

RELATIVE ENTROPIES AND CONTRACTION UP TO SHIFTS FOR A NONLOCAL BURGERS EQUATION

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ABSTRACT. We study L^2 -type contraction properties, up to a spatial shift, for a nonlocal Burgers equation with a one-sided exponential kernel. For a shock connecting a positive state to vacuum, we exhibit an explicit two-bump perturbation of the initial datum for which no Lipschitz shift starting at the initial shock position makes the squared L^2 distance between the averaged densities non-increasing, even for short times. We also identify two classes of initial data that admit a contractive shift locally in time, and construct data for which such a shift exists locally but cannot be continued to all positive times. For small shocks connecting two positive states, we prove that a logarithmic relative entropy is globally non-increasing along a suitable shift, for all initial data that take values between the two end states and differ from the traveling wave by a function in $L^1(\mathbb{R})$. The resulting estimates control both the distance to the local entropy shock and the size of the shift. Finally, near a Lipschitz solution of the local equation, and for identical initial data, we obtain an $O(\sqrt{\varepsilon})$ convergence rate locally in L^2 : for both the density and its average in the case of the exponential kernel, and for the average alone in the case of general monotone kernels satisfying a quantitative decay condition.

1. INTRODUCTION

We study the Cauchy problem for a nonlocal scalar conservation law,

$$(1.1) \quad \begin{cases} \partial_t u^\varepsilon(t, x) + \partial_x (V((u^\varepsilon * \gamma_\varepsilon)(t, x)) u^\varepsilon(t, x)) = 0, & (t, x) \in (0, \infty) \times \mathbb{R}, \\ u^\varepsilon(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

with a Lipschitz non-decreasing velocity $V: [0, 1] \rightarrow \mathbb{R}$, a parameter $\varepsilon > 0$, an initial datum

$$(1.2) \quad u_0 \in L^\infty(\mathbb{R}), \quad 0 \leq u_0 \leq 1, \quad \text{TV}(u_0) < +\infty,$$

where $\text{TV}(u_0)$ denotes the total variation of u_0 on $\mathbb{R}$, and the nonlocal average

$$w^\varepsilon := u^\varepsilon * \gamma_\varepsilon,$$

given by a weighted average of the density over the half-line $(-\infty, x]$ with kernel $\gamma_\varepsilon(\cdot) := \varepsilon^{-1} \gamma(\cdot/\varepsilon)$, where

$$(1.3) \quad \begin{aligned} \gamma &\in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \quad \text{supp } \gamma \subset [0, +\infty), \quad \int_0^{+\infty} \gamma(z) \, dz = 1, \\ \gamma &\geq 0, \quad \gamma \text{ is non-increasing on } [0, +\infty). \end{aligned}$$

Convolving (1.1) with γ_ε , we obtain, in the sense of distributions,

$$(1.4) \quad \partial_t w^\varepsilon + \partial_x (V(w^\varepsilon) w^\varepsilon) = \partial_x (V(w^\varepsilon) w^\varepsilon - (V(w^\varepsilon) u^\varepsilon) * \gamma_\varepsilon).$$

Under (1.3), the Cauchy problem (1.1) has a unique distributional solution with values in $[0, 1]$, and w^ε is Lipschitz in x (Section 2).

We focus on the case in which γ is exponential, that is,

$$(1.5) \quad \gamma := \mathbf{1}_{[0, +\infty)}(\cdot) \exp(-\cdot).$$

In this case, the average satisfies the resolvent identity

$$(1.6) \quad \varepsilon \partial_x w^\varepsilon = u^\varepsilon - w^\varepsilon.$$

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The convergence of u^ε and w^ε to the entropy solution of the corresponding local conservation law

$$(1.7) \quad \begin{cases} \partial_t u(t, x) + \partial_x (V(u(t, x))u(t, x)) = 0, & (t, x) \in (0, \infty) \times \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R} \end{cases}$$

has been studied for several classes of kernels and initial data. The numerical experiments of Amorim, Colombo, and Teixeira [ACT15, Section 3.3], using a family of even, compactly supported kernels, suggested convergence. Convergence does not hold in general: Colombo, Crippa, and Spinolo [CCS19, Section 5] constructed counterexamples. For the quadratic flux, Zumbrun [Zum99, Proposition 4.1] proved convergence for even kernels as long as the local solution remains smooth.

For monotone initial data, Keimer and Pflug [KP19] proved strong convergence under additional assumptions on the velocity and kernel. Colombo, Crippa, Marconi, and Spinolo [Col+21, Corollary 4 and Theorem 6] treated data of bounded variation that are bounded away from zero and satisfy a one-sided Lipschitz condition, again under structural assumptions on the velocity and on the kernel. They also showed that, when vacuum is allowed, the total variation of the nonlocal densities can become unbounded as $\varepsilon \rightarrow 0$. Bressan and Shen [BS20, Theorems 5.1 and 6.1] and [BS21] proved convergence for exponential kernels and initial data of bounded variation that are bounded away from zero. For exponential kernels, Coclite, Coron, De Nitti, Keimer, and Pflug [Coc+23, Theorems 3.2 and 4.2] removed the positive lower bound by estimating the total variation of w^ε . Their convergence theorem assumes strict concavity of the local flux with the sign convention used for traffic models.

For kernels that are convex on the half-line on which they are supported and initial data of bounded variation, [Col+23, Theorems 1.1–1.3] established strong convergence of the averages. If the kernel also has a finite first moment, then, by [Col+23, Theorem 1.3],

$$(1.8) \quad \|w^\varepsilon(t, \cdot) - u(t, \cdot)\|_{L^1(\mathbb{R})} \leq C(\gamma, V)(\varepsilon + \sqrt{\varepsilon t}) \text{TV}(u_0)$$

for every $\varepsilon > 0$ and almost every $t > 0$. De Nitti and Huang [DH26, Theorem 1.1] obtained strong convergence of the densities themselves for this class of kernels under an additional L^1 assumption on the initial datum. In [DH25, Theorem 2.1], the same authors introduced a Godunov-type discretization whose discrete averages converge to the local entropy solution as the mesh size and the nonlocal scale tend to zero, without a restriction on their relative sizes.

Coclite, Coron, De Nitti, Keimer, and Pflug [Coc+24, Corollary 2.6] proved convergence for the exponential kernel and bounded initial data under additional assumptions on V , using Oleĭnik-type estimates. Coclite, De Nitti, and Huang [CDH25, Theorem 1.2] proved strong convergence of the averages for certain non-convex kernels by compensated compactness, without assuming finite total variation of the initial datum. Keimer and Pflug [KP26, Theorem 4.5 and Remark 4.6] proved weak-* convergence for bounded initial data and monotone one-sided kernels. Their proof uses viscosity solutions of Hamilton–Jacobi equations for the spatial primitives. De Nitti and Prasad [DP26, Theorem 1.1] proved strong local convergence of the averages for bounded initial data, Lipschitz non-increasing velocities, and one-sided kernels of bounded variation that are non-negative, monotone, and have integral one. They use the entropy balance of [CDH25, Proposition 3.1] and uniqueness of entropy measure-valued solutions.

The rate in (1.8) is analogous to Kuznetsov’s estimate [Kuz76, Theorem 3] for the viscous regularization

$$(1.9) \quad \begin{cases} \partial_t u_\mu + \partial_x (V(u_\mu)u_\mu) = \mu \partial_{xx} u_\mu, & (t, x) \in (0, \infty) \times \mathbb{R}, \\ u_\mu(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

namely

$$\|u_\mu(t, \cdot) - u(t, \cdot)\|_{L^1(\mathbb{R})} \leq C\sqrt{\mu t} \text{TV}(u_0).$$

Both estimates concern the $L^1(\mathbb{R})$ distance, for which the local entropy semigroup is contractive [Kru70]. On the other hand, the entropy semigroup is in general not a contraction in $L^p(\mathbb{R})$ for $p > 1$, even for the quadratic flux. We study L^2 -type stability using the relative entropy method. Relative entropy gives stability of Lipschitz solutions among bounded entropy solutions [Daf79; DiP79; Daf96]: for a bounded entropy solution u and a Lipschitz solution $\bar{u}$ of the same scalar

conservation law with a C^2 flux,

$$\|u(t, \cdot) - \bar{u}(t, \cdot)\|_{L^2([-R, R])} \leq C_1 e^{C_2 t} \|u(0, \cdot) - \bar{u}(0, \cdot)\|_{L^2([-R - \Lambda t, R + \Lambda t])},$$

where Λ bounds the characteristic speeds on the common range of u and $\bar{u}$, and C_1, C_2 depend on that range, on the flux, and on the Lipschitz norm of $\bar{u}$ on the time interval considered. At a shock, a shift must also be allowed: for a strictly convex scalar flux, Leger [Leg11, Theorem 1.1] proved that every bounded entropy solution at finite L^2 distance from a shock remains within its initial L^2 distance after a suitable Lipschitz shift of the shock.¹

Building on [Leg11, Theorem 1.1], Choi and Vasseur [CV15, Theorem 1.1] proved a corresponding estimate for the vanishing-viscosity limit (1.9) with a strictly convex C^2 flux: for bounded initial data of locally bounded variation, at finite L^2 distance from the shock $\bar{u}$ and with the positive part of the initial derivative in $L^2(\mathbb{R})$, some Lipschitz curve $t \mapsto X(t)$ satisfies

$$\|u_\mu(t, \cdot) - \bar{u}(t, \cdot - X(t))\|_{L^2(\mathbb{R})}^2 \leq \|u_0 - \bar{u}(0, \cdot)\|_{L^2(\mathbb{R})}^2 + C\mu \log(1/\mu)$$

for μ sufficiently small, with a constant C depending on the length of the time interval and on the bounds on the data. For quadratic fluxes and sufficiently small C^2 perturbations of them, Kang and Vasseur [KV17, Theorem 1.3] removed the logarithmic loss and the one-sided regularity assumption: for an inviscid shock $\bar{u}$, some Lipschitz shift X satisfies

$$(1.10) \quad \|u_\mu(t, \cdot) - \bar{u}(t, \cdot - X(t))\|_{L^2(\mathbb{R})} \leq \|u_0 - \bar{u}(0, \cdot)\|_{L^2(\mathbb{R})} + C\sqrt{\mu}.$$

Kang and Vasseur also constructed strictly convex fluxes whose viscous shocks fail to be L^2 -contractive up to a shift [KV17, Theorem 1.2]. For initial data equal to the shock, (1.10) gives a squared distance of order μ , and the estimates for identical initial data in Corollaries 3.7 and 3.9 and Theorem 3.10 are its analogues for the nonlocal regularization.

For systems endowed with a convex entropy, Leger and Vasseur [LV11, Theorem 2.1], Vasseur [Vas16, Theorem 2.1], and Serre and Vasseur [SV14, Theorem 2.1] (see also [SV16, Sections 3.1 and 4]) developed stability estimates for discontinuities using relative entropy. Allowing different positive weights for the relative entropy on the two sides of the shock gives a weighted form of contraction. Kang and Vasseur [KV16, Theorem 3.1] established criteria for such contraction, and Golding, Krupa, and Vasseur [GKV23, Theorem 1.2] proved that, for small extremal shocks, the weight can be chosen to differ from 1 by a quantity proportional to the shock strength. Applications include convergence to contact discontinuities as heat conductivity tends to zero [VW15, Theorem 1.1], stability of two-shock Riemann solutions of the isentropic Euler system among limits obtained as the physical viscosity tends to zero [KV22, Theorem 1.1], L^2 -contraction of large planar shocks of multidimensional scalar viscous conservation laws [KVV19, Theorem 1.1], and stability of small-BV solutions of 2×2 systems among a larger class of weak solutions [CKV22, Theorem 1.3]. A survey is given in [Vas23]. For spatially heterogeneous fluxes $F(x, u)$, Ghoshal and Venkatesh [GV25, Theorem 2.2] proved stability up to a shift under structural assumptions that include $\partial_x \partial_u F \geq 0$, and [GV25, Section 5.3] constructs examples in which the L^2 distance increases even after a shift when this condition is violated.

Main results. We ask whether the relative entropy method with shifts extends from the viscous regularization (1.9) to the nonlocal regularization (1.1). The answer depends on whether vacuum is present.

¹For example, we consider the entropic shock $\bar{u}(t, x) = \frac{1}{2} \mathbb{1}_{(-\infty, t/2)}(x)$ of (1.7) with $V(\xi) = \xi$, and let u be the entropy solution with initial datum $u_0(x) = \frac{1}{2} \mathbb{1}_{(-\infty, 0)}(x) + \frac{1}{2} \mathbb{1}_{[-2, 0)}(x)$. At $t = 0$, the L^2 distance between the two initial data is $\|u_0 - \bar{u}(0, \cdot)\|_{L^2(\mathbb{R})} = 1/\sqrt{2}$, and u contains a shock also emanating from $x = 0$, but traveling at twice the speed of the shock in $\bar{u}$. At $t = 1/2$,

$$\bar{u}(1/2, x) = \begin{cases} \frac{1}{2}, & \text{if } x < \frac{1}{4}, \\ 0, & \text{if } x \geq \frac{1}{4}, \end{cases} \quad u(1/2, x) = \begin{cases} \frac{1}{2}, & \text{if } x < -\frac{3}{2}, \\ x + 2, & \text{if } -\frac{3}{2} \leq x < -1, \\ 1, & \text{if } -1 \leq x < \frac{1}{2}, \\ 0, & \text{if } x \geq \frac{1}{2}. \end{cases}$$

The squared distance therefore satisfies

$$\|u(1/2, \cdot) - \bar{u}(1/2, \cdot)\|_{L^2(\mathbb{R})}^2 = \frac{1}{24} + \frac{5}{16} + \frac{1}{4} = \frac{29}{48} > \frac{1}{2} = \|u_0 - \bar{u}(0, \cdot)\|_{L^2(\mathbb{R})}^2.$$

- (i) *Obstruction at vacuum* ([Theorem 3.1](#)). For $V(\xi) = \xi$, the shock $S_0(x - t)$, $S_0 = \mathbb{1}_{(-\infty, 0)}$, joining the state 1 to vacuum solves both the local and the nonlocal equation. Two explicit functionals J and K of the initial perturbation $u_0^\varepsilon - S_0 \in L^1(\mathbb{R})$ give a criterion for local existence or non-existence of a shift along which the $L^2(\mathbb{R})$ distance to the averaged profile is non-increasing (an L^2 -contractive shift, [Section 3.1](#)). In the time derivative at $t = 0$ of half the squared L^2 distance to the average of the shifted profile, J is the coefficient of the shift velocity and K is the remaining term. If $J[u_0^\varepsilon] \neq 0$ or $K[u_0^\varepsilon] < 0$, a Lipschitz shift X with $X(0) = 0$ makes the L^2 distance between w^ε and the average of the shifted profile non-increasing for short times. If $J[u_0^\varepsilon] = 0$ and $K[u_0^\varepsilon] > 0$, this distance is strictly increasing for short times along *every* Lipschitz shift starting at zero. An explicit two-bump perturbation of S_0 lies in the second class, and, for this datum, local contractive shifts starting near zero exist but cannot be continued to all positive times ([Section 4.1](#)). In contrast with the viscous case [[KV17](#), Theorem 1.1], L^2 -contraction up to a shift therefore fails at vacuum.
- (ii) *Contraction for small shocks with positive end states* ([Theorems 3.2 and 3.3](#)). For $V(\xi) = v_0 + \beta\xi$ with $\beta > 0$ and end states $0 < u_+ < u_- \leq 1$ satisfying $(u_- - u_+)/u_+ \leq \theta_* < 1/2$, the nonlocal equation admits an explicit smooth decreasing traveling wave S^ε ([Proposition 2.3](#)), and the logarithmic relative entropy $\int_{\mathbb{R}} h(u^\varepsilon | S^\varepsilon) dx$ is non-increasing along a shift whose velocity is determined by a weighted integral of the perturbation, for all initial data with values in $[u_+, u_-]$ and $u_0^\varepsilon - S^\varepsilon \in L^1(\mathbb{R})$. The proof uses a coercivity inequality ([Theorem 3.2](#)). The dissipation from the nonlocal averaging does not by itself control the term caused by the variation of the profile. The shift adds the square of a weighted integral of the perturbation, which makes the quadratic form non-negative. As consequences, we obtain quantitative L^2 bounds on the distance to the local entropy shock and on the size of the shift ([Theorem 3.6](#)), and explicit rates in ε for Riemann data ([Corollary 3.7](#)).
- (iii) *Stability near Lipschitz solutions* ([Theorems 3.8 and 3.10 and Corollary 3.9](#)). For C^2 velocities, we prove relative entropy estimates with respect to Lipschitz solutions of (1.7): for the density u^ε with the exponential kernel, and for the average w^ε with general monotone one-sided kernels satisfying a quantitative decay condition.

Outline of the paper. The paper is organized as follows. In [Section 2](#), we collect the preliminary material: the maximum principle, the logarithmic relative entropy and its comparison with the squared distance, the explicit traveling wave profiles, and the resolvent identity for the exponential kernel. In [Section 3](#), we state the main results: the vacuum obstruction, the coercivity inequality, the logarithmic contraction with a shift and its consequences, and the estimates near Lipschitz solutions. [Section 4](#) contains the proof of the vacuum obstruction, together with examples of local shifts that cannot be continued for all times, the case $J = K = 0$, and a discussion of more general kernels. [Section 5](#) contains the proofs of the results on the logarithmic relative entropy, including the coercivity inequality. [Section 6](#) contains the proofs of the estimates near Lipschitz solutions, for the exponential kernel first and for general kernels afterwards. We conclude in [Section 7](#) with some open questions.

2. PRELIMINARIES

In this section, we summarize the well-posedness theory for (1.1), introduce the Riemann datum and the maximum principle, define the logarithmic relative entropy used in the contraction estimates, collect a few elementary properties of the exponential kernel, and construct the traveling wave profiles of the nonlocal equation.

Let $M > 0$, let $0 \leq u_0^\varepsilon \leq M$, and let V be locally Lipschitz and non-decreasing on $[0, M]$. We extend $V|_{[0, M]}$ to a globally Lipschitz non-decreasing function, which does not affect the solution, since the solution takes values in $[0, M]$. The results of [[Col+23](#); [Coc+24](#); [CDH25](#)] are stated for kernels supported on the negative half-line, and they transfer to our setting through the substitution $x \mapsto -x$, $V \mapsto -V$, which we apply without further mention. After scaling the density by M , we apply [[Col+23](#), Proposition 2.1 and Corollary 2.2] for the kernels of (1.3). For the exponential kernel, if also $V \in W_{\text{loc}}^{2, \infty}(\mathbb{R})$, we apply [[Coc+24](#), Theorem 2.7]. These results give, for every $T > 0$, a unique solution in the sense of distributions with

$$u^\varepsilon \in C([0, T]; L_{\text{loc}}^1(\mathbb{R})) \cap L^\infty((0, T) \times \mathbb{R}), \quad 0 \leq u^\varepsilon, w^\varepsilon \leq M.$$

In the rest of the paper, the solution of (1.1) for data with values in $[0, M]$ is this solution, and the total variation bound in (1.2) is assumed only where stated explicitly. We write $f(\xi) := \xi V(\xi)$ for the local flux. For the results on shocks, we restrict our attention to an affine velocity law, so that the local flux is quadratic:

$$(2.1) \quad V(\xi) = v_0 + \beta\xi, \quad \beta > 0, \quad f(\xi) = v_0\xi + \beta\xi^2,$$

and, for the results on shocks with positive end states, we fix the end states

$$(2.2) \quad 0 < u_+ < u_- \leq 1, \quad \theta := \frac{u_- - u_+}{u_+}.$$

The case $u_+ = 0$ is allowed in Proposition 2.3 and in Section 3.1. For $0 \leq u_+ < u_- \leq 1$, we define the Riemann datum

$$(2.3) \quad S^0(x) := \begin{cases} u_- & \text{if } x < 0, \\ u_+ & \text{if } x \geq 0. \end{cases}$$

By the Rankine–Hugoniot condition, the function

$$(2.4) \quad U(t, x) := S^0(x - \sigma t), \quad \text{with } \sigma := \frac{f(u_-) - f(u_+)}{u_- - u_+},$$

is a weak solution of (1.7) with the quadratic flux f of (2.1). Moreover, the condition $u_- > u_+$ implies that it satisfies the entropy inequality

$$\partial_t \eta(U) + \partial_x q(U) \leq 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}$$

for any convex entropy $\eta \in C^2(\mathbb{R})$ and entropy flux q with $q' = \eta' f'$. The maximum principle in the next lemma follows from [Coc+24, Theorem 2.7], after extending V monotonically outside $[m, M]$.

Lemma 2.1 (Maximum principle). *Let $0 \leq m < M$, and let $V \in W_{\text{loc}}^{2,\infty}(\mathbb{R})$ be non-decreasing on a neighborhood of $[m, M]$. If $m \leq u_0^\varepsilon \leq M$, then the solution of (1.1) with the exponential kernel (1.5) satisfies*

$$m \leq u^\varepsilon(t, x), w^\varepsilon(t, x) \leq M \quad \text{for almost every } (t, x).$$

2.1. Logarithmic relative entropy. At vacuum, where the density may vanish, we use the quadratic relative entropy. Away from vacuum, we work with the logarithmic relative entropy

$$(2.5) \quad h(r \mid s) := r \log \frac{r}{s} - r + s, \quad r, s > 0.$$

On $[u_+, u_-]$, it is uniformly equivalent to the squared distance:

$$(2.6) \quad \frac{1}{2u_-} |r - s|^2 \leq h(r \mid s) \leq \frac{1}{2u_+} |r - s|^2, \quad r, s \in [u_+, u_-].$$

This follows from the integral representation given by Taylor's theorem,

$$h(r \mid s) = (r - s)^2 \int_0^1 \frac{1 - \vartheta}{s + \vartheta(r - s)} d\vartheta.$$

2.2. The exponential resolvent. We collect the elementary properties of the exponential kernel used in the proofs. For $1 \leq p \leq \infty$, convolution with γ_ε is a positive contraction on $L^p(\mathbb{R})$, maps constants to themselves, commutes with translations and with distributional spatial derivatives, and inverts the first-order differential operator $\varepsilon \partial_x + 1$.

Lemma 2.2 (Exponential resolvent). *Let γ_ε be the exponential kernel (1.5). For every $g \in L^p(\mathbb{R})$, $1 \leq p \leq \infty$, the function $G = \gamma_\varepsilon * g$ belongs to $W^{1,p}(\mathbb{R})$, and*

$$(2.7) \quad \varepsilon \partial_x G = g - G$$

in the sense of distributions. In particular, the resolvent identity (1.6) holds. Moreover, $G = \gamma_\varepsilon * g$ is the unique solution of (2.7) in $L^p(\mathbb{R})$ when $p < \infty$ and the unique bounded solution when $p = \infty$. Finally, if $g \in L^\infty(\mathbb{R})$ is locally absolutely continuous and $\partial_x g \in L^p(\mathbb{R})$, then

$$(2.8) \quad \|\gamma_\varepsilon * g - g\|_{L^p(\mathbb{R})} \leq \varepsilon \|\partial_x g\|_{L^p(\mathbb{R})}.$$

Proof. The bounds in $L^p(\mathbb{R})$ follow from Young's inequality. Since $\varepsilon \partial_x \gamma_\varepsilon = \delta_0 - \gamma_\varepsilon$ in the sense of distributions, (2.7) follows, and with it the Sobolev regularity. The difference of two solutions of the resolvent equation is a multiple of $e^{-x/\varepsilon}$, which cannot belong to $L^p(\mathbb{R})$ or remain bounded on $\mathbb{R}$ unless that multiple is zero. Finally, Minkowski's inequality and the fundamental theorem of calculus yield

$$\begin{aligned} \|\gamma_\varepsilon * g - g\|_{L^p(\mathbb{R})} &\leq \frac{1}{\varepsilon} \int_0^\infty e^{-z/\varepsilon} \|g(\cdot - z) - g\|_{L^p(\mathbb{R})} dz \\ &\leq \frac{1}{\varepsilon} \int_0^\infty z e^{-z/\varepsilon} dz \|\partial_x g\|_{L^p(\mathbb{R})} = \varepsilon \|\partial_x g\|_{L^p(\mathbb{R})}. \end{aligned}$$

□

2.3. Traveling wave solutions. For the exponential kernel and an affine velocity, the nonlocal equation (1.1) admits traveling waves connecting u_- to u_+ , and the profile can be computed explicitly through the resolvent identity (1.6). When $u_+ = 0$, the profile is the Riemann datum itself, and when $u_+ > 0$, the profile is smooth and its width is of order ε . These two cases lead to the different contraction results in Section 3.

