

REGULARITY OF SOLUTIONS TO A FREE BOUNDARY p -LAPLACIAN PROBLEM WITH A SINGULAR TERM

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ABSTRACT. In this paper, we study the higher Hölder regularity of a solution to the following p -Laplacian minimization problem with singular term

$$(0.1) \quad \min \left\{ \int_{\Omega} w |\nabla u|^p + f u^{\gamma} : u \in W^{1,p}(\Omega), u \geq 0, u = \phi \text{ on } \partial\Omega \right\}$$

where $0 < \gamma < 1$, $1 < p < \infty$, f is a positive function in $L^q(\Omega)$ with $q > \max\{\frac{p}{p-\gamma}, \frac{N}{p}\}$, ϕ is a nonnegative functions in $W^{1,p}(\Omega)$, and $w \in C^2(\Omega)$ with $w \geq c_w > 0$. More precisely, we will show the following statements:

$$\begin{cases} f \in L^q(\Omega) \Rightarrow u \in C_{loc}^{0, \frac{p-\frac{N}{q}}{p-\gamma}}(\Omega), & q \leq \frac{N}{\gamma}, \\ f \in L^q(\Omega) \Rightarrow u \in C_{loc}^{1, \min\left\{\alpha_*, \frac{\gamma-\frac{N}{q}}{p-\gamma}\right\}}(\Omega), & q > \frac{N}{\gamma}, \quad p \geq 2, \\ f \in L^q(\Omega) \Rightarrow u \in C_{loc}^{1, \min\left\{\alpha_*, \frac{\gamma-\frac{N}{q}}{2-\gamma}\right\}}(\Omega), & q > \frac{N}{\gamma}, \quad 1 < p < 2. \end{cases}$$

Here, we denotes by α_* a constant such that any solution to the weighted p -Laplacian equation $\nabla \cdot [w |\nabla u|^{p-2} \nabla u] = 0$ is locally of class C^{1, α_*} .

If $q > \frac{N}{\gamma}$, we will show that u has more regularity on the free boundary $\partial\{u > 0\}$, it is of class $C^{1, \alpha}$ with $\alpha = \min\left\{\frac{\gamma-\frac{N}{q}}{p-\gamma}, 1\right\}$.

1. INTRODUCTION

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$. Let w and f be two positive functions on Ω . Fix $0 < \gamma < 1$, then we consider the problem of minimizing the energy functional

$$u \mapsto \int_{\Omega} (w |\nabla u|^p + f u^{\gamma}) dx, \quad (1 < p < \infty)$$

among all nonnegative functions $u \in W^{1,p}(\Omega)$ such that $u - \phi \in W_0^{1,p}(\Omega)$, where ϕ is a given fixed nonnegative function in $W^{1,p}(\Omega)$. Assume u is a continuous minimizer, then u solves the following singular p -Laplacian equation:

$$\Delta_{p,w} u = \lambda f u^{\gamma-1} \text{ in } \{u > 0\}$$

with $\lambda = \frac{\gamma}{p}$. Here $\Delta_{p,w} u = \nabla \cdot [w |\nabla u|^{p-2} \nabla u]$ is the weighted p -Laplace operator. The limit problem $\gamma = 0$ (with uniform weight $w = 1$) has been considered in several papers [1, 2, 7, 11, 12]. For the linear case $p = 2$, the problem has been studied in the pioneer papers [1, 2] of Alt and Caffarelli. Then, the nonlinear case $p \neq 2$ was studied in [7]. In all these works, the nonhomogeneous term f was assumed to be constant. One of the main goals in these papers was to study the regularity of minimizers. More precisely, they showed that any minimizer u is locally Lipschitz continuous in Ω . In [11], we investigated the regularity of a minimizer u in the more general setting where the nonhomogeneous term f belongs

only to $L^q(\Omega)$ and, the weight w is smooth and bounded away from zero. Assume $q > \frac{N}{p}$, then we proved the following statement:

$$f \in L^q(\Omega) \Rightarrow u \in C_{loc}^{0,1-\frac{N}{pq}}(\Omega).$$

The case when $\gamma = 1$ has been also studied in several papers. In [6, 16], the authors showed under various assumptions on the regularity of w and the summability of f that the solution u is locally of class $C^{1,\alpha}$. While in [3], the authors proved that on the free boundary $\partial\{u > 0\}$, the minimizer u is locally $C^{1,\min\{\frac{1}{p-1},1\}}$. Moreover, this result turns to be optimal if $p > 2$. Under weaker assumptions on w and f (i.e. w needs to be only σ -Hölder and f belongs to $L^q(\Omega)$ but not necessarily bounded), we proved in [13], by an approximation argument and thanks to the regularity results in [5, 8, 9, 10], that the minimizer u is locally of class $C^{1,\alpha}$ with

$$\alpha = \min \left\{ \alpha_*, \min \left\{ \sigma, 1 - \frac{N}{q} \right\} \cdot \min \left\{ \frac{1}{p-1}, 1 \right\} \right\}$$

where α_* is any positive number such that any p -harmonic function (i.e. solution to the p -Laplace equation $-\Delta_p u = 0$) is locally of class C^{1,α_*} .

In [19], the regularity of minimizers to Problem (0.1) was studied under the assumptions that $p = 2$, w and f are constant, and $0 < \gamma < 1$. More precisely, it was shown that $u \in C_{loc}^{1,\frac{\gamma}{2-\gamma}}(\Omega)$. While in [15], the authors studied the regularity of minimizers for a two-phase free boundary problem involving the p -Laplacian. More precisely, they considered the following minimization problem:

$$(1.1) \quad \min \left\{ \int_{\Omega} |\nabla u|^p + \lambda_+[u^+]^\gamma + \lambda_-[u^-]^\gamma + f u : u \in W^{1,p}(\Omega), u = \phi \text{ on } \partial\Omega \right\}$$

where $0 \leq \lambda_- < \lambda_+ < \infty$, $u^\pm = \max\{\pm u, 0\}$, and $f \in L^q(\Omega)$ with $q > N$. Under the assumption that $p \geq 2$, they showed that if u is a minimizer for Problem (1.1) then u must be locally of class $C^{1,\alpha}$ in Ω with

$$\alpha = \min \left\{ \alpha_*, \frac{\gamma}{p-\gamma}, \frac{1-\frac{N}{q}}{p-1} \right\}.$$

We also note that in [4], the authors extended the results of [15] to the case of a nonzero obstacle φ (i.e. $u \geq \varphi$ in Ω but $\varphi \neq 0$). In [17, 18], the authors also studied the variational problem (0.1) under more general Orlicz growth conditions on the energy functional. The key idea in all these works is based on comparison with harmonic replacements, a technique originally introduced by Alt and Caffarelli in [1, 2] (see also [7]).

