Boundary regularity results for variational integrals

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Abstract: We prove partial Hölder continuity for the gradient of minimizers $u \in W^{1,p}(\Omega, \mathbb{R}^N)$, $\Omega \subset \mathbb{R}^n$ a bounded domain, of variational integrals of the form

$$\mathcal{F}[u;\Omega] := \int_\Omega \big[f(x,u,Du) + h(x,u)\big]\,dx\,,$$

where f is strictly quasi-convex and satisfies standard continuity and growth conditions, but where h is only a Caratheodory function of subcritical growth. The main focus is set on the presentation of a unified approach for the interior and the boundary estimates (provided that the boundary data are sufficiently regular) for all $p \in (1,\infty)$. Furthermore, a corresponding lower order Hölder regularity result for u is given in dimensions $n \leq p+2$ under the stronger assumption that f is strictly convex.

1 Introduction and statement of the results

We deal with variational integrals of the form

$$\mathcal{F}[u;\Omega] := \int_\Omega \big[f(x,u,Du) + h(x,u)\big]\,dx\,, \tag{1.1}$$

where the integrand $f\colon \Omega \times \mathbb{R}^N \times R^{nN} \to \mathbb{R}$ (with $n, N \geq 2$) is strictly quasi-convex, continuous and grows polynomially, and where $h\colon \Omega \times \mathbb{R}^N \to \mathbb{R}$ is only a Carathéodory function with subcritical growth, i. e., measurable with respect to the first variable (in particular, it may even be discontinuous) and (Hölder-) continuous with respect to the second variable. More precisely, we first of all assume that $z \mapsto f(\cdot,\cdot,z)$ is of class C^2 with jointly continuous second order derivatives, and that f is coercive, strictly quasi-convex and of p-growth, i. e.

$$\begin{cases} D_{zz}f(x,u,z) \text{ is continuous on } \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \\ \nu\,|z|^p \;\leq\; f(x,u,z) \;\leq\; L\,(1+|z|)^p \\ \nu \int_{(0,1)^n} (1+|z|+|D\varphi(y)|)^{p-2}|D\varphi(y)|^2\,dy \;\leq\; \int_{(0,1)^n} \big[f(x,u,z+D\varphi(y)) - f(x,u,z)\big]\,dy \end{cases} \tag{1.2}$$

for all $\varphi \in C_0^\infty((0,1)^n, \mathbb{R}^N)$. Secondly, we assume one of the following Hölder continuity conditions with respect to the first and the second variable: namely a mixed condition on f with respect to x and on $D_z f$ with respect to (x,u)

$$\begin{cases} \big|D_z f(x,u,z) - D_z f(\bar{x},\bar{u},z)\big| \;\leq\; L\,(1+|z|)^{p-1}\,\omega_1\big(|x-\bar{x}|+|u-\bar{u}|\big) \\ \big|f(x,u,z) - f(x,\bar{u},z)\big| \;\leq\; L\,(1+|z|)^p\,\omega_2\big(|u-\bar{u}|\big) \end{cases} \tag{1.3}$$

or a condition on f with respect to (x,u)

$$\big|f(x,u,z) - f(\bar{x},\bar{u},z)\big| \;\leq\; L\,(1+|z|)^p\,\omega_1\big(|x-\bar{x}|+|u-\bar{u}|\big)\,. \tag{1.4}$$

A connection between conditions (1.3) and (1.4) will be discussed at the end of the introduction. All these inequalities in (1.2)-(1.4) are assumed to hold for fixed $0 < \nu \leq L$, some $p \in (1,\infty)$ and all $x, \bar{x} \in \Omega$, $u, \bar{u} \in \mathbb{R}^N$, and $z \in \mathbb{R}^{nN}$. Here, $\omega_1, \omega_2\colon \mathbb{R}^+ \to \mathbb{R}^+$ are two modulus of continuity, i. e. bounded by 1 (without loss of generality), concave and non-decreasing such that $\lim_{\rho\to 0^+}\omega_1(\rho) = 0 = \lim_{\rho\to 0^+}\omega_2(\rho)$.

We note that quasi-convexity (a notion which was introduced by Morrey [52]) is an extension of convexity to a global property and is essentially equivalent to lower semicontinuity, cf. [1, 41]. Applying [48, p. 6, step 2], we may also assume a growth condition on the first derivatives of the form $|D_z f(x,u,z)| \leq L(1+|z|)^{p-1}$.

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Furthermore, it can be verified that the conditions (1.2) above, see [52, Theorem 4.3], imply the strict ellipticity of the matrix $D_{zz}f$ in the sense of Legendre-Hadamard, and therefore we may also assume

$$\nu\,(1+|z|)^{p-2}|\xi|^2\,|\eta|^2 \;\le\; D_{zz}f(x,u,z)\,\xi\otimes\eta\cdot\xi\otimes\eta$$

for all $\xi\in\mathbb{R}^N, \eta\in\mathbb{R}^n$. As observed by Acerbi and Fusco [2], a growth condition for the second derivatives is not needed, and the continuity of $D_{zz}f$ is sufficient to prove a partial regularity result. This latter condition implies in particular that second order derivatives are bounded on compact subsets of $\bar\Omega\times\mathbb{R}^N\times\mathbb{R}^{nN}$, i. e. for every $M>0$ there exists a constant K_M such that there holds

$$\big|D_{zz}f(x,u,z)\big| \;\le\; K_M$$

for all $x\in\Omega$, $u\in\mathbb{R}^N$ and $z\in\mathbb{R}^{nN}$ with $|u|,|z|\le M$. Due to the continuity of $D_{zz}f$ we may hence assume the existence of a modulus of continuity for $D_{zz}f$ on compact subsets. More precisely, we may assume that to a given $M>0$ there exists a monotone nondecreasing concave function $\mu_M\colon[0,\infty)\to[0,\infty)$ continuous at 0 such that $\mu_M(0)=0$ and

$$\big|D_{zz}f(x,u,z_1)-D_{zz}f(x,u,z_2)\big| \;\le\; \mu_M(|z_1-z_2|) \tag{1.5}$$

for all $x\in\Omega$, $u\in\mathbb{R}^N$ and $z_1,z_2\in\mathbb{R}^{nN}$ with $|u|\le M$ and $|z_1|,|z_1|\le M+1$. For the second integrand h we further assume

$$\begin{cases} 0 \;\le\; h(x,u) \;\le\; L\,(1+|u|)^{\gamma} \\ \big|h(x,u)-h(\bar x,\bar u)\big| \;\le\; L\,(1+|u|+|\bar u|)^{\gamma}\,\omega_2\big(|u-\bar u|\big)\,, \end{cases} \tag{1.6}$$

for all $x,\bar x\in\Omega$, $u,\bar u\in\mathbb{R}^N$, and some $\gamma\in(0,p^*)$. Here we denote by p^* the exponent from the Sobolev embedding, i. e. $p^*=\frac{np}{n-p}$ if $p<n$ and $p^*\in(p,\infty)$ arbitrary if $p\ge n$. We emphasize that taking the same modulus of continuity $\omega_2(\cdot)$ for h as in condition (1.3) is not restrictive.

The above conditions concerning the growth and the quasi-convexity of the integrand guarantee that the integral in (1.1) is well defined for all maps $u\in W^{1,p}(\Omega,\mathbb{R}^N)$, and moreover, the existence of minimizers follows in a standard way from the direct method of the calculus of variations and the sequentially weak lower semicontinuity of $\mathcal{F}[u;\Omega]$ in $W^{1,p}(\Omega,\mathbb{R}^N)$. We recall that u is a minimizer of the functional $\mathcal{F}[\,\cdot\,;\Omega]$ with boundary values $g\in W^{1,p}(\Omega,\mathbb{R}^N)$ if u is a $g+W_0^{1,p}(\Omega,\mathbb{R}^N)$-map such that

$$\mathcal{F}[u;\Omega] \;\le\; \mathcal{F}[v;\Omega] \qquad \text{for every } v\in g+W_0^{1,p}(\Omega,\mathbb{R}^N)\,.$$

For dealing with partial regularity of Du it is further a standard condition to assume not only continuity, but Hölder continuity of the integrand with respect to the first and second variable, i. e.

$$\omega_1(t) \;\le\; \min\{1,t^{\alpha_1}\} \quad\text{and}\quad \omega_2(t) \;\le\; \min\{1,t^{\alpha_2}\} \qquad \text{for some } \alpha_1,\alpha_2\in(0,1) \tag{1.7}$$

and all $t\in\mathbb{R}^+$; for shortness of notation we shall also use $\omega_{12}(t):=\max\{\omega_1(t),\omega_2(t)\}$ for all $t\in\mathbb{R}$. Under these assumptions we shall now investigate the regularity properties of minimizers u under different circumstances. Restricting ourselves temporarily to the situation where the additional integrand h does not occur, we comment on some well-known partial regularity results: The first results were obtained by Morrey, Giusti and Miranda [53, 35] for weak solutions of elliptic systems employing techniques originally developed by De Giorgi and Almgren [15, 5] in the setting of geometric measure theory and the theory of minimal surfaces. We here recall that in general we cannot expect full regularity result, see the examples in [16, 36, 54, 29, 55, 61]; hence, the objective in the regularity results discussed here is to study the set of points where the gradient Du is regular (in the sense where it is locally in a relative neighbourhood known to be continuous) and then to obtain higher regularity on this set via the fact that u is actually a weak solution respectively a minimizer. This is usually achieved by a comparison of the original solution or minimizer u to the solution of a linearized system which enjoys good a priori estimates. Up to now this comparison principle can be implemented by the direct approach, the indirect approach via blow up, or the method of $\mathcal{A}$-harmonic approximation. Furthermore, the regularity proof comes along with a characterization of the regular set which then results in a bound on the singular set (the complement of the regular set), e.g. that its Lebesgue measure is zero or even that the Hausdorff dimension is less than the space dimension. Taking the theory for elliptic systems for granted, then if the minimization problem (1.1) admits to write down the Euler-Lagrange system for the functional $\mathcal{F}[\,\cdot\,;\Omega]$

$$\operatorname{div} D_zf(x,u,Du) \;=\; D_u\big(f(x,u,Du)+h(x,u)\big) \qquad \text{in } \Omega \tag{1.8}$$

and if f is convex (see $(1.9)_2$ further below), the partial regularity of minimizers of the variational problem follows immediately. Nevertheless, following this approach of exploiting the fact that the minimizer is an extremal of the Euler-Lagrange system (1.8) does often not lead to the desired results, because this technique cannot distinguish between a minimizer and an extremal, and therefore, it usually requires further assumptions on the solution which are generally not satisfied. To clarify this issue we recall that in the case of quadratic type problems such as $f(x,u,z) = a(x,u)|z|^2$ the Euler-Lagrange system has an inhomogeneity of critical growth (at least provided that $D_u a(x,u)$ does not vanish identically), and partial Hölder continuity of weak solutions is only known in this case for a priori bounded solutions satisfying an additional smallness condition. Furthermore, in order to obtain the existence of the Euler-Lagrange system we have to require integrands which are differentiable with respect to the u-variable – a further restriction not fulfilled by most interesting examples. This illustrates the need to find a partial regularity proof relying heavily on the minimization property of u and hence on the functional $\mathcal{F}[\,\cdot\,;\Omega]$, and not any more on the Euler-Lagrange system (1.8). Under the assumption of quasi-convexity this was first accomplished by Evans for integrands depending only on the gradient variable and then extended to the general case by Fusco, Hutchinson, Giaquinta and Modica [25, 28, 33], ending up with a partial Hölder continuity result for the gradient Du. Some extensions of this result which are of interest for the present paper include [2, 14, 17, 19, 46, 40]. Furthermore, we refer to Mingione's survey paper [51] which provides an extensive list of references and detailed comments on various results (and also the techniques) in this directions. We now define

$$\beta \,=\, \min\Big\{\frac{\alpha_2}{2-\alpha_2}, \frac{\alpha_1}{2}\Big\} \,<\, \frac{1}{2}$$

and recall the V-function given by $V(\xi) = (1+|\xi|^2)^{(p-2)/4}\xi$ for all $\xi \in R^k$ and $p > 1$. In the context of minimizers of quasi-convex integrals including the possibly discontinuous part h we then can formulate our first partial regularity result which on the one hand gives a necessary and sufficient criterion – both in the interior and at the boundary – for a point to be a regular one for Du (this characterization implies in particular that the regular set is of full Lebesgue measure) and which on the other hand states the local optimal Hölder regularity for Du on the regular set. This result will be established via the method of $\mathcal{A}$-harmonic approximation.

Theorem 1.1: *Consider $n \geq 2$, $p \in (1,\infty)$, $\gamma \in (0,p^*)$, $\Omega \subset \mathbb{R}^n$ a bounded domain of class $C^{1,2\beta}$ and a map $g \in C^{1,2\beta}(\overline{\Omega},\mathbb{R}^N)$. Let $u \in W^{1,p}(\Omega,\mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{F}[\,\cdot\,;\Omega]$ in* (1.1) *under the assumptions* (1.2), (1.3) *or* (1.4), (1.6), (1.7) *and boundary values $u = g$ on $\partial\Omega$. Then there exist an open set $\mathrm{Reg}_\Omega(Du) \subset \Omega$ and a relatively open set $\mathrm{Reg}_{\partial\Omega}(Du) \subset \partial\Omega$ such that*

$$|\overline{\Omega} \setminus (\mathrm{Reg}_\Omega(Du) \cup \mathrm{Reg}_{\partial\Omega}(Du))| \,=\, 0 \qquad \textit{and} \qquad Du \in C^{0,\beta}_{\mathrm{loc}}(\mathrm{Reg}_\Omega(Du) \cup \mathrm{Reg}_{\partial\Omega}(Du),\mathbb{R}^{nN}).$$

Moreover, the regular points of Du in the interior and at the boundary are characterized by

$$\begin{aligned}\mathrm{Reg}_\Omega(Du) = \Big\{x_0 \in \Omega\colon\, &\limsup_{\rho\to 0^+} \big(|(u)_{\Omega\cap B_\rho(x_0)}| + |(Du)_{\Omega\cap B_\rho(x_0)}|\big) < \infty \\ &\text{and } \liminf_{\rho\to 0^+} \fint_{\Omega\cap B_\rho(x_0)} \big|V\big(Du - (Du)_{\Omega\cap B_\rho(x_0)}\big)\big|^2\,dx = 0\Big\}\,,\\ \mathrm{Reg}_{\partial\Omega}(Du) = \Big\{x_0 \in \partial\Omega\colon\, &\limsup_{\rho\to 0^+} |(D_{\nu_{\partial\Omega}(x_0)}u)_{\Omega\cap B_\rho(x_0)}| < \infty \\ &\text{and } \liminf_{\rho\to 0^+} \fint_{\Omega\cap B_\rho(x_0)} \big|V\big(D(u-g) - (D_{\nu_{\partial\Omega}(x_0)}(u-g))_{\Omega\cap B_\rho(x_0)} \otimes \nu_{\partial\Omega}(x_0)\big)\big|^2\,dx = 0\Big\}\,.\end{aligned}$$

Here $\nu_{\partial\Omega}(x_0)$ denotes the inward-pointing unit normal vector to $\partial\Omega$ in x_0.

