THE BOURGAIN, BREZIS, MIRONESCU LOCALIZATION FOR DETERMINISTIC OBSTACLE PROBLEMS

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Abstract. We study the asymptotic behaviour of families of fractional energies normalized according to the Bourgain–Brezis–Mironescu formula and defined on functions satisfying a pinning condition on possibly aperiodic families of small obstacles. The result of the simultaneous limit depends crucially on the critical rescaling chosen for the obstacles.

This work is dedicated to Paolo Marcellini, a pioneer in homogenization, on the occasion of his 80th birthday

1. Introduction

In the seminal paper [4], Bourgain, Brezis, and Mironescu established a connection between classical Sobolev spaces and their fractional counterparts as the order of fractional differentiability tends to one (cf. also [10]). In particular, the appropriate normalization of the fractional seminorm restores continuity in the differentiation index. More precisely, let $U \subset \mathbb{R}^n$ be a bounded, connected Lipschitz open set, $n \geq 2$, and $p \in (1, \infty)$. For $u \in L^p(U)$ and $s \in (0, 1)$, set

$$[u]^p_{s,p;U} := \int_{U\times U} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}}\, dx\, dy.$$

Then [4, Corollary 2] gives

$$\lim_{s\to 1^-} (1 - s)[u]^p_{s,p;U} = \begin{cases} K_{n,p} \displaystyle\int_U |\nabla u|^p\, dx, & u \in W^{1,p}(U), \\ +\infty, & u \in L^p(U) \setminus W^{1,p}(U). \end{cases}$$

The constant $K_{n,p}$ is given explicitly in (2.8).

Subsequent developments have extended this result in several directions: to the whole space $\mathbb{R}^n$, to variational convergence in the sense of Γ-convergence, to problems with boundary conditions, and to magnetic Sobolev spaces. Necessary and sufficient conditions on more general families of kernels have also been investigated (see, among others, [10, 40, 36, 16, 31, 21, 29]).

Recent contributions have analyzed the interaction of the Bourgain, Brezis, and Mironescu limit (BBM limit in short) with another singular limit involving microscopic scales: homogenization. For quadratic fractional energies with periodic coefficients, Braides, Brusca, and Donati [6] established a regime in which localization and homogenization can be separated (see also [26] for a preliminary result).

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Braides, Mannella, and Schiavoni Piazza [8] identified the transition scale between homogenized and averaged energies, governed by the relative rates of the fractional order approaching one and the period tending to zero.

In this paper we pursue a related but different path by studying the asymptotics via Γ-convergence of the rescaled Gagliardo seminorms under the additional constraint that the admissible functions vanish on a family of small obstacles.

The limiting behavior of solutions to local Dirichlet problems in periodically perforated domains has been an active area of research since the 1970s, following the seminal papers of Marchenko and Khruslov [33], Rauch and Taylor [41, 42], Cioranescu and Murat [15, 14], and De Giorgi, Dal Maso and Longo [22] (cf. [17, 18, 19] for a complete solution of the problem). These problems are motivated by applications in electrostatic screening, fluid mechanics, elasticity and acoustics, among others. It is well known that when the perforations have a critical size the limit equation or functional contains, in addition to the original operator or energy, a zero-order term reflecting the cumulative capacitary cost of the obstacles. We do not attempt to survey the vast bibliography on the subject and refer to the comprehensive accounts in [33, 20, 3, 7, 5]. Here, we only stress that the capacity of the individual obstacles, together with their distribution, determines their contribution to the limit problem.

The interest in the homogenization of obstacle problems resurfaced more recently thanks to the work of Caffarelli and Mellet [11] in the nonlocal setting, with interest driven by applications in contact mechanics, flow through semipermeable membranes, and mathematical finance among others. Caffarelli and Mellet studied these problems for the fractional Laplacian via Tartar's oscillating test functions method and taking advantage of the celebrated Caffarelli–Silvestre extension argument [12] to reduce the nonlocal homogenization problem to one involving a local but degenerate energy. An intrinsic variational approach working directly with the nonlocal energy was later proposed by the author, using Γ-convergence techniques for an aperiodic distribution of obstacles [25] (see also [24] for an earlier result using Γ-convergence and the above mentioned extension method, and [37, 38] for soft Robin-type obstacle constraints and a different intrinsic approach if $p = 2$).

Stochastic distributions of obstacles satisfying suitable stationarity and integrability conditions, allowing dependence between centres and marks as well as clustering of the obstacles have been recently dealt with in the work of Giunti, Höfer and Velázquez [30] in the linear local case using PDEs methods. The nonlinear counterpart using the variational approach has been studied in Scardia, Zemas and Zeppieri [43] still in the local setting. Deangelis, Zeppieri and the author [23] have established more recently an extension to fractional energies under natural ergodicity assumptions.

1.1. **Overview of the main results.** The focus of this paper is to consider simultaneously the BBM limit and the homogenization of an obstacle condition, with deterministic though possibly aperiodic distributions. For these problems, as customary, we assume $p \in (1, n)$. We stress that both asymptotic processes involve a localization mechanism, so their interaction requires separating two contributions which concentrate near the diagonal.

For $p = 2$, the associated minimum problems describe diffusion in a medium containing small absorbing inclusions. The simultaneous limit asks whether the

collective absorption due to these inclusions persists as the diffusion energy becomes local, and how its strength depends on their size.

The outcome of our analysis is that the interaction between the two phenomena depends subtly on the chosen scaling of the obstacles. In particular, if $s_j \to 1^-$ denotes the fractional order of differentiability and $\varepsilon_j \to 0^+$ the characteristic distance between the obstacles, at the critical scale $\varepsilon_j^{\frac{n}{n-s_jp}}$, dictated by the homogeneity of the fractional capacity, the bulk and capacitary terms are additive and their coefficients are independent of the relative rates at which $1-s_j$ and ε_j vanish (Theorem 5.1). Instead, at the critical scale $\varepsilon_j^{\frac{n}{n-p}}$, dictated by the capacity of the limiting local energy, three different regimes arise (Theorem 5.5). For further details on the interaction between the two concentration mechanisms we refer to the discussion in Subsection 1.3.

For the sake of simplicity, in this introduction we describe the results we obtain for a periodic distribution of obstacles, referring to Theorems 5.1 and 5.5 for the general, possibly aperiodic, statements. Extensions to thin obstacles and to translation-invariant homogeneous kernels are discussed in Remarks 5.6 and 5.7, respectively. Furthermore, the ideas behind the proof of the results described below will be discussed heuristically in Subsection 1.3.

Let then $T \Subset B_1$ be a nonempty open set, and let $s_j \uparrow 1$ and $\varepsilon_j \downarrow 0$. Set

$$\lambda_j := \varepsilon_j^{\frac{n}{n-s_jp}}, \qquad \mathcal{K}_j := \bigcup_{\mathtt{i}\in\mathbb{Z}^n} (\varepsilon_j \mathtt{i} + \lambda_j T), \tag{1.1}$$

and define $\mathcal{F}_j : L^p(U) \to [0,+\infty]$ by

$$\mathcal{F}_j(u) := \begin{cases} (1-s_j)[u]^p_{s_j,p;U}, & u \in W^{s_j,p}(U), \quad u = 0 \text{ a.e. on } \mathcal{K}_j \cap U, \\ +\infty, & \text{otherwise.} \end{cases} \tag{1.2}$$

Under the above assumptions $(\mathcal{F}_j)_j$ Γ-converges in $L^p(U)$ to

$$\mathcal{F}(u) := \begin{cases} K_{n,p} \displaystyle\int_U |\nabla u|^p \, dx + K_{n,p} \operatorname{cap}_p(T) \int_U |u|^p \, dx, & u \in W^{1,p}(U), \\ +\infty, & \text{otherwise.} \end{cases} \tag{1.3}$$

Here, $\operatorname{cap}_p(T)$ denotes the homogeneous p-capacity defined in (3.11).

Theorem 5.5 deals instead with the rescaling associated with the local capacity. In the periodic case, set

$$\bar\lambda_j := \varepsilon_j^{\frac{n}{n-p}}, \qquad \bar{\mathcal{K}}_j := \bigcup_{\mathtt{i}\in\mathbb{Z}^n} (\varepsilon_j \mathtt{i} + \bar\lambda_j T), \tag{1.4}$$

and define $\bar{\mathcal{F}}_j : L^p(U) \to [0,+\infty]$ by

$$\bar{\mathcal{F}}_j(u) := \begin{cases} (1-s_j)[u]^p_{s_j,p;U}, & u \in W^{s_j,p}(U), \quad u = 0 \quad \text{a.e. on } \bar{\mathcal{K}}_j \cap U, \\ +\infty, & \text{otherwise,} \end{cases} \tag{1.5}$$

If

$$(1-s_j)|\log \varepsilon_j| \longrightarrow \ell \qquad \text{for some } \ell \in [0,+\infty], \tag{1.6}$$

then $(\bar{\mathcal{F}}_j)_j$ $\Gamma(L^p(U))$-converges to

$$\bar{\mathcal{F}}^{(\ell)}(u) := \begin{cases} K_{n,p} \displaystyle\int_U |\nabla u|^p \, dx + \kappa_\ell K_{n,p} \operatorname{cap}_p(T) \int_U |u|^p \, dx, & u \in W^{1,p}(U), \\ +\infty, & \text{otherwise,} \end{cases} \tag{1.7}$$

where

$$\kappa_\ell := \begin{cases} \exp\left(-\frac{np}{n-p}\ell\right), & \ell < +\infty, \\ 0, & \ell = +\infty. \end{cases} \tag{1.8}$$

Thus the capacitary term is unchanged if $\ell = 0$, is modulated by $\exp(-np\ell/(n-p))$ if $\ell \in (0, +\infty)$, and disappears if $\ell = +\infty$.

Note that the logarithmic parameter in (1.6) also governs the interaction between the BBM limit and periodic homogenization as established in [8]. In that quadratic setting, the limit is a convex combination of the homogenized energy and the energy with averaged coefficients, with weights $e^{-2\ell}$ and $1 - e^{-2\ell}$, respectively. In formula (1.7) instead κ_ℓ affects only the capacitary coefficient, while the bulk term remains unchanged.

In the companion paper [28] we will study the analogous problem in the stochastic setting following the approach in [23]. On the one hand results similar to Theorems 5.1 and 5.5 can be obtained, on the other hand several new additional phenomena intimately related to the stochastic setting arise.

1.2. **Key results for the proofs.** The proofs are inspired by the Γ-convergence analysis at fixed s in [25], though several new ingredients are needed to deal with the simultaneous BBM localization and the homogenization of the obstacle sets.

To begin with, as the parameter of fractional differentiation s tends to 1 we need to revisit several well-known results and check their validity with uniform constants with respect to s itself (cf. Section 2).

In addition, upon rescaling the fractional capacities by $1 - s$, we establish the convergence of their values on bounded open sets to the corresponding value of the local capacity (cf. Corollary 3.2). To this aim we revisit the BBM argument, and provide an extension of its Γ-convergence version with respect to the $L^p_{\rm loc}(\mathbb{R}^n)$ topology (Proposition 3.1). In Remark 3.3 the validity of such a convergence statement for general bounded sets is discussed at length.

The second key technical tool is an extension of the nonlocal joining lemma of [25] with estimates uniform as $s_j \uparrow 1$ together with a further filling procedure (cf. Lemma 4.1). We recall that [25, Lemmas 3.8–3.9] build upon the ideas introduced by Ansini and Braides [2] for local energies on varying domains. The uniform joining lemma with filling in Lemma 4.1 below shows that given a sequence $u_j \to u$ in $L^p(U)$ satisfying the obstacle constraints and with bounded normalized fractional energy, the joining construction modifies u_j on annuli surrounding separated obstacles. The resulting function w_j agrees with u_j outside these annuli and is constant on a smaller concentric annulus, with value equal to the mean of u_j over that annulus. The modification preserves the constraint on the enclosed obstacles and changes the energy by an arbitrarily small relative error, uniformly in j. These constant annular values allow one to compare the internal energies with the normalized relative fractional capacities of the enclosed obstacles, multiplied by the p-th powers of the absolute values of the corresponding means. Summing these estimates and passing to the limit yields the capacitary contribution.

We complement this construction with a filling procedure: inside each ball enclosed by a constant annulus, we replace w_j by the same constant. The filled sequence v_j still converges to u in $L^p(U)$. The filling estimates separate the energy of w_j into the full energy of v_j and the internal energies accounting for the obstacles, with an error tending to zero as $s_j \uparrow 1$. The full energy of v_j then recovers

the bulk Dirichlet term through the BBM lower bound. This separation is the additional ingredient needed to obtain both contributions in the simultaneous limit; its relation to the concentration mechanisms is explained next.

1.3. **Heuristic ideas of the asymptotics.** Having described the main technical tools employed in what follows, we discuss next the mechanisms underlying the asymptotic analysis on an intuitive level. As already mentioned above, both terms in (1.3) arise from concentration of the nonlocal energies near the diagonal. For the gradient term this is the BBM limit, whereas the capacitary term is obtained by summing the energies needed to impose the constraint near each obstacle. Its coefficient is identified by the BBM limit of the fractional capacities after rescaling around the obstacles. One might therefore expect different limits according to the relative speed at which $\varepsilon_j \downarrow 0$ and $s_j \uparrow 1$. No such additional regime occurs at the scale (1.1) (cf. Theorem 5.1). Instead, at the scale (1.4) such a trichotomy takes place (cf. Theorem 5.5) affecting the coefficient of the capacitray term.

Let us first explain why the coefficient of the gradient term is unchanged at both scales (1.1) and (1.4). To this aim, we distinguish concentration around the obstacles from concentration at small interaction distances. Indeed, both the fractional and local scales λ_j, $\bar\lambda_j$ are $o(\varepsilon_j)$ as $j \to \infty$. One can therefore surround the obstacles by regions much smaller than the spacing of their centres but much larger than the obstacles themselves. These regions occupy an overall vanishing volume, and for a fixed Lipschitz function carry a vanishing part of the normalized energy. The capacitary correctors can thus contribute a nonvanishing energy on regions carrying a vanishing part of the energy of the fixed target function. Removing these regions from the latter does not change its BBM limit. The joining and filling estimates (4.15)–(4.16) in Lemma 4.1 below give the corresponding separation for general sequences of bounded energy.

The concentration of BBM energies near the diagonal involves a different scale. If the lower cutoff for the interaction distance is ε_j, then the normalized radial integral for an affine function satisfies, for every fixed $\delta > 0$ and all sufficiently large j,

$$(1-s_j)\int_{\varepsilon_j}^{\delta} t^{p(1-s_j)-1}\,dt = \frac{1}{p}(\delta^{p(1-s_j)} - \varepsilon_j^{p(1-s_j)}). \tag{1.9}$$

If $\varepsilon_j^{p(1-s_j)} \to a$, taking first $j \to \infty$ and then $\delta \downarrow 0$ gives the limit value $(1-a)/p$. Note that the energy contribution at distances greater than a fixed δ vanishes. On the other hand, the BBM mass need not be concentrated at distances below ε_j. The parameter

$$\ell_j := (1-s_j)|\log \varepsilon_j| \tag{1.10}$$

controls this radial distribution through $\varepsilon_j^{p(1-s_j)} = e^{-p\ell_j}$. It does not alter the spatial separation or the full bulk coefficient: the spatial regions occupied by the capacitary corrections can be separated from the energy of the limiting function without cutting the interaction kernel at ε_j as discussed above.

