$$|\nabla P_c(x)| = \frac{h |x - c|}{((x - c) \cdot \nu)^2} = \frac{|P_c(x) - c|^2}{h |x - c|} \leq \frac{9}{2} \frac{1}{|x - c|},$$

since $|P_c(x) - c| \leq \frac{3\sqrt{2}}{4}$.

Lemma 2.2 (projection on $\varepsilon\mathcal{Z}$). *For all $u \in W^{1,p}(\Omega, \mathbb{R}^2)$ there exists a function $P_\varepsilon(u) \in W^{1,p}(\Omega, \varepsilon\mathcal{Z})$ such that $\|P_\varepsilon(u) - u\|_\infty \leq \varepsilon$, $P_\varepsilon(u)(x) = u(x)$ if $u(x) \in \varepsilon\mathcal{Z}$, and*

$$\|\nabla P_\varepsilon(u)\|_p \leq C \|\nabla u\|_p,$$

with $C = C(p)$ depending only on p .

Proof. With fixed $c \in \frac{1}{2}Q$ we define

$$P_c^\varepsilon(u(x)) = P_c^{\varepsilon,i}(u(x)) \text{ if } u(x) \in Q_i^\varepsilon \setminus \{x_c^{\varepsilon,i}\}.$$

Note that there is no ambiguity if $\partial Q_i^\varepsilon \cap \partial Q_j^\varepsilon$ since in that case $P_c^\varepsilon(u(x)) = u(x)$. Moreover, if $u(x) \in Q_i^\varepsilon \setminus \{x_c^{\varepsilon,i}\}$ then

$$|\nabla P_c^\varepsilon(u)(x)| \leq \varepsilon \frac{C}{|u(x) - x_c^{\varepsilon,i}|} |\nabla u(x)|. \quad (10)$$

Let $\Omega_i^\varepsilon = \{x \in \Omega : u(x) \in Q_i^\varepsilon\}$. Then, since $|\frac{1}{2}Q| = \frac{1}{4}$,

$$\begin{aligned} 4 \int_{\frac{1}{2}Q} \int_{\Omega_i^\varepsilon} |\nabla P_c^\varepsilon(u)|^p dx dc &\leq C \varepsilon^p \int_{\frac{1}{2}Q} \int_{\Omega_i^\varepsilon} \frac{1}{|u(x) - x_c^{\varepsilon,i}|^p} |\nabla u|^p dx dc \\ &= C \varepsilon^p \int_{\Omega_i^\varepsilon} \int_{\frac{1}{2}Q} \frac{1}{|u(x) - x_c^{\varepsilon,i}|^p} dc |\nabla u|^p dx \leq C \int_{\Omega_i^\varepsilon} |\nabla u|^p dx. \end{aligned}$$

Note that here we use $p < 2$ so that $c \mapsto \frac{1}{|c|^p}$ is integrable and

$$\int_{\frac{1}{2}Q} \frac{1}{|u(x) - x_c^{\varepsilon,i}|^p} dc \leq C\varepsilon^{-p}$$

for all x . We then deduce that for all $i \in \mathbb{Z}^2$ there exists $c_i \in \frac{1}{2}Q$ such that

$$\int_{\Omega_i^\varepsilon} |\nabla P_{c_i}^\varepsilon(u)|^p dx \leq 4 \int_{\frac{1}{2}Q} \int_{\Omega_i^\varepsilon} |\nabla P_c^\varepsilon|^p dx dc \leq C \int_{\Omega_i^\varepsilon} |\nabla u|^p dx.$$

If we set

$$P_\varepsilon(u)(x) = P_{c_i}^{\varepsilon,i}(u(x)) \text{ if } u(x) \in Q_i^\varepsilon \quad (11)$$

then $P_\varepsilon(u)$ satisfies the desired estimate. $\square$

Definition 2.3 (projection of a linear function). *For all $A \in \mathbb{R}^{2 \times 2}$, we define the projection $u_A(x) = P_0(Ax)$, where P_0 is constructed as in (11) with $\varepsilon = 1$ and $c_i = 0$ for all i . In particular, denoting $Q_z := z + Q$, $z \in \mathbb{Z}^2$, the explicit form of P_0 reads*

$$P_0(x) = z + \frac{1}{2} \frac{x - z}{|x - z|_\infty}, \quad x \in \overline{Q}_z \setminus z. \quad (12)$$

We will also define the projection $\psi(x) := x/|x|_\infty$, $x \in \overline{Q} \setminus \{0\}$, so that

$$P_0(x) = z + \frac{1}{2} \psi(x - z), \quad x \in \overline{Q}_z \setminus z.$$

If $x \in \partial Q_z$ then we have $|x - z|_\infty = 1/2$, so that $P_0(x) = x$, P_0 coincides with the identity on the net, and P_0 is $\mathcal{Z}$ -valued. Moreover, for every $z \in \mathbb{Z}^2$ the bound $|\nabla P_0(x)| \leq |x - z|^{-1}$ holds for almost every $x \in z + Q$. Note that $P_0(Ax) \neq P_1(Ax)$ as given by Lemma 2.2, since the latter is defined with varying centers c_i with respect to the squares Q_i . The former is just given by the composition of P_0 and the linear function.

Lemma 2.4. *Given u_A as in Definition 2.3, and set $C_A := \int_Q |(\nabla P_0)(x)A|^p dx$, the rescaled function $\bar{u}_\varepsilon(x) := \varepsilon u_A(x/\varepsilon)$ satisfies*

$$\lim_{\varepsilon \rightarrow 0} \int_U |\nabla \bar{u}_\varepsilon|^p dx = C_A |U|$$

for all fixed bounded measurable subsets U .

Proof. By the change of variables

$$\int_U |\nabla \bar{u}_\varepsilon(x)|^p dx = \int_U |(\nabla P_0)(Ax/\varepsilon)A|^p dx = \frac{1}{|\det A|} \int_{AU} |(\nabla P_0)(x/\varepsilon)A|^p dx.$$

Since $x \mapsto |(\nabla P_0)(x/\varepsilon)A|^p$ is the rescaling of a $\mathbb{Z}^2$ -periodic function in $L^1(Q)$ (since $p < 2$), it converges weakly in L^1_{loc} to its average C_A . Hence,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_U |\nabla \bar{u}_\varepsilon|^p dx &= \lim_{\varepsilon \rightarrow 0} \frac{1}{|\det A|} \int_{AU} |(\nabla P_0)(x/\varepsilon)A|^p dx \\ &= \frac{|AU|}{|\det A|} \int_Q |(\nabla P_0)(x)A|^p dx = C_A |U| \end{aligned}$$

by the Riemann-Lebesgue lemma. $\square$

2.2 Modification of boundary conditions

As often is the case in variational problems, it will be useful to construct converging sequences satisfying suitable boundary conditions and with controlled energy. Usually, such sequences match the same boundary conditions as their limit. In this case, this is not possible, since we have varying constraints, and the result must be stated correspondingly.

Lemma 2.5. *Let $\bar{u}_\varepsilon \rightarrow u$ be such that $\bar{u}_\varepsilon \in W^{1,p}(\Omega, \varepsilon \mathcal{Z})$ and there exists a constant C such that*

$$\limsup_{\varepsilon \rightarrow 0} \int_U |\nabla \bar{u}_\varepsilon|^p dx \leq C|U|$$

for all fixed open sets U with Lipschitz boundary. Then for all $u_\varepsilon \in W^{1,p}(\Omega, \varepsilon \mathcal{Z})$ with $u_\varepsilon \rightarrow u$ there exists $v_\varepsilon \in W^{1,p}(\Omega, \varepsilon \mathcal{Z})$ with

$$\liminf_{\varepsilon \rightarrow 0} F_\varepsilon(v_\varepsilon) \leq \liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon),$$

and $v_\varepsilon = \bar{u}_\varepsilon$ on a neighbourhood of $\partial\Omega$.

Proof. It suffices to consider the case $\liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) < +\infty$, and prove that for all $\eta > 0$ we may find $v_\varepsilon \in W^{1,p}(\Omega, \varepsilon \mathcal{Z})$ with

$$\liminf_{\varepsilon \rightarrow 0} F_\varepsilon(v_\varepsilon) \leq (1 + \eta) \liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) + \eta$$

and $v_\varepsilon = \bar{u}_\varepsilon$ on a neighbourhood of $\partial\Omega$. For the construction of v_ε we may follow a well-known procedure by De Giorgi. With fixed $\delta > 0$ and $N \in \mathbb{N}$, $N > 0$, we consider the sets

$$\Omega_i^\delta = \left\{ x \in \Omega : \text{dist}(x, \partial\Omega) \leq \frac{i}{N} \delta \right\}$$

for $i \in \mathbb{N}$, and the functions

$$\varphi_i(x) = \left(\frac{N}{\delta} \text{dist}(x, \Omega_i^\delta) \right) \wedge 1.$$

We have $\varphi_i \in W^{1,\infty}(\Omega)$, $0 \leq \varphi_i \leq 1$, $\varphi_i = 0$ on Ω_i^δ and $\varphi_i = 1$ on $\Omega \setminus \Omega_{i+1}^\delta$; in particular, $\nabla \varphi_i \neq 0$ only in $\Omega_{i+1}^\delta \setminus \Omega_i^\delta$. We set $\Omega^\delta = \Omega_N^\delta$.

We consider the functions

$$w_\varepsilon^i = P_\varepsilon(\varphi_i u_\varepsilon + (1 - \varphi_i) \bar{u}_\varepsilon),$$

where P_ε is defined in Lemma 2.2. Note that $w_\varepsilon^i = u_\varepsilon$ on $\Omega \setminus \Omega_{i+1}^\delta$ and $w_\varepsilon^i = \bar{u}_\varepsilon$ on Ω_i^δ . Using Lemma 2.2, we can first estimate

$$\begin{aligned} \int_\Omega |\nabla w_\varepsilon^i|^p dx &= \int_{\Omega_i^\delta} |\nabla \bar{u}_\varepsilon|^p dx + \int_{\Omega_{i+1}^\delta \setminus \Omega_i^\delta} |\nabla w_\varepsilon^i|^p dx + \int_{\Omega \setminus \Omega_{i+1}^\delta} |\nabla u_\varepsilon|^p dx \\ &\leq \int_{\Omega_i^\delta} |\nabla \bar{u}_\varepsilon|^p dx + C_p \int_{\Omega_{i+1}^\delta \setminus \Omega_i^\delta} |\nabla(\varphi_i u_\varepsilon + (1 - \varphi_i) \bar{u}_\varepsilon)|^p dx \\ &\quad + \int_{\Omega \setminus \Omega_{i+1}^\delta} |\nabla u_\varepsilon|^p dx \\ &\leq \int_{\Omega_i^\delta} |\nabla \bar{u}_\varepsilon|^p dx + C_p \int_{\Omega_{i+1}^\delta \setminus \Omega_i^\delta} (|\nabla \varphi_i|^p |u_\varepsilon - \bar{u}_\varepsilon|^p + |\nabla \bar{u}_\varepsilon|^p + |\nabla u_\varepsilon|^p) dx \\ &\quad + \int_{\Omega \setminus \Omega_{i+1}^\delta} |\nabla u_\varepsilon|^p dx \\ &\leq C_p \int_{\Omega^\delta} |\nabla \bar{u}_\varepsilon|^p dx + C_p \int_{\Omega_{i+1}^\delta \setminus \Omega_i^\delta} |\nabla u_\varepsilon|^p dx \\ &\quad + C_p \int_{\Omega_{i+1}^\delta \setminus \Omega_i^\delta} \left(\frac{N}{\delta}\right)^p |u_\varepsilon - \bar{u}_\varepsilon|^p dx + \int_\Omega |\nabla u_\varepsilon|^p dx, \end{aligned}$$

with C_p a positive constant depending only on p . We can now define $w_\varepsilon = w_\varepsilon^i$, where $i \in \{1, \dots, N-1\}$ is such that

$$\int_{\Omega_{i+1}^\delta \setminus \Omega_i^\delta} |\nabla u_\varepsilon|^p dx \leq \frac{1}{N-1} \int_{\Omega^\delta} |\nabla u_\varepsilon|^p dx \leq \frac{1}{N-1} \int_\Omega |\nabla u_\varepsilon|^p dx,$$

so that we have

$$\begin{aligned} \int_\Omega |\nabla w_\varepsilon|^p dx &\leq C_p \int_{\Omega^\delta} |\nabla \bar{u}_\varepsilon|^p dx \\ &\quad + \left(1 + \frac{C_p}{N-1}\right) \int_\Omega |\nabla u_\varepsilon|^p dx + C_p \int_\Omega \left(\frac{N}{\delta}\right)^p |u_\varepsilon - \bar{u}_\varepsilon|^p dx. \end{aligned}$$

By the hypotheses on $\bar{u}_\varepsilon$, passing to the liminf as $\varepsilon \rightarrow 0$ we obtain

$$\liminf_{\varepsilon \rightarrow 0} \int_\Omega |\nabla w_\varepsilon|^p dx \leq C|\Omega^\delta| + \left(1 + \frac{C_p}{N-1}\right) \liminf_{\varepsilon \rightarrow 0} \int_\Omega |\nabla u_\varepsilon|^p dx,$$

which produces the desired estimate, choosing δ and N appropriately. $\square$

Note that the assumption on $\bar{u}_\varepsilon$ of Lemma 2.5 is satisfied by the rescaled function $\varepsilon u_A(x/\varepsilon)$ in Lemma 2.4. This fact will be used repeatedly in the following.

2.3 Paths in the network

In this section $\varepsilon > 0$ is fixed, and we analyze paths with values in $\varepsilon\mathcal{Z}$. Such paths will be interpreted as sections of linear functions in the rank-one case.

We call *vertices* of $\varepsilon\mathcal{Z}$ the points of $\varepsilon(\frac{1}{2} + \mathbb{Z})^2$, and for $w \in \mathbb{R}^2$ we write $|w|_1 := |w_1| + |w_2|$, so that $|w| \leq |w|_1 \leq \sqrt{2}|w|$. The network $\varepsilon\mathcal{Z}$ contains the horizontal and the vertical line through each of its vertices; every point of $\mathbb{R}^2$ has a vertex at $\cdot|_1$ -distance at most ε ; and, for every vertex β , the reflections $\xi \mapsto (2\beta_1 - \xi_1, \xi_2)$ and $\xi \mapsto (\xi_1, 2\beta_2 - \xi_2)$ map $\varepsilon\mathcal{Z}$ onto itself. Every path in $\varepsilon\mathcal{Z}$ from a vertex β to a vertex β' has length at least $|\beta' - \beta|_1$. We construct paths that attain this length and stay at distance of order ε from the segment $[\beta, \beta']$, with no condition on its direction. They are used with $\varepsilon = 1$ in Section 2, for the continuity of f_{hom}^p and for the construction of laminates, and in Section 4 for the recovery sequences.

Lemma 2.6. *Let β and β' be vertices of $\varepsilon\mathcal{Z}$, let $a := \beta' - \beta$, and let $L > 0$. There exists a piecewise-affine map $v: [0, L] \rightarrow \varepsilon\mathcal{Z}$, with monotone components, $v(0) = \beta$ and $v(L) = \beta'$, such that:*

(i) *for almost every $t \in (0, L)$ either $v'(t) = \frac{|a|_1}{L} \text{sign}(a_1) e_1$ or $v'(t) = \frac{|a|_1}{L} \text{sign}(a_2) e_2$, and, if $a \neq 0$, the first alternative holds on a set of measure $\frac{|a_1|}{|a|_1} L$; in particular,*

$$|v(s) - v(t)| \leq |v(s) - v(t)|_1 = \frac{|a|_1}{L} |s - t| \quad \text{for all } s, t \in [0, L];$$

(ii) $\left|v(t) - \beta - \frac{t}{L} a\right| \leq \sqrt{2}\varepsilon$ for all $t \in [0, L]$;

(iii) $\int_0^L |v'(t)|^p dt = \frac{|a|_1^p}{L^{p-1}}$ for every $p \geq 1$.

Proof. If $a_1 a_2 = 0$, the segment $[\beta, \beta']$ lies on one of the lines of $\varepsilon\mathcal{Z}$ through β , and $v(t) := \beta + \frac{t}{L} a$ has all the properties, with 0 in place of $\sqrt{2}\varepsilon$ in (ii). Otherwise, after one or two reflections about β , we may suppose that $a_1, a_2 > 0$, and we write $a = \varepsilon(m_1, m_2)$, with $m_1, m_2 \in \mathbb{N}$.

For $k \in \{0, 1, \dots, m_1\}$, let $x_k := k$ and let y_k be the integer closest to km_2/m_1 (the smaller one in the case of a tie), so that $y_0 = 0$, $y_{m_1} = m_2$, $k \mapsto y_k$ is nondecreasing, and $|m_1 y_k - m_2 x_k| \leq m_1/2$. Let γ be the polygonal path through the points

$$(x_0, y_0), (x_1, y_0), (x_1, y_1), (x_2, y_1), \dots, (x_{m_1}, y_{m_1-1}), (x_{m_1}, y_{m_1}) = (m_1, m_2),$$

which alternates horizontal steps of length 1 with vertical steps of length $y_k - y_{k-1} \geq 0$ (see Fig. 1). The path $\beta + \varepsilon\gamma$ lies on the horizontal and vertical lines through the vertices $\beta + \varepsilon\mathbb{Z}^2$, hence in $\varepsilon\mathcal{Z}$. Since the components of γ are nondecreasing, its length is $m_1 + m_2$; we parameterize it by arclength, $\sigma \mapsto \gamma(\sigma)$, $\sigma \in [0, m_1 + m_2]$, and set

$$v(t) := \beta + \varepsilon\gamma\left(\frac{m_1 + m_2}{L} t\right), \quad t \in [0, L].$$

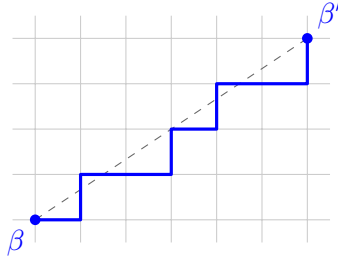


Figure 1: The staircase of Lemma 2.6 for $\beta' - \beta = \varepsilon(6, 4)$. The grid lines, at mutual distance ε , belong to $\varepsilon\mathcal{Z}$; the dashed line is the segment $[\beta, \beta']$.

Then $v' = \frac{|a|_1}{L} e_1$ on the horizontal steps and $v' = \frac{|a|_1}{L} e_2$ on the vertical ones, and integrating the first component over $[0, L]$ shows that the horizontal steps take a time of measure $\frac{|a|_1}{|a|_1} L$. Since the components of v are monotone and, at almost every time, only one of them changes, $|v(t) - v(s)|_1 = \int_s^t |v'| d\tau = \frac{|a|_1}{L} (t - s)$ for $s \leq t$. This proves (i), and (iii) follows from $|v'| = \frac{|a|_1}{L}$ almost everywhere.

As for (ii), let $t \in [0, L]$ and $(x, y) := \gamma((m_1 + m_2)t/L)$. Since γ has nondecreasing components, $x + y = (m_1 + m_2)t/L$, so that

$$\beta + \frac{t}{L} a - v(t) = \varepsilon \left(\frac{x + y}{m_1 + m_2} (m_1, m_2) - (x, y) \right) = \frac{\varepsilon (m_1 y - m_2 x)}{m_1 + m_2} (1, -1).$$

On each step of γ the quantity $m_1 y - m_2 x$ is an affine function of t , so that it suffices to estimate it at the points listed above. At (x_k, y_k) we have $|m_1 y_k - m_2 x_k| \leq m_1/2$, and at (x_k, y_{k-1}) , $k \geq 1$, we have $|m_1 y_{k-1} - m_2 x_k| = |m_1 y_{k-1} - m_2 x_{k-1} - m_2| \leq m_1/2 + m_2$. Hence $|m_1 y - m_2 x| \leq m_1 + m_2$, and (ii) follows. $\square$

2.4 Homogenized energy functions

We now introduce the homogenized energy density that will allow us to describe the Γ -limit. It is defined differently on matrices of rank 2 and on matrices of rank 1.

Definition 2.7. For all $A \in \mathbb{R}^{2 \times 2}$ with $\text{rank}(A) = 2$, we define

$$f_{\text{hom}}^p(A) = \liminf_{T \rightarrow +\infty} \frac{|\det A|}{T^2} \inf \left\{ \int_{TQ_A} |\nabla u|^p dx : u \in W^{1,p}(TQ_A, \mathcal{Z}), \right. \\ \left. u = u_A \text{ in a neighbourhood of } \partial TQ_A \right\}, \quad (13)$$

where u_A is as in Definition 2.3 and $Q_A = A^{-1}(Q)$ (see Fig. 2).

If $\text{rank}(A) \leq 1$ then we set

$$f_{\text{hom}}^p(A) = (|a_1| + |a_2|)^p,$$

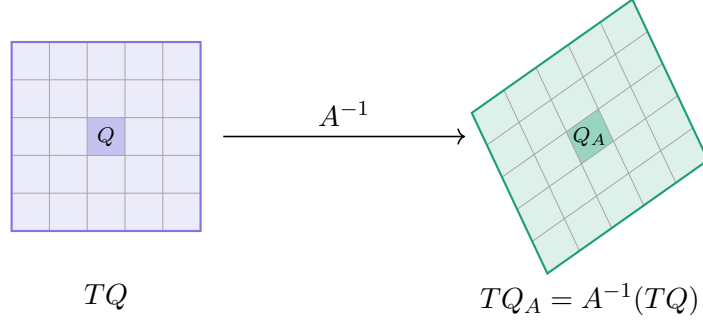


Figure 2: The domain $TQ_A = A^{-1}(TQ)$ of the test functions in the definition of f_{hom}^p .

where $A = a \otimes \nu$, with $\nu \in S^1$ and $a = (a_1, a_2)$. Equivalently, for any rank-one matrix $A = a \otimes \xi$ we set

$$f_{\text{hom}}^p(A) = (|A_1| + |A_2|)^p = (|a_1| + |a_2|)^p |\xi|^p,$$

where $A_i = a_i \xi$ are the rows of A .

Although this property is not immediate, f_{hom}^p is p -homogeneous, as proved in Lemma 2.12 below.

In the following, when necessary to shorten the notation, for fixed $T > 0$ and A non-singular, $\mathcal{A}_T(A)$ will denote the set of competitors in the infimum problem in (13).

Remark 2.8 (Estimate of f_{hom}^p on the identity map $A = I$). Take $u^T(x) = P_0(x)$ to test $f_{\text{hom}}^p(I)$, for fixed $T > 0$. Then

$$\nabla u^T(x) = (\nabla P_0)(x) = \frac{1}{2}(\nabla \psi)(x - z), \quad x = z + \left[-\frac{1}{2}, \frac{1}{2}\right]^2 \setminus \{0\}.$$

Suppose for simplicity that T is odd, so that we cover an exact number of cells without some partial cells. Then, since $A = I$ gives $TQ_A = TQ$,

$$\begin{aligned} f_{\text{hom}}^p(I) &\leq \frac{1}{T^2} \int_{TQ} |\nabla u_\varepsilon|^p dx = \frac{1}{T^2} \sum_{i=1}^{T^2} \int_{Q_z} \frac{1}{2^p} |\nabla \psi(x - z)|^p dx \\ &= \frac{1}{T^2} \sum_{i=1}^{T^2} \int_Q \frac{1}{2^p} |\nabla \psi(y)|^p dy = \frac{1}{2^p} \int_Q |\nabla \psi(y)|^p dy. \end{aligned}$$

By symmetry, we can divide Q into the four triangles each spanned by an angle of $\pi/2$, and compute the integral only for one of them. Let $\Delta := \{0 < y_1 < 1/2, |y_2| < y_1\}$ be the first triangle, then

$$\psi(y_1, y_2) = \left(1, \frac{y_2}{y_1}\right), \quad \nabla \psi(y_1, y_2) = \begin{pmatrix} 0 & 0 \\ -\frac{y_2}{y_1^2} & \frac{1}{y_1} \end{pmatrix},$$

(note $\nabla\psi \in K$ has $\det \nabla\psi = 0$, see Lemma 2.18), and

$$|\nabla\psi| = \frac{|y|}{y_1^2}.$$

In polar coordinates $|\nabla\psi|(r, \theta) = 1/(r \cos^2 \theta)$, with domain $|\theta| < \pi/4$ and $0 < r < \frac{1}{2 \cos \theta}$. In this domain $\cos^2 \theta > 1/2$, so that $|\nabla\psi| \leq 1/r \in L_{\text{loc}}^p$ for $p < 2$. We can now compute

$$\frac{1}{2^p} \int_Q |\nabla\psi(y)|^p dy = \frac{4}{2^p} \int_{\Delta} |\nabla\psi(y)|^p dy$$

$$\int_{\Delta} |\nabla\psi(y)|^p dy = \int_{-\pi/4}^{\pi/4} \frac{1}{\cos^{2p} \theta} \int_0^{\frac{1}{2 \cos \theta}} r^{1-p} dr d\theta = \frac{1}{(2-p)2^{2-p}} \int_{-\pi/4}^{\pi/4} \cos^{-(p+2)} \theta d\theta.$$

As $1 < p < 2$ and in $|\theta| < \pi/4$ it holds $1/\sqrt{2} < \cos \theta \leq 1$, the integral above is strictly increasing in p and we have the bounds (attained at the endpoints $p = 1$ and $p = 2$)

$$\sqrt{2} + \ln(1 + \sqrt{2}) < \int_{-\pi/4}^{\pi/4} \cos^{-(p+2)} \theta d\theta < \frac{8}{3}, \quad 1 < p < 2.$$

Therefore

$$f_{\text{hom}}^p(I) \leq \frac{4}{2^p} \int_{\Delta} |\nabla\psi(y)|^p dy \leq \frac{8}{3(2-p)}. \quad (14)$$

This estimate will be useful for deriving an upper bound of f_{hom}^p in Proposition 3.5.