Remark 1.2: It is worth noting that the characterization of regular boundary points can be slightly improved for $\gamma \in (0,p)$, see Remark 5.2, if f does not depend explicitly on u or if f satisfies condition (1.9) of strict convexity instead of merely the strict quasi-convexity condition (1.2). In these cases we have

$$\begin{aligned}\mathrm{Reg}_{\partial\Omega}(Du) = \Big\{x_0 \in \partial\Omega\colon\, &\limsup_{\rho\to 0^+} |(D_{\nu_{\partial\Omega}(x_0)}u)_{\Omega\cap B_\rho(x_0)}| < \infty \\ &\text{and } \liminf_{\rho\to 0^+} \fint_{\Omega\cap B_\rho(x_0)} \big|V\big(D_{\nu_{\partial\Omega}(x_0)}u - (D_{\nu_{\partial\Omega}(x_0)}u)_{\Omega\cap B_\rho(x_0)}\big)\big|^2\,dx = 0\Big\}\,.\end{aligned}$$

This means that in these situations only the normal component of Du is of importance for the verification whether a given boundary point is regular or not. Actually, this holds also for weak solutions of elliptic systems, but it is not clear whether this characterization is also available in the general quasi-convex case.

Related results were for example given by Giaquinta, Giusti [32] in the convex case and by Hamburger [39] in the quasi-convex case which however differ in several points: they only considered quadratic principal parts respectively superquadratic principal parts, and they were concerned with interior partial regularity and left the characterization of regular boundary points open. Furthermore, in particular in Hamburger's paper, there are slight differences in the assumption on the integrands: rewriting Hamburger's condition on $F(x,u,z)$ in the case $F(x,u,z) = f(x,u,z) + g(x,u)$ reveals that $g(x,u)$ would have to be assumed bounded without any possible growth assumption with respect to u (even if formulating the assumptions only on F and not separately on f and g is more general).

Example: Typical examples for quasi-convex integrals arising in particular in the theory of elasticity are $f(z) = |z|^p + |\det z|$ for $p \geq n = N$. Variational integrals of this kind have been studied widely, and Theorem 1.1 now also provides a partial Hölder regularity result for the model functional

$$\int_\Omega \big[|Du(x)|^2 + \big(1 + |\det Du(x)|^2\big)^{\frac{1}{2}} + h(x)\,\big(1 + |u(x)|^{\gamma-\alpha_2}\big)\,|u(x) - u_0|^{\alpha_2}\big]\,dx\,,$$

where $h(x)$ is a bounded, non-negative and measurable function and $u_0 \in \mathbb{R}^N$ arbitrary (this is an example which is not covered by [39]). We highlight that this provides a class of examples which do not admit uniformly bounded second order derivatives in the gradient variable, see [2]. We further refer to [60] for the existence of quasi-convex function with subquadratic growth.

Remark (possible extensions): In fact, even the continuity condition on h with respect to u is in certain cases dispensable, namely if $(1.6)_1$ is replaced by $0 \leq h(x,u) \leq L|u|^\gamma$, which allows to consider other interesting examples. This could be treated in the context of almost-minimizers, see e.g. [17, 20, 42]. We further note that for a general partial regularity result as in Theorem 1.1 for the gradient of u a condition like (1.7) (or also the weaker condition of Dini-continuous integrands) seems to be crucial. For the case of merely continuous integrands (which can be understood as the limit case $\alpha_1, \alpha_2 \searrow 0$) we refer to the recent paper of Foss and Mingione [27] where the authors tackled this long-standing open problem (assuming $h = 0$) and succeeded in showing that the minimizer is then partially Hölder continuous with any exponent $\beta \in (0,1)$ (which was so far only known to hold globally on $\bar{\Omega}$ in the scalar case $N = 1$). We believe that an extension of this result to the case of possibly discontinuous integrand as considered above is possible (but due to the delicate choice of the excess the proof will be quite technical). Furthermore, we believe that problems of nonstandard $p(x)$-growth should be obtained in a similar way by passing from the exponent $p(x)$ at a point x to a constant exponent p via a higher integrability result and the localization technique.

We now give some explanations regarding the Hölder exponent β: it turned out that β is the optimal exponent under the assumptions made on the Hölder continuity of f and h, see the example of Phillips [56, p.5]. The proof that this exponent is actually attained is demonstrated below in two steps, where in a first step $C^{1,\beta_0}_{\mathrm{loc}}$-regularity on the regular set is proved for some $\beta_0 > 0$, and then in a second step the improvement of the exponent is achieved taking advantage of the local boundedness of Du, see also [32, 40]. Unless the functional does not depend on u we don't know whether it is possible to obtain the optimal regularity in only one step. The regularity required on the boundary data instead is not optimal and is needed for technical reasons (we use a transformation allowing us to prove the result in the model situation of a half-ball which is easier to handle). This loss could probably be avoided (in the sense that g and Ω of class $C^{1,\beta}$ should be sufficient) if we had worked on intersection of balls with Ω without transforming the system, as it was performed by Kronz [47].

Lastly we give a short outlook on results of Kristensen and Mingione regarding the Hausdorff dimension of the singular set for Du. The characterization of regular points applies a priori almost everywhere in the Lebesgue sense, in other words the theorem makes no improvement on the bound of the Hausdorff dimension of the singular set. In fact, as a consequence of a higher integrability result for Du, the bound $\dim_{\mathcal{H}}(\overline{\Omega} \setminus (\mathrm{Reg}_\Omega(Du) \cup \mathrm{Reg}_{\partial\Omega}(Du))) \leq n$ given above is never optimal [43, Theorem 1.1], but it is strictly less than the space dimension. However, this does not yield the existence of even one single regular boundary point, since the boundary $\partial\Omega$ itself is a set of Lebesgue measure zero (and due to the counterexample in [29] we know that singularities may occur at the boundary even if the boundary data is smooth). So a further dimension reduction of the singular set is needed in order to show that almost every boundary point is in fact a regular one. For weak solutions of elliptic systems this reduction was achieved in the interior and then up to the boundary [50, 49, 21], ending up with conditions (on n, p and the Hölder continuity exponent of the coefficients) guaranteeing that almost every boundary point is a regular one. For minimizers of variational functionals it is again not possible to recover this conclusion from the Euler-Lagrange system (1.8), but it has to be obtained directly from the functional and the minimization property. First steps in this direction were taken in [43] for a dimension reduction in the interior, and the boundary regularity for various functionals was accomplished recently in [45, 44].

The second aim of this paper is to study lower order regularity of minimizers $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ to the variational problem (1.1). This means that we are no longer interested in the regular set of the gradient Du, but rather in the regular set of u itself (i.e. the set where u is locally continuous). We note that similarly to the regularity result for Du above, the situation in the scalar case is much easier. In fact condition $(1.2)_2$ is sufficient to show a full Hölder continuity result [31]. In the vectorial case instead we now replace in the assumptions above the strict quasi-convexity condition $(1.2)_3$ by the stronger condition of strict convexity and an additional upper bound on the second order derivatives of f with respect to the gradient variable. More precisely, we assume for fixed $0 < \nu \le L$:

$$\begin{cases} \nu\,|z|^p \,\le\, f(x,u,z) \,\le\, L\,(1+|z|)^p \\ \nu\,(1+|z|)^{p-2}|\lambda|^2 \,\le\, D_{zz}f(x,u,z)\,\lambda\cdot\lambda \,\le\, L\,(1+|z|)^{p-2}|\lambda|^2 \end{cases} \tag{1.9}$$

for all $x \in \Omega$, $u \in \mathbb{R}^N$, and $z, \lambda \in \mathbb{R}^{nN}$. Assuming a continuity condition in the (x,u)-variable as in (1.3) or (1.4), the task is now to obtain Morrey-type estimates up to the boundary by a direct comparison principle. This in turn allows us to conclude Hölder regularity of u on the regular set via the Meyers-Campanato embedding, provided that the assumption $n \le p+2$ is satisfied (this is usually called the assumption of low dimensions). Furthermore, we shall conclude a bound on the Hausdorff dimension of the singular set from the characterization of the regular points obtained in the course of these estimates. This Morrey space regularity theory traces back to several results of Campanato (for instance [11, 12, 13]) on weak solutions to elliptic systems, and until today it was extended in various respects. In particular, also the corresponding interior results for minimizers were obtained [43, Theorem 8.1], again by only taking advantage of the minimization property of u and not of the related Euler-Lagrange system (1.8) (which is only used for the frozen system to perform a comparison principle). Further extensions to non-standard $p(x)$-growth, obstacle problems or to asymptotically convex problems can be found in the recent publications [24, 26]. We now pass again to the functionals of form (1.1) including the possibly discontinuous part h. Provided that the boundary datum is regular, our second theorem states that u is partially Hölder continuous up to the boundary outside a set of Hausdorff dimension $n-p$ (in particular, this means that a partial boundary regularity holds). Here it is worth mentioning that f is only assumed to be continuous, but not necessarily Hölder continuous with respect to the first and second variable.

Theorem 1.3: *Consider $n \ge 2$, $p \in (1,\infty)$, $\gamma \in (0,p^*)$, $\Omega \subset \mathbb{R}^n$ a bounded domain of class C^1 and a map $g \in C^1(\overline{\Omega}, \mathbb{R}^N)$. Let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{F}[\,\cdot\,;\Omega]$ in (1.1) under the assumptions (1.9), (1.3), $(1.6)_1$ and boundary values $u = g$ on $\partial\Omega$. Then there exists $\varepsilon > 0$ depending only on n, N, p and $\frac{L}{\nu}$ such that for every $\lambda \in (0, \min\{1-(n-2-\varepsilon)/p, 1\})$ there hold:*

1. *for $n \in (p, p+2+\varepsilon)$ we have*

$$\dim_{\mathcal{H}}\big(\overline{\Omega} \setminus (\mathrm{Reg}_{\Omega}(u) \cup \mathrm{Reg}_{\partial\Omega}(u))\big) < n-p \qquad \text{and} \qquad u \in C^{0,\lambda}_{\mathrm{loc}}\big(\mathrm{Reg}_{\Omega}(u) \cup \mathrm{Reg}_{\partial\Omega}(u), \mathbb{R}^N\big)\,.$$

Moreover, the regular points of u in the interior and at the boundary are given by

$$\mathrm{Reg}_{\Omega}(u) = \Big\{x_0 \in \Omega\colon \liminf_{\rho\to 0^+} \rho^{p-n} \int_{\Omega\cap B_\rho(x_0)} \big(1+|Du|\big)^p\,dx = 0 \text{ and } \limsup_{\rho\to 0^+} |(u)_{\Omega\cap B_\rho(x_0)}| < \infty\Big\}\,,$$

$$\mathrm{Reg}_{\partial\Omega}(u) = \Big\{x_0 \in \partial\Omega\colon \liminf_{\rho\to 0^+} \rho^{p-n} \int_{\Omega\cap B_\rho(x_0)} \big(1+|Du|\big)^p\,dx \,=\, 0\Big\}\,;$$

2. *for $n \in [2,p]$ or for f and h not depending explicitly on u we have $u \in C^{0,\lambda}(\overline{\Omega}, \mathbb{R}^N)$.*

Remark: In the second statement of the theorem we observe that for $n \in [2,p]$ (only to be considered in the superquadratic case) a first global Hölder continuity result up to the boundary is inferred from the Sobolev embedding and higher integrability of Du. The a priori Hölder exponent is then given by $\lambda = 1-\frac{n}{q}$ for some $q > p$ arising from the application of Gehring's Lemma and depending only on structure constants. Therefore, Theorem 1.3 improves the Hölder exponent from the standard Sobolev embedding using the fact that u is actually a minimizer of a sufficiently regular variational problem (except of course if q happens to be very large).

We here want to mention the related up to the boundary result [57] also including possibly discontinuities in the x-variable, namely functionals with integrand $f(x,u,z)$, satisfying essentially the same conditions as in our statement, but with $x \mapsto f(x,u,z)$ belonging to the space VMO of functions of vanishing mean oscillation (uniformly with a compatible p-growth condition in the gradient variable). This seems to be the weakest possible dependency on the x-variable in the leading term under which a partial regularity result can still be proved. We also emphasize that the results of Theorems 1.3 and the following Theorem 1.4 are similar to the results in [6, Theorem 1 and 2] obtained in the setting of weak solutions to nonlinear elliptic systems (and $p \geq 2$).

So far, we have gained for all low dimensional cases $n \in [2, p+2]$ an almost everywhere Hölder regularity result for the minimizer u to $\mathcal{F}[\,\cdot\,;\Omega]$ and its gradient. However, weak solutions of elliptic systems or minimizers to variational problems in the two-dimensional case $n = 2$ are known to enjoy much better regularity properties (notice that all the counterexamples to full regularity named above only work for $n \geq 3$), see for example an interior result for minimizers in the superquadratic case given by Kristensen and Mingione [43, Theorem 1.7]. It turns out that we can follow this line of argument also for variational integrals of type (1.1): in dimensions $n = 2$ we exploit the fact from Theorem 1.3 that u is Hölder continuous either everywhere for $p \geq 2$ or at least outside a set of Hausdorff dimension less than $n-p$ for $p \in (1,2)$ – provided that we still assume the strict convexity assumption. This allows to show in a direct proof (based on the comparison principle applied for the Morrey-type estimates) that the regular sets of u and of Du actually coincide:

Theorem 1.4: *Let $n = 2$, $p \in (1,\infty)$, $\gamma \in (0,p^*)$, $\Omega \subset \mathbb{R}^2$ a bounded domain of class $C^{1,2\beta}$, and a map $g \in C^{1,2\beta}(\overline{\Omega},\mathbb{R}^N)$. Let $u \in W^{1,p}(\Omega,\mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{F}[\,\cdot\,;\Omega]$ in* (1.1) *under the assumptions* (1.9), (1.3), (1.6), (1.7), *and boundary values $u = g$ on $\partial\Omega$. Then there holds*

$$u \in C^{1,\beta}\big(\overline{\Omega},\mathbb{R}^N\big)$$

for the superquadratic case $p \geq 2$, whereas in the subquadratic case $p \in (1,2)$ there holds

$$u \in C^{1,\beta}_{\mathrm{loc}}\big(\operatorname{Reg}_{\Omega}(u) \cup \operatorname{Reg}_{\partial\Omega}(u),\mathbb{R}^N\big)\,,$$

with $\operatorname{Reg}_{\Omega}(u)$ and $\operatorname{Reg}_{\partial\Omega}(u)$ defined in Theorem 1.3. In particular, $\mathcal{H}^{\min\{0,n-p\}}$-almost every boundary point is a regular one for Du. Moreover, we have full regularity $u \in C^{1,\beta}(\overline{\Omega},\mathbb{R}^N)$ also for $p \in (1,2)$ if f and h do not depend explicitly on u.

We further deduce an immediate consequence of the previous two theorems: we first recall that in the previous two theorem we have only considered integrands f under the mixed continuity respectively Hölder continuity condition (1.3), namely on f with respect to x and on $D_z f$ with respect to (x,u). The corresponding result under the continuity respectively Hölder continuity condition (1.4) on f with respect to (x,u) now follows by an observation of Giaquinta and Giusti [32, p. 247], see also [59, Appendix A]: it was shown that if $\omega(\cdot)$ denotes the modulus of continuity of f, then the growth condition on $D_{zz}f$ in $(1.9)_2$ allows to conclude that $D_z f$ has the corresponding (and optimal) modulus of continuity $\sqrt{\omega(\cdot)}$ (and no further assumption on ω is needed). Hence, we also have:

Corollary 1.5: *The results of Theorem 1.3 and 1.4 remain true if the continuity conditions* (1.3) *in the assumptions of each are replaced by the continuity condition* (1.4).

In case of Theorem 1.1 it is however not clear whether the result under the assumption (1.4) follows from the one under the the assumption (1.3), since in the statement it was not required that second order derivatives of f are bounded uniformly (with a suitable growth condition in the gradient variable).

To conclude the introduction we emphasize that parts of the results (in particular the subquadratic case and the extensions to regularity up to the boundary) are new for minimizers of functionals with $h = 0$, even if the techniques of the proof are in line with older and by now classical results. The outline of the paper is the following: in Sections 2, 3 we collect some preliminary material and standard properties of minimizers. Section 4 is devoted to comparison estimates used in the Morrey-estimates for the low dimensional situation, and in Sections 5-7 we finally combine the previous facts and give the proofs of Theorems 1.1, 1.3 and 1.4.

