

Nonlocal-to-local variational limits of energies on partitions

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Abstract. We study, through a Γ -convergence procedure, the asymptotic behaviour of a general class of nonlocal energies defined on functions taking values in a finite set $Y = \{y_1, \dots, y_N\} \subset \mathbb{R}^m$. These energies are written in terms of double integrals accounting for pairwise interactions in an open set Ω , weighted by a kernel K_ε . If the latter is controlled from above and below by suitable non-negative radial kernels with unit mass concentrating at the origin, then, up to subsequences, the Γ -limit has domain $BV(\Omega; Y)$ and is given by a local surface energy. We show that the locality of the Γ -limit is not guaranteed if the concentration of mass fails. For a class of reference energies, we obtain a more explicit representation of the limit energy, also for non-radial kernels, recovering the limits of isotropic and anisotropic fractional perimeters. We also prove a homogenization result for periodic kernels.

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1 Introduction

Over the last two decades, increasing attention has been devoted to variational models involving nonlocal energies, which depend on interactions between points at a finite distance, providing a natural framework for phenomena that cannot be adequately described by purely local energies. Such phenomena arise, for instance, in the description of phase transitions with long-range interactions [1, 21, 28, 32], in nonlocal minimal surfaces [20], in image processing [17, 26], and in continuum mechanics through peridynamic theory [15].

A prototypical example of such energies is given by functionals of the form

$$\int_{\Omega} \int_{\Omega} f(x, y, u(x), u(y)) \, dx \, dy$$

where Ω is an open set of $\mathbb{R}^d$ and f accounts for the interaction between pairs of points and depends on their relative position and on the states of the system at both points.

A central question is whether, and in which form, a local description is recovered when the range of the interactions becomes infinitesimal. The problem was first investigated by Bourgain, Brezis, and Mironescu in their celebrated paper [18], where they studied the asymptotic behaviour of the Gagliardo seminorm $[u]_{W^{s,p}(\Omega)}$, as $s \rightarrow 1^-$, showing that

$$(1-s)[u]_{W^{s,p}(\Omega)} = (1-s) \int_{\Omega} \int_{\Omega} \frac{|u(y) - u(x)|^p}{|y-x|^{d+sp}} dx dy$$

approximates, up to a multiplicative constant, the p -th power of the L^p norm of ∇u for $p > 1$ and $u \in W^{1,p}(\Omega)$, and the total variation $|Du|(\Omega)$ for $p = 1$ and $u \in BV(\Omega)$ (see also [23]). The result was subsequently extended in [30] in terms of Γ -convergence.

For energies with superlinear growth, a general asymptotic analysis as $\varepsilon \rightarrow 0$ of families of functionals of the form

$$\int_{\Omega} \int_{\Omega} f_{\varepsilon}(x, y, u(y) - u(x)) dx dy, \quad (1.1)$$

under concentration of the energies on $x = y$, has been recently provided in [2], leading to a general class of energies whose Γ -limits are local integral functionals on Sobolev space of the form

$$\int_{\Omega} f(x, \nabla u) dx.$$

This result has been applied, among others, to periodic and stochastic homogenization, to perforated domains (see for instance [10, 14, 24, 33]), and, very recently, to the derivation of lower-dimensional models for nonlocal thin films [11]. The same approach has also been extended to energies leading to free discontinuity functionals [13].

In this paper we consider the case in which the order parameter u takes only finitely many values; i.e. $u: \Omega \rightarrow Y = \{y_1, \dots, y_N\} \subset \mathbb{R}^m$, so that every admissible configuration is a partition of Ω into the phases $\{u = y_i\}$, for every $i = 1, \dots, N$. Such a description is the natural setting whenever the system can only be represented by a finite number of states. Typical examples are the components of a mixture of immiscible fluids, as in the vector-valued Cahn–Hilliard theory [12], the admissible orientations of a small droplet of a liquid crystal [34], and the spin values of a magnetic system of Ising type [3, 4]. The local theory for this kind of phenomena typically involves interfacial energies of the form

$$\int_{S_u} \varphi(x, u^+, u^-, \nu_u) d\mathcal{H}^{d-1}, \quad (1.2)$$

where S_u is the jump set of u , ν_u is the measure-theoretic normal to S_u , and $u^{\pm}$ are the traces of u on the two sides of S_u . A general variational analysis, studying integral representation, lower semicontinuity, relaxation, and homogenization of this kind of energies, was developed by Ambrosio and Braides [6, 7].