Proposition 2.3 (Nonlocal traveling wave). *Let (2.1) hold with the exponential kernel (1.5), let $0 \leq u_+ < u_- \leq 1$, and let $\sigma = v_0 + \beta(u_- + u_+)$ be the speed (2.4). If $u_+ > 0$, then for every $\varepsilon > 0$ there exists a smooth decreasing profile S^ε , unique up to translation among smooth decreasing profiles with the stated limits, such that $S^\varepsilon(x - \sigma t)$ solves (1.1) and*

$$(2.9) \quad \lim_{x \rightarrow -\infty} S^\varepsilon(x) = u_-, \quad \lim_{x \rightarrow +\infty} S^\varepsilon(x) = u_+.$$

We set $W^\varepsilon := \gamma_\varepsilon * S^\varepsilon$ and choose the translation for which $W^\varepsilon(0) = (u_- + u_+)/2$. The profiles have exponential tails and satisfy

$$(2.10) \quad \|S^\varepsilon - S^0\|_{L^1(\mathbb{R})} = C_{S,1}\varepsilon,$$

$$(2.11) \quad \|S^\varepsilon - S^0\|_{L^2(\mathbb{R})}^2 = C_{S,2}\varepsilon,$$

$$(2.12) \quad \|W^\varepsilon - S^0\|_{L^2(\mathbb{R})}^2 = C_{W,2}\varepsilon,$$

$$(2.13) \quad \int_{\mathbb{R}} h(S^0 | S^\varepsilon) dx = C_h \varepsilon.$$

Here $C_{S,1}$, $C_{S,2}$, $C_{W,2}$, and C_h are finite positive constants depending only on $u_\pm$, and S^0 and h are defined in (2.3) and (2.5). If $u_+ = 0$, then $S^\varepsilon := S^0$ is a traveling wave profile: $S^0(x - \sigma t)$ solves (1.1).

Proof. Step 1: The case $u_+ = 0$. If $u_+ = 0$, then $\gamma_\varepsilon * S^0 = u_-$ on $(-\infty, 0)$ and $S^0 = 0$ on $(0, \infty)$. Consequently,

$$V(\gamma_\varepsilon * S^0)S^0 = V(S^0)S^0,$$

and the Rankine-Hugoniot condition shows that $S^0(x - \sigma t)$ solves (1.1). In the rest of the proof, we assume that $u_+ > 0$.

Step 2: Profile equation. Let S^ε be a smooth decreasing profile with (2.9) such that $S^\varepsilon(x - \sigma t)$ solves (1.1), and let $W^\varepsilon := \gamma_\varepsilon * S^\varepsilon$. Then $u_+ \leq S^\varepsilon \leq u_-$, hence $u_+ \leq W^\varepsilon \leq u_-$, and Lebesgue's dominated convergence theorem in the convolution integral gives $\lim_{x \rightarrow \pm\infty} W^\varepsilon(x) = u_\pm$. By the traveling wave equation, $S^\varepsilon(V(W^\varepsilon) - \sigma)$ is constant in the sense of distributions, and

$$V(u_\pm) - \sigma = -\beta u_\mp, \quad V(W^\varepsilon) - \sigma = -\beta(u_- + u_+ - W^\varepsilon), \quad u_- + u_+ - W^\varepsilon \geq u_+ > 0.$$

Taking the limit as $x \rightarrow -\infty$ and using (1.6), we obtain

$$(2.14) \quad \begin{aligned} S^\varepsilon(V(W^\varepsilon) - \sigma) &= -\beta u_- u_+, \\ S^\varepsilon &= \frac{u_- u_+}{u_- + u_+ - W^\varepsilon}, \quad \varepsilon \partial_x W^\varepsilon = S^\varepsilon - W^\varepsilon. \end{aligned}$$

Thus

$$\varepsilon \partial_x W^\varepsilon = \frac{(W^\varepsilon - u_-)(W^\varepsilon - u_+)}{u_- + u_+ - W^\varepsilon}.$$

The normalized coordinate

$$y^\varepsilon(x) := \frac{W^\varepsilon(x) - u_+}{u_- - u_+}$$

satisfies

$$(2.15) \quad \varepsilon \partial_x y^\varepsilon = -\frac{\theta y^\varepsilon (1 - y^\varepsilon)}{1 + \theta(1 - y^\varepsilon)}, \quad y^\varepsilon(0) = \frac{1}{2}.$$

Separation of variables, using

$$\frac{1 + \theta(1 - y)}{y(1 - y)} = \frac{1 + \theta}{y} + \frac{1}{1 - y},$$

gives

$$(2.16) \quad \frac{x}{\varepsilon} = -\frac{1 + \theta}{\theta} \log(2y^\varepsilon(x)) + \frac{1}{\theta} \log(2(1 - y^\varepsilon(x))).$$

Step 3: Existence and uniqueness. The right-hand side of (2.16), viewed as a function of $y \in (0, 1)$, has negative derivative and limits $+\infty$ at 0 and $-\infty$ at 1. Its inverse defines a smooth decreasing function y^ε with limits 1 at $-\infty$ and 0 at $+\infty$. We define

$$W^\varepsilon := u_+ + (u_- - u_+)y^\varepsilon, \quad S^\varepsilon := \frac{u_- u_+}{u_- + u_+ - W^\varepsilon}.$$

Then (2.14) holds. The bounded solution of $\varepsilon \partial_x W^\varepsilon = S^\varepsilon - W^\varepsilon$ is $\gamma_\varepsilon * S^\varepsilon$ by Lemma 2.2. Therefore $S^\varepsilon(x - \sigma t)$ solves (1.1).

Conversely, let $\tilde{S}^\varepsilon$ be a smooth decreasing profile with (2.9) such that $\tilde{S}^\varepsilon(x - \sigma t)$ solves (1.1), and let $\tilde{W}^\varepsilon := \gamma_\varepsilon * \tilde{S}^\varepsilon$. By Step 2, $\tilde{W}^\varepsilon$ takes values in $[u_+, u_-]$, has the limits $u_\pm$ at $\pm\infty$, and solves the autonomous equation $\varepsilon \partial_x \tilde{W}^\varepsilon = (\tilde{W}^\varepsilon - u_-)(\tilde{W}^\varepsilon - u_+)/(u_- + u_+ - \tilde{W}^\varepsilon)$, whose right-hand side is locally Lipschitz on $(-\infty, u_- + u_+)$. By continuity, some $\bar{x} \in \mathbb{R}$ satisfies $\tilde{W}^\varepsilon(\bar{x}) = (u_- + u_+)/2$, and uniqueness for the initial value problem gives $\tilde{W}^\varepsilon(\cdot + \bar{x}) = W^\varepsilon$. The first integral in (2.14) then gives $\tilde{S}^\varepsilon(\cdot + \bar{x}) = S^\varepsilon$. This proves uniqueness of the profile up to translation.

Step 4: Decay and dependence on ε . Since $0 < y^\varepsilon < 1$, (2.16) gives, for every $x \in \mathbb{R}$,

$$y^\varepsilon(x) \leq 2^{-\theta/(1+\theta)} \exp\left(-\frac{\theta x}{(1+\theta)\varepsilon}\right), \quad 1 - y^\varepsilon(x) \leq 2^\theta \exp\left(\frac{\theta x}{\varepsilon}\right),$$

and (2.14) gives

$$S^\varepsilon - u_+ = \frac{(u_- - u_+) y^\varepsilon}{1 + \theta(1 - y^\varepsilon)} \leq (u_- - u_+) y^\varepsilon, \quad u_- - S^\varepsilon = \frac{u_- \theta(1 - y^\varepsilon)}{1 + \theta(1 - y^\varepsilon)} \leq u_- \theta(1 - y^\varepsilon),$$

so that $S^\varepsilon - u_+$ decays exponentially as $x \rightarrow +\infty$ and $u_- - S^\varepsilon$ decays exponentially as $x \rightarrow -\infty$. With the normalization $W^\varepsilon(0) = (u_- + u_+)/2$,

$$S^\varepsilon(x) = S^1(x/\varepsilon), \quad W^\varepsilon(x) = W^1(x/\varepsilon).$$

Since $S^0(\varepsilon x) = S^0(x)$ for $\varepsilon > 0$, a change of variables proves (2.10), (2.11), (2.12) and (2.13), with each constant equal to its left-hand side at $\varepsilon = 1$. Finiteness follows from the exponential tails and (2.6), and positivity from strict monotonicity. The constants can be made explicit by the further change of variables $y = y^1(x)$, whose Jacobian is $|dx/dy| = [1 + \theta(1 - y)]/(\theta y(1 - y))$ by (2.15),

and under which $x > 0$ corresponds to $0 < y < 1/2$ and $x < 0$ to $1/2 < y < 1$:

$$\begin{aligned}
C_{S,1} &= \int_0^{1/2} \left(\frac{u_-}{1 + \theta(1-y)} - u_+ \right) \frac{1 + \theta(1-y)}{\theta y(1-y)} dy \\
&\quad + \int_{1/2}^1 \left(u_- - \frac{u_-}{1 + \theta(1-y)} \right) \frac{1 + \theta(1-y)}{\theta y(1-y)} dy, \\
C_{S,2} &= \int_0^{1/2} \left(\frac{u_-}{1 + \theta(1-y)} - u_+ \right)^2 \frac{1 + \theta(1-y)}{\theta y(1-y)} dy \\
&\quad + \int_{1/2}^1 \left(u_- - \frac{u_-}{1 + \theta(1-y)} \right)^2 \frac{1 + \theta(1-y)}{\theta y(1-y)} dy, \\
C_{W,2} &= (u_- - u_+)^2 \int_0^{1/2} y^2 \frac{1 + \theta(1-y)}{\theta y(1-y)} dy \\
&\quad + (u_- - u_+)^2 \int_{1/2}^1 (1-y)^2 \frac{1 + \theta(1-y)}{\theta y(1-y)} dy, \\
C_h &= \int_0^{1/2} h\left(u_+ \mid \frac{u_-}{1 + \theta(1-y)}\right) \frac{1 + \theta(1-y)}{\theta y(1-y)} dy \\
&\quad + \int_{1/2}^1 h\left(u_- \mid \frac{u_-}{1 + \theta(1-y)}\right) \frac{1 + \theta(1-y)}{\theta y(1-y)} dy.
\end{aligned}$$

Here $S^1 = u_-/[1 + \theta(1-y)]$ by (2.14), and the integrands are bounded, because $S^1 - u_+$ and $u_- - S^1$ vanish to first order at $y = 0$ and $y = 1$, respectively, which compensates the simple poles of the Jacobian. $\square$

3. MAIN RESULTS

We state our main results, first for shocks (Section 3.1) and then near Lipschitz solutions (Section 3.2).

3.1. Shock solutions. We begin with the case of a shock connecting a positive state to vacuum, for which we take the velocity $V(\xi) = \xi$ and the Riemann datum

$$S_0(x) := \mathbf{1}_{(-\infty, 0)}(x),$$

that is, $S_0 = S^0$ of (2.3) with $u_- = 1$ and $u_+ = 0$. By Proposition 2.3, $S_0(x-t)$ solves both (1.1) and (1.7). Its average is

$$(3.1) \quad W_\varepsilon(x) := (\gamma_\varepsilon * S_0)(x) = \begin{cases} 1, & x \leq 0, \\ e^{-x/\varepsilon}, & x > 0. \end{cases}$$

For every $t \geq 0$,

$$(3.2) \quad \begin{aligned} \|W_\varepsilon(\cdot - t) - S_0(\cdot - t)\|_{L^1(\mathbb{R})} &= \varepsilon, \\ \|W_\varepsilon(\cdot - t) - S_0(\cdot - t)\|_{L^2(\mathbb{R})} &= \sqrt{\varepsilon/2}. \end{aligned}$$

The question is whether, for a perturbation of S_0 , the $L^2(\mathbb{R})$ distance between the average w^ε and a suitably shifted copy of W_ε can be made non-increasing by choosing the shift. We call a Lipschitz curve $X: [0, T] \rightarrow \mathbb{R}$ an L^2 -contractive shift on $[0, T]$ if the distance $\|w^\varepsilon(t, \cdot) - W_\varepsilon(\cdot - X(t))\|_{L^2(\mathbb{R})}$ is non-increasing on $[0, T]$.

Let $\mathcal{B}$ denote the class of initial data u_0^ε satisfying (1.2) and

$$u_0^\varepsilon - S_0 \in L^1(\mathbb{R}).$$

Since both functions are bounded, $u_0^\varepsilon - S_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. We define

$$\begin{aligned}
J[u_0^\varepsilon] &:= \int_{\mathbb{R}} (u_0^\varepsilon - S_0) (\gamma_\varepsilon^\vee * \partial_x W_\varepsilon) dx, \\
K[u_0^\varepsilon] &:= \int_{\mathbb{R}} [(w_0^\varepsilon u_0^\varepsilon) * \gamma_\varepsilon] \partial_x (w_0^\varepsilon - W_\varepsilon) dx,
\end{aligned}$$

where $g^\vee(x) := g(-x)$ and $w_0^\varepsilon := \gamma_\varepsilon * u_0^\varepsilon$. We also consider the subsets

$$\begin{aligned}\mathcal{A} &:= \{u \in \mathcal{B} : J[u] = 0, K[u] > 0\}, \\ \mathcal{C} &:= \{u \in \mathcal{B} : J[u] \neq 0\} \cup \{u \in \mathcal{B} : K[u] < 0\}.\end{aligned}$$

The criterion is given by the two functionals J and K : for a shift X with $X(0) = 0$ that is differentiable at $t = 0$, they are the coefficient of $\dot{X}(0)$ and the term independent of $\dot{X}(0)$ in the right derivative

$$(3.3) \quad \frac{d}{dt} \Big|_{t=0^+} \frac{1}{2} \|w^\varepsilon(t, \cdot) - W_\varepsilon(\cdot - X(t))\|_{L^2(\mathbb{R})}^2 = \dot{X}(0)J[u_0^\varepsilon] + K[u_0^\varepsilon].$$

Identity (3.3) is proved in Step 2 of the proof of the next theorem, whose second part covers all Lipschitz shifts starting at zero, including shifts that are not differentiable at $t = 0$.

Theorem 3.1 (Existence and non-existence of contractive shifts at vacuum). *Let $\varepsilon > 0$, let $V(\xi) = \xi$, and let γ be the exponential kernel (1.5).*

- (i) *If $u_0^\varepsilon \in \mathcal{C}$, then there exist $T > 0$ and a Lipschitz shift $X : [0, T] \rightarrow \mathbb{R}$ depending on u^ε , with $X(0) = 0$, such that, for all $0 \leq s < t \leq T$,*

$$(3.4) \quad \|w^\varepsilon(t, \cdot) - W_\varepsilon(\cdot - X(t))\|_{L^2(\mathbb{R})} \leq \|w^\varepsilon(s, \cdot) - W_\varepsilon(\cdot - X(s))\|_{L^2(\mathbb{R})},$$

$$(3.5) \quad \|w^\varepsilon(t, \cdot) - S_0(\cdot - X(t))\|_{L^2(\mathbb{R})} \leq \|w^\varepsilon(s, \cdot) - S_0(\cdot - X(s))\|_{L^2(\mathbb{R})} + \sqrt{2\varepsilon}.$$

- (ii) *If $u_0^\varepsilon \in \mathcal{A}$, then no L^2 -contractive shift with $X(0) = 0$ exists on any interval $[0, T]$, $T > 0$. More precisely, for every Lipschitz curve X with $X(0) = 0$, the $L^2(\mathbb{R})$ distance in (3.4) is strictly increasing on some time interval $[0, \delta]$ depending on u_0^ε , ε , and the Lipschitz constant of X . The set $\mathcal{A}$ is non-empty for every $\varepsilon > 0$.*

The remaining data satisfy

$$\mathcal{B} \setminus (\mathcal{A} \cup \mathcal{C}) = \{u \in \mathcal{B} : J[u] = K[u] = 0\}.$$

For these data, the behavior depends on the time evolution of the two functionals. This set contains the unperturbed datum S_0 , as well as non-trivial data, as shown in Section 4.2. The same criterion based on the time derivative applies to general kernels and convex entropies, under regularity hypotheses that justify the relative entropy identity, as discussed in Section 4.3. The existence result in Theorem 3.1 is local in time, and Section 4.1 shows that this cannot be improved.

We turn to shocks with positive end states. For the affine velocity (2.1) and end states as in (2.2), we assume the smallness condition

$$(3.6) \quad 0 < \theta \leq \theta_* < \frac{1}{2}$$

and we set

$$(3.7) \quad \lambda := 1 + \frac{(1 + \theta_*)(4 - 8\theta_* + \theta_*^2)}{4(1 - 2\theta_*)}.$$

The contraction result uses a lower bound for the quadratic form in (3.8). Its first term comes from the nonlocal averaging and is non-negative, whereas the second comes from the variation of the profile and is negative for test functions that are nearly constant across the transition layer, so that the sum of the first two terms need not be non-negative. This sum vanishes for $g = W^\varepsilon(\cdot - \zeta) - W^\varepsilon$, for every $\zeta \in \mathbb{R}$, because the translated profile is stationary in the moving frame. The shift in (3.9) contributes a negative multiple of the square of the integral appearing there. By the resolvent identity (1.6), this is the same squared integral as in (3.8) when the test function is the difference between the shifted average and W^ε . The smallness condition (3.6) guarantees that this correction controls the negative part of the quadratic form. The coefficient $\lambda - 1$ depends only on θ_* and tends to infinity as $\theta_* \uparrow 1/2$.

Theorem 3.2 (Coercivity inequality). *Let $\varepsilon > 0$, assume the affine velocity (2.1) and the exponential kernel (1.5), let $0 < u_+ < u_- \leq 1$ satisfy (3.6), and let λ be given by (3.7). Let S^ε be the*

profile of [Proposition 2.3](#), with any translation. Then every real-valued $g \in H^1(\mathbb{R})$ satisfies

$$(3.8) \quad \begin{aligned} & \varepsilon \int_{\mathbb{R}} |\partial_x g|^2 dx + \int_{\mathbb{R}} \left(\frac{\partial_x S^\varepsilon}{S^\varepsilon} - \frac{\varepsilon}{2} \partial_x \left(\frac{\partial_x S^\varepsilon}{S^\varepsilon} \right) \right) g^2 dx \\ & + \frac{\lambda - 1}{u_- - u_+} \left[\int_{\mathbb{R}} \left(\frac{\partial_x S^\varepsilon}{S^\varepsilon} - \varepsilon \partial_x \left(\frac{\partial_x S^\varepsilon}{S^\varepsilon} \right) \right) g dx \right]^2 \geq 0. \end{aligned}$$

The coefficient λ is independent of ε and of the translation of the profile.

[Theorem 3.2](#) is proved in [Section 5.2](#). The proof uses the change of variables $y = (W^\varepsilon - u_+)/ (u_- - u_+) \in (0, 1)$, in which all weights become explicit rational functions of y independent of ε , and then applies a weighted Poincaré inequality on $(0, 1)$ from [\[KV21, Lemma 2.9\]](#) to the difference between g and a suitable weighted mean.

The next theorem is the main result of this subsection: along a shift whose velocity is a weighted mean of the perturbation, the logarithmic relative entropy is non-increasing for all times.

Theorem 3.3 (Contraction of the logarithmic relative entropy with a shift). *Let [\(2.1\)](#), [\(2.2\)](#) and [\(3.6\)](#) hold with the exponential kernel [\(1.5\)](#), let λ be given by [\(3.7\)](#), and, for each $\varepsilon > 0$, let u^ε be the solution of [\(1.1\)](#) with*

$$u_+ \leq u_0^\varepsilon \leq u_-, \quad u_0^\varepsilon - S^\varepsilon \in L^1(\mathbb{R}),$$

where S^ε is the profile of [Proposition 2.3](#), normalized by $W^\varepsilon(0) = (u_- + u_+)/2$. The equation

$$(3.9) \quad \begin{aligned} \dot{X}^\varepsilon(t) &= -\frac{\lambda\beta}{u_- - u_+} \int_{\mathbb{R}} \frac{\partial_x S^\varepsilon}{S^\varepsilon}(x) (u^\varepsilon(t, x + \sigma t + X^\varepsilon(t)) - S^\varepsilon(x)) dx, \\ X^\varepsilon(0) &= 0 \end{aligned}$$

has a unique global solution $X^\varepsilon \in W_{\text{loc}}^{1,\infty}([0, \infty))$, which is globally Lipschitz, with $|\dot{X}^\varepsilon(t)| \leq \lambda\beta \log(u_-/u_+)$ for almost every $t \geq 0$. The shifted relative entropy has a locally absolutely continuous representative, with initial value $\int_{\mathbb{R}} h(u_0^\varepsilon | S^\varepsilon) dx$, which satisfies, for every $0 \leq s \leq t$,

$$(3.10) \quad \begin{aligned} & \int_{\mathbb{R}} h(u^\varepsilon(t, x + \sigma t + X^\varepsilon(t)) | S^\varepsilon(x)) dx \\ & + \frac{\beta}{u_- - u_+} \int_s^t \left[\int_{\mathbb{R}} \frac{\partial_x S^\varepsilon}{S^\varepsilon}(x) (u^\varepsilon(\tau, x + \sigma\tau + X^\varepsilon(\tau)) - S^\varepsilon(x)) dx \right]^2 d\tau \\ & \leq \int_{\mathbb{R}} h(u^\varepsilon(s, x + \sigma s + X^\varepsilon(s)) | S^\varepsilon(x)) dx. \end{aligned}$$

The distance estimate holds for almost every $t \geq 0$, and the bounds on the shift hold for every $t \geq 0$:

$$(3.11) \quad \|u^\varepsilon(t, \cdot + \sigma t + X^\varepsilon(t)) - S^\varepsilon\|_{L^2(\mathbb{R})}^2 \leq 2u_- \int_{\mathbb{R}} h(u_0^\varepsilon | S^\varepsilon) dx,$$

$$(3.12) \quad |X^\varepsilon(t)| \leq \lambda \left(\frac{\beta t}{u_- - u_+} \int_{\mathbb{R}} h(u_0^\varepsilon | S^\varepsilon) dx \right)^{1/2},$$

$$(3.13) \quad \int_0^\infty |\dot{X}^\varepsilon(s)|^2 ds \leq \frac{\lambda^2 \beta}{u_- - u_+} \int_{\mathbb{R}} h(u_0^\varepsilon | S^\varepsilon) dx.$$

The smallness condition enters through the coercivity inequality, whereas the shift exists for all positive end states.

Remark 3.4 (Existence of the shift for arbitrary positive end states). The existence and uniqueness of the shift hold for any end states $0 < u_+ < u_- \leq 1$ and any positive constant in place of $\lambda\beta/(u_- - u_+)$, since the right-hand side of [\(3.9\)](#) is bounded and Lipschitz in the shift variable, so that [\(3.9\)](#) has a unique global Lipschitz solution.

The estimates on X^ε control its size but not its limit as $t \rightarrow \infty$.

Remark 3.5 (Growth of the shift). The bound [\(3.13\)](#) controls $\dot{X}^\varepsilon$ in $L^2((0, \infty))$ only. Integrability of $\dot{X}^\varepsilon$ on $(0, \infty)$ would imply that $X^\varepsilon(t)$ has a finite limit as $t \rightarrow \infty$, but square integrability on $(0, \infty)$ does not imply integrability: the function $t \mapsto \log(1+t)$ has square-integrable derivative and is unbounded. Thus the only information on the growth of the shift is [\(3.12\)](#), which excludes growth faster than $\sqrt{t}$.

Combining [Theorem 3.3](#) with the estimates [\(2.11\)](#) and [\(2.12\)](#) on the distance between the profile S^ε and the local shock, we obtain quantitative bounds on the distance between the nonlocal solution and the local entropy shock, both with and without the shift X^ε .