2 Notation and preliminaries

We start with some remarks on the notation used below: we write $B_\rho(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < \rho\}$ and $B_\rho^+(x_0) = \{x \in \mathbb{R}^n : x_n > 0,\, |x - x_0| < \rho\}$ for a ball or an upper half-ball, respectively, centered on a point x_0 ($\in \mathbb{R}^{n-1} \times \{0\}$ in the latter case) with radius $\rho > 0$. Sometimes it will be convenient to treat the n-th component of $x \in \mathbb{R}^n$ separately; therefore, we set $x = (x', x_n)$ where $x' = (x_1, \dots, x_{n-1})$. Furthermore, we write

$$\Gamma_\rho(x_0) \,=\, \{x \in \mathbb{R}^n : |x - x_0| < \rho,\, x_n = 0\}\,,$$

for $x_0 \in \mathbb{R}^{n-1} \times \{0\}$. In the case $x_0 = 0$ we set $B_\rho := B_\rho(0)$, $B := B_1$ as well as $B_\rho^+ := B_\rho^+(0)$, $B^+ := B_1^+$ with $\Gamma_\rho := \Gamma_\rho(0)$, $\Gamma := \Gamma_1$. For ease of notation we will sometimes write $B_\rho^{(+)}(x_0)$ to cover at the same time the cases $B_\rho(x_0)$ of full balls and $B_\rho^+(x_0)$ of half-balls.

Let $\mathcal{L}^n$ denote the n-dimensional Lebesgue measure. For any bounded, measurable set $X \subset \mathbb{R}^n$ with $\mathcal{L}^n(X) =: |X| > 0$, we denote the mean value of a function $h \in L^1(X, \mathbb{R}^N)$ by $(h)_X = \fint_X h\,dx$, and, in particular, we use the abbreviation $(h)_{x_0,\rho}$ for the mean value on $B_\rho(x_0)$ or on $B_\rho^+(x_0)$, respectively. The constants c appearing in the different estimates will all be chosen greater than or equal to 1, and they may vary from line to line.

In this paper we shall use several function spaces: we denote by $W^{1,p}(\Omega, \mathbb{R}^N)$, $p \in (1, \infty)$ the usual Sobolev spaces of functions mapping from a domain $\Omega \subset \mathbb{R}^n$ to $\mathbb{R}^N$, and $W_0^{1,p}(\Omega, \mathbb{R}^N)$ stands for the subspace of functions with zero-boundary values. We also introduce the following notation for $W^{1,p}$-functions defined on a half-ball $B_\rho^+(x_0)$ which vanish in the sense of traces on the flat part of the boundary:

$$W_\Gamma^{1,p}(B_\rho^+(x_0), \mathbb{R}^N) := \{u \in W^{1,p}(B_\rho^+(x_0), \mathbb{R}^N) : u = 0 \text{ on } \Gamma_\rho(x_0)\}\,.$$

Furthermore, the Hölder spaces $C^{k,\alpha}(\Omega, \mathbb{R}^N)$, $\alpha \in (0,1)$, $k \in \mathbb{N}_0$, consist of all functions in $C^k(\Omega, \mathbb{R}^N)$, i. e. k times continuously differentiable, for which all derivatives of order k are Hölder continuous with exponent α, and by $C^{k,\alpha}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ we denote all function which are locally of class $C^{k,\alpha}$. We will further use the Morrey spaces $L^{p,\varsigma}(\Omega, \mathbb{R}^N)$, $\varsigma > 0$, which is the linear space of all functions $u \in L^p(\Omega, \mathbb{R}^N)$ such that

$$\|u\|^p_{L^{p,\varsigma}(\Omega,\mathbb{R}^N)} := \sup_{y \in \Omega, 0 < \rho \le \operatorname{diam} \Omega} \rho^{-\varsigma} \int_{B_\rho(y) \cap \Omega} |u|^p\,dx \,<\, \infty\,.$$

We consider a bounded domain Ω in $\mathbb{R}^n$, $n \ge 2$. The boundary of Ω is assumed to be of class $C^{1,\alpha}$ or of class C^1; this means that for every point $x_0 \in \partial\Omega$ there exist a radius $r > 0$ and a function $b : \mathbb{R}^{n-1} \to \mathbb{R}$ of class $C^{1,\alpha}$ (or C^1) such that (up to an isometry) Ω is locally represented by $\Omega \cap B_r(x_0) = \{x \in B_r(x_0) : x_n > b(x')\}$. Thus we can locally straighten the boundary $\partial\Omega$ via a $C^{1,\alpha}$- (or C^1-) transformation $(x', x_n) \mapsto (x', x_n - h(x'))$.

In what follows we will often use the V-function: For $\xi \in \mathbb{R}^k$, $k \in \mathbb{N}$ and $p > 1$ it is defined by

$$V(\xi) = \big(1 + |\xi|^2\big)^{\frac{p-2}{4}}\,\xi\,,$$

which is a locally bi-Lipschitz bijection on $\mathbb{R}^k$. It behaves linearly for $|\xi|$ very small, but grows like $|\xi|^{p/2}$ for $|\xi| \to \infty$. This is used in particular when we deal with the subquadratic case, whereas it is often sufficient to remark that $|V(\xi)|$ and $|\xi| + |\xi|^{p/2}$ are equivalent up to a constant depending only on p in the superquadratic case. We here list some useful algebraic properties of V we shall frequently use:

Lemma 2.1 (cf. [14], Lemma 2.1): *Let $p \in [1, \infty)$ and $V : \mathbb{R}^k \to \mathbb{R}^k$ be the function defined above. Then for all $\xi, \eta \in \mathbb{R}^k$ and $t > 0$ there hold:*

(i) $2^{\frac{p-2}{4}} \min\{|\xi|, |\xi|^{\frac{p}{2}}\} \,\le\, |V(\xi)| \,\le\, \min\{|\xi|, |\xi|^{\frac{p}{2}}\}$ *for* $p \in (1,2)$,
$\max\{|\xi|, |\xi|^{\frac{p}{2}}\} \,\le\, |V(\xi)| \,\le\, 2^{\frac{p-2}{4}} \max\{|\xi|, |\xi|^{\frac{p}{2}}\}$ *for* $p \ge 2$,

(ii) $|V(t\xi)| \,\le\, \max\{t, t^{\frac{p}{2}}\} |V(\xi)|$,

(iii) $|V(\xi + \eta)| \,\le\, c(p)\,\big(|V(\xi)| + |V(\eta)|\big)$,

(iv) $c(p)\,|\xi-\eta| \,\le\, \dfrac{|V(\xi)-V(\eta)|}{(1+|\xi|^2+|\eta|^2)^{\frac{p-2}{4}}} \,\le\, c(k,p)\,|\xi-\eta|$,

(v) $|V(\xi)-V(\eta)| \,\le\, c(k,p)\,|V(\xi-\eta)|$ *for* $p\in(1,2)$,
$|V(\xi)-V(\eta)| \,\le\, c(k,p,M)\,|V(\xi-\eta)|$, *provided* $|\eta|\le M$, *for* $p\ge 2$,

(vi) $|V(\xi-\eta)| \,\le\, c(p,M)\,|V(\xi)-V(\eta)|$, *provided* $|\eta|\le M$, *for* $p\in(1,2)$,
$|V(\xi-\eta)| \,\le\, c(p)\,|V(\xi)-V(\eta)|$ *for* $p\ge 2$.

A useful property needed below was observed in [58, Lemma 6.2]. We need the following version (the proof remains essentially unchanged), applied later for the special choices $I=\{1,\dots,n\}$ with $\sum_{k=1}^n (D_k u)_\Omega \otimes e_k = (Du)_\Omega$ and $I=\{n\}$:

Lemma 2.2: *Let* $p\in[1,\infty)$ *and* $u\in W^{1,p}(\Omega,\mathbb{R}^N)$*. Then we have:*

$$\int_\Omega \big|V\big(Du-\sum_{k\in I}(D_k u)_\Omega\otimes e_k\big)\big|^2\,dx \,\le\, c\int_\Omega \big|V\big(Du-\sum_{k\in I}A_k\otimes e_k\big)\big|^2\,dx$$

for all $I\subset\{1,\dots,n\}$ *and all* $A_k\in\mathbb{R}^N$ *for* $k\in I$*, and the constant* c *depends only on* p*.*

Furthermore, we state the following results concerning $\mathcal{A}$-harmonic approximation, a method which was first used in this context in [23]. For the version given here we refer to the proofs [20, Lemma 6] and [7, Lemma 4.3] for the interior and the boundary situation (combined with the a priori estimates for $\mathcal{A}$-harmonic functions to be found in the same papers) for the subquadratic case, and for the superquadratic case to [58, Lemma 6.8] and [38, Lemma 2.1].

Lemma 2.3 ($\mathcal{A}$-harmonic approximation): *Let* ν, L *be positive constants. Then for every* $\varepsilon>0$ *there exists* $\delta=\delta(n,N,\,p,\frac{L}{\nu},\varepsilon)$ *with the following property: For all* $s\in[0,1]$*, every bilinear form* $\mathcal{A}$ *on* $\mathbb{R}^{nN}$ *which is elliptic in the sense of Legendre-Hadamard with ellipticity constant* ν *and upper bound* L *and*

a) for every $u\in W^{1,p}(B_\rho(x_0),\mathbb{R}^N)$ *satisfying:*

$$\fint_{B_\rho(x_0)} |V(Du)|^2\,dx \,\le\, s^2\,,$$
$$\Big|\fint_{B_\rho(x_0)} \mathcal{A}(Du,D\varphi)\,dx\Big| \,\le\, \delta\, s\sup_{B_\rho(x_0)}|D\varphi| \qquad \forall\,\varphi\in C^1_0(B_\rho(x_0),\mathbb{R}^N)\,,\ \textit{or}$$

b) for every $u\in W^{1,p}_\Gamma(B^+_\rho(x_0),\mathbb{R}^N)$ *(with some* $\rho>0$*,* $x_0\in\mathbb{R}^{n-1}\times\{0\}$*) satisfying:*

$$\fint_{B^+_\rho(x_0)} |V(Du)|^2\,dx \,\le\, s^2\,,$$
$$\Big|\fint_{B^+_\rho(x_0)} \mathcal{A}(Du,D\varphi)\,dx\Big| \,\le\, \delta\, s\sup_{B^+_\rho(x_0)}|D\varphi| \qquad \forall\,\varphi\in C^1_0(B^+_\rho(x_0),\mathbb{R}^N)\,,$$

there exists an $\mathcal{A}$*-harmonic function* $h\in W^{1,p}(B_\rho(x_0),\mathbb{R}^N)$ *(i. e. for all* $\varphi\in C^1_0(B_\rho(x_0),\mathbb{R}^N)$ *there holds* $\int_{B_\rho(x_0)}\mathcal{A}(Dh,D\varphi)\,dx=0$*) or an* $\mathcal{A}$*-harmonic function* $h\in W^{1,p}_\Gamma(B^+_{\rho/2}(x_0),\mathbb{R}^N)$ *which satisfies*

$$\sup_{B^{(+)}_{\rho/2}(x_0)}\big(|Du|+\rho\,|D^2u|\big) \,\le\, c \qquad \textit{and} \qquad \fint_{B^{(+)}_{\rho/2}(x_0)}\Big|V\Big(\frac{u-sh}{\rho}\Big)\Big|^2\,dx \,\le\, s^2\varepsilon\,.$$

In both situations the constant c *depends only on* n,N,p *and* $\frac{L}{\nu}$*.*

3 Caccioppoli's inequality and higher integrability of Du

We shall here collect some well-known facts about minimizers u. In what follows we will concentrate on the characterization of regular boundary points, but the interior regularity is obtained along our way since we also have to prove estimates on balls in the interior in order to end up with regularity in a neighbourhood of a given boundary point. This will be made clearer further below. For establishing boundary estimates we now consider the model case, i. e., we deal with minimizers $u \in W^{1,p}_\Gamma(B^+,\mathbb{R}^N)$ of variational integrals of the form

$$\mathcal{F}[u;B^+] = \int_{B^+} \big[f(x,u,Du) + h(x,u)\big]\,dx\,, \tag{3.1}$$

where the integrands f and h satisfy the assumptions (1.2), (1.3) or (1.4), (1.6) on B^+ instead of on a more general domain Ω. As we will see, this model case (combined with the interior estimates) is sufficient to obtain the desired characterization of regular points on domains Ω of class $C^{1,\alpha}$.

The first step in proving a regularity theorem for minimizers u of variational integrals is to establish a suitable reverse-Poincaré or Caccioppoli inequality:

Lemma 3.1 (Caccioppoli inequality): *a) Let $u \in W^{1,p}(\Omega,\mathbb{R}^N)$ be a local minimizer of (1.1) under the assumptions (1.2), (1.3) or (1.4), and (1.6) with $\gamma \in (0,p^*)$. Furthermore, let $M>0$, $\Lambda \in \mathbb{R}^{nN}$ and $B_\rho(x_0) \Subset \Omega$ such that $|\Lambda|, |(u)_{B_\rho(x_0)}| \le M$. Then we have*

$$\begin{aligned}
\fint_{B_{\rho/2}(x_0)} \big|V\big(Du-\Lambda\big)\big|^2\,dx \le c \fint_{B_\rho(x_0)} \Big|V\Big(\frac{u-(u)_{B_\rho(x_0)}-\Lambda\,(x-x_0)}{\rho}\Big)\Big|^2\,dx \\
+ c \fint_{B_\rho(x_0)} \omega_1\big(\rho + |u-(u)_{B_\rho(x_0)}-\Lambda\,(x-x_0)|\big)\,(1+|Du|)^p\,dx \\
+ c \fint_{B_\rho(x_0)} \omega_2\big(|u-(u)_{B_\rho(x_0)}-\Lambda\,(x-x_0)|\big)\,\big(1+|u-(u)_{B_\rho(x_0)}-\Lambda\,(x-x_0)|\big)^\gamma\,dx\,.
\end{aligned}$$

b) Let $u \in W^{1,p}_\Gamma(B^+,\mathbb{R}^N)$ be a local minimizer of (3.1) under the assumptions (1.2), (1.3) or (1.4), and (1.6). Furthermore, let $M>0$, $\xi \in \mathbb{R}^N$ and $B^+_\rho(x_0)$, $x_0 \in \Gamma$, $\rho < 1-|x_0|$ such that $|\xi| \le M$. Then we have

$$\begin{aligned}
\fint_{B^+_{\rho/2}(x_0)} \big|V\big(Du-\xi\otimes e_n\big)\big|^2\,dx \le c \fint_{B^+_\rho(x_0)} \Big|V\Big(\frac{u-\xi\,x_n}{\rho}\Big)\Big|^2\,dx \\
+ c \fint_{B^+_\rho(x_0)} \omega_1\big(\rho+|u-\xi\,x_n|\big)\,(1+|Du|)^p\,dx \\
+ c \fint_{B^+_\rho(x_0)} \omega_2\big(|u-\xi\,x_n|\big)\,\big(1+|u-\xi\,x_n|\big)^\gamma\,dx\,.
\end{aligned}$$

In both inequalities the constant c depends only on n,N,p,ν,L,γ,M and K_M.