The question we address here is to understand how such interfacial energies emerge from nonlocal models. In the discrete setting, this problem was investigated in [4] for Ising-type energies of spin systems on a lattice. In the continuum setting, this kind of nonlocal-to-local analysis was developed in [8], where the asymptotic behaviour of the s -fractional perimeters as $s \rightarrow 1^-$ was studied. In particular, it was shown that $(1-s)[\chi_E]_{W^{s,1}(\Omega)}$ Γ -converges to $\omega_{d-1}P(E, \Omega)$, where $P(\cdot, \Omega)$ denotes the classical De Giorgi perimeter. More recently, in [5] the authors considered energies on partitions depending on the Riesz s -fractional gradient, for real-valued functions taking finitely many values, and proved that, as $s \rightarrow 1^-$, the Γ -limit is a local interfacial energy as in (1.2).

In this paper, we consider a general class of functionals of the form (1.1) defined on $L^1(\Omega; Y)$, where $f_\varepsilon: \Omega \times \Omega \times Y \times Y \rightarrow [0, +\infty]$ is a Borel function such that $f_\varepsilon(x, y, u, u) = 0$. The last condition encodes the fact that uniform configurations minimize the energy. Setting

$$K_\varepsilon(x, y, u, v) := \begin{cases} 0 & \text{if } u = v \\ f_\varepsilon(x, y, u, v) \frac{|y-x|}{|v-u|} & \text{if } u \neq v, \end{cases}$$

the integrand f_ε can be written as

$$f_\varepsilon(x, y, u, v) = K_\varepsilon(x, y, u, v) \frac{|v-u|}{|y-x|}$$

and, in turn,

$$F_\varepsilon(u) = \int_{\Omega} \int_{\Omega} K_\varepsilon(x, y, u(x), u(y)) \frac{|u(y) - u(x)|}{|y-x|} dx dy.$$

We assume that the kernels K_ε are comparable with a family of radial kernels ρ_ε with unit mass concentrating at the origin, namely

$$C_1 \rho_\varepsilon(y-x) \leq K_\varepsilon(x, y, u, v) \leq C_2 \rho_\varepsilon(y-x) \quad \text{for every } u, v \in Y, u \neq v$$

(see Section 3 for the precise assumptions). Our main result, Theorem 3.1, establishes compactness and integral representation for this class of energies. Namely, we prove that, up to subsequences, the functionals F_ε Γ -converge, as $\varepsilon \rightarrow 0^+$, to a functional of the form

$$F(u) = \begin{cases} \int_{S_u} \varphi(x, u^+, u^-, \nu_u) d\mathcal{H}^{d-1} & \text{if } u \in BV(\Omega; Y), \\ +\infty & \text{otherwise.} \end{cases}$$

When $K_\varepsilon(x, y, u, v) = \rho_\varepsilon(y-x)$, the Γ -limit has a more explicit representation in terms of a measure μ associated to the kernels ρ_ε (see Theorem 4.1). Our analysis covers both kernels of fractional type, isotropic and anisotropic, thus including the fractional perimeters considered in [8] and their anisotropic versions studied in [29, 25] (see Examples 4.7–4.10). As a consequence of our Γ -convergence results, in Section 6 we prove the convergence of minimum problems with Dirichlet boundary conditions and with prescribed volume fractions of the phases (Theorems 6.3 and 6.4).

The proof of Theorem 3.1, carried out in Section 5, relies on an abstract method originally exploited in the homogenization theory for multiple integrals [22, 16]. It amounts to applying a localization argument which allows us to regard our functionals and their Γ -limits as defined on functions and sets and then to prove that all the assumptions of an integral representation result for functionals defined on partitions proved in [19] are fulfilled (see Theorem 2.4). A key point is to show that the Γ -limit is the restriction of a measure, and in particular that it is subadditive (see Proposition 5.3). In this respect, the hypothesis of mass concentration at the origin (3.4) plays a crucial role in showing that the nonlocality of the approximating functionals disappears in the limit.

An application of our result is the homogenization of nonlocal energies on partitions, which we address in Section 7. We consider kernels of the form

$$K_\varepsilon(x, y, u, v) = \varepsilon^{-d} K(x/\varepsilon, y/\varepsilon, u, v),$$

with K 1-periodic with respect to simultaneous translations of the two spatial variables (see (7.1)), so that the heterogeneity of the medium and the range of the interactions are of the same

order ε . We prove that the family of functionals F_ε Γ -converges, as $\varepsilon \rightarrow 0^+$, to a homogeneous interfacial energy, whose density is defined by an asymptotic nonlocal cell formula (see Theorem 7.2).

Finally, in Section 8 we show that the condition (3.4) is essential to recover the locality of the Γ -limit. Indeed we exhibit two examples where (3.4) fails and the Γ -limit is the sum of an interfacial energy and of a genuinely nonlocal term.