The capacitary coefficient requires a separate analysis. It is independent of the relative rates if the scaling of the obstacles is done according to the fractional critical choice in (1.1), as shown by the identity

$$\lambda_j^{n-s_jp} = \varepsilon_j^n \tag{1.11}$$

valid for every j. Indeed, by homogeneity of the fractional capacity, the normalized cost of one obstacle is

$$(1-s_j)\,\mathrm{cap}_{s_j,p}(\lambda_j T) = (1-s_j)\varepsilon_j^n\,\mathrm{cap}_{s_j,p}(T),$$

and Corollary 3.2 gives

$$(1-s_j)\,\mathrm{cap}_{s_j,p}(T) \longrightarrow K_{n,p}\,\mathrm{cap}_p(T).$$

Thus, as the number of obstacles per unit volume is of order ε_j^{-n}, the capacitary coefficient contains no residual dependence on ℓ_j. The exact balance (1.11) removes the rate dependence from the capacitary coefficient in Theorem 5.1. The uniform joining and filling estimates separate this contribution from the bulk BBM energy without imposing any relation between ε_j and s_j.

Three regimes arise instead if one uses the s-independent local critical scale $\bar\lambda_j := \varepsilon_j^{n/(n-p)}$ instead of λ_j. In that case the total normalized capacitary cost contains the additional factor

$$\varepsilon_j^{-n}\bar\lambda_j^{n-s_jp} = \varepsilon_j^{\frac{np(1-s_j)}{n-p}} = \exp\left(-\frac{np}{n-p}\ell_j\right). \tag{1.12}$$

It converges to 1, to $\exp(-np\ell/(n-p))$, or to 0 according as $\ell_j \to 0$, $\ell_j \to \ell \in (0,+\infty)$, or $\ell_j \to +\infty$. Thus spatial separation accounts for the retention of the full bulk term, while (1.12) gives the factor κ_ℓ in (1.8).

1.4. **Outline of the paper.** Section 2 contains the notation and the needed preliminaries on fractional Sobolev spaces, and the related uniform inequalities as $s \uparrow 1$. Section 3 establishes the convergence results for the normalized fractional capacities. Section 4 is dedicated to the proof of the uniform joining and filling lemma. In Section 5 we state and prove the main results in the general aperiodic setting modelled upon Delone sets of points. Appendix A collects some useful technical results used throughout the paper.

2. Preliminaries and uniform inequalities as $s \uparrow 1$

2.1. **Basic notation.** We denote by $\mathcal{L}^n(E)$ the Lebesgue measure of a measurable set E. We use the notation χ_A for the characteristic function of a set A, and $(u)_A := \fint_A u(x)\,dx$ for the mean value of an integrable function u if A is measurable with $0 < \mathcal{L}^n(A) < +\infty$. We write $B_r(x) := \{y \in \mathbb{R}^n : |y-x| < r\}$ and denote its closure by $\overline{B}_r(x)$. We also write $B_r := B_r(0)$, $\omega_n := \mathcal{L}^n(B_1)$. Equalities between functions are understood $\mathcal{L}^n$-almost everywhere (abbreviated as a.e.).

Standard Sobolev and fractional Sobolev spaces are denoted as usual. In particular, throughout this paper $n \ge 2$ and $p \in (1,n)$. For $s \in (0,1)$ with $sp < n$, set

$$p_s^* := \frac{np}{n-sp}, \qquad [u]_{s,p}^p := \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy. \tag{2.1}$$

We denote by $\dot W^{s,p}(\mathbb{R}^n)$ the completion of $C_c^\infty(\mathbb{R}^n)$ with respect to $[\,\cdot\,]_{s,p}$. By [9, Theorem 3.1], in the subconformal range $sp < n$ this abstract completion is canonically identified with

$$\dot W^{s,p}(\mathbb{R}^n) = \left\{u \in L^{p_s^*}(\mathbb{R}^n) : [u]_{s,p} < +\infty\right\}. \tag{2.2}$$

The same statement for $s = 1$ gives

$$\dot{W}^{1,p}(\mathbb{R}^n) = \left\{ u \in L^{p^*}(\mathbb{R}^n) : \nabla u \in L^p(\mathbb{R}^n;\mathbb{R}^n) \right\}, \qquad p^* = \frac{np}{n-p}, \tag{2.3}$$

and $C_c^\infty(\mathbb{R}^n)$ is dense in this space for the norm $\|\nabla u\|_{L^p(\mathbb{R}^n)}$.

2.2. **Uniform inequalities as $s \uparrow 1$.** We first recall the form of the fractional Poincaré–Wirtinger inequality which will be used below. Let $A \subset \mathbb{R}^n$ be a bounded, connected Lipschitz open set and let $O \subset A$ be measurable with $\mathcal{L}^n(O) > 0$. For every $s_0 \in (0,1)$ there exists $c = c(n,p,s_0,A,O) > 0$ such that

$$\|u - (u)_O\|^p_{L^p(A)} \le c(1-s)[u]^p_{s,p;A} \qquad \text{for every } s \in [s_0,1),\ u \in W^{s,p}(A). \tag{2.4}$$

For cubes and the mean on the whole cube this is precisely [39, Corollary 2.1]; the bounded Lipschitz-domain version follows from [39, Theorem 1.1], and the passage from the mean on A to the mean on O follows from Hölder's inequality. By scaling, if $z \in \mathbb{R}^n$ and $r > 0$, then

$$\|u - (u)_{z+rO}\|^p_{L^p(z+rA)} \le c(1-s) r^{sp} [u]^p_{s,p;z+rA}. \tag{2.5}$$

An elementary change of variables shows that the constant is uniform for families of pairs (A,O) obtained from a fixed pair by bi-Lipschitz transformations whose Lipschitz constants, together with those of their inverses, are uniformly bounded. This is the uniform version of the estimate used in the joining lemma.

The second global estimate which will be crucial in what follows is the sharp-order fractional Sobolev–Gagliardo–Nirenberg inequality of Maz'ya and Shaposhnikova. With the correction recorded in [35], [34, Theorem 1] states that

$$\|u\|^p_{L^{p^*_s}(\mathbb{R}^n)} \le c(n,p) \frac{s(1-s)}{(n-sp)^{p-1}} [u]^p_{s,p}, \qquad u \in \dot{W}^{s,p}(\mathbb{R}^n). \tag{2.6}$$

In particular, if $s \in [s_0,1)$ and $p \in (1,n)$, then

$$\|u\|^p_{L^{p^*_s}(\mathbb{R}^n)} \le c(n,p,s_0)(1-s)[u]^p_{s,p}. \tag{2.7}$$

Unlike (2.4), estimate (2.7) fixes the additive constant at infinity and is therefore the coercive estimate appropriate to homogeneous spaces.

We finally set

$$K_{n,p} := \frac{1}{p} \int_{\mathbb{S}^{n-1}} |e \cdot \sigma|^p \, d\mathcal{H}^{n-1}(\sigma), \tag{2.8}$$

where $e \in \mathbb{S}^{n-1}$ is arbitrary. With the convention (2.1), the Bourgain–Brezis–Mironescu formula [4] reads

$$\lim_{s\uparrow 1} (1-s)[u]^p_{s,p} = K_{n,p} \int_{\mathbb{R}^n} |\nabla u|^p \, dx \qquad \text{for } u \in W^{1,p}(\mathbb{R}^n). \tag{2.9}$$

3. Homogeneous Γ-convergence and capacities

In this section we prove the convergence of the rescaled fractional capacities of bounded open sets to the corresponding local capacity. For this purpose, we need a BBM formula in homogeneous spaces with respect to the $L^p_{\mathrm{loc}}(\mathbb{R}^n)$ topology. The results available in the literature concern related, but different, settings: the Gagliardo seminorm on a bounded Lipschitz open set, with the $L^p(U)$ topology, in [40, Theorem 8]; whole-space energies, with the $L^p(\mathbb{R}^n)$ topology, in [29, Theorem 3.1]; and energies with homogeneous Dirichlet conditions on a bounded Lipschitz open set, again with the global L^p topology, in [16, Theorem 2.1] for

$p = 2$ and in [31, Theorem 1.1] for general p and weighted kernels. For our purposes we need convergence in the local topology together with control at infinity. The following proposition provides this homogeneous version of the BBM result.

Proposition 3.1 (Homogeneous BBM Γ-convergence)**.** *Let $s_j \uparrow 1$ and define $F_j : L^p_{\rm loc}(\mathbb{R}^n) \to [0, +\infty]$ by*

$$F_j(u) := \begin{cases} (1 - s_j)[u]^p_{s_j,p}, & u \in \dot W^{s_j,p}(\mathbb{R}^n), \\ +\infty, & \text{otherwise.} \end{cases} \tag{3.1}$$

Then

$$\Gamma(L^p_{\rm loc}(\mathbb{R}^n))\text{-}\lim_{j\to+\infty} F_j = F, \tag{3.2}$$

where

$$F(u) := \begin{cases} K_{n,p} \displaystyle\int_{\mathbb{R}^n} |\nabla u|^p \, dx, & u \in \dot W^{1,p}(\mathbb{R}^n), \\ +\infty, & \text{otherwise.} \end{cases} \tag{3.3}$$

Moreover, $(F_j)_j$ is equicoercive in $L^p_{\rm loc}(\mathbb{R}^n)$: every sequence $(u_j)_j$ such that

$$\sup_j F_j(u_j) < +\infty$$

is relatively compact in $L^p_{\rm loc}(\mathbb{R}^n)$.

Proof. We start with compactness. By (2.7), whenever $\sup_j F_j(u_j) < +\infty$

$$\sup_j \|u_j\|_{L^{p^*_{s_j}}(\mathbb{R}^n)} < +\infty. \tag{3.4}$$

Consequently $(u_j)_j$ is bounded in $L^p(B_R)$ for every $R > 0$. The local compactness theorem of Bourgain–Brezis–Mironescu, in the form proved in [39, Theorem 1.2], applied on B_R and followed by a diagonal argument, gives relative compactness in $L^p_{\rm loc}(\mathbb{R}^n)$.

Let now $u_j \to u$ in $L^p_{\rm loc}(\mathbb{R}^n)$ and suppose, without loss of generality, that $\sup_j F_j(u_j) < +\infty$. For every $R > 0$, the local Γ-liminf inequality in [40, Theorem 8] gives

$$\liminf_{j\to+\infty} F_j(u_j) \ge K_{n,p} \int_{B_R} |\nabla u|^p \, dx. \tag{3.5}$$

Hence $u \in W^{1,p}_{\rm loc}(\mathbb{R}^n)$ and, by letting $R \to +\infty$,

$$\liminf_{j\to+\infty} F_j(u_j) \ge K_{n,p} \int_{\mathbb{R}^n} |\nabla u|^p \, dx. \tag{3.6}$$

It remains to identify the behaviour at infinity. Up to a subsequence, $u_j \to u$ almost everywhere in $\mathbb{R}^n$; since $p^*_{s_j} \to p^*$, Fatou's lemma and (3.4) imply

$$\int_{\mathbb{R}^n} |u|^{p^*} \, dx \le \liminf_{j\to+\infty} \int_{\mathbb{R}^n} |u_j|^{p^*_{s_j}} \, dx < +\infty. \tag{3.7}$$

Thus $u \in \dot W^{1,p}(\mathbb{R}^n)$ by (2.3), and (3.6) is the required Γ-liminf inequality.

To prove the Γ-limsup inequality, let $u \in \dot W^{1,p}(\mathbb{R}^n)$. By [9, Theorem 3.1], there exist $\varphi_k \in C^\infty_c(\mathbb{R}^n)$ such that

$$\varphi_k \to u \quad \text{in } L^{p^*}(\mathbb{R}^n), \qquad \nabla\varphi_k \to \nabla u \quad \text{in } L^p(\mathbb{R}^n; \mathbb{R}^n). \tag{3.8}$$

For every fixed k, the BBM formula yields

$$\lim_{j\to+\infty} F_j(\varphi_k) = K_{n,p}\int_{\mathbb{R}^n} |\nabla\varphi_k|^p\,dx. \tag{3.9}$$

A standard diagonal choice $k = k(j) \uparrow +\infty$ in (3.8)–(3.9) provides a sequence converging to u in $L^p_{\mathrm{loc}}(\mathbb{R}^n)$ and proves the Γ-limsup inequality. □

We record the capacitary consequence needed below. If $T \subset \mathbb{R}^n$ is a bounded open set, define

$$\mathrm{cap}_{s,p}(T) := \inf\Big\{[u]^p_{s,p} : u \in \dot W^{s,p}(\mathbb{R}^n),\ u \geq 1 \text{ a.e. on } T\Big\} \tag{3.10}$$

and

$$\mathrm{cap}_{p}(T) := \inf\left\{\int_{\mathbb{R}^n} |\nabla u|^p\,dx : u \in \dot W^{1,p}(\mathbb{R}^n),\ u \geq 1 \text{ a.e. on } T\right\}. \tag{3.11}$$

If $T \Subset B_R$, the corresponding relative capacities are

$$\begin{aligned} C_{s,p}(T,B_R) := \inf\big\{[u]^p_{s,p} : u \in \dot W^{s,p}(\mathbb{R}^n),\ u = 0 \text{ a.e. on } B_R^c,\\ u \geq 1 \text{ a.e. on } T\big\}. \end{aligned} \tag{3.12}$$

$$\begin{aligned} C_{p}(T,B_R) := \inf\left\{\int_{\mathbb{R}^n} |\nabla u|^p\,dx : u \in \dot W^{1,p}(\mathbb{R}^n),\ u = 0 \text{ a.e. on } B_R^c,\right.\\ \left. u \geq 1 \text{ a.e. on } T\right\}. \end{aligned} \tag{3.13}$$

Corollary 3.2 (Convergence of the minimum problems)**.** *Let $T \subset \mathbb{R}^n$ be a bounded open set and let $s_j \uparrow 1$. Then*

$$\lim_{j\to+\infty} (1-s_j)\,\mathrm{cap}_{s_j,p}(T) = K_{n,p}\,\mathrm{cap}_p(T). \tag{3.14}$$

If $T \Subset B_R$, then

$$\lim_{j\to+\infty} (1-s_j) C_{s_j,p}(T,B_R) = K_{n,p} C_p(T,B_R). \tag{3.15}$$

Finally, if $L_j \to +\infty$ and $T \Subset B_{L_j}$ eventually, then

$$\lim_{j\to+\infty} (1-s_j) C_{s_j,p}(T,B_{L_j}) = K_{n,p}\,\mathrm{cap}_p(T). \tag{3.16}$$

Proof. Formula (3.14) follows from Proposition 3.1 and the fundamental theorem of Γ-convergence (cf. [20, Theorem 7.8]), applied to the functionals F_j, F restricted to the closed set

$$\mathcal{A}_T := \{v \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : v \geq 1 \text{ a.e. on } T\}.$$

Equicoercivity and the lower bound are inherited from Proposition 3.1. To check that recovery sequences can be chosen in $\mathcal{A}_T$, let $v \in \mathcal{A}_T \cap \dot W^{1,p}(\mathbb{R}^n)$ and take $v_R = \eta_R v$, where $\eta_R \in C^\infty_c(B_{2R})$, $0 \leq \eta_R \leq 1$, $\eta_R = 1$ on B_R, $|\nabla\eta_R| \leq c/R$, and $T \Subset B_R$. Then $v_R \in \mathcal{A}_T \cap W^{1,p}(\mathbb{R}^n)$, and

$$\int_{\mathbb{R}^n} |v|^p |\nabla\eta_R|^p\,dx \leq c\left(\int_{B_{2R}\setminus B_R} |v|^{p^*}\,dx\right)^{p/p^*} \longrightarrow 0. \tag{3.17}$$

Thus $\nabla v_R \to \nabla v$ in $L^p(\mathbb{R}^n;\mathbb{R}^n)$. The BBM formula for fixed R, followed by a diagonal choice $R = R(j) \to +\infty$, gives the required recovery sequence. The convergence of the minimum values now follows from the cited theorem.

A similar argument works for the relative capacities. We give the details for the reader's convenience.