In this paper, we will study the Hölder regularity of minimizers to Problem (0.1) in the case where $f \in L^q(\Omega)$. More precisely, we will see how the Hölder regularity of these minimizers depends on the summability of f as well as the parameter γ . To the best of our knowledge, this generalization has not been addressed in the literature. We note that our approach is again based on comparing minimizers of (0.1) with their p -harmonic replacements. If $f \in L^q(\Omega)$ with $\max\{\frac{p}{p-\gamma}, \frac{N}{p}\} < q \leq \frac{N}{\gamma}$, then we will show that any minimizer u belongs to $C_{loc}^{0,\alpha}(\Omega)$ with $\alpha = \frac{p-\frac{N}{q}}{p-\gamma}$. In addition, u is locally of class $C^{1,\alpha}$ in Ω as soon as $q > \frac{N}{\gamma}$, with $0 < \alpha \leq \min \left\{ \frac{\gamma-\frac{N}{q}}{\max\{p,2\}-\gamma}, 1 \right\}$. Notice that this regularity result in the case $1 < p < 2$ is also new even when f is bounded, since it has not been addressed in the previous works [15, 4]. Moreover, the proof presented here seems to be much simpler than the one given in [15, 4]. Finally, we will show that any minimizer u is more regular near the free boundary than away from it, u is locally of class $C^{1,\beta}$ on $\partial\{u > 0\}$ with $\beta = \min \left\{ \frac{\gamma-\frac{N}{q}}{\max\{p,2\}-\gamma}, 1 \right\}$.

2. EXISTENCE AND REGULARITY OF MINIMIZERS

Throughout this section, we assume that w is a positive bounded C^2 function in Ω , and that f is a positive function in $L^q(\Omega)$ with $q \geq \frac{p}{p-\gamma}$. Assume $\phi \in W^{1,p}(\Omega)$ and ϕ is nonnegative. Then, we consider the following minimization problem:

$$(2.1) \quad \min \left\{ \int_{\Omega} w |\nabla u|^p + f |u|^\gamma : u \in W^{1,p}(\Omega), u - \phi \in W_0^{1,p}(\Omega) \right\}.$$

The existence of a minimizer for Problem (2.1) follows immediately from the direct method in the calculus of variations. However, we provide the proof here for the sake of completeness.

Proposition 2.1. *Assume $w \geq c_w > 0$. Then, there exists a nonnegative function u that minimizes Problem (2.1).*

Proof. Let $\{u_n\}_n$ be a minimizing sequence in Problem (2.1). For every $n \in \mathbb{N}$, we may assume that $u_n \geq 0$, since otherwise we replace u_n by its positive part $u_n^+ = \max\{u_n, 0\}$ which belongs to $W^{1,p}(\Omega)$ with $u_n^+ = \phi$ on $\partial\Omega$ and, it is also clear that

$$\int_{\Omega} w |\nabla u_n^+|^p + f [u_n^+]^\gamma \leq \int_{\Omega} w |\nabla u_n|^p + f |u_n|^\gamma.$$

Let $M < \infty$ be a uniform (that is independent of n) constant such that for all $n \in \mathbb{N}$, we have

$$\int_{\Omega} w |\nabla u_n|^p + f u_n^\gamma \leq M.$$

But, $w \geq c_w > 0$ and f is nonnegative. Hence,

$$\int_{\Omega} |\nabla u_n|^p \leq \frac{M}{c_w}.$$

Since $u_n - \phi \in W_0^{1,p}(\Omega)$, the Poincaré inequality yields that the sequence $\{u_n\}$ is bounded in $W^{1,p}(\Omega)$. Then, there is a function $u \in W^{1,p}(\Omega)$ such that up to a subsequence, $u_n \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$. In particular, one has $u = \phi$ on $\partial\Omega$ and u is nonnegative. Due to the weak lower semicontinuity of the L^p -norm, we have

$$(2.2) \quad \int_{\Omega} w |\nabla u|^p \leq \liminf_n \int_{\Omega} w |\nabla u_n|^p.$$

On the other hand, $u_n \rightarrow u$ in $L^p(\Omega)$ and so, $u_n(x) \rightarrow u(x)$ at almost everywhere point $x \in \Omega$. By Fatou's Lemma, we get

$$(2.3) \quad \int_{\Omega} f u^\gamma = \int_{\Omega} \liminf_n f u_n^\gamma \leq \liminf_n \int_{\Omega} f u_n^\gamma.$$

By (2.2) & (2.3), we infer that

$$\int_{\Omega} w |\nabla u|^p + \int_{\Omega} f u^\gamma \leq \liminf_n \left[\int_{\Omega} w |\nabla u_n|^p + \int_{\Omega} f u_n^\gamma \right].$$

Hence, u minimizes Problem (0.1). $\square$

For any ball $B \subset \Omega$, we will denote by $v = v_B$ the unique solution to the following p -Laplacian problem:

$$\begin{cases} \nabla \cdot [w |\nabla v|^{p-2} \nabla v] = 0 & \text{in } B \\ v = u & \text{on } \partial B \end{cases}$$

or equivalently, v is the unique minimizer to (we will call this function v the weighted p -harmonic replacement of u in B)

$$(2.4) \quad \min \left\{ \int_B w |\nabla v|^p : v \in W^{1,p}(B), v = u \text{ on } \partial B \right\}.$$

Clearly, v is nonnegative. In the next proposition, we show that u is weighted p -subharmonic in Ω . This result extends the one in [19, Theorem 1.2] to the case $p \neq 2$. We also note that the proof used there is different than the one we present here.