The following lemma states that the lower limit in (13) is indeed a limit. In the rest of the section we use the shorthand $u = u_A \sim \partial TQ$ to mean $u = u_A$ in a neighbourhood of ∂TQ .

Lemma 2.9. *The liminf in definition (13) is actually a limit; that is,*

$$f_{\text{hom}}^p(A) = \lim_{T \rightarrow +\infty} \frac{|\det A|}{T^2} \inf \left\{ \int_{TQ_A} |\nabla u|^p dx : u \in W^{1,p}(TQ_A, \mathcal{Z}), \ u = u_A \sim \partial TQ_A \right\}.$$

Proof. Fix $A \in \mathbb{R}^{2 \times 2}$ nonsingular and define

$$g_T := \frac{|\det A|}{T^2} \inf \left\{ \int_{TQ_A} |\nabla u|^p dx : u \in W^{1,p}(TQ_A, \mathcal{Z}), \ u = u_A \sim \partial TQ_A \right\}.$$

Let u_T satisfy the boundary conditions of the infimum and be such that

$$\frac{|\det A|}{T^2} \int_{TQ_A} |\nabla u_T|^p dx \leq g_T + \frac{1}{T}.$$

Set $N_T := \lfloor T \rfloor + 1$, $w_T := N_T - T \in (0, 1]$, and, for $S \geq T$, $m := \lfloor \frac{S-T}{2N_T} \rfloor$, and $\mathbf{I} := \{-m, \dots, m\}^2$. Then $\#\mathbf{I} = (2m+1)^2$, and the cells $A^{-1}(N_T \mathbf{i}) + TQ_A$, $\mathbf{i} \in \mathbf{I}$, are pairwise disjoint and contained in SQ_A . Define

$$u_S(x) = \begin{cases} u_T(x - A^{-1}(N_T \mathbf{i})) + N_T \mathbf{i} & \text{if } x \in A^{-1}(N_T \mathbf{i}) + TQ_A, \mathbf{i} \in \mathbf{I} \\ u_A(x) & \text{otherwise in } SQ_A. \end{cases}$$

Since $P_0(y+z) = P_0(y) + z$ for $z \in \mathbb{Z}^2$, we have $u_A(x - A^{-1}(N_T \mathbf{i})) + N_T \mathbf{i} = u_A(x)$, so $u_S = u_A$ in a neighbourhood of the boundary of each cell; as $\mathcal{Z} + \mathbb{Z}^2 = \mathcal{Z}$, it follows that $u_S \in W^{1,p}(SQ_A, \mathcal{Z})$ and $u_S = u_A$ near ∂SQ_A .

We now estimate the energy of u_A on $L_{S,T} := SQ_A \setminus \bigcup_{\mathbf{i} \in \mathbf{I}} (A^{-1}(N_T \mathbf{i}) + TQ_A)$. Since $P_0(y+z) = P_0(y) + z$, the function $|(\nabla P_0)(y)A|^p$ is $\mathbb{Z}^2$ -periodic, so that

$$\int_{z+Q} |(\nabla P_0)(y)A|^p dy = C_A \quad \text{for every } z \in \mathbb{Z}^2,$$

with C_A as in Lemma 2.4. In the variable $y = Ax$ we have $A(L_{S,T}) \subset (G \times I_S) \cup (I_S \times G)$, where $I_S := (-\frac{S}{2}, \frac{S}{2})$ and

$$G := I_S \setminus \bigcup_{i=-m}^m \left(N_T i - \frac{T}{2}, N_T i + \frac{T}{2} \right)$$

is the union of $2m$ intervals of length $w_T \leq 1$ and of two intervals of length less than N_T . Since an interval of length L is covered by at most $\lceil L \rceil + 1$ intervals $[l - \frac{1}{2}, l + \frac{1}{2}]$, $l \in \mathbb{Z}$ (it may straddle their endpoints), G is covered by at most $2 \cdot 2m + 2(N_T + 1) = 4m + 2N_T + 2$ of them and I_S by at most $\lceil S \rceil + 1 \leq S + 2$, so that $A(L_{S,T})$ is covered by at most $2(4m + 2N_T + 2)(S + 2)$ squares $z + Q$, $z \in \mathbb{Z}^2$. Therefore

$$\begin{aligned} \frac{|\det A|}{S^2} \int_{L_{S,T}} |\nabla u_A|^p dx &= \frac{1}{S^2} \int_{A(L_{S,T})} |(\nabla P_0)(y)A|^p dy \\ &\leq r(S, T) := 2C_A \frac{(4m + 2N_T + 2)(S + 2)}{S^2}. \end{aligned}$$

We can now estimate g_S using u_S as a test function; since the constants $N_T \mathbf{i}$ do not affect

the gradient,

$$\begin{aligned}
g_S &\leq \frac{|\det A|}{S^2} \int_{SQ_A} |\nabla u_S|^p dx \\
&= \frac{|\det A|}{S^2} \left(\sum_{\mathbf{i} \in \mathbf{I}} \int_{A^{-1}(N_T \mathbf{i}) + TQ_A} |\nabla u_T(x - A^{-1}(N_T \mathbf{i}))|^p dx + \int_{L_{S,T}} |\nabla u_A|^p dx \right) \\
&\leq \frac{|\det A|}{S^2} (2m+1)^2 \int_{TQ_A} |\nabla u_T|^p dx + r(S, T) \\
&\leq \frac{T^2(2m+1)^2}{S^2} \left(g_T + \frac{1}{T} \right) + r(S, T) \\
&\leq g_T + \frac{1}{T} + r(S, T),
\end{aligned}$$

where in the last inequality we used $T(2m+1) \leq \frac{T}{N_T}(S-T)+T \leq S$. Since $m/S \rightarrow 1/(2N_T)$ as $S \rightarrow +\infty$,

$$\limsup_{S \rightarrow +\infty} g_S \leq g_T + \frac{1}{T} + \frac{4C_A}{N_T} \quad \text{for every } T > 0.$$

Taking the lower limit as $T \rightarrow +\infty$, and recalling that $N_T > T$, we obtain $\limsup_{S \rightarrow +\infty} g_S \leq \liminf_{T \rightarrow +\infty} g_T$, which proves the claim. $\square$

Remark 2.10. We use the tilted squares $TQ_A = A^{-1}(TQ)$ in the cell formula (13) because of the upper bound in Proposition 2.13, where we need a recovery sequence whose gradient is periodic, in order to apply the Riemann–Lebesgue lemma. The reason is the boundary datum $u_A(x) = P_0(Ax)$, which projects the affine function Ax on the $\mathbb{Z}^2$ -periodic grid $\mathcal{Z}$ in the target space: since $P_0(y+z) = P_0(y) + z$ for $z \in \mathbb{Z}^2$, we have $u_A(x + A^{-1}z) = u_A(x) + z$, so that $u_A(x) - Ax$ is periodic with respect to the tilted lattice $A^{-1}\mathbb{Z}^2$ in the domain, whose cells $A^{-1}(z + Q)$ are mapped by A onto the squares $z + Q$ of the grid. Hence, for $T \in \mathbb{N}$, a competitor on TQ_A that equals u_A near ∂TQ_A can be translated by the vectors $A^{-1}(T\mathbf{k})$, $\mathbf{k} \in \mathbb{Z}^2$, adding $T\mathbf{k}$ to its values, and the copies glue into a function whose gradient is periodic with periodicity cell TQ_A . By Lemma 2.11, the squares TQ give the same value $f_{\text{hom}}^p(A)$, so the tilted squares are only a matter of convenience.

Lemma 2.11. *Let $\text{rank}(A) = 2$ and define*

$$f_{\text{hom}}^{\#,p}(A) := \liminf_{T \rightarrow +\infty} \frac{1}{T^2} \inf \left\{ \int_{TQ} |\nabla u|^p dx : u \in W^{1,p}(TQ, \mathcal{Z}), u = u_A \sim \partial TQ \right\}.$$

Then $f_{\text{hom}}^{\#,p}(A) = f_{\text{hom}}^p(A)$.

Proof. Denote

$$g_T^\#(A) = \frac{1}{T^2} \inf \left\{ \int_{TQ} |\nabla u|^p dx : u \in W^{1,p}(TQ, \mathcal{Z}), u = u_A \sim \partial TQ \right\},$$

$$g_T(A) = \frac{|\det A|}{T^2} \inf \left\{ \int_{TQ_A} |\nabla u|^p dx : u \in W^{1,p}(TQ_A, \mathcal{Z}), u = u_A \sim \partial TQ_A \right\}.$$

We now prove that for $T \in \mathbb{N}$ and $S \gg T$

$$g_S^\#(A) \leq g_T(A) + o_T(1) + C \frac{T}{S},$$

where C depends on A and p . Since the liminf defining $f_{\text{hom}}^p(A)$ is a limit, this will give, taking the liminf first for $S \rightarrow \infty$ and then for $T \rightarrow \infty$ along the integers, the first inequality $f_{\text{hom}}^{\#,p}(A) \leq f_{\text{hom}}^p(A)$.

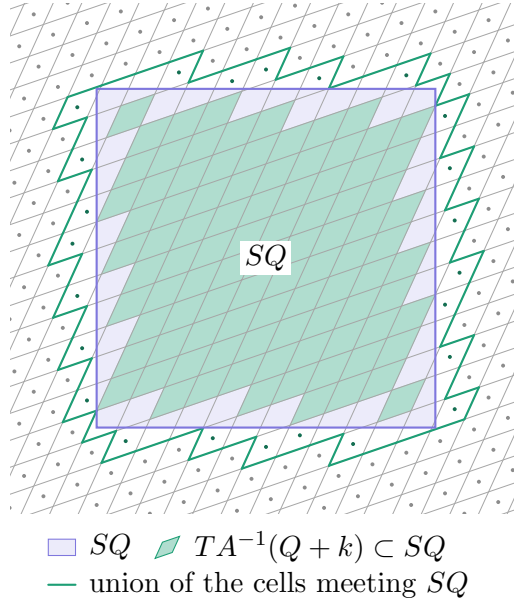


Figure 3: From lozenges to squares

Let u_T be a quasi-minimum for g_T

$$\frac{|\det A|}{T^2} \int_{TQ_A} |\nabla u_T|^p dx \leq g_T + \frac{1}{T}.$$

Referring to Fig. 3, we want to fill in the big square of size S with parallelograms TQ_A and estimate the energy of u_A on the remaining set, where some parallelograms do not

fit completely in the square; the thickness of this boundary layer is at most $\delta_T := T \cdot \text{diam}(Q_A) \leq T\sqrt{2}|A^{-1}|$, since $\sqrt{2} = \text{diam}(Q)$.

Let $\mathbf{I}$ be the set of indices $\mathbf{i} = (i_1, i_2) \in T\mathbb{Z}^2$ for which $A^{-1}(\mathbf{i}) + TQ_A \subset SQ$ (that is, the T -parallelograms are well contained). We define u_S as

$$u_S(x) = \begin{cases} u_T(x - A^{-1}(\mathbf{i})) + \mathbf{i} & \text{if } x \in A^{-1}(\mathbf{i}) + TQ_A \text{ for some } \mathbf{i} \in \mathbf{I} \\ u_A(x) = P_0(Ax) & \text{if } x \in L_{S,T} := SQ \setminus \bigcup_{\mathbf{i} \in \mathbf{I}} (A^{-1}(\mathbf{i}) + TQ_A). \end{cases}$$

Since $\mathbf{i} \in T\mathbb{Z}^2 \subset \mathbb{Z}^2$ and $P_0(y+z) = P_0(y) + z$ for $z \in \mathbb{Z}^2$, we have $u_A(x - A^{-1}(\mathbf{i})) + \mathbf{i} = u_A(x)$, so that $u_S = u_A$ in a neighbourhood of the boundary of each parallelogram; as $\mathbb{Z} + \mathbb{Z}^2 = \mathbb{Z}$, it follows that $u_S \in W^{1,p}(SQ, \mathbb{Z})$ and $u_S = u_A$ near ∂SQ .

Since ∇u_A is not bounded, we estimate its energy on $L_{S,T}$ by covering $L_{S,T}$ with whole parallelograms. The parallelograms $A^{-1}(\mathbf{i}) + TQ_A$, $\mathbf{i} \in T\mathbb{Z}^2$, are pairwise disjoint, so that $L_{S,T}$ is contained, up to a null set, in the union of those that meet SQ without being contained in it. Each of these meets ∂SQ ; hence it lies in $\{x : \text{dist}(x, \partial SQ) \leq \delta_T\}$, whose measure is at most $8S\delta_T$ for $S \geq 2\delta_T$ (as it is made by two concentric squares); since $|TQ_A| = T^2/|\det A|$, their number is at most

$$\frac{8S\delta_T}{|TQ_A|} \leq 8\sqrt{2}|\det A| |A^{-1}| \frac{S}{T}.$$

Moreover, since $T \in \mathbb{N}$, each square $\mathbf{i} + TQ$ is, up to a null set, the union of T^2 unit squares, on each of which the $\mathbb{Z}^2$ -periodic function $|(\nabla P_0)(y)A|^p$ has integral C_A , with C_A as in Lemma 2.4. Hence

$$\int_{A^{-1}(\mathbf{i}) + TQ_A} |\nabla u_A|^p dx = \frac{1}{|\det A|} \int_{\mathbf{i} + TQ} |(\nabla P_0)(y)A|^p dy = \frac{T^2}{|\det A|} C_A,$$

and therefore

$$\int_{L_{S,T}} |\nabla u_A|^p dx \leq 8\sqrt{2} C_A |A^{-1}| S T.$$

We can also estimate the cardinality of $\mathbf{I}$ as $\#\mathbf{I} \leq \frac{S^2}{|TQ_A|} = |\det A| \frac{S^2}{T^2}$. Then

$$\begin{aligned} g_S^\# &\leq \frac{1}{S^2} \int_{SQ} |\nabla u_S|^p dx \\ &= \frac{1}{S^2} \left(\sum_{\mathbf{i} \in \mathbf{I}} \int_{A^{-1}(\mathbf{i}) + TQ_A} |\nabla u_T(x - A^{-1}(\mathbf{i}))|^p dx + \int_{L_{S,T}} |\nabla u_A|^p dx \right) \\ &\leq \frac{1}{S^2} \left(\#\mathbf{I} \int_{TQ_A} |\nabla u_T|^p dx + 8\sqrt{2} C_A |A^{-1}| S T \right) \\ &\leq \frac{|\det A|}{T^2} \int_{TQ_A} |\nabla u_T|^p dx + 8\sqrt{2} C_A |A^{-1}| \frac{T}{S} \\ &\leq g_T + \frac{1}{T} + 8\sqrt{2} C_A |A^{-1}| \frac{T}{S}, \end{aligned}$$

which gives the claim with $C = 8\sqrt{2}C_A|A^{-1}|$.

For the opposite inequality we reverse the roles of squares and parallelograms, with one difference: a translated copy $u_T^\#(x - \tau) + A\tau$ of a quasi-minimum $u_T^\#$ for $g_T^\#$ agrees with u_A near the boundary of $\tau + TQ$ only if $\tau \in A^{-1}\mathbb{Z}^2$, and the translates of TQ by the lattice $A^{-1}\mathbb{Z}^2$ do not tile the plane. We therefore set $d := \text{diam}(Q_A)$ and, for every $\mathbf{j} \in \mathbb{Z}^2$, choose $\mathbf{k}_\mathbf{j} \in \mathbb{Z}^2$ with $|A^{-1}\mathbf{k}_\mathbf{j} - (T + d)\mathbf{j}| \leq d/2$, which is possible since the cells $A^{-1}(\mathbf{k} + Q)$ cover the plane and have diameter d . The squares $A^{-1}\mathbf{k}_\mathbf{j} + TQ$ are then contained in $(T + d)(\mathbf{j} + Q)$, hence pairwise disjoint, and we place in them the copies $u_T^\#(x - A^{-1}\mathbf{k}_\mathbf{j}) + \mathbf{k}_\mathbf{j}$ for all $\mathbf{j}$ with $(T + d)(\mathbf{j} + Q) \subset SQ_A$, setting $u_S = u_A$ elsewhere in SQ_A . The frames $(T + d)(\mathbf{j} + Q) \setminus (A^{-1}\mathbf{k}_\mathbf{j} + TQ)$ have width at most d , so covering them by cells $A^{-1}(z + Q)$, $z \in \mathbb{Z}^2$, and using periodicity as above, their total energy is at most CdS^2/T , while the boundary layer of SQ_A is treated as before. This gives

$$g_S \leq g_T^\# + \frac{1}{T} + C\left(\frac{d}{T} + \frac{T}{S}\right),$$

with C depending on A and p ; letting first $S \rightarrow +\infty$ and then taking the lower limit as $T \rightarrow +\infty$, we obtain $f_{\text{hom}}^p(A) \leq f_{\text{hom}}^{\#,p}(A)$. $\square$

Denote for simplicity $\mathcal{R} := \{A \in \mathbb{R}^{2 \times 2} : \det A \neq 0\}$. For $A \in \mathcal{R}$, we recall the definition of f_{hom}^p in (2.7) in a shortened notation,

$$f_{\text{hom}}^p(A) = \lim_{T \rightarrow \infty} g_T(A), \quad g_T(A) := \inf \left\{ \frac{1}{|TQ_A|} \int_{TQ_A} |\nabla v|^p dx : v \in \mathcal{A}_T(A) \right\},$$

where $\mathcal{A}_T(A)$ is the already defined set of competitors $v \in W^{1,p}(TQ_A, \mathbb{R}^2)$ with $v(x) \in \mathcal{Z}$ for almost every x and $v(x) = P_0(Ax)$ on ∂TQ_A . For $\det A = 0$, $f_{\text{hom}}^p(A) = N(A)^p$, where $N(A) = |A_1| + |A_2|$ with A_i the rows of A .

Lemma 2.12 (p -homogeneity). *For every $A \in \mathbb{R}^{2 \times 2}$ and $s \in \mathbb{R} \setminus \{0\}$,*

$$f_{\text{hom}}^p(sA) = |s|^p f_{\text{hom}}^p(A).$$

Proof. If $\det A = 0$, then $f_{\text{hom}}^p(sA) = N(sA)^p = |s|^p N(A)^p = |s|^p f_{\text{hom}}^p(A)$.

Let $\det A \neq 0$ and $s > 0$. Since $TQ_{sA} = s^{-1}TQ_A$ and dilations preserve traces, the map $v \mapsto w$, $w(y) := v(y/s)$, is a bijection from $\mathcal{A}_T(sA)$ onto $\mathcal{A}_T(A)$: indeed, w takes values in $\mathcal{Z}$, and on ∂TQ_A we have $w(y) = P_0(sAy/s) = P_0(Ay)$. Moreover $\nabla w(y) = s^{-1}\nabla v(y/s)$ and $|TQ_A| = s^2|TQ_{sA}|$, so that

$$\frac{1}{|TQ_A|} \int_{TQ_A} |\nabla w|^p dy = s^{-p} \frac{1}{|TQ_{sA}|} \int_{TQ_{sA}} |\nabla v|^p dx.$$

Hence $g_T(sA) = s^p g_T(A)$ for every T , and letting $T \rightarrow \infty$ gives the claim for $s > 0$.

For $s = -1$: since $Q = -Q$, we have $TQ_{-A} = TQ_A$. Since $\mathcal{Z} = -\mathcal{Z}$ and $P_0(-y) = -P_0(y)$, the map $v \mapsto -v$ is an energy-preserving bijection from $\mathcal{A}_T(A)$ onto $\mathcal{A}_T(-A)$. Thus $g_T(-A) = g_T(A)$, and the case $s < 0$ follows by combining the two. $\square$

2.5 Proof of the lower bound in the homogenization theorem

The proof of Theorem 2.1 is subdivided into this section and the following one for the lower and the upper limits, respectively.

The proof of the lower bound can be achieved by the blow-up argument of Fonseca and Müller [33]. Its formalization adapted to homogenization problems can be found in [20], to which we refer for details. In this particular case, we note that the domain of the Γ -limit is trivially contained in $W^{1,p}(\Omega, \mathbb{R}^2)$. In order to prove the lower bound, we fix $u_\varepsilon \rightarrow u$ weakly in $W^{1,p}(\Omega, \mathbb{R}^2)$ with $\liminf_{\varepsilon \rightarrow 0} F_\varepsilon^p(u_\varepsilon) < +\infty$. Upon extracting subsequences, we can suppose that this liminf is actually a limit, and that the measures

$$\mu_\varepsilon(B) = \int_B |\nabla u_\varepsilon|^p dx$$

weakly*-converge to some μ . It suffices then to prove that for $\mathcal{L}^2$ -almost every x_0 we have

$$\frac{d\mu}{dx}(x_0) \geq f_{\text{hom}}^p(\nabla u(x_0)). \quad (15)$$

Select a point $x_0 \in \Omega$ which is a Lebesgue point of u and ∇u , a point of approximate differentiability of u , a point of almost everywhere convergence of $u_\varepsilon \rightarrow u$, and such that the Radon-Nikodým derivative of μ with respect to $\mathcal{L}^2$ exists and is finite. Note that $\mathcal{L}^2$ -almost every $x_0 \in \Omega$ satisfy these properties. For simplicity of notation, upon a translation, we can suppose that $x_0 = 0$. Denote $A = \nabla u(0)$.

We first treat the case of $\text{rank}(A) = 2$. Write

$$\frac{d\mu}{dx}(0) = \lim_{\rho \rightarrow 0} \frac{\mu(\rho Q)}{\rho^2} = \lim_{\rho \rightarrow 0} \liminf_{\varepsilon \rightarrow 0} \frac{\mu_\varepsilon(\rho Q)}{\rho^2} = \lim_{\rho \rightarrow 0} \liminf_{\varepsilon \rightarrow 0} \frac{1}{\rho^2} \int_{\rho Q} |\nabla u_\varepsilon|^p dx,$$

noting that for this liminf is actually a limit except for a countable number of ρ . We can then find $\varepsilon = \varepsilon(\rho)$ such that

$$\frac{d\mu}{dx}(0) = \lim_{\rho \rightarrow 0} \frac{1}{\rho^2} \int_{\rho Q} |\nabla u_\varepsilon|^p dx.$$

We can then consider the scaled functions

$$v_\varepsilon(y) = \frac{1}{\rho} u_\varepsilon(\rho y) - \frac{\varepsilon}{\rho} \left\lfloor \frac{u(0)}{\varepsilon} \right\rfloor, \quad y \in Q$$

so that v_ε takes values in $\frac{\varepsilon}{\rho} \mathbb{Z}$ (and the integer translation is irrelevant) and

$$\frac{d\mu}{dx}(0) = \lim_{\rho \rightarrow 0} \int_Q |\nabla v_\varepsilon|^p dy.$$

We have that v_ε converges to $v(y) = Ay$ in $L^p(Q, \mathbb{R}^2)$, since 0 is a point of approximate differentiability so that

$$\tilde{v}_\varepsilon(y) = \frac{u_\varepsilon(\rho y) - u(0)}{\rho}$$

tends to Ay in $L^p(Q, \mathbb{R}^2)$, see e.g. Remark 3.72 in [6], and $v_\varepsilon - \tilde{v}_\varepsilon = \frac{\varepsilon}{\rho} \lambda_\varepsilon \rightarrow 0$ uniformly, where $0 \leq \lambda_\varepsilon \leq 1$. It is then not restrictive to suppose also that ∇v_ε weakly converges to the constant value $A = \nabla u(0)$, since $\nabla v_\varepsilon(y) = \nabla u_\varepsilon(\rho y)$ giving that the gradients $|\nabla v_\varepsilon|$ are equi-bounded in $L^p(Q, \mathbb{R}^{2 \times 2})$. If we set $\eta = \frac{\varepsilon}{\rho}$ we can apply Lemma 2.5 with $\Omega = Q$, η in place of ε , v_ε in place of u_ε , v in place of u and the functions $\bar{u}_\varepsilon$ given by $\eta u_A(y/\eta)$, where u_A is as in Definition 2.7. We check that the assumption on such $\bar{u}_\varepsilon$ holds. Since $\bar{u}_\varepsilon(y) = \eta u_A(\frac{y}{\eta})$, then by Lemma 2.4 we have

$$\lim_{\varepsilon \rightarrow 0} \int_U |\nabla \bar{u}_\varepsilon|^p dy = C_A |U| \quad (16)$$

for all fixed open sets $U \subset Q$ with Lipschitz boundary. Also, $\bar{u}_\varepsilon \rightarrow v(y) = Ay$ in $L^p(Q, \mathbb{R}^2)$ since

$$\int_Q |Ay - \eta P_0(A \frac{y}{\eta})|^p dy = \int_Q \eta^p \underbrace{\left| A \frac{y}{\eta} - P_0(A \frac{y}{\eta}) \right|^p}_{\leq 1} dy \leq \eta^p \rightarrow 0,$$

by the definition of P_0 . By (16) with Q in place of U , the gradients are clearly equi-bounded, which gives the weak convergence of $\bar{u}_\varepsilon$ to v in $W^{1,p}(Q, \mathbb{R}^2)$. We then deduce that we may suppose that $v_\varepsilon(y) = \eta u_A(y/\eta)$ in a neighbourhood of ∂Q , and that

$$\begin{aligned} \frac{d\mu}{dx}(0) &\geq \liminf_{\eta \rightarrow 0} \inf \left\{ \int_Q |\nabla w|^p dy : w \in W^{1,p}(Q, \eta \mathcal{Z}), \right. \\ &\quad \left. w(y) = \eta u_A(y/\eta) \text{ in a neighbourhood of } \partial Q \right\}. \end{aligned}$$

By taking $T = \frac{1}{\eta}$ and rewriting the right-hand side of this inequality as a minimization problem on TQ ; that is, by using the transformation

$$u(x) = Tw\left(\frac{x}{T}\right), \quad \nabla u(x) = \nabla w\left(\frac{x}{T}\right),$$

changing variables in the integral, we obtain the lower bound (15). Note that we chose coordinate cubes Q rather than the tilted cubes Q_A since the lower bound is indifferent to the choice; the two formulations agree by Lemma 2.11.