Fix R with $T \Subset B_R$ and choose admissible u_j such that

$$(1-s_j)[u_j]^p_{s_j,p} \le (1-s_j)C_{s_j,p}(T,B_R) + \frac{1}{j}.$$

By truncation we may assume $0 \le u_j \le 1$. A fixed function in $C^\infty_c(B_R)$ equal to one near $\overline{T}$, together with the BBM formula, bounds the right-hand side uniformly in j. Along a subsequence realizing the lower limit, Proposition 3.1 therefore gives $u_j \to u$ in $L^p_{\mathrm{loc}}(\mathbb{R}^n)$, with $u \in \dot W^{1,p}(\mathbb{R}^n)$. Since $u_j = 1$ a.e. on T and $u_j = 0$ a.e. on B_R^c, the same conditions hold for u. The lower bound in that proposition yields

$$\liminf_{j\to+\infty}(1-s_j)C_{s_j,p}(T,B_R) \ge K_{n,p}\int_{\mathbb{R}^n}|\nabla u|^p\,dx \ge K_{n,p}C_p(T,B_R).$$

Conversely, every v admissible for $C_p(T,B_R)$ belongs to $W^{1,p}(\mathbb{R}^n)$, since it has bounded support and belongs to $L^{p^*}(\mathbb{R}^n)$. It is therefore admissible for $C_{s_j,p}(T,B_R)$ for every j, and

$$\limsup_{j\to+\infty}(1-s_j)C_{s_j,p}(T,B_R) \le \lim_{j\to+\infty}(1-s_j)[v]^p_{s_j,p} = K_{n,p}\int_{\mathbb{R}^n}|\nabla v|^p\,dx.$$

Taking the infimum over v proves (3.15).

Finally, fix R with $T \Subset B_R$. For all sufficiently large j we have $L_j \ge R$, and monotonicity with respect to the outer domain gives

$$\mathrm{cap}_{s_j,p}(T) \le C_{s_j,p}(T,B_{L_j}) \le C_{s_j,p}(T,B_R). \tag{3.18}$$

First letting $j \to +\infty$ and then $R \to +\infty$, and using $C_p(T,B_R) \downarrow \mathrm{cap}_p(T)$ (which follows from the same cut-off argument as in (3.17)), proves (3.16). □

Remark 3.3 (Capacity of thin obstacles)**.** *The convergence in* (3.14) *does not hold for every bounded set. Recall that capacity is extended by outer regularity to general bounded sets $T \subset \mathbb{R}^n$, i.e.*

$$\mathrm{cap}_{s,p}(T) := \inf\{\mathrm{cap}_{s,p}(O) : T \subset O,\ O \text{ bounded and open}\}.$$

Outer regularity and Corollary 3.2 give only

$$\limsup_{j\to\infty}(1-s_j)\,\mathrm{cap}_{s_j,p}(T) \le K_{n,p}\,\mathrm{cap}_p(T). \tag{3.19}$$

Indeed, the inequality may be strict: critical Cantor-type compact sets can have Hausdorff dimension $n-p$ and positive p-capacity, while their fractional capacities vanish for every $s \in (0,1)$ as $n-p < n-sp$ (see [1, Section 5.3 and Theorem 5.1.13]*).*

For $p = 2$, the upper bound for compact sets and the full limit for closures of bounded Lipschitz open sets are proved in [13, Proposition 1.2]*.*

On the other hand, using the monotonicity property of capacities under set inclusion, the equality in (3.14) *can be established more generally if* $\mathrm{cap}_p(T) = \mathrm{cap}_p(\mathrm{int}T)$.

Following [25, p. 3510], for $m \ge 2$, $R > 1$, and $T \Subset B_R$, we also introduce the interior relative capacity

$$\mathrm{cap}_{s,p}(T,B_{mR};R) := \inf\big\{[\varphi]^p_{s,p;B_{mR}} : \varphi \in \dot W^{s,p}(\mathbb{R}^n),\quad \varphi = 0 \text{ a.e. on } \mathbb{R}^n \setminus B_R,\ \varphi \ge 1 \text{ a.e. on } T\big\}. \tag{3.20}$$

The following lemma gives the uniform-in-s comparison of the interior and global capacities, together with the complete limit for expanding domains. It is the counterpart of [25, Lemma 2.12] in the present normalization.

Lemma 3.4 (Uniform comparison and convergence of the interior capacity)**.** *Let $T \subset \mathbb{R}^n$ be bounded and open and let $m \geq 2$ be fixed. There is a constant $c_m = c(n,p,s_0,m) > 0$ such that, for every $s \in [s_0,1)$ and $R > 1$ with $T \Subset B_R$,*

$$\frac{\operatorname{cap}_{s,p}(T)}{1+c_m(1-s)} \leq \operatorname{cap}_{s,p}(T, B_{mR}; R) \leq C_{s,p}(T, B_R). \tag{3.21}$$

Consequently, if $L_j \to +\infty$ and $s_j \uparrow 1$, then

$$\lim_{j\to+\infty} (1-s_j) \operatorname{cap}_{s_j,p}(T, B_{mL_j}; L_j) = K_{n,p} \operatorname{cap}_p(T). \tag{3.22}$$

Proof. Let φ be admissible in (3.20). Since $\varphi = 0$ on $B_{mR} \setminus B_R$, the scaled uniform Poincaré–Wirtinger inequality (2.5), applied to the fixed pair $(B_m, B_m \setminus B_1)$, gives

$$\|\varphi\|^p_{L^p(B_{mR})} \leq c_m(1-s) R^{sp} [\varphi]^p_{s,p;B_{mR}}. \tag{3.23}$$

A direct integration of the tail, as in (ii) of Lemma A.1, and (3.23) therefore yield

$$\begin{aligned} [\varphi]^p_{s,p} &= [\varphi]^p_{s,p;B_{mR}} + 2\int_{B_R}\int_{B^c_{mR}} \frac{|\varphi(x)|^p}{|x-y|^{n+sp}}\,dy\,dx \\ &\leq [\varphi]^p_{s,p;B_{mR}} + \frac{2n\omega_n}{sp}((m-1)R)^{-sp}\|\varphi\|^p_{L^p(B_R)} \\ &\leq \big(1+c_m(1-s)\big)[\varphi]^p_{s,p;B_{mR}}. \end{aligned} \tag{3.24}$$

Taking the infimum proves the first inequality in (3.21). The second follows by restricting the energy of each competitor for $C_{s,p}(T,B_R)$ to $B_{mR}\times B_{mR}$. Multiplying the resulting inequalities by $1-s_j$, with $R = L_j$, and applying Corollary 3.2 proves (3.22). □

Remark 3.5. *The use of the scaled uniform Poincaré–Wirtinger inequality leading to (3.23) is not necessary. Indeed, let $C_{n,p}$ denote the constant $c(n,p)$ in (2.6). Since $\varphi = 0$ outside B_R, Hölder's inequality and (2.6) give*

$$\|\varphi\|^p_{L^p(B_R)} \leq \omega_n^{sp/n} R^{sp} \|\varphi\|^p_{L^{p^*_s}(\mathbb{R}^n)} \leq C_{n,p}\omega_n^{sp/n} R^{sp} \frac{s(1-s)}{(n-sp)^{p-1}} [\varphi]^p_{s,p}.$$

Inserting this estimate in the second line of (3.24) yields

$$\left(1 - \frac{2nC_{n,p}\omega_n^{1+sp/n}}{p(n-sp)^{p-1}(m-1)^{sp}}(1-s)\right)[\varphi]^p_{s,p} \leq [\varphi]^p_{s,p;B_{mR}}.$$

Having fixed $p \in (1,n)$, the coefficient multiplying $1-s$ is uniformly bounded for $s \in (0,1)$ and $m \geq 2$. Thus the argument above gives the same asymptotically exact comparison.

Finally, we point out that the gain with respect to the fixed-order comparison is that the relative error in (3.21) tends to zero while m remains fixed. In [25, Lemma 2.12], the corresponding conclusion is obtained by letting the ratio of the outer and inner radii diverge. Here the uniform Poincaré inequality supplies the additional factor $1-s_j$, so a fixed ratio m suffices and no relation between L_j and s_j is needed. This is a consequence of the relative tail estimate, rather than merely of multiplying a fixed-order capacity estimate by $1-s_j$.

4. The uniform joining lemma with filling

In this section we establish the main technical tool to prove the asymptotics of the rescaled Gagliardo seminorms. More precisely, we prove a joining and filling lemma for sequences with bounded rescaled fractional energy. The construction follows [25, Lemmas 3.8–3.9], up to checking that the relevant constants there are uniform as $s_j \uparrow 1$. Only separation of the centres is needed to make the functions constant on suitable annuli. Filling the enclosed balls with the same constants then separates the energy of the filled sequence from the internal energies of the original one, with an error controlled by the ratio of the two radii. These estimates are collected in Lemma 4.1. The conclusion concerning averages on Voronoï cells related to Delone sets of points is stated separately in Corollary 4.3.

In this section only, the restriction $p < n$ is dropped. Let $U \subset \mathbb{R}^n$ be a bounded open set, let $p \in (1, +\infty)$, and let $s_j \uparrow 1$. After discarding finitely many terms, fix $s_0 \in (\max\{0, 1 - n/p\}, 1)$ such that $s_j \in [s_0, 1)$. For $u \in L^p(U)$ and every measurable set $E \subseteq U \times U$ define the normalized interaction energy by

$$\mathcal{D}_j(u, E) := (1 - s_j) \int_E \frac{|u(x) - u(y)|^p}{|x - y|^{n + s_j p}} \, dx \, dy. \tag{4.1}$$

For every $j \in \mathbb{N}$, let $\mathcal{I}_j$ be a finite index set, let $\{x_j^{\mathtt{i}}\}_{\mathtt{i} \in \mathcal{I}_j} \subset U$, and let $\varepsilon_j > 0$ be such that

$$|x_j^{\mathtt{i}} - x_j^{\mathtt{k}}| \geq 2\varepsilon_j \quad \text{if } \mathtt{i} \neq \mathtt{k}, \qquad B_{\varepsilon_j}(x_j^{\mathtt{i}}) \subset U \quad \text{for every } \mathtt{i} \in \mathcal{I}_j. \tag{4.2}$$

We assume throughout that $\varepsilon_j \to 0$. With fixed $m \in \mathbb{N}$, $m \geq 2$, for integers $h \geq 1$ and $\mathtt{i} \in \mathcal{I}_j$ set, as in [25, Lemma 3.9],

$$B_j^{\mathtt{i},h} := B_{m^{-3h}\varepsilon_j}(x_j^{\mathtt{i}}), \qquad C_j^{\mathtt{i},h} := B_{m^{-3h-1}\varepsilon_j}(x_j^{\mathtt{i}}) \setminus \overline{B}_{m^{-3h-2}\varepsilon_j}(x_j^{\mathtt{i}}). \tag{4.3}$$

We complement the joining lemma of [25, Lemma 3.9] needed to extract the underlying capacitary contribution on sequences $(u_j)_j$ with equi-bounded $\mathcal{D}_j(u_j, U \times U)$ energies, with a filling procedure to recover the BBM localization. In addition, on those sequences, filling separates the two contributions with an error tending to zero without sending m to infinity.

Lemma 4.1 (Uniform joining lemma with filling)**.** *Let $u_j \to u$ strongly in $L^p(U)$ and assume that*

$$\sup_j \mathcal{D}_j(u_j, U \times U) < +\infty. \tag{4.4}$$

For every fixed pair of integers $m \geq 2$, $N \geq 1$, there exist $h_j \in \{1, \dots, N\}$ and $w_j \in W^{s_j,p}(U)$ such that

$$w_j = u_j \quad \text{on } U \setminus \bigcup_{\mathtt{i} \in \mathcal{I}_j} \big(\overline{B}_j^{\mathtt{i},h_j} \setminus B_j^{\mathtt{i},h_j+1}\big), \tag{4.5}$$

$$w_j = (u_j)_{C_j^{\mathtt{i},h_j}} \quad \text{on } C_j^{\mathtt{i},h_j}, \tag{4.6}$$

and, for every measurable set $E \subseteq U \times U$,

$$|\mathcal{D}_j(u_j, E) - \mathcal{D}_j(w_j, E)| \leq \frac{c_m}{N} \mathcal{D}_j(u_j, U \times U). \tag{4.7}$$

Here and below $c_m = c(n, p, s_0, m) > 0$ is independent of j and N. Moreover,

$$w_j \longrightarrow u \qquad \text{strongly in } L^p(U). \tag{4.8}$$

For the filling construction, let

$$\rho_j := m^{-3h_j-2}\varepsilon_j, \qquad D_j^{\mathtt{i}} := B_{\rho_j}(x_j^{\mathtt{i}}), \qquad P_j^{\mathtt{i}} := B_{m\rho_j}(x_j^{\mathtt{i}}). \tag{4.9}$$

Then $C_j^{\mathtt{i},h_j} = P_j^{\mathtt{i}} \setminus \overline{D_j^{\mathtt{i}}}$. *Define the filled sequence by*

$$v_j(x) := \begin{cases} (u_j)_{C_j^{\mathtt{i},h_j}}, & x \in D_j^{\mathtt{i}}, \quad \mathtt{i} \in \mathcal{I}_j, \\ w_j(x), & x \in U \setminus \bigcup_{\mathtt{i}\in\mathcal{I}_j} D_j^{\mathtt{i}}. \end{cases} \tag{4.10}$$

Then $v_j \in W^{s_j,p}(U)$,

$$v_j \longrightarrow u \qquad \text{strongly in } L^p(U), \tag{4.11}$$

and the filling satisfies the following estimates. For every measurable $E \subseteq U \times U$, *set*

$$\mathcal{S}_j(E) := \sum_{\mathtt{i}\in\mathcal{I}_j} \mathcal{D}_j\big(w_j, E \cap (P_j^{\mathtt{i}} \times P_j^{\mathtt{i}})\big), \qquad \mathcal{S}_j := \mathcal{S}_j(U \times U). \tag{4.12}$$

There is a constant $c = c(n,p,s_0) > 0$, *independent of* j, m, N, *such that, on putting*

$$\delta_{j,m} := c(1-s_j)^{1/p} m^{-s_j}, \tag{4.13}$$

we have

$$|\mathcal{D}_j(w_j,E) - \mathcal{D}_j(v_j,E) - \mathcal{S}_j(E)| \le \delta_{j,m}\big(\mathcal{D}_j(v_j, U\times U) + \mathcal{S}_j\big). \tag{4.14}$$

In particular,

$$\mathcal{D}_j(w_j, U\times U) \ge (1-\delta_{j,m})\big(\mathcal{D}_j(v_j,U\times U) + \mathcal{S}_j\big) \tag{4.15}$$

and

$$\mathcal{D}_j(w_j, U\times U) \le (1+\delta_{j,m})\big(\mathcal{D}_j(v_j,U\times U) + \mathcal{S}_j\big). \tag{4.16}$$

For each fixed $m \ge 2$, $\delta_{j,m} \to 0$ *as* $j \to \infty$.