Proof: The proof of part a) in the superquadratic case is standard and can be derived by an obvious modification (concerning the integrand $g(\cdot,\cdot)$) of [33, Proof of Proposition 4.1]. We note that further changes are needed due to the fact that we do not assume second order derivatives of f to be uniformly bounded, but only to be bounded on compact subsets of $\Omega\times\mathbb{R}^N\times\mathbb{R}^{nN}$, see e.g. [3]. The latter changes can be found in [20, Proof of Lemma 3] where the subquadratic case is discussed in detail. For the proof of part b) we note that $u-\xi\,x_n$ has zero boundary-data on Γ, and hence, the Caccioppoli inequality follows exactly as in the interior situation. □

Remark 3.2: If instead of strict quasi-convexity we require strict convexity of f, see (1.9), then it turns out that all terms containing the gradient Du can be absorbed by the hole-filling argument. This means e. g. for the boundary inequality that under the assumption given above we obtain

$$\begin{aligned}
\fint_{B^+_{\rho/2}(x_0)} \big|V\big(Du-\xi\otimes e_n\big)\big|^2\,dx \le c \fint_{B^+_\rho(x_0)} \Big|V\Big(\frac{u-\xi\,x_n}{\rho}\Big)\Big|^2\,dx \\
+ c \fint_{B^+_\rho(x_0)} \big[\omega_1\big(\rho+|u-\xi\,x_n|\big) + \omega_2\big(|u-\xi\,x_n|\big)\,\big(1+|u-\xi\,x_n|\big)^\gamma\big]\,dx\,,
\end{aligned}$$

and c depends only on n,N,p,ν,L,γ and M (cf. [59, Lemma 4.3] for the convex situation in the superquadratic case). The same inequality holds true for quasi-convex functions f not depending explicitly on u, provided that the radius is sufficiently small in dependency of p,ν,L and $\omega_1(\cdot)$.

The second ingredient is a higher integrability result (in order to enable an appropriate estimate for the second-last integral arising on the right-hand side of the Caccioppoli inequality): we refer to the result [34, Theorem 6.8] for quasi-minimizers, where higher integrability of Du was obtained via a Caccioppoli-type inequality (under slightly weaker assumptions) and the application of a Gehring-Lemma:

Lemma 3.3: *a) Let $u \in W^{1,p}(\Omega,\mathbb{R}^N)$ be a local minimizer of* (1.1) *under the assumptions* (1.2), $(1.6)_1$. *Then there exists a higher integrability exponent $q_1 > p$ and a constant c both depending only on n, N, p, ν, L and γ such that for every ball $B_\rho(x_0)$ with centre $x_0 \in \Omega$ and there holds*

$$\Big(\fint_{B_{\rho/2}(x_0)} |Du|^{q_1}\,dx\Big)^{\frac{1}{q_1}} \le c\Big(\fint_{B_\rho(x_0)} \big(1+|Du|\big)^p\,dx\Big)^{\frac{1}{p}} + c\,\rho^{\frac{n\gamma}{p}(\frac{1}{p}-\frac{1}{p^*})}\Big(\fint_{B_\rho(x_0)} \big(1+|Du|\big)^p\,dx\Big)^{\frac{\gamma}{p^2}}.$$

b) Let $u \in W^{1,p}_\Gamma(B^+,\mathbb{R}^N)$ be a local minimizer of (3.1) *under the assumptions* (1.2), $(1.6)_1$. *Then there exists a higher integrability exponent $q_1 > p$ and a constant c both depending only on n, N, p, ν, L and γ such that for every half-ball $B^+_\rho(x_0)$ with centre $x_0 \in \Gamma$ and $\rho < 1-|x_0|$ there holds*

$$\Big(\fint_{B^+_{\rho/2}(x_0)} |Du|^{q_1}\,dx\Big)^{\frac{1}{q_1}} \le c\Big(\fint_{B^+_\rho(x_0)} \big(1+|Du|\big)^p\,dx\Big)^{\frac{1}{p}} + c\,\rho^{\frac{n\gamma}{p}(\frac{1}{p}-\frac{1}{p^*})}\Big(\fint_{B^+_\rho(x_0)} \big(1+|Du|\big)^p\,dx\Big)^{\frac{\gamma}{p^2}}.$$

4 Comparison estimates

We next sketch the proofs to some more or less standard results which will be needed later in order to apply a comparison principle. The strategy is to define a suitable frozen problem of the variational problem (1.1), which is of easier structure than the original problem and thus will allow to derive good a priori estimates of Morrey-type for its minimizers. In our case it is useful to consider functionals on half-balls or on balls, where f is frozen in the first two variables

$$\mathcal{F}_0[w; B^{(+)}_R(x_0)] := \int_{B^{(+)}_R(x_0)} f(x_0, (u)_{B^{(+)}_R(x_0)}, Dw)\,dx\,, \tag{4.1}$$

and to study minimizers of the functional $\mathcal{F}_0[\,\cdot\,; B^{(+)}_R(x_0)]$ subject to boundary values of the minimizer u to the original variational problem. We first state a higher integrability result up to the boundary (provided that the boundary values are higher integrable). For a proof we refer to the similar result [19, Lemma 3.2].

Lemma 4.1: *Assume $u \in W^{1,q_1}_\Gamma(B^+,\mathbb{R}^N)$, $q_1 > p$ and let $v_0 \in u + W^{1,p}_0(B^+_{\rho/2}(x_0),\mathbb{R}^N)$ be a minimizer of $\mathcal{F}_0[\,\cdot\,; B^+_{\rho/2}(x_0)]$ among all functions in the class $u + W^{1,p}_0(B^+_{\rho/2}(x_0),\mathbb{R}^N)$ under the assumptions* (1.2), *where $B^+_{\rho/2}(x_0) \subset B^+$ is a half-ball with $x_0 \in \Gamma$. Then there exists another higher integrability exponent $q \in (p, q_1]$ depending only on n, N, p, ν and L such that*

$$\Big(\fint_{B^+_{\rho/2}(x_0)} |Dv_0|^q\,dx\Big)^{\frac{1}{q}} \le c\Big(\fint_{B^+_{\rho/2}(x_0)} |Dv_0|^p\,dx\Big)^{\frac{1}{p}} + c\Big(\fint_{B^+_{\rho/2}(x_0)} \big(1+|Du|\big)^{q_1}\,dx\Big)^{\frac{1}{q_1}}.$$

The corresponding results holds true on balls $B_{\rho/2}(x_0)$ in the interior of B^+.

Furthermore, since the formulation admits an Euler-Lagrange system, we then observe that standard theory for weak solutions may be applied in order to obtain the following Morrey-type estimates for minimizers of the frozen problem:

Proposition 4.2: *Let $v_0 \in W^{1,p}_\Gamma(B^+_R(x_0),\mathbb{R}^N)$, with $x_0 \in \mathbb{R}^{n-1}\times\{0\}$, $R<1$, be the minimizer of $\mathcal{F}_0[\,\cdot\,; B^+_R(x_0)]$ among all functions in the class $u + W^{1,p}_0(B^+_R(x_0),\mathbb{R}^N)$ under the assumptions* (1.9). *Then there exists a positive number ε depending only on n, N, p and $\frac{L}{\nu}$ such that for every $\rho \le R$ we have*

$$\int_{B^+_\rho(x_0)} \big(1+|Dv_0|\big)^p\,dx \le c\Big(\frac{\rho}{R}\Big)^{\mu_0}\int_{B^+_R(x_0)} \big(1+|Dv_0|\big)^p\,dx$$

with $\mu_0 = \min\{2+2\varepsilon, n\}$, and

$$\int_{B^+_\rho(x_0)} \big|V(Dv_0) - \big(V(Dv_0)\big)_{\rho,x_0}\big|^2\,dx \le c\Big(\frac{\rho}{R}\Big)^{2+2\varepsilon}\int_{B^+_R(x_0)} \big(1+|Dv_0|\big)^p\,dx\,,$$

where the constants c depends only on n, N, p and $\frac{L}{\nu}$. Under the same assumption, the corresponding results holds true on full balls for local minimizers $v_0 \in W^{1,p}(B_R(x_0), \mathbb{R}^N)$, $x_0 \in \mathbb{R}^n$, $R < 1$, of the functional $\mathcal{F}_0[\,\cdot\,; B_R(x_0)]$ among all functions in the class $u + W_0^{1,p}(B_R(x_0), \mathbb{R}^N)$.

PROOF: The proofs for the interior and the boundary situation are very similar. Therefore, we will give the arguments only for half-balls, but state the steps needed in between also for the interior situation. We first note that the minimizer $v_0 \in W_\Gamma^{1,p}(B_R^+(x_0), \mathbb{R}^N)$ is a weak solution to the Euler-Lagrange system

$$\operatorname{div} D_z f(x_0, (u)_{B_R^+(x_0)}, Dv_0) \;=\; 0 \qquad \text{in } B_R^+(x_0)\,.$$

Hence, in view of the ellipticity and growth condition on the coefficients $A(z) := D_z f(x_0, (u)_{B_R^+(x_0)}, z)$, the first inequality follows immediately from [13, Theorem 6.2 and Theorem 3.I] in the superquadratic case and [9, Corollary 3.4 and the Remark thereafter] in the subquadratic case. The second inequality is an immediate consequence of [13, Theorem 6.1 and estimate (3.12)] for $p \geq 2$ [9, estimate (3.17) and the Remark thereafter] combined with the Caccioppoli's and Poincare's inequalities in [13, 9]. □

Remark 4.3: This result would be desirable without requiring a bound on the second order derivatives of f, so we comment briefly on the cited results for weak solutions of autonomous systems: in order to prove these results the existence of second order derivatives is shown in a first step, more precisely that $DV(Dv_0)$ exists in L^2. The interior estimates and also the tangential derivatives $(D_1 V(Dv_0), \ldots, D_{n-1} V(Dv_0))$ follow from finite difference techniques and do not use the boundedness of $D_{zz} f(\cdot)$ (or of the derivatives of first order of the coefficients for the corresponding Euler system). In contrast, the bound seems to be needed when deriving the missing normal derivative of $V(Dv_0)$ by exploiting the system equation. The Morrey-type result then essentially follows from a higher integrability result and the Sobolev-Poincaré-inequality (note that for these steps it is no longer necessary to return to the system equation).

The last result is a comparison estimate which allows to bound the L^p-difference of the gradient of the minimizer u to the original problem to the gradient of the (unique) solution to the frozen problem (which is of less complicated structure) with boundary values u. We note that similar estimates in the interior were achieved in [43, Lemma 4.9] in the case of a convex integrand f not depending explicitly on u.

Lemma 4.4: *Let $u \in W_\Gamma^{1,q_1}(B^+, \mathbb{R}^N)$, $q_1 > p$, be a minimizer to the functional $\mathcal{F}[\,\cdot\,; B^+]$ in (1.1) under the assumptions (1.9), (1.3), $(1.6)_1$ with $\gamma \in [0, p^*)$, and let $v_0 \in u + W_0^{1,p}(B_R^+(x_0), \mathbb{R}^N)$, $x_0 \in \Gamma$, $2R < 1 - |x_0|$, be the unique minimizer to the functional $\mathcal{F}_0[\,\cdot\,; B_R^+(x_0)]$ in (4.1) among all functions in the class $u + W_0^{1,p}(B_R^+(x_0), \mathbb{R}^N)$. Then we have*

$$\begin{aligned}
&\int_{B_R^+(x_0)} \big(1 + |Du| + |Dv_0|\big)^{p-2} \, |Du - Dv_0|^2 \, dx \\
&\quad \leq \; c\,\omega_{12} \Big(\Big(R^{p-n} \int_{B_R^+(x_0)} (1 + |Du|)^p \, dx \Big)^{\frac{1}{p}} \Big)^{\frac{q-p}{q}} \int_{B_{2R}^+(x_0)} (1 + |Du|)^p \, dx \\
&\qquad + c\, R^{n(1 - \frac{\gamma}{p^*})} \Big(\int_{B_R^+(x_0)} (1 + |Du|)^p \, dx \Big)^{\frac{\gamma}{p}} + c\, R^n \,.
\end{aligned}$$

If we assume the continuity condition $(1.6)_2$, we have the stronger inequality

$$\begin{aligned}
&\int_{B_R^+(x_0)} \big(1 + |Du| + |Dv_0|\big)^{p-2} \, |Du - Dv_0|^2 \, dx \\
&\quad \leq \; c\,\omega_{12} \Big(\Big(R^{p-n} \int_{B_R^+(x_0)} (1 + |Du|)^p \, dx \Big)^{\frac{1}{p}} \Big)^{\frac{q-p}{q}} \int_{B_{2R}^+(x_0)} (1 + |Du|)^p \, dx \\
&\qquad + c\, R^n \, \omega_2 \Big(\Big(R^{p-n} \int_{B_R^+(x_0)} (1 + |Du|)^p \, dx \Big)^{\frac{1}{p}} \Big)^{1 - \frac{\gamma}{p^*}} \Big(1 + R^{p-n} \int_{B_R^+(x_0)} (1 + |Du|)^p \, dx \Big)^{\frac{\gamma}{p}} \\
&\qquad + c\, R^{n(1 - \frac{\gamma}{p^*})} \Big(\int_{B_R^+(x_0)} (1 + |Du|)^p \, dx \Big)^{\frac{\gamma}{p}} \,.
\end{aligned}$$

The constants c depend only on n, N, p, ν and L, the number q denotes the higher integrability exponent from Lemma 4.1. The corresponding results hold true for local minimizers $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ to the functional $\mathcal{F}[\,\cdot\,; U]$, $U \subset \mathbb{R}^n$ a bounded domain, on full balls $B_R(x_0)$ in the interior of U, provided that $|(u)_{B_R(x_0)}| \leq M$ for some $M > 0$. The constants c then depends additionally on M.

