

SPECTRAL INEQUALITIES FOR WEIGHTED p -LAPLACIANS VIA TALENTI SYMMETRIZATION

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ABSTRACT. We consider the weighted p -Laplacian associated with a measure μ having a continuous positive density with respect to the Lebesgue measure on an open convex subset $X \subset \mathbb{R}^N$. We prove that Talenti's weighted Pólya–Szegő inequality—originally established for Lipschitz functions on X —extends to the weighted Sobolev space of functions vanishing on the relative boundary of an open set $\Omega \subset X$, defined as the closure of Lipschitz functions supported away from it. This yields Faber–Krahn-type inequalities for the first (p, q) -eigenvalue of the weighted Dirichlet p -Laplacian. We also show that the assumptions of Talenti's theorem force a rigid structure on the function driving the symmetrization: its level sets are quantitatively separated, the modulus of its gradient is constant on each of them, and, under a mild additional assumption, they are parallel hyper-surfaces, so that the function depends only on the distance either from a hyperplane or from a focal set. We present several examples fitting this abstract framework, in which the isoperimetric sets are balls or half-spaces.

1. INTRODUCTION

Let $\mathcal{L}^N$ be the standard N -dimensional Lebesgue measure on $\mathbb{R}^N$ and let $X \subset \mathbb{R}^N$ be an open and connected set. Given a continuous function $W : X \rightarrow \mathbb{R}$, we consider the strictly positive measure

$$\mu = e^{W(x)} \mathcal{L}^N. \tag{1.1}$$

This measure is mutually absolutely continuous with respect to the Lebesgue measure and its Radon–Nikodym derivative is e^W , which we shall call the *density* or *weight*, while we shall refer to W as the *potential*. Measures of this kind are Borel and encompass, for instance, log-convex and log-concave measures that arise in connection with Witten Laplacians and Gaussian-type diffusions.

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The functional framework. For an open set $\Omega \subset X$ and $p \in [1, +\infty)$, we define the space $L_\mu^p(\Omega)$ as the collection of all μ -measurable functions u on Ω such that

$$\|u\|_{p,\mu,\Omega}^p = \int_\Omega |u(x)|^p d\mu < +\infty.$$

This space, equipped with the norm $\|\cdot\|_{p,\mu,\Omega}$, is a Banach space. Further, if $p \in (1, +\infty)$, by Clarkson's inequality it is uniformly convex, hence reflexive by the Milman–Pettis Theorem. From now on, unless otherwise stated, we assume $p \in (1, +\infty)$.

Since W is continuous, the weight e^W is bounded from above and from below by positive constants on every compact subset of X . In particular,

$$e^{-W/(p-1)} \in L_{\text{loc}}^1(X), \quad (1.2)$$

so that, by Hölder's inequality, $L_\mu^p(\Omega) \subset L_{\text{loc}}^1(\Omega)$, and distributional derivatives of functions in $L_\mu^p(\Omega)$ make sense. We define the weighted Sobolev space $W_\mu^{1,p}(\Omega)$ as the space of functions $u \in L_\mu^p(\Omega)$ whose distributional derivatives $\partial_i u$ belong to $L_\mu^p(\Omega)$. Thanks to (1.2), see [30], this space is a Banach space when endowed with the norm

$$\|u\|_{W_\mu^{1,p}(\Omega)} = \left(\|u\|_{p,\mu,\Omega}^p + \|\nabla u\|_{p,\mu,\Omega}^p \right)^{\frac{1}{p}}, \quad (1.3)$$

where $\|\nabla u\|_{p,\mu,\Omega} = \left\| |\nabla u| \right\|_{p,\mu,\Omega}$ and $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^N$. Moreover, it is reflexive. Indeed, since all norms on $\mathbb{R}^N$ are equivalent, the map $u \mapsto (u, \partial_1 u, \dots, \partial_N u)$ is an isomorphism of $W_\mu^{1,p}(\Omega)$ onto a closed subspace of the reflexive space $L_\mu^p(\Omega) \times [L_\mu^p(\Omega)]^N$, endowed with the ℓ^p -product norm, and reflexivity is invariant under isomorphisms.

We denote by $\text{Lip}_0(\Omega; X)$ the class of Lipschitz functions u on X such that

- (i) $\text{supp}_X(u) \subset \Omega$ is bounded;
- (ii) $\text{dist}(\text{supp}_X(u); \partial\Omega \cap X) > 0$;
- (iii) $\mu(\{u \neq 0\}) < +\infty$;

where $\text{supp}_X(u) = \overline{\{x \in X : u(x) \neq 0\}}^X$ and $\text{dist}(\cdot; \cdot)$ is the distance on $\mathbb{R}^N$. Here and in the following, $\overline{\Omega}^A$ denotes the closure of Ω relative to the set A , while $\overline{\Omega}$ denotes its closure in $\mathbb{R}^N$. The last condition is automatically satisfied if $\mu(\Omega) < +\infty$, which is the case in our main results, or if μ is finite on bounded subsets of X , that is, $\mu(X \cap B_R) < +\infty$ for all $R > 0$, as in all the examples of Section 4. We stress that the latter does not follow from the continuity of W , since the weight may blow up at ∂X .

Every $u \in \text{Lip}_0(\Omega; X)$ is bounded, and both u and ∇u vanish a.e. outside $\{u \neq 0\}$, so that $\text{Lip}_0(\Omega; X) \subset W_\mu^{1,p}(\Omega)$. We define the weighted Sobolev space $W_{\mu,0}^{1,p}(\Omega; X)$ as the closure, with respect to the norm (1.3), of (the restrictions

to Ω of the functions in $\text{Lip}_0(\Omega; X)$. Being a closed subspace of $W_\mu^{1,p}(\Omega)$, it inherits reflexivity. No condition is imposed on $\partial\Omega \cap \partial X$.

Shape optimization problems. The *first Dirichlet eigenvalue* of the weighted p -Laplacian on the open set Ω is defined variationally by

$$\lambda_{1,p}^\mu(\Omega) = \inf \left\{ \frac{\|\nabla u\|_{p,\mu,\Omega}^p}{\|u\|_{p,\mu,\Omega}^p} : u \in W_{\mu,0}^{1,p}(\Omega; X), u \neq 0 \right\}.$$

The admissible class is nonempty: given $x_0 \in \Omega$ and $r > 0$ with $\overline{B_r(x_0)} \subset \Omega$, it is easy to see that the function $u(x) = (r - \|x - x_0\|)_+$ belongs to $\text{Lip}_0(\Omega; X)$. Thus, $\lambda_{1,p}^\mu(\Omega) < +\infty$.

For a continuous potential W , the associated eigenvalue problem is understood in the weak sense: $u \in W_{\mu,0}^{1,p}(\Omega; X) \setminus \{0\}$ is an eigenfunction with eigenvalue λ if

$$\int_\Omega |\nabla u|^{p-2} \langle \nabla u, \nabla \varphi \rangle d\mu = \lambda \int_\Omega |u|^{p-2} u \varphi d\mu, \quad \forall \varphi \in W_{\mu,0}^{1,p}(\Omega; X). \quad (1.4)$$

This is the Euler–Lagrange equation of the Rayleigh quotient above, so that its minimizers, when they exist, are eigenfunctions with eigenvalue $\lambda_{1,p}^\mu(\Omega)$. The weak formulation (1.4) encodes a Dirichlet boundary condition on $\partial\Omega \cap X$, through the space $W_{\mu,0}^{1,p}(\Omega; X)$, and a natural boundary condition on $\partial\Omega \cap \partial X$. If, in addition, the potential W is of class C^1 , an integration by parts shows that (1.4) is the weak form of the principal eigenvalue problem

$$-\Delta_{p,\mu} u = \lambda |u|^{p-2} u \quad \text{in } \Omega,$$

where the weighted (or Witten-type) p -Laplacian of u is given by the formal expression

$$\begin{aligned} -\Delta_{p,\mu}(u) &= -e^{-W} \operatorname{div}(e^W |\nabla u|^{p-2} \nabla u) \\ &= -\operatorname{div}(|\nabla u|^{p-2} \nabla u) - |\nabla u|^{p-2} \langle \nabla W, \nabla u \rangle, \end{aligned}$$

that is, the usual p -Laplacian paired with a drift term, see, *e.g.*, [20].

A classical problem in shape optimization is identifying sets $\Omega^* \subset X$ such that the *Faber–Krahn inequality*

$$\lambda_{1,p}^\mu(\Omega^*) \leq \lambda_{1,p}^\mu(\Omega), \quad \forall \text{ open sets } \Omega \subset X \text{ with } \mu(\Omega) = \mu(\Omega^*), \quad (1.5)$$

holds.

A similar shape optimization problem revolves around the *p -torsional rigidity* of the weighted p -Laplacian on the open set Ω , variationally defined as

$$T_p^\mu(\Omega) = \sup \left\{ \frac{\|u\|_{1,\mu,\Omega}^p}{\|\nabla u\|_{p,\mu,\Omega}^p} : u \in W_{\mu,0}^{1,p}(\Omega; X), \nabla u \neq 0 \right\}.$$

Two remarks on this definition are in order. First, different conventions exist: in place of the p -th power of the ratio, some take its p' -th power, where p' is the conjugate exponent of p . Second, usually one requires $u \neq 0$ rather than $\nabla u \neq 0$. For the Lebesgue measure on $X = \mathbb{R}^N$, the two requirements are equivalent for functions in $W_{\mu,0}^{1,p}(\Omega; X)$. For a general measure, instead, nonzero constant functions may belong to $W_{\mu,0}^{1,p}(\Omega; X)$, and for these the ratio within the supremum would be ill-posed. This occurs, for instance, when $\Omega = X = \mathbb{R}^N$ and μ is the Gaussian measure.

By the example above, the admissible class is nonempty, so that $T_p^\mu(\Omega) \in (0, +\infty]$. The p -torsional rigidity is naturally associated with the torsion problem, whose weak formulation reads

$$\int_{\Omega} |\nabla u|^{p-2} \langle \nabla u, \nabla \varphi \rangle d\mu = \int_{\Omega} \varphi d\mu, \quad \forall \varphi \in W_{\mu,0}^{1,p}(\Omega; X),$$

and which, if W is of class C^1 , is the weak form of the PDE

$$-\Delta_{p,\mu} u = 1 \quad \text{in } \Omega,$$

with the same boundary conditions as above. If it admits a solution u_Ω , the so-called *torsion function*, then $T_p^\mu(\Omega) = (\int_{\Omega} u_\Omega d\mu)^{p-1}$, as follows by testing the equation with u_Ω and from the fact that u_Ω minimizes the functional $v \mapsto \frac{1}{p} \int_{\Omega} |\nabla v|^p d\mu - \int_{\Omega} v d\mu$.

The p -torsional rigidity analog of the Faber–Krahn inequality (1.5) is the *Saint-Venant inequality*, and one looks for sets $\Omega^* \subset X$, if they exist, such that

$$T_p^\mu(\Omega^*) \geq T_p^\mu(\Omega), \quad \forall \text{ open sets } \Omega \subset X \text{ with } \mu(\Omega) = \mu(\Omega^*), \quad (1.6)$$

holds. These two inequalities are particular instances of Faber–Krahn-type inequalities for the first (p, q) -eigenvalue of the weighted p -Laplacian, see [Remark 3.10](#).

Abstract Talenti symmetrization. In proving these inequalities, a valuable tool is given by symmetrization techniques, which entail the *Pólya–Szegő principle*: a competitor u on Ω is transformed into a competitor $u^\sharp$ on a set $\Omega^\sharp$, decreasing the L^p -norm of the gradient while keeping unchanged the L^p -norm of the function. In the Euclidean case, the superlevel set of $u^\sharp$ at a given height is the ball centered at a fixed point having the same measure as the corresponding superlevel set of u .

In this note we revisit the Faber–Krahn and Saint-Venant inequalities in the weighted setting of measures of the form (1.1) by exploiting an abstract symmetrization, first introduced by Talenti in [42], which, again, replaces each superlevel set of u with the set having the same measure in a fixed nested family of isoperimetric sets. Namely, it is driven by an underlying smooth function

$f : X \rightarrow [0, 1]$ without critical points in $\{0 < f < 1\}$, see [Definition 2.4](#): the Talenti symmetrization $u^\sharp$ of u is the function whose superlevel sets are the superlevel sets of f having the same μ -measure as the corresponding superlevel sets of $|u|$, see [\(2.4\)](#). Talenti's weighted Pólya–Szegő principle [[42](#), Main Thm.] states that

$$\int_X |\nabla u^\sharp|^p d\mu \leq \int_X |\nabla u|^p d\mu$$

for Lipschitz functions u on X with $\mu(\{u \neq 0\}) < +\infty$, provided the level sets of f are connected, the superlevel sets of f saturate a suitable weighted relative isoperimetric inequality, and the lines of steepest ascent of f are straight; see [Theorem 3.1](#) and the subsequent discussion. The advantage of Talenti's approach is that, once such an f is available, the Pólya–Szegő principle, the Faber–Krahn and Saint-Venant inequalities follow in a unified way, without having to construct an ad hoc rearrangement for each measure. On the other hand, the assumptions on f , and in particular the straightness of its lines of steepest ascent, are far from innocent, and understanding them is one of the purposes of this note.

Description of results. Our first main result, [Theorem 3.7](#), shows that Talenti's principle, originally formulated for Lipschitz functions on the whole of X , extends to the space $W_{\mu,0}^{1,p}(\Omega; X)$ of Sobolev functions vanishing on $\partial\Omega \cap X$, for open subsets $\Omega \subset X$ of finite measure, provided the ambient space X is convex. The extension is obtained via a density argument in weighted Sobolev spaces, and immediately yields as corollaries the corresponding Faber–Krahn and Saint-Venant inequalities, see [Corollaries 3.8](#) and [3.9](#), as well as Faber–Krahn-type inequalities for the first (p, q) -eigenvalue, see [Remark 3.10](#). In [Remark 3.11](#), we point out that it is enough to know that the superlevel sets of f are isoperimetric for volumes below a given threshold, at the price of restricting the Faber–Krahn and Saint-Venant inequalities to sets whose measure lies below the same threshold.