Finally, if $u_j \in L^\infty(U)$, *then*

$$\|w_j\|_{L^\infty(U)} \le \|u_j\|_{L^\infty(U)}, \qquad \|v_j\|_{L^\infty(U)} \le \|u_j\|_{L^\infty(U)}. \tag{4.17}$$

Proof. Given $m, N \in \mathbb{N}$, for every $j \in \mathbb{N}$ and $h \in \{1,\dots,N\}$ fixed, let

$$A_j^{\mathtt{i},h} := B_j^{\mathtt{i},h} \setminus \overline{B}_j^{\mathtt{i},h+1}, \qquad A_j^h := \bigcup_{\mathtt{i}\in\mathcal{I}_j} A_j^{\mathtt{i},h}, \qquad z_j^{\mathtt{i},h} := (u_j)_{C_j^{\mathtt{i},h}}. \tag{4.18}$$

Put $\rho_{j,h} := m^{-3h}\varepsilon_j$. For every $\mathtt{i} \in \mathcal{I}_j$, choose $\varphi_j^{\mathtt{i},h} \in W^{1,\infty}(\mathbb{R}^n)$ such that

$$\begin{gathered} 0 \le \varphi_j^{\mathtt{i},h} \le 1, \qquad \varphi_j^{\mathtt{i},h} = 1 \quad \text{on } C_j^{\mathtt{i},h}, \qquad \varphi_j^{\mathtt{i},h} = 0 \quad \text{on } \mathbb{R}^n \setminus A_j^{\mathtt{i},h}, \\ \operatorname{Lip}(\varphi_j^{\mathtt{i},h}) \le \frac{2m^2}{\rho_{j,h}}. \end{gathered} \tag{4.19}$$

Such a cut-off exists because $C_j^{\mathtt{i},h}$ is separated from both components of $\partial A_j^{\mathtt{i},h}$ by a distance bounded from below by $\rho_{j,h}/(2m^2)$. Since the sets $A_j^{\mathtt{i},h}$ are mutually disjoint, the joining modification is given explicitly by

$$\begin{aligned} w_j^h(x) &:= \textstyle\sum_{\mathtt{i}\in\mathcal{I}_j} \varphi_j^{\mathtt{i},h}(x) z_j^{\mathtt{i},h} + \Big(1 - \sum_{\mathtt{i}\in\mathcal{I}_j} \varphi_j^{\mathtt{i},h}(x)\Big) u_j(x) \\ &= u_j(x) + \textstyle\sum_{\mathtt{i}\in\mathcal{I}_j} \varphi_j^{\mathtt{i},h}(x)\big(z_j^{\mathtt{i},h} - u_j(x)\big). \end{aligned} \tag{4.20}$$

In particular, $w_j^h = z_j^{\mathtt{i},h}$ on $C_j^{\mathtt{i},h}$ and $w_j^h = u_j$ on $U \setminus A_j^h$.

Apply [25, Lemma 3.8] with $(A')^h_{\mathtt{i}} = C^{\mathtt{i},h}_j$, $A^h_{\mathtt{i}}$ as in (4.18), and $z_{\mathtt{i}} = z^{\mathtt{i},h}_j$. The radius ρ used in that lemma is $\rho_{j,h} = m^{-3h}\varepsilon_j$, and its resulting function is precisely (4.20). The normalized form of [25, Lemma 3.8, formula (3.8)] gives, for every measurable set $E \subseteq U \times U$,

$$\begin{aligned}\big|\mathcal{D}_j(u_j, E) - \mathcal{D}_j(w^h_j, E)\big| \\ \leq c\mathcal{D}_j(u_j, U \times A^h_j) + cm^{2p}\left(\frac{m^{3h}}{\varepsilon_j}\right)^{s_j p} \sum_{\mathtt{i}\in\mathcal{I}_j} \int_{A^{\mathtt{i},h}_j} |u_j - z^{\mathtt{i},h}_j|^p \, dx, \quad (4.21)\end{aligned}$$

where $c = c(n, p, s_0) > 0$ is independent of j and h.

Since the constant in [25, Lemma 3.8] is stated there for fixed s, we verify that its normalized version is uniform as $s \uparrow 1$. Fix j and h, and abbreviate

$$s := s_j, \qquad \rho := \rho_{j,h}, \qquad A_{\mathtt{i}} := A^{\mathtt{i},h}_j, \qquad z_{\mathtt{i}} := z^{\mathtt{i},h}_j, \qquad \varphi_{\mathtt{i}} := \varphi^{\mathtt{i},h}_j.$$

By (4.19), $0 \leq \varphi_{\mathtt{i}} \leq 1$, $\mathrm{Lip}(\varphi_{\mathtt{i}}) \leq 2m^2/\rho$, and the supports of the cut-offs are contained in the mutually disjoint sets $A_{\mathtt{i}}$. In the proof of [25, Lemma 3.8], the dependence on s occurs only in the singular-kernel estimates in formulas (3.11), (3.14)–(3.15), and (3.17)–(3.18); the constants coming from the elementary inequalities for the p-power are independent of s.

We first consider the interactions inside a single $A_{\mathtt{i}}$. The term produced by the cut-off is bounded, after multiplication by $1 - s$, by

$$(1-s)\left(\frac{2m^2}{\rho}\right)^p \sum_{\mathtt{i}\in\mathcal{I}_j} \int_{A_{\mathtt{i}}} |u_j(y) - z_{\mathtt{i}}|^p \left(\int_{A_{\mathtt{i}}} \frac{dx}{|x-y|^{n-(1-s)p}}\right) dy. \quad (4.22)$$

Here $\nu = n - (1-s)p$ belongs to $(0, n)$ by the choice of s_0. Since $\mathcal{L}^n(A_{\mathtt{i}}) \leq c_n\rho^n$, item (i) of Lemma A.1 gives

$$(1-s)\int_{A_{\mathtt{i}}} \frac{dx}{|x-y|^{n-(1-s)p}} \leq c(n,p)\rho^{p(1-s)}. \quad (4.23)$$

Indeed, the factor $(n-\nu)^{-1} = (p(1-s))^{-1}$ is exactly cancelled by the normalization $1 - s$. Consequently, (4.22) is at most

$$c(n,p)m^{2p}\rho^{-sp} \sum_{\mathtt{i}\in\mathcal{I}_j} \int_{A_{\mathtt{i}}} |u_j - z_{\mathtt{i}}|^p \, dx. \quad (4.24)$$

This is the uniform version of formula (3.11) in the cited proof.

We next consider the interactions between distinct annuli. The argument leading to [25, formula (3.14)] uses only the separation of the centres. More precisely, the packing estimate

$$\#\big\{\mathtt{k} \in \mathcal{I}_j : \ell\varepsilon_j < |x^{\mathtt{i}}_j - x^{\mathtt{k}}_j| \leq (\ell+1)\varepsilon_j\big\} \leq c_n\ell^{n-1}$$

holds for every integer $\ell \geq 1$ and every finite $2\varepsilon_j$-separated family; no relative-density assumption is needed. The shells with $\ell \geq 1$ then contain all the other centres.

Hence, for $x \in A_{\mathtt{i}}$,

$$\begin{aligned}\sum_{\mathtt{k}\neq\mathtt{i}} \int_{A_{\mathtt{k}}} \frac{dy}{|x-y|^{n+sp}} &\le c(n,p)\frac{\rho^n}{\varepsilon_j^{n+sp}} \sum_{\ell=1}^{+\infty} \ell^{-1-sp} \\ &\le c(n,p)\rho^{-sp} \sum_{\ell=1}^{+\infty} \ell^{-1-sp} \le c(n,p,s_0)\rho^{-sp}.\end{aligned} \tag{4.25}$$

In the second inequality we used $\rho = m^{-3h}\varepsilon_j \le \varepsilon_j$. Multiplication by $1-s \le 1$ preserves this bound. Thus the constant in formulas (3.14)–(3.15) is uniform for $s \in [s_0, 1)$.

It remains to check the locality-defect term, corresponding to formulas (3.16)–(3.19) of the cited proof. For $|x-y| < \rho$, the Lipschitz estimate for $\varphi_{\mathtt{i}}$ and item (i) of Lemma A.1 give, including the radial constant,

$$(1-s)\rho^{-p} \int_{B_\rho(x)} \frac{dy}{|x-y|^{n-(1-s)p}} = \frac{n\omega_n}{p}\rho^{-sp} \tag{4.26}$$

(this fact is already explicit in formula (3.17) of [25]). For $|x-y| \ge \rho$, we use $|\varphi_{\mathtt{i}}(x) - \varphi_{\mathtt{i}}(y)| \le 1$ and item (ii) of the same lemma to obtain

$$(1-s) \int_{\mathbb{R}^n \setminus B_\rho(x)} \frac{dy}{|x-y|^{n+sp}} = \frac{n\omega_n(1-s)}{sp}\rho^{-sp} \le \frac{n\omega_n}{s_0 p}\rho^{-sp}. \tag{4.27}$$

Therefore the normalized versions of (3.17) and (3.18), and hence of (3.19), are bounded by a constant of the form $c(n,p,s_0)m^{2p}$.

Finally, the passage from $E = U\times U$ to an arbitrary measurable $E \subseteq U\times U$ uses only that $w_j^h = u_j$ outside A_j^h and the last algebraic decomposition in the proof of [25, Lemma 3.8]; it introduces no further s-dependent constant. Combining (4.24), (4.25), (4.26), and (4.27) proves that the constant in (4.21) may indeed be chosen as $c = c(n,p,s_0)$, uniformly in j and h.

The uniform scaled Poincaré–Wirtinger inequality gives

$$\sum_{\mathtt{i}\in\mathcal{I}_j} \int_{A_j^{\mathtt{i},h}} |u_j - z_j^{\mathtt{i},h}|^p \, dx \le c_m (m^{-3h}\varepsilon_j)^{s_j p} \mathcal{D}_j(u_j, A_j^h \times A_j^h). \tag{4.28}$$

Therefore, (4.21) and (4.28) yield

$$\left|\mathcal{D}_j(u_j,E) - \mathcal{D}_j(w_j^h,E)\right| \le c_m \mathcal{D}_j(u_j, U\times A_j^h). \tag{4.29}$$

By summing up and averaging on h, being the A_j^h's disjoint, we find $h_j \in \{1,\dots,N\}$ such that

$$\mathcal{D}_j(u_j, U\times A_j^{h_j}) \le \frac{1}{N}\mathcal{D}_j\left(u_j, U\times \bigcup_{h=1}^N A_j^h\right) \le \frac{1}{N}\mathcal{D}_j(u_j, U\times U). \tag{4.30}$$

Set $w_j := w_j^{h_j}$. Then (4.5) and (4.6) are satisfied by construction, and (4.29)–(4.30) imply (4.7).

To prove that $(w_j)_{j\in\mathbb{N}}$ converges to u in $L^p(U)$, we use again (4.28), with radius $m^{-3h_j}\varepsilon_j$, and the very definition of w_j as a convex combination of u_j and the mean value $(u_j)_{C_j^{\mathtt{i},h_j}}$ on $B_j^{\mathtt{i},h_j} \setminus \overline{B}_j^{\mathtt{i},h_j+1}$. We obtain

$$\|u_j - w_j\|_{L^p(U)}^p = \sum_{\mathtt{i}\in\mathcal{I}_j} \|u_j - w_j\|_{L^p(B_j^{\mathtt{i},h_j}\setminus B_j^{\mathtt{i},h_j+1})}^p$$

$$\le \sum_{\mathtt{i}\in\mathcal{I}_j} \|u_j - (u_j)_{C_j^{\mathtt{i},h_j}}\|^p_{L^p(B_j^{\mathtt{i},h_j}\setminus B_j^{\mathtt{i},h_j+1})}$$
$$\le c_m \varepsilon_j^{s_j p} \mathcal{D}_j(u_j, U\times U). \tag{4.31}$$

Since $\varepsilon_j \to 0$, $s_j \ge s_0$, and (4.4) holds, the right-hand side is infinitesimal. Thus (4.8) follows from $u_j \to u$ in $L^p(U)$.

We now fill the balls $D_j^{\mathtt{i}}$ and check all the stated properties of v_j. Since $w_j = (u_j)_{C_j^{\mathtt{i},h_j}}$ on $C_j^{\mathtt{i},h_j} = P_j^{\mathtt{i}} \setminus \overline{D_j^{\mathtt{i}}}$, the function v_j is constant on the whole ball $P_j^{\mathtt{i}}$. Furthermore, by the construction of w_j and the uniform Poincaré–Wirtinger inequality,

$$\|v_j - w_j\|^p_{L^p(U)} = \sum_{\mathtt{i}\in\mathcal{I}_j} \int_{D_j^{\mathtt{i}}} |w_j - (u_j)_{C_j^{\mathtt{i},h_j}}|^p\,dx$$
$$\le c_m \varepsilon_j^{s_j p} \mathcal{D}_j(u_j, U\times U). \tag{4.32}$$

Thus (4.11) follows from (4.8).

We now prove the energy estimates directly by applying Poincaré–Wirtinger on a fixed annulus. This also establishes that $v_j \in W^{s_j,p}(U)$, without a separate geometric comparison of the interactions of the filled balls.

Set

$$G_j := (U\times U) \setminus \bigcup_{\mathtt{i}\in\mathcal{I}_j} (P_j^{\mathtt{i}} \times P_j^{\mathtt{i}}). \tag{4.33}$$

Since $m \ge 2$, the ball $B_{2\rho_j}(x_j^{\mathtt{i}})$ is contained in $P_j^{\mathtt{i}}$, and $w_j = (u_j)_{C_j^{\mathtt{i},h_j}}$ on $B_{2\rho_j}(x_j^{\mathtt{i}}) \setminus \overline{B}_{\rho_j}(x_j^{\mathtt{i}})$. Applying (2.5) to the pair $(B_2, B_2\setminus\overline{B}_1)$ and scaling by ρ_j gives

$$\rho_j^{-s_j p} \int_{D_j^{\mathtt{i}}} |w_j - (u_j)_{C_j^{\mathtt{i},h_j}}|^p\,dx \le c(1-s_j)[w_j]^p_{s_j,p;B_{2\rho_j}(x_j^{\mathtt{i}})}$$
$$\le c\mathcal{D}_j(w_j, P_j^{\mathtt{i}}\times P_j^{\mathtt{i}}). \tag{4.34}$$

Here $c = c(n,p,s_0)$ is independent of m, j, N and ρ_j.

For $x \in D_j^{\mathtt{i}}$, the distance to $U\setminus P_j^{\mathtt{i}}$ is at least $(m-1)\rho_j \ge m\rho_j/2$. Thus

$$\int_{U\setminus P_j^{\mathtt{i}}} \frac{dy}{|x-y|^{n+s_j p}} \le c(m\rho_j)^{-s_j p}. \tag{4.35}$$

Note that $w_j = v_j$ outside the inner balls. Using $|a-b|^p \le 2^{p-1}(|a|^p + |b|^p)$ for interactions between different inner balls, (4.34) and (4.35) therefore imply

$$\mathcal{D}_j(w_j - v_j, G_j) \le c(1-s_j)(m\rho_j)^{-s_j p} \sum_{\mathtt{i}\in\mathcal{I}_j} \int_{D_j^{\mathtt{i}}} |w_j - (u_j)_{C_j^{\mathtt{i},h_j}}|^p\,dx$$
$$\le c(1-s_j) m^{-s_j p} \mathcal{S}_j. \tag{4.36}$$

On each $P_j^{\mathtt{i}}\times P_j^{\mathtt{i}}$, the increments of $w_j - v_j$ equal those of w_j, because v_j is constant on $P_j^{\mathtt{i}}$. Consequently

$$\mathcal{D}_j\Big(w_j - v_j, \bigcup_{\mathtt{i}\in\mathcal{I}_j} (P_j^{\mathtt{i}}\times P_j^{\mathtt{i}})\Big) = \mathcal{S}_j.$$

Together with (4.36) and $w_j - v_j \in L^p(U)$, this proves $w_j - v_j \in W^{s_j,p}(U)$ and hence $v_j = w_j - (w_j - v_j) \in W^{s_j,p}(U)$.