Proof: From the convexity condition $(1.9)_2$, [4, Lemma 2.1] and the minimality of v_0 we first find:

$$\begin{aligned}
c(p,\nu) &\int_{B_R^+(x_0)} \big(1+|Du|+|Dv_0|\big)^{p-2}\,|Du-Dv_0|^2\,dx\\
&\le \int_{B_R^+(x_0)} \big[f(x_0,(u)_{B_R^+(x_0)},Du)-f(x_0,(u)_{B_R^+(x_0)},Dv_0)\\
&\qquad\qquad - D_zf(x_0,(u)_{B_R^+(x_0)},Dv_0)\cdot(Du-Dv_0)\big]\,dx\\
&= \int_{B_R^+(x_0)}\int_0^1 \big[D_zf\big(x_0,(u)_{B_R^+(x_0)},Dv_0+t(Du-Dv_0)\big)\\
&\qquad\qquad - D_zf\big(x,(u)_{B_R^+(x_0)},Dv_0+t(Du-Dv_0)\big)\big]\,dt\,(Du-Dv_0)\,dx\\
&\quad + \int_{B_R^+(x_0)}\int_0^1 \big[D_zf\big(x,(u)_{B_R^+(x_0)},Dv_0+t(Du-Dv_0)\big)\\
&\qquad\qquad - D_zf\big(x,u,Dv_0+t(Du-Dv_0)\big)\big]\,dt\,(Du-Dv_0)\,dx\\
&\quad + \int_{B_R^+(x_0)} \big[f(x,v_0,Dv_0)-f(x,u,Dv_0)\big]\,dx\\
&\quad + \int_{B_R^+(x_0)} \big[h(x,v_0)-h(x,u)\big]\,dx\\
&\quad + \mathcal{F}\big[u;B_R^+(x_0)\big]-\mathcal{F}\big[v_0;B_R^+(x_0)\big] \;=:\; I+II+III+IV+V\,.
\end{aligned}\tag{4.2}$$

We next recall an energy estimate for v_0 which will be used frequently in what follows: exploiting the minimality of v_0 and the growth condition $(1.2)_2$ yields

$$⨍_{B_R^+(x_0)} |Dv_0|^p\,dx \;\le\; \frac{L}{\nu}⨍_{B_R^+(x_0)} \big(1+|Du|\big)^p\,dx\,. \tag{4.3}$$

Since u and v_0 coincide on $\partial B_R^+(x_0)$ we now observe $V\le 0$. Then the continuity assumption $(1.3)_1$ on D_zf with respect to the first (and second) variable leads to

$$I \;\le\; c(\nu,L)\,\omega_1(R)\int_{B_R^+(x_0)} (1+|Du|)^p\,dx\,.$$

Taking into account the higher integrability results for Du and Dv_0 achieved in Lemma 3.3 and Lemma 4.1, respectively, we find after the application of Hölder's inequality, the concavity and boundedness of $\omega_3(\cdot)$, and Poincaré's inequality

$$\begin{aligned}
II+III \le\; c(n,N,p,\nu,L)\,\Big[&\omega_{12}\Big(\Big(R^{p-n}\int_{B_R^+(x_0)}(1+|Du|)^p\,dx\Big)^{\frac1p}\Big)^{\frac{q-p}{q}}\int_{B_{2R}^+(x_0)}(1+|Du|)^p\,dx\\
&+R^{n(1-\frac{\gamma}{p^*})}\Big(\int_{B_{2R}^+(x_0)}(1+|Du|)^p\,dx\Big)^{\frac{\gamma}{p}}\Big]\,.
\end{aligned}$$

Here, besides the continuity assumption $(1.3)_1$ on D_zf we have also used the continuity assumption $(1.3)_2$ on f (with respect to the second variable). Finally, the remaining term IV is estimated via $(1.6)_1$, Jensen's inequality and Sobolev-Poincaré (in the boundary version) by

$$IV \le\; c(n,N,p,\nu,L)\,\Big[R^n+R^{n(1-\frac{\gamma}{p^*})}\Big(\int_{B_R^+(x_0)}(1+|Du|)^p\,dx\Big)^{\frac{\gamma}{p}}\Big]\,.$$

If we assume also the continuity assumption $(1.6)_2$ on h with respect to the second variable, only the estimate for term IV changes: using Hölder's inequality, the concavity of ω_2 and the Sobolev-Poincaré inequality, we then find

$$\begin{aligned}
IV \le\; c(n,N,p,\nu,L)\,R^n\,\Big(&\omega_2\Big(\Big(R^{p-n}\int_{B_R^+(x_0)}(1+|Du|)^p\,dx\Big)^{\frac1p}\Big)\\
&+\omega_2\Big(\Big(R^{p-n}\int_{B_R^+(x_0)}(1+|Du|)^p\,dx\Big)^{\frac1p}\Big)^{1-\frac{\gamma}{p^*}}\Big(R^{p-n}\int_{B_R^+(x_0)}(1+|Du|)^p\,dx\Big)^{\frac{\gamma}{p}}\Big)\,.
\end{aligned}$$

Merging the estimates for I-VII together with (4.2) we find the asserted comparison estimates on half-balls stated in the lemma, namely the first inequality under the condition $(1.6)_1$ resp. the second one under both conditions in (1.6).

The proof for full balls in the interior is almost the same, but for the estimate of III in the last step we need the bound on the meanvalues of u in order to apply the interior version of the Sobolev-Poincaré inequality. For this reason the bound $|(u)_{B_R(x_0)}| \le M$ is required and the additional dependency on M in the constant c appears. $\square$

Remark 4.5: For variational functionals which do not depend explicitly on u, it is easy to see that the terms II, III and IV do not appear. Hence, the comparison estimate is then given by

$$\int_{B_R^{(+)}(x_0)} \big(1+|Du|+|Dv_0|\big)^{p-2}\,|Du-Dv_0|^2\,dx \;\le\; c(\nu,L)\,\omega_1(R)\int_{B_R^{(+)}(x_0)} \big(1+|Du|\big)^p\,dx\,.$$

5 Partial regularity for Du

The aim of this section is the proof of the partial regularity result for Du and the characterization of regular (boundary) points stated in Theorem 1.1. As explained above it is a standard technique to compare the minimizer of the original problem to a solution of an easier problem (in the sense that the solution enjoys good a priori estimates). We here compare the minimizer of the original variational functional $\mathcal{F}[\,\cdot\,;B^+]$ locally to the solution of a linearized system following the approach of Schmidt [59] for convex functionals.

In order to derive the characterization of a regular point, we define the excess functional: For any half-ball $B_\rho^+(x_0)\subset B^+$ with $x_0\in\Gamma$, a fixed function $u\in W^{1,p}_\Gamma(B^+,\mathbb{R}^N)$ and $\xi\in\mathbb{R}^N$ we denote the Campanato-type excess by

$$C(x_0,\rho) = \fint_{B_\rho^+(x_0)} \big|V\big(Du-(D_nu)_{x_0,\rho}\otimes e_n\big)\big|^2\,dx\,.$$

We note that the corresponding excess for the interior estimates is given by

$$C(x_0,\rho) = \fint_{B_\rho(x_0)} \big|V\big(Du-(Du)_{x_0,\rho}\big)\big|^2\,dx$$

for all balls $B_\rho(x_0)\subset\Omega$. Furthermore, we define

$$\beta_0 = \min\Big\{\frac{\alpha_2}{2-\alpha_2},\,\frac{\alpha_1}{2}\,\frac{q_1-p}{q_1}\Big\}$$

where q_1 is the number stemming from the higher integrability result from Lemma 3.3.

Next we establish a preliminary decay estimate provided that the mean values of u and Du are bounded by a given number and that the excess is sufficiently small:

Proposition 5.1: *For every $M\ge 2$ and $\widetilde\beta\in(\beta_0,1)$ there exist two positive numbers $\theta,\varepsilon_0\in(0,1)$ with*

$$\begin{aligned}\theta &= \theta\big(n,N,p,\nu,L,\gamma,M,K_M,\widetilde\beta\big)\\ \varepsilon_0 &= \varepsilon_0\big(n,N,p,\nu,L,\gamma,M,K_M,\beta_0,\widetilde\beta,\mu_M(\cdot)\big)\end{aligned} \tag{5.1}$$

such that the following is true: If $u\in W^{1,p}(B^+,\mathbb{R}^N)$ is a local minimizer to (3.1) under the assumptions (1.2), (1.3) *or* (1.4), (1.6), *and if $B_\rho^+(x_0)$, $x_0\in\Gamma$, $\rho<1-|x_0|$, is a half ball on which the smallness conditions*

$$\rho + C(x_0,\rho) < \varepsilon_0 \qquad\text{and}\qquad |(D_nu)_{x_0,\rho}| \le M \tag{5.2}$$

are satisfied, then we have

$$C(x_0,\theta\rho) \le \theta^{2\widetilde\beta}\,C(x_0,\rho) + c_*\,\rho^{2\beta_0}$$

for a constant c_* depending only on n,N,p,ν,L,γ,M,K_M and $\widetilde\beta$. The same excess decay estimate holds true for balls $B_\rho(x_0)$ in the interior of B^+ if we further assume $|(u)_{x_0,\rho}|,|(Du)_{x_0,\rho}|\le M$.

Proof: *Step 1a: Approximate $\mathcal{A}$-harmonicity under assumption* (1.4). Let $\varphi \in C_0^1(B^+_{\rho/2}(x_0),\mathbb{R}^N)$ with $\|D\varphi\|_{L^\infty(B^+_{\rho/2}(x_0),\mathbb{R}^N)} \leq 1$, which implies in particular $\sup_{x\in B^+_{\rho/2}(x_0)} |\varphi| \leq \rho$. The minimization property of u then gives

$$\begin{aligned}
0 \leq\ & \sigma^{-1} \fint_{B^+_{\rho/2}(x_0)} \big[f(\,\cdot\,,u-\sigma\varphi,Du-\sigma D\varphi) + h(\,\cdot\,,u-\sigma\varphi) - f(\,\cdot\,,u,Du) - h(\,\cdot\,,u)\big]\,dx\\
=\ & \sigma^{-1} \fint_{B^+_{\rho/2}(x_0)} \big[f(\,\cdot\,,u-\sigma\varphi,Du-\sigma D\varphi) - f(x_0,u-\sigma\varphi,Du-\sigma D\varphi)\big]\,dx\\
&+ \sigma^{-1} \fint_{B^+_{\rho/2}(x_0)} \big[f(x_0,u-\sigma\varphi,Du-\sigma D\varphi) - f(x_0,0,Du-\sigma D\varphi)\big]\,dx\\
&+ \sigma^{-1} \fint_{B^+_{\rho/2}(x_0)} \big[f(x_0,0,Du-\sigma D\varphi) - f(x_0,0,Du)\big]\,dx\\
&+ \sigma^{-1} \fint_{B^+_{\rho/2}(x_0)} \big[f(x_0,0,Du) - f(x_0,u,Du)\big]\,dx\\
&+ \sigma^{-1} \fint_{B^+_{\rho/2}(x_0)} \big[f(x_0,u,Du) - f(\,\cdot\,,u,Du)\big]\,dx\\
&+ \sigma^{-1} \fint_{B^+_{\rho/2}(x_0)} \big[h(\,\cdot\,,u-\sigma\varphi) - h(\,\cdot\,,u)\big]\,dx \ =:\ I + II + III + IV + V + VI
\end{aligned} \tag{5.3}$$

for every $\sigma \in (0,1]$ with the obvious abbreviations. We start by estimating term I and V via the continuity assumption (1.4) and the boundedness of $D\varphi$:

$$|I| + |V| \ \leq\ c(L)\,\sigma^{-1}\,\omega_1(\rho) \fint_{B^+_{\rho/2}(x_0)} \big(1+|Du|\big)^p\,dx\,.$$

Next, we observe from (1.4), Hölder's inequality, $\omega \leq 1$, Jensen's inequality and Lemma 3.3 that the terms II and IV are bounded by

$$\begin{aligned}
|II| + |IV| \leq\ & \sigma^{-1} \fint_{B^+_{\rho/2}(x_0)} \omega_1(|u|+\sigma|\varphi|)\big(1+|Du|+\sigma|D\varphi|\big)^p\,dx\\
\leq\ & c(p)\,\sigma^{-1} \Big(\fint_{B^+_{\rho/2}(x_0)} \omega_1(|u|+\rho)\,dx\Big)^{\frac{q_1-p}{q_1}} \Big(\fint_{B^+_{\rho/2}(x_0)} \big(1+|Du|\big)^{q_1}\,dx\Big)^{\frac{p}{q_1}}\\
\leq\ & c\,\sigma^{-1}\,\omega_1\Big(\rho \fint_{B^+_{\rho/2}(x_0)} \big(1+|D_n u|\big)\,dx\Big)^{\frac{q_1-p}{q_1}}\\
&\qquad \times \Big[\fint_{B^+_\rho(x_0)} \big(1+|Du|\big)^p\,dx + \rho^{n\gamma(\frac{1}{p}-\frac{1}{p^*})} \Big(\fint_{B^+_\rho(x_0)} \big(1+|Du|\big)^p\,dx\Big)^{\frac{\gamma}{p}}\Big]
\end{aligned}$$

where we have also used Poincaré's inequality in the last inequality, and the constant c depends only on n,N,p,ν,L and γ. In view of Poincaré's inequality (note $u=\varphi=0$ on $\partial\Gamma_{\rho/2}(x_0)$) and the continuity assumption $(1.6)_2$, we next observe

$$\begin{aligned}
|VI| \leq\ & \sigma^{-1}\,L \fint_{B^+_{\rho/2}(x_0)} \omega_2(|\sigma\varphi|)\,\big(1+|u|+|u-\varphi|\big)^\gamma\,dx\\
\leq\ & c\,\sigma^{-1}\,\omega_2(\sigma\rho)\,\Big[1+\rho^{n\gamma(\frac{1}{p}-\frac{1}{p^*})}\Big(\fint_{B^+_\rho(x_0)} \big(1+|Du|\big)^p\,dx\Big)^{\frac{\gamma}{p}}\Big]
\end{aligned}$$

with c depending only on n,N,p,ν,L and γ. In the next step we add

$$\fint_{B^+_{\rho/2}(x_0)} D_{zz}f\big(x_0,0,(D_n u)_{x_0,\rho}\otimes e_n\big)\,\big(Du-(D_n u)_{x_0,\rho}\otimes e_n, D\varphi\big)\,dx \tag{5.4}$$

to both sides of inequality (5.3), and so it only remains to estimate III and this new term. The sum of

(5.4) and III will be called III'. Taking into account

$$\begin{aligned}
\fint_{B^+_{\rho/2}(x_0)} D_z f\big(x_0,0,Du\big)\, D\varphi\, dx &= \fint_{B^+_{\rho/2}(x_0)} \big[D_z f\big(x_0,0,Du\big) - D_z f\big(x_0,0,(D_n u)_{x_0,\rho}\otimes e_n\big)\big]\, D\varphi\, dx\\
&= \fint_{B^+_{\rho/2}(x_0)} \int_0^1 D_{zz} f\big(x_0,0,(D_n u)_{x_0,\rho}\otimes e_n + t(Du-(D_n u)_{x_0,\rho}\otimes e_n)\big)\big]\, dt\\
&\qquad\qquad \times \big(Du-(D_n u)_{x_0,\rho}\otimes e_n, D\varphi\big)\, dx\,, \qquad (5.5)
\end{aligned}$$

term III' is now decomposed as follows:

$$\begin{aligned}
III' = & \fint_{B^+_{\rho/2}(x_0)} \int_0^1 \big[D_{zz} f\big(x_0,0,(D_n u)_{x_0,\rho}\otimes e_n\big) - D_{zz} f\big(x_0,0,(1-t)\,(D_n u)_{x_0,\rho}\otimes e_n + t\,Du\big)\big]\, dt\\
&\qquad \times \big(Du-(D_n u)_{x_0,\rho}\otimes e_n, D\varphi\big)\, dx\\
& + \fint_{B^+_{\rho/2}(x_0)} \fint_0^\sigma \big[D_z f\big(x_0,0,Du\big) - D_z f\big(x_0,0,Du-tD\varphi\big)\big]\, D\varphi\, dx \;=:\; III_1 + III_2\,.
\end{aligned}$$

The resulting terms will now be estimated: To bound III_1 we decompose the domain of integration: on $\{x\in B^+_{\rho/2}(x_0) : |Du-(D_n u)_{x_0,\rho}\otimes e_n|\le 1\}$ we use the fact that second derivatives $D_{zz}f$ are bounded and uniformly continuous, see (1.5), and the upper bound $|(D_n u)_{x_0,\rho}|\le M$ to deduce that the integrand of III_1 is controlled by

$$c(p,M)\,\mu_M\big(\big|V(Du-(D_n u)_{x_0,\rho}\otimes e_n)\big|\big)\,\big|V(Du-(D_n u)_{x_0,\rho}\otimes e_n)\big|\,.$$

On the complementary set $\{x\in B^+_{\rho/2}(x_0) : |Du-(D_n u)_{x_0,\rho}\otimes e_n|\ge 1\}$ we use the boundedness of $D_{zz}f$ for the first integrand in III_1 and the growth of $D_z f$ for the second integrand (rewritten as in (5.5) above), and we find that the integrand of III_1 is now bounded by

$$K_M + L\big(1+|Du|\big)^{p-1} \;\le\; c(p,L,M,K_M)\,\big|V(Du-(D_n u)_{x_0,\rho}\otimes e_n)\big|^2\,.$$