Proposition 2.2. *Let u be a minimizer for Problem (0.1) and v be the weighted p -harmonic replacement of u on a ball $B \subset \Omega$. Then, $u \leq v$ on B . In particular, u is locally bounded in Ω . Moreover, if u is upper semicontinuous then u is weighted p -subharmonic in the weak sense, that is for all $\varphi \in C_0^\infty(\Omega)$ with $\varphi \geq 0$, we have*

$$\int_\Omega w |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \leq 0.$$

Proof. Define $h = \min\{u, v\}$. We clearly have $h \in W^{1,p}(B)$ and $h = u$ on ∂B . First of all, we claim that

$$(2.5) \quad \int_B w |\nabla h|^p \leq \int_B w |\nabla u|^p.$$

Assume it is not the case. Then, we define $g = \max\{u, v\}$. Again, $g \in W^{1,p}(B)$ and $g = u$ on ∂B . Moreover, one has

$$(2.6) \quad \int_B w |\nabla g|^p = \int_{B \cap \{u > v\}} w |\nabla u|^p + \int_{B \cap \{u \leq v\}} w |\nabla v|^p.$$

On the other side, we have

$$\int_{B \cap \{u > v\}} w |\nabla u|^p + \int_{B \cap \{u \leq v\}} w |\nabla u|^p = \int_B w |\nabla u|^p < \int_B w |\nabla h|^p = \int_{B \cap \{u \leq v\}} w |\nabla u|^p + \int_{B \cap \{u > v\}} w |\nabla v|^p.$$

Hence,

$$\int_{B \cap \{u > v\}} w |\nabla u|^p < \int_{B \cap \{u > v\}} w |\nabla v|^p.$$

Recalling (2.6), we get that

$$(2.7) \quad \int_B w |\nabla g|^p < \int_B w |\nabla v|^p.$$

However, this equation (2.7) contradicts the minimality of v in Problem (2.4). Thus, the claim (2.5) holds. Since u minimizes Problem (0.1), then we also have

$$\int_\Omega w |\nabla u|^p + \int_\Omega f u^\gamma \leq \int_\Omega w |\nabla h|^p + \int_\Omega f h^\gamma \leq \int_\Omega w |\nabla u|^p + \int_\Omega f h^\gamma.$$

Hence,

$$\int_\Omega f [u^\gamma - h^\gamma] \leq 0.$$

Since $u \geq h$ and f is positive, we conclude that $u = h$ and so, $0 \leq u \leq v$. Thanks to [14, Theorem 3.41], we infer that u is locally bounded. Now, assume u is upper semi-continuous. Let us show that u is weighted p -subharmonic in Ω in the sense of [21, Definition 5.1]. For each domain $D \subset\subset \Omega$, we already proved that if v is the weighted p -harmonic replacement of u in D , then we must have $u \leq v$ in D . Consequently, one has the following comparison principle: if h is weighted p -harmonic in D with $u \leq h$ on ∂D , then $u \leq v \leq h$ in D . Finally, [21, Theorem 5.2] implies that u is weighted p -subharmonic in the weak sense. $\square$

Lemma 2.3. *Let u be a minimizer of Problem (0.1) and v be the weighted p -harmonic replacement of u in $B_r \subset \subset \Omega$. Assume $f \in L^q(\Omega)$ with $q > \frac{p}{p-\gamma}$, then there is a constant C depending on p, q, γ, N and, $\|f\|_{L^q(B_r)}$ such that*

$$\begin{cases} \|\nabla u - \nabla v\|_{L^p(B_r)} \leq C r^{\frac{1}{p-\gamma}[(1-\frac{N}{p})\gamma + \frac{N}{q}]} & p \geq 2, \\ \|\nabla u - \nabla v\|_{L^p(B_r)} \leq C \left(|B_r| + \int_{B_r} |\nabla u|^p \right)^{\frac{2-p}{p(2-\gamma)}} r^{\frac{1}{2-\gamma}[(1-\frac{N}{p})\gamma + \frac{N}{q}]} & 1 < p < 2, \end{cases}$$

where $q' = \frac{q}{q-1}$.

Proof. Let $\tilde{v}$ be the extension of v by u outside the ball B_r . Then, it is clear that $\tilde{v} \in W^{1,p}(\Omega)$ with $\tilde{v} = u$ on $\partial\Omega$. Therefore, we have

$$\int_{\Omega} w |\nabla u|^p + \int_{\Omega} f u^{\gamma} \leq \int_{\Omega} w |\nabla \tilde{v}|^p + \int_{\Omega} f \tilde{v}^{\gamma}.$$

Recalling the definition of $\tilde{v}$, one has

$$\int_{B_r} w |\nabla u|^p + \int_{\Omega \setminus B_r} w |\nabla u|^p + \int_{B_r} f u^{\gamma} + \int_{\Omega \setminus B_r} f u^{\gamma} \leq \int_{B_r} w |\nabla v|^p + \int_{\Omega \setminus B_r} w |\nabla u|^p + \int_{B_r} f v^{\gamma} + \int_{\Omega \setminus B_r} f u^{\gamma}.$$

Then,

$$\int_{B_r} w |\nabla u|^p + \int_{B_r} f u^{\gamma} \leq \int_{B_r} w |\nabla v|^p + \int_{B_r} f v^{\gamma}.$$

So, we get

$$(2.8) \quad \int_{B_r} w [|\nabla u|^p - |\nabla v|^p] \leq \int_{B_r} f v^{\gamma} - \int_{B_r} f u^{\gamma} = \int_{B_r} f [v^{\gamma} - u^{\gamma}].$$

Thanks to the γ -triangle inequality $(a+b)^{\gamma} \leq a^{\gamma} + b^{\gamma}$ for all $a, b \geq 0$ and the fact that $u \leq v$ on B_r by Proposition 2.2, then

$$\int_{B_r} f [v^{\gamma} - u^{\gamma}] \leq \int_{B_r} f [v - u]^{\gamma}.$$

Recalling (2.8), we infer that

$$(2.9) \quad \int_{B_r} w [|\nabla u|^p - |\nabla v|^p] \leq \int_{B_r} f [v - u]^{\gamma}.$$

Now, set

$$u_t = (1-t)v + tu, \quad \text{for every } 0 \leq t \leq 1.$$

We have

$$\int_{B_r} w [|\nabla u|^p - |\nabla v|^p] = \int_{B_r} w \left[\int_0^1 \frac{d}{dt} |\nabla u_t|^p \right] = p \int_0^1 \int_{B_r} w |\nabla u_t|^{p-2} \nabla u_t \cdot [\nabla u - \nabla v].$$

It is clear that $u_t = u = v$ on ∂B_r , for all $t \in [0, 1]$. Yet, v is p -harmonic in B_r . Then, we clearly have the following:

$$\int_{B_r} w |\nabla v|^{p-2} \nabla v \cdot [\nabla u_t - \nabla v] = 0.$$

Hence,

$$\begin{aligned} \int_{B_r} w [|\nabla u|^p - |\nabla v|^p] &= p \int_0^1 t^{-1} \int_{B_r} w |\nabla u_t|^{p-2} \nabla u_t \cdot [\nabla u_t - \nabla v] \\ &= p \int_0^1 t^{-1} \int_{B_r} w [|\nabla u_t|^{p-2} \nabla u_t - |\nabla v|^{p-2} \nabla v] \cdot [\nabla u_t - \nabla v]. \end{aligned}$$