It remains to compare the two energies on G_j. The inequality

$$\big||a+b|^p-|a|^p\big|\le c(p)\big(|a|^{p-1}|b|+|b|^p\big)$$

and Hölder's inequality give, for every measurable $E\subseteq U\times U$,

$$\begin{aligned}
&|\mathcal{D}_j(w_j,E\cap G_j)-\mathcal{D}_j(v_j,E\cap G_j)|\\
&\quad\le c\mathcal{D}_j(v_j,U\times U)^{(p-1)/p}\mathcal{D}_j(w_j-v_j,G_j)^{1/p}+c\mathcal{D}_j(w_j-v_j,G_j)\\
&\quad\le c(1-s_j)^{1/p}m^{-s_j}\mathcal{D}_j(v_j,U\times U)^{(p-1)/p}\mathcal{S}_j^{1/p}+c(1-s_j)m^{-s_jp}\mathcal{S}_j\\
&\quad\le c(1-s_j)^{1/p}m^{-s_j}\big(\mathcal{D}_j(v_j,U\times U)+\mathcal{S}_j\big).
\end{aligned}\tag{4.37}$$

The last step follows from Young's inequality. On the internal products $P_j^{\mathtt{i}}\times P_j^{\mathtt{i}}$ the energy of v_j is zero, and that of w_j gives exactly $\mathcal{S}_j(E)$. Adding these contributions proves (4.14); taking $E=U\times U$ gives (4.15) and (4.16).

Finally, the functions w_j are constructed as convex combinations of u_j and its mean values on the annuli $C_j^{\mathtt{i},h_j}$. This proves the first inequality in (4.17); the second follows immediately from the definition of v_j. □

Remark 4.2. *The additional factor $1-s_j$ in (4.36) is the gain specific to the present regime. It comes from combining the normalized tail with the uniform Poincaré estimate on the annulus. Thus m could be fixed throughout the applications. We leave it explicit to retain the comparison with the fixed-order arguments of [25].*

We introduce next the notation for Delone sets used in [25, Definition 2.2 and Propositions 2.4–2.5]. A set $\Lambda\subset\mathbb{R}^n$ is a Delone set if

$$r_\Lambda:=\frac12\inf_{\substack{x,y\in\Lambda\\ x\neq y}}|x-y|>0,\qquad R_\Lambda:=\sup_{x\in\mathbb{R}^n}\operatorname{dist}(x,\Lambda)<+\infty.\tag{4.38}$$

Here $\operatorname{dist}(x,\Lambda):=\inf_{y\in\Lambda}|x-y|$. Such a set is countably infinite; we write $\Lambda=\{x^{\mathtt{i}}\}_{\mathtt{i}\in\mathbb{Z}^n}$. The Voronoï cell of $x^{\mathtt{i}}$ is

$$V^{\mathtt{i}}:=\{x\in\mathbb{R}^n:|x-x^{\mathtt{i}}|\le|x-x^{\mathtt{k}}|\text{ for every }\mathtt{k}\in\mathbb{Z}^n\}.\tag{4.39}$$

The cells are convex polytopes with pairwise disjoint interiors, cover $\mathbb{R}^n$, and satisfy

$$B_{r_\Lambda}(x^{\mathtt{i}})\subseteq V^{\mathtt{i}}\subseteq\overline{B}_{R_\Lambda}(x^{\mathtt{i}}).\tag{4.40}$$

In particular, their boundaries have zero Lebesgue measure. For a bounded open set $A\subset\mathbb{R}^n$ set

$$\begin{aligned}
\mathcal{I}_\Lambda(A)&:=\{\mathtt{i}\in\mathbb{Z}^n:V^{\mathtt{i}}\subseteq A\},\\
\mathscr{I}_\Lambda(A)&:=\{\mathtt{i}\in\mathbb{Z}^n:V^{\mathtt{i}}\cap\partial A\neq\varnothing\}.
\end{aligned}\tag{4.41}$$

Both index sets are finite.

Given a sequence of Delone sets $\Lambda_j=\{x_j^{\mathtt{i}}\}_{\mathtt{i}\in\mathbb{Z}^n}$ denote its Voronoï cells by $V_j^{\mathtt{i}}$ and write $\mathcal{I}_j(A):=\mathcal{I}_{\Lambda_j}(A)$, $\mathscr{I}_j(A):=\mathscr{I}_{\Lambda_j}(A)$, and $R_j:=R_{\Lambda_j}$.

In the following result we will use that the constant in the Poincaré–Wirtinger inequality is uniform over all closed convex sets K satisfying $B_1\subset K\subset B_M$, for some $M>1$. This is the content of Proposition A.2 in the appendix.

Corollary 4.3. *Let U, p, s_j, and u_j be as in Lemma 4.1. Let $(\Lambda_j)_j$ be a sequence of Delone sets and let $0<\varepsilon_j\le r_{\Lambda_j}$ satisfy*

$$\varepsilon_j\longrightarrow0,\qquad\sup_j\frac{R_j}{\varepsilon_j}<+\infty.\tag{4.42}$$

Set $\mathcal{I}_j := \mathcal{I}_{\Lambda_j}(U)$ and fix integers $m \geq 2$, $N \geq 1$. Then Lemma 4.1 applies to the family $\{x_j^{\mathtt{i}}\}_{\mathtt{i}\in\mathcal{I}_j}$. If h_j is the sequence of indices given by that lemma, the functions

$$\zeta_j := \sum_{\mathtt{i}\in\mathcal{I}_j} (u_j)_{C_j^{\mathtt{i},h_j}} \chi_{V_j^{\mathtt{i}}} \tag{4.43}$$

converge to u strongly in $L^p(U)$.

Proof. By (4.40) and the definition of $\mathcal{I}_j$, the points $x_j^{\mathtt{i}}$ satisfy (4.2). It remains to prove the convergence of $(\zeta_j)_j$. Fix $M > \sup_j R_j/\varepsilon_j$ and set

$$K_j^{\mathtt{i}} := \varepsilon_j^{-1}(V_j^{\mathtt{i}} - x_j^{\mathtt{i}}).$$

These sets are convex and satisfy $B_1 \subset K_j^{\mathtt{i}} \subset B_M$. Applying inequality (A.3) to $K_j^{\mathtt{i}}$ and $v(x) = u_j(x_j^{\mathtt{i}} + \varepsilon_j x)$, and scaling back, we obtain

$$\|u_j - (u_j)_{V_j^{\mathtt{i}}}\|^p_{L^p(V_j^{\mathtt{i}})} \leq c\varepsilon_j^{s_j p}\mathcal{D}_j(u_j, V_j^{\mathtt{i}} \times V_j^{\mathtt{i}}), \tag{4.44}$$

where $c = c(n, p, s_0, M)$ is independent of j and $\mathtt{i}$. Since $C_j^{\mathtt{i},h_j} \subset V_j^{\mathtt{i}}$ and $h_j \leq N$, we also have $\mathcal{L}^n(V_j^{\mathtt{i}})/\mathcal{L}^n(C_j^{\mathtt{i},h_j}) \leq c(n, m, M, N)$. Jensen's inequality thus yields

$$\begin{aligned}\|u_j - (u_j)_{C_j^{\mathtt{i},h_j}}\|^p_{L^p(V_j^{\mathtt{i}})} &\leq 2^{p-1}\left(1 + \frac{\mathcal{L}^n(V_j^{\mathtt{i}})}{\mathcal{L}^n(C_j^{\mathtt{i},h_j})}\right)\|u_j - (u_j)_{V_j^{\mathtt{i}}}\|^p_{L^p(V_j^{\mathtt{i}})}\\ &\leq c\varepsilon_j^{s_j p}\mathcal{D}_j(u_j, V_j^{\mathtt{i}} \times V_j^{\mathtt{i}}),\end{aligned}$$

with $c = c(n, p, s_0, m, M, N)$. Summing over $\mathtt{i} \in \mathcal{I}_j$ gives

$$\sum_{\mathtt{i}\in\mathcal{I}_j} \|u_j - (u_j)_{C_j^{\mathtt{i},h_j}}\|^p_{L^p(V_j^{\mathtt{i}})} \leq c\varepsilon_j^{s_j p}\mathcal{D}_j(u_j, U \times U). \tag{4.45}$$

Set $E_j := U \setminus \bigcup_{\mathtt{i}\in\mathcal{I}_j} V_j^{\mathtt{i}}$. If $x \in E_j$, any Voronoï cell containing x also contains a point outside U. Since its diameter is at most $2R_j$,

$$E_j \subseteq \{x \in U : \operatorname{dist}(x, \partial U) \leq 2R_j\}.$$

As $R_j \to 0$ and U is bounded and open, $\mathcal{L}^n(E_j) \to 0$. By the definition of ζ_j,

$$\|\zeta_j - u_j\|^p_{L^p(U)} = \sum_{\mathtt{i}\in\mathcal{I}_j} \|u_j - (u_j)_{C_j^{\mathtt{i},h_j}}\|^p_{L^p(V_j^{\mathtt{i}})} + \|u_j\|^p_{L^p(E_j)}. \tag{4.46}$$

The first term tends to zero by (4.45) and (4.4); the second does so by the equi-integrability of $(|u_j|^p)_j$ due to the strong convergence $u_j \to u$ in $L^p(U)$. The conclusion then follows. □

Remark 4.4 (Separated families)**.** *Lemma 4.1 requires only (4.2). Relative density is used in Corollary 4.3 to prove the convergence of the cellwise averages (4.43).*

Corollary 4.5. *Under the assumptions and with the notation of Lemma 4.1, let c_m be the joining constant and let $\delta_{j,m}$ be as in (4.13). For all sufficiently large j,*

$$\mathcal{D}_j(u_j, U \times U) \geq \frac{1 - \delta_{j,m}}{1 + c_m/N}\big(\mathcal{D}_j(v_j, U \times U) + \mathcal{S}_j\big). \tag{4.47}$$

If $N > c_m$, then also

$$\mathcal{D}_j(u_j, U \times U) \leq \frac{1 + \delta_{j,m}}{1 - c_m/N}\big(\mathcal{D}_j(v_j, U \times U) + \mathcal{S}_j\big). \tag{4.48}$$

Proof. Taking $E = U \times U$ in (4.7) gives

$$\Big(1 - \frac{c_m}{N}\Big)\,\mathcal{D}_j(u_j, U\times U) \le \mathcal{D}_j(w_j, U\times U) \le \Big(1 + \frac{c_m}{N}\Big)\,\mathcal{D}_j(u_j, U\times U).$$

Combining these inequalities with (4.15)–(4.16) proves the claim. For each fixed m, the filling factors tend to one as $j \to \infty$; only the joining error requires the subsequent limit $N \to \infty$. □

5. Aperiodic obstacle homogenization in the BBM regime

Let $U \subset \mathbb{R}^n$ be bounded, connected, and Lipschitz, and let $\Lambda_j = \{x_j^{\mathtt{i}}\}_{\mathtt{i}\in\mathbb{Z}^n}$ be a sequence of Delone sets. We retain the notation r_{Λ_j}, $R_j = R_{\Lambda_j}$, $V_j^{\mathtt{i}}$, $\mathcal{I}_j(A)$, and $\mathscr{I}_j(A)$ introduced above. Choose $0 < \varepsilon_j \le r_{\Lambda_j}$ and assume

$$\varepsilon_j \longrightarrow 0, \qquad \sup_j \frac{R_j}{\varepsilon_j} < +\infty, \qquad \#\mathcal{I}_j(U)\,\varepsilon_j^n \longrightarrow \theta \in (0, +\infty). \tag{5.1}$$

Finally, suppose that the empirical measures of the interior centres satisfy

$$\mu_j := \frac{1}{\#\mathcal{I}_j(U)} \sum_{\mathtt{i}\in\mathcal{I}_j(U)} \delta_{x_j^{\mathtt{i}}} \overset{*}{\rightharpoonup} \beta\,\mathcal{L}^n\llcorner U \quad \text{in } (C_b(U))^*, \tag{5.2}$$

where $\beta \in L^\infty(U)$, $\beta > 0$ a.e., and $\int_U \beta\,dx = 1$. Conditions (5.1)–(5.2) are the assumptions in [25, Section 3.1, formulas (3.1)–(3.3)]; their compatibility with the Delone geometry is discussed precisely in [25, Remark 3.2] where it is shown that (5.2) is always satisfied up to subsequences under (5.1).

Let $s_j \uparrow 1$, with $s_j \in [s_0, 1)$, and retain $p \in (1, n)$. Fix a nonempty bounded open set $T \Subset B_1$. Define the critical radius and the obstacle set by

$$\lambda_j := \varepsilon_j^{\frac{n}{n - s_j p}}, \qquad T_j^{\mathtt{i}} := x_j^{\mathtt{i}} + \lambda_j T, \qquad \mathcal{K}_j := \bigcup_{\mathtt{i}\in\mathbb{Z}^n} T_j^{\mathtt{i}}. \tag{5.3}$$

Notice that

$$\lambda_j^{n - s_j p} = \varepsilon_j^n, \qquad \frac{\lambda_j}{\varepsilon_j} = \varepsilon_j^{\frac{s_j p}{n - s_j p}} \longrightarrow 0. \tag{5.4}$$

Thus, after discarding finitely many indices, every obstacle is contained in the corresponding Voronoï cell. The choice (5.3) is the simultaneous BBM version of [25, formula (3.5)].

Define $\mathcal{F}_j : L^p(U) \to [0, +\infty]$ by

$$\mathcal{F}_j(u) := \begin{cases} (1 - s_j)[u]^p_{s_j,p;U}, & u \in W^{s_j,p}(U), \quad u = 0 \quad \text{a.e. on } \mathcal{K}_j \cap U, \\ +\infty, & \text{otherwise.} \end{cases} \tag{5.5}$$

The assumption that T is open allows us to use Corollary 3.2. The extension to thin obstacles is discussed in Remarks 3.3 and 5.6.

Theorem 5.1 (Aperiodic BBM homogenization)**.** *Under assumptions* (5.1)– (5.2)*, the functionals* $(\mathcal{F}_j)_j$ *defined in* (5.5) Γ*-converge in* $L^p(U)$ *to*

$$\mathcal{F}(u) := \begin{cases} K_{n,p} \displaystyle\int_U |\nabla u|^p\,dx + \theta K_{n,p}\,\mathrm{cap}_p(T) \int_U |u|^p \beta\,dx, & u \in W^{1,p}(U), \\ +\infty, & \textit{otherwise.} \end{cases} \tag{5.6}$$

The proof follows the lines of [25, Propositions 3.10–3.11], though in the current setting both the bulk energy and the capacitary term now undergo the BBM limit so that the derivation of the limiting energy contributions changes. For the lower bound we replace formulas (3.28)–(3.29) therein by the lower filling estimate. For the upper bound we insert capacitary correctors in the balls where the filled sequence is constant and apply the upper filling estimate. More precisely, for sequences with bounded normalized energy, in the lower bound, the filled sequence still converges to u, and its full normalized fractional energy yields $K_{n,p}\int_U |\nabla u|^p\,dx$. The internal energies separated by filling yield the additional capacitary term. In the upper bound, the capacitary correctors are inserted into regions where the target function has been made constant, and the upper filling estimate controls the interaction error. After taking $j\to\infty$, letting the joining parameter N tend to infinity with m fixed leaves the full BBM coefficient in front of the gradient term, independently of the relative rates of ε_j and s_j.

We stress that the factor $1-s_j$ simplifies the comparison with the fixed-order arguments in [25]. Combined with the uniform Poincaré–Wirtinger inequality, it makes the tails of the capacitary competitors negligible compared with their internal energies, even when the ratio of the two concentric radii remains fixed. The same estimate improves the filling error to a quantity of order $(1-s_j)^{1/p}m^{-s_j}$. Thus the separation of the bulk and capacitary contributions is asymptotically exact for each fixed $m\geq 2$, and a further limit in m is unnecessary. We nevertheless keep m explicit to facilitate comparison with the fixed-order constructions.

Propositions 5.2 and 5.3 below prove Theorem 5.1.