Combining the latter estimates and applying Hölder's and Jensen's inequality (note that $\mu_M(\cdot)$ is concave) we infer

$$\begin{aligned}
III_1 \le\; & c\,\Big(\mu_M\Big(\fint_{B^+_{\rho/2}(x_0)} \big|V(Du-(D_n u)_{x_0,\rho}\otimes e_n)\big|^2\, dx\Big)^{\frac12}\Big)^{\frac12} \Big(\fint_{B^+_{\rho/2}(x_0)} \big|V(Du-(D_n u)_{x_0,\rho}\otimes e_n)\big|^2\, dx\Big)^{\frac12}\\
& + c\fint_{B^+_{\rho/2}(x_0)} \big|V(Du-(D_n u)_{x_0,\rho}\otimes e_n)\big|^2\, dx
\end{aligned}$$

with $c=c(p,L,M,K_M)$. For the remaining integral III_2 we only use the bounds of $D_{zz}f$ and $D_z f$, where we now distinguish the points where we have $|Du|\le M-1$ (note $M\ge 2$) or where the opposite inequality holds. This yields

$$III_2 \le\; \sigma\,K_M + c(p,L,M)\,\fint_{B^+_{\rho/2}(x_0)} \big|V(Du-(D_n u)_{x_0,\rho}\otimes e_n)\big|^2\, dx\,.$$

Taking into account the assumption (5.2) which implies in particular $\fint_{B^+_{\rho/2}(x_0)}(1+|Du|)^p dx\le c(p,M)$, we may combine the estimates for all terms with (5.3), and we finally arrive at

$$\begin{aligned}
&\fint_{B^+_{\rho/2}(x_0)} D_{zz} f\big(x_0,0,(D_n u)_{x_0,\rho}\otimes e_n\big)\,\big(Du-(D_n u)_{x_0,\rho}\otimes e_n, D\varphi\big)\, dx\\
&\quad\le\; c\,\sigma^{-1}\big(\omega_1(\rho)^{\frac{q_1-p}{q_1}} + \omega_2(\sigma\rho)\big) + c\,\mu_M\big(C(x_0,\rho)^{\frac12}\big)^{\frac12}\,C(x_0,\rho)^{\frac12} + c\,C(x_0,\rho) + c\,\sigma\\
&\quad\le\; c\,\big(\rho^{\beta_0} + \mu_M\big(C(x_0,\rho)^{\frac12}\big)^{\frac12}\,C(x_0,\rho)^{\frac12} + C(x_0,\rho)\big)\,, \qquad (5.6)
\end{aligned}$$

where we have already employed $C(x_0,\rho)<1$ assumed in (5.2). Moreover, in the last line we have used (1.7), the definition β_0 and we have chosen $\sigma=\rho^{\beta_0}$. The constant c here depends only on n,N,p,ν,L,γ,M

and K_M. The lower bound is established analogously, and using a rescaling argument, we then end up with

$$\Big| \fint_{B^+_{\rho/2}(x_0)} \mathcal{A}\left(Dw, D\varphi\right) dx \Big| \le c_1 \left(\rho^{\beta_0} + \mu_M\big(C(x_0,\rho)^{\frac{1}{2}}\big)^{\frac{1}{2}} C(x_0,\rho)^{\frac{1}{2}} + C(x_0,\rho)\right) \|D\varphi\|_{L^\infty(B^+_{\rho/2}(x_0))} \tag{5.7}$$

with $c_1 = c_1(n, N, p, \nu, L, \gamma, M, K_M) \ge 2^n$, where the bilinear form $\mathcal{A}$ and the function w are defined by

$$\mathcal{A} := D_{zz} f\big(x_0, 0, (D_n u)_{x_0,\rho} \otimes e_n\big) \qquad \text{and} \qquad w := u - (D_n u)_{x_0,\rho}\, x_n\,;$$

we further note that $\mathcal{A}$ is bounded from below in the sense of Legendre-Hadamard by a constant depending only on ν, p and M and from above by K_M.

Step 1b: Approximate $\mathcal{A}$-harmonicity under assumption (1.3). We only sketch briefly the modifications needed in contrast to the assumption of (1.4) in Step 1a (this is also the case discussed in detail in [59]). The inequality corresponding to (5.3) is then given by

$$\begin{aligned}
0 \le \sigma^{-1} & \fint_{B^+_{\rho/2}(x_0)} \big[f(\,\cdot\,, u - \sigma\varphi, Du - \sigma D\varphi) - f(\,\cdot\,, u, Du - \sigma D\varphi)\big]\, dx \\
&+ \fint_{B^+_{\rho/2}(x_0)} \int_0^\sigma \big[D_z f(x_0, 0, Du - \tau D\varphi) - D_z f(\,\cdot\,, u, Du - \tau D\varphi)\big]\, d\tau\, D\varphi\, dx \\
&- \fint_{B^+_{\rho/2}(x_0)} \int_0^\sigma D_z f(x_0, 0, Du - \tau D\varphi)\, d\tau\, D\varphi\, dx \\
&+ \sigma^{-1} \fint_{B^+_{\rho/2}(x_0)} \big[h(\,\cdot\,, u - \sigma\varphi) - h(\,\cdot\,, u)\big]\, dx =: I' + II' + III + VI\,,
\end{aligned}$$

for every $\sigma \in (0,1]$. Therefore, it only remains to estimate I' and II' via the continuity conditions (1.3), which is accomplished similarly to above or in [59, Proof of Lemma 5.1]. Taking into account the assumption $C(x_0,\rho)$ and proceeding as in Step 1a we then get the following estimate corresponding to (5.6):

$$\begin{aligned}
& \fint_{B^+_{\rho/2}(x_0)} D_{zz} f\big(x_0, 0, (D_n u)_{x_0,\rho} \otimes e_n\big) \left(Du - (D_n u)_{x_0,\rho} \otimes e_n, D\varphi\right) dx \\
&\quad \le c\, \omega_1(\rho) + c\, \sigma^{-1}\, \omega_2(\sigma\rho) + c\, \mu_M\big(C(x_0,\rho)^{\frac{1}{2}}\big)^{\frac{1}{2}} C(x_0,\rho)^{\frac{1}{2}} + c\, C(x_0,\rho) + c\,\sigma \\
&\quad \le c \left(\rho^{\beta_0} + \mu_M\big(C(x_0,\rho)^{\frac{1}{2}}\big)^{\frac{1}{2}} C(x_0,\rho)^{\frac{1}{2}} + C(x_0,\rho)\right),
\end{aligned}$$

where we again have chosen $\sigma = \rho^{\beta_0}$ and with c depending on $n, N, p, \nu, L, \gamma, M$ and K_M. Hence, we end up with the approximate $\mathcal{A}$-harmonicity result (5.7).

Step 2: Preliminary decay estimate via $\mathcal{A}$-harmonic approximation. For $\varepsilon > 0$ to be determined later, we now take $\delta = \delta(n, N, \nu, L, M, K_M, \varepsilon)$ to be the corresponding constant from the boundary version of the $\mathcal{A}$-harmonic approximation Lemma 2.3. We then define $s = c_1 C(x_0,\rho)^{1/2} + \frac{2c_1}{\delta}\rho^{\beta_0}$. The inequality (5.7) now reads as

$$\Big| \fint_{B^+_{\rho/2}(x_0)} \mathcal{A}\left(Dw, D\varphi\right) dx \Big| \le s \left(2^{-1}\delta + \mu_M\big(C(x_0,\rho)^{\frac{1}{2}}\big)^{\frac{1}{2}} + C(x_0,\rho)^{\frac{1}{2}}\right) \sup_{B^+_{\rho/2}(x_0)} |D\varphi|$$

for all $\varphi \in C^1_0(B^+_{\rho/2}(x_0), \mathbb{R}^N)$. Furthermore, by definition of the Campanato-type excess $C(x_0,\rho)$ and the function w, we observe $\fint_{B^+_{\rho/2}(x_0)} |V(Dw)|^2\, dx \le 2^n C(x_0,\rho) \le s^2$. Hence, assuming the smallness condition

$$s = c_1 C(x_0,\rho)^{1/2} + \frac{2c_1}{\delta}\rho^{\beta_0} \le 1\,, \tag{SC.1}$$

all assumptions of the $\mathcal{A}$-harmonic approximation Lemma are fulfilled, and thus we find an $\mathcal{A}$-harmonic map $h \in W^{1,2}_\Gamma(B^+_{\rho/2}(x_0), \mathbb{R}^N)$ satisfying

$$\sup_{B^+_{\rho/4}(x_0)} |Dh| + \rho \sup_{B^+_{\rho/4}(x_0)} |D^2 h| \le c_h \qquad \text{and} \qquad \fint_{B^+_{\rho/4}(x_0)} \Big| V\Big(\frac{w - sh}{\rho/2}\Big)\Big|^2 dx \le s^2\, \varepsilon$$

with a constant depending only on n, N, p, ν, L. We now consider a fixed number $\theta \in (0, 1/8)$ to be specified later, choose $\varepsilon = \theta^{n+\max\{2,p\}+2}$, and we define

$$\xi := (D_n u)_{x_0,\rho} + s\, D_n h(x_0) \in \mathbb{R}^N .$$

Next, the following estimate is deduced from Taylor's formula (keeping in mind that $h(x_0) = 0$):

$$\begin{aligned}
&\fint_{B^+_{2\theta\rho}(x_0)} \Big|V\Big(\frac{u - \xi\, x_n}{2\theta\rho}\Big)\Big|^2 dx \\
&\quad\le c(p) \fint_{B^+_{2\theta\rho}(x_0)} \Big|V\Big(\frac{w - sh}{2\theta\rho}\Big)\Big|^2 dx + c(p) \fint_{B^+_{2\theta\rho}(x_0)} \Big|V\Big(\frac{sh - s\, D_n h(x_0)\, x_n}{2\theta\rho}\Big)\Big|^2 dx \\
&\quad\le c(p)\, \theta^{-n-\max\{2,p\}} \fint_{B^+_{\rho/4}(x_0)} \Big|V\Big(\frac{w - sh}{\rho/2}\Big)\Big|^2 dx + c(n,N,p,\nu,L)\, \big(\theta^2 s^2 + \theta^{\max\{2,p\}}\, s^{\max\{2,p\}}\big) \\
&\quad\le c(n,N,p,\nu,L)\, \theta^2 s^2
\end{aligned} \tag{5.8}$$

where in the last line we have used the definition of ε and s, as well as the fact that $s, \theta \le 1$.

Step 3: Full decay estimate for the Campanato-type excess. Sharpening the smallness condition (SC.1) by assuming

$$s \le c_h^{-1}, \tag{SC.2}$$

we observe $|\xi| \le M + 1$. Therefore, the application of the quasi-minimizing property of the mean value of $D_n u$ for the map $\xi \mapsto \int_{B^+_{\theta\rho}(x_0)} |V(Du - \xi \otimes e_n)|^2 dx$ from Lemma 2.2 and of the Caccioppoli inequality in Lemma 3.1 yields

$$\begin{aligned}
C(x_0, \theta\rho) &= \fint_{B^+_{\theta\rho}(x_0)} \big|V\big(Du - (D_n u)_{x_0,\theta\rho} \otimes e_n\big)\big|^2 dx \\
&\le c(p) \fint_{B^+_{\theta\rho}(x_0)} \big|V\big(Du - \xi \otimes e_n\big)\big|^2 dx \\
&\le c \fint_{B^+_{2\theta\rho}(x_0)} \Big|V\Big(\frac{u - \xi\, x_n}{2\theta\rho}\Big)\Big|^2 dx + c \fint_{B^+_{2\theta\rho}(x_0)} \omega_1\big(2\theta\rho + |u - \xi\, x_n|\big)\, (1 + |Du|)^p\, dx \\
&\quad + c \fint_{B^+_{2\theta\rho}(x_0)} \omega_2\big(|u - \xi\, x_n|\big)\, \big(1 + |u - \xi\, x_n|\big)^\gamma\, dx .
\end{aligned}$$

The latter integral is estimated further via the use of Hölder's, Poincaré's and Sobolev's inequality, and we find

$$\begin{aligned}
C(x_0, \theta\rho) &\le c \fint_{B^+_{2\theta\rho}(x_0)} \Big|V\Big(\frac{u - \xi\, x_n}{2\theta\rho}\Big)\Big|^2 dx + c \fint_{B^+_{2\theta\rho}(x_0)} \omega_1\big(2\theta\rho + |u - \xi\, x_n|\big)\, (1 + |Du|)^p\, dx \\
&\quad + c\, \omega_2\Big(2\theta\rho \fint_{B^+_{2\theta\rho}(x_0)} |Du - \xi \otimes e_n|\, dx\Big)^{1-\frac{\gamma}{p^*}} \Big(\fint_{B^+_{2\theta\rho}(x_0)} \big(1 + |Du - \xi \otimes e_n|\big)^p\, dx\Big)^{\frac{\gamma}{p}} \\
&=: c \Big(\fint_{B^+_{2\theta\rho}(x_0)} \Big|V\Big(\frac{u - \xi\, x_n}{2\theta\rho}\Big)\Big|^2 dx + VII + VIII\Big)
\end{aligned} \tag{5.9}$$

with the obvious labelling and a constant c depending only on n, N, p, ν, L, M and K_M. To bound VII from above we use Hölder's inequality, the higher integrability result for Du (Lemma 3.3), the concavity of $\omega_1(\cdot)$, the Poincaré-inequality and the smallness condition (SC.2), and we find

$$\begin{aligned}
VII &\le \Big(\fint_{B^+_{2\theta\rho}(x_0)} \omega_1\big(2\theta\rho + |u - \xi\, x_n|\big)\, dx\Big)^{\frac{q_1 - p}{q_1}} \Big(\fint_{B^+_{2\theta\rho}(x_0)} (1 + |Du|)^{q_1}\, dx\Big)^{\frac{p}{q_1}} \\
&\le c\, \omega_1\Big(\fint_{B^+_{2\theta\rho}(x_0)} 2\theta\rho\, \big(1 + |Du - \xi \otimes e_n|\big)\, dx\Big)^{\frac{q_1 - p}{q_1}} \Big(\fint_{B^+_{4\theta\rho}(x_0)} (1 + |Du|)^p\, dx\Big)^{\max\{\frac{\gamma}{p}, 1\}} \\
&\le c\, \omega_1\big(\theta\rho + \theta^{\frac{p-n}{p}}\, \rho\, C(x_0, \rho)^{\frac{1}{p}}\big)^{\frac{q_1 - p}{q_1}} \big(1 + [\theta^{-n}\, C(x_0, \rho)]^{\max\{\frac{\gamma}{p}, 1\}}\big) \le c\, \rho^{2\beta_0} .
\end{aligned}$$

In the last step, we have also used the smallness condition

$$C(x_0,\rho) \,\le\, \theta^n\,. \tag{SC.3}$$

Moreover, the constant in the latter estimate depends only on n,N,p,ν,L,γ and M. Comparing $VIII$ with the second line in the inequalities for VII, it is easy to see that the same estimate holds true for $VIII$, i. e. we have

$$VIII \,\le\, c\,\rho^{2\beta_0}$$

with c depending on the same parameters. Combining (5.8), the previous estimates for VII and $VIII$ with (5.9), we thus end up with

$$C(x_0,\theta\rho) \,\le\, c_2(n,N,p,\nu,L,\gamma,M,K_M)\,\big[\theta^2\,C(x_0,\rho) + (1+\delta^{-2})\,\rho^{2\beta_0}\big]$$

where we have also used the definition of s given in (SC.1). In order to finish the proof of the proposition we now fix the parameters θ and ε_0: we first choose $\theta \in (0,\frac{1}{8})$ sufficiently small such that $c_2\theta^2 \le \theta^{2\widetilde{\beta}}$. This determines θ with the dependencies claimed in the proposition and also fixes the parameters ε and δ from the $\mathcal{A}$-harmonic approximation defined above. Then ε_0 is fixed such that the smallness conditions (SC.2) and (SC.3) hold true, and the dependencies of ε_0 are thus obtained by taking into consideration the dependencies in the smallness conditions (SC.2) and (SC.3), which were needed within the proof. Thus the excess decay estimate is proved with constant $c_* := c_2(1+\delta^{-2})$. The result in the interior follows analogously. We note that here the boundedness condition on the mean values of u is needed in order to find a corresponding approximate $\mathcal{A}$-harmonicity results (with the functional now frozen in the mean values of u and Du, i. e. $\mathcal{A} := D_{zz}f\big(x_0,(u)_{x_0,\rho},(Du)_{x_0,\rho}\big)$, and with $w := u-(u)_{x_0,\rho}-(Du)_{x_0,\rho}(x-x_0)$ the related approximately $\mathcal{A}$-harmonic map) and to apply Poincaré's and Caccioppoli's inequality in the interior. □

This decay estimate can now be iterated in a well-known manner in order to arrive at the desired partial regularity result and the characterization of regular points of Du. The proof is divided into two parts: we will first sketch briefly how the characterization of the regular set is obtained and that the solution is in fact more regular on this set, namely Hölder continuous for some (small) exponent. With this information we then enter the proof again, improve the previous excess-decay, and we then end up with the optimal Hölder regularity result.