Theorem 3.6 (Quantitative estimates). *Under the hypotheses of [Theorem 3.3](#), let X^ε be the unique solution of [\(3.9\)](#). For every $T > 0$,*

$$(3.14) \quad \operatorname{ess\,sup}_{0 \leq t \leq T} \|u^\varepsilon(t, \cdot) - S^0(\cdot - \sigma t - X^\varepsilon(t))\|_{L^2(\mathbb{R})}^2 \leq C \left(\int_{\mathbb{R}} h(u_0^\varepsilon | S^\varepsilon) dx + \varepsilon \right),$$

$$(3.15) \quad \operatorname{ess\,sup}_{0 \leq t \leq T} \|w^\varepsilon(t, \cdot) - S^0(\cdot - \sigma t - X^\varepsilon(t))\|_{L^2(\mathbb{R})}^2 \leq C \left(\int_{\mathbb{R}} h(u_0^\varepsilon | S^\varepsilon) dx + \varepsilon \right),$$

$$(3.16) \quad \begin{aligned} & \operatorname{ess\,sup}_{0 \leq t \leq T} \|u^\varepsilon(t, \cdot) - S^0(\cdot - \sigma t)\|_{L^2(\mathbb{R})}^2 \\ & \leq C \left(\int_{\mathbb{R}} h(u_0^\varepsilon | S^\varepsilon) dx + \varepsilon \right) + C \left(T \int_{\mathbb{R}} h(u_0^\varepsilon | S^\varepsilon) dx \right)^{1/2}, \end{aligned}$$

$$(3.17) \quad \begin{aligned} & \operatorname{ess\,sup}_{0 \leq t \leq T} \|w^\varepsilon(t, \cdot) - S^0(\cdot - \sigma t)\|_{L^2(\mathbb{R})}^2 \\ & \leq C \left(\int_{\mathbb{R}} h(u_0^\varepsilon | S^\varepsilon) dx + \varepsilon \right) + C \left(T \int_{\mathbb{R}} h(u_0^\varepsilon | S^\varepsilon) dx \right)^{1/2}, \end{aligned}$$

where C depends on $u_\pm$, β , and θ_* , but not on ε or T . If $\varepsilon \downarrow 0$ and $\int_{\mathbb{R}} h(u_0^\varepsilon | S^\varepsilon) dx \rightarrow 0$, then both u^ε and w^ε converge without a shift to the local entropy shock $S^0(\cdot - \sigma t)$ in $L^\infty(0, T; L^2(\mathbb{R}))$.

For Riemann initial data, the initial relative entropy $\int_{\mathbb{R}} h(u_0^\varepsilon | S^\varepsilon) dx$ is of order ε by [\(2.13\)](#), and the estimates in [Theorem 3.6](#) become explicit rates.

Corollary 3.7 (Cauchy problem for Riemann data). *Let $0 < u_+ < u_- \leq 1$, and assume [\(2.1\)](#) and [\(3.6\)](#) and the exponential kernel [\(1.5\)](#). Let $0 < \varepsilon \leq 1$, let $u_0^\varepsilon := S^0$, and let X^ε be the solution of [\(3.9\)](#). Then, for every $T > 0$,*

$$(3.18) \quad \operatorname{ess\,sup}_{0 \leq t \leq T} \|u^\varepsilon(t, \cdot) - S^0(\cdot - \sigma t - X^\varepsilon(t))\|_{L^2(\mathbb{R})} \leq C\sqrt{\varepsilon},$$

$$(3.19) \quad \sup_{0 \leq t \leq T} |X^\varepsilon(t)| \leq C\sqrt{\varepsilon T},$$

$$(3.20) \quad \operatorname{ess\,sup}_{0 \leq t \leq T} \|u^\varepsilon(t, \cdot) - S^0(\cdot - \sigma t)\|_{L^2(\mathbb{R})} \leq C(1 + T^{1/4})\varepsilon^{1/4}.$$

Estimates [\(3.18\)](#) and [\(3.20\)](#) hold with w^ε in place of u^ε , and [\(3.19\)](#) is unchanged. Here C depends only on $u_\pm$, β , and θ_* , and is independent of ε and T .

3.2. Lipschitz solutions. For the relative entropy estimates with respect to Lipschitz solutions of the local equation, we allow a general V with $V' \geq \nu > 0$ for the exponential kernel and $V' \geq 0$ for general kernels. For the exponential kernel, the resolvent identity [\(1.6\)](#) yields an exact entropy balance for the density u^ε itself, and we estimate u^ε . For a general kernel satisfying [\(1.3\)](#), we use the entropy balance for the average w^ε and estimate w^ε . We consider the nonlocal conservation law

$$(3.21) \quad \partial_t u^\varepsilon + \partial_x(u^\varepsilon V(w^\varepsilon)) = 0, \quad w^\varepsilon = \gamma_\varepsilon * u^\varepsilon$$

for $\varepsilon > 0$ and $(t, x) \in (0, \infty) \times \mathbb{R}$, with initial condition $u^\varepsilon(0, \cdot) = u_0^\varepsilon$. The monotonicity and regularity of V are specified in each result.

As $\varepsilon \downarrow 0$, convolution with γ_ε converges to the identity, and the local equation corresponding to [\(3.21\)](#) is

$$(3.22) \quad \partial_t u + \partial_x f(u) = 0, \quad f(u) = uV(u).$$

The next theorem concerns the exponential kernel and controls the density u^ε itself.

Theorem 3.8 (Stability on the real line near a Lipschitz solution). *Let $T, M > 0$. We assume*

$$(3.23) \quad V \in C^2([0, M]), \quad V'(\xi) \geq \nu > 0 \quad \text{for } \xi \in [0, M],$$

and let $\bar{u} \in W^{1,\infty}((0, T) \times \mathbb{R})$ be a solution of [\(3.22\)](#), extended continuously to $t = 0$, such that

$$\partial_x \bar{u} \in L^2((0, T) \times \mathbb{R}), \quad 0 \leq \bar{u} \leq M.$$

For each $\varepsilon > 0$, let u^ε be the solution of (3.21) with the exponential kernel and initial datum u_0^ε satisfying $0 \leq u_0^\varepsilon \leq M$ and $u_0^\varepsilon - \bar{u}(0, \cdot) \in L^2(\mathbb{R})$. Then, for almost every $t \in [0, T]$,

$$(3.24) \quad \begin{aligned} & \frac{1}{2} \|u^\varepsilon(t, \cdot) - \bar{u}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{\nu}{4\varepsilon} \int_0^t \int_{\mathbb{R}} (u^\varepsilon + w^\varepsilon) |w^\varepsilon - u^\varepsilon|^2 dx ds \\ & \leq \exp \left(\|f''\|_{L^\infty([0, M])} \int_0^t \|\partial_x \bar{u}(s, \cdot)\|_{L^\infty(\mathbb{R})} ds \right) \\ & \quad \times \left[\frac{1}{2} \|u_0^\varepsilon - \bar{u}(0, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{M \|V'\|_{L^\infty([0, M])}^2}{\nu} \varepsilon \int_0^t \|\partial_x \bar{u}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \right]. \end{aligned}$$

For almost every t ,

$$(3.25) \quad \|w^\varepsilon(t, \cdot) - \bar{u}(t, \cdot)\|_{L^2(\mathbb{R})} \leq \|u^\varepsilon(t, \cdot) - \bar{u}(t, \cdot)\|_{L^2(\mathbb{R})} + \varepsilon \|\partial_x \bar{u}(t, \cdot)\|_{L^2(\mathbb{R})}.$$

In particular, for identical initial data, both u^ε and w^ε converge to $\bar{u}$ in $L^\infty(0, T; L^2(\mathbb{R}))$ as $\varepsilon \downarrow 0$ at rate $O(\sqrt{\varepsilon})$, with a constant independent of ε , provided $\partial_x \bar{u} \in L^\infty(0, T; L^2(\mathbb{R}))$.

The estimate on $\mathbb{R}$ requires the gradient of the reference solution to be square integrable on $\mathbb{R}$, which excludes, for instance, non-constant spatially periodic Lipschitz solutions. Working with an integrable weight instead, we obtain the same rate locally in space without this restriction.

Corollary 3.9 (Stability near a Lipschitz solution on compact intervals). *Let $T, M > 0$. We assume (3.23) for some $\nu > 0$, and we let $\bar{u} \in W^{1, \infty}((0, T) \times \mathbb{R})$ solve (3.22), extended continuously to $t = 0$, with $0 \leq \bar{u} \leq M$. For every $\varepsilon > 0$, let u^ε be the solution of (3.21) with the exponential kernel and initial datum $\bar{u}(0, \cdot)$. Then every compact interval $I \subset \mathbb{R}$ satisfies*

$$(3.26) \quad \operatorname{ess\,sup}_{0 \leq t \leq T} (\|u^\varepsilon(t, \cdot) - \bar{u}(t, \cdot)\|_{L^2(I)} + \|w^\varepsilon(t, \cdot) - \bar{u}(t, \cdot)\|_{L^2(I)}) \leq C_{I, T, M, \nu, \bar{u}, V} \sqrt{\varepsilon}.$$

For a general kernel, we assume (1.3) and take the right-continuous representative of γ on $(0, \infty)$. Since γ is bounded, non-negative, and non-increasing on $(0, \infty)$, its total variation on $\mathbb{R}$ is at most $2\|\gamma\|_{L^\infty(\mathbb{R})}$, so that $\gamma \in \operatorname{BV}(\mathbb{R})$. We denote by

$$(3.27) \quad \mu_\gamma := -D(\gamma|_{(0, \infty)})$$

the (non-negative) measure obtained as the negative of the distributional derivative of γ on $(0, \infty)$. We impose the quantitative decay condition

$$(3.28) \quad \mu_\gamma \geq \kappa \gamma(z) dz \quad \text{on } (0, \infty)$$

for some $\kappa > 0$. Equivalently, the measure $\gamma(z) dz$ is absolutely continuous with respect to μ_γ , with

$$\frac{d(\gamma(z) dz)}{d\mu_\gamma} \leq \frac{1}{\kappa} \quad \mu_\gamma\text{-almost everywhere.}$$

Absolute continuity alone does not give this bound. For example, for $\gamma(z) = (1+z)^{-2}$ on $(0, \infty)$ and zero elsewhere,

$$d\mu_\gamma(z) = 2(1+z)^{-3} dz, \quad \frac{d(\gamma(z) dz)}{d\mu_\gamma}(z) = \frac{1+z}{2},$$

so no $\kappa > 0$ satisfies (3.28). On the other hand, (3.28) allows singular parts of μ_γ , including atoms at downward jumps of γ . The corresponding condition for kernels supported on the negative half-line appears in [Col+21, Assumption 2]. For $\varepsilon > 0$, we set

$$(3.29) \quad \begin{aligned} \gamma_\varepsilon(z) &= \frac{1}{\varepsilon} \gamma\left(\frac{z}{\varepsilon}\right), & \mu_{\gamma, \varepsilon} &:= \frac{1}{\varepsilon} (z \mapsto \varepsilon z)_\# \mu_\gamma, \\ w^\varepsilon(t, x) &= (\gamma_\varepsilon * u^\varepsilon)(t, x) = \int_{(0, \infty)} \gamma_\varepsilon(z) u^\varepsilon(t, x - z) dz. \end{aligned}$$

Then $\mu_{\gamma, \varepsilon} \geq (\kappa/\varepsilon) \gamma_\varepsilon(z) dz$ on $(0, \infty)$. For a non-decreasing V , we define

$$(3.30) \quad H(v | r) := \int_v^r (V(r) - V(\vartheta)) d\vartheta, \quad v, r \in [0, M].$$

Since V is non-decreasing, $H(v \mid r) \geq 0$ for all $v, r \in [0, M]$. For non-negative u^ε , the entropy balance for the average contains the non-negative term

$$(3.31) \quad \int_{(0,\infty)} u^\varepsilon(t, x-z) H(w^\varepsilon(t, x) \mid w^\varepsilon(t, x-z)) d\mu_{\gamma,\varepsilon}(z),$$

which gives dissipation in the estimate for w^ε . The corresponding term for u^ε in [Theorem 3.8](#) is $\frac{\nu}{4\varepsilon}(u^\varepsilon + w^\varepsilon)(w^\varepsilon - u^\varepsilon)^2$. For the exponential kernel, $\mu_{\gamma,\varepsilon} = \varepsilon^{-1}\gamma_\varepsilon(z) dz$, so that [\(3.28\)](#) holds with equality and $\kappa = 1$.

The next theorem is the analogue of [Theorem 3.8](#) for the average.

Theorem 3.10 (Quantitative stability of the averaged density). *Let $T, M > 0$. We assume [\(1.3\)](#) and [\(3.28\)](#) and*

$$V \in C^2([0, M]), \quad V'(\xi) \geq 0 \quad \text{for } \xi \in [0, M].$$

Let $\bar{u} \in W^{1,\infty}((0, T) \times \mathbb{R})$ solve [\(3.22\)](#), extended continuously to $t = 0$, with $0 \leq \bar{u} \leq M$. For each $\varepsilon > 0$, let u^ε be the solution of [\(3.21\)](#) with $0 \leq u_0^\varepsilon \leq M$.

(i) If

$$(3.32) \quad \partial_x \bar{u} \in L^2((0, T) \times \mathbb{R}), \quad w^\varepsilon(0, \cdot) - \bar{u}(0, \cdot) \in L^2(\mathbb{R}),$$

then, for almost every $t \in [0, T]$,

$$(3.33) \quad \begin{aligned} & \frac{1}{2} \|w^\varepsilon(t, \cdot) - \bar{u}(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ & + \frac{1}{2} \int_0^t \int_{\mathbb{R}} \int_{(0,\infty)} u^\varepsilon(s, x-z) H(w^\varepsilon(s, x) \mid w^\varepsilon(s, x-z)) d\mu_{\gamma,\varepsilon}(z) dx ds \\ & \leq \exp\left(\|f''\|_{L^\infty([0,M])} \int_0^t \|\partial_x \bar{u}(s, \cdot)\|_{L^\infty(\mathbb{R})} ds\right) \\ & \quad \times \left[\frac{1}{2} \|w^\varepsilon(0, \cdot) - \bar{u}(0, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{M\|V'\|_{L^\infty([0,M])}}{\kappa} \varepsilon \int_0^t \|\partial_x \bar{u}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \right]. \end{aligned}$$

(ii) If $u_0^\varepsilon = \bar{u}(0, \cdot)$, then every compact interval $I \subset \mathbb{R}$ satisfies

$$(3.34) \quad \operatorname{ess\,sup}_{0 \leq t \leq T} \|w^\varepsilon(t, \cdot) - \bar{u}(t, \cdot)\|_{L^2(I)} \leq C_{I,T,M,\kappa,\bar{u},V,\gamma} \sqrt{\varepsilon}.$$

Here $C_{I,T,M,\kappa,\bar{u},V,\gamma}$ is independent of ε .

For identical initial data, the initial error in [\(3.33\)](#) is of order ε .

Remark 3.11 (Identical initial data). Let $u_0^\varepsilon = \bar{u}(0, \cdot)$. If $\partial_x \bar{u}(0, \cdot) \in L^2(\mathbb{R})$, then, by Step 3 of the proof of [Theorem 3.10](#),

$$(3.35) \quad \|w^\varepsilon(0, \cdot) - \bar{u}(0, \cdot)\|_{L^2(\mathbb{R})} \leq \varepsilon \left(\int_0^\infty z\gamma(z) dz \right) \|\partial_x \bar{u}(0, \cdot)\|_{L^2(\mathbb{R})}.$$

The first moment is finite, since the product rule for functions of bounded variation and [\(3.28\)](#) give

$$\begin{aligned} D(e^{\kappa z} \gamma(z)) &= e^{\kappa z} (-\mu_\gamma + \kappa \gamma(z) dz) \leq 0 \quad \text{on } (0, \infty), \\ \gamma(z) &\leq \gamma(0+) e^{-\kappa z}, \quad \int_0^\infty z\gamma(z) dz \leq \frac{\gamma(0+)}{\kappa^2} < \infty. \end{aligned}$$

If $\partial_x \bar{u} \in L^\infty(0, T; L^2(\mathbb{R}))$, then $\partial_x \bar{u}(0, \cdot) \in L^2(\mathbb{R})$. We choose times $t_n \downarrow 0$ at which $\|\partial_x \bar{u}(t_n, \cdot)\|_{L^2(\mathbb{R})} \leq \|\partial_x \bar{u}\|_{L^\infty(0,T;L^2(\mathbb{R}))}$. By weak compactness, a subsequence of $\partial_x \bar{u}(t_n, \cdot)$ converges weakly in $L^2(\mathbb{R})$, and the limit is $\partial_x \bar{u}(0, \cdot)$ because $\bar{u}(t_n, \cdot) \rightarrow \bar{u}(0, \cdot)$ locally uniformly. Weak lower semicontinuity of the norm gives $\|\partial_x \bar{u}(0, \cdot)\|_{L^2(\mathbb{R})} \leq \|\partial_x \bar{u}\|_{L^\infty(0,T;L^2(\mathbb{R}))}$.

Under this additional hypothesis, [\(3.33\)](#) and [\(3.35\)](#) give

$$\|w^\varepsilon - \bar{u}\|_{L^\infty(0,T;L^2(\mathbb{R}))} \leq C\sqrt{\varepsilon}, \quad 0 < \varepsilon \leq 1,$$

where C is independent of ε .

4. PROOFS OF THE VACUUM OBSTRUCTION RESULTS

In this section, we prove [Theorem 3.1](#). The time derivative of half the squared $L^2(\mathbb{R})$ distance to the shifted profile is affine in the velocity of the shift, with coefficients J_ε and K_ε that reduce to J and K at $t = 0$ and shift zero, and the two parts of the theorem follow. The non-emptiness of $\mathcal{A}$ requires an explicit computation, for which we use the symmetric kernel $\gamma_\varepsilon^\vee * \gamma_\varepsilon$. We then show that these shifts need not exist for all times ([Section 4.1](#)), discuss the case $J = K = 0$ ([Section 4.2](#)), and extend the argument to other kernels and entropies ([Section 4.3](#)).

Proof of Theorem 3.1. Step 1: Regularity at fixed ε . For smooth solutions, differentiation of (1.1) and the identity (1.6) give

$$\begin{aligned} \partial_{tx} u^\varepsilon + \partial_x(w^\varepsilon \partial_x u^\varepsilon) &= - \left(\partial_x w^\varepsilon + \frac{u^\varepsilon}{\varepsilon} \right) \partial_x u^\varepsilon + \frac{u^\varepsilon}{\varepsilon} \partial_x w^\varepsilon, \\ \frac{d}{dt} \|\partial_x u^\varepsilon(t, \cdot)\|_{L^1(\mathbb{R})} &\leq \frac{3}{\varepsilon} \|\partial_x u^\varepsilon(t, \cdot)\|_{L^1(\mathbb{R})}. \end{aligned}$$

Here $\|\partial_x w^\varepsilon\|_{L^\infty(\mathbb{R})} \leq \varepsilon^{-1}$ and $\|\partial_x w^\varepsilon\|_{L^1(\mathbb{R})} \leq \|\partial_x u^\varepsilon\|_{L^1(\mathbb{R})}$. For mollified initial data, the solution is Lipschitz on bounded time intervals for fixed ε [[Col+23](#), Proposition 2.1(iii)], so that differentiation of (1.1) is justified. The solutions with mollified data converge to u^ε in $C([0, T]; L^1_{\text{loc}}(\mathbb{R}))$ by the L^1 stability with respect to the initial datum of the fixed- ε problem [[Coc+23](#), Theorem 3.1], and lower semicontinuity of the total variation yields

$$\text{TV}(u^\varepsilon(t, \cdot)) \leq e^{3t/\varepsilon} \text{TV}(u_0^\varepsilon).$$

Since $\text{TV}(w^\varepsilon u^\varepsilon) \leq 2 \text{TV}(u^\varepsilon)$, integration of (1.1) against bounded smooth test functions, followed by the dual characterization of the $L^1(\mathbb{R})$ norm, gives

$$(4.1) \quad \|u^\varepsilon(t, \cdot) - u^\varepsilon(s, \cdot)\|_{L^1(\mathbb{R})} \leq 2e^{3T/\varepsilon} \text{TV}(u_0^\varepsilon) |t - s|, \quad 0 \leq s, t \leq T.$$

Fatou's lemma allows us to pass (4.1) from smooth approximations to the $C([0, T]; L^1_{\text{loc}}(\mathbb{R}))$ representative. In particular, $u^\varepsilon(t, \cdot) - S_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ for every $t \geq 0$.

By [Lemma 2.2](#), $w^\varepsilon(t, \cdot) - W_\varepsilon \in W^{1,1}(\mathbb{R}) \cap H^1(\mathbb{R})$. Moreover,

$$\begin{aligned} \partial_t w^\varepsilon &= -\gamma_\varepsilon * \partial_x(w^\varepsilon u^\varepsilon), \\ \|\partial_t w^\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} &\leq 2\|\gamma_\varepsilon\|_{L^2(\mathbb{R})} \text{TV}(u^\varepsilon(t, \cdot)). \end{aligned}$$

Thus $w^\varepsilon - W_\varepsilon \in W^{1,\infty}([0, T]; L^2(\mathbb{R}))$. All constants in this step concern fixed $\varepsilon > 0$ and finite T .

Step 2: Squared distance to the shifted profile. We define

$$\begin{aligned} J_\varepsilon(t, \zeta) &:= \int_{\mathbb{R}} (w^\varepsilon(t, x) - W_\varepsilon(x - \zeta)) \partial_x W_\varepsilon(x - \zeta) dx, \\ K_\varepsilon(t, \zeta) &:= \int_{\mathbb{R}} [\gamma_\varepsilon * (w^\varepsilon u^\varepsilon)](t, x) \partial_x (w^\varepsilon(t, x) - W_\varepsilon(x - \zeta)) dx. \end{aligned}$$

These integrals are finite, since $w^\varepsilon(t, \cdot) - W_\varepsilon \in L^2(\mathbb{R})$ by [Lemma 2.2](#), $\partial_x W_\varepsilon \in L^2(\mathbb{R})$, and $\gamma_\varepsilon * (w^\varepsilon u^\varepsilon)$ is bounded while $\partial_x(w^\varepsilon - W_\varepsilon(\cdot - \zeta)) \in L^1(\mathbb{R})$. For a Lipschitz curve X , the map $t \mapsto W_\varepsilon(\cdot - X(t)) - W_\varepsilon$ is Lipschitz with values in $L^2(\mathbb{R})$, with derivative $-\dot{X}(t) \partial_x W_\varepsilon(\cdot - X(t))$ for almost every t , because $\|W_\varepsilon(\cdot - \zeta_1) - W_\varepsilon(\cdot - \zeta_2)\|_{L^2(\mathbb{R})} \leq |\zeta_1 - \zeta_2| \|\partial_x W_\varepsilon\|_{L^2(\mathbb{R})}$. Together with $w^\varepsilon - W_\varepsilon \in W^{1,\infty}([0, T]; L^2(\mathbb{R}))$, the chain rule in $L^2(\mathbb{R})$ and integration by parts give, for almost every t ,

$$\begin{aligned} (4.2) \quad & \frac{d}{dt} \frac{1}{2} \|w^\varepsilon(t, \cdot) - W_\varepsilon(\cdot - X(t))\|_{L^2(\mathbb{R})}^2 \\ &= \int_{\mathbb{R}} (w^\varepsilon - W_\varepsilon(\cdot - X(t))) (\partial_t w^\varepsilon + \dot{X}(t) \partial_x W_\varepsilon(\cdot - X(t))) dx \\ &= \dot{X}(t) J_\varepsilon(t, X(t)) + K_\varepsilon(t, X(t)). \end{aligned}$$

For the integration by parts, $w^\varepsilon - W_\varepsilon(\cdot - X(t)) \in W^{1,1}(\mathbb{R})$ and $\gamma_\varepsilon * (w^\varepsilon u^\varepsilon)$ is bounded. Multiplying by cutoffs with derivatives bounded by C/R makes the boundary term tend to zero as $R \rightarrow \infty$. By (4.1) and Young's inequality, J_ε and K_ε are continuous. Let X be a shift with $X(0) = 0$ that is differentiable at $t = 0$. Since $\dot{X}$ is bounded, $X(t)/t \rightarrow \dot{X}(0)$, and $J_\varepsilon(s, X(s)) \rightarrow J_\varepsilon(0, 0)$ as $s \downarrow 0$,

the integrated form of (4.2) gives the right derivative $\dot{X}(0)J_\varepsilon(0,0) + K_\varepsilon(0,0)$ at $t = 0$, which is (3.3).