Proposition 5.2 (Compactness and Γ-liminf inequality)**.** *Every sequence $(u_j)_j\subset L^p(U)$ such that*

$$\sup_j \mathcal{F}_j(u_j)<+\infty \tag{5.7}$$

is relatively compact in $L^p(U)$, and all its cluster points belong to $W^{1,p}(U)$. Moreover, for every sequence $u_j\to u$ strongly in $L^p(U)$,

$$\liminf_{j\to+\infty}\mathcal{F}_j(u_j)\geq\mathcal{F}(u). \tag{5.8}$$

Proof. We first prove the lower-bound inequality for a sequence $u_j\to u$ strongly in $L^p(U)$. It is enough to consider a subsequence, not relabelled, on which the left-hand side of (5.8) is finite and is attained as a limit. In particular,

$$\sup_j \mathcal{D}_j(u_j,U\times U)<+\infty. \tag{5.9}$$

Fix $m\geq 2$ and $N\in\mathbb{N}$. Apply Lemma 4.1 to the points $\{x_j^{\mathtt{i}}\}_{\mathtt{i}\in\mathcal{I}_j(U)}$. Denote by h_j, w_j, v_j, $P_j^{\mathtt{i}}$, $D_j^{\mathtt{i}}$, and $\mathcal{S}_j$ the quantities provided by that lemma, and put

$$z_j^{\mathtt{i}}:=(u_j)_{C_j^{\mathtt{i},h_j}},\qquad \zeta_j:=\sum_{\mathtt{i}\in\mathcal{I}_j(U)} z_j^{\mathtt{i}}\chi_{V_j^{\mathtt{i}}}. \tag{5.10}$$

For fixed m,N, scale separation (5.4) implies, for all sufficiently large j,

$$T_j^{\mathtt{i}}\subset B_{m^{-3h_j-3}\varepsilon_j}(x_j^{\mathtt{i}})\subset D_j^{\mathtt{i}}\quad\text{for every }\mathtt{i}\in\mathcal{I}_j(U). \tag{5.11}$$

The joining modification w_j in Lemma 4.1 does not change u_j near the obstacles; hence w_j remains admissible for $\mathcal{F}_j$. Obstacles associated with boundary

cells are untouched. From (4.7) we obtain the analogue of [25, Proposition 3.10, formula (3.27)]:

$$\Big(1+\frac{c_m}{N}\Big)\liminf_{j\to+\infty}\mathcal{F}_j(u_j)\geq\liminf_{j\to+\infty}\mathcal{D}_j(w_j,U\times U). \tag{5.12}$$

The essential change with respect to [25, Proposition 3.10] is the use of filling to retain the full bulk energy. For fixed m, the factor $\delta_{j,m}$ tends to zero. By (4.15) and the boundedness of the energy of w_j, the quantities $\mathcal{D}_j(v_j,U\times U)+\mathcal{S}_j$ remain bounded. Since $v_j\to u$ in $L^p(U)$, the regional BBM lower bound [40, Theorem 8] therefore gives

$$\begin{aligned}\liminf_{j\to+\infty}\mathcal{D}_j(w_j,U\times U)&\geq\liminf_{j\to+\infty}\mathcal{D}_j(v_j,U\times U)+\liminf_{j\to+\infty}\mathcal{S}_j\\&\geq K_{n,p}\int_U|\nabla u|^p\,dx+\liminf_{j\to+\infty}\mathcal{S}_j.\end{aligned} \tag{5.13}$$

This replaces [25, Proposition 3.10, formula (3.28)] and yields the complete BBM coefficient already at fixed m. The same lower bound shows that $u\in W^{1,p}(U)$.

We now estimate the internal energies $\mathcal{S}_j$. Recall that

$$\rho_j=m^{-3h_j-2}\varepsilon_j,\qquad D_j^{\mathtt{i}}=B_{\rho_j}(x_j^{\mathtt{i}}),\qquad P_j^{\mathtt{i}}=B_{m\rho_j}(x_j^{\mathtt{i}}),$$

and that $w_j=z_j^{\mathtt{i}}$ on $P_j^{\mathtt{i}}\setminus\overline{D_j^{\mathtt{i}}}$. Since $w_j=0$ a.e. on $T_j^{\mathtt{i}}$, translation and scaling in (3.20) give

$$\mathcal{D}_j(w_j,P_j^{\mathtt{i}}\times P_j^{\mathtt{i}})\geq(1-s_j)\lambda_j^{n-s_jp}|z_j^{\mathtt{i}}|^p\operatorname{cap}_{s_j,p}\Big(T,B_{m\rho_j/\lambda_j};\frac{\rho_j}{\lambda_j}\Big). \tag{5.14}$$

For $z_j^{\mathtt{i}}\neq0$, this follows by testing the relative capacity with the function $(z_j^{\mathtt{i}}-w_j)/z_j^{\mathtt{i}}$ on $P_j^{\mathtt{i}}$, extended by zero outside $P_j^{\mathtt{i}}$ and rescaled by λ_j. The zero extension has finite global seminorm because the function vanishes on $P_j^{\mathtt{i}}\setminus\overline{D_j^{\mathtt{i}}}$. For $z_j^{\mathtt{i}}=0$ the estimate is immediate. Because $h_j\in\{1,\dots,N\}$ and (5.4) holds,

$$\frac{\rho_j}{\lambda_j}\geq m^{-3N-2}\frac{\varepsilon_j}{\lambda_j}\longrightarrow+\infty. \tag{5.15}$$

Set

$$\Psi_j:=\sum_{\mathtt{i}\in\mathcal{I}_j(U)}\frac{\varepsilon_j^n}{\mathcal{L}^n(V_j^{\mathtt{i}})}\chi_{V_j^{\mathtt{i}}}. \tag{5.16}$$

The inclusions (4.40) imply that $(\Psi_j)_j$ is bounded in $L^\infty(U)$. Moreover,

$$\Psi_j\overset{*}{\rightharpoonup}\theta\beta\qquad\text{in }L^\infty(U). \tag{5.17}$$

Indeed, for a compactly contained Lipschitz set $A\subset U$, the difference between $\int_A\Psi_j\,dx$ and $\varepsilon_j^n\#\mathcal{I}_j(A)$ is bounded by a constant times $\varepsilon_j^n\#\mathscr{I}_j(A)$. The latter tends to zero by [25, Proposition 2.5, formulas (2.4)–(2.5)], while (5.1)–(5.2) identify the limit. This is exactly the counting argument following [25, Proposition 3.10, formula (3.31)].

By Corollary 4.3, $\zeta_j\to u$ strongly in $L^p(U)$. Therefore

$$\int_U|\zeta_j|^p\Psi_j\,dx\longrightarrow\theta\int_U|u|^p\beta\,dx. \tag{5.18}$$

Using (5.4), summing (5.14), and applying Lemma 3.4 with $L_j = \rho_j/\lambda_j$, we conclude that

$$\liminf_{j\to+\infty} \mathcal{S}_j \geq \theta K_{n,p} \operatorname{cap}_p(T) \int_U |u|^p \beta \, dx. \tag{5.19}$$

Estimate (5.19) replaces [25, Proposition 3.10, formula (3.29)]: the capacitary contribution is obtained from the internal energies separated by filling.

Combining (5.12), (5.13), and (5.19), we obtain

$$\Big(1 + \frac{c_m}{N}\Big) \liminf_{j\to+\infty} \mathcal{F}_j(u_j) \geq K_{n,p} \int_U |\nabla u|^p \, dx + \theta K_{n,p} \operatorname{cap}_p(T) \int_U |u|^p \beta \, dx. \tag{5.20}$$

Letting $N \to +\infty$ with m fixed proves (5.8). No further limit in m is needed, since both the filling error and the relative capacitary error have already vanished as $s_j \uparrow 1$.

It remains to prove the compactness assertion. We first claim that (5.7) implies

$$\sup_j \|u_j\|_{L^p(U)} < +\infty. \tag{5.21}$$

Otherwise, after passing to a subsequence, set $\widehat{u}_j := u_j/\|u_j\|_{L^p(U)}$. Then

$$\|\widehat{u}_j\|_{L^p(U)} = 1, \qquad \mathcal{F}_j(\widehat{u}_j) \longrightarrow 0.$$

The BBM compactness theorem [39, Theorem 1.2] yields, up to a further subsequence, $\widehat{u}_j \to \widehat{u}$ strongly in $L^p(U)$, with $\|\widehat{u}\|_{L^p(U)} = 1$. The already proved liminf inequality gives

$$0 \geq \mathcal{F}(\widehat{u}) = K_{n,p} \int_U |\nabla \widehat{u}|^p \, dx + \theta K_{n,p} \operatorname{cap}_p(T) \int_U |\widehat{u}|^p \beta \, dx.$$

Since T is nonempty and open, $\operatorname{cap}_p(T) > 0$; since $\beta > 0$ a.e., this forces $\widehat{u} = 0$ a.e. on U, a contradiction. Thus (5.21) holds. A second application of [39, Theorem 1.2], together with (5.7), gives relative compactness in $L^p(U)$. Finally, equations (5.9) and (5.13) show that every cluster point belongs to $W^{1,p}(U)$. □

Proposition 5.3 (The Γ-limsup inequality)**.** *For every $u \in L^p(U)$ there exists a sequence $(u_j)_j \subset L^p(U)$ such that $u_j \to u$ strongly in $L^p(U)$ and*

$$\limsup_{j\to+\infty} \mathcal{F}_j(u_j) \leq \mathcal{F}(u). \tag{5.22}$$

Proof. It is enough to prove the assertion for $u \in W^{1,\infty}(U)$; the general case follows by approximation in $W^{1,p}(U)$ and a diagonal argument, exactly as at the beginning of the proof of [25, Proposition 3.11]. Fix $m \geq 2$ and $N \in \mathbb{N}$. We apply Lemma 4.1 to the constant sequence $u_j = u$ and to the interior centres $\{x_j^{\mathtt{i}}\}_{\mathtt{i}\in\mathcal{I}_j(U)}$. We retain the notation $h_j, w_j, v_j, P_j^{\mathtt{i}}, D_j^{\mathtt{i}}$ and set

$$z_j^{\mathtt{i}} := (u)_{C_j^{\mathtt{i},h_j}}, \qquad \zeta_j := \sum_{\mathtt{i}\in\mathcal{I}_j(U)} z_j^{\mathtt{i}} \chi_{V_j^{\mathtt{i}}}. \tag{5.23}$$

By Lemma 4.1 and Corollary 4.3, $v_j \to u$ and $\zeta_j \to u$ strongly in $L^p(U)$. Moreover, (4.7) and (4.15) give

$$\limsup_{j\to+\infty} \mathcal{D}_j(v_j, U\times U) \leq \Big(1 + \frac{c_m}{N}\Big) K_{n,p} \int_U |\nabla u|^p \, dx, \tag{5.24}$$

where we used $\delta_{j,m} \to 0$ for fixed m and the regional BBM formula applied to the constant sequence $u_j = u$.

Set

$$\rho_j := m^{-3h_j-2}\varepsilon_j, \qquad L_j := \frac{\rho_j}{\lambda_j}. \tag{5.25}$$

By (5.4), $L_j \to +\infty$. Choose a $1/j$-minimizer φ_j for $C_{s_j,p}(T, B_{L_j})$ and truncate it so that

$$0 \le \varphi_j \le 1, \qquad \varphi_j = 0 \text{ a.e. on } B_{L_j}^c, \qquad \varphi_j = 1 \quad \text{a.e. on } T, \tag{5.26}$$

and

$$a_j := (1-s_j)[\varphi_j]_{s_j,p}^p \le (1-s_j)C_{s_j,p}(T, B_{L_j}) + \frac{1}{j}. \tag{5.27}$$

The convergence of rescaled relative fractional capacities to the global local counterpart in (3.16) of Corollary 3.2 yields

$$a_j \longrightarrow K_{n,p}\operatorname{cap}_p(T). \tag{5.28}$$

We insert the capacitary correctors into the joining modification w_j furnished by Lemma 4.1. The resulting function is denoted by $\widehat{u}_j$: it agrees with w_j outside the balls $D_j^{\mathtt{i}}$, while inside each such ball the profile is replaced by a capacitary correction of the constant $z_j^{\mathtt{i}}$. Its filling is still v_j. More precisely, define

$$\widehat{u}_j(x) := \begin{cases} z_j^{\mathtt{i}}\left(1-\varphi_j\left(\dfrac{x-x_j^{\mathtt{i}}}{\lambda_j}\right)\right), & x \in D_j^{\mathtt{i}}, \quad \mathtt{i} \in \mathcal{I}_j(U), \\ v_j(x), & x \in U \setminus \displaystyle\bigcup_{\mathtt{i}\in\mathcal{I}_j(U)} D_j^{\mathtt{i}}. \end{cases} \tag{5.29}$$

Since $v_j = z_j^{\mathtt{i}}$ on $D_j^{\mathtt{i}}$ and φ_j vanishes outside B_{L_j}, this definition gives a function in $W^{s_j,p}(U)$ which is constant on $P_j^{\mathtt{i}} \setminus \overline{D}_j^{\mathtt{i}}$ and vanishes a.e. on every interior obstacle $T_j^{\mathtt{i}}$. Its filling is precisely v_j. Therefore the proof of (4.16), applied to the pair $(\widehat{u}_j, v_j)$, gives

$$\mathcal{D}_j(\widehat{u}_j, U\times U) \le (1+\delta_{j,m})\left(\mathcal{D}_j(v_j, U\times U) + \widehat{\mathcal{S}}_j\right), \tag{5.30}$$

where

$$\widehat{\mathcal{S}}_j := \sum_{\mathtt{i}\in\mathcal{I}_j(U)} \mathcal{D}_j(\widehat{u}_j, P_j^{\mathtt{i}}\times P_j^{\mathtt{i}}).$$

A change of variables, (5.4), and (5.16) imply

$$\begin{aligned} \widehat{\mathcal{S}}_j &\le (1-s_j)\lambda_j^{n-s_jp}[\varphi_j]_{s_j,p}^p \sum_{\mathtt{i}\in\mathcal{I}_j(U)} |z_j^{\mathtt{i}}|^p \\ &= a_j \int_U |\zeta_j|^p \Psi_j \, dx. \end{aligned} \tag{5.31}$$

In view of (5.17), (5.18), and (5.28),

$$\limsup_{j\to+\infty} \widehat{\mathcal{S}}_j \le \theta K_{n,p}\operatorname{cap}_p(T) \int_U |u|^p \beta \, dx. \tag{5.32}$$

This proves the estimate corresponding to [25, Proposition 3.11, Step 1 and formula (3.35)].