We now deal with the singular case of $\text{rank}(A) = 1$. Denote $A = \nabla u(0)$, with $A = a \otimes \nu$. Let $Q_\nu := (0, 1)\nu + (0, 1)\nu^\perp$ be the unit cube with sides parallel to ν and $\nu^\perp$, and write

$$\begin{aligned} \frac{d\mu}{dx}(0) &= \lim_{\rho \rightarrow 0} \frac{\mu(\rho Q_\nu)}{|\rho Q_\nu|} = \lim_{\rho \rightarrow 0} \frac{1}{\rho^2} \mu(\rho Q_\nu) \\ &= \lim_{\rho \rightarrow 0} \liminf_{\varepsilon \rightarrow 0} \frac{1}{\rho^2} \mu_\varepsilon(\rho Q_\nu) = \lim_{\rho \rightarrow 0} \liminf_{\varepsilon \rightarrow 0} \frac{1}{\rho^2} \int_{\rho Q_\nu} |\nabla u_\varepsilon|^p dx, \end{aligned}$$

noting that this liminf is actually a limit except for a countable number of ρ . We can then find $\varepsilon = \varepsilon(\rho)$ such that

$$\frac{d\mu}{dx}(0) = \lim_{\rho \rightarrow 0} \frac{1}{\rho^2} \int_{\rho Q_\nu} |\nabla u_\varepsilon|^p dx.$$

Define now the one-dimensional restriction of u_ε on Q_ν

$$u_\varepsilon^y(t) = u_\varepsilon(y\nu^\perp + t\nu), \quad t \in (0, \rho),$$

which is absolutely continuous for $\mathcal{L}^1$ -almost every $y \in (0, \rho)$. For fixed ρ it holds

$$\frac{1}{\rho^2} \int_{\rho Q_\nu} |\nabla u_\varepsilon|^p dx \geq \frac{1}{\rho^2} \int_{\rho Q_\nu} |\nabla u_\varepsilon \cdot \nu|^p dx = \frac{1}{\rho^2} \int_0^\rho \int_0^\rho \left| \frac{du_\varepsilon^y(t)}{dt} \right|^p dt dy. \quad (17)$$

Since $u_\varepsilon^y(t) \in \varepsilon\mathcal{Z}$ for $\mathcal{L}^1$ -almost all y , taking into account that the image of u_ε^y is a polygonal in $\varepsilon\mathcal{Z}$, we now prove that for fixed ρ and $\mathcal{L}^1$ -almost all $y \in (0, \rho)$

$$\int_0^\rho \left| \frac{du_\varepsilon^y(t)}{dt} \right|^p dt \geq \rho \left(\left| \frac{a_1^{\varepsilon, \rho, y}}{\rho} \right| + \left| \frac{a_2^{\varepsilon, \rho, y}}{\rho} \right| \right)^p, \quad (18)$$

where $a^{\varepsilon, \rho, y} := (a_1^{\varepsilon, \rho, y}, a_2^{\varepsilon, \rho, y}) = u_\varepsilon^y(\rho) - u_\varepsilon^y(0)$.

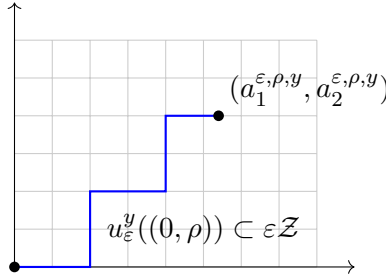


Figure 4: Path of u_ε^y in $\varepsilon\mathcal{Z}$

To this end, fix $\rho > 0$ and $y \in (0, \rho)$ a point of absolute continuity of $t \mapsto u_\varepsilon^y(t)$. Given $\frac{du_\varepsilon^y(t)}{dt} = (u_\varepsilon^y)'(t)$ and $|(u_\varepsilon^y)'(t)|^p = (|(u_\varepsilon^y)'_1(t)|^2 + |(u_\varepsilon^y)'_2(t)|^2)^{\frac{p}{2}}$, suppose first u_ε^y has piecewise-constant speed, either horizontally with (signed) speed v_1^ε or vertically with speed v_2^ε , say, in both cases in a finite union of intervals in $(0, \rho)$. There exists $\lambda \in (0, 1)$ such that $|(u_\varepsilon^y)'(t)|^p = |v_1^\varepsilon|^p$ in a set of t of measure $\lambda\rho$, and $|(u_\varepsilon^y)'(t)|^p = |v_2^\varepsilon|^p$ in a set of t of measure $(1 - \lambda)\rho$. Hence, we get

$$\int_0^\rho \left| \frac{du_\varepsilon^y(t)}{dt} \right|^p dt = \lambda\rho |v_1^\varepsilon|^p + (1 - \lambda)\rho |v_2^\varepsilon|^p$$

with the relations

$$\lambda \rho v_1^\varepsilon = a_1^{\varepsilon, \rho, y}, \quad (1 - \lambda) \rho v_2^\varepsilon = a_2^{\varepsilon, \rho, y}$$

(see Fig. 4). As depicted in the figure, the signs of v_1^ε and v_2^ε are given by the signs of the final point components $a_1^{\varepsilon, \rho, y}$ and $a_2^{\varepsilon, \rho, y}$. We consider the parameter λ as free, with the speeds adjust accordingly. Thus, the integral rewrites as

$$\begin{aligned} \int_0^\rho \left| \frac{du_\varepsilon^y(t)}{dt} \right|^p dt &= \lambda \rho |v_1^\varepsilon|^p + (1 - \lambda) \rho |v_2^\varepsilon|^p = \frac{1}{(\lambda \rho)^{p-1}} |a_1^{\varepsilon, \rho, y}|^p + \frac{1}{((1 - \lambda) \rho)^{p-1}} |a_2^{\varepsilon, \rho, y}|^p \\ &= \rho \left(\frac{1}{\lambda^{p-1}} \left| \frac{a_1^{\varepsilon, \rho, y}}{\rho} \right|^p + \frac{1}{(1 - \lambda)^{p-1}} \left| \frac{a_2^{\varepsilon, \rho, y}}{\rho} \right|^p \right). \end{aligned}$$

Minimizing the last quantity in λ ; that is, for

$$\lambda_{\min} = \frac{|a_1^{\varepsilon, \rho, y}|}{|a_1^{\varepsilon, \rho, y}| + |a_2^{\varepsilon, \rho, y}|},$$

this expression simplifies to (18).

In the general case, where $(u_\varepsilon^y)'(t)$ is not piecewise constant, we can use Jensen's inequality on each subinterval of $(0, \rho)$ where u_ε^y lies in one of the segments of $\varepsilon \mathcal{Z} \setminus \varepsilon \left(\frac{1}{2} + \mathbb{Z}\right)^2$

$$\begin{aligned} \int_0^\rho \left| \frac{du_\varepsilon^y(t)}{dt} \right|^p dt &= \int_0^{t_1} |(u_\varepsilon^y)'_1(t)|^p dt + \int_{t_1}^{t_2} |(u_\varepsilon^y)'_2(t)|^p dt + \dots + \int_{t_n}^\rho |(u_\varepsilon^y)'_1(t)|^p dt \\ &\geq \frac{1}{t_1^{p-1}} [(u_\varepsilon^y)_1(t_1) - (u_\varepsilon^y)_1(0)]^p + \frac{1}{(t_2 - t_1)^{p-1}} [(u_\varepsilon^y)_2(t_2) - (u_\varepsilon^y)_1(t_1)]^p \\ &\quad \dots + \frac{1}{(\rho - t_n)^{p-1}} [(u_\varepsilon^y)_1(\rho) - (u_\varepsilon^y)_1(t_n)]^p \\ &= \sum_{i=1}^{n+1} \frac{c_i^p}{\lambda_i^{p-1}}, \end{aligned}$$

where $\lambda_i := t_i - t_{i-1}$, (with $t_0 = 0, t_{n+1} = \rho$), $\sum_1^{n+1} \lambda_i = \rho$ and $c_i := (u_\varepsilon^y)_k(t_i) - (u_\varepsilon^y)_k(t_{i-1}) > 0$, for $k = 1$ or $k = 2$, with $\sum_1^{n+1} c_i = |a_1^{\rho, \varepsilon, y}| + |a_2^{\rho, \varepsilon, y}|$. This expression is minimized by

$$\lambda_i = \rho \frac{c_i}{\sum_{j=1}^{n+1} c_j},$$

and its minimum value is $\frac{1}{\rho^{p-1}} (\sum_{i=1}^{n+1} c_i)^p = \frac{1}{\rho^{p-1}} (|a_1^{\varepsilon, \rho, y}| + |a_2^{\varepsilon, \rho, y}|)^p$, again obtaining (18).

Finally, since $a^{\varepsilon, \rho, y}/\rho$ tends to a for almost all y as $\rho \rightarrow 0$, we can suppose it holds the limit

$$\lim_{\rho \rightarrow 0} \frac{1}{\rho} \int_0^\rho \left(\left| \frac{a_1^{\varepsilon, \rho, y}}{\rho} \right| + \left| \frac{a_2^{\varepsilon, \rho, y}}{\rho} \right| \right)^p dy = (|a_1| + |a_2|)^p, \quad (19)$$

which entails

$$\liminf_{\rho \rightarrow 0} \frac{1}{\rho^2} \int_{\rho Q_\nu} |\nabla u_\varepsilon|^p dx \geq (|a_1| + |a_2|)^p = f_{\text{hom}}^p(A), \quad (20)$$

as desired, by combining (17) and (18).

2.6 Proof of the upper bound in the homogenization theorem

In order to prove the upper bound, we can proceed by a density argument. First, we exhibit recovery sequences for a linear target function of gradient of maximum rank, and consequently by a gluing procedure, which makes use of Lemma 2.5, for piecewise-affine functions with gradients of maximum rank. After that, we prove in Proposition 2.14 that the function f_{hom}^p is continuous, so that the functional in (9) is strongly continuous in $W^{1,p}(\Omega, \mathbb{R}^2)$. Due to this fact, the upper bound is proved for a general $u \in W^{1,p}(\Omega, \mathbb{R}^2)$ by the density in norm of piecewise-affine functions.

Proposition 2.13. *The upper bound of the Γ -limit of F_ε^p with F_{hom}^p in (9) holds for piecewise-affine functions with non-singular gradients.*

Proof. Let u be the linear target function $u(x) = Ax$ with $A \in \mathbb{R}^{2 \times 2}$ nonsingular, fix $\eta > 0$, and let $u_A(x) = P_0(Ax)$ with P_0 as in Definition 2.3. Since the liminf in (13) is a limit, there exist $T \in \mathbb{N}$ and $u^T \in W^{1,p}(TQ_A, \mathcal{Z})$ with $u^T = u_A$ in a neighbourhood of ∂TQ_A such that

$$\frac{|\det A|}{T^2} \int_{TQ_A} |\nabla u^T|^p dx \leq f_{\text{hom}}^p(A) + \eta.$$

Note that we chose the tilted cubes TQ_A : since $T \in \mathbb{N}$, the parallelograms $A^{-1}(T\mathbf{k}) + TQ_A$, $\mathbf{k} \in \mathbb{Z}^2$, tile the plane, and the translations $A^{-1}(T\mathbf{k})$ are compatible with u_A , in the sense that $u_A(x - A^{-1}(T\mathbf{k})) + T\mathbf{k} = u_A(x)$, since $P_0(y + z) = P_0(y) + z$ for $z \in \mathbb{Z}^2$. Hence the function

$$\tilde{u}(x) := u^T(x - A^{-1}(T\mathbf{k})) + T\mathbf{k} \quad \text{if } x \in A^{-1}(T\mathbf{k}) + TQ_A, \mathbf{k} \in \mathbb{Z}^2,$$

coincides with u_A in a neighbourhood of the boundary of each parallelogram, so that, as $\mathcal{Z} + \mathbb{Z}^2 = \mathcal{Z}$, $\tilde{u} \in W_{\text{loc}}^{1,p}(\mathbb{R}^2, \mathcal{Z})$. Moreover, $\tilde{u}(x + A^{-1}(T\mathbf{k})) = \tilde{u}(x) + T\mathbf{k}$, so that $\psi(x) := \tilde{u}(x) - Ax$ is periodic with respect to the lattice $A^{-1}(T\mathbb{Z}^2)$, with periodicity cell TQ_A , and so are $\nabla \psi = \nabla \tilde{u} - A$ and $|\nabla \tilde{u}|^p$, as A is constant.

We claim that the functions $u_\varepsilon(x) := \varepsilon \tilde{u}(x/\varepsilon)$ weakly converge to u in $W^{1,p}(\Omega, \mathbb{R}^2)$ and

$$\lim_{\varepsilon \rightarrow 0} F_\varepsilon^p(u_\varepsilon) \leq (f_{\text{hom}}^p(A) + \eta)|\Omega|,$$

which yields the desired inequality. Indeed, $u_\varepsilon(x) = Ax + \varepsilon \psi(x/\varepsilon)$, and by the Riemann–Lebesgue lemma $\psi(\cdot/\varepsilon)$ is bounded in $L^p(\Omega, \mathbb{R}^2)$, so that $u_\varepsilon \rightarrow u$ strongly in $L^p(\Omega, \mathbb{R}^2)$; moreover

$$\nabla u_\varepsilon(x) = A + \nabla \psi\left(\frac{x}{\varepsilon}\right) \rightharpoonup A + \frac{|\det A|}{T^2} \int_{TQ_A} \nabla \psi dx = A \quad \text{weakly in } L^p(\Omega, \mathbb{R}^{2 \times 2}),$$

since the gradient of a periodic function has zero average on the periodicity cell. Finally, since $u_\varepsilon(x) \in \varepsilon\mathcal{Z}$ almost everywhere and $|\nabla u_\varepsilon(x)|^p = |\nabla \tilde{u}(x/\varepsilon)|^p$, again by the Riemann–Lebesgue lemma

$$\lim_{\varepsilon \rightarrow 0} F_\varepsilon^p(u_\varepsilon) = \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \left| \nabla \tilde{u}\left(\frac{x}{\varepsilon}\right) \right|^p dx = |\Omega| \frac{|\det A|}{T^2} \int_{TQ_A} |\nabla u^T|^p dx \leq (f_{\text{hom}}^p(A) + \eta)|\Omega|.$$

This proves the upper bound in the case of a linear target function.

Let now $u \in W^{1,p}(\Omega, \mathbb{R}^2)$ be piecewise affine; that is,

$$\Omega = \bigcup_{i=1}^m \Omega_i \cup N,$$

where Ω_i are pairwise disjoint open subsets of Ω , $|N| = 0$ and

$$u(x) = A_i x + b_i \quad \text{for } x \in \Omega_i, \quad i = 1, \dots, m,$$

with $A_i \in \mathbb{R}^{2 \times 2}$ non-singular and $b_i \in \mathbb{R}^2$. Since the liminf in (13) is a limit, there exist $T \in \mathbb{N}$ and $u_i^T \in W^{1,p}(TQ_{A_i}, \mathcal{Z})$ with $u_i^T = u_{A_i}$ in a neighbourhood of ∂TQ_{A_i} such that

$$\frac{|\det A_i|}{T^2} \int_{TQ_{A_i}} |\nabla u_i^T|^p dx \leq f_{\text{hom}}^p(A_i) + \eta,$$

for all $i = 1, \dots, m$. As in the case of one affine function, we extend u_i^T to

$$\tilde{u}_i(x) := u_i^T(x - A_i^{-1}(T\mathbf{k})) + T\mathbf{k} \quad \text{if } x \in A_i^{-1}(T\mathbf{k}) + TQ_{A_i}, \quad \mathbf{k} \in \mathbb{Z}^2,$$

so that $\tilde{u}_i \in W_{\text{loc}}^{1,p}(\mathbb{R}^2, \mathcal{Z})$ and $\tilde{u}_i(x) - A_i x$ is periodic with respect to the lattice $A_i^{-1}(T\mathbb{Z}^2)$. The functions

$$u_\varepsilon^i(x) := \varepsilon \tilde{u}_i\left(\frac{x}{\varepsilon}\right) + \varepsilon \left\lfloor \frac{b_i}{\varepsilon} \right\rfloor \in W_{\text{loc}}^{1,p}(\mathbb{R}^2, \varepsilon\mathcal{Z}),$$

where the integer part is taken componentwise, are good candidates when restricted on each Ω_i , since $\varepsilon \lfloor \frac{b_i}{\varepsilon} \rfloor - b_i = \varepsilon (\lfloor \frac{b_i}{\varepsilon} \rfloor - \frac{b_i}{\varepsilon}) \rightarrow 0$, so that $\varepsilon \lfloor \frac{b_i}{\varepsilon} \rfloor \rightarrow b_i$ also in $L^p(\Omega_i, \mathbb{R}^2)$ and trivially in $W^{1,p}(\Omega_i, \mathbb{R}^2)$, and

$$\varepsilon \tilde{u}_i\left(\frac{x}{\varepsilon}\right) \rightarrow A_i x \quad \text{weakly in } W^{1,p}(\Omega_i, \mathbb{R}^2)$$

by the same proof as before, performed for one affine function Ax , now applied to the functions $A_i x$ and $\varepsilon \tilde{u}_i\left(\frac{x}{\varepsilon}\right)$, on Ω_i , recalling that the Riemann–Lebesgue lemma works on every bounded open set. Thus, we have

$$u_\varepsilon^i(x) = \varepsilon \tilde{u}_i\left(\frac{x}{\varepsilon}\right) + \varepsilon \left\lfloor \frac{b_i}{\varepsilon} \right\rfloor \rightarrow A_i x + b_i \quad \text{weakly in } W^{1,p}(\Omega_i, \mathbb{R}^2)$$

for all $i = 1, \dots, m$. Moreover, since $\nabla u_\varepsilon^i(x) = \nabla \tilde{u}_i\left(\frac{x}{\varepsilon}\right)$ and $|\nabla \tilde{u}_i|^p$ is periodic with periodicity cell TQ_{A_i} , by the Riemann–Lebesgue lemma

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_i} |\nabla u_\varepsilon^i(x)|^p dx = |\Omega_i| \frac{|\det A_i|}{T^2} \int_{TQ_{A_i}} |\nabla u_i^T|^p dx \leq (f_{\text{hom}}^p(A_i) + \eta) |\Omega_i|. \quad (21)$$

The issue now is that we cannot simply merge the u_ε^i to a global u_ε in all Ω because we cannot guarantee that they coincide $\mathcal{H}^1$ almost everywhere on the interfaces across the different Ω_i . We need to modify u_ε^i suitably while also keeping the inequality (21) true. To this end, define

$$\bar{u}_\varepsilon(x) := \varepsilon P_0 \left(\frac{u(x)}{\varepsilon} \right).$$

We note that $\bar{u}_\varepsilon \in W^{1,p}(\Omega, \varepsilon \mathcal{Z})$. Also, by

$$\bar{u}_\varepsilon(x) = \varepsilon P_0 \left(\frac{A_i x + b_i}{\varepsilon} \right) = A_i x + b_i + \varepsilon \theta \left(\frac{A_i x + b_i}{\varepsilon} \right) \quad \text{for } x \in \Omega_i, \quad i = 1, \dots, m,$$

where $\theta(y) := P_0(y) - y$ is bounded and $\mathbb{Z}^2$ -periodic, we find that

$$\bar{u}_\varepsilon(x) \rightarrow A_i x + b_i \quad \text{weakly in } W^{1,p}(\Omega_i, \mathbb{R}^2), \quad i = 1, \dots, m,$$

using the same argument performed for the case of one affine function.

We now apply the variation of boundary conditions Lemma 2.5 on each Ω_i in place of Ω , to each u_ε^i and $\bar{u}_\varepsilon$ in common for all $i = 1, \dots, m$, to find a sequence $v_\varepsilon^i \in W^{1,p}(\Omega_i, \varepsilon \mathcal{Z})$ with

$$\limsup_{\varepsilon \rightarrow 0} \int_{\Omega_i} |\nabla v_\varepsilon^i|^p dx \leq \limsup_{\varepsilon \rightarrow 0} \int_{\Omega_i} |\nabla u_\varepsilon^i|^p dx$$

and $v_\varepsilon^i = \bar{u}_\varepsilon$ on a neighbourhood of $\partial\Omega_i$ (at the very end of the proof of Lemma 2.5 take the $\limsup$ instead of $\liminf$ on both sides). By the inequality above, the gradients of v_ε^i are equi-bounded in $L^p(\Omega_i, \mathbb{R}^{2 \times 2})$, as the gradients of u_ε^i are, thus we check only the convergence in $L^p(\Omega_i, \mathbb{R}^2)$ of the functions v_ε^i to $A_i x + b_i$. From the explicit form of v_ε^i in the proof of Lemma 2.5,

$$v_\varepsilon^i = P_\varepsilon(\varphi u_\varepsilon^i + (1 - \varphi)\bar{u}_\varepsilon),$$

where φ is a smooth cutoff function. Now we know that $\varphi u_\varepsilon^i + (1 - \varphi)\bar{u}_\varepsilon \rightarrow A_i x + b_i$ and v_ε^i converges to its same limit since

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_i} |v_\varepsilon^i - (\varphi u_\varepsilon^i + (1 - \varphi)\bar{u}_\varepsilon)|^p dx \leq \lim_{\varepsilon \rightarrow 0} (\sqrt{2}\varepsilon)^p |\Omega_i| = 0$$

by definition of P_ε .

Now we can merge v_ε^i , since they coincide with the common function $\bar{u}_\varepsilon$ close to the boundaries of Ω_i ; that is, we define in Ω

$$u_\varepsilon(x) := v_\varepsilon^i(x), \quad \text{if } x \in \Omega_i, \quad i \in \{1, \dots, m\},$$

which gives a function $u_\varepsilon \in W^{1,p}(\Omega, \varepsilon \mathcal{Z})$. We check that $u_\varepsilon \rightarrow u$ weakly in $W^{1,p}(\Omega, \mathbb{R}^2)$, by the linearity of the integral and the convergence of each v_ε^i . Finally,

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} F_\varepsilon^p(u_\varepsilon) &= \limsup_{\varepsilon \rightarrow 0} \int_{\Omega} |\nabla u_\varepsilon|^p dx = \limsup_{\varepsilon \rightarrow 0} \left(\sum_{i=1}^m \int_{\Omega_i} |\nabla v_\varepsilon^i|^p dx \right) \\ &\leq \sum_{i=1}^m \limsup_{\varepsilon \rightarrow 0} \int_{\Omega_i} |\nabla v_\varepsilon^i|^p dx \leq \sum_{i=1}^m \limsup_{\varepsilon \rightarrow 0} \int_{\Omega_i} |\nabla u_\varepsilon^i|^p dx \\ &\leq \sum_{i=1}^m (f_{\text{hom}}^p(A_i) + \eta) |\Omega_i| = \int_{\Omega} f_{\text{hom}}^p(\nabla u) dx + \eta |\Omega|, \end{aligned}$$

which concludes the proof by sending $\eta \rightarrow 0$. $\square$

Proposition 2.14. *The function f_{hom}^p as in Definition 2.7 is continuous.*

Proof. Since the formula defining $f_{\text{hom}}^p(A)$ for A of full rank is continuous with respect to the boundary conditions and the set of full-rank matrices is an open set, it suffices to prove the continuity for the limit matrices A that are singular. Due to the explicit form of $f_{\text{hom}}^p(A)$ when A is of rank 1, we may also limit the analysis to approximating matrices of rank 2.