The explicit profile gives

$$(4.3) \quad \begin{aligned} J_\varepsilon(t, \zeta) &= \frac{1}{2} - \frac{1}{\varepsilon} \int_\zeta^\infty w^\varepsilon(t, x) e^{(\zeta-x)/\varepsilon} dx, \\ \partial_\zeta J_\varepsilon(t, \zeta) &= \frac{1}{\varepsilon} \left(J_\varepsilon(t, \zeta) + w^\varepsilon(t, \zeta) - \frac{1}{2} \right). \end{aligned}$$

Hence J_ε is continuously differentiable in ζ , and $\partial_\zeta J_\varepsilon$ is jointly continuous in (t, ζ) , since w^ε is Lipschitz in x with constant $1/\varepsilon$ and, by (4.1) and $\|\gamma_\varepsilon\|_{L^\infty(\mathbb{R})} = \varepsilon^{-1}$, locally Lipschitz in t with values in $L^\infty(\mathbb{R})$. The ζ -dependent part of K_ε is

$$\frac{1}{\varepsilon} \int_\zeta^\infty [\gamma_\varepsilon * (w^\varepsilon u^\varepsilon)](t, x) e^{(\zeta-x)/\varepsilon} dx,$$

which is also continuously differentiable in ζ . By (4.1) and Young's inequality, w^ε and $\gamma_\varepsilon * (w^\varepsilon u^\varepsilon)$ are locally Lipschitz in time with values in $L^\infty(\mathbb{R})$, while $\partial_x w^\varepsilon$ is locally Lipschitz in time with values in $L^1(\mathbb{R})$. Thus J_ε and K_ε are locally Lipschitz in (t, ζ) .

Fubini's theorem, applied to the integrable perturbation $u_0^\varepsilon - S_0$, gives

$$\begin{aligned} J_\varepsilon(0, 0) &= \int_{\mathbb{R}} (u_0^\varepsilon - S_0)(\gamma_\varepsilon^\vee * \partial_x W_\varepsilon) dx = J[u_0^\varepsilon], \\ K_\varepsilon(0, 0) &= K[u_0^\varepsilon]. \end{aligned}$$

Step 3: Local existence and obstruction. We prove both parts of the theorem at an arbitrary point $(t_0, \zeta_0) \in [0, \infty) \times \mathbb{R}$, the theorem being the case $(t_0, \zeta_0) = (0, 0)$. If $K_\varepsilon(t_0, \zeta_0) < 0$, continuity gives $\delta > 0$ with $K_\varepsilon(t, \zeta_0) < 0$ for $t_0 \leq t \leq t_0 + \delta$, and (4.2), together with the absolute continuity of the squared distance, shows that $X \equiv \zeta_0$ is contractive on $[t_0, t_0 + \delta]$. If $J_\varepsilon(t_0, \zeta_0) \neq 0$, the Picard–Lindelöf theorem gives a local solution of

$$(4.4) \quad \dot{X}(t) = -\frac{K_\varepsilon(t, X(t))}{J_\varepsilon(t, X(t))}, \quad X(t_0) = \zeta_0,$$

which is Lipschitz on a closed interval $[t_0, t_0 + \delta]$. The squared distance in (4.2) is constant along this curve. For $(t_0, \zeta_0) = (0, 0)$, this proves Theorem 3.1(i).

For the obstruction, let (t_0, ζ_0) satisfy $J_\varepsilon(t_0, \zeta_0) = 0$ and $K_\varepsilon(t_0, \zeta_0) = k_0 > 0$, and let X be a Lipschitz curve with $X(t_0) = \zeta_0$ and Lipschitz constant L . Continuity gives a neighborhood of (t_0, ζ_0) on which

$$K_\varepsilon(t, \zeta) \geq \frac{3k_0}{4}, \quad |J_\varepsilon(t, \zeta)| \leq \frac{k_0}{4 \max\{L, 1\}}.$$

For sufficiently small $\delta > 0$, $(t, X(t))$ stays in this neighborhood for $t_0 \leq t \leq t_0 + \delta$. By (4.2),

$$(4.5) \quad \begin{aligned} &\frac{1}{2} \|w^\varepsilon(t, \cdot) - W_\varepsilon(\cdot - X(t))\|_{L^2(\mathbb{R})}^2 - \frac{1}{2} \|w^\varepsilon(s, \cdot) - W_\varepsilon(\cdot - X(s))\|_{L^2(\mathbb{R})}^2 \\ &\geq \frac{k_0}{2} (t - s) > 0, \quad t_0 \leq s < t \leq t_0 + \delta. \end{aligned}$$

Thus no contractive shift passes through a point (t_0, ζ_0) at which $J_\varepsilon = 0$ and $K_\varepsilon > 0$. For $(t_0, \zeta_0) = (0, 0)$, this proves Theorem 3.1(ii) for every Lipschitz shift, including curves without a derivative at $t = 0$.

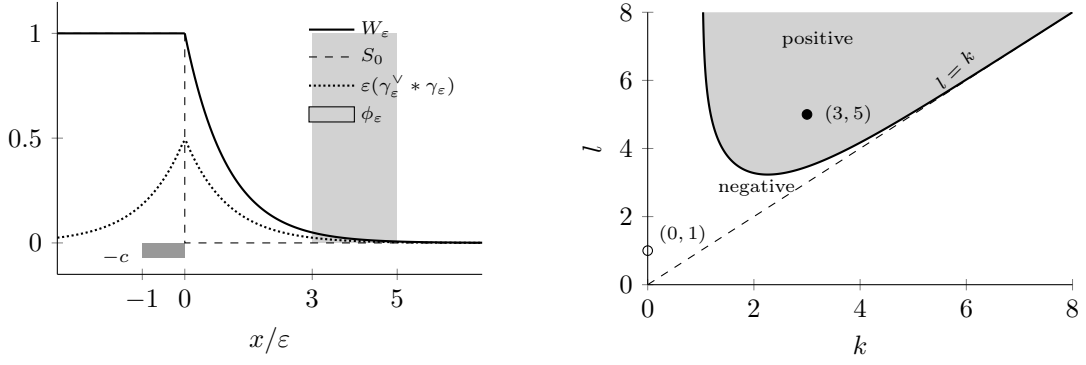
For every contractive shift and $0 \leq s < t \leq T$, (3.2) and the triangle inequality give

$$\begin{aligned} \|w^\varepsilon(t, \cdot) - S_0(\cdot - X(t))\|_{L^2(\mathbb{R})} &\leq \|w^\varepsilon(t, \cdot) - W_\varepsilon(\cdot - X(t))\|_{L^2(\mathbb{R})} + \sqrt{\varepsilon/2} \\ &\leq \|w^\varepsilon(s, \cdot) - W_\varepsilon(\cdot - X(s))\|_{L^2(\mathbb{R})} + \sqrt{\varepsilon/2} \\ &\leq \|w^\varepsilon(s, \cdot) - S_0(\cdot - X(s))\|_{L^2(\mathbb{R})} + \sqrt{2\varepsilon}. \end{aligned}$$

Thus (3.5) holds.

Step 4: A perturbation with no contractive shift starting at zero. The convolution identities

$$(4.6) \quad (\gamma_\varepsilon^\vee * \gamma_\varepsilon)(x) = \frac{1}{2\varepsilon} e^{-|x|/\varepsilon}, \quad (\gamma_\varepsilon^\vee * \partial_x W_\varepsilon)(x) = -(\gamma_\varepsilon^\vee * \gamma_\varepsilon)(x)$$



(A) The perturbation $\phi_\varepsilon = \mathbb{1}_{[3\varepsilon, 5\varepsilon]} - c\mathbb{1}_{(-\varepsilon, 0)}$ (shaded), the averaged profile W_ε , the step S_0 , and the rescaled symmetric kernel $\varepsilon(\gamma_\varepsilon^\vee * \gamma_\varepsilon)$. The integrals of the two terms defining ϕ_ε against $\gamma_\varepsilon^\vee * \gamma_\varepsilon$ sum to zero, so that $J[u_0^\varepsilon] = 0$ ((4.8)).

(B) Sign of the quadratic term in (4.11) in the (k, l) -plane. For $l > k$, it is positive in the shaded region and negative in the unshaded region, except on the solid curve, where it vanishes. In particular, it is negative for every $l > k$ when $k \leq k_* \simeq 1.0302$. The filled dot is the choice in (4.7). The open dot is the symmetric choice $k = 0, l = 1$.

FIGURE 1. The perturbation used in Step 4 of the proof of Theorem 3.1, and the sign of the quadratic term for different choices of its support.

follow by computing $\int_{\mathbb{R}} \gamma_\varepsilon(y - x) \gamma_\varepsilon(y) dy$, whose integrand is supported in $(\max\{x, 0\}, \infty)$, and from $\partial_x W_\varepsilon = -\gamma_\varepsilon$. We choose

$$(4.7) \quad \begin{aligned} \phi_\varepsilon(x) &:= \mathbb{1}_{[3\varepsilon, 5\varepsilon]}(x) - c\mathbb{1}_{(-\varepsilon, 0)}(x), \quad c := \frac{e^{-3} - e^{-5}}{1 - e^{-1}}, \\ u_0^\varepsilon(x) &:= S_0(x) + \alpha\phi_\varepsilon(x), \quad 0 < \alpha \leq 1. \end{aligned}$$

Since $0 < c < 1$, this datum belongs to $\mathcal{B}$ (Fig. 1a). Moreover, by (4.6),

$$(4.8) \quad J[u_0^\varepsilon] = -\alpha \int_{\mathbb{R}} (\gamma_\varepsilon^\vee * \gamma_\varepsilon) \phi_\varepsilon dx = -\frac{\alpha}{2} (e^{-3} - e^{-5} - c(1 - e^{-1})) = 0.$$

Expanding $w_0^\varepsilon u_0^\varepsilon$ and using that the linear term $\alpha(\gamma_\varepsilon^\vee * \gamma_\varepsilon * \phi_\varepsilon)(0)$ vanishes by (4.8), Fubini's theorem and (4.6) give

$$(4.9) \quad \begin{aligned} K[u_0^\varepsilon] &= \int_{\mathbb{R}} w_0^\varepsilon u_0^\varepsilon \partial_x (\gamma_\varepsilon^\vee * \gamma_\varepsilon * (u_0^\varepsilon - S_0)) dx \\ &= \alpha^2 \int_{-\infty}^0 (\gamma_\varepsilon * \phi_\varepsilon) \partial_x (\gamma_\varepsilon^\vee * \gamma_\varepsilon * \phi_\varepsilon) dx \\ &\quad + \alpha^2 \int_{\mathbb{R}} W_\varepsilon \phi_\varepsilon \partial_x (\gamma_\varepsilon^\vee * \gamma_\varepsilon * \phi_\varepsilon) dx \\ &\quad + \alpha^3 \int_{\mathbb{R}} (\gamma_\varepsilon * \phi_\varepsilon) \phi_\varepsilon \partial_x (\gamma_\varepsilon^\vee * \gamma_\varepsilon * \phi_\varepsilon) dx. \end{aligned}$$

For $z_1 < z_2$, the formulas

$$(4.10) \quad \begin{aligned} (\gamma_\varepsilon * \mathbb{1}_{[z_1, z_2]})(x) &= \begin{cases} 0, & x < z_1, \\ 1 - e^{(z_1 - x)/\varepsilon}, & z_1 \leq x \leq z_2, \\ e^{(z_2 - x)/\varepsilon} - e^{(z_1 - x)/\varepsilon}, & x > z_2, \end{cases} \\ \partial_x (\gamma_\varepsilon^\vee * \gamma_\varepsilon * \mathbb{1}_{[z_1, z_2]})(x) &= \frac{1}{2\varepsilon} \begin{cases} e^{(x - z_1)/\varepsilon} - e^{(x - z_2)/\varepsilon}, & x < z_1, \\ e^{(z_1 - x)/\varepsilon} - e^{(x - z_2)/\varepsilon}, & z_1 \leq x \leq z_2, \\ e^{(z_1 - x)/\varepsilon} - e^{(z_2 - x)/\varepsilon}, & x > z_2 \end{cases} \end{aligned}$$

allow us to compute the quadratic term explicitly. We do so for the whole family of two-bump perturbations

$$\phi_{\varepsilon,k,l} := \mathbf{1}_{[k\varepsilon,l\varepsilon]} - c(k,l)\mathbf{1}_{(-\varepsilon,0)}, \quad c(k,l) := \frac{e^{-k} - e^{-l}}{1 - e^{-1}}, \quad 0 \leq k < l.$$

The datum $S_0 + \alpha\phi_{\varepsilon,k,l}$ takes values in $[0,1]$ when $c(k,l) \leq 1$, that is, when $e^{-k} - e^{-l} \leq 1 - e^{-1}$, which holds for every $k \geq 1$ and for $k = 0, l = 1$, where $c(0,1) = 1$. The choice $\phi_\varepsilon = \phi_{\varepsilon,3,5}$, $c = c(3,5)$, is the one in (4.7). With $x = \varepsilon\varsigma$, the explicit convolutions give

$$\begin{aligned} & \int_{-\infty}^0 (\gamma_\varepsilon * \phi_{\varepsilon,k,l}) \partial_x (\gamma_\varepsilon^\vee * \gamma_\varepsilon * \phi_{\varepsilon,k,l}) dx \\ &= -\frac{c(k,l)}{2} \int_{-1}^0 (1 - e^{-1-\varsigma})(e^{\varsigma-k} - e^{\varsigma-l}) d\varsigma + \frac{c(k,l)^2}{2} \int_{-1}^0 (1 - e^{-1-\varsigma})(e^{-1-\varsigma} - e^\varsigma) d\varsigma \\ &= -\frac{c(k,l)}{2} (e^{-k} - e^{-l})(1 - 2e^{-1}) - \frac{c(k,l)^2}{4} (1 - 2e^{-1} - e^{-2}), \\ & \int_{\mathbb{R}} W_\varepsilon \phi_{\varepsilon,k,l} \partial_x (\gamma_\varepsilon^\vee * \gamma_\varepsilon * \phi_{\varepsilon,k,l}) dx \\ &= -\frac{c(k,l)}{2} \int_{-1}^0 (e^{\varsigma-k} - e^{\varsigma-l}) d\varsigma + \frac{c(k,l)^2}{2} \int_{-1}^0 (e^{-1-\varsigma} - e^\varsigma) d\varsigma \\ &\quad + \frac{1}{2} \int_k^l e^{-\varsigma} (e^{k-\varsigma} - e^{\varsigma-l}) d\varsigma - \frac{c(k,l)}{2} \int_k^l e^{-\varsigma} (e^{-1-\varsigma} - e^{-\varsigma}) d\varsigma \\ &= -\frac{c(k,l)}{2} (e^{-k} - e^{-l})(1 - e^{-1}) + \frac{e^{-k} - e^{k-2l}}{4} - \frac{l-k}{2} e^{-l} + \frac{(e^{-k} - e^{-l})(e^{-2k} - e^{-2l})}{4}, \end{aligned}$$

where $\int_{-1}^0 (e^{-1-\varsigma} - e^\varsigma) d\varsigma = 0$. We set

$$\begin{aligned} c_* &:= \frac{5 - 12e^{-1} + 5e^{-2}}{4(1 - e^{-1})^2} \in (1/2, 1), \\ k_* &:= -\log(2c_* - \sqrt{4c_*^2 - 1}) \simeq 1.0302. \end{aligned}$$

The bound $c_* < 1$ holds because $e^{-1} \in (1/3, 3/8)$ gives $5 - 12e^{-1} + 5e^{-2} > 5 - 9/2 > 0$ and $4(1 - e^{-1})^2 - (5 - 12e^{-1} + 5e^{-2}) = -(1 - 4e^{-1} + e^{-2}) > 0$, since $e^{-1} > 2 - \sqrt{3}$. The bound $c_* > 1/2$ holds because $5 - 12e^{-1} + 5e^{-2} - 2(1 - e^{-1})^2 = 3 - 8e^{-1} + 3e^{-2} > 3e^{-2} > 0$ for $e^{-1} < 3/8$. Since $c(k,l)(1 - e^{-1}) = e^{-k} - e^{-l}$, the sum of the two constant terms simplifies, and

$$(4.11) \quad \frac{1}{2} \partial_\alpha^2 K[S_0 + \alpha\phi_{\varepsilon,k,l}]|_{\alpha=0} = \frac{e^{-k} - e^{k-2l}}{4} - \frac{l-k}{2} e^{-l} + \frac{(e^{-k} - e^{-l})(e^{-2k} - e^{-2l})}{4} - c_*(e^{-k} - e^{-l})^2.$$

The sign of (4.11) depends on the placement of the bumps, as shown in Fig. 1b. The quantity (4.11) is negative for every $l > k$ when $k \leq k_*$, which we prove by an explicit estimate. For $\varrho = e^{-k}$ and $\varpi = e^{-(l-k)} \in (0, 1)$, the coefficient satisfies

$$(4.12) \quad \begin{aligned} \frac{\partial_\alpha^2 K[S_0 + \alpha\phi_{\varepsilon,k,l}]|_{\alpha=0}}{2\varrho(1-\varpi)^2} &= \frac{1 - \varpi^2 + 2\varpi \log \varpi}{4(1-\varpi)^2} + \frac{\varrho^2(1+\varpi)}{4} - c_*\varrho \\ &\leq \frac{1 + \varrho^2 - 4c_*\varrho + \varpi(\varrho^2 - 1)}{4}. \end{aligned}$$

Here $2\log \varpi \leq -3 + 4\varpi - \varpi^2$, since the difference of the left and right sides has derivative $2(1 - \varpi)^2/\varpi \geq 0$ and vanishes at $\varpi = 1$. Thus the right-hand side of (4.12) is strictly negative for every $l > k$ if $0 \leq k \leq k_*$: the first three terms in its numerator are non-positive for $2c_* - \sqrt{4c_*^2 - 1} \leq \varrho \leq 1$, and either $\varpi(\varrho^2 - 1) < 0$ or $\varrho = 1$, in which case $2 - 4c_* < 0$.

For $k = 0, l = 1$, the quantity (4.11) equals $(-3 + 9e^{-1} - 7e^{-2} + e^{-3})/4 < 0$, since the cubic $\varpi \mapsto -3 + 9\varpi - 7\varpi^2 + \varpi^3$ is increasing on $[0, 3/8]$ and equals $-285/512$ at $\varpi = 3/8$. For $k = 3$,

$l = 5$, (4.11) gives

$$(4.13) \quad \begin{aligned} \frac{1}{2} \partial_\alpha^2 K[S_0 + \alpha \phi_\varepsilon] \Big|_{\alpha=0} &= \frac{e^{-3} - e^{-7}}{4} - e^{-5} + \frac{(e^{-3} - e^{-5})(e^{-6} - e^{-10})}{4} \\ &\quad - c_*(e^{-3} - e^{-5})^2 \\ &> e^{-3} \left(\frac{1 - e^{-4}}{4} - e^{-2} - e^{-3} \right) > \frac{847}{16384e^3} > 0. \end{aligned}$$

The inequalities use $8/3 < e < 3$ and $0 < c_* < 1$. Young's inequality bounds the cubic term:

$$\left| \int_{\mathbb{R}} (\gamma_\varepsilon * \phi_\varepsilon) \phi_\varepsilon \partial_x (\gamma_\varepsilon^\vee * \gamma_\varepsilon * \phi_\varepsilon) dx \right| \leq \|\phi_\varepsilon\|_{L^\infty(\mathbb{R})}^2 \|\partial_x (\gamma_\varepsilon^\vee * \gamma_\varepsilon)\|_{L^1(\mathbb{R})} \|\phi_\varepsilon\|_{L^1(\mathbb{R})} \leq 2 + c.$$

The lower bound in (4.13) is independent of ε . Choosing

$$(4.14) \quad 0 < \alpha < \frac{847}{32768e^3(2+c)} < 1$$

therefore gives

$$K[u_0^\varepsilon] > \frac{847\alpha^2}{32768e^3} > 0$$

for every $\varepsilon > 0$, by (4.9). This proves $\mathcal{A} \neq \emptyset$. $\square$

4.1. Failure of global existence of the shift. The local-in-time existence statement in [Theorem 3.1](#) cannot, in general, be improved to global existence, as we show in this subsection. For the initial datum constructed in (4.7) and (4.14), local contractive shifts exist from every sufficiently small non-zero initial position, but none of them extends to all positive times. By translation invariance, this also produces initial data admitting a locally contractive shift with $X(0) = 0$ but no global contractive shift with that initial position. The argument uses the zero set of J_ε , which near $(0, 0)$ is a Lipschitz curve $t \mapsto Z(t)$ along which K_ε is positive. Any contractive shift starting close to $Z(0) = 0$ must meet this curve in finite time. At that point, (4.5) prevents the shift from continuing.

By (4.3) and (4.8),

$$\partial_\zeta J_\varepsilon(0, 0) = \frac{1}{\varepsilon} \left(\frac{1}{2} - \alpha(e^{-3} - e^{-5}) \right) > 0.$$

Continuity of K_ε and $\partial_\zeta J_\varepsilon$ gives positive constants T_0, d_1, k_1, j_1 , and j_2 such that

$$(4.15) \quad k_1 \leq K_\varepsilon(t, \zeta), \quad 0 < j_1 \leq \partial_\zeta J_\varepsilon(t, \zeta) \leq j_2, \quad (t, \zeta) \in [0, T_0] \times [-d_1, d_1].$$

After reducing T_0 , the intermediate value theorem and the strict monotonicity of J_ε in ζ give a unique function $Z: [0, T_0] \rightarrow (-d_1/4, d_1/4)$ with $Z(0) = 0$ and $J_\varepsilon(t, Z(t)) = 0$ for $0 \leq t \leq T_0$. The lower bound in (4.15) and the local Lipschitz bound in t for J_ε give

$$|Z(t) - Z(s)| \leq \frac{\text{Lip}_t(J_\varepsilon)}{j_1} |t - s|,$$

where $\text{Lip}_t(J_\varepsilon)$ is a Lipschitz constant of J_ε in t on $[0, T_0] \times [-d_1, d_1]$. Let $L_Z := \text{Lip}_t(J_\varepsilon)/j_1$. We choose $r_0 > 0$ such that

$$(4.16) \quad r_0 < \frac{d_1}{2}, \quad 2r_0 L_Z \leq \frac{k_1}{j_2},$$

and we choose $x_0 \neq 0$ with $|x_0| < r_0$ and $j_2 x_0^2 / k_1 < T_0$. Since $J_\varepsilon(0, 0) = 0$ and $\partial_\zeta J_\varepsilon(0, \cdot) \geq j_1 > 0$ on $[-d_1, d_1]$, we have $J_\varepsilon(0, x_0) \neq 0$, and (4.4) with $(t_0, \zeta_0) = (0, x_0)$ has a local solution, along which the squared distance in (4.2) is constant.

We assume, for contradiction, that a Lipschitz contractive shift with $X(0) = x_0$ is defined on $[0, T_0]$. At times when $0 < |X(t) - Z(t)| < r_0$, contractivity and (4.2) give $J_\varepsilon(t, X(t)) \dot{X}(t) \leq -K_\varepsilon(t, X(t)) \leq -k_1$, and $J_\varepsilon(t, Z(t)) = 0$ with (4.15) gives $(X(t) - Z(t))/J_\varepsilon(t, X(t)) \geq 1/j_2$. Together with (4.16), this yields

$$\begin{aligned} \frac{d}{dt} |X(t) - Z(t)|^2 &= 2 \frac{X(t) - Z(t)}{J_\varepsilon(t, X(t))} J_\varepsilon(t, X(t)) \dot{X}(t) - 2(X(t) - Z(t)) \dot{Z}(t) \\ &\leq -\frac{2k_1}{j_2} + 2r_0 L_Z \leq -\frac{k_1}{j_2} \end{aligned}$$

for almost every t . Thus $|X(t) - Z(t)|$ cannot reach r_0 , and X must meet Z by time $j_2 x_0^2 / k_1 < T_0$. At the first such time t_* , $J_\varepsilon(t_*, X(t_*)) = 0$ and $K_\varepsilon(t_*, X(t_*)) \geq k_1$. By (4.5), applied at $(t_0, \zeta_0) = (t_*, X(t_*))$, the squared distance increases on a right neighborhood of t_* , a contradiction.

To obtain an initial shift equal to zero, we replace $u_0^\varepsilon(x)$ by $u_0^\varepsilon(x + x_0)$. A shift X for the original datum corresponds to $X - x_0$ for the translated datum. This translation preserves membership in $\mathcal{B}$, and the translated datum belongs to $\mathcal{C}$, because the coefficient of $\dot{X}$ at $(0, 0)$ equals $J_\varepsilon(0, x_0) \neq 0$. Hence, for every $\varepsilon > 0$, some data in $\mathcal{C}$ admit a locally contractive Lipschitz shift with $X(0) = 0$ but no contractive Lipschitz shift on $[0, T_0]$, and in particular no global one.