We also have $\widehat{u}_j \to u$ strongly in $L^p(U)$. Indeed, the scaled uniform Poincaré inequality on $(B_{2L_j}, B_{2L_j}\setminus B_{L_j})$ gives

$$\|\varphi_j\|_{L^p(\mathbb{R}^n)}^p \le c(1-s_j)L_j^{s_jp}[\varphi_j]_{s_j,p}^p.$$

Consequently,

$$\|\widehat{u}_j - v_j\|^p_{L^p(U)} \le c\rho_j^{s_j p} a_j \varepsilon_j^n \sum_{\mathtt{i}\in\mathcal{I}_j(U)} |z_j^{\mathtt{i}}|^p$$
$$= c\rho_j^{s_j p} a_j \int_U |\zeta_j|^p \Psi_j \, dx \longrightarrow 0. \tag{5.33}$$

It remains to treat the obstacles whose Voronoï cells meet ∂U. Let $\eta \in C^\infty_c(B_2)$ satisfy $0 \le \eta \le 1$ and $\eta = 1$ on B_1. Since $\lambda_j/\varepsilon_j \to 0$, the balls $B_{2\lambda_j}(x_j^{\mathtt{i}})$, $\mathtt{i} \in \mathscr{I}_j(U)$, are mutually disjoint for large j and do not meet the interior joining and filling regions. In those balls $\widehat{u}_j = u$. Define

$$u_j(x) := \begin{cases} \left(1 - \eta\left(\dfrac{x - x_j^{\mathtt{i}}}{\lambda_j}\right)\right)\widehat{u}_j(x), & x \in B_{2\lambda_j}(x_j^{\mathtt{i}}) \cap U, \quad \mathtt{i} \in \mathscr{I}_j(U), \\ \widehat{u}_j(x), & \text{otherwise.} \end{cases} \tag{5.34}$$

Thus u_j is admissible for $\mathcal{F}_j$. Writing $(\partial U)_r := \{x \in \mathbb{R}^n : \operatorname{dist}(x, \partial U) < r\}$, the boundary counting estimate [25, Proposition 2.5, formula (2.5)] gives

$$\varepsilon_j^n \# \mathscr{I}_j(U) \le cLn((\partial U)_{2R_j}) \longrightarrow 0. \tag{5.35}$$

The boundary correction corresponds to Steps 4–6 of [25, Proposition 3.11]. To estimate its cost, set

$$\theta_j(x) := \max_{\mathtt{i}\in\mathscr{I}_j(U)} \eta\left(\frac{x - x_j^{\mathtt{i}}}{\lambda_j}\right),$$

with $\theta_j = 0$ if the index set is empty. Then $u_j = (1-\theta_j)\widehat{u}_j$ on U. Since $|\max_i a_i - \max_i b_i|^p \le \sum_i |a_i - b_i|^p$, scaling gives

$$(1 - s_j)[\theta_j]^p_{s_j,p} \le \varepsilon_j^n \# \mathscr{I}_j(U)\, (1 - s_j)[\eta]^p_{s_j,p} \longrightarrow 0 \tag{5.36}$$

by (5.35) and the BBM formula for η. The identity

$$u_j(x) - u_j(y) = (1 - \theta_j(x))(\widehat{u}_j(x) - \widehat{u}_j(y)) + \widehat{u}_j(y)(\theta_j(y) - \theta_j(x))$$

and Minkowski's inequality imply

$$\mathcal{D}_j(u_j, U \times U)^{1/p} \le \mathcal{D}_j(\widehat{u}_j, U \times U)^{1/p} + \|u\|_{L^\infty(U)} \big((1 - s_j)[\theta_j]^p_{s_j,p}\big)^{1/p}.$$

Therefore, by (5.36) we get

$$\mathcal{D}_j(u_j, U \times U)^{1/p} \le \mathcal{D}_j(\widehat{u}_j, U \times U)^{1/p} + \gamma_j, \qquad \gamma_j \longrightarrow 0. \tag{5.37}$$

Moreover, $\mathcal{L}^n(\operatorname{supp}\theta_j) \le c\lambda_j^n \# \mathscr{I}_j(U) \to 0$, and hence

$$\|u_j - \widehat{u}_j\|^p_{L^p(U)} \le \|u\|^p_{L^\infty(U)} \mathcal{L}^n(\operatorname{supp}\theta_j) \longrightarrow 0.$$

Combining (5.24), (5.30), (5.32), and (5.37), and using $\delta_{j,m} \to 0$, we obtain

$$\limsup_{j\to+\infty} \mathcal{F}_j(u_j) \le \left(1 + \frac{c_m}{N}\right) K_{n,p} \int_U |\nabla u|^p \, dx$$
$$+ \theta K_{n,p} \operatorname{cap}_p(T) \int_U |u|^p \beta \, dx. \tag{5.38}$$

Letting $N \to +\infty$ at fixed m and making a diagonal choice proves (5.22) for $u \in W^{1,\infty}(U)$. Since U is Lipschitz and $\beta \in L^\infty(U)$, approximation in $W^{1,p}(U)$ and a second diagonal argument prove the claim for every $u \in W^{1,p}(U)$. If $u \notin W^{1,p}(U)$, the right-hand side is infinite and there is nothing to prove. □

We deduce next the convergence of minimizers for a suitable family of perturbed functionals related to $\mathcal{F}_j$'s.

Corollary 5.4. *Let $g \in L^{p'}(U)$, p' being the conjugate exponent of p, and*

$$\mathcal{G}_j(u) := \mathcal{F}_j(u) + \int_U g u dx \qquad \mathcal{G}(u) := \mathcal{F}(u) + \int_U g u dx\,,$$

where $\mathcal{F}_j$ is defined in (5.5), and $\mathcal{F}$ in (5.6), respectively. Then $(\mathcal{G}_j)_j$ $\Gamma(L^p(U))$-converges to $\mathcal{G}$.

Moreover, if u_j minimizes $\mathcal{G}_j$ over $W^{s_j,p}(U)$, the sequence $(u_j)_j$ converges to the unique minimizer of $\mathcal{G}$, and moreover $\mathcal{G}_j(u_j) \to \mathcal{G}(u)$ as $j \to +\infty$.

Proof. The Γ-convergence of $(\mathcal{G}_j)_j$ to $\mathcal{G}$ is standard in view of Theorem 5.1 and the linearity of the perturbation.

We claim next that there is $\lambda_0 > 0$ such that for every $\lambda \in (0, \lambda_0)$ there is j_λ such that for $j \geq j_\lambda$

$$\mathcal{F}_j(u) \geq \lambda \|u\|^p_{L^p(U)} \tag{5.39}$$

for every $u \in L^p(U)$. The proof is similar to that of (5.21). We detail it for the reader's convenience. Assume not, then there are a subsequence of integers j_k and $\lambda_k \to 0^+$ such that

$$\mathcal{F}_{j_k}(v_{j_k}) < \lambda_k \|v_{j_k}\|^p_{L^p(U)}$$

for some $v_{j_k} \in L^p(U)$. Moreover, up to a normalization, we may also assume $\|v_{j_k}\|_{L^p(U)} = 1$. Then, Proposition 5.2 implies that a subsequence (not relabeled) converges strongly in $L^p(U)$ to some function v, with $\|v\|_{L^p(U)} = 1$, and

$$\theta K_{n,p} \operatorname{cap}_p(T) \int_U |v|^p \beta dx \leq \mathcal{F}(v) \leq \liminf_k \mathcal{F}_{j_k}(v_{j_k}) \leq 0\,.$$

Being $\beta > 0$ a.e. on U (cf. the comments right after (5.2)) and $\theta K_{n,p} \operatorname{cap}_p(T) > 0$, we conclude $v = 0$ a.e. on U, a contradiction.

With (5.39) at hand, the equi-coercivity of the family $(\mathcal{G}_j)_j$ follows easily from Young's inequality and Proposition 5.2. The rest of the statement is an easy consequence of the fundamental theorem of Γ-convergence (cf. [20, Theorem 7.8]), the strict convexity of $\mathcal{G}$ and the Urysohn's property. □

We record next the corresponding result when the obstacles are scaled according to the critical radius of the local problem. Set

$$\bar\lambda_j := \varepsilon_j^{\frac{n}{n-p}}, \qquad \bar T_j^{\mathtt{i}} := x_j^{\mathtt{i}} + \bar\lambda_j T, \qquad \bar{\mathcal{K}}_j := \bigcup_{\mathtt{i}\in\mathbb{Z}^n} \bar T_j^{\mathtt{i}}, \tag{5.40}$$

and define $\bar{\mathcal{F}}_j : L^p(U) \to [0, +\infty]$ by

$$\bar{\mathcal{F}}_j(u) := \begin{cases} (1-s_j)[u]^p_{s_j,p;U}, & u \in W^{s_j,p}(U), \quad u = 0 \quad \text{a.e. on } \bar{\mathcal{K}}_j \cap U, \\ +\infty, & \text{otherwise.} \end{cases} \tag{5.41}$$

Theorem 5.5 (The three regimes at the local critical scale)**.** *Assume (5.1)– (5.2), and suppose that*

$$(1-s_j)|\log \varepsilon_j| \longrightarrow \ell \qquad \text{for some } \ell \in [0, +\infty]. \tag{5.42}$$

Then $(\bar{\mathcal{F}}_j)_j$ Γ*-converges in* $L^p(U)$ *to*

$$\bar{\mathcal{F}}^{(\ell)}(u) := \begin{cases} K_{n,p} \displaystyle\int_U |\nabla u|^p \, dx + \theta \kappa_\ell K_{n,p} \operatorname{cap}_p(T) \int_U |u|^p \beta \, dx, & u \in W^{1,p}(U), \\ +\infty, & \text{otherwise}, \end{cases} \tag{5.43}$$

where

$$\kappa_\ell := \begin{cases} \exp\left(-\dfrac{np}{n-p}\ell\right), & \ell < +\infty, \\ 0, & \ell = +\infty. \end{cases} \tag{5.44}$$

For $\ell < +\infty$ *the family* $(\bar{\mathcal{F}}_j)_j$ *is equicoercive in* $L^p(U)$ *after discarding finitely many indices.*

We stress again that the capacitary term is unchanged if $\ell = 0$, is modulated by $\exp(-np\ell/(n-p))$ if $\ell \in (0, +\infty)$, and disappears if $\ell = +\infty$.

The proof is omitted, since it follows from that of Theorem 5.1 by replacing λ_j with $\bar{\lambda}_j$ and observing that by the very definition in (5.44)

$$\varepsilon_j^{-n} \bar{\lambda}_j^{n-s_j p} = \varepsilon_j^{\frac{np(1-s_j)}{n-p}} \longrightarrow \kappa_\ell. \tag{5.45}$$

Few remarsk are in order.

Remark 5.6 (The case of thin obstacles)**.** *For open obstacles all constraints in this paper are imposed almost everywhere with respect to Lebesgue measure. The asymptotic result in Theorem 5.1 also extends to compact sets* $T \Subset B_1$*, possibly of zero Lebesgue measure, provided that*

$$(1 - s_j) \operatorname{cap}_{s_j,p}(T) \longrightarrow K_{n,p} \operatorname{cap}_p(T), \tag{5.46}$$

in view of the observations in Remark 3.3 and the proof of Theorem 5.1. Recall that for such sets the capacities are understood in the usual outer sense:

$$\operatorname{cap}_{s,p}(T) := \inf\{\operatorname{cap}_{s,p}(O) : T \subset O, \ O \text{ bounded and open}\}.$$

The obstacle condition must then be imposed on the quasicontinuous representative $\widetilde{u}$*, namely*

$$\widetilde{u} = 0 \quad \operatorname{cap}_{s_j,p}\text{-q.e. on } \mathcal{K}_j \cap U.$$

Here q.e. (quasi-everywhere) means outside a set of zero capacity of the indicated order, and $\widetilde{u}$ *is the quasicontinuous representative of* u *(cf.* [27, Section 2 and Appendix A] *for the relevant definitions and for a concise account of the theory of fractional capacities). Likewise, the admissibility condition in the capacitary problems is* $\widetilde{\varphi} \geq 1$ *q.e. on* T*, with respect to* $\operatorname{cap}_{s,p}$ *or* cap_p *as appropriate. We stress that on open sets these conditions agree with the almost everywhere formulation used in the text. Note that for compact thin* T*, smooth compactly supported capacitary competitors replace the almost everywhere competitors used for open sets.*

The compactness conclusion in Proposition 5.2 requires a positive limiting capacitary coefficient. In particular, one must assume $\operatorname{cap}_p(T) > 0$*. Instead, the* Γ*-convergence statements remain valid even when this capacity is zero.*

To see that the homogenization proofs still apply, let $L_j \to +\infty$*. The lower bound* $C_{s_j,p}(T, B_{L_j}) \geq \operatorname{cap}_{s_j,p}(T)$ *and (5.46) give the required capacitary liminf. For the upper bound, choose smooth compactly supported functions that are at least one in a*

neighbourhood of T and whose local energies approach $\mathrm{cap}_p(T)$. Each such function is admissible in B_{L_j} for large j, and the BBM formula gives

$$(1-s_j)C_{s_j,p}(T,B_{L_j}) \longrightarrow K_{n,p}\,\mathrm{cap}_p(T).$$

The interior comparison (3.21) also holds with the q.e. constraint. These are the capacity estimates used in the proofs of Theorem 5.1.

As for open sets, the joining modification in Lemma 4.1 preserves the constraint on the selected obstacles. Filling instead replaces the function by the annular constant inside each inner ball and removes that constraint, thereby separating the bulk energy from the capacitary cost. In the upper bound the capacitary correctors impose the constraint again.

The same argument applies to Theorem 5.5; at the local critical scale, compactness also requires $\ell < +\infty$.

Remark 5.7 (Translation-invariant homogeneous kernels)**.** *Theorems 5.1 and 5.5 extend to energies with translation-invariant kernels, homogeneous of degree $-(n+s_jp)$ and uniformly comparable to the Gagliardo kernel. More precisely, let*

$$\mathscr{K}_j(z) := \frac{a_j(z/|z|)}{|z|^{n+s_jp}}, \qquad z\in\mathbb{R}^n\setminus\{0\}, \qquad 0<c\le a_j\le C<+\infty, \tag{5.47}$$

where $a_j \in L^\infty(\mathbb{S}^{n-1})$ are even and the bounds are independent of j. Note that since replacing a kernel by its even part leaves the energy unchanged, the symmetry requirement is no restriction. Assume in addition that

$$a_j \overset{*}{\rightharpoonup} a \quad \text{in } L^\infty(\mathbb{S}^{n-1}). \tag{5.48}$$

This includes, in particular, a fixed angular profile $a_j = a$. Define the local density and its homogeneous capacity by

$$\Phi_a(\xi) := \frac{1}{p}\int_{\mathbb{S}^{n-1}} a(\sigma)|\xi\cdot\sigma|^p\, d\mathcal{H}^{n-1}(\sigma), \qquad \xi\in\mathbb{R}^n, \tag{5.49}$$

$$\mathrm{cap}_{\Phi_a}(T) := \inf\left\{\int_{\mathbb{R}^n}\Phi_a(\nabla v)\,dx : v\in\dot W^{1,p}(\mathbb{R}^n),\ v\ge 1 \text{ a.e. on } T\right\}. \tag{5.50}$$

If the energies in (5.5) are replaced by

$$(1-s_j)\int_{U\times U}|u(x)-u(y)|^p\mathscr{K}_j(x-y)\,dx\,dy,$$

with the same obstacle constraints, their Γ-limit is finite on $W^{1,p}(U)$ with value

$$\int_U \Phi_a(\nabla u)\,dx + \theta\,\mathrm{cap}_{\Phi_a}(T)\int_U |u|^p\beta\,dx,$$

and is $+\infty$ otherwise on $L^p(U)$. At the local critical scale $\bar\lambda_j$, assuming (5.42), the limit on $W^{1,p}(U)$ is instead

$$\int_U \Phi_a(\nabla u)\,dx + \theta\kappa_\ell\,\mathrm{cap}_{\Phi_a}(T)\int_U |u|^p\beta\,dx.$$

For $a=1$ we recover $\Phi_a(\xi)=K_{n,p}|\xi|^p$ and $\mathrm{cap}_{\Phi_a}(T)=K_{n,p}\,\mathrm{cap}_p(T)$.

Indeed, uniform comparison in (5.47) preserves the Sobolev and Poincaré estimates and the joining and filling estimates, with constants depending also on c and C. The BBM limit in this case is the anisotropic density Φ_a (cf. [29, Theorems 1.1–1.2]*). Applying the argument of Proposition 3.1 and Corollary 3.2 to these*

energies gives the corresponding homogeneous and relative capacity limits. Homogeneity yields the same factor $\lambda_j^{n-s_jp}$ under rescaling, so the critical obstacle scales and the capacitary weights are unchanged. These are the ingredients needed in the two homogenization proofs.

Under the homogeneity assumption, condition (5.48) identifies a single limit. If this condition is omitted, the uniform bounds on a_j give the result along subsequences with weakly-$*$ convergent angular profiles, whose limits may differ.