2 Notation and Preliminaries

In what follows, given $x, y \in \mathbb{R}^d$ we denote by $\langle x, y \rangle$ the standard scalar product in $\mathbb{R}^d$ and we set $|x| = \sqrt{\langle x, x \rangle}$. Given $k \in \mathbb{N}$, we denote by $\mathcal{H}^k$ the k -dimensional Hausdorff measure, by σ_k the k -dimensional Hausdorff measure of the unit sphere $S^k := \{x \in \mathbb{R}^{k+1} : |x| = 1\}$ in $\mathbb{R}^{k+1}$, and by ω_k the volume of the unit ball in $\mathbb{R}^k$. We denote by $\mathcal{A}(\Omega)$ the class of open subsets in Ω and by $\mathcal{A}^{\text{reg}}(\Omega)$ the family of open subsets of Ω with Lipschitz boundary. For any $\nu \in S^{d-1}$, fix an orthonormal basis $\{\nu^1, \dots, \nu^{d-1}, \nu\}$ of $\mathbb{R}^d$, and let $Q_\nu := \{x \in \mathbb{R}^d : |\langle x, \nu^i \rangle| < 1/2, \text{ for } i = 1, \dots, d-1, |\langle x, \nu \rangle| < 1/2\}$ be the unit cube centered at the origin with two of its faces orthogonal to ν . Given $r > 0$ and $x_0 \in \mathbb{R}^d$, we set $Q_\nu(x_0, r) := x_0 + rQ_\nu$, the cube centered at x_0 , with side length r , and oriented in the direction ν . Unless otherwise stated, C will always denote a generic strictly positive constant that may change from line to line.

For the reader's convenience, we recall the definition of $BV(\Omega; Y)$ and the compactness result related to this space, which can be deduced by the corresponding properties of sets of finite perimeter (see [6, 9, 27]).

Definition 2.1. Let $Y \subset \mathbb{R}^m$ be a finite set whose elements are denoted by $y_1, \dots, y_N$. We denote by $BV(\Omega; Y)$ the class of Borel functions $u : \Omega \rightarrow Y$ whose level sets $\{u = y_i\}$ are sets of finite perimeter in Ω . We denote by S_u the Borel set

$$S_u = \bigcup_{y_i \in Y} \partial^* \{u = y_i\} = \bigcup_{y_i, y_j \in Y, y_i \neq y_j} \partial^* \{u = y_i\} \cap \partial^* \{u = y_j\},$$

where, for a set $E \subset \mathbb{R}^d$, of finite perimeter we denote by $\partial^* E$ the essential boundary of E .

Moreover, for $\mathcal{H}^{d-1}$ -almost every $x \in S_u$ there exists a unit vector $\nu_u(x) \in S^{d-1}$ such that

$$\lim_{r \rightarrow 0^+} r^{-d} |\{z \in B_r(x) : \langle z - x, \nu_u(x) \rangle > 0, u(z) \neq y_i\}| = 0$$

and

$$\lim_{r \rightarrow 0^+} r^{-d} |\{z \in B_r(x) : \langle z - x, \nu_u(x) \rangle < 0, u(z) \neq y_j\}| = 0.$$

We set $y_i = u^+(x)$ and $y_j = u^-(x)$. The triplet (u^+, u^-, ν_u) is defined $\mathcal{H}^{d-1}$ -almost everywhere on S_u , and it is uniquely determined, up to a change of sign of ν_u and an interchange of u^+ , u^- .

Theorem 2.2 (Compactness). *Let $u_k \in BV(\Omega; Y)$ such that*

$$\sup_k \mathcal{H}^{d-1}(S_{u_k}) < +\infty.$$

Then there exists a subsequence (not relabeled) and $u \in BV(\Omega; Y)$ such that $u_k \rightarrow u$ in L^1 .

We recall a characterization of measures as set functions due to De Giorgi and Letta (1977) (see [16, Theorem 10.2]).

Theorem 2.3. *Let $\alpha : \mathcal{A}(\Omega) \rightarrow [0, +\infty]$ be an increasing set function. The following statements are equivalent:*

- (i) α is the restriction to $\mathcal{A}(\Omega)$ of a Borel measure on Ω ;
- (ii) α is subadditive, superadditive and inner regular;
- (iii) the set function

$$\beta(E) = \inf\{\alpha(U) : U \in \mathcal{A}(\Omega), E \subseteq U\}$$

defines a Borel measure on Ω .

The following integral representation result for functionals defined on partitions has been proved in [19, Theorem 3].