4.2. The case $J = K = 0$. The case $J[u_0^\varepsilon] = K[u_0^\varepsilon] = 0$ lies outside $\mathcal{A} \cup \mathcal{C}$, and the time derivative at $t = 0$ does not decide whether a contractive shift exists: the quotient in (4.4) is undefined at $(t, \zeta) = (0, 0)$. The simplest example is the unperturbed datum $u_0^\varepsilon = S_0$, for which $J[u_0^\varepsilon] = K[u_0^\varepsilon] = 0$. The solution is $u^\varepsilon(t, x) = S_0(x - t)$, and $X(t) = t$ gives zero distance in (3.4). In particular, $\mathcal{B} \setminus (\mathcal{A} \cup \mathcal{C})$ is non-empty.

This class also contains non-trivial data. For ϕ_ε from (4.7), the convolution formulas (4.10), with $x = \varepsilon\zeta$ on $(-\varepsilon, 0)$ and $x = \varepsilon(3 + \zeta)$ on $[3\varepsilon, 5\varepsilon]$, give $\gamma_\varepsilon * \phi_\varepsilon = -c(1 - e^{-1-\zeta})$ and $\partial_x(\gamma_\varepsilon^\vee * \gamma_\varepsilon * \phi_\varepsilon) = \frac{c}{2\varepsilon}((2 - e^{-1})e^\zeta - e^{-1-\zeta})$ on the first interval, and $\gamma_\varepsilon * \phi_\varepsilon = 1 - (1 + e^{-6} - e^{-8})e^{-\zeta}$ and $\partial_x(\gamma_\varepsilon^\vee * \gamma_\varepsilon * \phi_\varepsilon) = \frac{1}{2\varepsilon}((1 + e^{-6} - e^{-8})e^{-\zeta} - e^{\zeta-2})$ on the second. Hence

$$\begin{aligned}
 (4.17) \quad & \int_{\mathbb{R}} (\gamma_\varepsilon * \phi_\varepsilon) \phi_\varepsilon \partial_x (\gamma_\varepsilon^\vee * \gamma_\varepsilon * \phi_\varepsilon) dx \\
 &= \frac{c^3}{2} \int_{-1}^0 (1 - e^{-1-\zeta}) ((2 - e^{-1})e^\zeta - e^{-1-\zeta}) d\zeta \\
 &+ \frac{1}{2} \int_0^2 (1 - (1 + e^{-6} - e^{-8})e^{-\zeta}) ((1 + e^{-6} - e^{-8})e^{-\zeta} - e^{\zeta-2}) d\zeta \\
 &= \frac{c^3}{4} (3 - 8e^{-1} + 3e^{-2}) - \frac{1}{4} + e^{-2} + \frac{e^{-4}}{4} \\
 &+ \frac{(e^{-6} - e^{-8})(e^{-2} + e^{-4})}{2} - \frac{(e^{-6} - e^{-8})^2(1 - e^{-4})}{4}.
 \end{aligned}$$

Since $8/3 < e < 3$, we have $c < 1/10$, and (4.13) and (4.17) imply

$$0 < \frac{1}{2} \partial_\alpha^2 K[S_0 + \alpha \phi_\varepsilon] \big|_{\alpha=0} < \frac{e^{-3} + e^{-9}}{4} < \frac{1}{64},$$

and

$$\int_{\mathbb{R}} (\gamma_\varepsilon * \phi_\varepsilon) \phi_\varepsilon \partial_x (\gamma_\varepsilon^\vee * \gamma_\varepsilon * \phi_\varepsilon) dx < -\frac{1}{4} + \frac{9}{64} + \frac{1}{200} + \frac{3}{4000} + \frac{1}{2000} = -\frac{33}{320} < -\frac{1}{10},$$

where we used $e^{-4}/4 < 1/200$, $c^3(3 - 8e^{-1} + 3e^{-2})/4 < 3/4000$, and $(e^{-6} - e^{-8})(e^{-2} + e^{-4})/2 < 1/2000$. Thus

$$K[S_0 + \frac{5}{32} \phi_\varepsilon] < \left(\frac{5}{32}\right)^2 \left(\frac{1}{64} - \frac{5}{32} \cdot \frac{1}{10}\right) = 0.$$

By (4.9), the map $\alpha \mapsto \alpha^{-2} K[S_0 + \alpha \phi_\varepsilon]$ is affine for $\alpha > 0$, with positive limit as $\alpha \downarrow 0$ and negative slope. It therefore has a unique zero $\alpha_* \in (0, 5/32)$. The explicit expressions in (4.13) and (4.17) show that this zero is independent of ε . By (4.8), the datum $u_0^\varepsilon = S_0 + \alpha_* \phi_\varepsilon \in \mathcal{B}$ satisfies $J[u_0^\varepsilon] = K[u_0^\varepsilon] = 0$.

By (4.5), a contractive shift starting from data with $J = K = 0$ cannot be continued beyond a time t at which $J_\varepsilon(t, X(t)) = 0$ and $K_\varepsilon(t, X(t)) > 0$.

More generally, we fix a solution u^ε and let $\mathcal{S}_\varepsilon \subset [0, \infty) \times \mathbb{R}$ be the set of points (t_0, ζ_0) from which a Lipschitz contractive shift exists on some interval $[t_0, t_0 + \delta]$, $\delta > 0$, with $X(t_0) = \zeta_0$, that is, a Lipschitz curve for which (3.4) holds for $t_0 \leq s < t \leq t_0 + \delta$. Since J_ε and K_ε are locally Lipschitz in (t, ζ) , Step 3 of the proof of Theorem 3.1 applies at every (t_0, ζ_0) and gives

$$\{J_\varepsilon \neq 0\} \cup \{J_\varepsilon = 0, K_\varepsilon < 0\} \subset \mathcal{S}_\varepsilon \subset \{J_\varepsilon \neq 0\} \cup \{J_\varepsilon = 0, K_\varepsilon \leq 0\}.$$

This argument does not decide whether a contractive shift exists from points where $J_\varepsilon = K_\varepsilon = 0$.

4.3. More general nonlocal conservation laws. The local criterion of [Theorem 3.1](#) applies to any kernel and any convex entropy, provided the solution is regular enough to justify the relative entropy identity. Let $\gamma_\varepsilon \in L^1(\mathbb{R}) \cap \text{BV}(\mathbb{R})$ be non-negative with integral one, not necessarily of the scaled form $\varepsilon^{-1}\gamma(\cdot/\varepsilon)$. Let $V \in C^1(\mathbb{R})$, and assume that [\(1.1\)](#) has a bounded solution u^ε such that

$$(4.18) \quad u^\varepsilon \in C([0, T]; L^1_{\text{loc}}(\mathbb{R})), \quad \sup_{0 \leq t \leq T} \text{TV}(u^\varepsilon(t, \cdot)) < \infty.$$

We fix a bounded reference function $\tilde{w}_0^\varepsilon \in \text{BV}(\mathbb{R})$ with $u_0^\varepsilon - \tilde{w}_0^\varepsilon \in L^1(\mathbb{R})$, and define $\tilde{w}_0^\varepsilon := \gamma_\varepsilon * \tilde{u}_0^\varepsilon$. Let $\eta \in C^3(\mathbb{R})$ be convex, with $\eta(r | s) := \eta(r) - \eta(s) - \eta'(s)(r - s)$. For every Lipschitz curve X ,

$$(4.19) \quad \begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}} \eta(w^\varepsilon(t, x) | \tilde{w}_0^\varepsilon(x - X(t))) dx \\ &= \dot{X}(t) \int_{\mathbb{R}} \eta''(\tilde{w}_0^\varepsilon(x - X(t))) (w^\varepsilon(t, x) - \tilde{w}_0^\varepsilon(x - X(t))) \partial_x \tilde{w}_0^\varepsilon(x - X(t)) dx \\ & \quad + \int_{\mathbb{R}} \partial_x (\eta'(w^\varepsilon(t, x)) - \eta'(\tilde{w}_0^\varepsilon(x - X(t)))) [\gamma_\varepsilon * (V(w^\varepsilon)u^\varepsilon)](t, x) dx \end{aligned}$$

for almost every t . The bounded variation of the flux and [\(4.18\)](#) imply $u^\varepsilon - \tilde{w}_0^\varepsilon \in C^{0,1}([0, T]; L^1(\mathbb{R}))$: the duality argument for [\(4.1\)](#) applies with a constant depending on the bounds for V , u^ε , and its total variation. Convolution gives $w^\varepsilon - \tilde{w}_0^\varepsilon \in W^{1,\infty}([0, T]; L^2(\mathbb{R}))$. Moreover, $\partial_x w^\varepsilon, \partial_x \tilde{w}_0^\varepsilon \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, since $\partial_x \tilde{w}_0^\varepsilon = (D\gamma_\varepsilon) * \tilde{u}_0^\varepsilon = \gamma_\varepsilon * D\tilde{u}_0^\varepsilon$, with $D\gamma_\varepsilon$ and $D\tilde{u}_0^\varepsilon$ finite measures, and likewise $\partial_x w^\varepsilon = (D\gamma_\varepsilon) * u^\varepsilon = \gamma_\varepsilon * Du^\varepsilon$ by [\(4.18\)](#), and $\partial_x \tilde{w}_0^\varepsilon$ has bounded variation, since $D\partial_x \tilde{w}_0^\varepsilon = (D\gamma_\varepsilon) * (D\tilde{u}_0^\varepsilon)$ is a finite measure. Since $w^\varepsilon - \tilde{w}_0^\varepsilon(\cdot - X(t)) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $\eta \in C^3(\mathbb{R})$, the chain rule applies, and the boundary term introduced by the cutoff tends to zero because $\eta'(w^\varepsilon) - \eta'(\tilde{w}_0^\varepsilon(\cdot - X(t))) \in L^1(\mathbb{R})$.

The two integrals in [\(4.19\)](#) are continuous in t and locally Lipschitz as functions of the shift. To prove the Lipschitz property, we apply the $L^1(\mathbb{R})$ translation estimate for functions of bounded variation to $\eta''(\tilde{w}_0^\varepsilon)\partial_x \tilde{w}_0^\varepsilon$ (the term involving $\tilde{w}_0^\varepsilon(\cdot - X(t))$ alone does not depend on the shift, by translation invariance of the integral). Step 3 of the proof of [Theorem 3.1](#) then applies: a non-zero first integral or a negative second integral at the initial point gives a local contractive shift. If the first integral is zero and the second is positive, no contractive Lipschitz shift starts from that point. The reference function $\tilde{u}_0^\varepsilon$ is arbitrary here. The criterion applies even when this function is neither a traveling wave nor a solution of the nonlocal conservation law. The examples in [Theorem 3.1](#) and [Section 4.1](#) show that, even for an exact traveling wave, such a shift may fail to exist, or may exist locally but not globally in time.

5. PROOFS OF THE LOGARITHMIC RELATIVE ENTROPY RESULTS

This section contains the proofs of the results of [Section 3.1](#) on shocks with positive end states. We first show, in [Section 5.1](#), that an integrable perturbation of the profile remains integrable for all times, so that the relative entropy and all the integrals appearing in its time derivative are well defined. We then prove the coercivity inequality of [Theorem 3.2](#) in [Section 5.2](#). We use these results to prove the contraction theorem in [Section 5.3](#) and its quantitative consequences in [Sections 5.4](#) and [5.5](#).

The reason for the logarithmic relative entropy [\(2.5\)](#) is visible in the formal entropy identity. For a smooth perturbation, the time derivative of half the squared $L^2(\mathbb{R})$ distance in a frame moving with a shift X contains the cubic term

$$-\frac{\beta}{2} \int_{\mathbb{R}} |u^\varepsilon(t, x + \sigma t + X(t)) - S^\varepsilon(x)|^2 \cdot \partial_x (w^\varepsilon(t, x + \sigma t + X(t)) - W^\varepsilon(x)) dx.$$

This term changes sign when the perturbation is reversed. For the logarithmic entropy, the second derivative $1/r$ cancels the density factor in the nonlocal flux. The resulting identity [\(5.18\)](#) is quadratic in the perturbation and is the basis of the proof of [Theorem 3.3](#).

5.1. Integrability of perturbations. The next lemma also gives the $L^2(\mathbb{R})$ and $H^1(\mathbb{R})$ regularity of the shifted perturbation and of its average that the entropy identity requires.

Lemma 5.1 (Propagation of an integrable perturbation). *We assume (2.1) and (2.2) and the exponential kernel (1.5). Let $U^\varepsilon(t, x) := S^\varepsilon(x - \sigma t)$, and let u^ε be a bounded solution of (1.1) with $u_+ \leq u^\varepsilon \leq u_-$. If $u_0^\varepsilon - S^\varepsilon \in L^1(\mathbb{R})$, then for every $t \geq 0$,*

$$(5.1) \quad \|u^\varepsilon(t, \cdot) - U^\varepsilon(t, \cdot)\|_{L^1(\mathbb{R})} \leq \exp\left(\frac{\beta(3u_- - u_+)}{\varepsilon}t\right) \|u_0^\varepsilon - S^\varepsilon\|_{L^1(\mathbb{R})}.$$

For every locally absolutely continuous $X: [0, \infty) \rightarrow \mathbb{R}$, the difference $u^\varepsilon(t, \cdot + \sigma t + X(t)) - S^\varepsilon$ belongs to $L_{\text{loc}}^\infty([0, \infty); L^1(\mathbb{R}) \cap L^2(\mathbb{R}))$. Its convolution with γ_ε belongs to $L_{\text{loc}}^\infty([0, \infty); W^{1,1}(\mathbb{R}) \cap H^1(\mathbb{R}))$.

Proof. Step 1: Equation for the difference and its renormalization. Since $\gamma_\varepsilon * U^\varepsilon(t, x) = W^\varepsilon(x - \sigma t)$, the affine form of V gives $u^\varepsilon V(w^\varepsilon) - U^\varepsilon V(\gamma_\varepsilon * U^\varepsilon) = V(w^\varepsilon)(u^\varepsilon - U^\varepsilon) + \beta U^\varepsilon \gamma_\varepsilon * (u^\varepsilon - U^\varepsilon)$. Hence, subtracting the equations satisfied by u^ε and U^ε gives

$$\partial_t(u^\varepsilon - U^\varepsilon) + \partial_x(V(w^\varepsilon)(u^\varepsilon - U^\varepsilon)) = -\beta \partial_x(U^\varepsilon \gamma_\varepsilon * (u^\varepsilon - U^\varepsilon)).$$

For fixed ε , the velocity $V(w^\varepsilon)$ is bounded and Lipschitz in (t, x) on bounded time intervals: $\partial_x w^\varepsilon$ is bounded by (2.7), and $\partial_t w^\varepsilon = -(D\gamma_\varepsilon) * (u^\varepsilon V(w^\varepsilon))$ is bounded because $D\gamma_\varepsilon$ is a finite measure. Moreover, $\partial_x(U^\varepsilon \gamma_\varepsilon * (u^\varepsilon - U^\varepsilon))$ is bounded. Hence $u^\varepsilon - U^\varepsilon$ is the unique bounded distributional solution of a linear continuity equation with Lipschitz velocity and bounded source. It is given by the Lagrangian formula along the characteristic flow of $V(w^\varepsilon)$, and it is renormalized [DL89, Theorem II.2 and Corollaries II.1–II.2]. In particular, we have

$$\partial_t|u^\varepsilon - U^\varepsilon| + \partial_x(V(w^\varepsilon)|u^\varepsilon - U^\varepsilon|) = -\beta \operatorname{sgn}(u^\varepsilon - U^\varepsilon) \partial_x(U^\varepsilon \gamma_\varepsilon * (u^\varepsilon - U^\varepsilon))$$

in the sense of distributions, with initial trace $|u_0^\varepsilon - S^\varepsilon|$ in $L_{\text{loc}}^1(\mathbb{R})$, that is, $|u^\varepsilon(t, \cdot) - U^\varepsilon(t, \cdot)| \rightarrow |u_0^\varepsilon - S^\varepsilon|$ in $L_{\text{loc}}^1(\mathbb{R})$ as $t \downarrow 0$. Consequently,

$$(5.2) \quad \partial_t|u^\varepsilon - U^\varepsilon| + \partial_x(V(w^\varepsilon)|u^\varepsilon - U^\varepsilon|) \leq \beta |\partial_x(U^\varepsilon \gamma_\varepsilon * (u^\varepsilon - U^\varepsilon))|.$$

Step 2: Weighted integrability. For $a > 0$ with $a\varepsilon < 1/2$, we set

$$(5.3) \quad \rho_a(x) := e^{-a\sqrt{1+x^2}}.$$

Then $|\partial_x \rho_a| \leq a\rho_a$ and $\int_{\mathbb{R}} \rho_a dx \leq 2/a$. The inequality $\rho_a(x+z) \leq e^{az}\rho_a(x)$ for $z \geq 0$ and Tonelli's theorem give

$$\int_{\mathbb{R}} \rho_a |\gamma_\varepsilon * (u^\varepsilon - U^\varepsilon)| dx \leq \frac{1}{1-a\varepsilon} \int_{\mathbb{R}} \rho_a |u^\varepsilon - U^\varepsilon| dx,$$

while (2.7) gives

$$\varepsilon \partial_x(\gamma_\varepsilon * (u^\varepsilon - U^\varepsilon)) = (u^\varepsilon - U^\varepsilon) - \gamma_\varepsilon * (u^\varepsilon - U^\varepsilon).$$

We multiply the renormalized equation for $|u^\varepsilon - U^\varepsilon|$ from Step 1 by $\rho_a \chi_R$, where χ_R is a smooth cutoff equal to one on $[-R, R]$, zero outside $[-2R, 2R]$, and $|\partial_x \chi_R| \leq C/R$. The flux and source are integrable against ρ_a on every bounded time interval. After integration in time, we remove the cutoff, including from the initial term, by Lebesgue's dominated convergence theorem as $R \rightarrow \infty$. Thus $t \mapsto \int_{\mathbb{R}} \rho_a |u^\varepsilon(t, \cdot) - U^\varepsilon(t, \cdot)| dx$ has an absolutely continuous representative. Its value at zero is $\int_{\mathbb{R}} \rho_a |u_0^\varepsilon - S^\varepsilon| dx$, by the initial trace in $L_{\text{loc}}^1(\mathbb{R})$ and the bound $(u_- - u_+)\rho_a$ on the integrand. By (5.2),

$$\frac{d}{dt} \int_{\mathbb{R}} \rho_a |u^\varepsilon - U^\varepsilon| dx \leq \left[a \sup_{[u_+, u_-]} |V| + \frac{\beta \|\partial_x S^\varepsilon\|_{L^\infty(\mathbb{R})}}{1-a\varepsilon} + \frac{\beta u_-}{\varepsilon} \left(1 + \frac{1}{1-a\varepsilon}\right) \right] \int_{\mathbb{R}} \rho_a |u^\varepsilon - U^\varepsilon| dx.$$

For fixed ε , the bracket is bounded uniformly as $a \downarrow 0$. Grönwall's inequality, followed by the monotone convergence theorem along a fixed sequence $a \downarrow 0$, proves $u^\varepsilon - U^\varepsilon \in L_{\text{loc}}^\infty([0, \infty); L^1(\mathbb{R}))$.

Step 3: Unweighted estimate. Young's inequality, (1.6), and the monotonicity of S^ε give

$$\begin{aligned} \|\gamma_\varepsilon * (u^\varepsilon - U^\varepsilon)\|_{L^\infty(\mathbb{R})} &\leq \varepsilon^{-1} \|u^\varepsilon - U^\varepsilon\|_{L^1(\mathbb{R})}, \\ \|\partial_x(\gamma_\varepsilon * (u^\varepsilon - U^\varepsilon))\|_{L^1(\mathbb{R})} &\leq 2\varepsilon^{-1} \|u^\varepsilon - U^\varepsilon\|_{L^1(\mathbb{R})}, \end{aligned}$$

and therefore

$$\begin{aligned} \|\partial_x(U^\varepsilon \gamma_\varepsilon * (u^\varepsilon - U^\varepsilon))\|_{L^1(\mathbb{R})} &\leq \|\partial_x U^\varepsilon\|_{L^1(\mathbb{R})} \|\gamma_\varepsilon * (u^\varepsilon - U^\varepsilon)\|_{L^\infty(\mathbb{R})} + u_- \|\partial_x(\gamma_\varepsilon * (u^\varepsilon - U^\varepsilon))\|_{L^1(\mathbb{R})} \\ &\leq \frac{(u_- - u_+) + 2u_-}{\varepsilon} \|u^\varepsilon - U^\varepsilon\|_{L^1(\mathbb{R})} = \frac{3u_- - u_+}{\varepsilon} \|u^\varepsilon - U^\varepsilon\|_{L^1(\mathbb{R})}. \end{aligned}$$

The flux and the source in (5.2) now belong to $L^1(\mathbb{R})$ on bounded time intervals. Testing with χ_R , integrating from time zero, and applying Lebesgue's dominated convergence theorem as $R \rightarrow \infty$, we obtain

$$\|u^\varepsilon(t, \cdot) - U^\varepsilon(t, \cdot)\|_{L^1(\mathbb{R})} \leq \|u_0^\varepsilon - S^\varepsilon\|_{L^1(\mathbb{R})} + \frac{\beta(3u_- - u_+)}{\varepsilon} \int_0^t \|u^\varepsilon(s, \cdot) - U^\varepsilon(s, \cdot)\|_{L^1(\mathbb{R})} ds,$$

and Grönwall's inequality proves (5.1) for almost every t . Since $u^\varepsilon \in C([0, T]; L^1_{\text{loc}}(\mathbb{R}))$ and the $L^1(\mathbb{R})$ norm is lower semicontinuous under convergence in $L^1_{\text{loc}}(\mathbb{R})$, while the right-hand side is continuous, the estimate holds for every $t \geq 0$.

Step 4: Shifted perturbations. By translation invariance and the monotonicity of S^ε ,

$$(5.4) \quad \|u^\varepsilon(t, \cdot + \sigma t + X(t)) - S^\varepsilon\|_{L^1(\mathbb{R})} \leq \|u^\varepsilon(t, \cdot) - U^\varepsilon(t, \cdot)\|_{L^1(\mathbb{R})} + (u_- - u_+)|X(t)|.$$

The shifted difference is bounded in absolute value by $u_- - u_+$, so its squared $L^2(\mathbb{R})$ norm is at most $u_- - u_+$ times its $L^1(\mathbb{R})$ norm. Young's inequality and (2.7) then give the regularity of its convolution with γ_ε stated in Lemma 5.1. $\square$

5.2. Proof of the coercivity inequality. The proof of Theorem 3.2 reduces (3.8) to a weighted Poincaré inequality on $(0, 1)$.

Proof of Theorem 3.2. We first prove (3.8) for $g \in C_c^\infty(\mathbb{R})$.