Our second group of results, [Theorems 3.3](#) and [3.5](#), concerns the structure of the functions f satisfying the assumptions of Talenti's theorem, and shows that these assumptions are much more rigid than they may appear. These results do not involve the isoperimetric properties of μ , but only the geometry of f .

In [Theorem 3.3](#), we prove that $|\nabla f| = g(f)$ for a continuous profile g , that is, $|\nabla f|$ is constant on each level set of f , and that the level sets of f are quantitatively separated: if G is a primitive of $1/g$, then

$$\text{dist}(\{f \leq t_1\}; \{f \geq t_2\}) \geq G(t_2) - G(t_1) > 0, \quad 0 < t_1 < t_2 < 1.$$

This separation is the key ingredient in the proof of [Theorem 3.7](#): it guarantees that the symmetrization of a Lipschitz function supported away from $\partial\Omega \cap X$ is supported away from $\partial\Omega^\# \cap X$, so that it can be approximated within $W_{\mu,0}^{1,p}(\Omega^\#; X)$.

In [Theorem 3.5](#), under the mild additional assumption that f takes all values in $(0, 1)$ along each of its lines of steepest ascent, we show that the level sets of f are parallel hypersurfaces, and that $f = G^{-1} \circ d$, where d is the signed distance from any fixed level set of f . Moreover, either the level sets of f are parallel hyperplanes, or f is a function of the distance from a focal set. In geometric terms, the superlevel sets of f , *i.e.*, the isoperimetric sets used in the symmetrization, are mutually parallel sets: relative to X , they are either parallel half-spaces, or metric neighborhoods of a focal set, or complements of such neighborhoods, see [Remark 3.6](#). Parallel half-spaces and concentric balls, whose focal set is their common center, are the simplest instances of these alternatives, and they are the isoperimetric sets in all the examples of [Section 4](#). Thus, [Theorems 3.3](#) and [3.5](#) explain why balls and half-spaces naturally arise in this approach, and provide a simple test to decide whether a given family of isoperimetric sets can be handled by Talenti symmetrization.

State of the art. The abstract framework in which we prove the Pólya–Szegő principle allows one to prove the Faber–Krahn and Saint-Venant inequalities (and, more generally, inequalities for the first (p, q) -eigenvalues) all at once. This recovers several known results and also yields these inequalities in a few cases which, to the best of our knowledge, have not been treated in the literature. Here we give a short account, as complete as possible.

In the Euclidean setting, corresponding to $X = \mathbb{R}^N$ and $W \equiv c \in \mathbb{R}$, inequality [\(1.5\)](#) was proved independently by Faber [\[23\]](#) and Krahn [\[29\]](#), whereas inequality [\(1.6\)](#) was proved by Pólya [\[37\]](#). In both cases, $\Omega^* = B$ is a Euclidean ball, and the proof relies on the *Schwarz symmetrization* [\[25, 40\]](#), which entails the (Euclidean) Pólya–Szegő principle [\[38\]](#).

In the Gaussian setting, corresponding to $X = \mathbb{R}^N$ and $W(x) = -\|x\|^2/2$, the Faber–Krahn inequality [\(1.5\)](#) has been proved in [\[22\]](#) via a different symmetrization, known as *Ehrhard symmetrization* [\[21\]](#). This analysis has been extended in [\[1\]](#) to a broader class of elliptic operators associated with the Gaussian measure, with refined estimates for the first eigenfunction. The same symmetrization allows one to prove the Saint-Venant inequality [\(1.6\)](#) in this setting [\[32, Sect. 5\]](#) for $p = 2$; see also [\[15\]](#) for general p .

While the Euclidean and Gaussian settings were the first to attract attention, several other weighted settings have been addressed over the years. The

first key step is proving the relevant isoperimetric inequality. In convex cones, for the Lebesgue measure, the relative isoperimetric inequality, with balls centered at the vertex as optimal sets, is due to Lions and Pacella [31]. It has been extended by Cabré, Ros-Oton, and Serra to homogeneous weights satisfying a suitable concavity condition [10, 11], see also [9] for monomial weights. Nonhomogeneous weights have been addressed by Brock, Chiacchio, and Mercaldo, in half-spaces [6], extending a two-dimensional result of Maderna and Salsa [33], and in orthants [7]. Further, the same authors derived necessary conditions on a weight for balls centered at the vertex of a cone to be isoperimetric [6]. Balls centered at the origin are also isoperimetric for smooth radial log-convex densities, as proved by Chambers [13].

Transitioning to settings where the isoperimetric sets are half-spaces, beyond the usual Gaussian, we mention that Brock, Chiacchio, and Mercaldo also gave necessary conditions on a weight for coordinate half-spaces of a product domain to be isoperimetric, and proved that they are indeed isoperimetric for a broad class of Gauss-like product measures [8], see also [5]. Half-spaces also are the isoperimetric sets for anisotropic Gaussian measures [44] and for log-concave one-dimensional perturbations of the Gaussian measure [39].

In several of these settings, the isoperimetric inequality has then been combined with suitable rearrangements to derive Pólya–Szegő principles, comparison results for degenerate elliptic equations and, in some cases, Faber–Krahn or Saint-Venant inequalities, mostly for $p = 2$, see [5, 6, 8, 14, 33, 36]. Our framework covers all these settings at once, and yields the Faber–Krahn, Saint-Venant, and (p, q) -eigenvalue inequalities for every $p \in (1, +\infty)$; several of them, in particular for $p \neq 2$, appear not to have been stated before. A precise account of the known results and of the novelties is given in Section 4, setting by setting.

Whether the isoperimetric sets are balls or half-spaces, all these settings are handled by a single argument whose only input is the relative isoperimetric inequality, and Theorems 3.3 and 3.5 clarify to which extent such an argument can be pushed: by Remark 3.6, the fact that the isoperimetric sets are balls or half-spaces in all these settings is not a coincidence.

Organization of the paper. In Section 2, we recall Talenti symmetrization in the present setting, introduce the associated symmetrized sets, and prove some basic properties. In Section 3, we state Talenti’s weighted Pólya–Szegő inequality, prove our structural results on the admissible functions f in Section 3.1, and extend the principle from Lipschitz functions to $W_{\mu,0}^{1,p}(\Omega; X)$,

thereby deriving the corresponding Faber–Krahn inequality (1.5) and Saint-Venant inequality (1.6). In Section 4, we exhibit several classes of measures for which the hypotheses of Talenti’s theorem are satisfied. Finally, in Section 5, we list future directions of research.

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2. TALENTI SYMMETRIZATION

In this section we recall the symmetrization defined in [42]. We let X be an open and connected subset of $\mathbb{R}^N$ and $\mu = e^W \mathcal{L}^N$ be as in (1.1), with W continuous on X . This latter assumption can be considerably weakened regarding the results of this section, as the next remark explains.

Remark 2.1. The constructions and the results of this section only use two properties of μ : μ -negligible and $\mathcal{L}^N$ -negligible sets coincide, and nonempty open subsets of X have positive μ -measure. Thus, they hold, with the same proofs, for every σ -finite measure μ on X mutually absolutely continuous with $\mathcal{L}^N$. Any such measure can be written as $\mu = e^W \mathcal{L}^N$, where $W = \ln w$ and $w > 0$ is its Radon–Nikodym derivative, but then W is merely measurable and defined $\mathcal{L}^N$ -a.e.; in particular, no integrability of negative powers of the weight, such as (1.2), can be expected. The continuity of W is used in a critical way from Section 3 on: it gives a meaning to the weighted perimeter (3.1), which integrates e^W over an $\mathcal{L}^N$ -negligible set, and it ensures, via (1.2), that the weighted Sobolev spaces of the introduction have the usual properties.

Definition 2.2 (μ -distribution function). Let $v : X \rightarrow \mathbb{R}$ be a real-valued μ -measurable function. Assume either that the superlevel sets $\{|v(x)| > t\}$ have finite μ -measure for all $t > 0$ or that $\mu(X) < +\infty$. The μ -distribution function of v , denoted by Φ_v^μ , is the map that associates to any $t > 0$ the μ -measure of the superlevel set $\{|v| > t\}$, i.e., the map from $[0, +\infty)$ into $[0, +\infty]$ defined by

$$\Phi_v^\mu(t) = \mu(\{x \in X : |v(x)| > t\}). \quad (2.1)$$

If no confusion can arise, we will drop the superscript μ and denote it simply by Φ_v and refer to it simply as the distribution function. It is clear that Φ_v :

decreases monotonically; is right continuous¹. Moreover, one has

$$\Phi_v(0) = \mu(\{x \in X : v(x) \neq 0\})$$

and

$$\{t \geq 0 : \Phi_v(t) = 0\} = [\|v\|_\infty, +\infty),$$

where the essential supremum is with respect to the measure μ .

Definition 2.3 (μ -decreasing rearrangement). Let $v : X \rightarrow \mathbb{R}$ be a real-valued μ -measurable function. Assume either that the superlevel sets $\{|v(x)| > t\}$ have finite μ -measure for all $t > 0$ or that $\mu(X) < +\infty$. The μ -decreasing rearrangement of v is defined by

$$v^{\mu,*}(s) = \inf\{t \geq 0 : \Phi_v^\mu(t) \leq s\}, \quad (2.2)$$

that is, $v^{\mu,*}$ is the (right-continuous) generalized inverse of the nonincreasing function Φ_v^μ .

As for the distribution function, if no confusion can arise, we will drop the superscript μ and denote the μ -decreasing rearrangement of v with v^* and refer to it simply as the decreasing rearrangement.

The decreasing rearrangement v^* can also be interpreted as the distribution function of Φ_v with respect to the Lebesgue measure. Hence, the function v^* is a decreasing, right-continuous map from $[0, +\infty)$ into $[0, +\infty]$, satisfying $v^*(0) = \|v\|_\infty$ and

$$\{s \geq 0 : v^*(s) = 0\} = [\mu(\{x \in X : v(x) \neq 0\}), +\infty).$$

Moreover, for every $y > 0$ we have

$$\{s \geq 0 : v^*(s) > y\} = [0, \Phi_v(y)). \quad (2.3)$$

Definition 2.4 ((μ, f) -Talenti symmetrization). Let $f : X \rightarrow [0, 1]$ be a smooth function such that

$$\inf f = 0, \quad \sup f = 1, \quad \text{with} \quad \mu(f^{-1}(0)) = \mu(f^{-1}(1)) = 0, \quad (\text{H1})$$

and

$$\nabla f(x) \neq 0, \quad \text{when } 0 < f(x) < 1. \quad (\text{H2})$$

Assume further that

$$\mu(\{x \in X : f(x) > t\}) < +\infty, \quad \text{if } t \in (0, 1), \quad \text{or } \mu(X) < +\infty. \quad (\text{H3})$$

¹If, for all $t > 0$, $\mu(\{x \in X : |v(x)| = t\}) = 0$, it improves to continuity.

Given a nonempty, proper open subset $\Omega \subsetneq X$ and a μ -measurable function $u : \Omega \rightarrow \mathbb{R}$ such that either the superlevel sets $\{|u(x)| > t\}$ have finite μ -measure for all $t > 0$ or $\mu(X) < +\infty$, the (μ, f) -Talenti symmetrization of u is the function $u^{\mu, \sharp} : X \rightarrow [0, +\infty]$ given by

$$u^{\mu, \sharp}(x) = \tilde{u}^{\mu, *}(\Phi_f^\mu(f(x))), \quad (2.4)$$

where $\tilde{u}$ is the extension to zero of u to $X \setminus \Omega$.

Notice that $u^\sharp$ may take the value $+\infty$ only on the μ -negligible set $\{f = 1\}$ (see [Lemma 2.5](#) below), where $u^\sharp = \tilde{u}^*(0) = \|u\|_\infty$. If no confusion can arise, we will drop the superscript μ and denote it simply by $u^\sharp$, and also drop the tilde on $\tilde{u}$ and simply write u . For the sake of ease, we will say Talenti symmetrization, without making explicit the dependence on the measure μ and on the function f . We start collecting the basic properties of the distribution function of f .

Lemma 2.5. *Let μ and f be as in [Definition 2.4](#). Then Φ_f is continuous and strictly decreasing on $[0, 1]$, with $\Phi_f(0) = \mu(X)$ and $\Phi_f(1) = 0$. In other words, Φ_f is a homeomorphism of $[0, 1]$ onto $[0, \mu(X)]$ (onto $[0, +\infty]$, if $\mu(X) = +\infty$), and its inverse is the restriction of f^* to $[0, \mu(X)]$.*

Proof. First, by [\(H1\)](#), $\Phi_f(0) = \mu(\{f > 0\}) = \mu(X)$ and $\Phi_f(1) = \mu(\emptyset) = 0$.

Second, the function Φ_f is right-continuous and, owing to [\(H3\)](#), its jump at $t \in (0, 1]$ equals $\mu(\{f = t\})$. This quantity vanishes: for $t = 1$ by [\(H1\)](#); for $t \in (0, 1)$ because, by [\(H2\)](#), $\{f = t\}$ is a smooth hypersurface, hence it is $\mathcal{L}^N$ -negligible, thus μ -negligible.

Third, let $0 \leq t_1 < t_2 \leq 1$. Since f is continuous on the connected set X with $\inf f = 0$ and $\sup f = 1$, one has $f(X) \supset (0, 1)$, so that $\{t_1 < f < t_2\}$ is a nonempty open set. Since μ is strictly positive,

$$\Phi_f(t_1) - \Phi_f(t_2) = \mu(\{t_1 < f \leq t_2\}) \geq \mu(\{t_1 < f < t_2\}) > 0,$$

with the obvious meaning if $\Phi_f(t_1) = +\infty$.

Finally, for a continuous, strictly decreasing function, the generalized inverse [\(2.2\)](#) coincides with the inverse. $\square$

Definition 2.6 (Symmetrization of a set). Let μ , f , and Ω be as in [Definition 2.4](#), with $\mu(\Omega) < +\infty$. We define the Talenti-symmetrized set of Ω as $\Omega^\sharp = \{x \in X : \chi_\Omega^\sharp(x) > 0\}$.