Proof (of Theorem 1.1): We consider a bounded domain $\Omega \subset \mathbb{R}^n$, $n \ge 2$, of class $C^{1,2\beta}$ and boundary values $g \in C^{1,2\beta}(\bar{\Omega},\mathbb{R}^N)$. This means that we can locally straighten the boundary $\partial\Omega$ by a $C^{1,2\beta}$-transformation, and hence, via a covering and a transformation argument, the proof of Theorem 1.1 is reduced in a standard way to the proof of the corresponding partial regularity result for functions minimizing the variational functional in the model situations of the unit half-ball B^+ and vanishing on Γ (for the boundary regularity) and the proof of the partial regularity result in the interior of Ω. It is easy to calculate that under the flattening procedure and the reduction to zero boundary values on Γ the structure assumptions on the new integrand are preserved, apart from a possibly decreased Hölder modulus of continuity ω_1 if $2\beta < \alpha_1$, which nevertheless does not change the result in Proposition 5.1. Therefore we will use the same notation f and h even if we are now working with the transformed integrands. For detailed calculations we refer to [8, Section 4.2] for a similar problem or to [37, Section 3.7] in the case of the transformation of a system. At this stage we should also emphasize that it is easy to calculate that also regular points are mapped to regular points by the transformed functional.

An iteration of Proposition 5.1 reveals that for every $M \ge 2$ and every $\widetilde{\beta} \in (\beta_0,1)$ there exists $\varepsilon_1 \in (0,1)$ depending on the same parameter as ε_0 in (5.1) such that the smallness assumptions

$$R + C(x_0,R) \,<\, \varepsilon_1 \qquad \text{and} \qquad |(u)_{x_0,R}|, |(Du)_{x_0,R}| \,<\, \frac{M}{2} \tag{5.10}$$

(or $|(D_n u)_{x_0,R}| < \frac{M}{2}$ for $x_0 \in \Gamma$) imply

$$C(x_0,\rho) \,\le\, c\Big(\Big(\frac{\rho}{R}\Big)^{2\widetilde{\beta}} C(x_0,R) + \rho^{2\beta_0}\Big) \tag{5.11}$$

for all $\rho \le R$ and a constant c depending only on $n,N,p,\nu,L,\gamma,M,K_M,\beta_0$ and $\widetilde{\beta}$. Here, $B_R(x_0)$ is either an arbitrary full ball in the interior of Ω (or in the interior of B^+), or it is a half-ball $B_R^+(x_0)$ with $x_0 \in \Gamma$

and $R < 1-|x_0|$. In the latter case the boundedness of the mean values of u is not required. For a proof of continuous excess-decay estimate (5.11) we omit detailed calculations and refer to several papers to cover the different cases of super-/sub-/quadratic growth for the interior and the boundary situation (we note that the citations include partial regularity results for weak solution of elliptic system, because once the excess-decay estimate is obtained there is no need to return to the system or the functional in order to end up with the regularity result): the quadratic case follows from a similar formula as [18, (3.37)] and [38, (3-58)], the subquadratic case is proved as [7, (6.26) and (6.20)], the superquadratic case as in [59, Lemma 6.3] (and the boundary result is obtained analogously).

Now we take a point $x_0 \in \mathrm{Reg}_\Omega(Du)$ and observe that (5.10) is satisfied for some radius $R > 0$. Then due to the continuity of the maps $z \mapsto C(z,R)$, $z \mapsto (u)_{z,R}$ and $z \mapsto (Du)_{z,R}$, we observe that (5.11) holds for all points z in a small neighbourhood of x_0. Nevertheless, for a point $x_0 \in \mathrm{Reg}_\Gamma(Du)$ (i. e. a transformation of a point in $\mathrm{Reg}_{\partial\Omega}(Du)$), this conclusion of the local growth estimate does not immediately follow because there are only estimates on full balls and on half-balls available so far, but not on arbitrary intersections. Therefore, the estimates in the interior and at the boundary have to be combined similarly as in [38, Section 3.6] or [9, Section 5] to see that (5.11) holds in a relative neighbourhood of x_0. Hence, Lemma 2.1 (v) and Campanato's characterization of Hölder continuous functions [10, Teorema I.2], [38, Theorem 2.3] then yield that $V(Du)$ and thus Du is of class C^{0,β_0} locally around x_0 (for $p \geq 2$ the latter implication is clear from the definition of the V-function, for $p \in (1,2)$ it is obtained by the properties of the V-function as in [22, Lemma 3]). Transforming back to the original problem this gives a first regularity result $u \in C^{1,\beta_0}_{\mathrm{loc}}(\mathrm{Reg}_\Omega(Du) \cup \mathrm{Reg}_{\partial\Omega}(Du), \mathbb{R}^N)$.

In the last step the optimal Hölder continuity is established by revising the decay-estimate from Proposition 5.1 taking into account the additional information that we already have demonstrated C^1-regularity of u on the regular set of Du, which is by definition relatively open. For a similar line of arguments we refer to [32, Proof of Theorem 3.1] or [40, Remark 2]. Terms II and IV in Proposition 5.1, Step 1, are then estimated without using higher integrability of Du:

$$|II| + |IV| \,\leq\, c\,\sigma^{-1} \fint_{B^+_{\rho/2}(x_0)} \omega_1(\rho)\big(1+|Du|\big)^p\,dx$$

(with ρ sufficiently small such that we remain in the regular set of Du), for a constant c depending only on the local Lipschitz-norm of u, hence, on all structure parameters. Therefore, inequality (5.7) now holds with β_0 replaced by $\beta = \min\{\frac{\alpha_2}{2-\alpha_2}, \frac{\alpha_1}{2}\}$. Moreover, revising the Caccioppoli inequality from Lemma 3.1, it easily follows that VII and $VIII$ arising in Step 3 can be bounded by $c\rho^{2\beta}$. This shows that Proposition 5.1 now holds on (half-)balls contained in the regular set of Du with β_0 replaced by β. Then, repeating the iteration as above, we arrive at the desired conclusion $u \in C^{1,\beta}_{\mathrm{loc}}(\mathrm{Reg}_\Omega(Du) \cup \mathrm{Reg}_{\partial\Omega}(Du), \mathbb{R}^N)$. $\square$

Remark 5.2: If we assume additionally to the assumptions of the theorem that $\gamma \in (0,p)$ and that f does not depend explicitly on u or that it is strictly convex, then the characterization of x_0 being a regular point can be rewritten in an easier form. For this purpose we first recall in the model situation of the half-ball the inequality

$$\fint_{B^+_\rho(x_0)} \Big|V\Big(\frac{u-(D_n u)_{x_0,\rho}\,x_n}{\rho}\Big)\Big|^2\,dx \,\leq\, c(p) \fint_{B^+_\rho(x_0)} \big|V\big(D_n u - (D_n u)_{x_0,\rho}\big)\big|^2\,dx \,=:\, c(p)\,\widetilde{C}(x_0,\rho)$$

for all $x_0 \in \Gamma$, $\rho < 1-|x_0|$, which is concluded by the boundary version of Poincaré's inequality (we here refer to [7, Lemma 3.4] and Lemma 2.1 (i) in the superquadratic case, and to [7, Lemma 3.6] in the subquadratic case). Via Lemma 2.2, Caccioppoli's inequality from Remark 3.2, and Hölder's inequality we then get the following line of inequalities:

$$\begin{aligned}
C(x_0,\rho/2) \leq\; & c(p) \fint_{B^+_{\rho/2}(x_0)} \big|V\big(Du - (D_n u)_{x_0,\rho}\otimes e_n\big)\big|^2\,dx \\
\leq\; & c \fint_{B^+_\rho(x_0)} \Big|V\Big(\frac{u-(D_n u)_{x_0,\rho}\,x_n}{\rho}\Big)\Big|^2\,dx \\
& + c \fint_{B^+_\rho(x_0)} \omega_{12}\big(|u-(D_n u)_{x_0,\rho}\,x_n|\big)\,\big(1+|u-(D_n u)_{x_0,\rho}\,x_n|\big)^\gamma\,dx \\
\leq\; & c\,\widetilde{C}(x_0,\rho) + c\,\omega_{12}\big(\widetilde{C}(x_0,\rho)^{\frac{1}{p}}\big)^{\frac{p-\gamma}{p}}\,\big(1+\widetilde{C}(x_0,\rho)\big)^{\frac{\gamma}{p}}
\end{aligned}$$

for a constant depending only on n, N, p, ν, L, γ and M, provided that $|(D_n u)_{x_0,\rho}| \leq M$ and that ρ is sufficiently small in dependency of p, ν, L and $\omega_1(\cdot)$. This shows (after transforming back to the original domain Ω) that the full derivative of u in the characterization $\mathrm{Reg}_{\partial\Omega}(Du)$ of the set of regular boundary points for Du can be replaced by only the normal derivative as stated in Remark 1.2.

6 Morrey estimates in dimensions $\mathbf{n \leq p+2}$

In this section we are concerned with lower order regularity of minimizers of convex integrals in low dimensions. Combining the comparison estimates of Section 4 and using a standard transformation argument we can proceed to the proof of the lower order regularity result:

Proof (of Theorem 1.3): *Step 1: Reduction to the proof of interior regularity and of boundary regularity in the model case.* We consider a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, of class C^1. The transformation follows the scheme described at the beginning of the proof of Theorem 1.1. Therefore we will give here the arguments for minimizers of variational functionals in the model situations of the unit half-ball B^+ and vanishing on Γ (for the boundary regularity), and we will derive the result in the interior of Ω.

Step 2: A Morrey estimate. In order to find a Morrey-type decay estimate for Du we note

$$(1+|\xi|)^p \;\leq\; c(p)\,(1+|\eta|)^p + c(p)\,(1+|\xi|+|\eta|)^p|\xi-\eta|^2$$

for all $\xi, \eta \in \mathbb{R}^k$, $k \in \mathbb{N}$, and $p \in (1,\infty)$. For an arbitrary $x_0 \in \Gamma$ and $R < (1-|x_0|)/2$ we now apply the previous inequality for ξ, η replaced by Du and Dv_0 where v_0 is the unique minimizer to the frozen functional $\mathcal{F}_0[\,\cdot\,; B_R^+(x_0)]$ (introduced in Section 4) among all functions in $u + W_0^{1,p}(B_R^+(x_0), \mathbb{R}^N)$. Thus, Proposition 4.2, Lemma 4.4 and the energy estimate (4.3) yield for every $\rho \leq R$:

$$\begin{aligned}
\int_{B_\rho^+(x_0)} \big(1+|Du|\big)^p\,dx &\leq c\Big(\frac{\rho}{2R}\Big)^{\mu_0} \int_{B_{2R}^+(x_0)} \big(1+|Du|\big)^p\,dx + c\,R^{n(1-\frac{\gamma}{p^*})}\Big(\int_{B_R^+(x_0)} (1+|Du|)^p\,dx\Big)^{\frac{\gamma}{p}}\\
&\quad + c\,\omega_{12}\Big(\Big(R^{p-n}\int_{B_R^+(x_0)} (1+|Du|)^p\,dx\Big)^{\frac{1}{p}}\Big)^{\frac{q-p}{q}} \int_{B_{2R}^+(x_0)} (1+|Du|)^p\,dx + c\,R^n\\
&\leq c_b\Big(\Big(\frac{\rho}{2R}\Big)^{\mu_0} + R^{n\gamma(\frac{1}{p}-\frac{1}{p^*})}\,[R^{-p}\,M(x_0,2R)]^{\max\{\frac{\gamma-p}{p},0\}} + \omega_{12}\big(M(x_0,2R)^{\frac{1}{p}}\big)^{\frac{q-p}{q}}\Big)\\
&\qquad \times \int_{B_{2R}^+(x_0)} \big(1+|Du|\big)^p\,dx + c_b\,R^n
\end{aligned} \tag{6.1}$$

for a constant depending only on n, N, p, ν and L (note that this estimate trivially holds true for radii $R \leq \rho \leq 2R$), where in the last equation we have defined the Morrey-type excess via

$$M(y,r) \;:=\; r^{p-n}\int_{B_r(y)\cap B^+} (1+|Du|)^p\,dx$$

for any $y \in B^+ \cup \Gamma$ and $r > 0$, and we have distinguished the cases $\gamma \leq p$ and $\gamma > p$. Under suitable smallness assumptions we are now in a position to apply the iteration scheme [30, Chapter III, Lemma 2.1] with the following quantities: choosing the left-hand side of (6.1) as the nonnegative and nondecreasing function $\phi(\rho)$, exponents μ_0, $\mu_0 - \varepsilon$ instead of α_1, α_2 and c_b instead of A, we determine the number $\varepsilon_b > 0$ according to this lemma in dependency of n, N, p, ν and L. We emphasize that ε is the number stemming from the a priori estimate in Proposition 4.2 for dimensions $n \geq 3$ and sufficiently small in the dimensions $n = 2$.

Step 2a: The case $n > p$. We first note $n\gamma(\frac{1}{p}-\frac{1}{p^*}) = \gamma$. We next determine a number $\sigma_b > 0$ in dependency of $n, N, p, \nu, L, \gamma, \omega_1(\cdot)$ and $\omega_3(\cdot)$ such that

$$\begin{aligned}
&2\,\sigma_b^{\frac{\gamma-p}{p}} + 2\,\omega_{12}\big(\sigma_b^{\frac{1}{p}}\big)^{\frac{q-p}{q}} \;\leq\; \varepsilon_b \qquad && \text{if } \gamma > p\,,\\
&4\,\omega_{12}\big(\sigma_b^{\frac{1}{p}}\big)^{\frac{q-p}{q}} \;\leq\; \varepsilon_b && \text{otherwise.}
\end{aligned}$$

Furthermore, let $M > 1$. Under the assumption $|(u)_{B_R(x_0)}| < M$ we know that an estimate corresponding to (6.1) holds true for full balls in the interior of the original domain Ω or of the transformed relative neighbourhood B^+ of a given boundary point on $\partial\Omega$. The associated constant is denoted by c_i and now depends also on M, and the related constants determined by [30, Chapter III, Lemma 2.1] by $\varepsilon_i = \varepsilon_i(n,N,p,\nu,L,M)$ and $\sigma_i = \sigma_i(n,N,p,\nu,L,M,\omega_1(\cdot),\omega_3(\cdot)) \in (0,1)$.