Step 1: Change of variables. We have

$$\begin{aligned} S^\varepsilon(x) &= \frac{u_+(1 + \theta)}{1 + \theta(1 - y^\varepsilon(x))}, \\ -\frac{\partial_x S^\varepsilon(x)}{S^\varepsilon(x)} &= \frac{\theta^2 y^\varepsilon(x)(1 - y^\varepsilon(x))}{\varepsilon[1 + \theta(1 - y^\varepsilon(x))]^2}. \end{aligned}$$

Since $W^\varepsilon = u_+(1 + \theta y^\varepsilon)$ and $u_- + u_+ - W^\varepsilon = u_+[1 + \theta(1 - y^\varepsilon)]$, the formula for S^ε follows from (2.14). Hence $-\partial_x \log S^\varepsilon = \partial_x \log[1 + \theta(1 - y^\varepsilon)]$, and (2.15) gives the explicit expression. Let $\mathcal{X}^\varepsilon: (0, 1) \rightarrow \mathbb{R}$ be the inverse of y^ε , and let $\mathcal{G}^\varepsilon(y) := g(\mathcal{X}^\varepsilon(y))$ for $0 < y < 1$. Then $\mathcal{G}^\varepsilon \in C_c^\infty((0, 1))$ and

$$\partial_x g(\mathcal{X}^\varepsilon(y)) = -\frac{\theta y(1 - y)}{\varepsilon[1 + \theta(1 - y)]} \partial_y \mathcal{G}^\varepsilon(y), \quad \partial_y \mathcal{X}^\varepsilon(y) = -\frac{\varepsilon[1 + \theta(1 - y)]}{\theta y(1 - y)}.$$

After this change of variables, the integrals containing g^2 and g have the respective weights

$$\begin{aligned} \psi_\theta(y) &:= \frac{\theta}{1 + \theta(1 - y)} + \frac{\theta^2[(2 + \theta)(1 - y) - 1]}{2[1 + \theta(1 - y)]^3}, \\ \omega_\theta(y) &:= \frac{\theta}{1 + \theta(1 - y)} + \frac{\theta^2[(2 + \theta)(1 - y) - 1]}{[1 + \theta(1 - y)]^3}. \end{aligned}$$

Explicitly,

$$(5.5) \quad \varepsilon \int_{\mathbb{R}} |\partial_x g|^2 dx = \int_0^1 \frac{\theta y(1 - y)}{1 + \theta(1 - y)} |\partial_y \mathcal{G}^\varepsilon|^2 dy,$$

$$(5.6) \quad \int_{\mathbb{R}} \left(-\frac{\partial_x S^\varepsilon}{S^\varepsilon} + \frac{\varepsilon}{2} \partial_x \left(\frac{\partial_x S^\varepsilon}{S^\varepsilon} \right) \right) g^2 dx = \int_0^1 \psi_\theta |\mathcal{G}^\varepsilon|^2 dy,$$

$$(5.7) \quad \int_{\mathbb{R}} \left(-\frac{\partial_x S^\varepsilon}{S^\varepsilon} + \varepsilon \partial_x \left(\frac{\partial_x S^\varepsilon}{S^\varepsilon} \right) \right) g dx = \int_0^1 \omega_\theta \mathcal{G}^\varepsilon dy.$$

Step 2: Bounds on the weights. For $0 \leq y \leq 1$,

$$\begin{aligned} [1 + \theta(1 - y)]^3 [\theta - \omega_\theta(y)] &= \theta^2 [y(1 - \theta(1 - y)) + \theta(1 - y)^2 [1 + \theta(1 - y)]] \geq 0, \\ [1 + \theta(1 - y)]^3 [\omega_\theta(y) - \theta(1 - \theta)] &= \theta^2(1 - y) [1 + 4\theta + (-2 + 3\theta)\theta(1 - y) - (1 - \theta)\theta^2(1 - y)^2] \\ &\geq \theta^2(1 - y)(1 + 2\theta - \theta^2) \geq 0, \\ [1 + \theta(1 - y)]^3 [\theta - \psi_\theta(y)] &= \frac{\theta^2}{2} [1 - \theta(1 - y) + 4\theta(1 - y)^2 + 2\theta^2(1 - y)^3] \geq 0. \end{aligned}$$

Hence

$$(5.8) \quad \theta(1 - \theta) \leq \omega_\theta(y) \leq \theta,$$

$$(5.9) \quad \psi_\theta(y) \leq \theta.$$

The numerator $(2 + \theta)(1 - y) - 1$ is at least -1 , the denominator $[1 + \theta(1 - y)]^3$ is at least 1, and

$$\begin{aligned} & [1 + \theta(1 - y)]^3 - [(2 + \theta)(1 - y) - 1] \\ &= 2 - 2(1 - \theta)(1 - y) + 3\theta^2(1 - y)^2 + \theta^3(1 - y)^3 \geq 0. \end{aligned}$$

Therefore

$$(5.10) \quad |\psi_\theta(y) - \omega_\theta(y)| \leq \frac{\theta^2}{2}.$$

Since

$$\partial_y \left(\frac{\theta^2 y(1 - y)}{[1 + \theta(1 - y)]^2} \right) = \frac{\theta^2 [(2 + \theta)(1 - y) - 1]}{[1 + \theta(1 - y)]^3},$$

integrating over $(0, 1)$ yields

$$(5.11) \quad \int_0^1 \psi_\theta \, dy = \int_0^1 \omega_\theta \, dy = \log(1 + \theta),$$

and $\theta/(1 + \theta) \leq \log(1 + \theta) \leq \theta$.

Step 3: Weighted mean and Poincaré inequality. We define

$$(5.12) \quad \bar{\mathcal{G}}_\theta^\varepsilon := \frac{\int_0^1 \omega_\theta \mathcal{G}^\varepsilon \, dy}{\log(1 + \theta)}.$$

Then $\int_0^1 \omega_\theta (\mathcal{G}^\varepsilon - \bar{\mathcal{G}}_\theta^\varepsilon) \, dy = 0$, and (5.8), which gives $0 \leq 1 - \omega_\theta/\theta \leq \theta$, implies

$$(5.13) \quad \begin{aligned} \left| \int_0^1 (\mathcal{G}^\varepsilon - \bar{\mathcal{G}}_\theta^\varepsilon) \, dy \right| &= \left| \int_0^1 \left(1 - \frac{\omega_\theta}{\theta} \right) (\mathcal{G}^\varepsilon - \bar{\mathcal{G}}_\theta^\varepsilon) \, dy \right| \\ &\leq \theta \|\mathcal{G}^\varepsilon - \bar{\mathcal{G}}_\theta^\varepsilon\|_{L^2((0,1))}. \end{aligned}$$

The weighted Poincaré inequality [KV21, Lemma 2.9] states that

$$\int_0^1 \left| F - \int_0^1 F \, dy \right|^2 \, dy \leq \frac{1}{2} \int_0^1 y(1 - y) |\partial_y F|^2 \, dy$$

for every $F \in H^1((0, 1))$. We apply it to $\mathcal{G}^\varepsilon - \bar{\mathcal{G}}_\theta^\varepsilon \in H^1((0, 1))$. By (5.13),

$$(5.14) \quad \begin{aligned} \int_0^1 \frac{\theta y(1 - y)}{1 + \theta(1 - y)} |\partial_y \mathcal{G}^\varepsilon|^2 \, dy &\geq \frac{2\theta}{1 + \theta} \left[\|\mathcal{G}^\varepsilon - \bar{\mathcal{G}}_\theta^\varepsilon\|_{L^2((0,1))}^2 - \left| \int_0^1 (\mathcal{G}^\varepsilon - \bar{\mathcal{G}}_\theta^\varepsilon) \, dy \right|^2 \right] \\ &\geq \frac{2\theta(1 - \theta^2)}{1 + \theta} \|\mathcal{G}^\varepsilon - \bar{\mathcal{G}}_\theta^\varepsilon\|_{L^2((0,1))}^2 \\ &= 2\theta(1 - \theta) \|\mathcal{G}^\varepsilon - \bar{\mathcal{G}}_\theta^\varepsilon\|_{L^2((0,1))}^2. \end{aligned}$$

Step 4: Quadratic form. Orthogonality and (5.10) give

$$\left| \int_0^1 \psi_\theta (\mathcal{G}^\varepsilon - \bar{\mathcal{G}}_\theta^\varepsilon) \, dy \right| \leq \frac{\theta^2}{2} \|\mathcal{G}^\varepsilon - \bar{\mathcal{G}}_\theta^\varepsilon\|_{L^2((0,1))}.$$

Expanding about the weighted mean and using (5.9), (5.11) and (5.14) yields

$$(5.15) \quad \begin{aligned} & \int_0^1 \frac{\theta y(1 - y)}{1 + \theta(1 - y)} |\partial_y \mathcal{G}^\varepsilon|^2 \, dy - \int_0^1 \psi_\theta |\mathcal{G}^\varepsilon|^2 \, dy \\ & \geq \theta(1 - 2\theta) \|\mathcal{G}^\varepsilon - \bar{\mathcal{G}}_\theta^\varepsilon\|_{L^2((0,1))}^2 - \theta^2 |\bar{\mathcal{G}}_\theta^\varepsilon| \|\mathcal{G}^\varepsilon - \bar{\mathcal{G}}_\theta^\varepsilon\|_{L^2((0,1))} - \theta |\bar{\mathcal{G}}_\theta^\varepsilon|^2 \\ & \geq -\theta \left(1 + \frac{\theta^2}{4(1 - 2\theta)} \right) |\bar{\mathcal{G}}_\theta^\varepsilon|^2 = -\frac{\theta(4 - 8\theta + \theta^2)}{4(1 - 2\theta)} |\bar{\mathcal{G}}_\theta^\varepsilon|^2. \end{aligned}$$

Step 5: Completion of the proof. The normalization $u_- \leq 1$ gives $u_- - u_+ = u_+ \theta \leq \theta/(1 + \theta)$. Thus (5.7), (5.11) and (5.12) imply

$$(5.16) \quad \begin{aligned} \frac{1}{u_- - u_+} \left[\int_{\mathbb{R}} \left(-\frac{\partial_x S^\varepsilon}{S^\varepsilon} + \varepsilon \partial_x \left(\frac{\partial_x S^\varepsilon}{S^\varepsilon} \right) \right) g \, dx \right]^2 &= \frac{\log^2(1 + \theta)}{u_- - u_+} |\bar{\mathcal{G}}_\theta^\varepsilon|^2 \\ &\geq \frac{\theta}{1 + \theta} |\bar{\mathcal{G}}_\theta^\varepsilon|^2. \end{aligned}$$

For $0 < \theta < 1/2$,

$$\partial_\theta \left[\frac{(1 + \theta)(4 - 8\theta + \theta^2)}{4(1 - 2\theta)} \right] = \frac{4 - 14\theta + 17\theta^2 - 4\theta^3}{4(1 - 2\theta)^2} > 0,$$

since the numerator is decreasing on $(0, 1/2)$ with value $3/4$ at $\theta = 1/2$. Therefore, by (3.7) and $\theta \leq \theta_*$,

$$\lambda - 1 \geq \frac{(1 + \theta)(4 - 8\theta + \theta^2)}{4(1 - 2\theta)}, \quad -\frac{\theta(4 - 8\theta + \theta^2)}{4(1 - 2\theta)} + (\lambda - 1) \frac{\theta}{1 + \theta} \geq 0.$$

By (5.5), (5.6), (5.15) and (5.16), this proves (3.8) for $g \in C_c^\infty(\mathbb{R})$.

For fixed ε , the exponential tails give $-\frac{\partial_x S^\varepsilon}{S^\varepsilon} + \frac{\varepsilon}{2} \partial_x \left(\frac{\partial_x S^\varepsilon}{S^\varepsilon} \right) \in L^\infty(\mathbb{R})$ and $-\frac{\partial_x S^\varepsilon}{S^\varepsilon} + \varepsilon \partial_x \left(\frac{\partial_x S^\varepsilon}{S^\varepsilon} \right) \in L^2(\mathbb{R})$. Every term in (3.8) is therefore continuous under convergence in $H^1(\mathbb{R})$. Density of $C_c^\infty(\mathbb{R})$ extends the inequality to every $g \in H^1(\mathbb{R})$. Translation invariance proves the stated independence of the profile normalization. $\square$

5.3. Contraction of the logarithmic relative entropy with a shift. The proof of Theorem 3.3 has three steps.

Proof of Theorem 3.3. Since the affine velocity (2.1) is non-decreasing and belongs to $W_{\text{loc}}^{2,\infty}(\mathbb{R})$, Lemma 2.1 with $m = u_+$ and $M = u_-$ gives $u_+ \leq u^\varepsilon, u^\varepsilon \leq u_-$ almost everywhere. In particular, Lemma 5.1 applies.

Step 1: Existence of the shift. We prove that the right-hand side of (3.9) is bounded and globally Lipschitz in the shift. The exponential tails and the end states of the profile give

$$\begin{aligned} \frac{\partial_x S^\varepsilon}{S^\varepsilon}, \partial_x \left(\frac{\partial_x S^\varepsilon}{S^\varepsilon} \right) &\in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \\ -\int_{\mathbb{R}} \frac{\partial_x S^\varepsilon}{S^\varepsilon} S^\varepsilon \, dx &= u_- - u_+, \quad -\int_{\mathbb{R}} \frac{\partial_x S^\varepsilon}{S^\varepsilon} \, dx = \log \frac{u_-}{u_+}. \end{aligned}$$

A change of variables gives, for every $\zeta \in \mathbb{R}$,

$$\begin{aligned} \int_{\mathbb{R}} \frac{\partial_x S^\varepsilon}{S^\varepsilon}(x) (u^\varepsilon(t, x + \sigma t + \zeta) - S^\varepsilon(x)) \, dx \\ = \int_{\mathbb{R}} \frac{\partial_x S^\varepsilon}{S^\varepsilon}(z - \sigma t - \zeta) u^\varepsilon(t, z) \, dz + (u_- - u_+). \end{aligned}$$

The translation estimate in $W^{1,1}(\mathbb{R})$ therefore bounds the Lipschitz constant of the right-hand side of (3.9) by

$$\frac{\lambda \beta u_-}{u_- - u_+} \left\| \partial_x \left(\frac{\partial_x S^\varepsilon}{S^\varepsilon} \right) \right\|_{L^1(\mathbb{R})}.$$

Since $|u^\varepsilon - S^\varepsilon| \leq u_- - u_+$,

$$\begin{aligned} \left| \int_{\mathbb{R}} \frac{\partial_x S^\varepsilon}{S^\varepsilon}(x) (u^\varepsilon(t, x + \sigma t + \zeta) - S^\varepsilon(x)) \, dx \right| \\ \leq (u_- - u_+) \log \frac{u_-}{u_+}. \end{aligned}$$

The right-hand side of (3.9) is continuous in time by the continuity of u^ε in $L_{\text{loc}}^1(\mathbb{R})$, its boundedness, the integrability of $\partial_x S^\varepsilon/S^\varepsilon$, and Lebesgue's dominated convergence theorem on the tails. The Picard–Lindelöf theorem and the global Lipschitz bound give a unique global solution X^ε , with

$$|\dot{X}^\varepsilon(t)| \leq \lambda \beta \log \frac{u_-}{u_+}$$

for almost every t .

Step 2: *Relative entropy identity.* The initial relative entropy is finite by (2.6):

$$0 \leq \int_{\mathbb{R}} h(u_0^\varepsilon \mid S^\varepsilon) dx \leq \frac{1}{2u_+} \|u_0^\varepsilon - S^\varepsilon\|_{L^2(\mathbb{R})}^2 \leq \frac{u_- - u_+}{2u_+} \|u_0^\varepsilon - S^\varepsilon\|_{L^1(\mathbb{R})} < \infty.$$

The change of variables $x \mapsto x + \sigma t + X^\varepsilon(t)$ is Lipschitz, and the flux is bounded. Thus the shifted solution satisfies

$$(5.17) \quad \begin{aligned} & \partial_t [u^\varepsilon(t, x + \sigma t + X^\varepsilon(t))] \\ & + \partial_x \left[(V(w^\varepsilon(t, x + \sigma t + X^\varepsilon(t))) - \sigma - \dot{X}^\varepsilon(t)) u^\varepsilon(t, x + \sigma t + X^\varepsilon(t)) \right] = 0 \end{aligned}$$

in the sense of distributions. For fixed ε , the velocity in (5.17) is bounded, uniformly Lipschitz in space by (2.7), and measurable in time. Bounded distributional solutions of this linear continuity equation are renormalized [DL89, Theorem II.2 and Corollaries II.1–II.2]. Since the density stays in $[u_+, u_-]$, we may use the entropy $\xi \mapsto \xi \log \xi - \xi$. The identity

$$\partial_x (V(W^\varepsilon) - \sigma) = -\frac{\partial_x S^\varepsilon}{S^\varepsilon} (V(W^\varepsilon) - \sigma)$$

follows from (2.14). Together with the affine form of V , it gives

$$\begin{aligned} & \partial_t h(u^\varepsilon(t, x + \sigma t + X^\varepsilon(t)) \mid S^\varepsilon(x)) \\ & + \partial_x \left[(V(w^\varepsilon(t, x + \sigma t + X^\varepsilon(t))) - \sigma - \dot{X}^\varepsilon(t)) h(u^\varepsilon(t, x + \sigma t + X^\varepsilon(t)) \mid S^\varepsilon(x)) \right] \\ & = -\beta(u^\varepsilon(t, x + \sigma t + X^\varepsilon(t)) - S^\varepsilon(x)) \partial_x (w^\varepsilon(t, x + \sigma t + X^\varepsilon(t)) - W^\varepsilon(x)) \\ & \quad - \beta \frac{\partial_x S^\varepsilon}{S^\varepsilon} (u^\varepsilon(t, x + \sigma t + X^\varepsilon(t)) - S^\varepsilon(x)) (w^\varepsilon(t, x + \sigma t + X^\varepsilon(t)) - W^\varepsilon(x)) \\ & \quad + \dot{X}^\varepsilon(t) \frac{\partial_x S^\varepsilon}{S^\varepsilon} (u^\varepsilon(t, x + \sigma t + X^\varepsilon(t)) - S^\varepsilon(x)). \end{aligned}$$

By Lemma 5.1, on every bounded time interval the difference between the shifted solution and the profile is bounded in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, and the corresponding difference of averages is bounded in $W^{1,1}(\mathbb{R}) \cap H^1(\mathbb{R})$. Thus all source terms and the entropy flux are integrable on bounded time intervals. We integrate against spatial cutoffs from time zero. We identify the initial entropy in each cutoff integral by the initial trace of u^ε in $L^1_{\text{loc}}(\mathbb{R})$, the condition $X^\varepsilon(0) = 0$, and the Lipschitz continuity of h on $[u_+, u_-]^2$. We remove the cutoffs by Lebesgue's dominated convergence theorem, using (2.6) and (5.4) and the integrability of the source terms. Consequently, the shifted relative entropy has a locally absolutely continuous representative with initial value $\int_{\mathbb{R}} h(u_0^\varepsilon \mid S^\varepsilon) dx$.

The resolvent identity (2.7) gives

$$\begin{aligned} u^\varepsilon(t, x + \sigma t + X^\varepsilon(t)) - S^\varepsilon(x) &= w^\varepsilon(t, x + \sigma t + X^\varepsilon(t)) - W^\varepsilon(x) \\ &\quad + \varepsilon \partial_x (w^\varepsilon(t, x + \sigma t + X^\varepsilon(t)) - W^\varepsilon(x)). \end{aligned}$$

Integration by parts therefore yields

$$\begin{aligned} & \int_{\mathbb{R}} (u^\varepsilon(t, x + \sigma t + X^\varepsilon(t)) - S^\varepsilon(x)) \partial_x (w^\varepsilon(t, x + \sigma t + X^\varepsilon(t)) - W^\varepsilon(x)) dx \\ & = \varepsilon \int_{\mathbb{R}} |\partial_x (w^\varepsilon(t, x + \sigma t + X^\varepsilon(t)) - W^\varepsilon(x))|^2 dx, \\ & \int_{\mathbb{R}} \frac{\partial_x S^\varepsilon}{S^\varepsilon} (u^\varepsilon(t, x + \sigma t + X^\varepsilon(t)) - S^\varepsilon(x)) (w^\varepsilon(t, x + \sigma t + X^\varepsilon(t)) - W^\varepsilon(x)) dx \\ & = \int_{\mathbb{R}} \left(\frac{\partial_x S^\varepsilon}{S^\varepsilon} - \frac{\varepsilon}{2} \partial_x \left(\frac{\partial_x S^\varepsilon}{S^\varepsilon} \right) \right) |w^\varepsilon(t, x + \sigma t + X^\varepsilon(t)) - W^\varepsilon(x)|^2 dx, \end{aligned}$$

and

$$\begin{aligned} & \int_{\mathbb{R}} \frac{\partial_x S^\varepsilon}{S^\varepsilon} (u^\varepsilon(t, x + \sigma t + X^\varepsilon(t)) - S^\varepsilon(x)) dx \\ & = \int_{\mathbb{R}} \left(\frac{\partial_x S^\varepsilon}{S^\varepsilon} - \varepsilon \partial_x \left(\frac{\partial_x S^\varepsilon}{S^\varepsilon} \right) \right) (w^\varepsilon(t, x + \sigma t + X^\varepsilon(t)) - W^\varepsilon(x)) dx. \end{aligned}$$

The boundary terms vanish because the difference of averages belongs to $H^1(\mathbb{R})$, its square belongs to $W^{1,1}(\mathbb{R})$, and $\partial_x S^\varepsilon / S^\varepsilon \in W^{1,\infty}(\mathbb{R})$. After integration of the relative entropy identity and substitution of (3.9), we obtain

$$\begin{aligned}
 (5.18) \quad & \frac{d}{dt} \int_{\mathbb{R}} h(u^\varepsilon(t, x + \sigma t + X^\varepsilon(t)) \mid S^\varepsilon(x)) dx \\
 &= -\beta \varepsilon \int_{\mathbb{R}} |\partial_x (w^\varepsilon(t, x + \sigma t + X^\varepsilon(t)) - W^\varepsilon(x))|^2 dx \\
 &\quad - \beta \int_{\mathbb{R}} \left(\frac{\partial_x S^\varepsilon}{S^\varepsilon} - \frac{\varepsilon}{2} \partial_x \left(\frac{\partial_x S^\varepsilon}{S^\varepsilon} \right) \right) |w^\varepsilon(t, x + \sigma t + X^\varepsilon(t)) - W^\varepsilon(x)|^2 dx \\
 &\quad - \frac{\lambda \beta}{u_- - u_+} \left[\int_{\mathbb{R}} \frac{\partial_x S^\varepsilon}{S^\varepsilon} (u^\varepsilon(t, x + \sigma t + X^\varepsilon(t)) - S^\varepsilon(x)) dx \right]^2.
 \end{aligned}$$

We apply (3.8) to $g = w^\varepsilon(t, \cdot + \sigma t + X^\varepsilon(t)) - W^\varepsilon \in H^1(\mathbb{R})$. Then (5.18) gives

$$\begin{aligned}
 & \frac{d}{dt} \int_{\mathbb{R}} h(u^\varepsilon(t, x + \sigma t + X^\varepsilon(t)) \mid S^\varepsilon(x)) dx \\
 & \leq -\frac{\beta}{u_- - u_+} \left[\int_{\mathbb{R}} \frac{\partial_x S^\varepsilon}{S^\varepsilon} (u^\varepsilon(t, x + \sigma t + X^\varepsilon(t)) - S^\varepsilon(x)) dx \right]^2.
 \end{aligned}$$

Integration on $[s, t]$ proves (3.10) for every $0 \leq s \leq t$.

Step 3: Distance and shift estimates. We obtain the remaining bounds from (3.10). By (2.6),

$$\begin{aligned}
 & \|u^\varepsilon(t, \cdot + \sigma t + X^\varepsilon(t)) - S^\varepsilon\|_{L^2(\mathbb{R})}^2 \\
 & \leq 2u_- \int_{\mathbb{R}} h(u^\varepsilon(t, x + \sigma t + X^\varepsilon(t)) \mid S^\varepsilon(x)) dx \leq 2u_- \int_{\mathbb{R}} h(u_0^\varepsilon \mid S^\varepsilon) dx
 \end{aligned}$$

for almost every t . The shift equation and (3.10) give

$$\int_0^t |\dot{X}^\varepsilon(s)|^2 ds \leq \frac{\lambda^2 \beta}{u_- - u_+} \int_{\mathbb{R}} h(u_0^\varepsilon \mid S^\varepsilon) dx.$$

Since $X^\varepsilon(0) = 0$, the Cauchy-Schwarz inequality yields

$$|X^\varepsilon(t)| \leq \sqrt{t} \left(\int_0^t |\dot{X}^\varepsilon(s)|^2 ds \right)^{1/2} \leq \lambda \left(\frac{\beta t}{u_- - u_+} \int_{\mathbb{R}} h(u_0^\varepsilon \mid S^\varepsilon) dx \right)^{1/2}.$$

Using the monotone convergence theorem as $t \rightarrow \infty$, we obtain (3.13). $\square$

5.4. Quantitative estimates. The estimates of Theorem 3.6 follow from Theorem 3.3 by comparing the profile S^ε with the local shock through Proposition 2.3 and by controlling the effect of the shift through the elementary identity (5.19).

Proof of Theorem 3.6. Step 1: Distance to the shifted shock. We compare the solution with the local shock after applying the shift. By (2.11) and (3.11) and the inequality $(a + b)^2 \leq 2a^2 + 2b^2$,

$$\|u^\varepsilon(t, \cdot + \sigma t + X^\varepsilon(t)) - S^0\|_{L^2(\mathbb{R})}^2 \leq 4u_- \int_{\mathbb{R}} h(u_0^\varepsilon \mid S^\varepsilon) dx + 2C_{S,2}\varepsilon.$$

Young's convolution inequality and (2.12) give

$$\|w^\varepsilon(t, \cdot + \sigma t + X^\varepsilon(t)) - S^0\|_{L^2(\mathbb{R})}^2 \leq 4u_- \int_{\mathbb{R}} h(u_0^\varepsilon \mid S^\varepsilon) dx + 2C_{W,2}\varepsilon.$$

These prove (3.14) and (3.15).