We now prove the following lemma, which characterizes the symmetrized set $\Omega^\sharp$ as the union of the positivity sets of the Talenti symmetrizations of all μ -measurable functions defined on Ω . In view of it, given a μ -measurable u

defined on an open set Ω as in [Definition 2.4](#), we can treat its Talenti symmetrization $u^\sharp$ as defined on $\Omega^\sharp$.

Lemma 2.7. *Let μ , f , and Ω be as in [Definition 2.4](#). Assume further that $\mu(\Omega) < +\infty$. We have that*

$$\bigcup \{x : u^\sharp(x) > 0\} = \Omega^\sharp,$$

where the union is taken among all μ -measurable functions u defined on Ω .

Proof. Let us show the two opposite inclusions. On the one hand, since χ_Ω is μ -measurable, one inclusion is trivial, by definition of $\Omega^\sharp$.

On the other hand, let us start writing down the positivity set of $\chi_\Omega^\sharp$. By definition [\(2.4\)](#), setting $\sigma = \sigma(x) = \Phi_f(f(x))$, we have

$$\chi_\Omega^\sharp(x) = \chi_\Omega^*(\sigma(x)).$$

Recall that by definition [\(2.2\)](#)

$$\chi_\Omega^*(s) = \inf\{t \geq 0 : \Phi_{\chi_\Omega}(t) \leq s\},$$

where the distribution function of χ_Ω , refer to [\(2.1\)](#), is

$$\Phi_{\chi_\Omega}(t) = \mu(\{z \in X : \chi_\Omega(z) > t\}).$$

Thus, we have

$$\Phi_{\chi_\Omega}(t) = \begin{cases} \mu(\Omega), & \text{if } t \in [0, 1), \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, we get

$$\chi_\Omega^*(s) = \begin{cases} 0, & \text{if } s \geq \mu(\Omega), \\ 1, & \text{otherwise.} \end{cases} \quad (2.5)$$

Let now u be any μ -measurable function defined on Ω , let $x \in \{u^\sharp > 0\}$, and $\sigma(x) = \Phi_f(f(x))$ as before. By definition of $u^\sharp$ and the characterization of u^* given in [\(2.3\)](#), $\sigma(x) \in [0, \Phi_u(0))$. Therefore, we have $\sigma(x) < \Phi_u(0) \leq \mu(\Omega)$. Hence, by [\(2.5\)](#), $x \in \Omega^\sharp$. $\square$

Further, the symmetrized set $\Omega^\sharp$ is a superlevel set of f with the same measure as Ω and, whenever $\mu(\Omega) \neq \mu(X)$, its boundary relative to X coincides with the corresponding level set of f .

Lemma 2.8. *Let μ , f , and Ω be as in [Definition 2.4](#), with $\mu(\Omega) < +\infty$, and let $t_\Omega = \Phi_f^{-1}(\mu(\Omega))$. Then $t_\Omega \in [0, 1)$,*

$$\Omega^\sharp = \{f > t_\Omega\}, \quad X \setminus \Omega^\sharp = \{f \leq t_\Omega\}, \quad \mu(\Omega^\sharp) = \mu(\Omega).$$

In particular, $\Omega^\sharp$ is open and it depends on Ω only through $\mu(\Omega)$. If, moreover, $\mu(\Omega) < \mu(X)$, then $t_\Omega \in (0, 1)$ and $\partial\Omega^\sharp \cap X = \{f = t_\Omega\}$.

Proof. Since Ω is a nonempty open set and μ is strictly positive, we have $0 < \mu(\Omega) \leq \mu(X)$. Thus, by Lemma 2.5, t_Ω is well-defined and $t_\Omega < 1$. By (2.5), recalling that in the proof of Lemma 2.7 we set $\sigma(x) = \Phi_f(f(x))$, we have $\chi_\Omega^\sharp(x) > 0$ if and only if $\Phi_f(f(x)) < \mu(\Omega) = \Phi_f(t_\Omega)$. By Lemma 2.5, Φ_f is strictly decreasing, thus this is equivalent to $f(x) > t_\Omega$. This proves the first two equalities, and gives $\mu(\Omega^\sharp) = \Phi_f(t_\Omega) = \mu(\Omega)$.

If moreover $\mu(\Omega) < \mu(X) = \Phi_f(0)$, then $t_\Omega > 0$. By continuity of f , one has the set inclusion $\partial\Omega^\sharp \cap X \subset \{f = t_\Omega\}$. Conversely, let $x \in X$ with $f(x) = t_\Omega \in (0, 1)$, so that by the first part of the proof $x \notin \Omega^\sharp$. Recall that by assumption (H2), we have $\nabla f(x) \neq 0$. Since X is open, the function

$$\phi(\varepsilon) = f(x + \varepsilon \nabla f(x))$$

is well-defined for ε small enough. Moreover, it satisfies

$$\phi(0) = t_\Omega \quad \text{and} \quad \phi'(0) = |\nabla f(x)|^2 > 0.$$

Hence for all $0 < \varepsilon \ll 1$, we have $\phi(\varepsilon) > t_\Omega$, that is, $x + \varepsilon \nabla f(x) \in \Omega^\sharp$. Letting $\varepsilon \rightarrow 0^+$, we get $x \in \overline{\Omega^\sharp}^X \setminus \Omega^\sharp = \partial\Omega^\sharp \cap X$. $\square$

Proposition 2.9 (Continuity of the symmetrization). *Let μ , f , and Ω be as in Definition 2.4, with $\mu(\Omega) < +\infty$. For every $u \in L_\mu^p(\Omega)$, its Talenti symmetrization $u^\sharp$ belongs to $L_\mu^p(\Omega^\sharp)$ and*

$$\|u\|_{p,\mu,\Omega} = \|u^\sharp\|_{p,\mu,\Omega^\sharp}. \quad (2.6)$$

Moreover, the Talenti symmetrization map $T : u \mapsto u^\sharp$ is an L^p -nonexpansive map on the cone of nonnegative functions in $L_\mu^p(\Omega)$, i.e.,

$$\|u^\sharp - v^\sharp\|_{p,\mu,\Omega^\sharp} \leq \|u - v\|_{p,\mu,\Omega}, \quad \forall u, v \geq 0, \quad \mu\text{-a.e.} \quad (2.7)$$

Therefore, it is continuous on such cone.

Proof. The construction of the Talenti symmetrization ensures that every superlevel set of $u^\sharp$ coincides with a superlevel set of f . Indeed, for every $t > 0$,

$$\{x \in X : u^\sharp(x) > t\} = \{x \in X : f(x) > f^*(\Phi_u(t))\},$$

also owing to the fact that, by Lemma 2.5, Φ_f is a homeomorphism of $[0, 1]$ onto $[0, \mu(X)]$, whose inverse is the restriction of f^* to $[0, \mu(X)]$. Moreover, $\Phi_{u^\sharp}(t) = \Phi_u(t)$. Then, by the layer-cake formula (see [34, Ex. 1.13]), the definition (2.1) of μ -distribution of u , the previous equality, the layer-cake formula once again, and Lemma 2.7, we have

$$\|u\|_{p,\mu,\Omega}^p = \|u\|_{p,\mu,X}^p = p \int_0^{+\infty} t^{p-1} \mu(\{x \in X : |u(x)| > t\}) \, dt$$

$$\begin{aligned}
&= p \int_0^{+\infty} t^{p-1} \Phi_u(t) \, dt = p \int_0^{+\infty} t^{p-1} \Phi_{u^\sharp}(t) \, dt \\
&= \|u^\sharp\|_{p,\mu,X}^p = \|u^\sharp\|_{p,\mu,\Omega^\sharp}^p.
\end{aligned}$$

The nonexpansivity of the Talenti symmetrization from $L_\mu^p(\Omega)$ to $L_\mu^p(\Omega^\sharp)$ is an immediate consequence of [19]. The cited results are stated for maps from an $L_\mu^p(\Omega)$ space into itself, with μ a nonnegative measure, but proofs can be repeated verbatim for maps $L_\mu^p(\Omega_\mu) \rightarrow L_\nu^p(\Omega_\nu)$, where the spaces are taken with respect to different nonnegative measures, up to additionally requiring $\mu(\Omega_\mu) = \nu(\Omega_\nu)$, which in our case holds by Lemma 2.8. We also refer to [19, Ex. 2] where rearrangements are explicitly considered.

Since the Talenti symmetrization $T : u \mapsto u^\sharp$ is integral preserving and order preserving for nonnegative functions, by [19, Prop. 1(a) and (c)], it is an L^1 -nonexpansive map between the cone of nonnegative functions in $L_\mu^1(\Omega)$ and $L_\mu^1(\Omega^\sharp)$.

Further, the symmetrization commutes with the addition of nonnegative constants on the cone of nonnegative functions: if $u \geq 0$ and $c \geq 0$, where $u + c$ is understood on Ω and extended by zero outside, then $(u + c)^* = u^* + c$ on $[0, \mu(\Omega))$, so that

$$T(u + c) = T(u) + c \quad \text{on } \Omega^\sharp.$$

Hence, given $u, v \geq 0$ and $k = \text{ess sup}_\Omega |u - v|$, from $u \leq v + k$ and the order preservation we get $T(u) \leq T(v + k) = T(v) + k$ on $\Omega^\sharp$. Exchanging the roles of u and v , the symmetrization is an L^∞ -nonexpansive map between the cone of nonnegative functions in $L_\mu^\infty(\Omega)$ and $L_\mu^\infty(\Omega^\sharp)$, see also [19, Prop. 2].

By standard nonlinear interpolation, it is an L^p -nonexpansive map between the cone of nonnegative functions in $L_\mu^p(\Omega)$ and $L_\mu^p(\Omega^\sharp)$, refer to [19, Ex. 2, Eqs. (9)–(10)] or to [43], so that (2.7) follows. $\square$

3. PÓLYA–SZEGŐ PRINCIPLE IN SOBOLEV SPACES

Pólya–Szegő principles arising from symmetrizations ensure that integrals of Young functions $F(\cdot)$, depending only on $|\nabla u|$, decrease when evaluated on $|\nabla u^\sharp|$, and we refer the interested reader to [38] for its first appearance, in the Euclidean context. The usual symmetrizations rely on transforming level sets into isoperimetric sets. Talenti showed in [42] that this principle stays true for Lipschitz functions also for his symmetrization provided that: the measure space (X, μ) possesses a relative isoperimetric inequality with respect to the notions of weighted volume and perimeter, see (i); f is chosen so that its level

sets are connected and its superlevel sets are isoperimetric sets, see (ii), and it satisfies a suitable steepest ascent condition, see (iii).

We recall below the theorem, tailored to the case of our interest, $F(z) = |z|^p$. In order to properly state it, we recall that the μ -perimeter, with $\mu = e^W \mathcal{L}^N$ and W continuous, of a Borel set Ω relative to X is

$$\text{Per}_\mu(\Omega; X) = \int_{\partial^* \Omega \cap X} e^{W(x)} d\mathcal{H}^{N-1}(x), \quad (3.1)$$

where $\partial^* \Omega$ is the reduced boundary of Ω , refer to [34].

Theorem 3.1 (Pólya–Szegő principle of [42]). *Let μ and f be as in Definition 2.4. Suppose that the following conditions hold:*

(i) *there exists a continuous function $q : [0, \mu(X)) \rightarrow [0, +\infty)$, with $q(s) > 0$ for $0 < s < \mu(X)$, such that*

$$q(\mu(\Omega)) \leq \text{Per}_\mu(\Omega; X), \quad \text{for all Borel subsets } \Omega \subset X \text{ with } \mu(\Omega) < \mu(X);$$

(ii) *the level sets $\{x \in X : f(x) = t\}$, $t \in (0, 1)$, are connected, and the superlevel sets of f saturate the isoperimetric inequality of point (i), i.e.,*

$$\text{Per}_\mu(\{x \in X : f(x) > t\}; X) = q(\Phi_f^\mu(t)), \quad \forall t \in (0, 1);$$

(iii) *the lines of steepest ascent of f are straight, that is, if we define the curves $s \mapsto x(s) \in \mathbb{R}^N$ such that*

$$\frac{dx}{ds} = \frac{\nabla f}{|\nabla f|}(x(s)),$$

where s is the arclength parameter, then they satisfy

$$\frac{d^2 x}{ds^2} = 0.$$

Given any Lipschitz function $u : X \rightarrow \mathbb{R}$ such that $\mu(\{u \neq 0\}) < +\infty$, the Talenti symmetrization $u^\sharp$ of u (defined by (2.4) with $\tilde{u} = u$) is locally Lipschitz and, for every $p \in (1, +\infty)$, it holds

$$\int_X |\nabla u|^p d\mu \geq \int_X |\nabla u^\sharp|^p d\mu. \quad (3.2)$$

We remark that Talenti does not explicitly mention that the symmetrized function is locally Lipschitz, but this can be derived by closely inspecting his proof. A global Lipschitz bound, which is what is needed in the sequel, is a consequence of (3.2), see Lemma 3.2 below.

Further, he does not use as perimeter the weighted $(N - 1)$ -dimensional Hausdorff measure (3.1) but rather the relative dual-sided Minkowski content [42, Eq. (1.7)]. Nevertheless, in the proof he uses the weighted variational

relative perimeter to exploit coarea formula. This is well-known to be equivalent to (3.1), which coincides with the relative dual-sided Minkowski content for almost every superlevel set of a Lipschitz function.

Moreover, in [42, Main Thm.] the function q is required to be smooth on $[0, \mu(X))$. However, only its continuity and positivity on $(0, \mu(X))$ are used in the proof, see [42, Eqs. (3.4) and (3.10)], while smoothness at 0 fails already for the Euclidean isoperimetric profile $q(s) = c_N s^{1-1/N}$, considered in [42, Ex. 1]. We have also made explicit in (ii) the requirement that the level sets of f are connected. Talenti's argument uses the fact that $|\nabla f|$ is constant on each level set of f , where, in its proof, this is derived from (iii) by means of [35]. As shown in Step 1 of the proof of Theorem 3.3 below, condition (iii) only yields that $|\nabla f|$ is constant on each *connected component* of the level sets of f , whence the need of the connectedness assumption.

As a consequence of (3.2) holding for every p , the Talenti symmetrization does not increase the Lipschitz constant, provided X is convex.