Thanks to [21, Chapter 12], we have the following inequalities:

$$\begin{cases} |a-b|^2(1+|a|^2+|b|^2)^{\frac{p-2}{2}} \leq \frac{1}{p-1} [|a|^{p-2}a - |b|^{p-2}b] \cdot [a-b] & 1 < p < 2, \\ 2^{2-p}|a-b|^p \leq [|a|^{p-2}a - |b|^{p-2}b] \cdot [a-b] & p \geq 2. \end{cases}$$

Hence,

$$\begin{cases} \int_{B_r} w[|\nabla u|^p - |\nabla v|^p] \geq p(p-1) \int_0^1 t^{-1} \int_{B_r} w|\nabla u_t - \nabla v|^2(1+|\nabla u_t|^2+|\nabla v|^2)^{\frac{p-2}{2}}, & 1 < p < 2, \\ \int_{B_r} w[|\nabla u|^p - |\nabla v|^p] \geq p2^{2-p} \int_0^1 t^{-1} \int_{B_r} w|\nabla u_t - \nabla v|^p, & p \geq 2. \end{cases}$$

If $p \geq 2$, then

$$\int_{B_r} w[|\nabla u|^p - |\nabla v|^p] \geq p2^{2-p} \int_0^1 t^{p-1} \int_{B_r} w|\nabla u - \nabla v|^p = 2^{2-p} \int_{B_r} w|\nabla u - \nabla v|^p.$$

Now, (2.9) yields that

$$\begin{aligned} \int_{B_r} w|\nabla u - \nabla v|^p &\leq 2^{p-2} \int_{B_r} w[|\nabla u|^p - |\nabla v|^p] \leq 2^{p-2} \int_{B_r} f[v-u]^\gamma \\ (2.10) \quad &\leq 2^{p-2} \|f\|_{L^q(B_r)} \left(\int_{B_r} [v-u]^{\gamma q'} \right)^{\frac{1}{q'}}. \end{aligned}$$

Thanks to the Hölder inequality as well as the Poincaré inequality, we clearly have the following estimate:

$$(2.11) \quad \left(\int_{B_r} [v-u]^{\gamma q'} \right)^{\frac{1}{q'}} \leq \|v-u\|_{L^p(B_r)}^\gamma |B_r|^{\frac{p-\gamma q'}{q'p}} \leq C \|\nabla v - \nabla u\|_{L^p(B_r)}^\gamma r^{\gamma + \frac{N(p-\gamma q')}{q'p}},$$

where C is a constant that depends only on N , p , q , and γ ; we note that in the last inequality we have used the fact that the Poincaré constant on B_r scales linearly with r . Recalling (2.10), we get

$$(2.12) \quad \|\nabla v - \nabla u\|_{L^p(B_r)} \leq \frac{C}{c_w} \|f\|_{L^q(B_r)}^{\frac{1}{p-\gamma}} r^{\frac{1}{p-\gamma}[(1-\frac{N}{p})\gamma + \frac{N}{q}]}.$$

Assume $1 < p < 2$. Then, one has

$$\int_{B_r} w[|\nabla u|^p - |\nabla v|^p] \geq C(p) \int_{B_r} w|\nabla u - \nabla v|^2(1+|\nabla u|^2+|\nabla v|^2)^{\frac{p-2}{2}}.$$

Combining (2.9) & (2.11), we have

$$\begin{aligned} \int_{B_r} |\nabla u - \nabla v|^2(1+|\nabla u|^2+|\nabla v|^2)^{\frac{p-2}{2}} &\leq \frac{C}{c_w} \int_{B_r} f[v-u]^\gamma \leq C \|f\|_{L^q(B_r)} \left(\int_{B_r} [v-u]^{\gamma q'} \right)^{\frac{1}{q'}} \\ (2.13) \quad &\leq C \|f\|_{L^q(B_r)} \|\nabla v - \nabla u\|_{L^p(B_r)}^\gamma r^{\gamma + \frac{N(p-\gamma q')}{q'p}}. \end{aligned}$$

On the other side, using Hölder inequality, one has

$$\int_{B_r} |\nabla u - \nabla v|^p \leq \left(\int_{B_r} |\nabla u - \nabla v|^2(1+|\nabla u|^2+|\nabla v|^2)^{\frac{p-2}{2}} \right)^{\frac{p}{2}} \left(\int_{B_r} (1+|\nabla u|^2+|\nabla v|^2)^{\frac{p}{2}} \right)^{1-\frac{p}{2}}.$$

Then, by (2.13), we get

$$\begin{aligned} \int_{B_r} |\nabla u - \nabla v|^p &\leq \left[C \|f\|_{L^q(B_r)} \|\nabla v - \nabla u\|_{L^p(B_r)}^\gamma r^{\gamma + \frac{N(p-\gamma q')}{q'p}} \right]^{\frac{p}{2}} \left(\int_{B_r} (1+|\nabla u|^2+|\nabla v|^2)^{\frac{p}{2}} \right)^{1-\frac{p}{2}} \\ &\leq C \left(\int_{B_r} 1+|\nabla u|^p+|\nabla v|^p \right)^{1-\frac{p}{2}} \|\nabla v - \nabla u\|_{L^p(B_r)}^{\frac{p\gamma}{2}} r^{\frac{p\gamma}{2} + \frac{N(p-\gamma q')}{2q'}} \end{aligned}$$

$$\leq C \left(|B_r| + \int_{B_r} |\nabla u|^p \right)^{1-\frac{p}{2}} \|\nabla v - \nabla u\|_{L^p(B_r)}^{\frac{p\gamma}{2}} r^{\frac{p\gamma}{2} + \frac{N(p-\gamma q')}{2q'}},$$

where the last inequality follows from the fact that v minimizes (2.4) (so, $\int_{B_r} w |\nabla v|^p \leq \int_{B_r} w |\nabla u|^p$) and $w \geq c_w > 0$. Consequently,

$$\|\nabla u - \nabla v\|_{L^p(B_r)} \leq C \left(|B_r| + \int_{B_r} |\nabla u|^p \right)^{\frac{2-p}{p(2-\gamma)}} r^{\frac{1}{2-\gamma}[(1-\frac{N}{p})\gamma + \frac{N}{q'}]}. \quad \square$$

Now, we are ready to prove α -Hölder regularity (with $0 < \alpha \leq 1$) of minimizers to Problem (0.1).