Theorem 2.4. *Let $F : BV(\Omega; Y) \times \mathcal{A}(\Omega) \rightarrow [0, +\infty)$ satisfy for every $(u, A) \in BV(\Omega; Y) \times \mathcal{A}(\Omega)$ the following hypotheses:*

- (i) $F(u, \cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure;
- (ii) $F(u, A) = F(v, A)$ whenever $u = v$ a.e. on $A \in \mathcal{A}(\Omega)$;
- (iii) $F(\cdot, A)$ is $L^1(\Omega; \mathbb{R}^m)$ lower semicontinuous;
- (iv) there exist $C_1, C_2 > 0$ such that

$$C_1 \mathcal{H}^{d-1}(S_u \cap A) \leq F(u, A) \leq C_2 \mathcal{H}^{d-1}(S_u \cap A).$$

Then for every $u \in BV(\Omega; Y)$ and $A \in \mathcal{A}(\Omega)$

$$F(u, A) = \int_{S_u \cap A} \varphi(x, u^+, u^-, \nu_u) d\mathcal{H}^{d-1},$$

with

$$\varphi(x_0, a, b, \nu) = \limsup_{r \rightarrow 0^+} \frac{m(u_{x_0, a, b, \nu}, Q_\nu(x_0, r))}{r^{d-1}} \quad (2.1)$$

for all $x_0 \in \Omega, a, b \in Y, \nu \in S^{d-1}$, where

$$u_{x_0, a, b, \nu}(x) := \begin{cases} a & \text{if } \langle x - x_0, \nu \rangle > 0 \\ b & \text{if } \langle x - x_0, \nu \rangle \leq 0 \end{cases} \quad (2.2)$$

and for any $(v, A) \in BV(\Omega; Y) \times \mathcal{A}(\Omega)$,

$$m(v, A) := \inf\{F(w, A) : w \in BV(\Omega; Y), w = v \text{ in a neighborhood of } \partial A\}.$$

In the sequel, we will use the notation $u_{a, b, \nu}$ to denote the function defined in (2.2) with $x_0 = 0$.

3 Setting of the problem and main result

Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary and $Y \subset \mathbb{R}^m$ be a finite set endowed with the discrete topology. Given $\varepsilon > 0$, let $K_\varepsilon : \Omega \times \Omega \times Y \times Y \rightarrow [0, +\infty]$ be a Borel function and let $F_\varepsilon : L^1(\Omega; Y) \rightarrow [0, +\infty]$ be defined by

$$F_\varepsilon(u) = \int_{\Omega} \int_{\Omega} K_\varepsilon(x, y, u(x), u(y)) \frac{|u(y) - u(x)|}{|y - x|} dx dy. \quad (3.1)$$

We fix the following set of hypotheses on $K_\varepsilon(x, y, u, v)$: there exists a family of Borel functions $\rho_\varepsilon : \mathbb{R}^d \rightarrow [0, +\infty)$ satisfying

$$\begin{cases} \rho_\varepsilon(\cdot) \text{ is radial} & \forall \varepsilon > 0, \end{cases} \quad (3.2)$$

$$\begin{cases} \int_{\mathbb{R}^d} \rho_\varepsilon(z) dz = 1 & \forall \varepsilon > 0, \end{cases} \quad (3.3)$$

$$\begin{cases} \lim_{\varepsilon \rightarrow 0} \int_{|z| > \eta} \rho_\varepsilon(z) dz = 0 & \forall \eta > 0, \end{cases} \quad (3.4)$$

such that for a.e. $x, y \in \Omega$ and for every $(u, v) \in Y \times Y$ with $u \neq v$, it holds:

(H1) (*Coerciveness*) $\exists C_1 > 0$ such that

$$K_\varepsilon(x, y, u, v) \geq C_1 \rho_\varepsilon(y - x),$$

(H2) (*Growth*) $\exists C_2 > 0$ such that

$$K_\varepsilon(x, y, u, v) \leq C_2 \rho_\varepsilon(y - x).$$

We also define a localized version of our energies. For any $A \in \mathcal{A}(\Omega)$ and for $u \in L^1(A; Y)$ we set

$$F_\varepsilon(u, A) := \int_A \int_A K_\varepsilon(x, y, u(x), u(y)) \frac{|u(y) - u(x)|}{|y - x|} dx dy.$$

Notice that, when $A = \Omega$, the localized functional agrees with the functional defined in (3.1), namely $F_\varepsilon(u, \Omega) = F_\varepsilon(u)$. We will denote by $F'(\cdot, A)$ and $F''(\cdot, A)$ the Γ -lim inf and Γ -lim sup of $F_\varepsilon(\cdot, A)$ with respect to the $L^1(A; \mathbb{R}^m)$ strong topology, respectively.