The same asymptotics also hold imposing homogeneous Dirichlet boundary conditions. We introduce standard notation: for $s \in (0,1)$, let $W^{s,p}_0(U)$ denote the norm closure of $C^\infty_c(U)$ in $W^{s,p}(U)$.

Corollary 5.8 (On homogeneous Dirichlet boundary conditions)**.** *Let $\mathcal{F}^0_j$ be defined as $\mathcal{F}_j$ in (5.5), with $W^{s_j,p}(U)$ there replaced by $W^{s_j,p}_0(U)$. Then the analogous conclusion of Theorem 5.1 holds with $\Gamma(L^p(U))$-limit*

$$\mathcal{F}^0(u) := \begin{cases} \mathcal{F}(u), & u \in W^{1,p}_0(U), \\ +\infty, & \text{otherwise in } L^p(U). \end{cases}$$

Proof. First note that since $p \in (1,n)$ and $s_j \to 1^-$, we have $s_jp \in (1,n)$ for all sufficiently large j. In this range, $W^{s_j,p}_0(U)$ is the subspace of functions with zero trace on ∂U.

The only difficulty is to check that the boundary condition is preserved in the limit. To this aim fix $s_0 \in (1/p,1)$, then the fractional Hardy inequality on a bounded Lipschitz set gives for every $s \in [s_0,1)$ and $v \in W^{s,p}_0(U)$

$$\int_U \frac{|v(x)|^p}{\operatorname{dist}^{sp}(x,\partial U)}\,dx \le c(1-s)[v]^p_{s,p;U}\,, \tag{5.51}$$

where $c = c(n,p,s_0,U)$ is independent of s. Indeed, this follows from [32, Theorem 1.2]: we claim that the Lipschitz regularity of U allows the distance weight there to be compared with $\operatorname{dist}^{-sp}(\cdot,\partial U)$, while the explicit Hardy constant in [32, formula (15)] is of order $(1-s)^{-1}$ as $s \uparrow 1$.

To prove the claim, we examine first the constant

$$D_{n,p,sp} = 2\pi^{\frac{n-1}{2}}\frac{\Gamma(\frac{1+sp}{2})}{\Gamma(\frac{n+sp}{2})}\int_0^1 \frac{|1-r^{s-\frac{1}{p}}|^p}{(1-r)^{1+sp}}dr$$

there. In the formula above Γ denotes the Euler Gamma function. By an elementary calculation it is easy to check that

$$\lim_{s\to 1^-}(1-s)D_{n,p,sp} = 2\pi^{\frac{n-1}{2}}\frac{\Gamma(\frac{1+p}{2})}{\Gamma(\frac{n+p}{2})}\frac{1}{p}\Big(1-\frac{1}{p}\Big)^p.$$

Instead, to compare the weight in that theorem with the distance to the boundary, write $d_U(x) := \operatorname{dist}(x,\partial U)$ and, for $\sigma \in \mathbb{S}^{n-1}$, set

$$d_{\sigma,U}(x) := \inf\{|t| : x + t\sigma \notin U\}.$$

For $\alpha \in [s_0p, p]$, the weight in [32, formula (14)] is defined by

$$A_{n,\alpha} := \int_{\mathbb{S}^{n-1}} |\sigma_n|^\alpha\, d\mathcal{H}^{n-1}(\sigma),$$
$$m^{-\alpha}_\alpha(x) := \frac{1}{A_{n,\alpha}}\int_{\mathbb{S}^{n-1}} d_{\sigma,U}(x)^{-\alpha}\, d\mathcal{H}^{n-1}(\sigma).$$

Since $d_{\sigma,U}(x) \geq d_U(x)$, we immediately obtain $m_\alpha^{-\alpha}(x) \leq \mathcal{H}^{n-1}(\mathbb{S}^{n-1}) A_{n,\alpha}^{-1} d_U^{-\alpha}(x)$.

The reverse inequality follows from the Lipschitz regularity of U. In particular, we recall that for every $z \in \partial U$ there are $\rho > 0$ and a closed cone K_z, with vertex in z and opening independent of z, such that $B_\rho(z) \cap \partial U \cap K_z = \{z\}$, and $K_z \subseteq B_\rho(z) \cap U^c$.

Now fix $x \in U$, and choose $z \in \partial U$ with $|x - z| = d_U(x)$. There are $\delta > 0$ and $\eta \in (0,1)$, depending only on U, such that, if $d_U(x) < \delta$, there is $q \in \mathbb{R}^n$ with $|q - z| = 2d_U(x)$ and $B_{\eta d_U(x)}(q) \subset K_z$. In particular, $B_{\eta d_U(x)}(q) \subset U^c$ and $|q - x| \leq 3d_U(x)$. Let $\Sigma_x \subset \mathbb{S}^{n-1}$ be the set of directions whose rays from x meet this ball. Since its radius divided by $|q - x|$ is at least $\eta/3$, we have $\mathcal{H}^{n-1}(\Sigma_x) \geq a > 0$, with a independent of x. For every $\sigma \in \Sigma_x$, $d_{\sigma,U}(x) \leq |q - x| + \eta d_U(x) \leq 4 d_U(x)$, and hence

$$m_\alpha(x)^{-\alpha} \geq \frac{a\, 4^{-\alpha}}{A_{n,\alpha}} d_U(x)^{-\alpha} \geq c\, d_U(x)^{-\alpha}.$$

If $d_U(x) \geq d_*$, the same conclusion follows from $d_{\sigma,U}(x) \leq \operatorname{diam} U$ for every σ. Thus the two weights are comparable on all of U, uniformly in $\alpha \in [s_0 p, p]$, since $A_{n,\alpha}$ is continuous and strictly positive on this interval. Taking $\alpha = sp$, [32, Theorem 1.2] gives, for $v \in C_c^\infty(U)$,

$$\int_U \frac{|v|^p}{\operatorname{dist}^{sp}(x, \partial U)}\, dx \leq C D_{n,p,sp}^{-1} [v]_{s,p;U}^p \leq C(1-s)[v]_{s,p;U}^p.$$

The latter estimate extends from $C_c^\infty(U)$ to $W_0^{s,p}(U)$ by density.

Thus, if $\sup_j \mathcal{F}_j^0(u_j) < +\infty$, Proposition 5.2 gives, up to a subsequence, strong convergence in $L^p(U)$ to some $u \in W^{1,p}(U)$. By (5.51) and Fatou's lemma, $u/\operatorname{dist}(\cdot, \partial U) \in L^p(U)$, which implies that u has zero trace. Thus $u \in W_0^{1,p}(U)$, and the liminf inequality follows from Proposition 5.2.

For the limsup inequality, by density and a diagonal argument it suffices to consider $u \in C_c^\infty(U)$. We show that the recovery sequence constructed in Proposition 5.3 belongs to $W_0^{s_j,p}(U)$. Indeed, the joining and filling constructions in Lemma 4.1 modify u only in the balls B_j^{i,h_j}, whose radii are at most ε_j. If such a ball is disjoint from $\operatorname{supp} u$, both u and the corresponding annular average vanish there. Therefore, for all large j, the resulting functions have support in a fixed compact subset of U. The capacitary corrections in (5.29) preserve this property, since they are multiplied by the corresponding annular averages, which vanish near ∂U. In this setting the boundary correction is actually unnecessary. □

Remark 5.9. *We stress that the same argument applies to the functionals $\bar{\mathcal{F}}_j^0$ defined as $\bar{\mathcal{F}}_j$ in (5.41) on $W_0^{s_j,p}(U)$ and $+\infty$ otherwise. The Γ-limit coincides with the functional $\bar{\mathcal{F}}$ in Theorem 5.5 restricted to $W_0^{1,p}(U)$ and $+\infty$ elsewhere.*

Note that in this Dirichlet setting, equicoercivity also holds when $\ell = +\infty$. Indeed, (5.51) and the boundedness of U give a uniform $L^p(U)$ bound from the energy bound, so the BBM compactness theorem applies in every regime.

Remark 5.10. *The same asymptotics hold for the Gagliardo seminorm on the full space. More precisely, let*

$$\hat{\mathcal{F}}_j(u) := (1 - s_j) \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n + s_j p}}\, dx\, dy$$

if $u \in W_0^{s_j,p}(U)$, $u = 0$ a.e. on $\mathcal{K}_j \cap U$, and $+\infty$ otherwise on $L^p(U)$.

Noting that

$$\hat{\mathcal{F}}_j(u) = \mathcal{F}_j^0(u) + 2(1-s_j)\int_U \int_{U^c} \frac{|u(x)|^p}{|x-y|^{n+s_jp}}\,dydx$$

Corollary 5.8 easily implies that $(\hat{\mathcal{F}}_j)_j$ $\Gamma(L^p(U))$*-converges to* $\mathcal{F}^0$*. Indeed, by splitting* U^c *into* $U^c \cap B_{2R}$ *and the complementary set, where* $U \subseteq B_R$*, it suffices to note that the uniform Hardy inequality in (5.51) yields*

$$\begin{aligned}
&\int_U \int_{U^c} \frac{|u(x)|^p}{|x-y|^{n+s_jp}}\,dydx\\
&\le \frac{2^n n\mathcal{L}^n(B_1)}{s_jpR^{s_jp}} \int_U |u(x)|^p dx + (2R)^n \mathcal{L}^n(B_1) \int_U \frac{|u(x)|^p}{\operatorname{dist}^{n+s_jp}(x,\partial U)}\,dx\\
&\le \frac{C}{s_jpR^{s_jp}} \int_U |u(x)|^p dx + CR^n(1-s_j)\mathcal{F}_j^0(u)
\end{aligned}$$

for some dimensional constant $C > 0$*.*

Appendix A. Elementary results

We record the elementary estimates used in the proof of the uniform joining lemma. They are the explicit-constant version of [25, Lemma A.1].

Lemma A.1. *Let* $\nu > 0$*,* $z \in \mathbb{R}^n$*, and let* $O \subset \mathbb{R}^n$ *be measurable. Then the following statements hold:*

(i) *if* $\nu \in (0,n)$*, then*

$$\int_O \frac{dx}{|x-z|^\nu} \le \frac{n}{n-\nu}\omega_n^{\nu/n}\big(\mathcal{L}^n(O)\big)^{1-\nu/n}; \tag{A.1}$$

(ii) *if* $\nu \in (n,+\infty)$ *and* $d := \operatorname{dist}(z,O) > 0$*, then*

$$\int_O \frac{dx}{|x-z|^\nu} \le \frac{n\omega_n}{\nu-n} d^{n-\nu}. \tag{A.2}$$

Proof. We may assume $0 < \mathcal{L}^n(O) < +\infty$ in item (i), since the other cases are immediate. By Cavalieri's formula and the change of variables $t = r^{-\nu}$,

$$\int_O |x-z|^{-\nu}\,dx = \nu \int_0^{+\infty} \frac{\mathcal{L}^n(O\cap B_r(z))}{r^{1+\nu}}\,dr.$$

Set $\bar r = (\mathcal{L}^n(O)/\omega_n)^{1/n}$. Using $\mathcal{L}^n(O\cap B_r(z)) \le \min\{\omega_n r^n, \mathcal{L}^n(O)\}$ and splitting the last integral at $\bar r$, we obtain

$$\begin{aligned}
\int_O |x-z|^{-\nu}\,dx &\le \nu\omega_n \int_0^{\bar r} r^{n-\nu-1}\,dr + \nu\mathcal{L}^n(O)\int_{\bar r}^{+\infty} r^{-\nu-1}\,dr\\
&= \frac{n}{n-\nu}\omega_n^{\nu/n}\big(\mathcal{L}^n(O)\big)^{1-\nu/n}.
\end{aligned}$$

This proves item (i). For item (ii), $O \subset \mathbb{R}^n \setminus B_d(z)$ up to a negligible subset of the sphere $\partial B_d(z)$, and hence

$$\int_O |x-z|^{-\nu}\,dx \le \int_{\mathbb{R}^n\setminus B_d(z)} |x-z|^{-\nu}\,dx = \frac{n\omega_n}{\nu-n} d^{n-\nu}.$$

□

We show next that the constant in the Poincaré–Wirtinger inequality is uniform for the large class of closed convex sets K satisfying $B_1 \subset K \subset B_M$.

Proposition A.2. *Let $p \in (1,+\infty)$, $s_0 \in (0,1)$, and $M > 1$. There exists a constant $c = c(n,p,s_0,M) > 0$ such that, for every closed convex set $K \subset \mathbb{R}^n$ with $B_1 \subset K \subset B_M$, every $s \in [s_0,1)$, and every $v \in W^{s,p}(\operatorname{int} K)$,*

$$\|v - (v)_K\|^p_{L^p(K)} \le c(1-s) \int_{K\times K} \frac{|v(x)-v(y)|^p}{|x-y|^{n+sp}}\,dx\,dy. \tag{A.3}$$

Proof. Let

$$g_K(x) := \inf\{t > 0 : x \in tK\}$$

be the Minkowski functional of K. Convexity gives subadditivity, i.e. $g_K(x+y) \le g_K(x) + g_K(y)$, whereas the two inclusions $B_1 \subset K \subset B_M$ imply

$$\frac{|x|}{M} \le g_K(x) \le |x|.$$

Applying subadditivity to $x = y+(x-y)$ and then interchanging x and y, we obtain $|g_K(x) - g_K(y)| \le |x-y|$. Thus, on the unit sphere $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$, the function $\rho_K := 1/g_K$ satisfies

$$1 \le \rho_K \le M, \qquad |\rho_K(\omega) - \rho_K(\eta)| \le M^2|\omega - \eta|.$$

Define $F_K(0) = 0$ and

$$F_K(t\omega) := t\rho_K(\omega)\omega, \qquad 0 < t < 1, \quad \omega \in \mathbb{S}^{n-1}.$$

This map is a bijection from B_1 onto $\operatorname{int} K$, with inverse $F_K^{-1}(y) = g_K(y/|y|)y$ for $y \neq 0$. For nonzero x, y, the inequality

$$\min\{|x|,|y|\}\left|\frac{x}{|x|} - \frac{y}{|y|}\right| \le 2|x-y|$$

gives

$$\operatorname{Lip}(F_K) \le M + 2M^2, \qquad \operatorname{Lip}(F_K^{-1}) \le 3.$$

Consequently both constants are bounded by a number $L = L(M) \ge 1$, independently of K.

Given $s \in [s_0,1)$ and $v \in W^{s,p}(\operatorname{int} K)$, put $w = v \circ F_K$. For every constant $b \in \mathbb{R}$, Jensen's inequality gives $\|v - (v)_K\|_{L^p(K)} \le 2\|v - b\|_{L^p(K)}$. Choose $b = (w)_{B_1}$ and apply (2.4) on B_1. Changing variables, and using the bounds on the Jacobians of F_K and F_K^{-1}, yields

$$\begin{aligned}
\|v - (v)_K\|^p_{L^p(K)} &\le 2^p L^n \|w - (w)_{B_1}\|^p_{L^p(B_1)} \\
&\le c(n,p,s_0) L^n (1-s) \int_{B_1\times B_1} \frac{|w(x)-w(y)|^p}{|x-y|^{n+sp}}\,dx\,dy \\
&\le c(n,p,s_0) L^{4n+sp} (1-s) \int_{K\times K} \frac{|v(x)-v(y)|^p}{|x-y|^{n+sp}}\,dx\,dy.
\end{aligned}$$

Since $sp \le p$, the last constant depends only on n,p,s_0,M. □

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Use of artificial intelligence

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OpenAI's ChatGPT was used in the preparation of this manuscript for drafting, revising and simplifying parts of the proofs, and for comparing them with the source paper [25]. It also assisted with checking calculations and references, reorganizing the exposition, and editing. This work involved repeated mathematical feedback and corrections from the author, who retains all the responsibility for the contents.

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