Proposition 2.4. *Assume $w \in C^2(\Omega)$, $w \geq c_w > 0$ and, $f \in L^q(\Omega)$ with $q > \frac{p}{p-\gamma}$. Let u be a minimizer of Problem (0.1). If $q > \frac{N}{p}$, then we have the following statement:*

$$f \in L^q(\Omega) \Rightarrow u \in C_{loc}^{0,\alpha}(\Omega)$$

with

$$\alpha := \min \left\{ \frac{p - \frac{N}{q}}{p - \gamma}, 1 \right\}.$$

More precisely, for any open subset $\Omega' \subset \subset \Omega$ and a compact set $K \subset \Omega'$, there exists a constant C depending on $p, q, \gamma, N, c_w, \|f\|_{L^q(\Omega')}, \|w\|_{L^\infty(\Omega')}, \|\nabla w\|_{L^\infty(\Omega')}, \|D^2 w\|_{L^\infty(\Omega')}, \|u\|_{L^\infty(\Omega')}$ and, $\text{dist}(K, \partial\Omega')$ such that

$$\|u\|_{C^{0,\alpha}(K)} \leq C.$$

Proof. Fix $x \in K$ and $0 < 2r < R < \text{dist}(K, \partial\Omega')$ (so, we have $B(x, r) \subset B(x, \frac{R}{2}) \subset B(x, R) \subset \Omega'$). Let v be the weighted p -harmonic replacement of u in $B_R = B(x, R)$. Thanks to [22, Theorem 1.1], we have

$$(2.14) \quad \|\nabla v\|_{L^p(B_r)} \leq |B_r|^{\frac{1}{p}} \|\nabla v\|_{L^\infty(B_r)} \leq C(p, N) [R^{-1} + \sqrt{\kappa}] \|v\|_{L^\infty(B_{\frac{R}{2}})} r^{\frac{N}{p}},$$

where κ is a nonnegative constant such that the Ricci tensor $\text{Ric}(w)$ of the Riemannian conformal metric $(\mathbb{R}^N, w)$ satisfies $\text{Ric}(w) \geq -(N-1)\kappa$. From [20], we note that

$$\text{Ric}(w) = -\frac{N-2}{2} \left[\frac{D^2 w}{w} - \frac{3}{2} \frac{\nabla w \otimes \nabla w}{w^2} \right] - \frac{1}{2} \left[\frac{\Delta w}{w} - \frac{(N-4)}{2} \frac{|\nabla w|^2}{w^2} \right].$$

Since $u = v$ on ∂B_R , then thanks to the comparison principle [14, Lemma 3.18], we have the following estimate:

$$\|v\|_{L^\infty(B_{\frac{R}{2}})} \leq \|v\|_{L^\infty(\partial B_R)} \leq \|u\|_{L^\infty(B_R)}.$$

Recalling (2.14), this yields that

$$(2.15) \quad \|\nabla v\|_{L^p(B_r)} \leq C [R^{-1} + \sqrt{\kappa}] \|u\|_{L^\infty(B_R)} r^{\frac{N}{p}} \leq C R^{-1} r^{\frac{N}{p}}.$$

Assume $p \geq 2$. Recalling Lemma 2.3, one has

$$\|\nabla v - \nabla u\|_{L^p(B_r)} \leq C r^{\frac{1}{p-\gamma}[(1-\frac{N}{p})\gamma + \frac{N}{q'}]}.$$

Hence,

$$\|\nabla u\|_{L^p(B_r)} \leq \|\nabla u - \nabla v\|_{L^p(B_r)} + \|\nabla v\|_{L^p(B_r)} \leq C r^{\frac{1}{p-\gamma}[(1-\frac{N}{p})\gamma + \frac{N}{q'}]} + C R^{-1} r^{\frac{N}{p}}.$$

In particular, one has

$$\int_{B_r} |\nabla u| \leq \|\nabla u\|_{L^p(B_r)} |B_r|^{1-\frac{1}{p}} \leq C r^{\frac{1}{p-\gamma}[(1-\frac{N}{p})\gamma + \frac{N}{q'}] + N(1-\frac{1}{p})} + C R^{-1} r^N \leq C r^{N-1+\alpha}$$

with

$$\alpha = \alpha(N, p, q, \gamma) = \min \left\{ \frac{p - \frac{N}{q}}{p - \gamma}, 1 \right\} > 0.$$

Thanks to the Morrey estimate (see [16, Theorem 1.53]), we conclude that $u \in C^{0,\alpha}(K)$. Moreover, we have

$$\|u\|_{C^{0,\alpha}(K)} \leq C(p, q, \gamma, N, c_w, \|f\|_{L^q(\Omega')}, \|w\|_{C^2(\Omega')}, \|u\|_{L^\infty(\Omega')}, \text{dist}(K, \partial\Omega')).$$

Finally, the case when $1 < p < 2$ is similar. In fact, thanks again to Lemma 2.3, we have the following estimate:

$$\begin{aligned} \|\nabla u - \nabla v\|_{L^p(B_r)} &\leq C \left(|B_r| + \int_{B_r} |\nabla u|^p \right)^{\frac{2-p}{p(2-\gamma)}} r^{\frac{1}{2-\gamma}[(1-\frac{N}{p})\gamma + \frac{N}{q}]} \\ &\leq C \left[r^{\frac{N(2-p)}{p(2-\gamma)}} + \left(\int_{B_r} |\nabla u|^p \right)^{\frac{2-p}{p(2-\gamma)}} \right] r^{\frac{1}{2-\gamma}[(1-\frac{N}{p})\gamma + \frac{N}{q}]} \end{aligned}$$

Therefore, by (2.15), we get

$$\begin{aligned} \|\nabla u\|_{L^p(B_r)} &\leq C \left[r^{\frac{N(2-p)}{p(2-\gamma)}} + \left(\int_{B_r} |\nabla u|^p \right)^{\frac{2-p}{p(2-\gamma)}} \right] r^{\frac{1}{2-\gamma}[(1-\frac{N}{p})\gamma + \frac{N}{q}]} + CR^{-1} r^{\frac{N}{p}} \\ &\leq C \left[r^{\frac{N(2-p)}{p(2-\gamma)} + \frac{1}{2-\gamma}[(1-\frac{N}{p})\gamma + \frac{N}{q}]} + r^{\frac{1}{2-\gamma}[(1-\frac{N}{p})\gamma + \frac{N}{q}]} \|\nabla u\|_{L^p(B_r)}^{\frac{2-p}{2-\gamma}} + r^{\frac{N}{p}} \right]. \end{aligned}$$

Using the ε -Young inequality, we infer that

$$(2.16) \quad \|\nabla u\|_{L^p(B_r)} \leq C \left[r^{\frac{N(2-p)}{p(2-\gamma)} + \frac{1}{2-\gamma}[(1-\frac{N}{p})\gamma + \frac{N}{q}]} + r^{\frac{1}{2-\gamma}[(1-\frac{N}{p})\gamma + \frac{N}{q}]} + r^{\frac{N}{p}} \right].$$