We now state the main result of the paper.

Theorem 3.1. *Let F_ε be defined by (3.1), with $K_\varepsilon(x, y, u, v)$ satisfying (H1)-(H2). Then for any sequence $\varepsilon_j \rightarrow 0$ there exist a subsequence ε_{j_k} and a function $\varphi : \Omega \times Y \times Y \times S^{d-1} \rightarrow [0, +\infty)$ such that the functionals $F_{\varepsilon_{j_k}}$ Γ -converge with respect to the $L^1(\Omega; \mathbb{R}^m)$ strong topology to the functional $F : L^1(\Omega; Y) \rightarrow [0, +\infty]$ defined by*

$$F(u) := \begin{cases} \int_{S_u} \varphi(x, u^+, u^-, \nu_u) d\mathcal{H}^{d-1} & \text{if } u \in BV(\Omega; Y) \\ +\infty & \text{otherwise.} \end{cases} \quad (3.5)$$

Moreover, for every $u \in BV(\Omega; Y)$ and $A \in \mathcal{A}^{\text{reg}}(\Omega)$, it holds

$$\Gamma(L^1) - \lim_k F_{\varepsilon_{j_k}}(u, A) = F(u, A) := \int_{S_u \cap A} \varphi(x, u^+, u^-, \nu_u) d\mathcal{H}^{d-1}.$$

Remark 3.2. Given $R > 0$ set

$$F_\varepsilon^R(u, A) = \int_{\{(x,y) \in A \times A : |y-x| \leq R\}} K_\varepsilon(x, y, u(x), u(y)) \frac{|u(y) - u(x)|}{|y - x|} dx dy.$$

We note that for every $u \in L^1(A; Y)$, by (H2) it holds

$$\begin{aligned} & \int_{\{(x,y) \in A \times A : |y-x| > R\}} K_\varepsilon(x, y, u(x), u(y)) \frac{|u(y) - u(x)|}{|y - x|} dx dy \\ & \leq C_2 \int_{\{(x,y) \in A \times A : |y-x| > R\}} \rho_\varepsilon(y - x) \frac{|u(y) - u(x)|}{|y - x|} dx dy \\ & \leq \frac{C}{R} |A| \int_{|z| > R} \rho_\varepsilon(z) dz. \end{aligned}$$

Hence, by (3.4),

$$0 \leq \lim_{\varepsilon \rightarrow 0} \sup_{u \in L^1(A; Y)} (F_\varepsilon(u, A) - F_\varepsilon^R(u, A)) \leq \lim_{\varepsilon \rightarrow 0} \frac{C}{R} |A| \int_{|z| > R} \rho_\varepsilon(z) dz = 0.$$

We deduce that F_ε and F_ε^R share the same asymptotic behavior. In particular, Theorem 3.1 still holds if we replace F_ε by F_ε^R .

We postpone the proof of Theorem 3.1, as it will be a consequence of some propositions which show that our limit functionals satisfy all the hypotheses of Theorem 2.4. To this end, we also introduce the special class of functionals $G_\varepsilon(\cdot, A) : L^1(A; Y) \rightarrow [0, +\infty]$ defined as

$$G_\varepsilon(u, A) := \int_A \int_A \rho_\varepsilon(y - x) \frac{|u(y) - u(x)|}{|y - x|} dx dy, \quad (3.6)$$

whose asymptotic behaviour as $\varepsilon \rightarrow 0$ will be examined in Section 4. When $A = \Omega$, we simply write $G_\varepsilon(u)$ in place of $G_\varepsilon(u, \Omega)$.

Note that, by (H1) and (H2), for any $A \in \mathcal{A}(\Omega)$ and $u \in L^1(A; Y)$ it holds

$$C_1 G_\varepsilon(u, A) \leq F_\varepsilon(u, A) \leq C_2 G_\varepsilon(u, A). \quad (3.7)$$

4 The Γ -limit of a class of reference energies

Since, by (3.7), our family of nonlocal functionals F_ε are estimated by the functionals G_ε defined in (3.6), we first examine the asymptotic behaviour of the latter. Our analysis strongly relies on the Γ -convergence analysis developed in [30] for functionals of the form (3.6), that in our case are defined in $L^1(\Omega; Y)$.