Proof of lower semicontinuity. Let $A = a \otimes \nu$ and $A_\eta = A + B_\eta$ with $B_\eta \rightarrow 0$ as $\eta \rightarrow 0$. We want to show

$$\liminf_{\eta \rightarrow 0} f_{\text{hom}}^p(A_\eta) \geq (|a_1| + |a_2|)^p = f_{\text{hom}}^p(A).$$

We preliminarily remark that in the definition of the homogenized energy density, we can use Q_ν with sides parallel to $\nu, \nu^\perp$ instead of Q since the proof of Lemma 2.11 works the same when an orthogonal transformation is applied; namely, we can write

$$f_{\text{hom}}^p(A_\eta) = \liminf_{T \rightarrow \infty} \frac{1}{T^2} \inf \left\{ \int_{TQ_\nu} |\nabla u|^p dx : u \in W^{1,p}(TQ_\nu, \mathcal{Z}), u = u_{A_\eta} \sim \partial TQ_\nu \right\}.$$

Using this definition of $f_{\text{hom}}^p(A_\eta)$, from any competitor of the infimum problem u define the rescaled function

$$v_T(x) = \frac{1}{T} u(Tx), \quad x \in Q_\nu,$$

which takes values in $\frac{1}{T} \mathcal{Z}$, and record

$$\frac{1}{T^2} \int_{TQ_\nu} |\nabla u|^p dx = \int_{Q_\nu} |\nabla v_T|^p dx, \quad v_T(x) = \frac{1}{T} u_{A_\eta}(Tx) \sim \partial Q.$$

Following the same slicing procedure used for the proof of the lower bound in the rank-1 case, we define the one-dimensional restriction of v_T on Q_ν

$$v_T^y(t) = v_T(y\nu^\perp + t\nu), \quad t \in (0, 1),$$

which is absolutely continuous for $\mathcal{L}^1$ -almost every $y \in (0, 1)$. It holds

$$\int_{Q_\nu} |\nabla v_T|^p dx \geq \int_{Q_\nu} |\nabla v_T \cdot \nu|^p dx = \int_0^1 \int_0^1 \left| \frac{dv_T^y(t)}{dt} \right|^p dt dy,$$

and by the same minimization performed to prove inequality (18), here with $\rho = 1$, we get

$$\int_0^1 \int_0^1 \left| \frac{dv_T^y(t)}{dt} \right|^p dt dy \geq \int_0^1 (|a_{\eta,1}^{T,y}| + |a_{\eta,2}^{T,y}|)^p dy,$$

where

$$a_\eta^{T,y} := (a_{\eta,1}^{T,y}, a_{\eta,2}^{T,y}) = v_T^y(1) - v_T^y(0).$$

Since we have the boundary constraint $v_T(x) = \frac{1}{T}u_{A_\eta}(Tx)$ close to ∂Q , and the points $(0, y)$ and $(1, y)$ are in ∂Q for all $y \in (0, 1)$, we have that

$$a_\eta^{T,y} = \frac{1}{T}u_{A_\eta}(T(y\nu^\perp + \nu)) - \frac{1}{T}u_{A_\eta}(T(y\nu^\perp)), \quad y \in (0, 1)$$

and, moreover, the uniform bound

$$\begin{aligned} |a_\eta^{T,y} - A_\eta \nu| &= \left| \frac{1}{T}u_{A_\eta}(T(y\nu^\perp + \nu)) - A_\eta(y\nu^\perp + \nu) - \frac{1}{T}u_{A_\eta}(T(y\nu^\perp)) + A_\eta(y\nu^\perp) \right| \\ &= \frac{1}{T} \left| u_{A_\eta}(T(y\nu^\perp + \nu)) - A_\eta(T(y\nu^\perp + \nu)) - u_{A_\eta}(T(y\nu^\perp)) + A_\eta(T(y\nu^\perp)) \right| \\ &\leq \frac{2}{T}, \end{aligned}$$

since $u_A(x) = P_0(Ax)$ and $|P_0(Ax) - Ax| \leq 1$ by the definition of P_0 . Therefore,

$$\lim_{T \rightarrow \infty} a_\eta^{T,y} = A_\eta \nu = a + B_\eta \nu,$$

uniformly in $y \in (0, 1)$, and hence also in $L_y^p(0, 1)$, yielding

$$\lim_{T \rightarrow \infty} \int_0^1 (|a_{\eta,1}^{T,y}| + |a_{\eta,2}^{T,y}|)^p dy = (|a_1 + (B_\eta \nu)_1| + |a_2 + (B_\eta \nu)_2|)^p.$$

By the arbitrariness of u , since the right-hand side does not depend on u , we get that

$$f_{\text{hom}}^p(A_\eta) \geq (|a_1 + (B_\eta \nu)_1| + |a_2 + (B_\eta \nu)_2|)^p,$$

and finally

$$\liminf_{\eta \rightarrow 0} f_{\text{hom}}^p(A_\eta) \geq (|a_1| + |a_2|)^p.$$

More specifically, take for each T an almost minimizer $\bar{u}$ such that

$$\frac{1}{T^2} \int_{TQ_\nu} |\nabla \bar{u}|^p dx - \frac{1}{T} \leq \inf \left\{ \frac{1}{T^2} \int_{TQ_\nu} |\nabla u|^p dx : u \in W^{1,p}(TQ, \mathcal{Z}), u = u_{A_\eta} \sim \partial TQ_\nu \right\}.$$

Then

$$\begin{aligned}
f_{\text{hom}}^p(A_\eta) &\geq \liminf_{T \rightarrow \infty} \left(\frac{1}{T^2} \int_{TQ_\nu} |\nabla \bar{u}|^p dx - \frac{1}{T} \right) = \liminf_{T \rightarrow \infty} \int_{Q_\nu} |\nabla \bar{v}_T|^p dx \\
&\geq \liminf_{T \rightarrow \infty} \int_0^1 \int_0^1 \left| \frac{dv_T^y(t)}{dt} \right|^p dt dy \geq \lim_{T \rightarrow \infty} \int_0^1 (|a_{\eta,1}^{T,y}| + |a_{\eta,2}^{T,y}|)^p dy \\
&= (|a_1 + (B_\eta \nu)_1| + |a_2 + (B_\eta \nu)_2|)^p,
\end{aligned}$$

and we conclude by taking the $\liminf$ as $\eta \rightarrow 0$ on both sides.

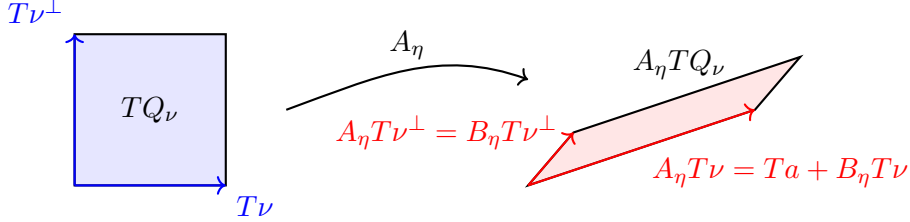


Figure 5: Stretched image of the cube TQ_ν under A_η .

Proof of upper semicontinuity. Let $A = a \otimes \nu$ and let $A_\eta = A + B_\eta$ with $B_\eta \rightarrow 0$ as $\eta \rightarrow 0$. The behaviour of A_η in terms of the deformation of the reference cube is pictured in Fig. 5: since A_η becomes singular as $\eta \rightarrow 0$, the parallelogram $A_\eta TQ_\nu$ is increasingly stretched and flattened. We want to show that

$$\limsup_{\eta \rightarrow 0} f_{\text{hom}}^p(A_\eta) \leq f_{\text{hom}}^p(A) = (|a_1| + |a_2|)^p.$$

It is sufficient to prove that for each $\eta > 0$ there exists $u_\eta^T \in W^{1,p}(TQ_\nu, \mathbb{Z})$, with $T \gg 1$ such that $\frac{1}{T} u_\eta^T(Tx) \rightarrow A_\eta x$ weakly in $W^{1,p}(Q_\nu, \mathbb{R}^2)$ as $T \rightarrow \infty$ and

$$\frac{1}{T^2} \int_{TQ_\nu} |\nabla u_\eta^T|^p dx \leq (|a_1| + |a_2|)^p + o_{\eta \rightarrow 0}(1).$$

The final step, after proving the inequality above, in order to conclude the upper semicontinuity will follow by using Lemma 2.5 to match the boundary conditions. Note that we are using Q_ν with sides parallel to $\nu, \nu^\perp$ instead of Q , but the proof of Lemma 2.11 works the same when an orthogonal transformation is applied.

We now describe the construction of u_η^T . This will be done by subdividing the reference cube in stripes, and repeating a construction stripe by stripe. We parameterize the cube TQ_ν with (t, y) ; that is,

$$x = t\nu + y\nu^\perp, \quad (t, y) \in (0, T) \times (0, T).$$

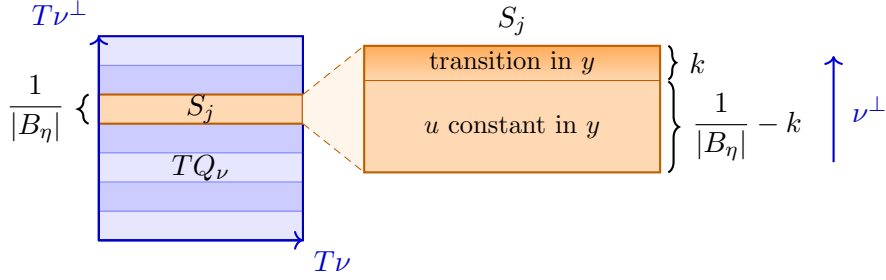


Figure 6: Stripe-by-stripe construction on the cube TQ_ν . Left: TQ_ν is sliced into stripes parallel to ν , of thickness $1/|B_\eta|$; S_j is one of them. Right: enlargement of S_j , split into a part of thickness $1/|B_\eta| - k$, where u_η^T is constant in y , and a thin layer of thickness k , where the transition in y takes place.

We want to construct the function

$$u_\eta^T(x) = u_\eta^T(t\nu + y\nu^\perp), \quad (t, y) \in (0, T) \times (0, T).$$

As shown in Figure 6, we divide the cube TQ_ν into a number of $|B_\eta|T$ stripes of thickness $1/|B_\eta|$ each, and without loss of generality we can assume $|B_\eta|T \in \mathbb{N}$ and $T \gg 1/|B_\eta|$, up to a negligible error. Then each stripe is subdivided in two regions of height $1/|B_\eta| - k$ and k , with $k \ll 1/|B_\eta|$. We build the function u_η^T on each stripe so that the derivative in y is concentrated only on the k -stripe, whose energy will be negligible at the limit.

Fix the stripe $S_j := (0, T) \times (y_j, y_{j+1})$, with $y_j = \frac{j}{|B_\eta|}$ and $j = 0, \dots, |B_\eta|T - 1$. In the lower and upper sides of the stripe we define

$$t \mapsto u_\eta^T(t, y_j) \text{ and } t \mapsto u_\eta^T(t, y_{j+1})$$

in such a way that it follows a “zigzag” path (on $\mathcal{Z}$) almost along the image of $(0, T) \times \{y_j\}$ and of $(0, T) \times \{y_{j+1}\}$ through the linear map A_η , up to a uniform error.

Define

$$\tilde{a}^{j,0} := \text{lattice point in } \left(\frac{1}{2} + \mathbb{Z}\right)^2 \text{ closest to } y_j B_\eta \nu^\perp, \quad (22)$$

$$\tilde{a}^{j,T} := \text{lattice point in } \left(\frac{1}{2} + \mathbb{Z}\right)^2 \text{ closest to } Ta + B_\eta T\nu + y_j B_\eta \nu^\perp. \quad (23)$$

Both points belong to $\mathcal{Z}$. These are (the projections of) the initial and final point of the image through the map A_η of the lower side of S_j . Note that we can estimate the length of the vector from $\tilde{a}^{j,0}$ to $\tilde{a}^{j,T}$, $a^j := \tilde{a}^{j,T} - \tilde{a}^{j,0}$ with

$$|a^j - Ta| = |\tilde{a}^{j,T} - (Ta + B_\eta T\nu + y_j B_\eta \nu^\perp) + y_j B_\eta \nu^\perp - \tilde{a}^{j,0} + B_\eta T\nu| \leq 2 + T|B_\eta|. \quad (24)$$

The above estimate holds component-wise,

$$|a_i^j - Ta_i| \leq |a^j - Ta| \leq 2 + T|B_\eta|, \quad i = 1, 2. \quad (25)$$

Let $v^j: [0, T] \rightarrow \mathcal{Z}$ be the path given by Lemma 2.6 (with $\varepsilon = 1$ and $L = T$) from $\tilde{a}^{j,0}$ to $\tilde{a}^{j,T}$, see Fig. 7, and set

$$u_\eta^T(t, y_j) := v^j(t), \quad t \in (0, T). \quad (26)$$

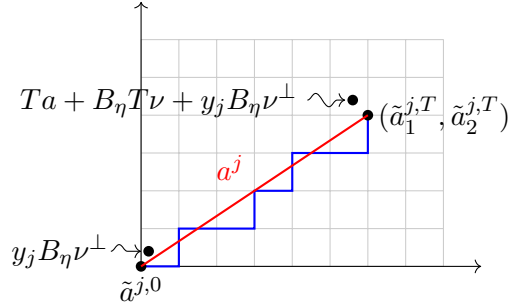


Figure 7: Construction of the path v^j at the step j

By Lemma 2.6(i), v^j moves horizontally, with velocity $v_1^j e_1$, on a set of measure λT , and vertically, with velocity $v_2^j e_2$, on the complementary set, where

$$\lambda := \frac{|a_1^j|}{|a_1^j| + |a_2^j|}, \quad v_i^j := \text{sign}(a_i^j) \frac{|a_1^j| + |a_2^j|}{T} \quad (i = 1, 2),$$

so that $\lambda T v_1^j = a_1^j$ and $(1 - \lambda) T v_2^j = a_2^j$. Moreover, by Lemma 2.6(iii) and (ii), with

$$z^j(t) := \tilde{a}^{j,0} + \frac{t}{T} a^j, \quad t \in (0, T), \quad (27)$$

we have

$$\int_0^T \left| \frac{dv^j}{dt} \right|^p dt = \frac{(|a_1^j| + |a_2^j|)^p}{T^{p-1}} \quad (28)$$

and

$$|z^j(t) - v^j(t)| \leq \sqrt{2} \quad \text{for all } t \in (0, T). \quad (29)$$

The definition of

$$t \mapsto u_\eta^T(t, y_{j+1}) =: v^{j+1}(t)$$

follows exactly the same steps as $v^j(t)$.

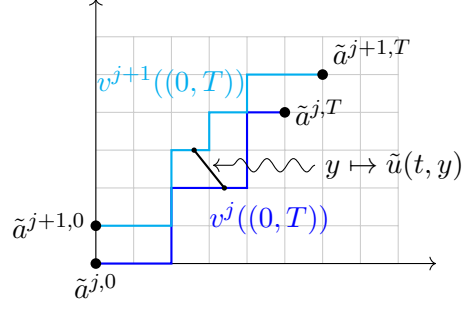


Figure 8: Schematic paths of v^j and v^{j+1}

Now, in the portion of the stripe $(0, T) \times (y_j, y_j + \frac{1}{|B_\eta|} - k)$, we let $u_\eta^T(t, y)$ be constant in y ,

$$u_\eta^T(t, y) = u_\eta^T(t, y_j), \quad (t, y) \in (0, T) \times \left(y_j, y_j + \frac{1}{|B_\eta|} - k\right).$$

In the thin stripe $y \in (y_j + \frac{1}{|B_\eta|} - k, y_{j+1})$ where $y_{j+1} = y_j + \frac{1}{|B_\eta|}$, the function u_η^T will also depend on y . For $y \in (y_j + \frac{1}{|B_\eta|} - k, y_j + \frac{1}{|B_\eta|})$ and $t \in (0, T)$ define the affine interpolation

$$\tilde{u}(t, y) = u_\eta^T(t, y_j) + (u_\eta^T(t, y_{j+1}) - u_\eta^T(t, y_j)) \frac{y - (y_j + \frac{1}{|B_\eta|} - k)}{k}$$

(see Fig. 8). Note that $\tilde{u} \notin \mathcal{Z}$ in general, thus we set

$$u_\eta^T(t, y) = P_1(\tilde{u})(t, y), \quad (t, y) \in (0, T) \times \left(y_j + \frac{1}{|B_\eta|} - k, y_j + \frac{1}{|B_\eta|}\right),$$

where $P_1(\tilde{u})$ is constructed as in Lemma 2.2, and we point out two facts:

(i) for all $t \in (0, T)$ we have $P_1(\tilde{u})(t, y_j) = u_\eta^T(t, y_j) \in \mathcal{Z}$ and $P_1(\tilde{u})(t, y_{j+1}) = u_\eta^T(t, y_{j+1}) \in \mathcal{Z}$, so u_η^T is well defined in the full stripe S_j when passing from the lower side to the upper side of it;

(ii) the proof and the estimate (with same constant $C(p)$) in Lemma 2.2 remain the same even if we change the domain Ω , in this case with the cube TQ_ν or its subdomains. Note that we are using the operator P_1 and not the simple composition with P_0 as in Definition 2.3.

Thanks to (ii), we need only to estimate the gradient of $\tilde{u}$ as defined on its domain after using the inequality of the type

$$\|\nabla u_\eta^T\|_p = \|\nabla P_1(\tilde{u})\|_p \leq C(p) \|\nabla \tilde{u}\|_p,$$

where the L^p norm is tacitly understood in the domain where $u_\eta^T = P_1(\tilde{u})$. Note that

$$\frac{\partial \tilde{u}}{\partial y}(t, y) = \frac{u_\eta^T(t, y_{j+1}) - u_\eta^T(t, y_j)}{k}.$$

We want to give a bound to $|u_\eta^T(t, y_{j+1}) - u_\eta^T(t, y_j)|$ uniform in t and j , in order to estimate the derivative above. By definition (27) and the bound (29),

$$\begin{aligned} & |u_\eta^T(t, y_{j+1}) - u_\eta^T(t, y_j)| \\ & \leq |z^j(t) - u_\eta^T(t, y_j)| + |z^{j+1}(t) - z^j(t)| + |u_\eta^T(t, y_{j+1}) - z^{j+1}(t)| \\ & \leq 2\sqrt{2} + |z^{j+1}(t) - z^j(t)| = 2\sqrt{2} + \left| (a_1^{j+1} - a_1^j) \frac{t}{T}, (a_2^{j+1} - a_2^j) \frac{t}{T} \right| \\ & \leq 2\sqrt{2} + |a^{j+1} - a^j| = 2\sqrt{2} + |(\tilde{a}^{j+1,T} - \tilde{a}^{j+1,0}) - (\tilde{a}^{j,T} - \tilde{a}^{j,0})| \\ & \leq 2\sqrt{2} + |\tilde{a}^{j+1,T} - \tilde{a}^{j,T}| + |\tilde{a}^{j+1,0} - \tilde{a}^{j,0}| \\ & = 2\sqrt{2} + \left| \tilde{a}^{j+1,T} - (Ta + B_\eta T\nu + y_{j+1}B_\eta\nu^\perp) \right. \\ & \quad \left. + (Ta + B_\eta T\nu + y_jB_\eta\nu^\perp - \tilde{a}^{j,T}) + \frac{B_\eta}{|B_\eta|}\nu^\perp \right| \\ & \quad + \left| \tilde{a}^{j+1,0} - y_{j+1}B_\eta\nu^\perp + (y_jB_\eta\nu^\perp - \tilde{a}^{j,0}) + \frac{B_\eta}{|B_\eta|}\nu^\perp \right| \\ & \leq 2\sqrt{2} + 6, \end{aligned}$$

where we used (22) and (23) for the last triangle inequality. Therefore, we find

$$\left| \frac{\partial \tilde{u}}{\partial y}(t, y) \right| = \frac{|u_\eta^T(t, y_{j+1}) - u_\eta^T(t, y_j)|}{k} \leq \frac{2\sqrt{2} + 6}{k} \leq \frac{9}{k}. \quad (30)$$

By $0 \leq \frac{y - (y_j + \frac{1}{|B_\eta|} - k)}{k} \leq 1$ we also see that

$$\left| \frac{\partial \tilde{u}}{\partial t}(t, y) \right| \leq 2 \left(\left| \frac{dv^j}{dt} \right| + \left| \frac{dv^{j+1}}{dt} \right| \right).$$

Finally, we compute the averaged integral:

$$\begin{aligned} \frac{1}{T^2} \int_{TQ_\nu} |\nabla u_\eta^T|^p dx &= \frac{1}{T^2} \int_0^T \int_0^T |\nabla u_\eta^T(t, y)|^p dt dy \\ &= \frac{1}{T^2} \sum_{j=0}^{|B_\eta|T-1} \int_{y_j}^{y_{j+1}} \int_0^T |\nabla u_\eta^T(t, y)|^p dt dy. \end{aligned}$$

For each j the summand is divided as

$$\begin{aligned}
& \int_{y_j}^{y_{j+1}} \int_0^T |\nabla u_\eta^T(t, y)|^p dt dy \\
&= \int_{y_j}^{y_j + \frac{1}{|B_\eta|} - k} \int_0^T \left| \frac{dv^j(t)}{dt} \right|^p dt dy + \int_{y_j + \frac{1}{|B_\eta|} - k}^{y_{j+1}} \int_0^T |\nabla P_1(\tilde{u})(t, y)|^p dt dy \\
&= \left(\frac{1}{|B_\eta|} - k \right) \left(\lambda T |v_1^j|^p + (1 - \lambda) T |v_2^j|^p \right) + \int_{y_j + \frac{1}{|B_\eta|} - k}^{y_{j+1}} \int_0^T |\nabla P_1(\tilde{u})(t, y)|^p dt dy \\
&= \left(\frac{1}{|B_\eta|} - k \right) \frac{(|a_1^j| + |a_2^j|)^p}{T^{p-1}} + \int_{y_j + \frac{1}{|B_\eta|} - k}^{y_{j+1}} \int_0^T |\nabla P_1(\tilde{u})(t, y)|^p dt dy \\
&\leq \left(\frac{1}{|B_\eta|} - k \right) \frac{(|a_1^j| + |a_2^j|)^p}{T^{p-1}} + C(p) \int_{y_j + \frac{1}{|B_\eta|} - k}^{y_{j+1}} \int_0^T |\nabla \tilde{u}(t, y)|^p dt dy \\
&\leq \left(\frac{1}{|B_\eta|} - k \right) \frac{(|a_1^j| + |a_2^j|)^p}{T^{p-1}} + C(p) \int_{y_j + \frac{1}{|B_\eta|} - k}^{y_{j+1}} \int_0^T \left| \frac{\partial \tilde{u}}{\partial y}(t, y) \right|^p dt dy \\
&\quad + C(p) \int_{y_j + \frac{1}{|B_\eta|} - k}^{y_{j+1}} \int_0^T \left| \frac{\partial \tilde{u}}{\partial t}(t, y) \right|^p dt dy \\
&= I_1^j + I_2^j + I_3^j,
\end{aligned}$$

where we used $p/2 < 1$ to split the full gradient norm $|\nabla \tilde{u}|^p$ and (28) in the third equality. We handle the three terms separately.

First term I_1^j . We recall estimate (25) which gives

$$\begin{aligned}
(|a_1^j| + |a_2^j|)^p &= \left(|a_1^j| - T|a_1| + |a_2^j| - T|a_2| + T|a_1| + T|a_2| \right)^p \\
&\leq (1 + \delta)(T|a_1| + T|a_2|)^p + C_\delta \left| |a_1^j| - T|a_1| + |a_2^j| - T|a_2| \right|^p \\
&\leq (1 + \delta)T^p(|a_1| + |a_2|)^p + C_\delta 2^{p-1} \left(\left| |a_1^j| - T|a_1| \right|^p + \left| |a_2^j| - T|a_2| \right|^p \right) \\
&\leq (1 + \delta)T^p(|a_1| + |a_2|)^p + C_\delta 2^p(2 + T|B_\eta|)^p.
\end{aligned}$$

and therefore

$$\begin{aligned}
& \frac{1}{T^2} \sum_{j=0}^{|B_\eta|T-1} \left(\frac{1}{|B_\eta|} - k \right) \frac{(|a_1^j| + |a_2^j|)^p}{T^{p-1}} \\
& \leq \frac{1}{T^2} (T - |B_\eta|Tk) \left((1 + \delta)T(|a_1| + |a_2|)^p + \frac{C_\delta 2^p}{T^{p-1}} (2 + T|B_\eta|)^p \right) \\
& = (1 - |B_\eta|k)(1 + \delta)(|a_1| + |a_2|)^p + (1 - |B_\eta|k)C_\delta 2^p \left(\frac{2}{T} + |B_\eta| \right)^p \\
& = (1 + \delta)(|a_1| + |a_2|)^p + C_\delta o_{\eta \rightarrow 0}(1).
\end{aligned}$$

Second term I_2^j .

$$I_2^j = C(p) \int_{y_j + \frac{1}{|B_\eta|} - k}^{y_{j+1}} \int_0^T \left| \frac{\partial \tilde{u}}{\partial y}(t, y) \right|^p dt dy \leq C(p)kT \left(\frac{9}{k} \right)^p = C(p)9^p \frac{T}{k^{p-1}},$$

and thus

$$\frac{1}{T^2} \sum_{j=0}^{|B_\eta|T-1} C(p) \int_{y_j + \frac{1}{|B_\eta|} - k}^{y_{j+1}} \int_0^T \left| \frac{\partial \tilde{u}}{\partial y}(t, y) \right|^p dt dy \leq \frac{1}{T^2} |B_\eta|T C(p)9^p \frac{T}{k^{p-1}} = C(p)9^p \frac{|B_\eta|}{k^{p-1}},$$

which goes to 0 as $\eta \rightarrow 0$.