Thanks again to the Morrey estimate, it is not difficult to see that this estimate (2.16) yields that when $1 < p < 2$, any minimizer u of Problem (0.1) is locally α -Hölder continuous in Ω with

$$\alpha = \min \left\{ \frac{p - \frac{N}{q}}{p - \gamma}, 1 \right\}. \quad \square$$

From Proposition 2.4, it follows that if $q \geq \frac{N}{\gamma}$, then any minimizer u of Problem (0.1) is locally Lipschitz continuous in Ω . This naturally leads to the question of whether the regularity of minimizers can be further improved when $q > \frac{N}{\gamma}$, namely whether one can prove local Hölder continuity estimates for the gradient of u , and consequently obtain

$$u \in C_{loc}^{1,\beta}(\Omega).$$

The case $f \in L^\infty(\Omega)$ with $p \geq 2$ was previously considered in [15, 4]. In the next proposition, we provide a simpler proof and we also extend the result to the general case $p > 1$ and $f \in L^q(\Omega)$. More precisely, we have the following regularity result:

Proposition 2.5. *Assume $f \in L^q(\Omega)$ with $q > \frac{N}{\gamma}$. Then, any minimizer u of Problem (0.1) is locally of class $C^{1,\beta_\star}$ with $\beta_\star := \min\{\alpha_\star, \beta\}$ and,*

$$(2.17) \quad \beta = \begin{cases} \frac{\gamma - \frac{N}{q}}{p - \gamma} & p \geq 2, \\ \frac{\gamma - \frac{N}{q}}{2 - \gamma} & 1 < p < 2. \end{cases}$$

We recall that here $\alpha_\star \in (0, 1]$ is such that any p -harmonic function is locally of class $C^{1,\alpha_\star}$. In addition, for any compact set K contained in an open set $\Omega' \subset \subset \Omega$, there exists a constant $C = C(p, q, \gamma, N, c_w, \|f\|_{L^q(\Omega')}, \|w\|_{C^2(\Omega')}, \|u\|_{L^\infty(\Omega')}, \text{dist}(K, \partial\Omega'))$ such that

$$\|\nabla u\|_{C^{0,\beta_\star}(K)} \leq C.$$

Proof. Let K be a compact set in Ω . For a ball $B_r \subset K \subset \Omega$, we denote again by v the weighted p -harmonic replacement of u in B_r . Our goal is to show that there exist an exponent $\alpha \in (0, 1]$ depending only on p, q, N , and γ , and a constant $C < \infty$, also depending on $\|f\|_{L^q(K)}$ and $\text{dist}(K, \partial\Omega)$, such that the following estimate holds:

$$(2.18) \quad \int_{B_r} |\nabla u - \overline{\nabla} u_r| \leq C r^{N+\alpha}$$

where $\overline{\nabla} u_r := \int_{B_r} \nabla u$. Then, by the Campanato estimate (see [16, Theorem 1.54]), we infer that $\nabla u \in C^{0,\alpha}(K)$. First of all, it is clear that

$$\|\nabla u - \overline{\nabla} u_r\|_{L^p(B_r)} \leq \|\nabla u - \nabla v\|_{L^p(B_r)} + \|\nabla v - \overline{\nabla} v_r\|_{L^p(B_r)} + \|\overline{\nabla} v_r - \overline{\nabla} u_r\|_{L^p(B_r)}.$$

Since v is weighted p -harmonic in B_r , then by [21], v is locally of class $C^{1,\alpha_\star}$ in Ω . As a direct consequence of this, we obtain the following estimate:

$$(2.19) \quad \|\nabla v - \overline{\nabla} v_r\|_{L^p(B_r)} \leq C r^{\alpha_\star + \frac{N}{p}}.$$

Moreover,

$$(2.20) \quad \|\overline{\nabla} v_r - \overline{\nabla} u_r\|_{L^p(B_r)} \leq |\overline{\nabla} v_r - \overline{\nabla} u_r| |B_r|^{\frac{1}{p}} \leq |B_r|^{\frac{1}{p}-1} \int_{B_r} |\nabla v - \nabla u| \leq \|\nabla u - \nabla v\|_{L^p(B_r)}.$$

Now, assume $p \geq 2$. Recalling Lemma 2.3, we have

$$\|\nabla u - \nabla v\|_{L^p(B_r)} \leq C r^{\frac{1}{p-\gamma}[(1-\frac{N}{p})\gamma + \frac{N}{q}]}.$$

Hence,

$$\|\nabla u - \overline{\nabla} u_r\|_{L^p(B_r)} \leq C r^{\frac{1}{p-\gamma}[(1-\frac{N}{p})\gamma + \frac{N}{q}]} + C r^{\frac{N}{p} + \alpha_\star}.$$

Consequently, we get

$$\|\nabla u - \overline{\nabla} u_r\|_{L^1(B_r)} \leq C r^{N + \frac{\gamma q - N}{q(p-\gamma)}} + C r^{N+\alpha_\star} \leq C r^{N+\beta_\star}$$

with

$$\beta_\star := \min \left\{ \alpha_\star, \frac{\gamma - \frac{N}{q}}{p - \gamma} \right\}.$$

Assume $1 < p < 2$. From Lemma 2.3, one has

$$\|\nabla u - \nabla v\|_{L^p(B_r)} \leq C \left(|B_r| + \int_{B_r} |\nabla u|^p \right)^{\frac{2-p}{p(2-\gamma)}} r^{\frac{1}{2-\gamma}[(1-\frac{N}{p})\gamma + \frac{N}{q}]}.$$

Yet, thanks to Proposition 2.4 and the fact that $q > \frac{N}{\gamma}$, then any minimizer u of Problem (0.1) is locally Lipschitz and so, we have

$$\|\nabla u\|_{L^p(B_r)} \leq C r^{\frac{N}{p}}.$$

Hence, we get

$$\|\nabla u - \nabla v\|_{L^p(B_r)} \leq C r^{\frac{N(2-p)}{p(2-\gamma)} + \frac{1}{2-\gamma}[(1-\frac{N}{p})\gamma + \frac{N}{q}]}.$$

Recalling (2.19) & (2.20), we infer that

$$\|\nabla u - \overline{\nabla} u_r\|_{L^p(B_r)} \leq C r^{\frac{N}{p} + \frac{\gamma - \frac{N}{q}}{2-\gamma}} + C r^{\frac{N}{p} + \alpha_\star}.$$

Consequently, $u \in C_{loc}^{1,\beta_\star}(\Omega)$ with

$$\beta_\star := \min \left\{ \alpha_\star, \frac{\gamma - \frac{N}{q}}{2-\gamma} \right\}. \quad \square$$

3. REGULARITY OF MINIMIZERS ON THE FREE BOUNDARY

From Proposition 2.5, we already know that any minimizer u of Problem (0.1) is locally of class $C^{1,\beta_\star}$, where $\min\{\alpha_\star, \beta\}$ and β is defined in (2.17). However, $\alpha_\star$ denotes the sharp exponent for the local $C^{1,\alpha_\star}$ -regularity of p -harmonic functions, whose exact value remains unknown when $N \geq 3$. In this section, we show that any minimizer u of Problem (0.1) is locally of class $C^{1,\beta}$ on the free boundary $\partial\{u > 0\}$, independently of the value of $\alpha_\star$. The proof of the next proposition is inspired by the arguments in [3, 4].