We associate with the kernel $\rho_\varepsilon : \mathbb{R}^d \rightarrow [0, +\infty)$ the measure μ_ε on S^{d-1} defined by

$$\mu_\varepsilon(E) := \int_{R_+ E} \rho_\varepsilon(z) dz \quad \text{for each } E \subset S^{d-1},$$

where $R_+ E := \{qx : q \geq 0 \text{ and } x \in E\}$ is the cone generated by E with respect to the origin. Note that

$$\mu_\varepsilon(E) = \int_E \int_0^{+\infty} \rho_\varepsilon(t\sigma) t^{d-1} dt d\mathcal{H}^{d-1}(\sigma).$$

Therefore,

$$\mu_\varepsilon = a_\varepsilon(\cdot) \mathcal{H}^{d-1} \llcorner_{S^{d-1}}, \quad a_\varepsilon(\sigma) := \int_0^{+\infty} \rho_\varepsilon(t\sigma) t^{d-1} dt. \quad (4.1)$$

Moreover, if ρ_ε is radial, then $\mu_\varepsilon = \frac{\mu_\varepsilon(S^{d-1})}{\sigma_{d-1}} \mathcal{H}^{d-1} \llcorner_{S^{d-1}}$.

If ρ_ε satisfies (3.3), the family (μ_ε) is bounded in $\mathcal{M}(S^{d-1})$ (the space of Radon measures on S^{d-1}), so there exists a sequence $\varepsilon_j \rightarrow 0$ and $\mu \in \mathcal{M}(S^{d-1})$ such that $\mu_{\varepsilon_j} \xrightarrow{*} \mu$ in $\mathcal{M}(S^{d-1})$. In particular, $\mu \geq 0$ on S^{d-1} and $\mu(S^{d-1}) = 1$.

Theorem 4.1. *Let $A \in \mathcal{A}^{\text{reg}}(\Omega)$ and $G_\varepsilon(u, A)$ be defined by (3.6) with $\rho_\varepsilon : \mathbb{R}^d \rightarrow [0, +\infty)$ satisfying (3.4). Suppose that the measures μ_ε weakly* converge to a measure $\mu \in \mathcal{M}(S^{d-1})$ such that $\text{span}(\text{supp}(\mu)) = \mathbb{R}^d$. Then*

$$\Gamma(L^1) - \lim_{\varepsilon \rightarrow 0} G_\varepsilon(u, A) = \begin{cases} \int_{S_u \cap A} |u^+ - u^-| \varphi(\nu_u) d\mathcal{H}^{d-1} & \text{if } u \in BV(A; Y) \\ +\infty & \text{otherwise,} \end{cases} \quad (4.2)$$

where

$$\varphi(\nu) = \int_{S^{d-1}} |\langle \nu, \sigma \rangle| d\mu(\sigma). \quad (4.3)$$

Moreover, if $u \in BV(A; Y)$,

$$\Gamma(L^1) - \lim_{\varepsilon \rightarrow 0} G_\varepsilon(u, A) = \lim_{\varepsilon \rightarrow 0} G_\varepsilon(u, A). \quad (4.4)$$

Remark 4.2. Under the assumptions of Theorem 4.1, we see that, for ε small enough,

$$0 < c_1 \leq \int_{\mathbb{R}^d} \rho_\varepsilon(z) dz \leq c_2,$$

for some positive constants c_1, c_2 independent on ε . Indeed,

$$\int_{\mathbb{R}^d} \rho_\varepsilon(z) dz = \mu_\varepsilon(S^{d-1}) \rightarrow \mu(S^{d-1}) \in (0, +\infty).$$

It would not be restrictive to assume that ρ_ε satisfies (3.3), up to replacing ρ_ε with $\tilde{\rho}_\varepsilon := \frac{1}{\mu_\varepsilon(S^{d-1})} \rho_\varepsilon$.

Remark 4.3. Given μ as in Theorem 4.1, consider the function $f_\mu : \mathbb{R}^{m \times d} \rightarrow [0, +\infty)$ defined by

$$f_\mu(M) := \int_{S^{d-1}} |M\sigma| d\mu(\sigma). \quad (4.5)$$

We note that f_μ is convex, continuous, and positively one-homogeneous. Moreover, if $f_\mu(M) = 0$ then $M\sigma = 0$ for μ -a.e. σ . By continuity, this holds for every $\sigma \in \text{supp } \mu$. Since $\text{span}(\text{supp } \mu) = \mathbb{R}^d$, it follows that $M = 0$. Therefore, by the compactness of the unit sphere in $\mathbb{R}^{m \times d}$, there exists $\alpha > 0$ such that

$$\alpha |M| \leq \int_{S^{d-1}} |M\sigma| d\mu(\sigma) \quad \forall M \in \mathbb{R}^{m \times d}. \quad (4.6)$$

In particular

$$0 < \alpha \leq \varphi(\nu) \quad \forall \nu \in S^{d-1}.$$

Note, moreover, that

$$\varphi(\nu) \leq \mu(S^{d-1}) \quad \forall \nu \in S^{d-1}.$$

Remark 4.4. If ρ_ε is radial and (3.3) is satisfied, then $\text{supp } \mu = S^{d-1}$. Hence, the spanning condition in Theorem 4.1 is automatically satisfied, and (4.6) follows.