Lemma 3.2 (Global Lipschitz bound). *Let μ and f be as in Theorem 3.1, and let $u : X \rightarrow \mathbb{R}$ be a Lipschitz function with $\mu(\{u \neq 0\}) < +\infty$. Then*

$$\|\nabla u^\sharp\|_{L^\infty(X)} \leq \|\nabla u\|_{L^\infty(X)}.$$

In particular, if X is convex, then $u^\sharp$ is Lipschitz on X , and its Lipschitz constant does not exceed that of u .

Proof. Let $m = \mu(\{u \neq 0\})$. If $m = 0$, then $u^\sharp \equiv 0$ and there is nothing to prove. Assume now $m \neq 0$. By (3.2), and since $\nabla u = 0$ a.e. on $\{u = 0\}$, for every $p \in (1, +\infty)$, we have

$$\|\nabla u^\sharp\|_{p,\mu,X} \leq \|\nabla u\|_{p,\mu,X} \leq \|\nabla u\|_{L^\infty(X)} m^{\frac{1}{p}}. \quad (3.3)$$

Let now $s < \|\nabla u^\sharp\|_{L^\infty(X)}$, where the essential supremum is equivalently taken with respect to μ or $\mathcal{L}^N$, these measures being mutually absolutely continuous. Then, by definition of essential supremum, $S = \{|\nabla u^\sharp| > s\}$ has positive μ -measure, so that by (3.3) and by trivial lower estimates, for all $p \in (1, +\infty)$, we get

$$\|\nabla u\|_{L^\infty(X)} m^{\frac{1}{p}} \geq \|\nabla u^\sharp\|_{p,\mu,X} \geq s \mu(S)^{\frac{1}{p}}.$$

In particular, $\mu(S) < +\infty$ whenever $s > 0$. Letting now $p \rightarrow +\infty$, we get $s \leq \|\nabla u\|_{L^\infty(X)}$, and the claimed bound follows by arbitrariness of s .

Assume now that X is convex, and let $x, y \in X$. The segment $[x, y]$ is a compact subset of the open set X , thus it has positive distance from ∂X . Since $u^\sharp$ is locally Lipschitz by Theorem 3.1, its distributional gradient coincides with

its a.e. gradient, and, for ε small enough, the mollifications $u^\sharp * \rho_\varepsilon$ are well-defined and smooth in a neighborhood of $[x, y]$, where they satisfy

$$|\nabla(u^\sharp * \rho_\varepsilon)| = |(\nabla u^\sharp) * \rho_\varepsilon| \leq \|\nabla u^\sharp\|_{L^\infty(X)} \leq \|\nabla u\|_{L^\infty(X)},$$

by the first part of the statement. Moreover, $u^\sharp * \rho_\varepsilon \rightarrow u^\sharp$ uniformly on $[x, y]$. Hence,

$$|u^\sharp(x) - u^\sharp(y)| = \lim_{\varepsilon \rightarrow 0^+} |u^\sharp * \rho_\varepsilon(x) - u^\sharp * \rho_\varepsilon(y)| \leq \|\nabla u\|_{L^\infty(X)} |x - y|. \quad \square$$

3.1. Analytic and geometric properties of f . In this subsection, assuming convexity of the ambient space X , we establish some structural properties of the functions f as in [Definition 2.4](#), which additionally satisfy the connectedness condition of (ii) and condition (iii) of [Theorem 3.1](#). Specifically, in [Theorem 3.3](#) we prove a quantitative separation of their level sets, which in particular yields the uniform continuity of f , and which is the key ingredient in the proof of the Pólya–Szegő principle below, see [Theorem 3.7](#). In [Theorem 3.5](#), under a mild additional assumption, we obtain a representation of f in terms of a distance function, whose geometric consequences on the family of superlevel sets of f are collected in [Remark 3.6](#).

Theorem 3.3. *Let μ and f be as in [Definition 2.4](#). Assume that X is convex, that f satisfies condition (iii) of [Theorem 3.1](#), and that its level sets*

$$M_c = \{x \in X : f(x) = c\}, \quad c \in (0, 1),$$

are connected. Let $X_{\text{reg}} = \{x \in X : 0 < f(x) < 1\}$. Then, the following hold.

- (a) *There exists a continuous function $g : (0, 1) \rightarrow (0, +\infty)$ such that $\|\nabla f(x)\| = g(f(x))$ for all $x \in X_{\text{reg}}$.*
- (b) *Let $c_0 \in (0, 1)$ and*

$$G(c) = \int_{c_0}^c \frac{dt}{g(t)}, \quad \text{for } c \in (0, 1).$$

If $y_1, y_2 \in X$ and $0 < \tau_1 < \tau_2 < 1$ are such that $f(y_1) \leq \tau_1$ and $f(y_2) \geq \tau_2$, then

$$\|y_1 - y_2\| \geq G(\tau_2) - G(\tau_1) > 0. \quad (3.4)$$

In particular,

$$\text{dist}(\{f \leq t_1\}; \{f \geq t_2\}) > 0 \quad \text{whenever } 0 \leq t_1 < t_2 \leq 1.$$

- (c) *The function f is uniformly continuous on X ; thus, it uniquely extends to a continuous function on $\overline{X}$.*

Proof. We divide the proof into several steps. By (H2), $\nabla f \neq 0$ on X_{reg} , thus each level set M_c , $c \in (0, 1)$, is a smooth hypersurface, connected by hypothesis.

Step 1: Proof of (a). This is mentioned en passant in [42, Property (viii)]; though, the key request of level sets being connected is missing there. We give here the proof. Let $\mathbf{N}$ denote the unit normal vector field to the level sets of f in X_{reg} , that is,

$$\mathbf{N}(x) = \frac{\nabla f(x)}{\|\nabla f(x)\|}. \quad (3.5)$$

A trajectory of steepest ascent $\gamma(s)$ parameterized by arclength s satisfies the ODE $\gamma'(s) = \mathbf{N}(\gamma(s))$. Since these trajectories are straight lines, $\mathbf{N}(\gamma(s))$ is constant along each of them, giving $\gamma(s) = x_0 + s\mathbf{N}(x_0)$, where $x_0 = \gamma(0)$ is the initial datum. By constancy of $\mathbf{N}$, differentiating we get

$$(D\mathbf{N})\mathbf{N} = 0. \quad (3.6)$$

For $x \in X_{\text{reg}}$, we let $w(x) = \|\nabla f(x)\| > 0$. Thus, by definition,

$$\mathbf{N}(x) = \frac{\nabla f(x)}{w(x)}. \quad (3.7)$$

Differentiating the above equality, we find

$$D\mathbf{N}(x) = \frac{1}{w}D\nabla f(x) - \frac{\nabla f(x) \otimes \nabla w(x)}{w^2} = \frac{1}{w}D^2f(x) - \frac{1}{w}\mathbf{N}(x) \otimes \nabla w(x).$$

Applying the above to $\mathbf{N}(x)$, using (3.6), rearranging, and owing to tensor algebra rules $((a \otimes b)c = (b \cdot c)a)$, we find

$$D^2f \mathbf{N} = (\mathbf{N} \otimes \nabla w)\mathbf{N} = (\nabla w \cdot \mathbf{N})\mathbf{N}.$$

On the other hand, differentiating the equality $w^2 = \|\nabla f\|^2$, rearranging, and using the relation (3.7), we get $\nabla w = D^2f \mathbf{N}$. Putting these two last equalities together and recalling the definition of w and of $\mathbf{N}$, we have

$$\nabla \|\nabla f\| = \left(\nabla \|\nabla f\| \cdot \frac{\nabla f}{\|\nabla f\|} \right) \frac{\nabla f}{\|\nabla f\|},$$

that is, $\nabla \|\nabla f\|$ is parallel to ∇f . Hence, the tangential gradient of $\|\nabla f\|$ along each level set M_c , $c \in (0, 1)$, vanishes, so that $\|\nabla f\|$ is constant on each connected component of M_c . Since the level sets are connected by assumption, one has

$$\|\nabla f(x)\| = g(f(x)), \quad \forall x \in X_{\text{reg}}, \quad (3.8)$$

for some function $g : (0, 1) \rightarrow (0, +\infty)$. To see that g is continuous, we first notice that f maps the open set X_{reg} onto $(0, 1)$, since X is connected,

$\inf f = 0$, and $\sup f = 1$. Moreover, f is an open map on X_{reg} . Indeed, given $x \in X_{\text{reg}}$ and $r > 0$ with $B_r(x) \subset X_{\text{reg}}$, the continuous function

$$\phi(\varepsilon) = f(x + \varepsilon \nabla f(x))$$

satisfies $\phi'(0) = |\nabla f(x)|^2 > 0$, so that it is strictly increasing on $[-\delta, \delta]$ for some $0 < \delta \ll 1$. Up to choosing a smaller δ , the segment $\{x + \varepsilon \nabla f(x) : |\varepsilon| \leq \delta\}$ is contained in $B_r(x)$. Then, by the intermediate value theorem, we get

$$f(B_r(x)) \supset \{f(x + \varepsilon \nabla f(x)) : |\varepsilon| \leq \delta\} = [\phi(-\delta), \phi(\delta)] \supset (\phi(-\delta), \phi(\delta)).$$

Therefore, for every open set $U \subset (0, +\infty)$, owing to (3.8) and to the surjectivity of f , we have

$$g^{-1}(U) = f(\{x \in X_{\text{reg}} : \|\nabla f(x)\| \in U\}),$$

which is open, being the image under the open map f of an open set, as $x \mapsto \|\nabla f(x)\|$ is continuous. Thus, g is continuous.

Step 2: Proof of (b) and (c). Let y_1, y_2, τ_1 , and τ_2 be as in (b). Clearly $y_1 \neq y_2$; let

$$\ell = \|y_2 - y_1\| \quad \text{and} \quad e = \frac{y_2 - y_1}{\ell}.$$

Since X is convex, the function $\phi(\tau) = f(y_1 + \tau e)$ is well-defined; since f is smooth it also is continuous on $[0, \ell]$. Since, by assumption,

$$\phi(0) = f(y_1) \leq \tau_1 < \tau_2 \leq f(y_2) = \phi(\ell),$$

the numbers

$$\sigma_2 = \min\{\tau \in [0, \ell] : \phi(\tau) = \tau_2\}, \quad \sigma_1 = \max\{\tau \in [0, \sigma_2] : \phi(\tau) = \tau_1\},$$

are well-defined, and by the intermediate value theorem $\tau_1 < \phi < \tau_2$ on (σ_1, σ_2) . Hence, the segment $\{y_1 + \tau e : \tau \in [\sigma_1, \sigma_2]\}$ lies in X_{reg} . Moreover, the function ϕ is C^1 on $[\sigma_1, \sigma_2]$ and, by (3.8),

$$|\phi'(\tau)| \leq \|\nabla f(y_1 + \tau e)\| = g(\phi(\tau)).$$

Therefore, by definition of ℓ , by definition of σ_i , by definition of G , the above inequality, and the chain rule on $(G \circ \phi)'$ which shows that $(G \circ \phi)' \leq 1$ on $[\sigma_1, \sigma_2]$, once again by definition of σ_i , and finally by definition of G and the positivity of g , we have

$$\begin{aligned} \|y_2 - y_1\| = \ell &\geq \sigma_2 - \sigma_1 \geq \int_{\sigma_1}^{\sigma_2} (G \circ \phi)'(\tau) \, d\tau \\ &= G(\tau_2) - G(\tau_1) = \int_{\tau_1}^{\tau_2} \frac{1}{g(t)} \, dt > 0, \end{aligned}$$

which proves (3.4).

Let now $0 \leq t_1 < t_2 \leq 1$ be fixed and let y_1, y_2 be any two points in, respectively, $\{f \leq t_1\}$ and $\{f \geq t_2\}$. Let us define

$$\tau_1 = \frac{2t_1 + t_2}{3} \quad \text{and} \quad \tau_2 = \frac{t_1 + 2t_2}{3},$$

which satisfy $0 \leq t_1 < \tau_1 < \tau_2 < t_2 \leq 1$. Hence, applying (3.4) with τ_1, τ_2 as above gives the claim, by the arbitrariness of y_1, y_2 .

Let us now turn our attention to point (c). Start fixing $\varepsilon \in (0, 1)$, then let $I = [\varepsilon/3, 1 - \varepsilon/3]$, and let

$$\eta_\varepsilon = \frac{\varepsilon}{3} \left(\max_I g \right)^{-1} > 0.$$

Let $y_1, y_2 \in X$ be such that $f(y_2) - f(y_1) \geq \varepsilon$. Then the pair

$$\tau_1 = f(y_1) + \frac{\varepsilon}{3} \quad \text{and} \quad \tau_2 = f(y_1) + \frac{2\varepsilon}{3},$$

owing to $0 \leq f \leq 1$ on X , satisfies

$$\frac{\varepsilon}{3} \leq \tau_1 < \tau_2 \leq f(y_2) - \frac{\varepsilon}{3} \leq 1 - \frac{\varepsilon}{3},$$

so that, by (3.4),

$$\|y_1 - y_2\| \geq \int_{\tau_1}^{\tau_2} \frac{dt}{g(t)} \geq \eta_\varepsilon.$$

Therefore, whenever $\|y_1 - y_2\| < \eta_\varepsilon$, we have $|f(y_1) - f(y_2)| < \varepsilon$, that is, f is uniformly continuous on X , and thus it uniquely extends to a continuous function on $\overline{X}$. $\square$

Remark 3.4 (Nested isoperimetric family). If, in addition to the assumptions of Theorem 3.3, condition (ii) of Theorem 3.1 holds, then the superlevel sets $E_t = \{f > t\}$, $t \in (0, 1)$, form a nested one-parameter family of open isoperimetric sets. By Lemma 2.5, the map $t \mapsto \mu(E_t) = \Phi_f(t)$ is a continuous, strictly decreasing bijection of $(0, 1)$ onto $(0, \mu(X))$, so that every volume $m \in (0, \mu(X))$ is attained by exactly one set E_t . Moreover, by Theorem 3.3(b), if $t_1 < t_2$, then the relative closure of E_{t_2} in X , which is contained in $\{f \geq t_2\}$, has positive distance from $X \setminus E_{t_1} = \{f \leq t_1\}$. By Lemma 2.8, for every Ω as in Definition 2.4 with $\mu(\Omega) < +\infty$, the symmetrized set $\Omega^\sharp$ is the member of this family with $\mu(\Omega^\sharp) = \mu(\Omega)$.