Third term I_3^j . Since we have

$$\left| \frac{\partial \tilde{u}}{\partial t}(t, y) \right|^p \leq 2^p \left(\left| \frac{dv^j}{dt} \right| + \left| \frac{dv^{j+1}}{dt} \right| \right)^p \leq 2^{2p-1} \left| \frac{dv^j}{dt} \right|^p + 2^{2p-1} \left| \frac{dv^{j+1}}{dt} \right|^p,$$

we can follow the same computations performed for the first term, equally for j and $j + 1$,

$$\begin{aligned}
I_3^j &= C(p) \int_{y_j + \frac{1}{|B_\eta|} - k}^{y_{j+1}} \int_0^T \left| \frac{\partial \tilde{u}}{\partial t}(t, y) \right|^p dt dy \\
&\leq C(p)2^{2p-1} \int_{y_j + \frac{1}{|B_\eta|} - k}^{y_{j+1}} \int_0^T \left(\left| \frac{dv^j}{dt} \right|^p + \left| \frac{dv^{j+1}}{dt} \right|^p \right) dt dy \\
&= C(p)2^{2p-1}k \left(\frac{(|a_1^j| + |a_2^j|)^p}{T^{p-1}} + \frac{(|a_1^{j+1}| + |a_2^{j+1}|)^p}{T^{p-1}} \right) \\
&\leq C(p)2^{2p-1}k \left(2(1 + \delta)T(|a_1| + |a_2|)^p + 2\frac{C_\delta 2^p}{T^{p-1}}(2 + T|B_\eta|)^p \right),
\end{aligned}$$

applying the summation

$$\begin{aligned}
\frac{1}{T^2} \sum_{j=0}^{|B_\eta|T-1} I_3^j &\leq \frac{|B_\eta|C(p)2^{2p-1}k}{T} \left(2(1 + \delta)T(|a_1| + |a_2|)^p + 2\frac{C_\delta 2^p}{T^{p-1}}(2 + T|B_\eta|)^p \right) \\
&= C_1|B_\eta|k(1 + \delta)(|a_1| + |a_2|)^p + C_2C_\delta|B_\eta|k \left(\frac{2}{T} + |B_\eta| \right)^p,
\end{aligned}$$

and everything tends to 0 as $\eta \rightarrow 0$ since $|B_\eta| \rightarrow 0$. In conclusion,

$$\frac{1}{T^2} \int_{TQ_\nu} |\nabla u_\eta^T|^p dx \leq (1 + \delta)(|a_1| + |a_2|)^p + C_\delta o_{\eta \rightarrow 0}(1),$$

which yields the claim sending first $\eta \rightarrow 0$ and then $\delta \rightarrow 0$.

We check that the weak-convergence statement holds. Define

$$u_\varepsilon(x) = \varepsilon u_\eta^T\left(\frac{x}{\varepsilon}\right), \quad x \in Q_\nu,$$

where $\varepsilon = 1/T$. Note that $u_\varepsilon \in W^{1,p}(Q_\nu, \varepsilon \mathcal{Z})$. Then, for η fixed, it holds

$$\lim_{\varepsilon \rightarrow 0} u_\varepsilon(x) = A_\eta x \quad \text{weakly in } W^{1,p}(Q_\nu, \mathbb{R}^2).$$

We need to prove that the gradients are equi-bounded and that the functions converge in $L^p(Q_\nu, \mathbb{R}^2)$. We note that

$$\int_{Q_\nu} |\nabla u_\varepsilon(x)|^p dx = \int_{Q_\nu} |\nabla u_\eta^T\left(\frac{x}{\varepsilon}\right)|^p dx = \frac{1}{T^2} \int_{TQ_\nu} |\nabla u_\eta^T|^p dx \leq (1 + \delta)(|a_1| + |a_2|)^p + C_\delta o_\eta(1).$$

Thus, the gradients are equi-bounded in L^p . To prove the convergence of u_ε , we note that

$$\begin{aligned} \int_{Q_\nu} |u_\varepsilon(x) - A_\eta x|^p dx &= \int_{Q_\nu} \left| \varepsilon u_\eta^T\left(\frac{x}{\varepsilon}\right) - \varepsilon A_\eta \frac{x}{\varepsilon} \right|^p dx \\ &= \varepsilon^{2+p} \int_{\frac{1}{\varepsilon} Q_\nu} |u_\eta^T(x) - A_\eta x|^p dx = \frac{1}{T^{2+p}} \int_{TQ_\nu} |u_\eta^T(x) - A_\eta x|^p dx. \end{aligned}$$

Generally speaking, the integrand $|u_\eta^T(x) - A_\eta x|$ is uniformly (in x) close to a constant, since u_η^T is a zigzag almost running along the image lines through A_η , and therefore the integral is of order $O(T^{-p}) \rightarrow 0$ as $T \rightarrow \infty$. Indeed, for $x = t\nu + y\nu^\perp$, $(t, y) \in (0, T) \times (0, T)$, and $A_\eta = a \otimes \nu + B_\eta$,

$$|u_\eta^T(x) - A_\eta x| = |u_\eta^T(t, y) - at - tB_\eta\nu - yB_\eta\nu^\perp|,$$

if $y \in (y_j, y_j + \frac{1}{|B_\eta|} - k)$ for some $j = 0, \dots, |B_\eta|T - 1$; that is, where $u_\eta^T(t, y) = u_\eta^T(t, y_j)$; then, recalling (27),

$$\begin{aligned} &|u_\eta^T(t, y_j) - at - tB_\eta\nu - yB_\eta\nu^\perp| \\ &\leq |u_\eta^T(t, y_j) - z^j(t)| + |z^j(t) - at - tB_\eta\nu - yB_\eta\nu^\perp| \\ &\leq \sqrt{2} + |a^j \frac{t}{T} - at - tB_\eta\nu| + |\tilde{a}^{j,0} - yB_\eta\nu^\perp| \\ &\leq \sqrt{2} + |\frac{t}{T}(a^j - Ta - B_\eta T\nu)| + |\tilde{a}^{j,0} - y_j B_\eta\nu^\perp| + |y_j B_\eta\nu^\perp - yB_\eta\nu^\perp| \\ &\leq \sqrt{2} + \frac{t}{T}2 + 1 + |y - y_j||B_\eta| \leq C, \end{aligned}$$

where we used the (rearranged) inequality (24), the bound (29) and definition (22). If instead $y \in (y_j + \frac{1}{|B_\eta|} - k, y_{j+1})$, where we have the transition, we recall that

$$u_\eta^T(t, y) = P_1(\tilde{u})(t, y), \quad \tilde{u}(t, y) = u_\eta^T(t, y_j) + (u_\eta^T(t, y_{j+1}) - u_\eta^T(t, y_j)) \frac{y - (y_j + \frac{1}{|B_\eta|} - k)}{k}.$$

By the estimates

$$\begin{aligned} |u_\eta^T(t, y) - \tilde{u}(t, y)| &= |P_1(\tilde{u})(t, y) - \tilde{u}(t, y)| \leq 1, \\ |\tilde{u}(t, y) - u_\eta^T(t, y_j)| &= \left| (u_\eta^T(t, y_{j+1}) - u_\eta^T(t, y_j)) \frac{y - (y_j + \frac{1}{|B_\eta|} - k)}{k} \right| \leq 9, \end{aligned}$$

the second coming from (30), through a concatenated triangle inequality, we get

$$\begin{aligned} &|u_\eta^T(t, y) - at - tB_\eta\nu - yB_\eta\nu^\perp| \\ &\leq |u_\eta^T(t, y) - \tilde{u}(t, y)| + |\tilde{u}(t, y) - u_\eta^T(t, y_j)| + |u_\eta^T(t, y_j) - at - tB_\eta\nu - yB_\eta\nu^\perp| \\ &\leq 1 + 9 + C. \end{aligned}$$

Therefore,

$$|u_\eta^T(x) - A_\eta x| = |u_\eta^T(t, y) - at - tB_\eta\nu - yB_\eta\nu^\perp| \leq C$$

for all $x \in TQ_\nu$, and

$$\int_{Q_\nu} |u_\varepsilon(x) - A_\eta x|^p dx = \frac{1}{T^{2+p}} \int_{TQ_\nu} |u_\eta^T(x) - A_\eta x|^p dx \leq C^p \frac{1}{T^p} = C^p \varepsilon^p,$$

which tends to 0. We can apply Lemma 2.5 with $\Omega = Q_\nu$, the functions $u_\varepsilon \rightarrow u(x) = A_\eta(x)$, and the function $\bar{u}_\varepsilon$ given by $\varepsilon u_{A_\eta}(x/\varepsilon)$, where u_{A_η} is as in Definition 2.7. We check that the assumption on such $\bar{u}_\varepsilon$ holds: since $\bar{u}_\varepsilon(x) = \varepsilon u_{A_\eta}(\frac{x}{\varepsilon})$, then for any fixed $U \subset Q_\nu$ with Lipschitz boundary, we have

$$\lim_{\varepsilon \rightarrow 0} \int_U |\nabla \bar{u}_\varepsilon|^p dx = C_{A_\eta} |U| \quad (31)$$

by Lemma 2.4. Also, $\bar{u}_\varepsilon \rightarrow u(x) = A_\eta x$ in $L^p(\Omega, \mathbb{R}^2)$ since

$$\lim_{\varepsilon \rightarrow 0} \int_{Q_\nu} |A_\eta x - \varepsilon P_0(A_\eta \frac{x}{\varepsilon})|^p dx = \lim_{\varepsilon \rightarrow 0} \int_{Q_\nu} \varepsilon^p |A_\eta \frac{x}{\varepsilon} - P_0(A_\eta \frac{x}{\varepsilon})|^p dy = 0,$$

as $|A_\eta \frac{x}{\varepsilon} - P_0(A_\eta \frac{x}{\varepsilon})| \leq 1$ by the definition of P_0 . By (31) above with $\Omega = Q_\nu$ in place of U , we deduce that the gradients are equi-bounded, which gives the weak convergence of $\bar{u}_\varepsilon$ to u in $W^{1,p}(Q_\nu, \mathbb{R}^2)$. Thus, thanks to Lemma 2.5 we find $v_\varepsilon \in W^{1,p}(Q_\nu, \varepsilon \mathcal{Z})$ with

$$\liminf_{\varepsilon \rightarrow 0} \int_{Q_\nu} |\nabla v_\varepsilon|^p dx \leq \liminf_{\varepsilon \rightarrow 0} \int_{Q_\nu} |\nabla u_\varepsilon|^p dx$$

and $v_\varepsilon = \bar{u}_\varepsilon = \varepsilon u_{A_\eta}(\frac{x}{\varepsilon})$ in a neighbourhood of Q_ν . Scaling back to TQ_ν , with $T = 1/\varepsilon$,

$$w_T(x) := Tv_\varepsilon\left(\frac{x}{T}\right) \in W^{1,p}(TQ_\nu, \mathcal{Z}),$$

and $w_T = u_{A_\eta}$ in a neighbourhood of ∂TQ_ν . Therefore

$$\begin{aligned} f_{\text{hom}}^p(A_\eta) &= \liminf_{T \rightarrow +\infty} \frac{1}{T^2} \inf \left\{ \int_{TQ_\nu} |\nabla u|^p dx : u \in W^{1,p}(TQ_\nu, \mathcal{Z}), u = u_{A_\eta} \sim \partial TQ_\nu \right\} \\ &\leq \liminf_{T \rightarrow +\infty} \frac{1}{T^2} \int_{TQ_\nu} |\nabla w_T|^p dx = \liminf_{\varepsilon \rightarrow 0} \int_{Q_\nu} |\nabla v_\varepsilon|^p dx \\ &\leq \liminf_{\varepsilon \rightarrow 0} \int_{Q_\nu} |\nabla u_\varepsilon|^p dx = \liminf_{T \rightarrow +\infty} \frac{1}{T^2} \int_{TQ_\nu} |\nabla u_\eta^T|^p dx \\ &\leq (1 + \delta)(|a_1| + |a_2|)^p + C_\delta o_\eta(1), \end{aligned}$$

and we find

$$\limsup_{\eta \rightarrow 0} f_{\text{hom}}^p(A_\eta) \leq (|a_1| + |a_2|)^p,$$

by first sending $\eta \rightarrow 0$ and then $\delta \rightarrow 0$. $\square$

Proposition 2.15. *The Γ -limsup inequality for F_ε^p with (9) holds for all functions $u \in W^{1,p}(\Omega, \mathbb{R}^2)$.*

Proof. By Proposition 2.14 the function $A \mapsto f_{\text{hom}}^p(A)$ is continuous, so that

$$u \mapsto \int_{\Omega} f_{\text{hom}}^p(\nabla u(x)) dx$$

is continuous in $W^{1,p}(\Omega, \mathbb{R}^2)$. Fix $u \in W^{1,p}(\Omega, \mathbb{R}^2)$; then there exists a sequence $u_k \in W^{1,p}(\Omega, \mathbb{R}^2)$ piecewise affine, with non-singular gradients, such that $u_k \rightarrow u$ strongly as $k \rightarrow \infty$. By Proposition 2.13, for each k the upper bound holds

$$\Gamma\text{-}\limsup_{\varepsilon \rightarrow 0} F_\varepsilon^p(u_k) \leq \int_{\Omega} f_{\text{hom}}^p(\nabla u_k) dx.$$

By the lower semicontinuity of the Γ -lim sup we get

$$\begin{aligned} \Gamma\text{-}\limsup_{\varepsilon \rightarrow 0} F_\varepsilon^p(u) &\leq \liminf_{k \rightarrow \infty} \left(\Gamma\text{-}\limsup_{\varepsilon \rightarrow 0} F_\varepsilon^p(u_k) \right) \\ &\leq \liminf_{k \rightarrow \infty} \int_{\Omega} f_{\text{hom}}^p(\nabla u_k) dx = \int_{\Omega} f_{\text{hom}}^p(\nabla u) dx, \end{aligned}$$

which gives the upper bound for the prescribed u . $\square$

2.7 The Young measure setting

Although the Γ -limit in Theorem A has already been proved without their use, we now introduce Young measures, as they make the geometric constraint of the problem transparent. Indeed, since u_ε takes values in $\varepsilon\mathcal{Z}$, its gradient lies almost everywhere in

$$K := (e_1 \otimes \mathbb{R}^2) \cup (e_2 \otimes \mathbb{R}^2),$$

every Young measure generated by such gradients is supported in K . This support condition is the starting point for the bounds on f_{hom}^p in Section 3, which in particular show that f_{hom}^p is not convex. In addition, Young measure give an alternative derivation of the rank-one formula $f_{\text{hom}}^p(a \otimes \nu) = (|a_1| + |a_2|)^p$ to the construction of optimized taxicab paths in Section 2.5.

We write $\mathcal{P}(\mathbb{R}^N)$ for the set of Borel probability measures on $\mathbb{R}^N$. The space $\mathbf{Y}^p(\Omega, \mathbb{R}^N)$ of p -Young measures consists of all parametrized families $\nu = (\nu_x)_{x \in \Omega} \subset \mathcal{P}(\mathbb{R}^N)$ that are weakly* measurable: that is, $x \mapsto \int_{\mathbb{R}^N} h \, d\nu_x$ is Lebesgue measurable for every $h \in C_0(\mathbb{R}^N)$, and satisfy

$$\int_{\Omega} \int_{\mathbb{R}^N} |z|^p \, d\nu_x(z) \, dx < \infty.$$

A bounded sequence $(v_j) \subset L^p(\Omega, \mathbb{R}^N)$ generates $\nu \in \mathbf{Y}^p(\Omega, \mathbb{R}^N)$, in symbols $v_j \xrightarrow{\mathbf{Y}} \nu$ as $j \rightarrow \infty$, if

$$h(v_j) \rightharpoonup \int_{\mathbb{R}^N} h \, d\nu_x \quad \text{weakly in } L^1(\Omega) \quad \text{for every } h \in C_0(\mathbb{R}^N),$$

that is, $\int_{\Omega} \varphi(x) h(v_j(x)) \, dx \rightarrow \int_{\Omega} \varphi(x) \int_{\mathbb{R}^N} h \, d\nu_x \, dx$ for all $\varphi \in L^\infty(\Omega)$ and $h \in C_0(\mathbb{R}^N)$. We say that the Young measure ν is *homogeneous* when there exists $\mu \in \mathcal{P}(\mathbb{R}^N)$ such that $\nu_x = \mu$ for almost every $x \in \Omega$. A *gradient* Young measure is generated by sequences of L^p gradients $v_j = \nabla u_j$, for some sequence of Sobolev functions $(u_j) \subset W^{1,p}$.

We record here two results from the general theory of Young measures, see [61, Lemma 4.12] and [32, Theorem 8.6]; for a general Young-measure approach to lower semicontinuity, covering normal integrands and arbitrary tight (e.g. L^p -bounded) sequences, we refer the reader to Balder [9].

Lemma 2.16. *Let $\nu = (\nu_x)_{x \in \Omega} \in \mathbf{Y}^p(\Omega, \mathbb{R}^N)$ be the Young measure generated by a L^p -bounded sequence $(v_j) \subset L^p(\Omega, \mathbb{R}^N)$, and let $C \subset \mathbb{R}^N$ be a closed set. Then*

$$\text{dist}(v_j, C) \rightarrow 0 \text{ in measure} \iff \text{supp } \nu_x \subset C \text{ for almost every } x \in \Omega.$$

We actually need the following weaker statement of the lemma above: if $v_j(x) \in C$ for almost every $x \in \Omega$ then $\text{supp } \nu_x \subset C$ for almost every $x \in \Omega$.

Proposition 2.17. *Let $1 < p < \infty$ and $\nu = (\nu_x)_{x \in \Omega} \in \mathbf{Y}^p(\Omega, \mathbb{R}^N)$ be the Young measure generated by a L^p -bounded sequence $(v_j) \subset L^p(\Omega, \mathbb{R}^N)$. Let $F: \Omega \times \mathbb{R}^N \rightarrow [0, \infty)$ be a Carathéodory integrand of p -growth. Then*

$$(i) \quad \liminf_{j \rightarrow \infty} \int_{\Omega} F(x, v_j(x)) \, dx \geq \int_{\Omega} \int_{\mathbb{R}^N} F(x, M) \, d\nu_x(M) \, dx.$$

Moreover, $F(\cdot, v_j(\cdot))$ is equi-integrable if and only if

$$(ii) \quad \lim_{j \rightarrow \infty} \int_{\Omega} F(x, v_j(x)) \, dx = \int_{\Omega} \int_{\mathbb{R}^N} F(x, M) \, d\nu_x(M) \, dx.$$

Now we point out a pointwise differential constraint satisfied by the sequences u_{ε} which are $\varepsilon\mathcal{Z}$ -valued. Define the closed set $K \subset \mathbb{R}^{2 \times 2}$ as

$$K := (e_1 \otimes \mathbb{R}^2) \cup (e_2 \otimes \mathbb{R}^2). \quad (32)$$

So $M \in K$ takes the row-wise form $M = \begin{pmatrix} M_1 \\ 0 \end{pmatrix}$ or $M = \begin{pmatrix} 0 \\ M_2 \end{pmatrix}$.

Lemma 2.18. *If $u_{\varepsilon} \in W^{1,p}(\Omega, \varepsilon\mathcal{Z})$, then $\nabla u_{\varepsilon}(x) \in K$ for almost every $x \in \Omega$.*

Proof. From $u_{\varepsilon}(x) = (u_{\varepsilon,1}(x), u_{\varepsilon,2}(x)) \in \varepsilon\mathcal{Z}$ for almost every $x \in \Omega$, we have that for almost every $x \in \Omega$

$$u_{\varepsilon,1}(x) \in \frac{\varepsilon}{2} + \varepsilon\mathbb{Z} \quad \text{or} \quad u_{\varepsilon,2}(x) \in \frac{\varepsilon}{2} + \varepsilon\mathbb{Z}.$$

Since

$$\frac{\varepsilon}{2} + \varepsilon\mathbb{Z} = \bigcup_{q \in \mathbb{Z}} \left\{ \frac{\varepsilon}{2} + \varepsilon q \right\}$$

we can use the well-known property for Sobolev functions f , $1 \leq p < \infty$, and $c \in \mathbb{R}$, that $\nabla f = 0$ almost everywhere on $\{x : f(x) = c\}$, yielding that on the sets

$$E_1^q := \left\{ x \in \Omega : u_{\varepsilon,1}(x) = \frac{\varepsilon}{2} + \varepsilon q \right\}, \quad E_2^q := \left\{ x \in \Omega : u_{\varepsilon,2}(x) = \frac{\varepsilon}{2} + \varepsilon q \right\},$$

we have $\nabla u_{\varepsilon,1} = 0$ for almost every $x \in E_1^q$, and $\nabla u_{\varepsilon,2} = 0$ for almost every $x \in E_2^q$. Therefore, also $\nabla u_{\varepsilon,1} = 0$ almost everywhere in $E_1 = \bigcup_{q \in \mathbb{Z}} E_1^q$ and the same for $\nabla u_{\varepsilon,2} = 0$ almost everywhere in $E_2 = \bigcup_{q \in \mathbb{Z}} E_2^q$. Since $N := \Omega \setminus (E_1 \cup E_2)$ is Lebesgue negligible by $u_{\varepsilon}(x) \in \varepsilon\mathcal{Z}$ for almost every x , we have that for almost every $x \in \Omega$ either

$$\nabla u_{\varepsilon,1} = 0 \implies \nabla u_{\varepsilon}(x) = \begin{pmatrix} 0 \\ \nabla u_{\varepsilon,2}(x) \end{pmatrix}, \quad \text{or} \quad \nabla u_{\varepsilon,2} = 0 \implies \nabla u_{\varepsilon}(x) = \begin{pmatrix} \nabla u_{\varepsilon,1}(x) \\ 0 \end{pmatrix},$$

which gives $\nabla u_{\varepsilon}(x) \in K$. □

A lower bound in terms of Young measures. We fix $u \in W^{1,p}(\Omega, \mathbb{R}^2)$ and $u_\varepsilon \rightharpoonup u$ weakly in $W^{1,p}(\Omega, \mathbb{R}^2)$ with $\liminf_{\varepsilon \rightarrow 0} F_\varepsilon^p(u_\varepsilon) < +\infty$. Upon extracting subsequences, we can suppose that this liminf is actually a limit, and that $u_\varepsilon(x) \in \varepsilon\mathcal{Z}$ almost everywhere in Ω . Since in particular $F_\varepsilon^p(u_\varepsilon) = \int_\Omega |\nabla u_\varepsilon|^p dx$, we can also assume that ∇u_ε , as a $L^p(\Omega, \mathbb{R}^{2 \times 2})$ -bounded sequence, generates the Young measure $\nu = (\nu_x)_{x \in \Omega} \in \mathbf{Y}^p(\Omega, \mathbb{R}^{2 \times 2})$. By condition $u_\varepsilon(x) \in \varepsilon\mathcal{Z}$ and Lemma 2.18 we infer $\nabla u_\varepsilon(x) \in K$ defined in (32) for almost every $x \in \Omega$. By Lemma 2.16 we find that

$$\text{supp } \nu_x \subset K \text{ for almost every } x \in \Omega. \quad (33)$$

Since $p > 1$, the center of mass $\bar{\nu}_x := \int_{\mathbb{R}^{2 \times 2}} M d\nu_x(M)$ equals $\nabla u(x)$ for almost every $x \in \Omega$. Indeed, for a given matrix M , the linear-growth function $\Phi(M) = M_{i,j}$ is such that $\Phi(\nabla u_\varepsilon)$ is p -uniformly integrable, since $p > 1$, and therefore

$$\int_\Omega \varphi(x) \int_{\mathbb{R}^{2 \times 2}} M_{i,j} d\nu_x(M) dx = \lim_{\varepsilon \rightarrow 0} \int_\Omega \varphi(x) (\nabla u_\varepsilon)_{i,j}(x) dx = \int_\Omega \varphi(x) (\nabla u)_{i,j}(x) dx,$$

for all $\varphi \in C(\bar{\Omega})$, where we have used the weak convergence $\nabla u_\varepsilon \rightharpoonup \nabla u$. Thanks to the properties above satisfied by ν , we can employ Proposition 2.17 (i) to find

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} F_\varepsilon^p(u_\varepsilon) &= \liminf_{\varepsilon \rightarrow 0} \int_\Omega |\nabla u_\varepsilon|^p dx \geq \int_\Omega \int_{\mathbb{R}^{2 \times 2}} |M|^p d\nu_x(M) dx \\ &\geq \int_\Omega \inf \left\{ \int_{\mathbb{R}^{2 \times 2}} |M|^p d\nu(M) : \nu \in \mathcal{P}(\mathbb{R}^{2 \times 2}), \text{supp } \nu \subset K, \bar{\nu} = \nabla u(x) \right\} dx. \end{aligned}$$

Thus, a lower bound $f_{\mathcal{Y}, \text{hom}}^p : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}$ ($\mathcal{Y}$ stands for Young measures) is defined as

$$f_{\mathcal{Y}, \text{hom}}^p(A) := \inf \left\{ \int_{\mathbb{R}^{2 \times 2}} |M|^p d\nu(M) : \nu \in \mathcal{P}(\mathbb{R}^{2 \times 2}), \text{supp } \nu \subset K, \bar{\nu} = A \right\}.$$