Motivated by the cone condition in [30, Section 1.3], we give a more general sufficient condition for $\text{span}(\text{supp } \mu) = \mathbb{R}^d$. Suppose that $\mu_\varepsilon \xrightarrow{*} \mu$ as $\varepsilon \rightarrow 0$. Assume that there exist linearly independent vectors $v_1, \dots, v_d \in \mathbb{R}^d$ and $\delta > 0$ such that

$$\limsup_{\varepsilon \rightarrow 0} \int_{C_\delta(v_i)} \rho_\varepsilon(z) dz > 0 \quad \text{for every } i = 1, \dots, d,$$

where, for $v \in \mathbb{R}^d \setminus \{0\}$,

$$C_\delta(v) := \left\{ w \in \mathbb{R}^d \setminus \{0\} : \left\langle \frac{v}{|v|}, \frac{w}{|w|} \right\rangle > 1 - \delta \right\}.$$

Set

$$E_\delta^i := C_\delta(v_i) \cap S^{d-1}, \quad i = 1, \dots, d.$$

We require δ to be sufficiently small that every choice $\sigma_i \in \overline{E_\delta^i}$, $i = 1, \dots, d$, yields linearly independent vectors. By weak* convergence,

$$\begin{aligned} \mu(\overline{E_\delta^i}) &\geq \limsup_{\varepsilon \rightarrow 0} \mu_\varepsilon(\overline{E_\delta^i}) \\ &\geq \limsup_{\varepsilon \rightarrow 0} \mu_\varepsilon(E_\delta^i) = \limsup_{\varepsilon \rightarrow 0} \int_{C_\delta(v_i)} \rho_\varepsilon(z) dz > 0. \end{aligned}$$

Thus, $\text{supp } \mu$ intersects each $\overline{E_\delta^i}$ and therefore contains a basis of $\mathbb{R}^d$. Consequently, the spanning condition holds, and (4.6) follows.

Remark 4.5. The previous theorem also includes the case $\rho_\varepsilon(z) = \frac{1}{\varepsilon^d} \rho\left(\frac{z}{\varepsilon}\right)$, where $\rho \in L^1(\mathbb{R}^d; [0, +\infty))$ and

$$\int_{\mathbb{R}^d} \rho(z) dz = 1.$$

Then

$$\varphi(\nu) = \int_{\mathbb{R}^d} \frac{\rho(z)}{|z|} |\langle \nu, z \rangle| dz,$$

Proof of Theorem 4.1.

Γ -liminf inequality. Let $u_\varepsilon \in L^1(A; Y)$ be such that $u_\varepsilon \rightarrow u$ in $L^1(A; \mathbb{R}^m)$ and

$$\liminf_{\varepsilon \rightarrow 0} G_\varepsilon(u_\varepsilon, A) < +\infty.$$

We have to prove that $u \in BV(A; Y)$ and

$$\liminf_{\varepsilon \rightarrow 0} G_\varepsilon(u_\varepsilon, A) \geq \int_{S_u \cap A} |u^+ - u^-| \int_{S^{d-1}} |\langle \nu_u, \sigma \rangle| d\mu(\sigma) d\mathcal{H}^{d-1}. \quad (4.7)$$

Let $\vartheta \in C_c^\infty(\mathbb{R}^d)$ be a nonnegative radial function such that

$$\int_{\mathbb{R}^d} \vartheta(x) dx = 1, \quad \text{supp } \vartheta \subset B_1,$$

and set

$$\vartheta_\delta(x) := \frac{1}{\delta^d} \vartheta\left(\frac{x}{\delta}\right), \quad (4.8)$$

for $\delta > 0$. We define

$$A_\delta := \{x \in A : \text{dist}(x, \partial A) > \delta\}$$

and

$$u_{\varepsilon, \delta}(x) := \vartheta_\delta * u_\varepsilon(x) = \int_A \vartheta_\delta(x - y) u_\varepsilon(y) dy, \quad x \in A_\delta.$$