If the lines of steepest ascent of f sweep the whole range of f , that is, the restriction of f to any of them is onto $(0, 1)$, then f can be determined by any of its level sets and by the profile g , as the composition of G^{-1} with the signed distance function from a fixed level set. In particular, the level sets of f are parallel hypersurfaces.

Theorem 3.5. *Let μ and f be as in [Theorem 3.3](#), and let g be as therein. Assume in addition that*

- (iv) *along every maximal line of steepest ascent of f in X_{reg} , the function f takes all values in $(0, 1)$.*

Let $c_0 \in (0, 1)$ be fixed, let G be defined as in [Theorem 3.3 \(b\)](#), and let

$$s_{\min} = \lim_{c \rightarrow 0^+} G(c) \in [-\infty, 0) \quad \text{and} \quad s_{\max} = \lim_{c \rightarrow 1^-} G(c) \in (0, +\infty],$$

so that we can extend the inverse $G^{-1} : (s_{\min}, s_{\max}) \rightarrow (0, 1)$ to the whole $\mathbb{R}$ by setting it equal to 0 on $(-\infty, s_{\min}]$ and to 1 on $[s_{\max}, +\infty)$. Let d be the signed distance from M_{c_0} , positive in $\{f > c_0\}$. Then, the following hold.

- (a) *Level sets of f are parallel hypersurfaces; specifically, one has*

$$f(y) = G^{-1}(d(y)), \quad \forall y \in X.$$

Moreover, $\{f = 0\} = \emptyset$ if $s_{\min} = -\infty$, and $\{f = 1\} = \emptyset$ if $s_{\max} = +\infty$.

- (b) *If the unit normal $\mathbf{N} = \nabla f / \|\nabla f\|$ is not constant on M_{c_0} , then either $s_{\min} > -\infty$ or $s_{\max} < +\infty$. If $s_{\max} < +\infty$, letting*

$$F_1 = \{x + s_{\max} \mathbf{N}(x) : x \in M_{c_0}\} \subset \overline{X},$$

one has

$$f(y) = G^{-1}(s_{\max} - \text{dist}(y; F_1)), \quad \forall y \in X.$$

Analogously, if $s_{\min} > -\infty$, f is represented via G^{-1} composed with the distance from F_0 , the corresponding analog of F_1 .

Proof. We divide the proof into several steps; $\mathbf{N}$ denotes the unit normal as in [\(3.5\)](#).

Step 1: Normal coordinates. We claim that:

- (1) for every $x \in M_{c_0}$, the segment $P_s(x) = x + s\mathbf{N}(x)$, $s \in (s_{\min}, s_{\max})$, is contained in X_{reg} , and

$$f(P_s(x)) = G^{-1}(s), \quad \forall s \in (s_{\min}, s_{\max}), \quad (3.9)$$

that is, for a fixed s , the image of the level set M_{c_0} via P_s is contained in another level set of f ;

- (2) every $y \in X_{\text{reg}}$ can be uniquely written as $y = P_s(x)$, with $x \in M_{c_0}$ and $s = G(f(y))$.

In other words, the level sets of f are the parallel hypersurfaces given by $M_{G^{-1}(s)} = P_s(M_{c_0})$.

Let γ be a maximal line of steepest ascent in X_{reg} , *i.e.*, a maximal solution in X_{reg} of $\gamma' = \mathbf{N}(\gamma)$, which exists and is uniquely determined by $\gamma(0)$, as $\mathbf{N}$ is of class C^1 on X_{reg} ; let I be its interval of definition. By (3.8),

$$\frac{d}{ds}f(\gamma(s)) = \nabla f(\gamma(s)) \cdot \mathbf{N}(\gamma(s)) = \|\nabla f(\gamma(s))\| = g(f(\gamma(s))),$$

so that $\frac{d}{ds}G(f(\gamma(s))) = 1$, that is,

$$G(f(\gamma(s))) = G(f(\gamma(0))) + s, \quad \forall s \in I. \quad (3.10)$$

Let $a = G(f(\gamma(0)))$. By (3.10), we have

$$G(f(\gamma(I))) = \{a + s : s \in I\} = a + I.$$

By assumption (iv), $f(\gamma(I)) = (0, 1)$, and G maps $(0, 1)$ onto $(s_{\min}, s_{\max})$. Hence, $a + I = (s_{\min}, s_{\max})$, that is,

$$I = (s_{\min} - a, s_{\max} - a).$$

If $\gamma(0) = x \in M_{c_0}$, then $G(f(x)) = G(c_0) = 0$, so that $I = (s_{\min}, s_{\max})$, $\gamma(s) = x + s\mathbf{N}(x) = P_s(x)$ by (iii), and (3.10) gives (3.9). This proves (1).

Let now $y \in X_{\text{reg}}$, let γ be the maximal line of steepest ascent with $\gamma(0) = y$, and set $s_y = G(f(y)) \in (s_{\min}, s_{\max})$. In this case $a = s_y$, so that

$$I = (s_{\min} - s_y, s_{\max} - s_y),$$

which contains $t = -s_y$, as $s_{\min} < 0 < s_{\max}$. Consider the point

$$x = \gamma(-s_y) = y - s_y\mathbf{N}(y).$$

By (3.10), $G(f(x)) = s_y - s_y = 0 = G(c_0)$, so that $f(x) = c_0$ by injectivity of G , that is, $x \in M_{c_0}$. Moreover, since γ is a straight line, its velocity $\gamma' = \mathbf{N}(\gamma)$ is constant, whence $\mathbf{N}(x) = \mathbf{N}(\gamma(-s_y)) = \mathbf{N}(\gamma(0)) = \mathbf{N}(y)$. Therefore,

$$y = x + s_y\mathbf{N}(y) = x + s_y\mathbf{N}(x) = P_{s_y}(x),$$

which proves the existence part of (2). As for uniqueness, if $y = P_s(x) = P_{s'}(x')$ with $x, x' \in M_{c_0}$, then $s = s' = G(f(y))$ by (3.9); moreover, y lies on the lines of steepest ascent through x and through x' , along which $\mathbf{N}$ is constant, so that $\mathbf{N}(x) = \mathbf{N}(y) = \mathbf{N}(x')$, whence $x = y - s\mathbf{N}(y) = x'$.

Step 2: Representation via the signed distance. Let $y \in X_{\text{reg}}$ and set also $s = G(f(y))$. Since G is strictly increasing and $G(c_0) = 0$, the sign of s is that of $f(y) - c_0$. We claim that

$$\text{dist}(y; M_{c_0}) = |s|. \quad (3.11)$$

On the one hand, by Step 1, there exists $x_0 \in M_{c_0}$ such that $y = x_0 + s\mathbf{N}(x_0)$; hence, $\text{dist}(y; M_{c_0}) \leq \|y - x_0\| = |s|$.

On the other hand, let $x \in M_{c_0}$. If $s = 0$, then trivially $\|y - x\| \geq 0 = |s|$. If $s > 0$, that is, $c_0 < f(y)$, we apply (3.4) with

$$y_1 = x, \quad y_2 = y, \quad \tau_1 = c_0, \quad \tau_2 = f(y),$$

which is allowed, since $0 < \tau_1 < \tau_2 < 1$, $f(y_1) = \tau_1$, and $f(y_2) = \tau_2$; this gives $\|y - x\| \geq G(f(y)) - G(c_0) = s$. If $s < 0$, that is, $f(y) < c_0$, we apply (3.4) as above, but with roles of x, y exchanged. Thus

$$\|y - x\| \geq |s|, \quad \forall x \in M_{c_0}.$$

Taking the infimum over $x \in M_{c_0}$ proves the opposite inequality, giving (3.11).

Finally, by definition, $d(y) = \text{dist}(y; M_{c_0})$ if $f(y) \geq c_0$, that is, if $s \geq 0$, and $d(y) = -\text{dist}(y; M_{c_0})$ otherwise. Hence, by (3.11), $d(y) = s = G(f(y))$, that is,

$$f(y) = G^{-1}(d(y)), \quad \forall y \in X_{\text{reg}}. \quad (3.12)$$

Step 3: Density of X_{reg} and proof of (a). We claim that X_{reg} is dense in X . Let $y \in X$ with $f(y) = 0$, and assume by contradiction that there exists $r > 0$ such that $B_r(y) \subset X$ and $B_r(y) \cap X_{\text{reg}} = \emptyset$. Then f only takes the values 0 and 1 on the connected open ball $B_r(y)$; by continuity, $f \equiv 0$ on $B_r(y)$. This implies

$$\mathcal{L}^N(\{f = 0\}) \geq \mathcal{L}^N(B_r(y)) > 0,$$

contradicting (H1), since μ and $\mathcal{L}^N$ are mutually absolutely continuous. The case $f(y) = 1$ is analogous.

The signed distance d is continuous on X , and so is the extended function G^{-1} on $\mathbb{R}$. Hence, both sides of (3.12) are continuous on X and, coinciding on the dense set X_{reg} , they coincide on X . Finally, if $f(y) = 0$ for some $y \in X$, then $G^{-1}(d(y)) = 0$, so that $s_{\min} \geq d(y) > -\infty$; analogously, if $f(y) = 1$ for some $y \in X$, then $s_{\max} \leq d(y) < +\infty$. This proves (a).

Step 4: Proof of (b). Assume that $\mathbf{N}$ is not constant on M_{c_0} . Since M_{c_0} is connected, its shape operator, *i.e.*, the tangential differential of $\mathbf{N}$, does not vanish identically; thus, there exist $x_0 \in M_{c_0}$ and an index j such that the j -th principal curvature $\kappa_j(x_0)$ of M_{c_0} at x_0 is not 0. Along the line $s \mapsto P_s(x_0)$, the principal curvatures (with respect to the arclength parameter) of the parallel hypersurfaces $P_s(M_{c_0}) = M_{G^{-1}(s)}$ satisfy the Riccati-type ODE

$$\dot{\kappa}_i = \kappa_i^2, \quad \kappa_i(0) = \kappa_i(x_0),$$

refer to [35, Thm. 3], whose solution is

$$\kappa_i(x_0, s) = \frac{\kappa_i(x_0)}{1 - s\kappa_i(x_0)}.$$

Since the level sets $M_{G^{-1}(s)}$ are smooth hypersurfaces for all $s \in (s_{\min}, s_{\max})$, their principal curvatures are finite, so that $1 - s\kappa_j(x_0) \neq 0$ on $(s_{\min}, s_{\max})$. Hence, depending on the sign of $\kappa_j(x_0)$,

$$\text{either } s_{\min} \geq \frac{1}{\kappa_j(x_0)} > -\infty, \quad \text{or } s_{\max} \leq \frac{1}{\kappa_j(x_0)} < +\infty.$$

Assume that $s_{\max} < +\infty$, and let F_1 be as in (b); notice that $F_1 \subset \overline{X}$, since $P_s(x) \in X_{\text{reg}}$ for all $s < s_{\max}$. Let $y \in X_{\text{reg}}$, and write $y = x_y + s\mathbf{N}(x_y)$ with $x_y \in M_{c_0}$ and $s = d(y)$, see Steps 1 and 2.

On the one hand, $y_1 = x_y + s_{\max}\mathbf{N}(x_y) \in F_1$ and $\|y - y_1\| = s_{\max} - s$, so that,

$$\text{dist}(y; F_1) \leq s_{\max} - s.$$

On the other hand, let $z = x + s_{\max}\mathbf{N}(x) \in F_1$ and, for $s_k \uparrow s_{\max}$, let $z_k = P_{s_k}(x)$, so that $z_k \rightarrow z$ and, by (3.9), $f(z_k) = G^{-1}(s_k) \rightarrow 1$. Given $\tau_2 \in (f(y), 1)$, for k large enough we have $f(z_k) \geq \tau_2$, and (3.4) yields

$$\|y - z_k\| \geq G(\tau_2) - G(f(y)).$$

Letting first $k \rightarrow \infty$ and then $\tau_2 \rightarrow 1^-$, we obtain $\|y - z\| \geq s_{\max} - s$, and passing to the infimum over $z \in F_1$, we recover a lower bound for the distance.

This proves that $\text{dist}(y; F_1) = s_{\max} - d(y)$ and, by (3.12),

$$f(y) = G^{-1}(s_{\max} - \text{dist}(y; F_1)), \quad \forall y \in X_{\text{reg}}.$$

As in Step 3, since both sides are continuous and X_{reg} is dense in X , the equality holds on the whole X . The case $s_{\min} > -\infty$ is analogous. $\square$

Remark 3.6 (Superlevel sets are parallel sets). Let μ and f be as in Theorem 3.5, and let $E_t = \{f > t\}$, for $t \in (0, 1)$. Then,

$$E_{t'} = \{x \in X : \text{dist}(x; X \setminus E_t) > G(t') - G(t)\}, \quad 0 < t < t' < 1, \quad (3.13)$$

that is, each superlevel set of f is an inner parallel set, relative to X , of any larger one.

Indeed, let $y \in E_{t'}$, and pick $\tau_2 \in (t', 1)$ with $\tau_2 \leq f(y)$. For every $z \in X \setminus E_t$, that is, with $f(z) \leq t$, (3.4) with $\tau_1 = t$ gives $\|y - z\| \geq G(\tau_2) - G(t)$. Hence, as G is strictly increasing,

$$\text{dist}(y; X \setminus E_t) \geq G(\tau_2) - G(t) > G(t') - G(t),$$

which proves that $E_{t'}$ is included in the set on the RHS of (3.13).

Conversely, let $y \in X \setminus E_{t'}$, that is, $f(y) \leq t'$. We aim to show that $\text{dist}(y; X \setminus E_t) \leq G(t') - G(t)$, which would prove the opposite inclusion.

If $f(y) \leq t$, then $y \in X \setminus E_t$, so that $\text{dist}(y; X \setminus E_t) = 0$. If otherwise $t < f(y) \leq t'$, then $y \in X_{\text{reg}}$ and, by Step 1 of the proof of Theorem 3.5, we

can write $y = P_s(x)$, with $x \in M_{c_0}$ and $s = G(f(y))$; the point $P_{G(t)}(x)$ belongs to $M_t \subset X \setminus E_t$ and has distance $s - G(t) \leq G(t') - G(t)$ from y .