Proposition 3.1. *Assume $f \in L^q(\Omega)$ with $q > \frac{N}{\gamma}$. Fix $x_0 \in \partial\{u > 0\} \cap \Omega$. Then, there exists a constant $C < \infty$ such that for every $r > 0$ small enough, we have*

$$\|u\|_{L^\infty(B(x_0, r))} \leq Cr^{\frac{p-\frac{N}{\gamma}}{p-\gamma}}.$$

Hence, u is locally of class $C^{1,\beta}$ on the free boundary $\partial\{u > 0\}$ with $\beta = \min\left\{\frac{\gamma-\frac{N}{\gamma}}{p-\gamma}, 1\right\}$.

Proof. Fix $0 < R < \text{dist}(x_0, \partial\Omega)$. For simplicity of exposition, we will assume that $x_0 = 0$ and $R = 1$. The proof is divided into several steps:

• *Step 1.* Let us assume that $\|u\|_{L^\infty(B_1)} \leq 1$. Then, we will show that for any $r \in (0, 1)$, there exists a $\delta_r > 0$ such that the following holds:

$$(3.1) \quad \|f\|_{L^q(B_1)} < \delta_r \Rightarrow \|u\|_{L^\infty(B_r)} \leq r^2.$$

Assume this is not the case. Then, there will be an $r \in (0, 1)$ such that for all $n \in \mathbb{N}^*$, there exist $f_n \in L^q(B_1)$ with $\|f_n\|_{L^q(B_1)} < \frac{1}{n}$ and, a minimizer u_n to the following problem

$$(3.2) \quad \min \left\{ \int_{B_1} w |\nabla u|^p + f_n u^\gamma : u \in W^{1,p}(B_1), u \geq 0, u = u_n \text{ on } \partial B_1 \right\}$$

such that $0 \in \partial\{u_n > 0\}$, $\|u_n\|_{L^\infty(B_1)} \leq 1$ and $\|u_n\|_{L^\infty(B_r)} \geq r^2$. Thanks to Propositions 2.4 & 2.5, one has the following estimates:

$$\|\nabla u_n\|_{L^\infty(B_{r'})}, \|\nabla u_n\|_{C^{0,\beta_\star}(B_{r'})} \leq C \left(p, q, \gamma, N, c_w, \|f_n\|_{L^q(B_1)}, \|w\|_{C^2(B_1)}, \|u_n\|_{L^\infty(B_1)}, \frac{1}{1-r'} \right),$$

for all $0 < r' < 1$. Consequently, $u_n \rightarrow u^*$ in $C^1(\overline{B_{r'}})$. From Proposition 2.2, we know that u_n is p -subharmonic in the weak sense in B_1 , i.e. for all $\varphi \in C_0^\infty(B_1)$ with $\varphi \geq 0$, we have

$$(3.3) \quad \int_{B_1} w |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \varphi \leq 0.$$

Since $\text{spt}(\varphi) \subset B_{r'}$, for some $0 < r' < 1$ and $\nabla u_n \rightarrow \nabla u^*$ uniformly in $\overline{B_{r'}}$, then passing to the limit in (3.3), we infer that

$$\int_{B_1} w |\nabla u^*|^{p-2} \nabla u^* \cdot \nabla \varphi \leq 0.$$

This means that u^* is weighted p -subharmonic. On the other hand, we claim that u^* is also weighted p -superharmonic. Thus, u^* will be weighted p -harmonic in B_1 . Let $\varphi \in C_0^\infty(B_1)$ with $\varphi \geq 0$ and $\text{spt}(\varphi) \subset B_{r'}$, for some $0 < r' < 1$. Then, we will show that

$$(3.4) \quad \int_{B_1} w |\nabla u^*|^{p-2} \nabla u^* \cdot \nabla \varphi \geq 0.$$

Let $v \in W^{1,p}(B_{r'})$, $v \geq 0$ and, $v = u^*$ on $\partial B_{r'}$. For every n , set $v_n = v + u_n - u^*$. Then, it is clear that $v_n \in W^{1,p}(B_{r'})$ with $v_n = u_n$ on $\partial B_{r'}$ (but, v_n is not necessarily nonnegative). However, we see

that u_n minimizes

$$\min \left\{ \int_{B_{r'}} w |\nabla h|^p + f_n |h|^\gamma : h \in W^{1,p}(B_{r'}), h = u_n \text{ on } \partial B_{r'} \right\}.$$

Thus, we have

$$\begin{aligned} \int_{B_{r'}} w |\nabla u_n|^p &\leq \int_{B_{r'}} w |\nabla u_n|^p + f_n u_n^\gamma \\ &\leq \int_{B_{r'}} w |\nabla v_n|^p + f_n |v_n|^\gamma \leq \int_{B_{r'}} w |\nabla v_n|^p + f_n (v + |u_n - u^*|)^\gamma \\ &\leq \int_{B_{r'}} w |\nabla v_n|^p + f_n (v^\gamma + |u_n - u^*|^\gamma) \leq \int_{B_{r'}} w |\nabla v_n|^p + f_n v^\gamma + \|f_n\|_{L^1(B_{r'})} \|u_n - u^*\|_{L^\infty(B_{r'})}^\gamma. \end{aligned}$$

But, we recall that $f_n \rightarrow 0$ in $L^q(B_1)$ and $\|u_n - u^*\|_{L^\infty(B_{r'})} \rightarrow 0$. Then, passing to the limit when $n \rightarrow \infty$, we get that

$$\int_{B_{r'}} w |\nabla u^*|^p \leq \int_{B_{r'}} w |\nabla v|^p.$$

Hence, u^* minimizes

$$\min \left\{ \int_{B_{r'}} |\nabla v|^p : v \in W^{1,p}(B_{r'}), v \geq 0, v = u^* \text{ on } \partial B_{r'} \right\}.$$

In particular, we have

$$\int_{B_{r'}} |\nabla u^*|^p \leq \int_{B_{r'}} |\nabla u^* + \varepsilon \nabla \varphi|^p \quad \text{for all } \varepsilon > 0 \text{ small enough.}$$

Thus, the claim (3.4) is proved. Yet, we recall that $u_n(0) = 0$ for all n , so $u^*(0) = 0$. Hence, thanks to the Harnack inequality in [14, Theorem 6.2], there exists a constant C such that

$$0 \leq \sup_{B_r} u^* \leq C \inf_{B_r} u^* \leq C u^*(0) = 0.$$

Hence, $u^* = 0$ on B_r . But, this yields to a contradiction since we recall that $\|u_n\|_{L^\infty(B_r)} \geq r^2$ and so, $\|u^*\|_{L^\infty(B_r)} \geq r^2$ (as $u_n \rightarrow u^*$ locally uniformly in B_1).