Step 2: Distance to the unshifted shock. To remove the shift, we use, for every $\zeta \in \mathbb{R}$,

$$(5.19) \quad \|S^0(\cdot - \zeta) - S^0\|_{L^2(\mathbb{R})}^2 = (u_- - u_+)^2 |\zeta|.$$

Combining (3.12) and (5.19) with the same inequality, we obtain

$$\begin{aligned}
 \|u^\varepsilon(t, \cdot) - S^0(\cdot - \sigma t)\|_{L^2(\mathbb{R})}^2 & \leq 8u_- \int_{\mathbb{R}} h(u_0^\varepsilon \mid S^\varepsilon) dx + 4C_{S,2}\varepsilon \\
 & \quad + 2\lambda(u_- - u_+)^{3/2} \left(\beta t \int_{\mathbb{R}} h(u_0^\varepsilon \mid S^\varepsilon) dx \right)^{1/2},
 \end{aligned}$$

$$\begin{aligned} \|w^\varepsilon(t, \cdot) - S^0(\cdot - \sigma t)\|_{L^2(\mathbb{R})}^2 &\leq 8u_- \int_{\mathbb{R}} h(u_0^\varepsilon | S^\varepsilon) dx + 4C_{W,2}\varepsilon \\ &\quad + 2\lambda(u_- - u_+)^{3/2} \left(\beta t \int_{\mathbb{R}} h(u_0^\varepsilon | S^\varepsilon) dx \right)^{1/2}. \end{aligned}$$

Taking the essential supremum for $0 \leq t \leq T$ proves (3.16) and (3.17). If $\int_{\mathbb{R}} h(u_0^\varepsilon | S^\varepsilon) dx \rightarrow 0$ as $\varepsilon \downarrow 0$, both right-hand sides tend to zero for each fixed T , proving convergence of the full family. $\square$

5.5. Cauchy problem for Riemann data. For Riemann data, we use the value of the initial relative entropy in Proposition 2.3.

Proof of Corollary 3.7. By (2.10) and (2.13), $S^0 - S^\varepsilon \in L^1(\mathbb{R})$ and $\int_{\mathbb{R}} h(u_0^\varepsilon | S^\varepsilon) dx = C_h \varepsilon$. Substitution into (3.12), (3.14) and (3.15) proves (3.18) and (3.19), including the estimate for w^ε . The squared distances to the shock without an additional shift are bounded by $C\varepsilon + C\sqrt{T\varepsilon}$. Since $0 < \varepsilon \leq 1$, taking square roots proves (3.20) for both u^ε and w^ε . $\square$

6. PROOFS OF THE ESTIMATES NEAR LIPSCHITZ SOLUTIONS

We prove the results of Section 3.2. We write the nonlocal equation as the local equation plus the spatial derivative of the difference between the local and nonlocal fluxes. The entropy balances of [CDH25, Propositions 3.1 and 3.3] bound the square of this flux difference by $C\varepsilon$ times the dissipation. Young's inequality then gives a relative entropy inequality with an error bounded by ε times the squared gradient of the reference solution. We conclude by Grönwall's inequality. We begin with the entropy balance and the pointwise bounds used in the proofs, and we then treat the exponential kernel in Section 6.1 and general kernels in Section 6.2.

Let $0 \leq m < M$, let $\eta \in C^2([m, M])$ be convex, and let $V \in C^1([m, M])$. We define P and choose q, Q , up to additive constants, by

$$(6.1) \quad q'(\xi) = \eta'(\xi)f'(\xi),$$

$$(6.2) \quad P(\xi) := \xi\eta'(\xi) - \eta(\xi), \quad Q'(\xi) = P(\xi)V'(\xi).$$

For the quadratic entropy $\eta(\xi) = \xi^2/2$, we have $P(\xi) = \xi^2/2$, and we fix the normalizations

$$(6.3) \quad q(\xi) := \int_m^\xi \vartheta f'(\vartheta) d\vartheta, \quad Q(\xi) := \int_m^\xi \frac{\vartheta^2}{2} V'(\vartheta) d\vartheta.$$

For integrable data and monotone velocities, the entropy balance of the next lemma is given in [CDH25, Proposition 3.3, Eq. (3.13)]. We include a proof for bounded data and general C^1 velocities.

Lemma 6.1 (Exact entropy balance). *Let $0 \leq m < M$, let $V \in C^1([m, M])$, and let $\eta \in C^2([m, M])$ be convex. Let P, q , and Q be as in (6.1) and (6.2). Every solution of (3.21) with the exponential kernel that takes values in $[m, M]$ satisfies*

$$(6.4) \quad \begin{aligned} \partial_t \eta(u^\varepsilon) + \partial_x q(u^\varepsilon) &= \partial_x \left[\eta(u^\varepsilon)(V(u^\varepsilon) - V(w^\varepsilon)) + Q(u^\varepsilon) - Q(w^\varepsilon) \right] \\ &\quad - \frac{1}{\varepsilon} V'(w^\varepsilon)(P(w^\varepsilon) - P(u^\varepsilon))(w^\varepsilon - u^\varepsilon) \end{aligned}$$

in the sense of distributions. For the quadratic entropy, this reads

$$(6.5) \quad \begin{aligned} \partial_t \frac{(u^\varepsilon)^2}{2} + \partial_x q(u^\varepsilon) &= \partial_x \left[\frac{(u^\varepsilon)^2}{2} (V(u^\varepsilon) - V(w^\varepsilon)) + Q(u^\varepsilon) - Q(w^\varepsilon) \right] \\ &\quad - \frac{1}{2\varepsilon} V'(w^\varepsilon)(w^\varepsilon + u^\varepsilon)(w^\varepsilon - u^\varepsilon)^2. \end{aligned}$$

In particular, if $0 \leq u^\varepsilon, w^\varepsilon \leq M$ and $V' \geq \nu > 0$ on $[0, M]$, then the last term is bounded above by $-\frac{\nu}{2\varepsilon}(w^\varepsilon + u^\varepsilon)(w^\varepsilon - u^\varepsilon)^2 \leq 0$.

Proof. Step 1: Renormalization. For fixed ε , the identities

$$\partial_x w^\varepsilon = \frac{u^\varepsilon - w^\varepsilon}{\varepsilon}, \quad \partial_t w^\varepsilon = -(D\gamma_\varepsilon) * (u^\varepsilon V(w^\varepsilon))$$

have bounded right-hand sides. Thus $V(w^\varepsilon)$ is Lipschitz in (t, x) , so that u^ε is the unique bounded distributional solution of the linear continuity equation with velocity $V(w^\varepsilon)$, and it is renormalized [DL89, Theorem II.2 and Corollaries II.1–II.2]. Hence

$$\partial_t \eta(u^\varepsilon) + \partial_x (\eta(u^\varepsilon) V(w^\varepsilon)) = -P(u^\varepsilon) V'(w^\varepsilon) \partial_x w^\varepsilon.$$

Step 2: Rewriting of the flux. Since $q' = (\eta V + Q)'$ by (6.1) and (6.2), and since the chain rule for $Q(w^\varepsilon)$ holds because w^ε is Lipschitz, we rewrite the identity of Step 1 as

$$\begin{aligned} \partial_t \eta(u^\varepsilon) + \partial_x q(u^\varepsilon) &= \partial_x \left[\eta(u^\varepsilon) (V(u^\varepsilon) - V(w^\varepsilon)) + Q(u^\varepsilon) - Q(w^\varepsilon) \right] \\ &\quad + (P(w^\varepsilon) - P(u^\varepsilon)) V'(w^\varepsilon) \frac{u^\varepsilon - w^\varepsilon}{\varepsilon}, \end{aligned}$$

which is (6.4). For the quadratic entropy, $P(w) - P(u) = \frac{1}{2}(w - u)(w + u)$, which gives (6.5). The bound on the last term follows from $V' \geq \nu$ and $u^\varepsilon, w^\varepsilon \geq 0$. $\square$

The sign of the dissipation term depends on the orientation of the kernel.

Remark 6.2 (Sign of the dissipation and orientation of the kernel). Since $P'(\xi) = \xi \eta''(\xi) \geq 0$ on $[m, M] \subset [0, \infty)$, the product $(P(w^\varepsilon) - P(u^\varepsilon))(w^\varepsilon - u^\varepsilon)$ is non-negative, so the last term in (6.4) is non-positive for every convex entropy whenever $V' \geq 0$ on $[m, M]$. The orientation of the kernel is essential: the kernel $\tilde{\gamma}_\varepsilon(z) := \varepsilon^{-1} e^{z/\varepsilon} \mathbb{1}_{(-\infty, 0]}(z)$ satisfies $\varepsilon \partial_x (\tilde{\gamma}_\varepsilon * g) = \tilde{\gamma}_\varepsilon * g - g$ and reverses this sign for a fixed velocity V , whereas making both substitutions $x \mapsto -x$ and $V \mapsto -V$, as in Section 2, preserves it.

For the quadratic entropy, we define the relative quantities

$$(6.6) \quad \mathcal{E}(u \mid v) := \frac{1}{2} |u - v|^2,$$

$$(6.7) \quad f(u \mid v) := f(u) - f(v) - f'(v)(u - v),$$

$$(6.8) \quad q(u; v) := q(u) - q(v) - v(f(u) - f(v)),$$

with q as in (6.3). Since $f(u) = uV(u)$, the nonlocal equation (3.21) can be written with the spatial derivative of the flux difference on the right-hand side:

$$(6.9) \quad \partial_t u^\varepsilon + \partial_x f(u^\varepsilon) = \partial_x \left[u^\varepsilon (V(u^\varepsilon) - V(w^\varepsilon)) \right].$$

The next lemma combines (6.5) and (6.9) with the equation for $\bar{u}$.

Lemma 6.3 (Relative entropy identity). *Let $0 \leq m < M$, let $V \in C^1([m, M])$, and let u^ε solve (3.21) with the exponential kernel, taking values in $[m, M]$. If $\bar{u}$ is a Lipschitz solution of (3.22) taking values in $[m, M]$, then*

$$\begin{aligned} (6.10) \quad \partial_t \mathcal{E}(u^\varepsilon \mid \bar{u}) + \partial_x \left[q(u^\varepsilon; \bar{u}) - \frac{(u^\varepsilon)^2}{2} (V(u^\varepsilon) - V(w^\varepsilon)) \right. \\ \left. - Q(u^\varepsilon) + Q(w^\varepsilon) + \bar{u} u^\varepsilon (V(u^\varepsilon) - V(w^\varepsilon)) \right] \\ + \frac{1}{2\varepsilon} V'(w^\varepsilon) (w^\varepsilon + u^\varepsilon) (w^\varepsilon - u^\varepsilon)^2 \\ = \partial_x \bar{u} \left[u^\varepsilon (V(u^\varepsilon) - V(w^\varepsilon)) - f(u^\varepsilon \mid \bar{u}) \right] \end{aligned}$$

in the sense of distributions.

Proof. Since $\mathcal{E}(u^\varepsilon \mid \bar{u}) = \frac{(u^\varepsilon)^2}{2} - \bar{u} u^\varepsilon + \frac{\bar{u}^2}{2}$, the identities (6.5) and (6.9) and $\partial_t \bar{u} = -f'(\bar{u}) \partial_x \bar{u}$ give

$$\begin{aligned} \partial_t \mathcal{E}(u^\varepsilon \mid \bar{u}) + \partial_x q(u^\varepsilon; \bar{u}) &= \partial_x \left[\frac{(u^\varepsilon)^2}{2} (V(u^\varepsilon) - V(w^\varepsilon)) + Q(u^\varepsilon) - Q(w^\varepsilon) \right] \\ &\quad - \frac{1}{2\varepsilon} V'(w^\varepsilon) (w^\varepsilon + u^\varepsilon) (w^\varepsilon - u^\varepsilon)^2 \\ &\quad - \bar{u} \partial_x \left[u^\varepsilon (V(u^\varepsilon) - V(w^\varepsilon)) \right] - \partial_x \bar{u} f(u^\varepsilon \mid \bar{u}). \end{aligned}$$

Applying the product rule to $-\bar{u} \partial_x [u^\varepsilon (V(u^\varepsilon) - V(w^\varepsilon))]$ yields (6.10). The products with $\bar{u}$ obey the distributional product rule because $\bar{u}$ is Lipschitz and the remaining factors are either locally

bounded or first-order distributional derivatives of locally bounded functions. For such a factor Φ , we set $\bar{u} \partial_x \Phi := \partial_x(\bar{u} \Phi) - \partial_x \bar{u} \Phi$, and likewise in t , and mollification shows that this definition agrees with the product rule. $\square$

In the remainder of this section, we take $m = 0$. The relative quantities (6.6), (6.7) and (6.8) and the difference $Q(u) - Q(w)$ do not depend on the additive constants chosen for q and Q .

The next lemma collects the bounds used to estimate the right-hand side and boundary terms of (6.10): the flux difference $u(V(u) - V(w))$ and the boundary fluxes are controlled by the dissipation term $V'(w)(w + u)(w - u)^2$, and the relative flux and relative entropy flux are controlled by the relative entropy itself.

Lemma 6.4 (Pointwise bounds). *Let $V \in C^2([0, M])$ and $0 \leq u, v, w \leq M$. Then*

$$(6.11) \quad |q(u; v)| \leq \|f'\|_{L^\infty([0, M])} \mathcal{E}(u | v),$$

$$(6.12) \quad |f(u | v)| \leq \|f''\|_{L^\infty([0, M])} \mathcal{E}(u | v).$$

If $V' \geq \nu > 0$ on $[0, M]$, then

$$\begin{aligned} |u(V(u) - V(w))|^2 &\leq \frac{M \|V'\|_{L^\infty([0, M])}^2}{\nu} V'(w)(w + u)(w - u)^2, \\ \left| \frac{u^2}{2} (V(u) - V(w)) + Q(u) - Q(w) \right| + |v u (V(u) - V(w))| \\ &\leq \frac{2 \|V'\|_{L^\infty([0, M])} M^{3/2}}{\sqrt{\nu}} \sqrt{V'(w)(w + u)(w - u)^2}. \end{aligned}$$

Proof. Step 1: Relative flux and relative entropy flux. The identities

$$q(u; v) = \int_v^u (r - v) f'(r) dr, \quad f(u | v) = \int_v^u (f'(r) - f'(v)) dr$$

give (6.11) and (6.12).

Step 2: Flux difference. Since $u^2 \leq M(u + w)$ and $\nu \leq V'(w)$,

$$|u(V(u) - V(w))|^2 \leq \|V'\|_{L^\infty([0, M])}^2 u^2 |u - w|^2 \leq \frac{M \|V'\|_{L^\infty([0, M])}^2}{\nu} V'(w)(w + u)(w - u)^2.$$

Step 3: Boundary fluxes. By (6.3),

$$\begin{aligned} |Q(u) - Q(w)| &\leq \frac{1}{2} \|V'\|_{L^\infty([0, M])} \max\{u, w\}^2 |u - w|, \\ u^2, \max\{u, w\}^2, vu &\leq M^{3/2} \sqrt{u + w}, \end{aligned}$$

and therefore

$$\begin{aligned} \left| \frac{u^2}{2} (V(u) - V(w)) + Q(u) - Q(w) \right| + |v u (V(u) - V(w))| \\ \leq 2 \|V'\|_{L^\infty([0, M])} M^{3/2} \sqrt{u + w} |u - w| \\ \leq \frac{2 \|V'\|_{L^\infty([0, M])} M^{3/2}}{\sqrt{\nu}} \sqrt{V'(w)(w + u)(w - u)^2}, \end{aligned}$$

where the last step uses $\nu \leq V'(w)$ once more. $\square$

6.1. Quantitative estimates for the exponential kernel. We prove Theorem 3.8 and Corollary 3.9. In both proofs, we integrate the relative entropy identity (6.10) against an integrable weight. The weight makes it possible to remove the spatial cutoffs without imposing decay at infinity. We estimate the terms containing the derivative of the weight. For the estimate on $\mathbb{R}$, we remove the weight by the monotone convergence theorem. For the estimate on a compact interval, we keep the weight and use its positive minimum on that interval. For $a > 0$, let ρ_a be the weight (5.3).

Proof of Theorem 3.8. Step 1: Weighted relative entropy estimate. We estimate the relative entropy integrated against ρ_a . We fix $\varepsilon > 0$ and $0 < a < 1/16$, so that $1/2 - \sqrt{a} > 1/4$. We test (6.10) of Lemma 6.3 with $\rho_a \chi_R$, where $0 \leq \chi_R \leq 1$ are compactly supported smooth cutoffs, $\chi_R \uparrow 1$, and $|\partial_x \chi_R| \leq C/R$. For fixed ε , the relative entropy, flux, dissipation, and right-hand side in (6.10)

are bounded on $(0, T) \times \mathbb{R}$. Because ρ_a is integrable, Lebesgue's dominated convergence theorem applies as $R \rightarrow \infty$, and the terms containing $\partial_x \chi_R$ tend to zero. Continuity of u^ε in $L^1_{\text{loc}}(\mathbb{R})$ with respect to time, its boundedness, and Lebesgue's dominated convergence theorem for the integrals with weight ρ_a outside bounded intervals give continuity of

$$t \mapsto \int_{\mathbb{R}} \rho_a(x) \mathcal{E}(u^\varepsilon(t, x) \mid \bar{u}(t, x)) dx.$$

The identity after integration against ρ_a shows that its time derivative belongs to $L^\infty((0, T))$. This integral is therefore Lipschitz on $[0, T]$, and its value at zero is obtained by substituting u_0^ε for $u^\varepsilon(t, \cdot)$ and $\bar{u}(0, \cdot)$ for $\bar{u}(t, \cdot)$.

By Lemma 6.4 and Young's inequality with weights $1/2 - \sqrt{a}$ and $\sqrt{a}$,

$$\begin{aligned} (6.13) \quad & \frac{d}{dt} \int_{\mathbb{R}} \rho_a \mathcal{E}(u^\varepsilon \mid \bar{u}) dx + \frac{1}{4\varepsilon} \int_{\mathbb{R}} \rho_a V'(w^\varepsilon)(w^\varepsilon + u^\varepsilon)(w^\varepsilon - u^\varepsilon)^2 dx \\ & \leq (\|f''\|_{L^\infty([0, M])} \|\partial_x \bar{u}\|_{L^\infty(\mathbb{R})} + a\|f'\|_{L^\infty([0, M])}) \int_{\mathbb{R}} \rho_a \mathcal{E}(u^\varepsilon \mid \bar{u}) dx \\ & \quad + \frac{M\|V'\|_{L^\infty([0, M])}^2 \varepsilon}{2\nu(1/2 - \sqrt{a})} \int_{\mathbb{R}} \rho_a |\partial_x \bar{u}|^2 dx + \frac{2\|V'\|_{L^\infty([0, M])}^2 M^3}{\nu} a^{3/2} \varepsilon \int_{\mathbb{R}} \rho_a dx \end{aligned}$$

for almost every time.

Step 2: Removal of the weight. We obtain the estimate on $\mathbb{R}$ by letting $a \downarrow 0$. The coefficient of the weighted relative entropy in (6.13) is non-negative. Grönwall's inequality, applied with an integrating factor, gives

$$\begin{aligned} (6.14) \quad & \int_{\mathbb{R}} \rho_a(x) \mathcal{E}(u^\varepsilon(t, x) \mid \bar{u}(t, x)) dx + \frac{1}{4\varepsilon} \int_0^t \int_{\mathbb{R}} \rho_a V'(w^\varepsilon)(w^\varepsilon + u^\varepsilon)(w^\varepsilon - u^\varepsilon)^2 dx ds \\ & \leq \exp\left(\|f''\|_{L^\infty([0, M])} \int_0^t \|\partial_x \bar{u}(s, \cdot)\|_{L^\infty(\mathbb{R})} ds + at\|f'\|_{L^\infty([0, M])}\right) \\ & \quad \times \left[\int_{\mathbb{R}} \rho_a(x) \mathcal{E}(u_0^\varepsilon(x) \mid \bar{u}(0, x)) dx \right. \\ & \quad \left. + \frac{M\|V'\|_{L^\infty([0, M])}^2 \varepsilon}{2\nu(1/2 - \sqrt{a})} \int_0^t \int_{\mathbb{R}} \rho_a |\partial_x \bar{u}|^2 dx ds + \frac{4\|V'\|_{L^\infty([0, M])}^2 M^3}{\nu} \sqrt{a} \varepsilon t \right]. \end{aligned}$$

Here the integrating factor on the dissipation is at least one, and $a^{3/2} \int_{\mathbb{R}} \rho_a dx \leq 2\sqrt{a}$. Since the weighted relative entropy is continuous, (6.14) holds for every $t \in [0, T]$. We choose a sequence $a \downarrow 0$. For fixed t , the monotone convergence theorem applies to the relative entropy, dissipation, and gradient integrals. The initial relative entropy and the gradient integral are finite by hypothesis, and the term proportional to $\sqrt{a}$ tends to zero. Thus (6.14), together with $V' \geq \nu$ in the dissipation term, implies (3.24).

Step 3: Estimate of the averaged density. We compare w^ε with $\bar{u}$ by separating the density error from the error due to averaging $\bar{u}$. By Young's inequality for convolution and (2.8),

$$\begin{aligned} \|w^\varepsilon(t, \cdot) - \bar{u}(t, \cdot)\|_{L^2(\mathbb{R})} & \leq \|\gamma_\varepsilon * (u^\varepsilon(t, \cdot) - \bar{u}(t, \cdot))\|_{L^2(\mathbb{R})} + \|\gamma_\varepsilon * \bar{u}(t, \cdot) - \bar{u}(t, \cdot)\|_{L^2(\mathbb{R})} \\ & \leq \|u^\varepsilon(t, \cdot) - \bar{u}(t, \cdot)\|_{L^2(\mathbb{R})} + \varepsilon \|\partial_x \bar{u}(t, \cdot)\|_{L^2(\mathbb{R})}. \end{aligned}$$

This proves (3.25). For identical initial data, (3.24) and (3.25) give the rate $O(\sqrt{\varepsilon})$ for the full family as $\varepsilon \downarrow 0$. $\square$

We turn to the local estimate.

Proof of Corollary 3.9. Step 1: Estimate with a fixed weight. We fix $a := 1/32$. We test (6.10) with $\rho_a \chi_R$ and let $R \rightarrow \infty$ by Lebesgue's dominated convergence theorem, which applies because all terms are bounded for fixed ε and $\rho_a \in L^1(\mathbb{R})$. Applying Young's inequality to each of the two

terms controlled by [Lemma 6.4](#) and absorbing $\frac{1}{8\varepsilon}$ of the dissipation each time, we obtain

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}} \rho_a \mathcal{E}(u^\varepsilon \mid \bar{u}) dx + \frac{1}{4\varepsilon} \int_{\mathbb{R}} \rho_a V'(w^\varepsilon)(w^\varepsilon + u^\varepsilon)(w^\varepsilon - u^\varepsilon)^2 dx \\ & \leq (\|f''\|_{L^\infty([0,M])} \|\partial_x \bar{u}\|_{L^\infty((0,T) \times \mathbb{R})} + a \|f'\|_{L^\infty([0,M])}) \int_{\mathbb{R}} \rho_a \mathcal{E}(u^\varepsilon \mid \bar{u}) dx \\ & \quad + \varepsilon \left(\frac{2M \|V'\|_{L^\infty([0,M])}^2}{\nu} \|\partial_x \bar{u}\|_{L^\infty((0,T) \times \mathbb{R})}^2 + \frac{8a^2 \|V'\|_{L^\infty([0,M])}^2 M^3}{\nu} \right) \int_{\mathbb{R}} \rho_a dx. \end{aligned}$$

Since the initial relative entropy vanishes, Grönwall's inequality yields

$$(6.15) \quad \operatorname{ess\,sup}_{0 \leq t \leq T} \int_{\mathbb{R}} \rho_a(x) \mathcal{E}(u^\varepsilon(t, x) \mid \bar{u}(t, x)) dx \leq C_{a,T,M,\nu,\bar{u},V} \varepsilon,$$

where the constant is independent of ε .