Since $\text{supp } \nu \subset K = K_1 \cup K_2$, with $K_1 = e_1 \otimes \mathbb{R}^2$ and $K_2 = e_2 \otimes \mathbb{R}^2$, we can write ν as a convex combination of two probability measures

$$\nu = \lambda \nu_1 + (1 - \lambda) \nu_2,$$

where ν_1 is supported on K_1 and ν_2 is supported on K_2 . Explicitly, ν_1 is defined as $\nu_1 = \frac{\nu \llcorner K_1}{\nu(K_1)} = \frac{\nu \llcorner K_1}{\lambda}$, and ν_2 accordingly. The moment condition $\bar{\nu} = A$ yields

$$A = \lambda \bar{\nu}_1 + (1 - \lambda) \bar{\nu}_2.$$

Since $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$ and ν_i are supported in K_i , which are convex subspaces of $\mathbb{R}^{2 \times 2}$, then also their center of masses $\bar{\nu}_i \in K_i$; hence, the moment condition row-wise reads

$$\bar{\nu}_1 = \frac{1}{\lambda} \begin{pmatrix} A_1 \\ 0 \end{pmatrix} = \frac{1}{\lambda} e_1 \otimes A_1 \quad \text{and} \quad \bar{\nu}_2 = \frac{1}{1 - \lambda} \begin{pmatrix} 0 \\ A_2 \end{pmatrix} = \frac{1}{1 - \lambda} e_2 \otimes A_2.$$

By Jensen's inequality, applied to each $\nu_i \in \mathcal{P}(\mathbb{R}^{2 \times 2})$,

$$\begin{aligned} \int_{\mathbb{R}^{2 \times 2}} |M|^p d\nu(M) &= \lambda \int_{\mathbb{R}^{2 \times 2}} |M|^p d\nu_1(M) + (1 - \lambda) \int_{\mathbb{R}^{2 \times 2}} |M|^p d\nu_2(M) \\ &\geq \lambda |\bar{\nu}_1|^p + (1 - \lambda) |\bar{\nu}_2|^p = \frac{|A_1|^p}{\lambda^{p-1}} + \frac{|A_2|^p}{(1 - \lambda)^{p-1}} \geq (|A_1| + |A_2|)^p. \end{aligned} \quad (34)$$

The minimum λ in the above expression is given by

$$\lambda_{\min} = \frac{|A_1|}{|A_1| + |A_2|}.$$

The probability measure minimizing the problem defining $f_{\mathcal{Y}, \text{hom}}^p(A)$ is given by the convex combination of Dirac deltas centered on the center of masses $\bar{\nu}_1$ and $\bar{\nu}_2$, namely

$$\nu_{\min} = \lambda \delta_{\frac{1}{\lambda} e_1 \otimes A_1} + (1 - \lambda) \delta_{\frac{1}{1-\lambda} e_2 \otimes A_2},$$

with $\lambda = \lambda_{\min}$. Indeed $\text{supp } \nu_{\min} = \{\frac{1}{\lambda} e_1 \otimes A_1\} \cup \{\frac{1}{1-\lambda} e_2 \otimes A_2\} \subset K$ and it holds

$$\bar{\nu}_{\min} = \lambda \frac{1}{\lambda} e_1 \otimes A_1 + (1 - \lambda) \frac{1}{1 - \lambda} e_2 \otimes A_2 = A.$$

The integral gives

$$\int_{\mathbb{R}^{2 \times 2}} |M|^p d\nu_{\min}(M) = \lambda \frac{|A_1|^p}{\lambda^p} + (1 - \lambda) \frac{|A_2|^p}{(1 - \lambda)^p} = (|A_1| + |A_2|)^p.$$

Remark 2.19. In the rank-one case $A = a \otimes n$, since $A_i = a_i n$,

$$(|A_1| + |A_2|)^p = (|a_1| + |a_2|)^p |n|^p = (|a_1| + |a_2|)^p,$$

if n is a unit vector. We thus recover the previous formula of f_{hom}^p .

The following proposition allows us to produce a recovery sequence for rank-1 matrices (see Corollary 2.21).

Proposition 2.20. *Let $A = a \otimes n \in \mathbb{R}^{2 \times 2}$ with $a \in \mathbb{R}^2$ and $n \in S^1$. Set $\lambda = \frac{|a_1|}{|a_1| + |a_2|}$ and*

$$B_1 := \frac{1}{\lambda} \begin{pmatrix} a_1 n \\ 0 \end{pmatrix} = \frac{a_1}{\lambda} e_1 \otimes n, \quad B_2 := \frac{1}{1 - \lambda} \begin{pmatrix} 0 \\ a_2 n \end{pmatrix} = \frac{a_2}{1 - \lambda} e_2 \otimes n.$$

The homogeneous Young measure

$$\nu_x = \lambda \delta_{B_1} + (1 - \lambda) \delta_{B_2} \quad \text{for almost every } x \in \Omega$$

is a gradient Young measure, with generating sequence ∇u_ε such that $u_\varepsilon(x) \in \varepsilon \mathcal{Z}$ almost everywhere and ∇u_ε is p -equi-integrable.

Proof. Given $a \in \mathbb{R}^2$, let $a_k \in \mathbb{Q}^2$ with $a_k \rightarrow a$ as $k \rightarrow \infty$, and for each k let $T_k \in \mathbb{N}$ be such that $T_k a_k \in \mathbb{Z}^2$. Set $\beta_0 := (\frac{1}{2}, \frac{1}{2}) \in \mathbb{R}^2$, $z_k(t) := \beta_0 + t a_k$, for $t \in \mathbb{R}$, and let $v_k : [0, T_k] \rightarrow \mathcal{Z}$ be the path given by Lemma 2.6 (with $\varepsilon = 1$ and $L = T_k$) from β_0 to $\beta_0 + T_k a_k$, which are vertices of $\mathcal{Z}$. With

$$\lambda_k := \frac{|a_{k,1}|}{|a_{k,1}| + |a_{k,2}|}, \quad v_{k,i} := \text{sign}(a_{k,i}) (|a_{k,1}| + |a_{k,2}|) \quad (i = 1, 2),$$

we have $\lambda_k v_{k,1} = a_{k,1}$ and $(1 - \lambda_k) v_{k,2} = a_{k,2}$, and, by Lemma 2.6(i), there exist disjoint sets $E_1^k, E_2^k \subset (0, T_k)$, with $|E_1^k| = \lambda_k T_k$ and $|E_2^k| = (1 - \lambda_k) T_k$, such that

$$v'_k(t) = \begin{cases} v_{k,1} e_1, & t \in E_1^k \\ v_{k,2} e_2, & t \in E_2^k. \end{cases}$$

The speeds $|v_{k,i}| = |a_{k,1}| + |a_{k,2}|$ converge to $|a_1| + |a_2|$, and $|z_k(t) - v_k(t)| \leq \sqrt{2}$ for all $t \in [0, T_k]$ by Lemma 2.6(ii). Since $v_k(0) = z_k(0)$ and $v_k(T_k) = z_k(T_k)$, the function $w_k := v_k - z_k$ belongs to $W_0^{1,\infty}((0, T_k), \mathbb{R}^2)$; we extend it T_k -periodically to $\mathbb{R}$ and set $v_k := z_k + w_k$ on $\mathbb{R}$. Then $v_k(t + T_k) = v_k(t) + T_k a_k$, and, since $T_k a_k \in \mathbb{Z}^2$ and $\mathcal{Z} + \mathbb{Z}^2 = \mathcal{Z}$, the extended v_k still takes values in $\mathcal{Z}$. We claim that, for a suitable sequence $\varepsilon(k) \rightarrow 0$ as $k \rightarrow \infty$, the maps $u_{\varepsilon(k)}(x) = \varepsilon(k) v_k(n \cdot x / \varepsilon(k))$ are the desired sequence. For fixed k we consider

$$u_\varepsilon(x) = \varepsilon v_k\left(\frac{n \cdot x}{\varepsilon}\right), \quad x \in \Omega,$$

and check the following properties.

1. Since v_k takes values in $\mathcal{Z}$, $u_\varepsilon(x) \in \varepsilon \mathcal{Z}$ for every $x \in \Omega$.
2. By the chain rule,

$$\nabla u_\varepsilon(x) = v'_k\left(\frac{n \cdot x}{\varepsilon}\right) \otimes n$$

and since v_k is piecewise affine with two alternating slope values $v_{k,1}$ and $v_{k,2}$ which are uniformly close (up to a sign) to $|a_1| + |a_2|$, its derivative is uniformly bounded and therefore ∇u_ε is bounded as well; this fact gives the p -equi-integrability.

3. It remains to check the generation of the Young measure ν . We note that $v'_k(t) = w'_k(t) + a_k$. Since w'_k is a periodic function as w_k , and a_k is a constant, we deduce that v'_k is also T_k -periodic. Therefore, by the Riemann-Lebesgue lemma, for a fixed measurable subset $E \subset \mathbb{R}$ we have

$$v'_k\left(\frac{t}{\varepsilon}\right) \rightharpoonup \frac{1}{T_k} \int_0^{T_k} v'_k(t) dt, \quad \text{in } L^p(E), \text{ as } \varepsilon \rightarrow 0.$$

Now take $\varphi \in C_0(\Omega)$ and $h \in C_0(\mathbb{R}^{2 \times 2})$, with the notation $x = x_1 n + x_2 n^\perp$ and for each $x_2 \in \mathbb{R}$, define $\Omega_{x_2} := \{x_1 \in \mathbb{R} : x = x_1 n + x_2 n^\perp \in \Omega\}$. By Fubini's theorem and the

Riemann–Lebesgue lemma in the variable x_1 , we have

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \varphi(x) h(\nabla u_{\varepsilon}(x)) dx \\
&= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \varphi(x) h\left(v'_k\left(\frac{n \cdot x}{\varepsilon}\right) \otimes n\right) dx \\
&= \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}} \int_{\Omega_{x_2}} \varphi(x_1, x_2) h\left(v'_k\left(\frac{x_1}{\varepsilon}\right) \otimes n\right) dx_1 dx_2 \\
&= \int_{\mathbb{R}} \int_{\Omega_{x_2}} \varphi(x_1, x_2) \left(\frac{1}{T_k} \int_0^{T_k} h\left(v'_k(t) \otimes n\right) dt\right) dx_1 dx_2 \\
&= \int_{\mathbb{R}} \int_{\Omega_{x_2}} \varphi(x_1, x_2) \left(\frac{|E_1^k|}{T_k} h(v_{k,1} e_1 \otimes n) + \frac{|E_2^k|}{T_k} h(v_{k,2} e_2 \otimes n)\right) dx_1 dx_2 \\
&= \int_{\mathbb{R}} \int_{\Omega_{x_2}} \varphi(x_1, x_2) \left(\lambda_k h\left(\frac{a_{k,1}}{\lambda_k} e_1 \otimes n\right) + (1 - \lambda_k) h\left(\frac{a_{k,2}}{1 - \lambda_k} e_2 \otimes n\right)\right) dx_1 dx_2 \\
&= \int_{\Omega} \varphi(x) dx \langle \lambda_k \delta_{B_1^k} + (1 - \lambda_k) \delta_{B_2^k}, h \rangle = \int_{\Omega} \varphi(x) dx \langle \nu_k, h \rangle,
\end{aligned}$$

where $\nu_k = \lambda_k \delta_{B_1^k} + (1 - \lambda_k) \delta_{B_2^k}$, $B_1^k = \frac{a_{k,1}}{\lambda_k} e_1 \otimes n$, and $B_2^k = \frac{a_{k,2}}{1 - \lambda_k} e_2 \otimes n$.

Since $\nu_k \xrightarrow{*} \nu$ in $C_0(\mathbb{R}^{2 \times 2})^*$ as $k \rightarrow \infty$, we can take a diagonal sequence $\varepsilon(k) \rightarrow 0$ as $k \rightarrow \infty$ such that $\nabla u_{\varepsilon(k)} \xrightarrow{\mathbf{Y}} \nu$ as $k \rightarrow \infty$. $\square$

Corollary 2.21. *In the notation of Proposition 2.20, the generating sequence ∇u_{ε} gives a recovery sequence for the linear function $u(x) = Ax$, namely*

$$\lim_{\varepsilon \rightarrow 0} F_{\varepsilon}^p(u_{\varepsilon}) = \lim_{\varepsilon \rightarrow 0} \int_{\Omega} |\nabla u_{\varepsilon}|^p dx = |\Omega|(|a_1| + |a_2|)^p = |\Omega| f_{\text{hom}}^p(A).$$

Proof. Since ∇u_{ε} is p -equi-integrable and generates $\nu_x = \lambda \delta_{B_1} + (1 - \lambda) \delta_{B_2}$ with B_1 and B_2 as in Proposition 2.20, by Proposition 2.17 (ii) applied to $F(M) = |M|^p$ we find

$$\begin{aligned}
\lim_{\varepsilon \rightarrow 0} \int_{\Omega} |\nabla u_{\varepsilon}|^p dx &= \int_{\Omega} \int_{\mathbb{R}^{2 \times 2}} |M|^p d\nu_x(M) dx = |\Omega| \left(\lambda \frac{|a_1|^p}{\lambda^p} + (1 - \lambda) \frac{|a_2|^p}{(1 - \lambda)^p} \right) \\
&= |\Omega|(|a_1| + |a_2|)^p,
\end{aligned}$$

and conclude the proof. $\square$

3 Quasiconvexity properties of f_{hom}^p

In this section we complete the proof of Theorem A, showing that the function f_{hom}^p is not convex. Since it satisfies a growth condition of order $p < 2$ from above, this implies

also that it is not polyconvex, since polyconvex non-convex functions satisfy grow at least quadratically in some direction. The goal of this section is then to find two constants $0 < c_p \leq C_p$ such that

$$(|A_1| + |A_2|)^p + c_p \frac{|\det A|}{|A|^{2-p}} \leq f_{\text{hom}}^p(A) \leq (|A_1| + |A_2|)^p + C_p \frac{|\det A|}{|A|^{2-p}},$$

with $c_p, C_p \simeq (2-p)^{-1}$ as $p \rightarrow 2^-$. In particular, the lower bound shows the non-convexity of f_{hom}^p , as remarked in the Introduction; see (6).

Lemma 3.1 (qualified Jensen estimate). *Let $\nu \in \mathcal{P}(\mathbb{R}^{2 \times 2})$ be a probability measure supported on K , defined in (32), with barycenter $\bar{\nu} = A$. Let $\Lambda \geq \kappa_p |A|$, where $\kappa_p := \sqrt{2}(2\sqrt{2}p)^{1/(p-1)}$. Then*

$$\int_{\mathbb{R}^{2 \times 2}} |M|^p d\nu \geq (|A_1| + |A_2|)^p + \frac{1}{2} \int_{|M| > \Lambda} |M|^p d\nu.$$

Proof. We recall the row-wise matrix norm $N(A) = (|A_1| + |A_2|)$ in (5), and the bound $N(A) \leq \sqrt{2}|A|$. We split ν at level Λ ; that is,

$$\nu = \nu|_{\{|M| \leq \Lambda\}} + \nu|_{\{|M| > \Lambda\}} =: \nu_b + \nu_t \quad (\text{bulk and tail}).$$

Let

$$T := \int_{\mathbb{R}^{2 \times 2}} M d\nu_t \in \mathbb{R}^{2 \times 2} \quad \text{and} \quad E_p := \int_{\mathbb{R}^{2 \times 2}} |M|^p d\nu_t = \int_{\{|M| > \Lambda\}} |M|^p d\nu$$

be the first and p -th moments of the tail. Hence the bulk has first moment $\bar{\nu}_b = A - T$. Both ν_b and ν_t are supported in K ; in particular, we can apply to ν_b the two-branch Jensen computation applied to ν above.

The first idea is to apply the double Jensen inequality as in (34). Split ν_b as a combination of two probability measures

$$\nu_b = \lambda \nu_{b,1} + (m_b - \lambda) \nu_{b,2},$$

with the mass $m_b := |\nu_b| \leq 1$ and $\lambda \in [0, m_b]$. Then by the moment condition $A - T = \bar{\nu}_b$ we get $A - T = \lambda \bar{\nu}_{b,1} + (m_b - \lambda) \bar{\nu}_{b,2}$. Since $\nu_{b,i}$ are supported in $e_i \otimes \mathbb{R}^2$ subspaces of $\mathbb{R}^{2 \times 2}$, the moment condition row-wise reads

$$\bar{\nu}_{b,1} = \frac{1}{\lambda} e_1 \otimes (A_1 - T_1) \quad \text{and} \quad \bar{\nu}_{b,2} = \frac{1}{m_b - \lambda} e_2 \otimes (A_2 - T_2).$$

By Jensen's inequality applied to each probability measure $\nu_{b,i}$

$$\begin{aligned}
\int_{\{|M| \leq \Lambda\}} |M|^p d\nu &= \int_{\mathbb{R}^{2 \times 2}} |M|^p d\nu_b(M) \\
&= \lambda \int_{\mathbb{R}^{2 \times 2}} |M|^p d\nu_{b,1}(M) + (m_b - \lambda) \int_{\mathbb{R}^{2 \times 2}} |M|^p d\nu_{b,2}(M) \\
&\geq \lambda |\bar{\nu}_{b,1}|^p + (m_b - \lambda) |\bar{\nu}_{b,2}|^p = \frac{|A_1 - T_1|^p}{\lambda^{p-1}} + \frac{|A_2 - T_2|^p}{(m_b - \lambda)^{p-1}} \\
&\geq m_b^{1-p} (|A_1 - T_1| + |A_2 - T_2|)^p = m_b^{1-p} N(A - T)^p \\
&\geq N(A - T)^p.
\end{aligned}$$

The minimum is given by $\lambda/m_b = |A_1 - T_1|/(|A_1 - T_1| + |A_2 - T_2|)$. Hence, the bulk contributes with at least $N(A - T)^p$.

On the tail $|M| > \Lambda$, since $1 < |M|/\Lambda \leq |M|^p/\Lambda^p$ as $p > 1$, it holds

$$|T| \leq \int_{\{|M| > \Lambda\}} |M| d\nu = \Lambda \int_{\{|M| > \Lambda\}} \frac{|M|}{\Lambda} d\nu \leq \Lambda \int_{\{|M| > \Lambda\}} \frac{|M|^p}{\Lambda^p} d\nu = \Lambda^{1-p} E_p.$$

We now want to estimate $N(A - T)^p$ with $N(A)^p$ plus other terms, using the convexity of $t \mapsto t^p$. Since N is a norm, it holds $N(A - T) \geq |N(A) - N(T)|$. Applying the convexity inequality $t^p \geq s^p + ps^{p-1}(t - s)$ with $t = |N(A) - N(T)| \geq 0$ and $s = N(A) \geq 0$, we get

$$N(A - T)^p \geq |N(A) - N(T)|^p \geq N(A)^p + pN(A)^{p-1}(|N(A) - N(T)| - N(A))$$

Moreover we have $(|N(A) - N(T)| - N(A)) \geq -N(T) \geq -\sqrt{2}|T| \geq -\sqrt{2}\Lambda^{1-p}E_p$, thus

$$N(A - T)^p \geq N(A)^p - \sqrt{2}pN(A)^{p-1}\Lambda^{1-p}E_p.$$

Finally, the choice of $\Lambda \geq \kappa_p|A|$ with κ_p as in the statement makes the second term on the right-hand side at most $E_p/2$, since by $N(A) \leq \sqrt{2}|A|$ we get $-\sqrt{2}pN(A)^{p-1}\Lambda^{1-p} \geq -\frac{1}{2}$. All the contributions now read

$$\int_{\mathbb{R}^{2 \times 2}} |M|^p d\nu = \int_{\{|M| \leq \Lambda\}} |M|^p d\nu + E_p \geq N(A - T)^p + E_p \geq N(A)^p + \frac{E_p}{2},$$

as desired, by the definition of E_p . □

For $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ we denote by

$$\mathcal{M}f(x) := \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)| dy, \quad x \in \mathbb{R}^d,$$

the Hardy–Littlewood maximal function of f . The function $\mathcal{M}f$ is lower semicontinuous, so its superlevel sets are open, and $\mathcal{M}$ is of weak type $(1, 1)$ [63, Chapter I, Theorem 1]:

there is $c = c(d)$ such that $|\{\mathcal{M}f > \lambda\}| \leq \frac{c}{\lambda} \|f\|_{L^1(\mathbb{R}^d)}$ for all $f \in L^1(\mathbb{R}^d)$ and $\lambda > 0$. Since $\mathcal{M}f \leq \mathcal{M}(f\mathbf{1}_{\{|f|>\lambda/2\}}) + \lambda/2$, applying this to $f\mathbf{1}_{\{|f|>\lambda/2\}}$ gives the refined estimate

$$|\{\mathcal{M}f > \lambda\}| \leq \frac{2c}{\lambda} \int_{\{|f|>\lambda/2\}} |f| dx \quad \text{for all } f \in L^1(\mathbb{R}^d), \lambda > 0. \quad (35)$$

We shall use the following Lipschitz truncation lemma, which goes back to Acerbi and Fusco [1, 2]; see also [6, Theorem 5.34].

Lemma 3.2 (Lipschitz truncation). *Let $1 \leq p < \infty$, $w \in W^{1,p}(\mathbb{R}^d, \mathbb{R}^m)$, $\lambda > 0$, and $E_\lambda := \{\mathcal{M}(|\nabla w|) > \lambda\}$. Then there exists a Lipschitz map $\tilde{w}: \mathbb{R}^d \rightarrow \mathbb{R}^m$ such that $\tilde{w} = w$ almost everywhere in $\mathbb{R}^d \setminus E_\lambda$, and $\text{Lip}(\tilde{w}) \leq c_d \lambda$, where c_d depends only on d ; in particular, c_d is independent of p , λ and w . If w has compact support, then E_λ is bounded and $\tilde{w}$ has compact support.*

Using this result we prove the estimate in the following lemma.

Lemma 3.3 (gradient estimates in terms of the Jacobian). *Let D be bounded open and $u - Ax \in W_0^{1,p}(D, \mathbb{R}^2)$, with $\det \nabla u = 0$ almost everywhere. Then for all $\lambda \geq 4|A|$,*

$$\int_{|\nabla u|>\lambda/4} |\nabla u|^p dx \geq c \frac{p-1}{2-p} \lambda^{p-2} |\det A| |D|,$$

with c universal constant.

Proof. Set $w := u - Ax \in W_0^{1,p}(D, \mathbb{R}^2)$, extended by zero to $\mathbb{R}^2$, and let $E_\lambda := \{\mathcal{M}(|\nabla w|) > \lambda\}$. By Lemma 3.2 there exists a compactly supported Lipschitz map $\tilde{w}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, in particular $\tilde{w} \in W^{1,\infty}(\mathbb{R}^2, \mathbb{R}^2)$, such that $\tilde{w} = w$ almost everywhere in $\mathbb{R}^2 \setminus E_\lambda$ and $\text{Lip}(\tilde{w}) \leq c_2 \lambda$, where c_2 is independent of p .

Since $\tilde{w}$ is Lipschitz with compact support, the null-Lagrangian property of the determinant (valid for compactly supported maps in $W^{1,q}$, $q \geq 2$) gives

$$\int_{\mathbb{R}^2} (\det(A + \nabla \tilde{w}) - \det A) dx = 0.$$

We split $\mathbb{R}^2$ into $D \setminus E_\lambda$, E_λ and $\mathbb{R}^2 \setminus (D \cup E_\lambda)$. Since $\tilde{w} = w$ on $\mathbb{R}^2 \setminus E_\lambda$, we have $\nabla \tilde{w} = \nabla w$ almost everywhere there. Hence on $D \setminus E_\lambda$ the integrand equals $\det \nabla u - \det A = -\det A$, because $\det \nabla u = 0$ almost everywhere; on $\mathbb{R}^2 \setminus (D \cup E_\lambda)$ we have $\tilde{w} = w = 0$, so the integrand vanishes. Therefore

$$|D \setminus E_\lambda| \det A = \int_{E_\lambda} (\det(A + \nabla \tilde{w}) - \det A) dx.$$

On E_λ we use the local Lipschitz estimate on the determinant, which is a polynomial of degree 2, $|\det(\xi_1) - \det(\xi_2)| \leq (|\xi_1| + |\xi_2|)|\xi_1 - \xi_2|$, which gives

$$|\det(A + \nabla \tilde{w}) - \det A| \leq (|A + \nabla \tilde{w}| + |A|)|\nabla \tilde{w}| \leq C\lambda^2,$$

as $|\nabla \tilde{w}| \leq c_2 \lambda$ and $|A| \leq \lambda/4$. The null-Lagrangian identity now reads

$$|D \setminus E_\lambda| |\det A| \leq C \lambda^2 |E_\lambda|.$$

Writing $|D| = |D \setminus E_\lambda| + |D \cap E_\lambda| \leq |D \setminus E_\lambda| + |E_\lambda|$, we arrive at the estimate

$$|D| |\det A| \leq |D \setminus E_\lambda| |\det A| + |E_\lambda| |\det A| \leq (C \lambda^2 + |\det A|) |E_\lambda| \leq C' \lambda^2 |E_\lambda|,$$

using $|\det A| \leq \frac{1}{2} |A|^2 \leq \frac{1}{2} \frac{\lambda^2}{16}$.