Moreover, the family $\{E_t\}_{t \in (0,1)}$ falls into one of the following alternatives.

- (a) If $\mathbf{N}$ is constant on M_{c_0} , say $\mathbf{N} \equiv \theta$, then M_{c_0} , being connected, is contained in a hyperplane $\{\langle x, \theta \rangle = r_0\}$. By Step 1 of the proof of [Theorem 3.5](#), $\langle y, \theta \rangle = r_0 + G(f(y))$ for all $y \in X_{\text{reg}}$, and, by continuity and density of X_{reg} in X , we get $E_t = H_{r_0+G(t)}(\theta) \cap X$, that is, the superlevel sets of f are parallel half-spaces intersected with X .
- (b) If $\mathbf{N}$ is not constant on M_{c_0} and $s_{\max} < +\infty$, then, by [Theorem 3.5\(b\)](#),

$$E_t = \{x \in X : \text{dist}(x; F_1) < s_{\max} - G(t)\}, \quad t \in (0, 1),$$

that is, the superlevel sets of f are metric neighborhoods, relative to X , of the focal set F_1 . Analogously, if $s_{\min} > -\infty$, they are the complements in X of the closed metric neighborhoods of F_0 .

For instance, concentric balls as in [Section 4.1](#) fall into (b), with $F_1 = \{0\}$, while parallel half-spaces as in [Section 4.2](#) fall into (a). Accordingly, Talenti symmetrization can only be performed with respect to nested families of mutually parallel isoperimetric sets.

3.2. Pólya–Szegő principle in $W_{\mu,0}^{1,p}(\Omega; X)$. We now prove that [\(3.2\)](#) extends from Lipschitz functions on X to the Sobolev space $W_{\mu,0}^{1,p}(\Omega; X)$, via a standard density argument.

Theorem 3.7 (Pólya–Szegő principle in $W_{\mu,0}^{1,p}(\Omega; X)$). *Let μ , f , and Ω be as in [Definition 2.4](#), with X convex and $\mu(\Omega) < +\infty$. Assume further conditions (i)–(iii) of [Theorem 3.1](#). Let $p \in (1, +\infty)$. Then, for every $u \in W_{\mu,0}^{1,p}(\Omega; X)$, the Talenti symmetrization $u^\sharp$ belongs to $W_{\mu,0}^{1,p}(\Omega^\sharp; X)$ and*

$$\int_{\Omega} |\nabla u|^p \, d\mu \geq \int_{\Omega^\sharp} |\nabla u^\sharp|^p \, d\mu. \quad (3.14)$$

Proof. For the sake of notation, we write v in place of u , as the letter u is booked below for auxiliary functions. Let $v \in W_{\mu,0}^{1,p}(\Omega; X)$. If $v = 0$, there is nothing to prove, thus we assume $v \neq 0$. Recall that $v^\sharp = |v|^\sharp$ is defined on the whole X ; by [Lemma 2.7](#), it vanishes outside $\Omega^\sharp$; and by [Lemma 2.8](#), $\Omega^\sharp = \{f > t_\Omega\}$, with $t_\Omega \in [0, 1)$ and $\mu(\Omega^\sharp) = \mu(\Omega)$.

Step 1: Approximation via Lipschitz functions. By definition, there exists a sequence $(v_n)_{n \in \mathbb{N}} \subset \text{Lip}_0(\Omega; X)$ such that $v_n \rightarrow v$ in the norm [\(1.3\)](#); without loss of generality, $v_n \neq 0$ for all n . Each v_n is Lipschitz on X , vanishes on $X \setminus \Omega$, has bounded support, and

$$d_n = \text{dist}(\text{supp}_X(v_n); \partial\Omega \cap X) > 0.$$

We stress that we do not need $v_n \geq 0$: since $u^\sharp = |u|^\sharp$ and $|\nabla|u|| = |\nabla u|$ a.e. for Lipschitz (and Sobolev) functions, all the quantities involved below only depend on $|v_n|$ and $|v|$.

Step 2: Symmetrization of the sequence. Since $\mu(\{v_n \neq 0\}) \leq \mu(\Omega) < +\infty$, [Theorem 3.1](#) applies to v_n , and $\nabla v_n = 0$ a.e. on $X \setminus \Omega$, so that

$$\int_{\Omega} |\nabla v_n|^p d\mu = \int_X |\nabla v_n|^p d\mu \geq \int_X |\nabla v_n^\sharp|^p d\mu \geq \int_{\Omega^\sharp} |\nabla v_n^\sharp|^p d\mu. \quad (3.15)$$

Together with [\(2.6\)](#), this gives $v_n^\sharp \in W_{\mu}^{1,p}(\Omega^\sharp)$. Moreover, since X is convex, by [Lemma 3.2](#), $v_n^\sharp$ is Lipschitz on X .

Step 3: Boundary buffer layer. We claim that

$$d_n^\sharp = \text{dist}(\text{supp}_X(v_n^\sharp); X \setminus \Omega^\sharp) > 0. \quad (3.16)$$

Since Ω is a nonempty proper open subset of the connected set X , there exists $b \in \partial\Omega \cap X$. Let $r_n \in (0, d_n)$ be such that $B_{r_n}(b) \subset X$. The set $U_n = B_{r_n}(b) \cap \Omega$ is open and nonempty, since $b \in \partial\Omega$, and it is disjoint from $\text{supp}_X(v_n)$, since its points have distance less than d_n from $\partial\Omega \cap X$. Since μ is strictly positive, $\mu(U_n) > 0$, and therefore

$$m_n = \mu(\{x : v_n(x) \neq 0\}) \leq \mu(\Omega \setminus U_n) = \mu(\Omega) - \mu(U_n) < \mu(\Omega).$$

By [Lemma 2.5](#), $\tau_n = \Phi_f^{-1}(m_n)$ satisfies $\tau_n > t_\Omega$, because Φ_f is strictly decreasing and $\Phi_f(t_\Omega) = \mu(\Omega) > m_n$. By the characterization of the zero set of v_n^* given after [Definition 2.3](#), $v_n^*(s) > 0$ if and only if $s < m_n$; hence,

$$\{v_n^\sharp \neq 0\} = \{x \in X : \Phi_f(f(x)) < m_n\} = \{f > \tau_n\},$$

so that $\text{supp}_X(v_n^\sharp) \subset \{f \geq \tau_n\}$, by continuity of f . Since $X \setminus \Omega^\sharp = \{f \leq t_\Omega\}$ and $t_\Omega < \tau_n$, [Theorem 3.3\(b\)](#) gives [\(3.16\)](#). In particular, $\text{supp}_X(v_n^\sharp) \subset \Omega^\sharp$ and, as $\Omega^\sharp$ is open, so that $\partial\Omega^\sharp \cap X \subset X \setminus \Omega^\sharp$, we have

$$\text{dist}(\text{supp}_X(v_n^\sharp); \partial\Omega^\sharp \cap X) \geq d_n^\sharp > 0.$$

Step 4: The symmetrized functions belong to $W_{\mu,0}^{1,p}(\Omega^\sharp; X)$. Let χ be a smooth cutoff function on the half-line $[0, +\infty)$, with $0 \leq \chi \leq 1$, $\chi(t) = 1$ on $[0, 1]$, $\chi(t) = 0$ for $t \geq 2$, and $|\chi'(t)| \leq 2$, and let $\chi_R(x) = \chi(\|x\|/R)$. The functions

$$u_R^n = v_n^\sharp \chi_R$$

are Lipschitz on X , as $v_n^\sharp$ is Lipschitz and bounded, have bounded support as χ_R has, and $\text{supp}_X(u_R^n) \subset \text{supp}_X(v_n^\sharp)$. By Step 3, also owing to the fact that $\mu(\{u_R^n \neq 0\}) \leq \mu(\{v_n^\sharp \neq 0\}) = m_n < +\infty$, $u_R^n \in \text{Lip}_0(\Omega^\sharp; X)$. Moreover,

$$|u_R^n - v_n^\sharp| \leq (1 - \chi_R)|v_n^\sharp|, \quad |\nabla u_R^n - \nabla v_n^\sharp| \leq (1 - \chi_R)|\nabla v_n^\sharp| + \frac{2}{R}|v_n^\sharp|,$$

so that, by the Dominated Convergence Theorem, $u_R^n \rightarrow v_n^\sharp$ in $W_\mu^{1,p}(\Omega^\sharp)$, as $R \rightarrow +\infty$. Hence, $v_n^\sharp \in W_{\mu,0}^{1,p}(\Omega^\sharp; X)$.

Step 5: Passing to the limit. By (2.7), applied to the nonnegative functions $|v_n|$ and $|v|$, and since $||v_n| - |v|| \leq |v_n - v|$, we have

$$\|v_n^\sharp - v^\sharp\|_{p,\mu,\Omega^\sharp} = \||v_n|^\sharp - |v|^\sharp\|_{p,\mu,\Omega^\sharp} \leq \||v_n| - |v|\|_{p,\mu,\Omega} \leq \|v_n - v\|_{p,\mu,\Omega} \rightarrow 0.$$

Moreover, the gradients $\nabla v_n^\sharp$ are uniformly bounded in $L_\mu^p(\Omega^\sharp)$, as a consequence of (3.15) together with the convergence of $(v_n)_n$ in the norm (1.3). Hence, $(v_n^\sharp)_{n \in \mathbb{N}}$ is bounded in $W_{\mu,0}^{1,p}(\Omega^\sharp; X)$.

Since $W_{\mu,0}^{1,p}(\Omega^\sharp; X)$ is reflexive, by the Eberlein–Šmulian Theorem there exist a subsequence $(v_{n_k}^\sharp)_{k \in \mathbb{N}}$ and a limit function $V \in W_{\mu,0}^{1,p}(\Omega^\sharp; X)$ (which is weakly closed, being a closed subspace) such that

$$v_{n_k}^\sharp \rightharpoonup V \quad \text{weakly in } L_\mu^p(\Omega^\sharp) \quad \text{and} \quad \nabla v_{n_k}^\sharp \rightharpoonup \nabla V \quad \text{weakly in } L_\mu^p(\Omega^\sharp).$$

From the uniqueness of the limit, $V = v^\sharp$, μ -almost everywhere; therefore, $v^\sharp \in W_{\mu,0}^{1,p}(\Omega^\sharp; X)$. Further, by the weak convergence of the gradients, by inequality (3.15), and by the convergence of $(v_n)_{n \in \mathbb{N}}$ in $W_\mu^{1,p}(\Omega)$, one has

$$\|\nabla v^\sharp\|_{p,\mu,\Omega^\sharp} \leq \liminf_{k \rightarrow \infty} \|\nabla v_{n_k}^\sharp\|_{p,\mu,\Omega^\sharp} \leq \liminf_{k \rightarrow \infty} \|\nabla v_{n_k}\|_{p,\mu,\Omega} = \|\nabla v\|_{p,\mu,\Omega}. \quad \square$$

As usual, the Pólya–Szegő principle on Sobolev spaces implies the corresponding Faber–Krahn and Saint-Venant inequalities.

Corollary 3.8 (Faber–Krahn inequality). *Let μ , f , Ω , and p be as in Theorem 3.7. Then,*

$$\lambda_{1,p}^\mu(\Omega^\sharp) \leq \lambda_{1,p}^\mu(\Omega).$$

Proof. Let $u \in W_{\mu,0}^{1,p}(\Omega; X)$. By Theorem 3.7, the Talenti symmetrization $u^\sharp$ belongs to $W_{\mu,0}^{1,p}(\Omega^\sharp; X)$. Moreover, by (2.6) and (3.14), we have

$$\lambda_{1,p}^\mu(\Omega^\sharp) \leq \frac{\int_{\Omega^\sharp} |\nabla u^\sharp|^p d\mu}{\int_{\Omega^\sharp} |u^\sharp|^p d\mu} \leq \frac{\int_{\Omega} |\nabla u|^p d\mu}{\int_{\Omega} |u|^p d\mu}.$$

Taking now the infimum on the right-hand side among non identically zero functions in $W_{\mu,0}^{1,p}(\Omega; X)$, the claim follows. $\square$

By Lemma 2.8, $\Omega^\sharp = \{f > t_\Omega\}$ depends on Ω only through $\mu(\Omega)$, and $\mu(\Omega^\sharp) = \mu(\Omega)$. Hence, Corollary 3.8 is precisely the Faber–Krahn inequality (1.5), with Ω^* the superlevel set of f having measure $\mu(\Omega)$; analogously, Corollary 3.9 below is the Saint-Venant inequality (1.6).

Corollary 3.9 (Saint-Venant inequality). *Let μ , f , Ω , and p be as in [Theorem 3.7](#). Then,*

$$T_p^\mu(\Omega^\sharp) \geq T_p^\mu(\Omega).$$

Proof. Let $u \in W_{\mu,0}^{1,p}(\Omega; X)$ with $\nabla u \neq 0$; since we are assuming $\mu(\Omega) < +\infty$, we have $u \in L_\mu^1(\Omega)$. By [Theorem 3.7](#), $u^\sharp \in W_{\mu,0}^{1,p}(\Omega^\sharp; X)$ and, by the proof of [Proposition 2.9](#), which holds verbatim for $p = 1$, $\|u^\sharp\|_{1,\mu,\Omega^\sharp} = \|u\|_{1,\mu,\Omega} > 0$. If $\nabla u^\sharp = 0$, then $T_p^\mu(\Omega^\sharp) = +\infty$: indeed, picking any $\psi \in \text{Lip}_0(\Omega^\sharp; X)$ with $\nabla \psi \neq 0$, the functions $u^\sharp + \varepsilon \psi$ are admissible and their ratios diverge as $\varepsilon \rightarrow 0^+$. Otherwise, by (3.14),

$$T_p^\mu(\Omega^\sharp) \geq \frac{\left(\int_{\Omega^\sharp} |u^\sharp| d\mu \right)^p}{\int_{\Omega^\sharp} |\nabla u^\sharp|^p d\mu} \geq \frac{\left(\int_{\Omega} |u| d\mu \right)^p}{\int_{\Omega} |\nabla u|^p d\mu}.$$

Taking the supremum over all such u , the claim follows. $\square$

Remark 3.10. [Corollaries 3.8](#) and [3.9](#) are particular cases of the more general Faber–Krahn-type inequality that one can prove on the first Dirichlet (p, q) -eigenvalue of the weighted p -Laplacian on Ω , $\lambda_{1,p,q}^\mu(\Omega)$. This is defined as

$$\lambda_{1,p,q}^\mu(\Omega) = \inf \left\{ \frac{\|\nabla u\|_{p,\mu,\Omega}^p}{\|u\|_{q,\mu,\Omega}^p} : u \in W_{\mu,0}^{1,p}(\Omega; X), u \neq 0 \right\},$$

where q is chosen so that the continuous embedding $W_{\mu,0}^{1,p}(\Omega; X) \hookrightarrow L_\mu^q(\Omega)$ holds. These first (p, q) -eigenvalues are linked to the optimal Poincaré–Sobolev constant in the above-mentioned embedding, refer to [\[3\]](#) in the Euclidean case.