• *Step 2.* We assume again that $\|u\|_{L^\infty(B_1)} \leq 1$. Fix $r_0 \in (0, 1)$. From *Step 1*, we recall that there exists a constant $\delta_{r_0} > 0$ such that if $\|f\|_{L^q(B_1)} < \delta_{r_0}$ then $\|u\|_{L^\infty(B_{r_0})} \leq r_0^2$. Now, assume $\|f\|_{L^q(B_1)} < \delta_{r_0}$. Then, we claim that

$$(3.5) \quad \|u\|_{L^\infty(B_{r_0^k})} \leq r_0^{k(1+\beta)}, \quad \text{for all } k \in \mathbb{N}.$$

For $k = 0$, the estimate in (3.5) is immediate. Suppose now that the estimate (3.5) holds for some $k \geq 0$. Then, we prove that it also holds for $k + 1$. For this aim, we define the rescaled function on the unit ball B_1 as follows:

$$u_{r_0}(z) = \frac{u(r_0^k z)}{r_0^{k(1+\beta)}}.$$

Thanks to (3.5), we have $\|u_{r_0}\|_{L^\infty(B_1)} \leq 1$. Let $v_{r_0} \in W^{1,p}(B_1)$ be such that $v_{r_0} = u_{r_0}$ on ∂B_1 . Set $v(x) = r_0^{k(1+\beta)} v_{r_0}(\frac{x}{r_0^k})$, for all $x \in B_{r_0^k}$. Hence, $v = u$ on $\partial B_{r_0^k}$. Thus, we have

$$\int_{B_{r_0^k}} [w(x) |\nabla u(x)|^p + f(x) u(x)^\gamma] dx \leq \int_{B_{r_0^k}} [w(x) |\nabla v(x)|^p + f(x) v(x)^\gamma] dx.$$

Hence,

$$\int_{B_1} [w(r_0^k z) |\nabla u(r_0^k z)|^p + f(r_0^k z) u(r_0^k z)^\gamma] dz \leq \int_{B_{r_0^k}} [w(r_0^k) |\nabla v(r_0^k)|^p + f(r_0^k) v(r_0^k z)^\gamma] dz.$$

But, $\nabla u(r_0^k z) = r_0^{k\beta} \nabla u_{r_0}(z)$. Then, we have

$$\int_{B_1} [w_{r_0}(z) |\nabla u_{r_0}(z)|^p + f_{r_0}(z) u_{r_0}(z)^\gamma] dz \leq \int_{B_1} [w_{r_0}(z) |\nabla v_{r_0}(z)|^p + f_{r_0}(z) v_{r_0}(z)^\gamma] dz$$

where

$$w_{r_0}(z) = w(r_0^k z) \quad \text{and} \quad f_{r_0}(z) = r_0^{k[\gamma+\beta(\gamma-p)]} f(r_0^k z).$$

Consequently, u_{r_0} minimizes the following problem:

$$\min \left\{ \int_{B_1} w_{r_0} |\nabla v|^p + f_{r_0} v^\gamma : v \in W^{1,p}(B_1), v \geq 0, v = u_{r_0} \text{ on } \partial B_1 \right\}.$$

Yet,

$$\|f_{r_0}\|_{L^q(B_1)} = r_0^{k[\gamma+\beta(\gamma-p)]} \|f(r_0^k z)\|_{L^q(B_1)} = r_0^{k[\gamma+\beta(\gamma-p)-\frac{N}{q}]} \|f\|_{L^q(B_{r_0^k})}.$$

Recalling the definition of β , we get that

$$\|f_{r_0}\|_{L^q(B_1)} = \|f\|_{L^q(B_{r_0^k})} \leq \|f\|_{L^q(B_1)} < \delta_{r_0}.$$

From (3.1) in *Step 1*, this yields that

$$\|u_{r_0}\|_{L^\infty(B_{r_0})} \leq r_0^2.$$

Thus,

$$\|u\|_{L^\infty(B_{r_0^{k+1}})} \leq r_0^2 r_0^{k(1+\beta)} \leq r_0^{(k+1)(1+\beta)}.$$

Hence, the claim (3.5) is proved.

• *Step 3.* Let $M > \max \left\{ \|u\|_{L^\infty(B_1)}, \left(\frac{\|f\|_{L^q(B_1)}}{\delta_{r_0}} \right)^{\frac{1}{p-\gamma}} \right\}$, where $\delta_{r_0} > 0$ is the constant in (3.1) with $r = r_0$. Then, we define the normalized function

$$u_M(x) = \frac{u(x)}{M}, \quad x \in B_1.$$

Clearly, one has $\|u_M\|_{L^\infty(B_1)} \leq 1$. In addition, it is easy to see that u_M minimizes the following problem:

$$(3.6) \quad \min \left\{ \int_{B_1} w |\nabla v|^p + M^{\gamma-p} f v^\gamma : v \in W^{1,p}(B_1), v \geq 0, v = u_M \text{ on } \partial B_1 \right\}.$$

But, $M^{\gamma-p} \|f\|_{L^q(B_1)} < \delta_{r_0}$. Hence, by (3.5) in *Step 2*, we have

$$(3.7) \quad \|u\|_{L^\infty(B_{r_0^k})} \leq M r_0^{k(1+\beta)}, \quad \text{for all } k \in \mathbb{N}.$$

Finally, let $r \in (0, 1)$. Then, there exists $k \in \mathbb{N}$ such that $r_0^{k+1} < r \leq r_0^k$. From the estimate (3.7), we get that

$$\|u\|_{L^\infty(B_r)} \leq \|u\|_{L^\infty(B_{r_0^k})} \leq M r_0^{k(1+\beta)} \leq M r_0^{-(1+\beta)} r^{1+\beta}. \quad \square$$

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