The weak $(1, 1)$ type estimate (35) gives

$$|E_\lambda| \leq \frac{C}{\lambda} \int_{\{|\nabla w| > \lambda/2\}} |\nabla w| dx,$$

so that

$$|D| |\det A| \leq C \lambda \int_{\{|\nabla w| > \lambda/2\}} |\nabla w| dx.$$

Fix $\lambda_0 \geq 4|A|$. The last inequality holds for every $\lambda \geq \lambda_0$, since then $\lambda \geq 4|A|$. Multiplying it by λ^{p-3} , integrating over $[\lambda_0, \infty)$ and using Fubini's theorem, we get

$$\begin{aligned} \frac{\lambda_0^{p-2}}{2-p} |D| |\det A| &\leq C \int_{\lambda_0}^{\infty} \lambda^{p-2} \int_{\{|\nabla w| > \lambda/2\}} |\nabla w| dx d\lambda \\ &= C \int_{\{|\nabla w| > \lambda_0/2\}} |\nabla w| \left(\int_{\lambda_0}^{2|\nabla w|} \lambda^{p-2} d\lambda \right) dx \\ &\leq C \int_{\{|\nabla w| > \lambda_0/2\}} |\nabla w| \left(\int_0^{2|\nabla w|} \lambda^{p-2} d\lambda \right) dx \\ &= C \frac{2^{p-1}}{p-1} \int_{\{|\nabla w| > \lambda_0/2\}} |\nabla w|^p dx. \end{aligned}$$

To convert the bounds on u rather than on w , first note that on $\{|\nabla w| > \lambda_0/2\}$ one has

$$\frac{\lambda_0}{2} < |\nabla w| = |\nabla u - A| \leq |\nabla u| + |A| \leq |\nabla u| + \frac{\lambda_0}{4},$$

so that $|\nabla u| > \lambda_0/4$. Moreover, by the same line above, $|\nabla w| \leq |\nabla u| + \frac{\lambda_0}{4} \leq |\nabla u| + \frac{|\nabla w|}{2}$, so that $|\nabla w| \leq 2|\nabla u|$. Therefore, we find

$$\frac{\lambda_0^{p-2}}{2-p} |D| |\det A| \leq C' \frac{2^{p-1}}{p-1} \int_{|\nabla u| > \lambda_0/4} |\nabla u|^p dx,$$

which we rewrite as

$$\int_{|\nabla u| > \lambda_0/4} |\nabla u|^p dx \geq c \frac{p-1}{2-p} \lambda_0^{p-2} |\det A| |D|;$$

that is, the claim with $\lambda = \lambda_0$ and $c = 1/C'$ universal constant independent of p . $\square$

Proposition 3.4 (lower bound on f_{hom}^p). *There exists a constant $c_p > 0$ such that for every A*

$$f_{\text{hom}}^p(A) \geq N(A)^p + c_p \frac{|\det A|}{|A|^{2-p}},$$

where N is the norm in (5). The constant $c_p \simeq (2-p)^{-1}$ as $p \rightarrow 2^-$.

Proof. If $\det A = 0$ the inequality is trivial. In the full-rank case, recall the cell formula (2.7)

$$f_{\text{hom}}^p(A) = \lim_{T \rightarrow +\infty} \frac{|\det A|}{T^2} \inf \left\{ \int_{TQ_A} |\nabla u|^p dx : u \in W^{1,p}(TQ_A, \mathcal{Z}), u = u_A \sim \partial TQ_A \right\}.$$

Fix T an odd integer, so that $\partial(TQ) \subset \mathcal{Z}$ and hence $P_0(Ax) = Ax$ on ∂D , $D = A^{-1}(TQ)$. Let u be a competitor, so that $u - Ax \in W_0^{1,p}(D, \mathbb{R}^2)$, and let ν_u be the gradient distribution, explicitly

$$\nu_u := (\nabla u)_{\#} \left(\frac{1}{|D|} \mathcal{L}^2 \llcorner D \right), \quad \int_{\mathbb{R}^{2 \times 2}} g(M) d\nu_u = \int_D g(\nabla u) dx$$

for every Borel $g \geq 0$. Then ν_u is a probability measure on $\mathbb{R}^{2 \times 2}$ with $\nu_u(\mathbb{R}^{2 \times 2} \setminus K) = 0$, since $\nabla u \in K$ almost everywhere by Lemma 2.18. Its first moment is finite, and

$$\int_{\mathbb{R}^{2 \times 2}} M d\nu_u = \int_D \nabla u dx = A,$$

because $\int_D \nabla(u - Ax) dx = 0$ for $u - Ax \in W_0^{1,p}(D)$. Finally, $\det \nabla u = 0$ almost everywhere in D , since every matrix in K has a zero row. Note that

$$\int_{\{|M| > \Lambda\}} |M|^p d\nu_u = \int_D |\nabla u|^p \chi_{\{|\nabla u| > \Lambda\}} dx,$$

so Lemma 3.1, applied to ν_u , and Lemma 3.3, applied to u , estimate the same quantity. We choose

$$\Lambda := \kappa_p |A|, \quad \kappa_p := \sqrt{2} (2\sqrt{2}p)^{\frac{1}{p-1}}.$$

As shown above, ν_u is a probability measure concentrated on K , with finite first moment and barycenter A , and $\Lambda \geq \kappa_p |A|$. Hence

$$\begin{aligned} \int_D |\nabla u|^p dx &= \int |M|^p d\nu_u \geq N(A)^p + \frac{1}{2} \int_{\{|M| > \Lambda\}} |M|^p d\nu_u \\ &= N(A)^p + \frac{1}{2} \int_D |\nabla u|^p \chi_{\{|\nabla u| > \Lambda\}} dx. \end{aligned}$$

Recall $\det \nabla u = 0$ almost everywhere in D and $u - Ax \in W_0^{1,p}(D, \mathbb{R}^2)$. We apply Lemma 3.3 with $\lambda := 4\Lambda = 4\kappa_p |A|$, which is admissible since $\lambda \geq 4|A|$, and obtain

$$\int_D |\nabla u|^p \chi_{\{|\nabla u| > \Lambda\}} dx = \frac{1}{|D|} \int_{\{|\nabla u| > \lambda/4\}} |\nabla u|^p dx \geq \frac{c(p-1)}{2-p} (4\kappa_p |A|)^{p-2} |\det A|.$$

Combining the two estimates,

$$\int_D |\nabla u|^p dx \geq N(A)^p + c_p \frac{|\det A|}{|A|^{2-p}}, \quad c_p := \frac{c(p-1)(4\kappa_p)^{p-2}}{2(2-p)}.$$

The right-hand side depends neither on u nor on T . Taking the infimum over all competitors and letting $T \rightarrow \infty$ along odd integers, we obtain

$$f_{\text{hom}}^p(A) \geq N(A)^p + c_p \frac{|\det A|}{|A|^{2-p}},$$

and conclude the proof. $\square$

Proposition 3.5 (upper bound on f_{hom}^p). *There exists a constant $C_p > 0$ such that for every A*

$$f_{\text{hom}}^p(A) \leq N(A)^p + C_p \frac{|\det A|}{|A|^{2-p}}.$$

The constant $C_p \simeq (2-p)^{-1}$ as $p \rightarrow 2^-$.

Proof. Decompose $A = \sigma_1 \eta_1 \otimes q_1 + \sigma_2 \eta_2 \otimes q_2$ with respect to its singular values σ_i , with $0 < \sigma_2 \leq \sigma_1$, $\eta_1 \perp \eta_2$ and $q_1 \perp q_2$ orthonormal. The idea of this decomposition into orthogonal rank-one tensors is to use the rank-one convexity of f_{hom}^p to find an upper bound. Write

$$R := \eta_1 \otimes q_1 + \eta_2 \otimes q_2 \in O(2), \quad A = \theta(\sigma_1 R) + (1-\theta)(\sigma_1 \eta_1 \otimes q_1), \quad \theta := \frac{\sigma_2}{\sigma_1} \in [0, 1].$$

The two matrices $\sigma_1 R$ and $\sigma_1 \eta_1 \otimes q_1$ differ by the rank-one matrix $\sigma_1 \eta_2 \otimes q_2$, so A is written as a convex combination along a rank-one direction. Moreover

$$f_{\text{hom}}^p(\sigma_1 R) = \sigma_1^p f_{\text{hom}}^p(I)$$

by the p -homogeneity (Lemma 2.12) and the frame indifference under $A \mapsto AQ$, $Q \in O(2)$. Using the rank-one convexity and $f_{\text{hom}}^p = N(\cdot)^p$ on the rank-one cone,

$$f_{\text{hom}}^p(A) \leq (1-\theta)(\sigma_1 |\eta_1|_{\ell^1})^p + \theta \sigma_1^p f_{\text{hom}}^p(I). \quad (36)$$

Denote for simplicity $A_0 := \sigma_1 \eta_1 \otimes q_1$ so that $N(A_0) = \sigma_1 |\eta_1|_{\ell^1}$.

Since $\theta \sigma_1^p = \sigma_2 \sigma_1^{p-1} = \sigma_1 \sigma_2 \sigma_1^{p-2} = |\det A| \sigma_1^{p-2}$, the right-hand side of (36) equals

$$N(A_0)^p + \theta \sigma_1^p (f_{\text{hom}}^p(I) - |\eta_1|_{\ell^1}^p) = N(A_0)^p + \frac{|\det A|}{\sigma_1^{2-p}} (f_{\text{hom}}^p(I) - |\eta_1|_{\ell^1}^p).$$

The bracket is nonnegative, since $f_{\text{hom}}^p(I) \geq N(I)^p = 2^p \geq 2^{p/2} \geq |\eta_1|_{\ell^1}^p$, and it is at most $f_{\text{hom}}^p(I) - 1$, since $|\eta_1|_{\ell^1} \geq |\eta_1| = 1$. We now want to bound $N(A_0)^p$ with $N(A)^p$ plus a term proportional to $|\det A| |A|^{p-2}$. First of all, using that N is a norm and $A - A_0 = \sigma_2 \eta_2 \otimes q_2$,

$$|N(A_0) - N(A)| \leq N(\sigma_2 \eta_2 \otimes q_2) = \sigma_2 |\eta_2|_{\ell^1} \leq \sqrt{2} \sigma_2,$$

because $|\eta_2|_{\ell^1} \leq \sqrt{2}|\eta_2| = \sqrt{2}$. For $s, h \geq 0$, the mean value theorem gives $(s+h)^p = s^p + p\xi^{p-1}h \leq s^p + p(s+h)^{p-1}h$. With $s = N(A)$ and $h = \sqrt{2}\sigma_2$,

$$N(A_0)^p \leq (N(A) + \sqrt{2}\sigma_2)^p \leq N(A)^p + \sqrt{2}p(N(A) + \sqrt{2}\sigma_2)^{p-1}\sigma_2.$$

Since $|A|^2 = \sigma_1^2 + \sigma_2^2$, we have $\sigma_2 \leq |A|$ and $\sigma_1 \geq |A|/\sqrt{2}$; moreover $N(A) \leq \sqrt{2}|A|$. Hence $N(A) + \sqrt{2}\sigma_2 \leq 2\sqrt{2}|A|$ and $\sigma_2 = |\det A|/\sigma_1 \leq \sqrt{2}|\det A|/|A|$, so that

$$N(A_0)^p \leq N(A)^p + 2p(2\sqrt{2})^{p-1} \frac{|\det A|}{|A|^{2-p}}.$$

To conclude, since $p-2 < 0$ and $\sigma_1 \geq |A|/\sqrt{2}$, we have $\sigma_1^{p-2} \leq 2^{\frac{2-p}{2}}|A|^{p-2}$. Therefore

$$f_{\text{hom}}^p(A) \leq N(A)^p + C_p \frac{|\det A|}{|A|^{2-p}}, \quad C_p := 2p(2\sqrt{2})^{p-1} + 2^{\frac{2-p}{2}}(f_{\text{hom}}^p(I) - 1).$$

The value of $f_{\text{hom}}^p(I)$ is bounded by $\frac{8}{3(2-p)}$ by (14). $\square$

Finally, we remark that from the explicit form of c_p and C_p , a direct computation allows one to rewrite the double estimate as in (4).

4 The supercritical case $p \geq 2$

In this section, we consider the case $p \geq 2$, in which the limit is represented as a simple but degenerate quasiconvex energy. For convenience, we recall the statement of the homogenization result, already given in the Introduction as Theorem B.

Theorem 4.1 (supercritical homogenization theorem). *Let $p \geq 2$. Then F_ε^p given in (1) Γ -converges, with respect to the strong L^p -topology, to*

$$F_{\text{hom}}^p(u) = \begin{cases} \int_{\Omega} (|\nabla u_1(x)| + |\nabla u_2(x)|)^p dx & \text{if } u \in W^{1,p}(\Omega, \mathbb{R}^2) \text{ and } \det \nabla u = 0 \text{ a.e.} \\ +\infty & \text{otherwise.} \end{cases} \quad (37)$$

The lower bound is obtained by noting that actually $F_\varepsilon^p = F_{\text{hom}}^p$ on $W^{1,p}(\Omega, \varepsilon \mathcal{Z})$.

Proposition 4.2 (lower bound). *The functional defined by (37) is a lower bound for the Γ -liminf of F_ε .*

Proof. We first note that the Γ -limit is finite only on functions with $\det \nabla u(x) = 0$ almost everywhere, by the weak continuity of the determinant in $W^{1,p}$, since $\det \nabla u_\varepsilon(x) = 0$ almost everywhere if $F_\varepsilon^p(u_\varepsilon) < +\infty$, and using the continuity of the determinant for $p \geq 2$.

Let $u_\varepsilon \rightarrow u$ be a sequence with $F_\varepsilon^p(u_\varepsilon) \leq C < +\infty$. We note that if $u_\varepsilon(x) \notin \frac{\varepsilon}{2} + \varepsilon\mathbb{Z}^2$ then for $|x - y|$ small we have either $(u_\varepsilon)_1(x) = (u_\varepsilon)_1(y)$ or $(u_\varepsilon)_2(x) = (u_\varepsilon)_2(y)$, so that we conclude that

$$|\nabla u_\varepsilon(x)| = |\nabla(u_\varepsilon)_1(x)| + |\nabla(u_\varepsilon)_2(x)|$$

for almost all $x \in \Omega$. By the convexity of the function $g(A) = (|A_1| + |A_2|)^p$, where A_1 and A_2 are the rows of A , we then have

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \int_{\Omega} |\nabla u_\varepsilon|^p dx &= \liminf_{\varepsilon \rightarrow 0} \int_{\Omega} g(\nabla u_\varepsilon) dx \\ &\geq \int_{\Omega} g(\nabla u) dx = \int_{\Omega} ||\nabla u_1(x)| + |\nabla u_2(x)||^p dx, \end{aligned}$$

and conclude the proof. $\square$

4.1 Recovery sequences for a class of special maps

In this section, we complete the proof of Theorem 4.1 for functions whose image is a curve. The proof generalizes the zig-zag construction for affine functions in the case $p < 2$. In the general case, the image is more complex, but for $p > 2$, thanks to the continuity of u , it can be described using some notions of analytic topology (next section). These allow us to consider a dense class of functions whose images are “trees”, on whose edges one can repeat the constructions below (or directly use Lemma 2.6).

Note that if $\det \nabla u = 0$, the two vectors $\nabla u_1(x)$ and $\nabla u_2(x)$ are linearly dependent, so that

$$||\nabla u_1(x)| + |\nabla u_2(x)|| = \max \left\{ |\nabla u_1(x) + \nabla u_2(x)|, |\nabla u_1(x) - \nabla u_2(x)| \right\}. \quad (38)$$

Proposition 4.3 (upper bound for special functions–I). *Let $u : \Omega \rightarrow \mathbb{R}^2$ be such that $u(\Omega)$ is a simple curve. Then there exists $u_\varepsilon \rightarrow u$ such that*

$$\lim_{\varepsilon \rightarrow 0} F_\varepsilon^p(u_\varepsilon) = \int_{\Omega} ||\nabla u_1(x)| + |\nabla u_2(x)||^p dx. \quad (39)$$

Proof. By approximation, we may consider the case when $u(\Omega)$ is composed of line segments. Furthermore, we may suppose that the direction τ of each such segment is rational (that is, $\tau \in \mathbb{R}^2$ with either $\tau_1 = 0$ or τ_2/τ_1 rational).

We first consider the case $u(\Omega) = [0, \alpha] =: S$ (where we have used the notation $[\alpha_1, \alpha_2]$ to denote the segment between the vectors α_1 and α_2 in $\mathbb{R}^2$) and suppose $\alpha_i = \varepsilon N \tau_i$ with $N \in \mathbb{N}$ and $\tau_i \in \mathbb{Z}$. This again is not restrictive up to an error of order ε .

We may approximate S by the family of line-segment curves S_ε defined by

$$S_\varepsilon := \bigcup_{k=0}^{N-1} \left(k\varepsilon\tau + \left([0, \varepsilon\tau_1] \times \{0\} \right) \cup \left(\{\varepsilon\tau_1\} \times [0, \varepsilon\tau_2] \right) \right).$$

Note that the extreme points of S_ε are the endpoints of S .

If $\tau_1\tau_2 \geq 0$ (so that we may suppose that both of them are non-negative) we consider the bijection $\Phi_\varepsilon : S \rightarrow S_\varepsilon$ defined by

$$\Phi_\varepsilon(\xi) = \begin{cases} k\varepsilon\tau + (\xi_1 + \xi_2 - k\varepsilon(\tau_1 + \tau_2), 0) & \text{if } \xi \in k\varepsilon\tau + \left[0, \frac{\tau_1}{\tau_1 + \tau_2}\varepsilon\tau\right] \\ k\varepsilon\tau + (\varepsilon\tau_1, \xi_1 + \xi_2 - \varepsilon((k+1)\tau_1 + k\tau_2)) & \text{if } \xi \in k\varepsilon\tau + \left[\frac{\tau_1}{\tau_1 + \tau_2}\varepsilon\tau, \varepsilon\tau\right], \end{cases}$$

and $k \in \{0, \dots, N-1\}$.

We then define $u_\varepsilon(x) = \Phi_\varepsilon(u(x))$, and note that

$$\left|\frac{\partial u_\varepsilon}{\partial x_i}\right| = \left|\frac{\partial u_1}{\partial x_i} + \frac{\partial u_2}{\partial x_i}\right| = \left|\frac{\partial u_1}{\partial x_i}\right| + \left|\frac{\partial u_2}{\partial x_i}\right|,$$

the last equality due to condition $\tau_2 \frac{\partial u_1}{\partial x_i} = \tau_1 \frac{\partial u_2}{\partial x_i}$ which is valid almost everywhere. This implies

$$\begin{aligned} |\nabla u_\varepsilon(x)|^2 &= \left|\frac{\partial u_\varepsilon}{\partial x_1}\right|^2 + \left|\frac{\partial u_\varepsilon}{\partial x_2}\right|^2 = \left|\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_1}\right|^2 + \left|\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_2}\right|^2 \\ &= \left(\frac{\tau_1 + \tau_2}{\tau_1}\right)^2 \left(\left|\frac{\partial u_1}{\partial x_1}\right|^2 + \left|\frac{\partial u_1}{\partial x_2}\right|^2\right) = \left(\frac{\tau_1 + \tau_2}{\tau_1}\right)^2 |\nabla u_1|^2. \end{aligned}$$

Hence, reasoning symmetrically for u_2 ,

$$|\nabla u_\varepsilon| = \frac{\tau_1 + \tau_2}{\tau_1} |\nabla u_1|, \quad |\nabla u_\varepsilon| = \frac{\tau_1 + \tau_2}{\tau_2} |\nabla u_2|.$$

We then have

$$|\nabla u_\varepsilon| = \frac{\tau_1}{\tau_1 + \tau_2} |\nabla u_\varepsilon| + \frac{\tau_2}{\tau_1 + \tau_2} |\nabla u_\varepsilon| = |\nabla u_1| + |\nabla u_2|.$$

If $\tau_1\tau_2 < 0$ then the bijection is defined analogously with $\xi_1 - \xi_2$ in place of $\xi_1 + \xi_2$, and the same argument gives

$$\left|\frac{\partial u_\varepsilon}{\partial x_i}\right| = \left|\frac{\partial u_1}{\partial x_i} - \frac{\partial u_2}{\partial x_i}\right| = \left|\frac{\partial u_1}{\partial x_i}\right| + \left|\frac{\partial u_2}{\partial x_i}\right|,$$

from which in all cases we derive $|\nabla u_\varepsilon(x)|^p = ||\nabla u_1(x)| + |\nabla u_2(x)||^p$, and we have the desired equality in the limit.

If $u(\Omega)$ is composed of more line segments, we can repeat the construction separately on each line segment. $\square$

Proposition 4.4 (upper bound for special functions-II). *Let $u : \Omega \rightarrow \mathbb{R}^2$ be a piecewise-affine function with $\det \nabla u = 0$ almost everywhere. Then there exist $u_\varepsilon \rightarrow u$ such that (39) holds.*

Proof. Up to an approximation argument, we may suppose that u takes values in $\varepsilon\mathbb{Z}$ at the vertices of each triangle T of its underlying triangulation. The condition $\det \nabla u = 0$ implies that $u(T)$ is a segment for all such T . Again by an approximation argument, we may also suppose that the direction of each such segment is rational and that we may separately apply the construction in the first part of the proof of Proposition 4.3 to the restriction of u to (the interior of) each T . We note now that if T and T' have as common side a segment with endpoints P and P' , then we have two cases: $u(P) \neq u(P')$, so that the image of T and that of T' lie in the same line. or $u(P) = u(P')$. In both cases, the constructions of Proposition 4.3 on T and T' agree on the segment $[P, P']$; in the second case, simply because $u_\varepsilon = u(P) = u(P')$ on the whole segment. $\square$

4.2 Recovery sequences for general maps

Throughout this subsection $\Omega \subset \mathbb{R}^2$ is a bounded convex open set, $p > 2$, and $u \in W^{1,p}(\Omega, \mathbb{R}^2)$ satisfies $\det \nabla u = 0$ almost everywhere; by Morrey's embedding we identify u with its continuous extension to $\bar{\Omega}$. We write $N(A) = |A_1| + |A_2|$, and for a continuous curve γ we let $\text{Var}_1(\gamma)$ denote its length with respect to $|\cdot|_1$.

The following result, together with Proposition 4.2, proves Theorem 4.1 in the case $p > 2$ and Ω convex.

Theorem 4.5. *Let $p > 2$, and let $u \in W^{1,p}(\Omega, \mathbb{R}^2)$ satisfy $\det \nabla u = 0$ almost everywhere. Then*

$$\Gamma\text{-}\limsup_{\varepsilon \rightarrow 0} F_\varepsilon^p(u) \leq \int_{\Omega} N(\nabla u)^p \, dx,$$

with respect to the strong L^p -topology.

Lemma 4.6. *Let $p > 2$, and let $u \in W^{1,p}(\Omega, \mathbb{R}^2)$ satisfy $\det \nabla u = 0$ almost everywhere. Then $|u(\bar{\Omega})| = 0$.*

Proof. Let $\tilde{u} \in W^{1,p}(\mathbb{R}^2, \mathbb{R}^2)$ be an extension of u . Since $p > 2$, $\tilde{u}$ is continuous and satisfies Lusin's condition (N) [53], so that the area formula holds. Hence

$$|u(\Omega)| \leq \int_{\mathbb{R}^2} \#(u^{-1}(y) \cap \Omega) \, dy = \int_{\Omega} |\det \nabla u| \, dx = 0.$$

Moreover, $|\partial\Omega| = 0$, so that $|u(\partial\Omega)| = |\tilde{u}(\partial\Omega)| = 0$ by (N). $\square$

Lemma 4.7 (factorization through a dendrite). *Let X be the set of connected components of the fibers $u^{-1}(y)$, $y \in u(\bar{\Omega})$, endowed with the quotient topology; let $q : \bar{\Omega} \rightarrow X$ map x to the component containing it and set $\psi(q(x)) := u(x)$. Then $u = \psi \circ q$, and:*

- (i) q is continuous, closed, surjective, and monotone; that is, its fibers are connected;
- (ii) ψ is continuous and light; that is, its fibers are totally disconnected;
- (iii) X is a dendrite; that is, a locally connected continuum containing no simple closed curve.

Proof. Items (i) and (ii) are standard, and can be directly derived from the monotone-light factorization [67, Theorem 4.2].

It remains to prove item (iii). Since $\bar{\Omega}$ is a locally connected continuum, so is its continuous image X . By Lemma 4.6 the compact set $u(\bar{\Omega})$ has an empty interior, hence $\dim u(\bar{\Omega}) \leq 1$. Since $\psi : X \rightarrow u(\bar{\Omega})$ is a closed surjection with zero-dimensional fibers, the Hurewicz formula for closed maps gives $\dim X \leq \dim u(\bar{\Omega}) \leq 1$ [43, Theorem VI.7].