Step 2: Estimate of the averaged density. We use the weighted estimate of $u^\varepsilon - \bar{u}$ to bound $w^\varepsilon - \bar{u}$ on compact intervals. For $a\varepsilon < 1/2$, Jensen's inequality, Tonelli's theorem, and $\rho_a(\varsigma + z) \leq e^{az} \rho_a(\varsigma)$ for $z \geq 0$ give

$$(6.16) \quad \|\gamma_\varepsilon * g\|_{L^2(\mathbb{R}, \rho_a dx)}^2 \leq \int_0^\infty \gamma_\varepsilon(z) \int_{\mathbb{R}} \rho_a(\varsigma + z) |g(\varsigma)|^2 d\varsigma dz \leq \frac{1}{1 - a\varepsilon} \|g\|_{L^2(\mathbb{R}, \rho_a dx)}^2.$$

Moreover,

$$|\gamma_\varepsilon * \bar{u}(t, x) - \bar{u}(t, x)| \leq \varepsilon \|\partial_x \bar{u}\|_{L^\infty((0,T) \times \mathbb{R})}.$$

Together with [\(6.16\)](#), this gives

$$\begin{aligned} \|w^\varepsilon(t, \cdot) - \bar{u}(t, \cdot)\|_{L^2(\mathbb{R}, \rho_a dx)} & \leq \sqrt{2} \|u^\varepsilon(t, \cdot) - \bar{u}(t, \cdot)\|_{L^2(\mathbb{R}, \rho_a dx)} \\ & \quad + \varepsilon \|\partial_x \bar{u}\|_{L^\infty((0,T) \times \mathbb{R})} \left(\int_{\mathbb{R}} \rho_a dx \right)^{1/2}. \end{aligned}$$

Since $\min_I \rho_a > 0$ and $\varepsilon \leq (2a)^{-1/2} \sqrt{\varepsilon}$ when $a\varepsilon < 1/2$, [\(6.15\)](#) and [\(6.16\)](#) prove [\(3.26\)](#) in this range. For $a\varepsilon \geq 1/2$, the sum of the two local norms is bounded by $2M|I|^{1/2} \leq 2M|I|^{1/2} \sqrt{2a} \sqrt{\varepsilon}$. $\square$

6.2. Estimates for the averaged density with general kernels. For a general kernel, we work with the entropy balance for the averaged density w^ε in place of the resolvent identity: convolving the equation with γ_ε gives the local equation plus the spatial derivative of the flux difference

$$(6.17) \quad f(w^\varepsilon) - \gamma_\varepsilon * (u^\varepsilon V(w^\varepsilon)),$$

and the entropy balance of [\[CDH25, Proposition 3.1, Eq. \(3.7\)\]](#) contains the non-negative term in [\(3.31\)](#), whose integral is taken with respect to $\mu_{\gamma,\varepsilon}$. Under [\(3.28\)](#), the square of the flux difference is bounded by a constant times ε times this dissipation. Throughout this subsection, we assume [\(1.3\)](#), and we use the notation [\(3.27\)](#) and [\(3.29\)](#) and the function H of [\(3.30\)](#). The decay condition [\(3.28\)](#) is assumed in the proof of [Theorem 3.10](#). Since $w^\varepsilon(t, x) = \int_{(0,\infty)} \gamma_\varepsilon(z) u^\varepsilon(t, x - z) dz$ by [\(3.29\)](#), the flux difference in [\(6.17\)](#) can also be written as

$$(6.18) \quad \begin{aligned} & f(w^\varepsilon)(t, x) - [\gamma_\varepsilon * (u^\varepsilon V(w^\varepsilon))](t, x) \\ & = \int_{(0,\infty)} \gamma_\varepsilon(z) u^\varepsilon(t, x - z) (V(w^\varepsilon(t, x)) - V(w^\varepsilon(t, x - z))) dz. \end{aligned}$$

The entropy balance for w^ε collects the contributions of the flux difference and of the measure $\mu_{\gamma,\varepsilon}$.

Lemma 6.5 (Entropy and relative entropy identities for the averaged density). *Let $M > 0$, and assume [\(1.3\)](#). Let $V \in C^1([0, M])$ be non-decreasing, and let u^ε solve [\(3.21\)](#), with $0 \leq u^\varepsilon, w^\varepsilon \leq M$. Then, in the sense of distributions,*

$$(6.19) \quad \partial_t w^\varepsilon + \partial_x f(w^\varepsilon) = \partial_x [f(w^\varepsilon) - \gamma_\varepsilon * (u^\varepsilon V(w^\varepsilon))],$$

and

$$\begin{aligned}
 \partial_t \frac{(w^\varepsilon)^2}{2} + \partial_x q(w^\varepsilon) &= \partial_x \left[w^\varepsilon \left(f(w^\varepsilon) - \gamma_\varepsilon * (u^\varepsilon V(w^\varepsilon)) \right) \right. \\
 &\quad \left. - \int_{(0,\infty)} \gamma_\varepsilon(z) u^\varepsilon(t, x-z) H(w^\varepsilon(t, x) \mid w^\varepsilon(t, x-z)) dz \right] \\
 &\quad - \int_{(0,\infty)} u^\varepsilon(t, x-z) H(w^\varepsilon(t, x) \mid w^\varepsilon(t, x-z)) d\mu_{\gamma,\varepsilon}(z).
 \end{aligned}
 \tag{6.20}$$

If $\bar{u}$ is a Lipschitz solution of (3.22) taking values in $[0, M]$, then

$$\begin{aligned}
 \partial_t \mathcal{E}(w^\varepsilon \mid \bar{u}) + \partial_x \left[q(w^\varepsilon; \bar{u}) - (w^\varepsilon - \bar{u}) \left(f(w^\varepsilon) - \gamma_\varepsilon * (u^\varepsilon V(w^\varepsilon)) \right) \right. \\
 \left. + \int_{(0,\infty)} \gamma_\varepsilon(z) u^\varepsilon(t, x-z) H(w^\varepsilon(t, x) \mid w^\varepsilon(t, x-z)) dz \right] \\
 + \int_{(0,\infty)} u^\varepsilon(t, x-z) H(w^\varepsilon(t, x) \mid w^\varepsilon(t, x-z)) d\mu_{\gamma,\varepsilon}(z) \\
 = \partial_x \bar{u} \left[f(w^\varepsilon) - \gamma_\varepsilon * (u^\varepsilon V(w^\varepsilon)) - f(w^\varepsilon \mid \bar{u}) \right]
 \end{aligned}
 \tag{6.21}$$

in the sense of distributions.

Proof. Step 1: Equation for the average. Convolution of (3.21) with γ_ε gives

$$\begin{aligned}
 \partial_t w^\varepsilon &= -\partial_x (\gamma_\varepsilon * (u^\varepsilon V(w^\varepsilon))) \\
 &= -\partial_x f(w^\varepsilon) + \partial_x \left[f(w^\varepsilon) - \gamma_\varepsilon * (u^\varepsilon V(w^\varepsilon)) \right],
 \end{aligned}$$

which is (6.19) (and coincides with (1.4)). Since $D\gamma_\varepsilon$ is a finite measure, $\partial_x w^\varepsilon$ and $\partial_t w^\varepsilon$ are bounded.

Step 2: Derivative of the integral involving H . We have

$$\begin{aligned}
 \partial_v H(v \mid r) &= V(v) - V(r), \quad \partial_r H(v \mid r) = (r - v)V'(r), \\
 D\gamma_\varepsilon &= \frac{\gamma(0+)}{\varepsilon} \delta_0 - \mu_{\gamma,\varepsilon}, \quad H(v \mid v) = 0.
 \end{aligned}$$

We claim that, for the integral involving H ,

$$\begin{aligned}
 \partial_x \int_{(0,\infty)} \gamma_\varepsilon(z) u^\varepsilon(t, x-z) H(w^\varepsilon(t, x) \mid w^\varepsilon(t, x-z)) dz \\
 = - \int_{(0,\infty)} u^\varepsilon(t, x-z) H(w^\varepsilon(t, x) \mid w^\varepsilon(t, x-z)) d\mu_{\gamma,\varepsilon}(z) \\
 + \partial_x w^\varepsilon(t, x) \int_{(0,\infty)} \gamma_\varepsilon(z) u^\varepsilon(t, x-z) (V(w^\varepsilon(t, x)) - V(w^\varepsilon(t, x-z))) dz \\
 = - \int_{(0,\infty)} u^\varepsilon(t, x-z) H(w^\varepsilon(t, x) \mid w^\varepsilon(t, x-z)) d\mu_{\gamma,\varepsilon}(z) \\
 + \partial_x w^\varepsilon \left[f(w^\varepsilon) - \gamma_\varepsilon * (u^\varepsilon V(w^\varepsilon)) \right],
 \end{aligned}
 \tag{6.22}$$

where the last step uses (6.18). To prove (6.22), we change variables to $\varsigma = x - z$:

$$\begin{aligned}
 \int_{(0,\infty)} \gamma_\varepsilon(z) u^\varepsilon(t, x-z) H(w^\varepsilon(t, x) \mid w^\varepsilon(t, x-z)) dz \\
 = \int_{\mathbb{R}} \gamma_\varepsilon(x - \varsigma) u^\varepsilon(t, \varsigma) H(w^\varepsilon(t, x) \mid w^\varepsilon(t, \varsigma)) d\varsigma,
 \end{aligned}$$

and $u^\varepsilon(t, \varsigma)$ is never differentiated. For fixed ς , the factor $x \mapsto \gamma_\varepsilon(x - \varsigma)$ has bounded variation, with derivative $\frac{\gamma(0+)}{\varepsilon} \delta_\varsigma - \mu_{\gamma,\varepsilon}(\cdot - \varsigma)$, and $x \mapsto H(w^\varepsilon(t, x) \mid w^\varepsilon(t, \varsigma))$ is Lipschitz, with derivative $(V(w^\varepsilon(t, x)) - V(w^\varepsilon(t, \varsigma))) \partial_x w^\varepsilon(t, x)$ by $\partial_v H(v \mid r) = V(v) - V(r)$. The product rule for a function of bounded variation times a Lipschitz function, integration by parts in x against a test function, and Fubini's theorem, which applies because all integrands are bounded and $\gamma_\varepsilon dz$ and $\mu_{\gamma,\varepsilon}$ are finite measures, give the identity. The atom at $x = \varsigma$ contributes $\frac{\gamma(0+)}{\varepsilon} u^\varepsilon(t, x) H(w^\varepsilon(t, x) \mid w^\varepsilon(t, x)) = 0$.

Step 3: Entropy and relative entropy identities. Multiplying (6.19) by w^ε , we obtain, in the sense of distributions,

$$\begin{aligned} \partial_t \frac{(w^\varepsilon)^2}{2} + \partial_x q(w^\varepsilon) &= w^\varepsilon \partial_x \left[f(w^\varepsilon) - \gamma_\varepsilon * (u^\varepsilon V(w^\varepsilon)) \right] \\ &= \partial_x \left[w^\varepsilon \left(f(w^\varepsilon) - \gamma_\varepsilon * (u^\varepsilon V(w^\varepsilon)) \right) \right] \\ &\quad - \partial_x w^\varepsilon \left[f(w^\varepsilon) - \gamma_\varepsilon * (u^\varepsilon V(w^\varepsilon)) \right], \end{aligned}$$

and substituting (6.22) for the last term gives (6.20). The product rule for w^ε and $\bar{u}$, applied to (6.19) and (6.20) and to $\partial_t \bar{u} = -f'(\bar{u})\partial_x \bar{u}$, gives (6.21). $\square$

We bound the flux terms in (6.21) by the dissipation, and we then argue as for the exponential kernel.

Proof of Theorem 3.10. Step 1: Bounds on the fluxes. Monotonicity and Lipschitz continuity of V give

$$\begin{aligned} |V(v) - V(r)|^2 &\leq 2\|V'\|_{L^\infty([0,M])} H(v | r), \\ (6.23) \quad 0 \leq H(v | r) &\leq \frac{\|V'\|_{L^\infty([0,M])}}{2} |v - r|^2, \quad v, r \in [0, M]. \end{aligned}$$

If V is constant, every term in (6.23) vanishes. Otherwise, for $v \leq r$,

$$H(v | r) \geq \int_v^r (V(r) - V(v) - \|V'\|_{L^\infty([0,M])}(\vartheta - v))_+ d\vartheta = \frac{|V(r) - V(v)|^2}{2\|V'\|_{L^\infty([0,M])}},$$

whereas for $r \leq v$,

$$H(v | r) \geq \int_r^v (V(v) - V(r) - \|V'\|_{L^\infty([0,M])}(v - \vartheta))_+ d\vartheta = \frac{|V(v) - V(r)|^2}{2\|V'\|_{L^\infty([0,M])}}.$$

The upper bound in (6.23) follows by integrating $|V(r) - V(\vartheta)| \leq \|V'\|_{L^\infty([0,M])}|r - \vartheta|$. Since $\mu_{\gamma,\varepsilon} \geq (\kappa/\varepsilon)\gamma_\varepsilon(z) dz$ by (3.29) and the integrand $u^\varepsilon(t, x - z)H(w^\varepsilon(t, x) | w^\varepsilon(t, x - z))$ is non-negative,

$$\begin{aligned} &\int_{(0,\infty)} \gamma_\varepsilon(z) u^\varepsilon(t, x - z) H(w^\varepsilon(t, x) | w^\varepsilon(t, x - z)) dz \\ &\leq \frac{\varepsilon}{\kappa} \int_{(0,\infty)} u^\varepsilon(t, x - z) H(w^\varepsilon(t, x) | w^\varepsilon(t, x - z)) d\mu_{\gamma,\varepsilon}(z). \end{aligned}$$

The Cauchy–Schwarz inequality with weight $\gamma_\varepsilon(z)u^\varepsilon(t, x - z)$ in (6.18), the first inequality in (6.23), and $0 \leq u^\varepsilon \leq M$ give, for the flux difference,

$$\begin{aligned} &\left| f(w^\varepsilon) - \gamma_\varepsilon * (u^\varepsilon V(w^\varepsilon)) \right|^2 \\ (6.24) \quad &\leq 2M\|V'\|_{L^\infty([0,M])} \int_{(0,\infty)} \gamma_\varepsilon(z) u^\varepsilon(t, x - z) H(w^\varepsilon(t, x) | w^\varepsilon(t, x - z)) dz \\ &\leq \frac{2M\|V'\|_{L^\infty([0,M])}\varepsilon}{\kappa} \int_{(0,\infty)} u^\varepsilon(t, x - z) H(w^\varepsilon(t, x) | w^\varepsilon(t, x - z)) d\mu_{\gamma,\varepsilon}(z), \end{aligned}$$

while the second inequality in (6.23) and $0 \leq u^\varepsilon, w^\varepsilon \leq M$ give, for the integral involving H ,

$$\begin{aligned} &\left| \int_{(0,\infty)} \gamma_\varepsilon(z) u^\varepsilon(t, x - z) H(w^\varepsilon(t, x) | w^\varepsilon(t, x - z)) dz \right|^2 \\ (6.25) \quad &\leq \frac{M^3\|V'\|_{L^\infty([0,M])}}{2} \int_{(0,\infty)} \gamma_\varepsilon(z) u^\varepsilon(t, x - z) H(w^\varepsilon(t, x) | w^\varepsilon(t, x - z)) dz \\ &\leq \frac{M^3\|V'\|_{L^\infty([0,M])}\varepsilon}{2\kappa} \int_{(0,\infty)} u^\varepsilon(t, x - z) H(w^\varepsilon(t, x) | w^\varepsilon(t, x - z)) d\mu_{\gamma,\varepsilon}(z). \end{aligned}$$

Here the first inequality in (6.25) follows from $0 \leq H \leq \frac{1}{2}\|V'\|_{L^\infty([0,M])}M^2$ and $0 \leq u^\varepsilon \leq M$, which bound the integral involving H by $\frac{1}{2}\|V'\|_{L^\infty([0,M])}M^3$.

Step 2: Weighted estimate and removal of the weight. We first integrate the relative entropy against ρ_a from (5.3), and then let $a \downarrow 0$. We take $0 < a < 1/16$, so that $1/2 - \sqrt{a} > 1/4$. For fixed ε , w^ε is Lipschitz by Step 1 of the proof of Lemma 6.5 or by [Col+23, Corollary 2.2], and the flux difference, the integral involving H , and the dissipation appearing in (6.21) are bounded because $u^\varepsilon, w^\varepsilon \in [0, M]$ and $\mu_{\gamma, \varepsilon}$ is finite. Since $\partial_t w^\varepsilon$ and $\partial_t \bar{u}$ are bounded, the function

$$t \mapsto \frac{1}{2} \int_{\mathbb{R}} \rho_a(x) |w^\varepsilon(t, x) - \bar{u}(t, x)|^2 dx$$

is Lipschitz on $[0, T]$, with its value at zero given by the initial traces. We test (6.21) with $\rho_a \chi_R$, where $0 \leq \chi_R \leq 1$, $\chi_R \uparrow 1$, and $|\partial_x \chi_R| \leq C/R$. We remove the cutoff as $R \rightarrow \infty$ by Lebesgue's dominated convergence theorem, since $\rho_a \in L^1(\mathbb{R})$. By (6.24) and (6.25) and $|w^\varepsilon - \bar{u}| \leq M$, the flux term multiplying $\partial_x \rho_a$ after integration against ρ_a satisfies

$$\begin{aligned} & \left| (w^\varepsilon - \bar{u}) \left(f(w^\varepsilon) - \gamma_\varepsilon * (u^\varepsilon V(w^\varepsilon)) \right) \right. \\ & \quad \left. - \int_{(0, \infty)} \gamma_\varepsilon(z) u^\varepsilon(t, x - z) H(w^\varepsilon(t, x) | w^\varepsilon(t, x - z)) dz \right| \\ & \leq 3 \sqrt{\frac{M^3 \|V'\|_{L^\infty([0, M])}}{2\kappa}} \left(\varepsilon \int_{(0, \infty)} u^\varepsilon(t, x - z) H(w^\varepsilon(t, x) | w^\varepsilon(t, x - z)) d\mu_{\gamma, \varepsilon}(z) \right)^{1/2}. \end{aligned}$$

Young's inequality with weights $1/2 - \sqrt{a}$ and $\sqrt{a}$, together with (6.11) and (6.12), gives

$$\begin{aligned} & \frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}} \rho_a |w^\varepsilon - \bar{u}|^2 dx \\ & + \frac{1}{2} \int_{\mathbb{R}} \int_{(0, \infty)} \rho_a(x) u^\varepsilon(t, x - z) H(w^\varepsilon(t, x) | w^\varepsilon(t, x - z)) d\mu_{\gamma, \varepsilon}(z) dx \\ & \leq (\|f''\|_{L^\infty([0, M])} \|\partial_x \bar{u}\|_{L^\infty(\mathbb{R})} + a \|f'\|_{L^\infty([0, M])}) \frac{1}{2} \int_{\mathbb{R}} \rho_a |w^\varepsilon - \bar{u}|^2 dx \\ & + \frac{M \|V'\|_{L^\infty([0, M])} \varepsilon}{2\kappa(1/2 - \sqrt{a})} \int_{\mathbb{R}} \rho_a |\partial_x \bar{u}|^2 dx + \frac{9M^3 \|V'\|_{L^\infty([0, M])}}{8\kappa} a^{3/2} \varepsilon \int_{\mathbb{R}} \rho_a dx. \end{aligned}$$

Grönwall's inequality, applied with an integrating factor, implies

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}} \rho_a(x) |w^\varepsilon(t, x) - \bar{u}(t, x)|^2 dx \\ & + \frac{1}{2} \int_0^t \int_{\mathbb{R}} \int_{(0, \infty)} \rho_a(x) u^\varepsilon(s, x - z) H(w^\varepsilon(s, x) | w^\varepsilon(s, x - z)) d\mu_{\gamma, \varepsilon}(z) dx ds \\ (6.26) \quad & \leq \exp \left(\|f''\|_{L^\infty([0, M])} \int_0^t \|\partial_x \bar{u}(s, \cdot)\|_{L^\infty(\mathbb{R})} ds + at \|f'\|_{L^\infty([0, M])} \right) \\ & \quad \times \left[\frac{1}{2} \int_{\mathbb{R}} \rho_a(x) |w^\varepsilon(0, x) - \bar{u}(0, x)|^2 dx \right. \\ & \quad \left. + \frac{M \|V'\|_{L^\infty([0, M])} \varepsilon}{2\kappa(1/2 - \sqrt{a})} \int_0^t \int_{\mathbb{R}} \rho_a |\partial_x \bar{u}|^2 dx ds + \frac{9M^3 \|V'\|_{L^\infty([0, M])}}{8\kappa} a^{3/2} \varepsilon t \int_{\mathbb{R}} \rho_a dx \right]. \end{aligned}$$

Here the factor multiplying the dissipation is at least one. Along any sequence $a \downarrow 0$, the monotone convergence theorem applies to the relative entropy, dissipation, initial relative entropy, and integrals of $|\partial_x \bar{u}|^2$ in (6.26). The limits of the initial relative entropy and the gradient terms are finite by (3.32). Since

$$a^{3/2} \int_{\mathbb{R}} \rho_a dx \leq 2\sqrt{a} \rightarrow 0,$$

estimate (6.26) gives (3.33), which proves Theorem 3.10(i).

Step 3: Initial error and local convergence. We estimate the error caused by averaging the initial datum and then deduce the rate of convergence on compact intervals. Minkowski's inequality and

the fundamental theorem of calculus give

$$\begin{aligned} \|\gamma_\varepsilon * \bar{u}(0, \cdot) - \bar{u}(0, \cdot)\|_{L^2(\mathbb{R})} &\leq \int_0^\infty \gamma(z) \|\bar{u}(0, \cdot - \varepsilon z) - \bar{u}(0, \cdot)\|_{L^2(\mathbb{R})} dz \\ &\leq \varepsilon \left(\int_0^\infty z \gamma(z) dz \right) \|\partial_x \bar{u}(0, \cdot)\|_{L^2(\mathbb{R})}. \end{aligned}$$

This proves (3.35). For Theorem 3.10(ii), we fix $a := 1/32$, so that $0 < a < 1/16$. Since $\bar{u}(t, \cdot) \rightarrow \bar{u}(0, \cdot)$ locally uniformly, the Lipschitz constant of $\bar{u}(0, \cdot)$ is at most $\|\partial_x \bar{u}\|_{L^\infty((0,T) \times \mathbb{R})}$, and therefore

$$|\gamma_\varepsilon * \bar{u}(0, \cdot) - \bar{u}(0, \cdot)| \leq \varepsilon \left(\int_0^\infty z \gamma(z) dz \right) \|\partial_x \bar{u}\|_{L^\infty((0,T) \times \mathbb{R})},$$

where the first moment is finite by Remark 3.11. Hence (6.26) gives

$$\operatorname{ess\,sup}_{0 \leq t \leq T} \frac{1}{2} \int_{\mathbb{R}} \rho_a(x) |w^\varepsilon(t, x) - \bar{u}(t, x)|^2 dx \leq C_{a,T,M,\kappa,\bar{u},V,\gamma}(\varepsilon^2 + \varepsilon).$$

Since $\min_I \rho_a > 0$, this proves (3.34) for $0 < \varepsilon \leq 1$. For $\varepsilon > 1$,

$$\|w^\varepsilon(t, \cdot) - \bar{u}(t, \cdot)\|_{L^2(I)} \leq M|I|^{1/2} \leq M|I|^{1/2} \sqrt{\varepsilon}.$$

□

7. CONCLUSIONS

The relative entropy method with shifts extends from the viscous to the nonlocal regularization away from vacuum (Theorems 3.3, 3.8 and 3.10), but fails at vacuum (Theorem 3.1).

Several questions remain open. First, we do not know which initial data in Theorem 3.1 admit a contractive shift for all positive times. This set is non-empty: for the translated step $u_0^\varepsilon = S_0(\cdot - x_0)$ with $x_0 > 0$, the solution is $S_0(x - x_0 - t)$, and $X(t) = t + \min\{t, x_0\}$ is a Lipschitz shift with $X(0) = 0$ along which the $L^2(\mathbb{R})$ distance in (3.4) is non-increasing, and vanishes for $t \geq x_0$. On the other hand, Section 4.1 shows that local existence of a contractive shift does not, in general, imply global existence, so the answer is not the set $\mathcal{C}$. Second, it remains to determine when the identity (4.19) yields contraction for other kernels and entropies. In particular, the profile construction in Proposition 2.3 does not apply to one-sided kernels that are not exponential. Third, the smallness condition $\theta < 1/2$ in Theorem 3.3 comes from the coercivity estimate of Theorem 3.2, and we do not know whether it is sharp. Whether contraction holds for large shocks with positive end states, possibly for a relative entropy with different weights on the two sides of the shock as in [KV16, Theorem 3.1], is open. Fourth, for general one-sided kernels, Theorem 3.10 controls only the averaged density w^ε . Whether the density u^ε itself converges at the same rate, and whether the decay condition (3.28) can be removed, remain open. Finally, we ask whether contraction estimates hold for symmetric kernels with finite second moment, with or without viscosity. Their formal expansion as $\varepsilon \rightarrow 0$ contains terms with third spatial derivatives, which suggests a comparison with equations containing viscosity and dispersion, as in [Zum99, Sections 3 and 5]. Recent contraction results for viscous-dispersive shocks include [EKK26, Theorem 1.1], [Che+26a, Theorem 1.1], and [Che+26b, Theorem 1.1].

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