The Talenti symmetrization keeps unchanged not only the L_μ^p -norm of u , but rather all of its L_μ^q -norms, as can be seen by following exactly the same proof as that of [Proposition 2.9](#). Then, Pólya–Szegő principle of [Theorem 3.7](#) entails, as in the proof of [Corollary 3.8](#), that

$$\lambda_{1,p,q}^\mu(\Omega) \geq \lambda_{1,p,q}^\mu(\Omega^\sharp).$$

In particular, [Corollary 3.8](#) corresponds to the case $q = p$, whereas [Corollary 3.9](#) corresponds to the case $q = 1$, since $T_p^\mu(\Omega) = 1/\lambda_{1,p,1}^\mu(\Omega)$, provided that nonzero constants do not belong to $W_{\mu,0}^{1,p}(\Omega; X)$, so that the admissible classes in the two definitions coincide.

Remark 3.11 (Isoperimetric sets known only for small volumes). In several situations, isoperimetric sets are known only for volumes below a certain threshold. The previous results are robust with respect to this restriction. Precisely, let $\bar{m} \in (0, \mu(X)]$, let $\bar{t} = \Phi_f^{-1}(\bar{m}) \in [0, 1)$, and replace conditions (i)–(ii) of [Theorem 3.1](#) with the following ones:

(i') there exists a continuous function $q : [0, \overline{m}] \rightarrow [0, +\infty)$, positive on $(0, \overline{m}]$, such that

$$q(\mu(E)) \leq \text{Per}_\mu(E; X)$$

for all Borel subsets $E \subset X$ with $\mu(E) \leq \overline{m}$;

(ii') the level sets $\{f = t\}$, $t \in (0, 1)$, are connected, and

$$\text{Per}_\mu(\{f > t\}; X) = q(\Phi_f^\mu(t))$$

for all $t \in [\bar{t}, 1)$.

Then, (3.2) holds for every Lipschitz function $u : X \rightarrow \mathbb{R}$ satisfying the request $\mu(\{u \neq 0\}) \leq \overline{m}$. Indeed, in the proof of [42], conditions (i)–(ii) only enter through the comparison, for a.e. $t > 0$, between $\text{Per}_\mu(\{|u| > t\}; X)$ and $\text{Per}_\mu(\{u^\sharp > t\}; X) = \text{Per}_\mu(\{f > \Phi_f^{-1}(\Phi_u(t))\}; X)$. Since, by assumption, $\Phi_u(t) \leq \mu(\{u \neq 0\}) \leq \overline{m}$ for all $t > 0$, only sets of measure at most $\overline{m}$ and superlevel sets $\{f > \tau\}$ with $\tau \geq \bar{t}$ are involved. Consequently, Lemma 3.2 holds for such functions u , and Theorem 3.7, Corollaries 3.8 and 3.9, and Remark 3.10 hold, with the same proofs, for every open set Ω as therein with $\mu(\Omega) \leq \overline{m}$, since the approximating functions v_n in the proof of Theorem 3.7 satisfy $\mu(\{v_n \neq 0\}) < \mu(\Omega) \leq \overline{m}$. In other words, one obtains Faber–Krahn and Saint-Venant inequalities among sets of small measure:

$$\lambda_{1,p}^\mu(\Omega^\sharp) \leq \lambda_{1,p}^\mu(\Omega) \quad \text{and} \quad T_p^\mu(\Omega^\sharp) \geq T_p^\mu(\Omega),$$

for all open sets $\Omega \subsetneq X$ with $\mu(\Omega) \leq \overline{m}$, where $\Omega^\sharp$ is the isoperimetric superlevel set of f with $\mu(\Omega^\sharp) = \mu(\Omega)$.

4. EXAMPLES

In this section we collect several examples that fall within the general framework presented, thus obtaining the corresponding Faber–Krahn and Saint-Venant inequalities. Actually, in all of the following settings, the more general inequality for the first (p, q) -eigenvalue of the weighted p -Laplacian mentioned in Remark 3.10 holds. In all the examples below, X is convex and the level sets of f are connected, as required in Theorem 3.7.

4.1. Isoperimetric sets are concentric balls. Let $X \subset \mathbb{R}^N$ with $N \geq 2$ be an open convex cone centered in the origin and let μ be a locally finite measure, that is, such that $\mu(B_r(0) \cap X) < +\infty$ for all $r > 0$. Assume that the one-parameter family of concentric balls $\{B_r(0) : r > 0\}$ is such that the family

$$\{B_r(0) \cap X\}_r, \tag{4.1}$$

is a complete family of isoperimetric sets, that is, for all $m \in (0, \mu(X))$ there exists $r = r(m) > 0$ such that $\mu(B_r(0) \cap X) = m$ and

$$\text{Per}_\mu(B_r(0) \cap X; X) \leq \text{Per}_\mu(\Omega; X), \quad \forall \Omega \subset X \text{ with } \mu(\Omega) = m, \quad (4.2)$$

that is, request (i) of [Theorem 3.1](#) holds with

$$q(m) = \text{Per}_\mu(B_{r(m)}(0) \cap X; X), \quad m \in (0, \mu(X)), \quad q(0) = 0,$$

provided q is continuous on $[0, \mu(X))$; its positivity on $(0, \mu(X))$ is automatic, as $\partial B_r(0) \cap X$ is a nonempty relatively open subset of the sphere $\partial B_r(0)$ and $e^W > 0$. In all the examples below, q is explicit and its continuity can be checked directly.

We now let

$$f(x) = \frac{1}{1 + \|x\|^2}, \quad \forall x \in X.$$

Trivially [\(H1\)](#) holds: $\inf f = 0$; $\sup f = 1$; and $\mathcal{L}^N(f^{-1}(0)) = \mathcal{L}^N(f^{-1}(1)) = 0$. Further, this function can be thought of as the composition $f = h(g)$, where

$$h(t) = \frac{1}{1+t} \quad \text{and} \quad g(x) = \|x\|^2.$$

Therefore, it is smooth and its gradient never vanishes since $\nabla f = h'(g)\nabla g$, with $h' < 0$ on $(0, +\infty)$ and $\nabla g \neq 0$ outside the origin. Hence, [\(H2\)](#) is met. Moreover, it is immediate that all superlevel sets of f are of the form in [\(4.1\)](#) so that: condition [\(H3\)](#) holds, having asked $\mu(B_r(0) \cap X) < +\infty$; condition [\(ii\)](#) holds, having assumed the one-parameter family $\{B_r(0) \cap X\}_r$ to be isoperimetric, and since the level sets $\{f = t\} = \partial B_r(0) \cap X$ are connected, as $N \geq 2$ and X is convex. Additionally, condition [\(iii\)](#) is met, since the lines of steepest ascent are rays pointing toward the origin and they sweep the whole interval $(0, 1)$, as f is radial and decreasing in $\|x\|$. In particular, f is represented via a composition with the distance from its focal set $F_1 = \{0\}$ as expected from [Theorem 3.5](#).

4.1.1. Standard Lebesgue measure. The standard Lebesgue measure falls under the above assumptions. It corresponds, up to a multiplicative prefactor, to the choices

$$X = \mathbb{R}^N \quad \text{and} \quad W(x) \equiv c \text{ with } c \in \mathbb{R}.$$

The classic isoperimetric inequality has been the object of extensive studies in the 20th century, eventually fully solved by De Giorgi, refer for instance to the book [\[34\]](#) for a proof of [\(4.2\)](#) in this setting.

The corresponding Faber–Krahn inequality had been proved independently by Faber [23] and Krahn [29], whereas the corresponding Saint-Venant inequality was proved by Pólya [37], both relying on the *Schwarz symmetrization* [25, 40]. Such a symmetrization is a special case of the one of Section 2.

4.1.2. *Radial log-convex measures.* A measure $\mu = e^W \mathcal{L}^N$ is said to be *log-convex* if the potential W is convex. A long-standing conjecture asserted that if W , besides being convex, is smooth and radial, then balls centered at the origin are isoperimetric sets, that is, (4.2) holds with $X = \mathbb{R}^N$. This has been positively settled in [13]. Choosing then $X = \mathbb{R}^N$ and $W(x)$ satisfying the above properties yields the corresponding Faber–Krahn and Saint-Venant inequalities. The former has been recently proved in [14, Thm. 1.1] and [36, Sect. 7.1].

4.1.3. *Convex cones and homogeneous densities.* Let $X \subset \mathbb{R}^N$ be an open and convex cone with vertex at the origin. If the potential W is constant, *i.e.*, the measure μ is the Lebesgue measure up to a prefactor, the relative isoperimetric inequality (4.2) was first proved in [31]. Thus, the Faber–Krahn and Saint-Venant inequalities follow in this case.

Relative weighted isoperimetric inequalities in open and convex cones with vertex at the origin—different from the whole space or a half-space—have also been proved by Cabré, Ros-Oton, and Serra, see [10] and [11, Thm. 1.3], under suitable assumptions on the density function $w = e^W$. Specifically, the density needs to be: continuous up to the boundary of X ; positively homogeneous of degree $\alpha \geq 0$; and such that $w^{1/\alpha}$ is concave, if $\alpha > 0$. Hence, the Faber–Krahn and Saint-Venant inequalities follow when the above assumptions are met.

In particular, we refer to [11, Sect. 2] for several choices of X and of w which fall within these hypotheses. We here only recall the case [11, Sect. 2(iv)], *i.e.*,

$$W(x) = \log \left(\prod_{i=1}^k x_i^{\alpha_i} \right),$$

with $\alpha_i > 0$ and $X = \{x \in \mathbb{R}^N : x_i > 0, \forall i = 1, \dots, k\}$, which leads to the density

$$w = e^W = \prod_{i=1}^k x_i^{\alpha_i}.$$

When $k = N$, such a density is usually called a *monomial weight*.

For monomial weights in the orthant, the isoperimetric inequality (4.2) was obtained simultaneously also by Brock, Chiacchio, and Mercaldo [7] (as the case $c = 0$ of the weights of Section 4.1.4). The theorem of [10, 11] is substantially more general for homogeneous weights in convex cones, but it does not

contain the full result of [7], which also covers the nonhomogeneous weights which we discuss in Section 4.1.4. We also mention that a Faber–Krahn inequality was first proved in the case $N = 2$ and $k = 1$, for the standard Laplacian ($p = 2$) in [33].

4.1.4. Monomial and anti-Gaussian weights. Consider the open orthant $X = \{x \in \mathbb{R}^N : x_i > 0, i = 1, \dots, N\}$ and the weights

$$w(x) = e^{W(x)} = e^{c\|x\|^2} \prod_{i=1}^N x_i^{k_i}, \quad c \geq 0, \quad k_1, \dots, k_N \geq 0, \quad (4.3)$$

so that

$$W(x) = c\|x\|^2 + \sum_{i=1}^N k_i \log x_i$$

is continuous on X . For $c = 0$, these are the monomial weights of Section 4.1.3; for $c > 0$, the factor $e^{c\|x\|^2}$ is anti-Gaussian, homogeneity and scaling are lost, and $\mu(X) = +\infty$, while $\mu(B_r(0) \cap X) < +\infty$ for all $r > 0$.

The sharp isoperimetric inequality (4.2) in this setting has been proved in [7]. We also refer to [6] for the case

$$w(x) = x_N^k e^{c\|x\|^2},$$

with $c, k \geq 0$, with X the half-space $\{x_N > 0\}$.

In all these cases, X is an open convex cone with vertex at the origin, μ is finite on bounded sets, and the intersections of X with the balls centered at the origin are isoperimetric. Hence, the framework of this section applies, and, for every $p \in (1, +\infty)$, Theorem 3.7, Corollaries 3.8 and 3.9, and Remark 3.10 hold, among open sets $\Omega \subset X$ with $\mu(\Omega) < +\infty$, the optimal sets being $B_r(0) \cap X$. As for the spectral consequences, the Faber–Krahn inequality for $p = 2$ in the half-plane with $c = 0$ was proved in [33], see Section 4.1.3, and the Saint-Venant inequality for $p = 2$ in the half-space case can be deduced from the comparison results of [6], choosing a constant datum. To the best of our knowledge, the other cases of the Faber–Krahn and Saint-Venant inequalities, as well as the Pólya–Szegő principle in $W_{\mu,0}^{1,p}(\Omega; X)$ and the inequalities for the (p, q) -eigenvalues, have not been stated before.