We are left to prove that X contains no simple closed curve. Let $g : X \rightarrow S^1$ be continuous. Since $\bar{\Omega}$ is convex, $g \circ q = e^{ih}$ for some continuous $h : \bar{\Omega} \rightarrow \mathbb{R}$. On each fiber $q^{-1}(a)$, which is connected, $g \circ q$ is constant; hence h takes values in a discrete set there, and then it is constant on the fiber. Thus, $h = \bar{h} \circ q$, with $\bar{h}$ continuous because q is a quotient map and $g = e^{i\bar{h}}$ is null-homotopic. If $S \subset X$ were a simple closed curve, then, since $\dim X \leq 1$, a homeomorphism $S \rightarrow S^1$ would extend to a continuous map $X \rightarrow S^1$ [43]. Since this extension, and hence the homeomorphism, would be null-homotopic, this is a contradiction. $\square$

The following result will be used to prove the Hölder continuity of d_1 .

Lemma 4.8. *Let $p > 2$ and $v \in W^{1,p}(\mathbb{R}^2, \mathbb{R}^2)$, identified with its continuous representative. Then for every $x \in \mathbb{R}^2$ and every $R > 0$ the following holds for almost every $\omega \in S^1$: the map $\rho \mapsto v(x + \rho\omega)$ is absolutely continuous on $[0, R]$, and*

$$v(x + \rho\omega) - v(x) = \int_0^\rho \nabla v(x + s\omega) \omega \, ds \quad \text{for every } \rho \in [0, R].$$

Proof. In polar coordinates centered at x , for every $f \in L^p(B_R(x))$,

$$\int_{S^1} \int_0^R |f(x + s\omega)| \, ds \, d\omega = \int_{B_R(x)} \frac{|f(w)|}{|w - x|} \, dw \leq \|f\|_{L^p(B_R(x))} \|\cdot - x|^{-1}\|_{L^{p'}(B_R(x))},$$

and the last norm is finite because $p' = p/(p-1) < 2$. Let $v_k \in C^\infty$ with $v_k \rightarrow v$ in $W^{1,p}(B_R(x))$. By Morrey's inequality, $v_k \rightarrow v$ uniformly on $B_R(x)$. Applying the estimate to $f = \nabla v$ and $f = \nabla v_k - \nabla v$, and passing to a subsequence, we find that for almost every $\omega \in S^1$

$$\int_0^R |\nabla v(x + s\omega)| \, ds < \infty \quad \text{and} \quad \int_0^R |\nabla v_k - \nabla v|(x + s\omega) \, ds \rightarrow 0.$$

For such ω and all $\rho \in [0, R]$ we pass to the limit in

$$v_k(x + \rho\omega) - v_k(x) = \int_0^\rho \nabla v_k(x + s\omega) \omega \, ds,$$

using the uniform convergence on the left-hand side. $\square$

By Lemma 4.7, any two points $a, b \in X$ are joined by a unique arc $[a, b] \subset X$ [58]. We set

$$d_1(a, b) := \text{Var}_1(\psi; [a, b]),$$

the $|\cdot|_1$ -length of ψ along $[a, b]$ and $d_1(a, a) := 0$.

Recall that a metric space (Y, d) is an $\mathbb{R}$ -tree if any two points $a, b \in Y$ are joined by a unique arc $[a, b] \subset Y$, and this arc is a geodesic segment: that is, it is isometric to the interval $[0, d(a, b)]$. In particular, $d(a, b) = d(a, c) + d(c, b)$ for every $c \in [a, b]$. Lemma 4.9(iii) below shows that the arcs $[a, b]$ of X are geodesic segments for d_1 , so that (X, d_1) is an $\mathbb{R}$ -tree.

Lemma 4.9 (the taxicab tree). *We have the following.*

- (i) *For every continuous curve γ in $\bar{\Omega}$ from x to y , $d_1(q(x), q(y)) \leq \text{Var}_1(u \circ \gamma)$.*
- (ii) *There is $C = C(\Omega, p)$ such that $d_1(q(x), q(y)) \leq C \|\nabla u\|_{L^p} |x - y|^{1-2/p}$ for all $x, y \in \bar{\Omega}$.*
- (iii) *(X, d_1) is a compact $\mathbb{R}$ -tree with the topology of X , and $\psi : (X, d_1) \rightarrow (\mathbb{R}^2, |\cdot|_1)$ is 1-Lipschitz.*

Proof. (i) If z is an interior point of the arc $[a, b]$, every path from a to b in X passes through z . Otherwise, a and b would be in the same component of the open set $X \setminus \{z\}$; this component is arcwise connected, and would contain a second arc from a to b , contradicting the uniqueness of $[a, b]$. Let $a = q(x)$, $b = q(y)$, and let $a = \alpha_0, \dots, \alpha_m = b$ be points in order along $[a, b]$. Choosing successive first-hitting times, we find $0 = t_0 \leq t_1 \leq \dots \leq t_m = 1$ with $q(\gamma(t_k)) = \alpha_k$, so that

$$\sum_k |\psi(\alpha_k) - \psi(\alpha_{k-1})|_1 = \sum_k |u(\gamma(t_k)) - u(\gamma(t_{k-1}))|_1 \leq \text{Var}_1(u \circ \gamma).$$

Taking the supremum over the points α_k gives (i).

(ii) Let $\tilde{u} \in W^{1,p}(\mathbb{R}^2, \mathbb{R}^2)$ be an extension of u . By Lemma 4.8 applied to $\tilde{u}$, for every $x \in \bar{\Omega}$ and almost every $z \in \Omega$ (the points lying on exceptional rays from x form a null set), u is absolutely continuous on the segment $[x, z] \subset \bar{\Omega}$, and

$$\text{Var}_1(u; [x, z]) = \int_0^{|z-x|} |\nabla u(x + s\omega)\omega|_1 ds, \quad \omega = \frac{z-x}{|z-x|}.$$

Let $r = |x - y|$ and $S = \Omega \cap B_r(x)$. Since Ω is convex, $|S| \geq c_\Omega r^2$. Apply (i) to the curves $x \rightarrow z \rightarrow y$, $z \in S$, and average over z as in the proof of Morrey's inequality [36, Lemma 7.16]. This gives

$$d_1(q(x), q(y)) \leq C \int_{\Omega \cap B_{2r}(x)} \frac{|\nabla u(w)|}{|w-x|} dw + C \int_{\Omega \cap B_{2r}(y)} \frac{|\nabla u(w)|}{|w-y|} dw,$$

and Hölder's inequality gives (ii).

(iii) By (ii), d_1 is finite and $q : \overline{\Omega} \rightarrow (X, d_1)$ is continuous. Moreover $d_1(a, b) > 0$ for $a \neq b$, since the light map ψ is not constant on $[a, b]$, and d_1 is additive along the arcs, so d_1 is a metric. Since q is a quotient map, the identity from X to (X, d_1) is a continuous bijection from a compact space onto a Hausdorff space, hence a homeomorphism. In each arc $[a, b]$ the function $t \mapsto d_1(a, t)$ is continuous and strictly increasing, so $[a, b]$ is isometric to the segment $[0, d_1(a, b)]$; therefore (X, d_1) is a tree $\mathbb{R}$. Finally, $|\psi(a) - \psi(b)|_1 \leq \text{Var}_1(\psi; [a, b]) = d_1(a, b)$. $\square$

We now recall the notions related to $\mathbb{R}$ -trees used in the construction of the recovery sequences, stated for the $\mathbb{R}$ -tree (X, d_1) ; see [22, Chapters II.1 and II.2]. A *subtree* of (X, d_1) is a subset T with $[a, b] \subset T$ for all $a, b \in T$; it is *finite* if it is spanned by finitely many points: that is, $T = \bigcup_{i=2}^m [a_1, a_i]$ for some $a_1, \dots, a_m \in X$. A finite subtree is compact, being a finite union of arcs. For $t \in T$, the *order* of t in T is the number of connected components of $T \setminus \{t\}$. If T is a finite subtree, the set V of its points of order different from 2 (endpoints and branch points) is finite, and the closures of the connected components of $T \setminus V$ are finitely many arcs $[v, v']$, with $v, v' \in V$, called the *edges* of T . Each edge is a geodesic segment of length $d_1(v, v') > 0$. Note that in general the edges are not the arcs $[a_1, a_i]$ that span T .

Finally, $\mathbb{R}$ -trees are CAT(0) spaces, and every subtree T is convex, since the geodesic segments joining its points are the arcs $[a, b]$. Hence, if T is a finite subtree, every $t \in X$ has a unique nearest point $r_T(t) \in T$, the *nearest-point retraction* $r_T : X \rightarrow T$ is 1-Lipschitz [22, Proposition II.2.4], and $d_1(r_T(t), t) = d_1(t, T)$.

Lemma 4.10. *Let $T \subset X$ be a finite subtree, and let $\lambda > 0$ be the minimal d_1 -length of its edges. For $0 < \varepsilon \leq h \leq \lambda$ there is a continuous map $\psi_{\varepsilon, h} : T \rightarrow \varepsilon\mathcal{Z}$ such that*

$$|\psi_{\varepsilon, h}(a) - \psi_{\varepsilon, h}(b)| \leq \left(1 + \frac{2\varepsilon}{h}\right) d_1(a, b), \quad |\psi_{\varepsilon, h}(a) - \psi(a)|_1 \leq h + 3\varepsilon$$

for all $a, b \in T$.

Proof. Subdivide each edge of T , of length $L \geq h$, into n arcs of equal d_1 -length $\ell := L/n$, where n is the integer part of L/h , so that $\ell \in [h, 2h)$. To each endpoint z of these arcs, associate a vertex β_z of $\varepsilon\mathcal{Z}$ with $|\beta_z - \psi(z)|_1 \leq \varepsilon$. For each arc $[z, z']$, let $v_{zz'} : [0, \ell] \rightarrow \varepsilon\mathcal{Z}$ be the path given by Lemma 2.6 from β_z to $\beta_{z'}$ on $[0, \ell]$, and set

$$\psi_{\varepsilon, h}(a) := v_{zz'}(d_1(z, a)) \quad \text{for } a \in [z, z'].$$

These definitions agree at the common endpoints of the arcs, so $\psi_{\varepsilon, h}$ is continuous. By Lemma 4.9(iii), $|\beta_{z'} - \beta_z|_1 \leq |\psi(z') - \psi(z)|_1 + 2\varepsilon \leq \ell + 2\varepsilon$; hence, by Lemma 2.6(i), for $a, b \in [z, z']$

$$|\psi_{\varepsilon, h}(a) - \psi_{\varepsilon, h}(b)| \leq \frac{\ell + 2\varepsilon}{\ell} |d_1(z, a) - d_1(z, b)| \leq \left(1 + \frac{2\varepsilon}{h}\right) d_1(a, b).$$

For general $a, b \in T$, the geodesic $[a, b]$ is the union of finitely many subarcs of the arcs $[z, z']$, and the first estimate follows from the additivity of d_1 along $[a, b]$.

For the second estimate, let $a \in [z, z']$, $s := d_1(z, a)$, and $\theta := s/\ell$. By Lemma 2.6(ii) and $|\cdot|_1 \leq \sqrt{2}|\cdot|$,

$$|\psi_{\varepsilon, h}(a) - (1 - \theta)\beta_z - \theta\beta_{z'}|_1 \leq 2\varepsilon, \quad |(1 - \theta)(\beta_z - \psi(z)) + \theta(\beta_{z'} - \psi(z'))|_1 \leq \varepsilon.$$

Moreover, ψ is 1-Lipschitz by Lemma 4.9(iii), and $d_1(z, a) = s$, $d_1(a, z') = \ell - s$, so that

$$|(1 - \theta)\psi(z) + \theta\psi(z') - \psi(a)|_1 \leq (1 - \theta)s + \theta(\ell - s) = 2\ell\theta(1 - \theta) \leq \frac{\ell}{2} < h.$$

Summing up, $|\psi_{\varepsilon, h}(a) - \psi(a)|_1 \leq h + 3\varepsilon$. $\square$

Proof of Theorem 4.5. The idea is to repeat the construction in Lemma 4.10 for suitable approximating trees T_δ . This allows one to construct recovery sequences for the corresponding u_δ , which are suitable approximations of u .

For $\delta > 0$ let $\{a_1, \dots, a_m\}$ be a δ -net of the compact space (X, d_1) , and let T_δ be the union of the arcs $[a_1, a_i]$, $i = 1, \dots, m$. Then T_δ is a finite subtree, and $d_1(t, T_\delta) \leq \delta$ for every $t \in X$. Let $r_\delta := r_{T_\delta}$ be the nearest-point retraction; it is 1-Lipschitz, and $d_1(r_\delta(t), t) \leq \delta$ for all $t \in X$.

We now construct recovery maps for the target function

$$u_\delta := \psi \circ r_\delta \circ q,$$

related to u and T_δ . Let λ_δ be the minimal edge length of T_δ . For ε small, choose $h = h(\varepsilon)$ with $\varepsilon \leq h \leq \lambda_\delta$, $h \rightarrow 0$ and $\varepsilon/h \rightarrow 0$, and set

$$v_\varepsilon := \psi_{\varepsilon, h} \circ r_\delta \circ q : \bar{\Omega} \rightarrow \varepsilon\mathcal{Z}.$$

By Lemmas 4.9 and 4.10, v_ε is continuous. Fix $\omega \in S^1$. For almost every line L parallel to ω , u is absolutely continuous on the segment $L \cap \Omega$. For $x, y \in L \cap \Omega$, Lemma 4.10, the 1-Lipschitz continuity of r_δ and Lemma 4.9(i) give

$$|v_\varepsilon(x) - v_\varepsilon(y)| \leq \left(1 + \frac{2\varepsilon}{h}\right) d_1(q(x), q(y)) \leq \left(1 + \frac{2\varepsilon}{h}\right) \int_{[x, y]} |\partial_\omega u|_1 \, d\mathcal{H}^1.$$

Hence, $v_\varepsilon \in W^{1,p}(\Omega, \varepsilon\mathcal{Z})$, and for every ω in a countable dense subset of S^1

$$|\nabla v_\varepsilon \omega| \leq \left(1 + \frac{2\varepsilon}{h}\right) |\nabla u \omega|_1 \quad \text{almost everywhere in } \Omega.$$

By Lemma 2.18, $\nabla v_\varepsilon = e_i \otimes w$ almost everywhere, so that $|\nabla v_\varepsilon \omega| = |w \cdot \omega|$, while

$$|\nabla u \omega|_1 = |\nabla u_1 \cdot \omega| + |\nabla u_2 \cdot \omega| \leq N(\nabla u).$$

Both sides are continuous in ω ; taking $\omega = w/|w|$ we obtain

$$|\nabla v_\varepsilon| \leq \left(1 + \frac{2\varepsilon}{h}\right) N(\nabla u)$$

almost everywhere in Ω . By Lemma 4.10, applied at the point $r_\delta(q(x)) \in T_\delta$, we have $|v_\varepsilon - u_\delta|_1 \leq h + 3\varepsilon$, so that $v_\varepsilon \rightarrow u_\delta$ uniformly as $\varepsilon \rightarrow 0$, and

$$\Gamma\text{-}\limsup_{\varepsilon \rightarrow 0} F_\varepsilon^p(u_\delta) \leq \limsup_{\varepsilon \rightarrow 0} \int_\Omega |\nabla v_\varepsilon|^p dx \leq \int_\Omega N(\nabla u)^p dx.$$

Finally, since ψ is 1-Lipschitz by Lemma 4.9(iii), $|u_\delta - u|_1 \leq d_1(r_\delta \circ q, q) \leq \delta$, so that $u_\delta \rightarrow u$ uniformly as $\delta \rightarrow 0$. Since the Γ -limsup is lower semicontinuous with respect to the L^p -convergence, the claim follows. $\square$

Remark 4.11. (a) Convexity is used to lift maps $\bar{\Omega} \rightarrow S^1$, and in the Morrey estimate of Lemma 4.9(ii). Lifting only requires that every continuous map $\bar{\Omega} \rightarrow S^1$ be null-homotopic. By Borsuk's separation theorem, this holds if and only if $\mathbb{R}^2 \setminus \bar{\Omega}$ is connected. The Morrey estimate holds for Lipschitz domains, with polygonal paths. The theorem therefore extends to bounded Lipschitz domains such that $\mathbb{R}^2 \setminus \bar{\Omega}$ is connected. If $\bar{\Omega}$ separates the plane, X may contain circles: for $u(x) = x/|x|$ on an annulus, X is a circle.

(b) For $p = 2$ the same proof applies to every continuous $u \in W^{1,2}$ with $|u(\bar{\Omega})| = 0$. Lemma 4.9(ii) is replaced by the finiteness of d_1 on a dense subset of X , which follows from Fubini's theorem. The nearest-point retractions are replaced by first-point retractions onto an increasing sequence of finite subtrees whose union is dense; these converge uniformly to the identity. However, $W^{1,2}$ maps need not be continuous, and continuous $W^{1,2}$ maps need not satisfy Lusin's condition (N). The general case $p = 2$ is open.

5 Concluding remarks: the metric point of view

We finally note that the convergence of families F_ε^p can be related to a convergence on varying metric spaces. Indeed, for every $\varepsilon > 0$ we consider the distance d_ε on $\varepsilon\mathcal{Z}$ defined by

$$d_\varepsilon(u, v) = \min \left\{ \int_0^1 |\phi'(t)| dt : \phi(t) \in \varepsilon\mathcal{Z} \text{ for all } t \in [0, 1], \phi(0) = u, \phi(1) = v \right\}. \quad (40)$$

Proposition 5.1. *The distances d_ε converge with uniform continuity to the distance*

$$d(u, v) = |u - v|_1 = |u_1 - v_1| + |u_2 - v_2|;$$

more precisely, if $(u_\varepsilon, v_\varepsilon) \rightarrow (u, v)$ then $|d_\varepsilon(u_\varepsilon, v_\varepsilon) - d(u, v)| \leq |u_\varepsilon - u| + |v_\varepsilon - v|$

Proof. For $u_\varepsilon, v_\varepsilon \in \varepsilon\mathcal{Z}$, there exists a path on $\varepsilon\mathcal{Z}$ connecting u_ε and v_ε and so that

$$d_\varepsilon(u_\varepsilon, v_\varepsilon) = d_1(u_\varepsilon, v_\varepsilon)$$

(e.g. by increasing or decreasing monotonically each component when going from u_ε to v_ε , or vice versa, the total sum of the horizontal and vertical lengths will be the same), thus for $u, v \in \mathbb{R}^2$

$$d_\varepsilon(u_\varepsilon, v_\varepsilon) = d_1(u_\varepsilon, v_\varepsilon) \leq d_1(u_\varepsilon, u) + d_1(u, v) + d_1(v, v_\varepsilon),$$

and this gives one inequality. The other is found in the same way:

$$d_1(u, v) \leq d_1(u, u_\varepsilon) + d_1(u_\varepsilon, v_\varepsilon) + d_1(v_\varepsilon, v) = d_1(u, u_\varepsilon) + d_\varepsilon(u_\varepsilon, v_\varepsilon) + d_1(v_\varepsilon, v),$$

and the claim is proved. $\square$

Let $u_\varepsilon \in W^{1,p}(\Omega, \varepsilon\mathcal{Z})$ with $p \geq 2$ be converging to u and let $z_\varepsilon \in \varepsilon\mathbb{Z}^2$ be converging to z_0 . We define the scalar function $v_\varepsilon \in W^{1,p}(\Omega)$ by

$$v_\varepsilon(x) = d_\varepsilon(u_\varepsilon(x), z_\varepsilon). \quad (41)$$

Then the convergence of u_ε can be related to that of these scalar functions v_ε , whose limit is $v(x) = d(u(x), z_0)$. Note that $|\nabla v_\varepsilon(x)| = |\nabla u_\varepsilon(x)|$, so that $F_\varepsilon(u_\varepsilon) = \int_\Omega |\nabla v_\varepsilon|^p dx$. The liminf inequality for F_ε^p in Proposition 4.2 can be recovered from the lower-semicontinuity inequality along the sequence v_ε by locally optimizing the choice of z_0 . This corresponds to the determination of the full Reshetnyak density, which is obtained by taking the least $g \in L^p$ dominating $|\nabla d_\varepsilon(u_\varepsilon(\cdot), z)|$ for z in a countable dense subset of $\varepsilon\mathcal{Z}$.

The limit in Theorem 4.1 has a natural metric interpretation. For $\varepsilon > 0$ and $v, w \in \varepsilon\mathcal{Z}$ let d_ε be the length distance of the network defined in (40). By Proposition 5.1, d_ε coincides with the restriction to $\varepsilon\mathcal{Z}$ of the ℓ^1 distance $|v - w|_1$, and $d_\varepsilon \rightarrow |\cdot|_1$ with uniform continuity. Hence $(\varepsilon\mathcal{Z}, d_\varepsilon)$ embeds isometrically in $(\mathbb{R}^2, |\cdot|_1)$ as an $\varepsilon/2$ -dense subset; in particular, the pointed Gromov–Hausdorff distance between $(\varepsilon\mathcal{Z}, d_\varepsilon)$ and $(\mathbb{R}^2, |\cdot|_1)$ is $O(\varepsilon)$, and the networks converge, as metric spaces, to the normed plane $(\mathbb{R}^2, |\cdot|_1)$. Since $|w| \leq |w|_1 \leq \sqrt{2}|w|$, the class $W^{1,p}(\Omega, \varepsilon\mathcal{Z})$ coincides, with energies comparable uniformly in ε , with the Sobolev space $W^{1,p}(\Omega, (\varepsilon\mathcal{Z}, d_\varepsilon))$ of maps into a metric space, which can be defined equivalently after Korevaar and Schoen [49], Reshetnyak [60] or Ambrosio [5]; the equivalence of these notions for Euclidean domains and complete separable targets is due to Chiron [25], and an explicit formula relating the Korevaar–Schoen p -energy to the Lipschitz-composition description is given by Logaritsch and Spadaro [50].

All these energies are built on the approximate metric derivative: for a Sobolev map u with values in a metric space (X, d) , at almost every x there is a seminorm $\text{ap md } u_x$ on $\mathbb{R}^2$ such that $d(u(y), u(x)) - \text{ap md } u_x(y - x) = o(|y - x|)$ as $y \rightarrow x$, in the approximate sense. Its existence is due to Kirchheim [46] for Lipschitz maps and to Karmanova [44] for Sobolev maps; we follow the notation of Lytchak and Wenger [51, Section 4]. Reshetnyak’s energy

density is the maximal stretch $\max_{|v|=1} \text{ap md } u_x(v)$, while the Korevaar–Schoen density is an average of $(\text{ap md } u_x(v))^p$ over directions; for general targets, already for $(\mathbb{R}^2, |\cdot|_1)$, the two energies are genuinely different, a phenomenon related to the competing notions of area in normed spaces [51, 52]. In our problem the ambiguity disappears. If $\nabla u(x) = a \otimes \nu$ with $|\nu| = 1$, then $\text{ap md } u_x(v) = |a|_1 |\nu \cdot v|$ for the ℓ^1 distance, a rank-one seminorm; moreover, every $u \in W^{1,p}(\Omega, \varepsilon\mathcal{Z})$ has $\nabla u(x) \in K$ almost everywhere, and a matrix with at most one non-zero row acts identically in the Euclidean and in the ℓ^1 norm. We obtain the following normalization statement.

Remark 5.2 (all energies agree on the constrained class). Let $u \in W^{1,p}(\Omega, \mathbb{R}^2)$ with $\det \nabla u = 0$ almost everywhere, regarded as a map into $(\mathbb{R}^2, |\cdot|_1)$. Then its Reshetnyak p -energy equals $\int_{\Omega} (|\nabla u_1| + |\nabla u_2|)^p dx$, and its Korevaar–Schoen p -energy equals γ_p times this quantity, where $\gamma_p > 0$ depends only on p and on the normalization chosen for the Korevaar–Schoen energy (with the averaged normalization, $\gamma_p = \int_{S^1} |v_1|^p d\mathcal{H}^1(v)$). If moreover $u \in W^{1,p}(\Omega, \varepsilon\mathcal{Z})$, then $|\nabla u_1| + |\nabla u_2| = |\nabla u|$ almost everywhere, so that the Reshetnyak and the Korevaar–Schoen p -energies of $u: \Omega \rightarrow (\varepsilon\mathcal{Z}, d_{\varepsilon})$ equal $F_{\varepsilon}^p(u)$ and $\gamma_p F_{\varepsilon}^p(u)$, respectively.

Consequently, F_{hom}^p is the Reshetnyak p -energy of u regarded as a map into $(\mathbb{R}^2, |\cdot|_1)$, the metric limit of the networks, and, the normalizing factor being the same for every competitor, none of the Γ -convergence statements depends on the chosen definition of energy. We also note that $(\mathbb{R}^2, |\cdot|_1)$ does not satisfy the Euclidean-tangent condition of [51], yet every map in the domain of F_{hom}^p satisfies it pointwise, its approximate metric derivative being induced by a degenerate scalar product: it is the constraint, and not the geometry of the target alone, that governs the limit.

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Use of AI tools. A large language model was used in the preparation of this article. Beyond general assistance with the exposition, the model provided valuable bibliographic suggestions that helped highlight the significance of the results, as well as the idea of employing tools from graph theory to construct recovery sequences for the case $p > 2$.

Apart from this, the overall approach, the ideas, and the mathematical framework of the article were developed prior to the use of artificial intelligence tools. All other results, arguments, and techniques presented here were developed exclusively by the authors. The authors assume full responsibility for the content of the article.

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