The objective is now to show that Du belongs to a suitable Morrey-space, provided that certain smallness conditions on the Morrey-excess and the mean values of u are satisfied. The procedure is standard in partial regularity proofs, so we will only sketch the remaining part of the proof: we start with the interior and assume that the smallness conditions

$$M(z_0,2R_0) < \sigma_i = \sigma_i\big(n,N,p,\nu,L,M,\omega_1(\cdot)\big) \qquad \text{and} \qquad |(u)_{B_{R_0}(z_0)}| < M \tag{6.2}$$

are fullfilled for some $R_0 \in (0,1)$ if $\gamma > p$ or for some $R_0 \in (0,[\varepsilon_i/2]^{1/\gamma})$ if $\gamma \le p$, $B_{2R_0}(z_0)$ compactly contained in B^+ or in Ω. Then, due to the continuity of the maps $z \mapsto M(z,2R_0)$ and $z \mapsto (u)_{B_{R_0}(z)}$, we observe that (6.2) holds for all points z in $B_r(z_0)$ for some $r > 0$ (chosen sufficiently small such that all $B_{2R_0}(z)$ are still contained in B^+ or in Ω). This allows us to deduce

$$\int_{B_\rho(z)} \big(1+|Du|\big)^p\,dx \le c_i \Big(\Big(\frac{\rho}{2R_0}\Big)^{\mu_0} + \frac{\varepsilon_i}{2}\Big) \int_{B^+_{2R_0}(z)} \big(1+|Du|\big)^p\,dx + c_i\,R^n$$

and then to apply [30, Chapter III, Lemma 2.1] to find

$$\int_{B_\rho(z)} \big(1+|Du|\big)^p\,dx \le \widetilde{c}_i \Big[\Big(\frac{\rho}{2R_0}\Big)^{\mu_0-\varepsilon} \int_{B^+_{2R_0}(z)} \big(1+|Du|\big)^p\,dx + \rho^{\mu_0-\varepsilon}\Big] \tag{6.3}$$

for all $\rho < 2R_0$ and a constant $\widetilde{c}_i$ depending only on n,N,p,ν,L and M. This means $Du \in L^{p,\mu_0-\varepsilon}$ in $B_r(z_0)$ by definition of the Morrey spaces, see Section 2.

To end up with the same embedding also in the boundary situation we assume analogously that for some $x_0 \in \Gamma$ and $R_0 \in (0,1)$ the Morrey-types excess $M(x_0,2R)$ satisfies

$$M(x_0,2R_0) < \sigma_b = \sigma_b\big(n,N,p,\nu,L,\omega_1(\cdot)\big) \tag{6.4}$$

and denote by $\widetilde{c}_b$ the constant appearing in the resulting estimate analogous to inequality (6.3). The continuity again yields that this smallness assumption holds in a relative neighbourhood $B_r(x_0) \cap (B^+ \cup \Gamma)$, but the Morrey embedding does not follow immediately from the smallness condition, because we only have estimates on balls or on half-ball, but not on intersections of arbitrary balls with B^+. To derive these estimates, the estimates on the boundary and in the interior now have to be combined similarly as in [38, Section 3.6] or [9, Section 5], and thus we have to assume a stronger smallness condition than in (6.4):

$$M(x_0,2R_0) < c(n,p)^{-1} \min\big\{\sigma_b, \widetilde{c}_b^{\,-1}\,\sigma_i\big\} =: \widetilde{\sigma}_b\big(n,N,p,\nu,L,\omega_1(\cdot)\big) \tag{6.5}$$

for a universal constant $c(n,p)$ and where σ_i is determined for the choice $M = 1$ (this choice is made because the mean value of balls $B_{z_n}(z)$ – which are needed in the combination of the estimates – are bounded by $|(u)_{B_{z_n}(z)}| \le c(n,p) M((z',0),2z_n)^{1/p}$ in view of Poincaré's inequality, and $M((z',0),2z_n)$ in turn is bounded by $\widetilde{c}_b$ times the Morrey excess on larger half-balls). Then, under the assumption (6.5), we conclude $Du \in L^{p,\mu_0-\varepsilon}$ in $B_r(x_0) \cap (B^+ \cup \Gamma)$.

Taking into account the smallness conditions (6.2) in the interior and (6.5) at the boundary, we hence conclude (after transforming back to the original domain Ω):

$$Du \in L^{p,\mu_0-\varepsilon}\big(\mathrm{Reg}_\Omega(u) \cup \mathrm{Reg}_{\partial\Omega}(u), \mathbb{R}^N\big)\,.$$

Step 2b: The case $n \le p$. Due to the higher integrability $Du \in L^q$ for some $q > p$ (Lemma 3.3), we know that u is a priori Hölder continuous. The improvement of the Hölder exponent now follows the line of arguments given in Step 2a, but taking into account $M(x_0,R) \le R^{q-n}\|1+|Du|\|_{L^q}$, which means that $M(x_0,R)$ can be made arbitrarily small for every x_0 only by choosing R sufficiently small (in dependence of $\|1+|Du|\|_{L^q}$ and the structure constants). We further emphasize that the term $R^{n\gamma(1/p-1/p^*)}\,[R^{-p}M(x_0,2R)]^{\gamma/p}$ possibly appearing in (6.1) is estimated by $cR^{n(1-\gamma/p^*)}$, which obviously tends to zero as $R \searrow 0$. Hence, $\mathrm{Reg}_\Omega(u) \cup \mathrm{Reg}_{\partial\Omega}(u)$ coincides with $\overline{\Omega}$, and we obtain analogously to above the Morrey-embedding of Du into $L^{p,\mu_0-\varepsilon}(\overline{\Omega},\mathbb{R}^N)$.

Step 2c: The case of integrands not depending explicitly on u. In this case the decay estimate (6.1) is replaced by

$$\int_{B^+_\rho(x_0)} \big(1+|Du|\big)^p\,dx \le c_b \Big(\Big(\frac{\rho}{2R}\Big)^{\mu_0} + \omega_1(R)\Big) \int_{B^+_{2R}(x_0)} \big(1+|Du|\big)^p\,dx$$

applying Remark 4.5 instead of Lemma 4.4. Noting that $\omega_1(R)$ vanishes as $R \searrow 0$, the iteration scheme [30, Chapter III, Lemma 2.1] can be applied exactly as in Step 2b independently of any smallness assumption on the Morrey excess $M(x_0, R)$ and $Du \in L^{p,\mu_0-\varepsilon}(\overline{\Omega}, \mathbb{R}^N)$ follows.

Step 3: Conclusion. We highlight that the definition of μ_0 and ε combined with the low-dimensional assumption $n \leq p+2$ ensures $\mu_0 - \varepsilon \in (n-p, n]$. Thus, according to the Campanato-Meyers embedding, see e. g. [43, Theorem 2.2], we arrive at the conclusion that u is locally Hölder continuous on the regular set $\mathrm{Reg}_\Omega(u) \cup \mathrm{Reg}_{\partial\Omega}(u)$ with Hölder exponent $\lambda = 1 - \frac{n-\mu_0+\varepsilon}{p}$. Returning to the definition of $\mu_0 = \min\{n, 2+2\varepsilon\}$ and noting that ε can be chosen arbitrarily small for $n = 2$, we thus end up with the Hölder continuity results for every exponent $\lambda \in (0, \min\{1 - \frac{n-2-\varepsilon}{p}, 1\})$ as asserted in the Theorem.

Furthermore, the higher integrability result Lemma 3.3 enables us to improve the condition of x_0 being a regular point. As a consequence we get that $\mathrm{Reg}_\Omega(u) \cup \mathrm{Reg}_{\partial\Omega}(u)$ is contained in the set

$$\Big\{x_0 \in \overline{\Omega} : \liminf_{R\to 0} R^{q_1-n} \int_{B_R(x_0)\cap\Omega} \big(1+|Du|\big)^{q_1}\,dx \,=\, 0 \text{ and } \limsup_{\rho\to 0^+} |(u)_{\Omega\cap B_\rho(x_0)}| \,<\, \infty\Big\}$$

which, in view of Giusti's measure density result [34, Proposition 2.7], proves the assertion on the upper bound for the Hausdorff dimension of the singular set and finishes the proof of Theorem 1.3. □

7 Improved regularity in dimensions n = 2

The result of the previous section now allows us to deduce a similar result as in Theorem 1.1 (assuming convexity instead of quasi-convexity), but which holds on a larger set. We here proceed close to the proof [43, Section 9] of a similar (interior) result for minimizers in the superquadratic case:

Proof (of Theorem 1.4): Analogously to the previous proofs of Theorem 1.1 and 1.3 a transformation and local flattening procedure allows us to restrict ourselves to the interior regularity and to the boundary regularity on Γ for the model situation of the upper unit half-ball.

In the sequel, we shall revise some of the previous estimates and improve them in dimensions $n = 2$. Taking into account $\mu_0 = n = 2$ we first want to comment on the Morrey embedding $Du \in L^{2-\tau,p}$ on the regular set of u for arbitrarily small $\tau > 0$ found in the previous section. We emphasize that in the superquadratic case $p \geq 2$ the regular set of u coincides with $\overline{\Omega}$, and the proof of Theorem 1.3 (in particular Step 2b) reveals: for every $\tau \in (0, p)$ there exists of a radius $R_0 = R_0(N, p, \nu, L, \alpha_1, \|u\|_{L^\infty}, \|Du\|_{L^q}, \tau) > 0$ (independently of the ball under consideration) such that we have

$$\int_{B_R^+(x_0)} \big(1+|Du|\big)^p\,dx \,<\, R^{2-\tau} \tag{7.1}$$

for all $x_0 \in \Gamma$ and $2R \leq \min\{R_0, (1-|x_0|)\}$ as well as the corresponding estimate on balls $B_R(x_0)$ in the interior of Ω. In view of $\omega_1(t) \leq \min\{1, t^{\alpha_1}\}$ and $\omega_2(t) \leq \min\{1, t^{\alpha_2}\}$ for all $t \geq 0$, the second statement in Lemma 4.4 thus yields the comparison estimate

$$\int_{B_R^+(x_0)} \big|V(Du) - V(Dv_0)\big|^2\,dx \,\leq\, c\,R^{2+\frac{p-\tau}{p}\frac{q-p}{q}\alpha_1-\tau} + c\,R^{2+\frac{p-\tau}{p}(1-\frac{\gamma}{p^*})\alpha_2} + c\,R^{2+2\gamma(\frac{1}{p}-\frac{1}{p^*})-\tau\frac{\gamma}{p}} \,\leq\, c\,R^{2+\delta}$$

for all $R \leq R_0/2$ with a constant $c = c(N, p, p^*, \nu, L, \gamma)$, and with v_0 the unique minimizer to the frozen functional $\mathcal{F}_0[\,\cdot\,; B_R^+(x_0)]$ in the class $u + W_0^{1,p}(B_R^+(x_0), \mathbb{R}^N)$. Moreover, we have defined

$$\delta \,=\, \min\big\{\tfrac{p-\tau}{p}\tfrac{q-p}{q}\alpha_1 - \tau, \tfrac{p-\tau}{p}(1-\tfrac{\gamma}{p^*})\alpha_2, 2\gamma(\tfrac{1}{p}-\tfrac{1}{p^*}) - \tau\tfrac{\gamma}{p}\big\}$$

which is strictly positive for τ chosen sufficiently small (in dependency of N, p, ν, L and α_1). Returning to a Campanato-type decay estimate, we then find via Proposition 4.2 for any $0 < \rho < R \leq R_0/2$:

$$\begin{aligned}
&\int_{B_\rho^+(x_0)} \big|V(Du) - \big(V(Du)\big)_{\rho,x_0}\big|^2\,dx \\
&\quad\leq\, 2\int_{B_\rho^+(x_0)} \big|V(Dv_0) - \big(V(Dv_0)\big)_{\rho,x_0}\big|^2\,dx + 2\int_{B_R^+(x_0)} \big|V(Du) - V(Dv_0)\big|^2\,dx \\
&\quad\leq\, c\Big(\frac{\rho}{R}\Big)^{2+\varepsilon}\int_{B_R^+(x_0)} \big(1+|Dv_0|\big)^p\,dx + c\,R^{2+\delta} \,\leq\, c\Big(\frac{\rho}{R}\Big)^{\varepsilon}\rho^2\,R^{-\tau} + c\,R^{2+\delta}\,.
\end{aligned}$$

Taking $R := \rho^\sigma$ with

$$\sigma := \frac{2+\varepsilon}{2+\delta+\varepsilon+\tau},$$

we obtain that the powers of ρ on the right-hand side of the previous inequality coincide and equal $\sigma(2+\delta)$. So in the next step we fix τ sufficiently small (which in turn fixes the radius R_0) such that this power of ρ is greater than the space dimension 2, i.e.

$$\begin{aligned} \sigma\,(2+\delta) &= \frac{4+2\varepsilon+2\delta+\varepsilon\delta}{2+\delta+\varepsilon+\tau} > 2 \\ \Leftrightarrow \quad \tau &< \min\Big\{\frac{\varepsilon(q-p)p\alpha_1}{2pq+\varepsilon pq+\varepsilon(q-p)\alpha_1}, \frac{\varepsilon p(1-\frac{\gamma}{p^*})\alpha_2}{2p+\varepsilon(1-\frac{\gamma}{p^*})\alpha_2}, \frac{\varepsilon(1+\frac{\gamma}{p}-\frac{\gamma}{p^*})}{1+\frac{\varepsilon\gamma}{2p}}\Big\} =: \tau_0 \end{aligned}$$

with $\tau_0 = \tau_0(N,p,p^*,\nu,L,\gamma,\alpha_1,\alpha_2)$. Hence, there exists an exponent $\lambda > 0$ depending on the same parameters such that

$$\int_{B_\rho^+(x_0)} \big|V(Du) - \big(V(Du)\big)_{\rho,x_0}\big|^2\,dx \le c(N,p,\nu,L)\,\rho^{n+2\lambda} \tag{7.2}$$

for $\rho \le \rho_0$, and ρ_0 sufficiently small in dependency of $N,p,\nu,L,\alpha_1,\|u\|_{L^\infty}$ and $\|Du\|_{L^q}$ (but ρ_0 is independent of x_0). For balls in the interior, the same estimate holds true, and combining the estimates for balls in the interior of B^+ and for half-balls with center on Γ, we again obtain this decay estimate for all intersects of balls with B^+. Scaling back, taking into account Campanato's characterization of Hölder continuous functions and using standard properties of the V-function, we thus arrive at the conclusion $Du \in C^{0,\lambda'}(\overline{\Omega},\mathbb{R}^N)$ in the superquadratic case $p \ge 2$ for some $\lambda' > 0$. Therefore, the set of regular points of Du given in Theorem 1.1 coincides with $\overline{\Omega}$, the optimal Hölder continuity follows, and the theorem is proved for $p \ge 2$.

Dealing with the subquadratic case, we first observe that the Morrey-embedding $Du \in L^{2-\tau,p}$ holds locally outside a set of Hausdorff dimension less than $n-p$, except in the case where f and h do not depend explicitly on u (in this case, the estimate (7.1) holds everywhere and the line of arguments above applies). Therefore, in the general situation the proof of Theorem 1.3 merely ensures that (7.1) holds locally on the regular set $\mathrm{Reg}_{B^+\cup\Gamma}(u)$ of u (and also on $\mathrm{Reg}_\Omega(u)$ in the interior). Then, as it was proved in Theorem 1.3, the radius $R_0 > 0$ determined in the superquadratic case now depends also on the point x_0, but due to the continuity of the map $x_0 \mapsto R^{p-n}\int_{B_R^+(x_0)}(1+|Du|)^p\,dx$ the estimate holds locally in a neighbourhood $U(x_0)$ of x_0. Therefore, for every x_0 in the regular set of u we have estimates of the type (7.1) at our disposal for a (relative) neighbourhood $U(x_0)$ of x_0, and we thus repeat the calculations restricting ourselves to (half-)balls contained in $U(x_0)$. This yields the inequality (7.2) on (half-)balls $B_\rho^{(+)}(x_0)$ for all $\rho > \rho_0$ with $\rho_0 > 0$ now depending also on x_0 in a continuous way. The conclusion then follows exactly as above. $\square$

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