4.2. Isoperimetric sets are half-spaces. Suppose the ambient space X is a product of nonempty open intervals (a_i, b_i) with $a_i, b_i \in [-\infty, +\infty]$. This covers, for instance, open slabs, open half-spaces, or all of $\mathbb{R}^N$. Let μ be a finite measure on X so that (H3) is satisfied, independently of f . Given

a direction $\nu \in \mathbb{S}^{N-1}$ we consider the one-parameter family of half-spaces $\{H_r(\nu) : r \in \mathbb{R}\}$, where

$$H_r(\nu) = \{x \in \mathbb{R}^N : \langle x, \nu \rangle > r\}.$$

Assume there exists a direction θ such that the family $\{H_r(\theta) \cap X\}_r$ is such that for all $m \in (0, \mu(X))$, there exists $r = r(m)$ with $\mu(H_r(\theta) \cap X) = m$ such that

$$\text{Per}_\mu(H_r(\theta) \cap X; X) \leq \text{Per}_\mu(\Omega; X), \quad \forall \Omega \subset X \text{ with } \mu(\Omega) = m, \quad (4.4)$$

that is, request (i) of Theorem 3.1 holds with

$$q(m) = \text{Per}_\mu(H_{r(m)}(\theta) \cap X; X), \quad m \in (0, \mu(X)), \quad q(0) = 0,$$

provided q is continuous on $[0, \mu(X))$ and positive on $(0, \mu(X))$, which, again, can be checked directly in all the examples below, where q is explicit or the measure has a product structure. We further require that $\langle x, \theta \rangle$ takes all real values as x ranges in X ; this is automatic if $X = \mathbb{R}^N$, while, if $X = \mathbb{R}^{N-1} \times (a, b)$ is a slab or a half-space, it holds if and only if $\theta \neq \pm e_N$, and in particular if $\langle \theta, e_N \rangle = 0$.

Given $h : \mathbb{R} \rightarrow (0, 1)$ any smooth map with $h' > 0$ on $\mathbb{R}$, $\inf h = 0$, and $\sup h = 1$, we let

$$f_\theta(x) = h(\langle x, \theta \rangle), \quad \forall x \in X. \quad (4.5)$$

Trivially: f_θ is smooth; $\inf f_\theta = 0$; $\sup f_\theta = 1$ (owing to the assumption on θ); and $\nabla f_\theta \neq 0$, since we have $\nabla f_\theta = h'(\langle x, \theta \rangle)\theta$ with $h' > 0$ in $\mathbb{R}$. Therefore, assumptions (H1) and (H2) are met.

By construction, the superlevel sets of f_θ are precisely half-spaces orthogonal to θ , intersected with X :

$$\{x \in X : f_\theta(x) > t\} = \{x \in X : \langle x, \theta \rangle > h^{-1}(t)\} = H_{h^{-1}(t)}(\theta) \cap X.$$

This, combined with our hypothesis on $\{H_r(\theta) \cap X\}_r$, guarantees that condition (ii) is satisfied. The lines of steepest ascent are straight lines parallel to θ , oriented as θ , hence also condition (iii) holds. In each of the examples below, we specify the direction θ , which is the inner unit normal of the boundary hyperplane $\{\langle x, \theta \rangle = r\}$ of the minimizing half-spaces $H_r(\theta)$. Finally, X is convex, and the level sets of f_θ , being intersections of X with hyperplanes, are connected. Here, the unit normal $\nabla f_\theta / |\nabla f_\theta| = \theta$ is constant, so that no focal set appears, in accordance with Remark 3.6 (a).

4.2.1. *Standard Gaussian measure.* The Gaussian setting is the first instance that comes to mind where half-spaces are the isoperimetric sets. It corresponds, up to a multiplicative prefactor, to the choices

$$X = \mathbb{R}^N \quad \text{and} \quad W(x) = -\frac{\|x\|^2}{2}, \quad (4.6)$$

that is, the Euclidean space weighted by $e^{-\|x\|^2/2}$.

The celebrated Gaussian isoperimetric inequality, independently proved by Borell [2] and Sudakov–Cirel’son [41], guarantees the validity of (4.4) for any choice of θ .

The associated Faber–Krahn inequality has been proved in [22, Thm. 4.7], and extended in [1] to a broader class of elliptic operators associated with the Gaussian measure, whereas the Saint-Venant inequality is proved in [32, Prop. 5.6] for $p = 2$ (for general p it is mentioned in [15, Thm. 2.3]). Their proofs rely on the *Ehrhard symmetrization* [21]. In our framework, Ehrhard’s construction is exactly equivalent to the (μ, f_θ) -Talenti symmetrization on $X = \mathbb{R}^N$, provided f_θ is defined according to (4.5) for an arbitrary unit direction θ . For completeness, we mention that in [17, Cor. 1.5] (for $p = 2$) and [18, Cor. 1.4] (for general p), a Faber–Krahn inequality under a *mean width constraint* (rather than a measure constraint) has been proved in the Gaussian setting: balls centered at the origin are optimal. While seemingly surprising, note that the mean width is measure-independent (and that half-spaces have infinite mean width!).

4.2.2. *Gauss-like product measures.* Gauss-like product measures are perturbations of the Gaussian measure on product domains, which have been studied in [8]. The ambient space is

$$X = S' \times \mathbb{R}, \quad S' = (a_1, b_1) \times \cdots \times (a_{N-1}, b_{N-1}), \quad -\infty \leq a_i < b_i \leq +\infty,$$

and the measure is of type (1.1) with potential

$$W(x) = -\frac{\|x\|^2}{2} - \sum_{i=1}^{N-1} B_i(x_i),$$

where each $B_i \in C^2(a_i, b_i)$ is convex, so that μ is finite. Clearly, if $S' = \mathbb{R}^{N-1}$ and $B_i \equiv 0$ for all i , we recover the standard Gaussian measure.

In [8], it is proved that the coordinate half-spaces $H_\lambda(e_N) \cap X = S' \times (\lambda, +\infty)$, $\lambda \in \mathbb{R}$, are the unique isoperimetric sets of their measure, that is, (4.4) holds with $\theta = e_N$, the direction carrying the Gaussian factor. Actually, this is proved for a more general class of finite measures with continuous, positive, and factorized densities, see [8, Thm. 1.1], to which the following discussion

applies verbatim. In [8, Sect. 2], necessary conditions, of stationarity and stability type, on a density for the coordinate half-spaces to be isoperimetric are also given. Notice that $\langle x, e_N \rangle = x_N$ takes all real values on X .

Therefore, by employing the (μ, f_{e_N}) -Talenti symmetrization on X with respect to the function f_{e_N} from (4.5), we obtain the Faber–Krahn and Saint-Venant inequalities, the optimal sets being the coordinate half-spaces given by $S' \times (\lambda, +\infty)$.

For $p = 2$, a weighted Pólya–Szegő principle for the Dirichlet integral, which relies on Talenti’s result [42] has been derived in [8, Sect. 5]. Hence, for $p = 2$, the Faber–Krahn inequality and the inequalities for the $(2, q)$ -eigenvalues follow from [8], and so does the Saint-Venant inequality, choosing a constant datum in the comparison results. To the best of our knowledge, for $p \neq 2$ these inequalities have not been stated before.

We point out that, up to a linear change of variables, this class contains both the anisotropic Gaussian measures of Section 4.2.3 and the smooth log-concave one-dimensional perturbations of the Gaussian measure of Section 4.2.4.

4.2.3. Anisotropic Gaussian measures. The anisotropic Gaussian space is a modification of the standard Gaussian setting of Section 4.2.1. It still corresponds to choosing $X = \mathbb{R}^N$, but we now equip it with an anisotropic Gaussian measure, that is, in place of the usual Euclidean norm appearing in (4.6), we use an anisotropic one, namely,

$$W(x) = -\frac{\langle Ax, x \rangle}{2},$$

where A is a symmetric and positive-definite matrix. Clearly, if $A = \text{Id}_N$, we recover the isotropic setting discussed previously. This space can be interpreted as the Gaussian one where certain directions carry more weight than others; thus, one would expect isoperimetric sets to be half-spaces oriented along specific directions. This has been proved by Yeh [44, Thm. 1.1]: the isoperimetric inequality (4.4) holds, suitably choosing θ as any unit vector in the eigenspace corresponding to the smallest eigenvalue of A (equivalently, of $\sqrt{A}$). In other words, the minimizing half-spaces are the sets $H_r(\theta)$, whose boundary hyperplanes are orthogonal to a direction θ of largest variance of μ , see also [8].

Therefore, by employing the (μ, f_θ) -Talenti symmetrization on $X = \mathbb{R}^N$ with respect to the function f_θ from (4.5), we obtain the Faber–Krahn and Saint-Venant inequalities.

4.2.4. Concave 1d-perturbations of the Gaussian measure. A log-concave one-dimensional perturbation of the Gaussian measure on $\mathbb{R}^N$, up to an isometry,

is a measure of type (1.1) with potential

$$W(x) = \varphi(\pi_N(x)) - c\|x\|^2,$$

where φ is a smooth concave function, π_N denotes the projection onto the vertical axis, and c is a positive constant. Clearly, if φ is constant, choosing $c = 1/2$, we recover the standard Gaussian measure.

As established in [39, Thm. 5.10], the nature of the isoperimetric sets depends on φ :

- (i) if φ is an affine function, then isoperimetric sets are half-spaces $H_r(\nu)$ oriented along any arbitrary direction ν ;
- (ii) if φ is not affine, then isoperimetric sets are exclusively vertical half-spaces, meaning they are of the form $H_r(\theta)$ with $\langle \theta, e_N \rangle = 0$.

Here, the half-space $H_r(\theta)$ with $\langle \theta, e_N \rangle = 0$ is called *vertical* since its boundary hyperplane contains the vertical direction e_N , along which the perturbation φ acts; the lines of steepest ascent of f_θ are then horizontal, and $\langle x, \theta \rangle$ takes all real values on X , also when X is a slab or a half-space. In case (i), the measure is, up to a multiplicative constant, a translated Gaussian measure, and any direction θ can be chosen. Consequently, by applying the (μ, f_θ) -Talenti symmetrization on the space $X = \mathbb{R}^N$ with an appropriately chosen direction θ , we obtain the Faber–Krahn and Saint-Venant inequalities. We mention that one can drop the smoothness assumption on φ , with the caveat that then one needs to take as family of isoperimetric sets vertical half-spaces, *i.e.*, as in point (ii) above, see [39, Thm. 5.13].

This result can be extended to the cases where the ambient space is a slab or a half-space of the form $X = \mathbb{R}^{N-1} \times (a, b)$, equipped with the same log-concave perturbed Gaussian measure. In this framework, provided φ is concave (and possibly non-smooth) on (a, b) , the isoperimetric sets of any given volume are precisely the intersections of X with vertical half-spaces $H_r(\theta)$ (see [39, Thm. 5.12]). Therefore, we can apply the theory developed above in the space $X = \mathbb{R}^{N-1} \times (a, b)$ with f_θ described in (4.5) to establish that the Faber–Krahn and Saint-Venant inequalities also hold in this restricted setting. If $\varphi \in C^2$, the isoperimetric property of vertical half-spaces, both in $\mathbb{R}^N$ and in slabs or half-spaces, also follows from [8]. For completeness, we refer to [5, Thm. 3.1], where a Pólya–Szegő principle has been proved in this framework for the choices $X = \mathbb{R}^{N-1} \times (0, +\infty)$ and $\varphi(t) = k \log(t)$ with $k \geq 0$.

5. FUTURE DIRECTIONS

5.1. Kohler-Jobin inequality. In the Euclidean setting of Section 4.1.1, Pólya conjectured that the ball minimizes the first eigenvalue of the Dirichlet

p -Laplacian among sets of given p -torsional rigidity, that is,

$$\lambda_{1,p}(B) \leq \lambda_{1,p}(\Omega), \quad \forall \Omega \subset \mathbb{R}^N \text{ with } T_p(\Omega) = T_p(B). \quad (5.1)$$

This inequality is named after Kohler-Jobin, who first proved it for $p = 2$ in [27, 28], whereas for general p it was proved much later in [3]. It can be thought of as a strengthening of the Faber–Krahn inequality, as the measure constraint is weaker than the torsional one. Indeed, due to the rescaling properties of the first Dirichlet p -eigenvalue and of the p -torsional rigidity, one can relax the constraint in (5.1) to $T_p(\Omega) \leq T_p(B)$. Then, the Faber–Krahn inequality easily follows from the Saint-Venant and Kohler-Jobin inequalities.

The proofs of (5.1) given in [3, 27, 28] exploit a sophisticated rearrangement, called *Dirichlet transplantation*, which ports to the *anisotropic* setting [3, Sect. 6] but not the *weighted* one we are concerned with.

The only other setting where the weighted analog of (5.1) has been proved is the standard Gaussian one of Section 4.2.1. This has been recently settled in [26] for $p = 2$ with an argument substantially different from the Euclidean one. The inequality has then been extended to general p in [15], with a more general method—still tailored to the Gaussian space though—which lacks the ability to completely characterize cases of the equality.

The techniques used in the Euclidean and Gaussian settings are highly dependent on the underlying ambient and there is no clear all-rounder method. Nevertheless, it is natural to wonder whether one can still prove the weighted analog of (5.1) in each of the settings discussed in Section 4 with ad hoc methods. Of course, it would be desirable to come up with an abstract new route—à la Theorem 3.1—which applies to all settings presented in Section 4.

5.2. Boosted Pólya–Szegő principle. The abstract framework presented here naturally lends itself to stability questions. Suppose that, in place of the classical (relative) isoperimetric inequality required in Theorem 3.1 (i), one has a quantitative version, *i.e.*,

$$q(\mu(\Omega))(1 + c(\mu(\Omega))\alpha_\mu^\beta(\Omega)) \leq \text{Per}_\mu(\Omega; X),$$

for all Borel subsets $\Omega \subset X$ with $\mu(\Omega) < \mu(X)$, where $\alpha_\mu(\cdot)$ is the μ -Fraenkel asymmetry index (with respect to the measure μ and the corresponding isoperimetric sets), β a positive constant, and c a function, depending on, at most, $\mu(\Omega)$. Would one be able to prove a boosted Pólya–Szegő principle as in [4, Lem. 7.10], which deals with the Euclidean case? Then, following ideas from Hansen–Nadirashvili [24], one could derive a (non-sharp) quantitative Faber–Krahn inequality as in [4, Thm. 7.11].

Apart from the above-mentioned Euclidean case, this route has been successfully followed in the standard Gaussian case [12], so that we expect it to be portable to this abstract setting. This would lead to proving a quantitative Faber–Krahn inequality in the setting covered in Section 4.1.3, since a quantitative inequality is available, refer to [16].

Finally, it is worth noting that, in order to get the sharp quantitative inequality, in the Euclidean case one reduces first to prove a quantitative version of Saint-Venant inequality [4, Thm. 7.20], then exploits the relevant Kohler–Jobin inequality, which we briefly touched upon in Section 5.1.

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