For simplicity, let us identify $G_\varepsilon(\cdot, A)$ with its extension to the whole $L^1(A; \mathbb{R}^m)$. By applying Jensen's inequality and a change of variable, the following inequality holds

$$G_\varepsilon(u_\varepsilon, A) \geq G_\varepsilon(u_{\varepsilon, \delta}, A_\delta),$$

(see [30, Lemma 4]). Let $A' \Subset A$. For $\delta > 0$ sufficiently small, we have $A' \Subset A_\delta$. Then

$$G_\varepsilon(u_\varepsilon, A) \geq G_\varepsilon(u_{\varepsilon, \delta}, A').$$

Since $u_\varepsilon \rightarrow u$ in $L^1(A; \mathbb{R}^m)$, we have

$$u_{\varepsilon, \delta} \rightarrow u_\delta := \vartheta_\delta * u \quad \text{as } \varepsilon \rightarrow 0 \quad \text{in } C^2(\overline{A'}; \mathbb{R}^m).$$

Therefore, applying [30, Proposition 1 and Remark 7], we get

$$\lim_{\varepsilon \rightarrow 0} G_\varepsilon(u_{\varepsilon, \delta}, A') = \int_{A'} \int_{S^{d-1}} |\nabla u_\delta(x) \sigma| d\mu(\sigma) dx = \int_{A'} f_\mu(\nabla u_\delta) dx \quad (4.9)$$

where f_μ is defined as in (4.5). Consequently, by (4.6),

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} G_\varepsilon(u_\varepsilon, A) &\geq \liminf_{\varepsilon \rightarrow 0} G_\varepsilon(u_{\varepsilon, \delta}, A') \\ &= \int_{A'} \int_{S^{d-1}} |\nabla u_\delta(x) \sigma| d\mu(\sigma) dx \\ &\geq \alpha \int_{A'} |\nabla u_\delta(x)| dx. \end{aligned}$$

Since $u_\delta \rightarrow u$ in $L^1(A'; \mathbb{R}^m)$, the lower semicontinuity of the total variation yields

$$|Du|(A') \leq \liminf_{\delta \rightarrow 0} \int_{A'} |\nabla u_\delta(x)| dx \leq \frac{1}{\alpha} \liminf_{\varepsilon \rightarrow 0} G_\varepsilon(u_\varepsilon, A) < +\infty,$$

from which we infer that $u \in BV(A; Y)$ by the arbitrariness of $A' \Subset A$. Since f_μ is convex and positively one-homogeneous, the L^1 -lower-semicontinuous envelope of the functional

$$F_\mu(u) = \begin{cases} \int_{A'} f_\mu(\nabla u) dx, & u \in C^2(\overline{A'}; \mathbb{R}^m) \\ +\infty, & \text{otherwise,} \end{cases}$$

is given by

$$\overline{F}_\mu(u) = \begin{cases} \int_{A'} f_\mu\left(\frac{dDu}{d|Du|}\right) d|Du| = \int_{S^{d-1}} |Du \sigma|(A') d\mu(\sigma) & u \in BV(A'; \mathbb{R}^m) \\ +\infty, & \text{otherwise,} \end{cases}$$

where Du denotes the distributional derivative of u . Hence, by (4.9), passing to the limit as $\delta \rightarrow 0$, we obtain

$$\liminf_{\varepsilon \rightarrow 0} G_\varepsilon(u_\varepsilon, A) \geq \int_{S^{d-1}} |Du \sigma|(A') d\mu(\sigma). \quad (4.10)$$

Since $u \in BV(A; Y)$, we have $Du = (u^+ - u^-) \otimes \nu_u \mathcal{H}^{d-1} \llcorner S_u$. Therefore, for every $\sigma \in S^{d-1}$,

$$|Du \sigma|(A') = \int_{S_u \cap A'} |u^+ - u^-| |\langle \nu_u, \sigma \rangle| d\mathcal{H}^{d-1}.$$

Hence, by Fubini's theorem,

$$\int_{S^{d-1}} |Du \sigma|(A') d\mu(\sigma) = \int_{S_u \cap A'} |u^+ - u^-| \int_{S^{d-1}} |\langle \nu_u, \sigma \rangle| d\mu(\sigma) d\mathcal{H}^{d-1}.$$

Combining this with (4.10), we obtain

$$\liminf_{\varepsilon \rightarrow 0} G_\varepsilon(u_\varepsilon, A) \geq \int_{S_u \cap A'} |u^+ - u^-| \int_{S^{d-1}} |\langle \nu_u, \sigma \rangle| d\mu(\sigma) d\mathcal{H}^{d-1},$$

from which we get (4.7), by the arbitrariness of $A' \Subset A$.

Γ -limsup inequality and pointwise limit. It suffices to prove that for every $u \in BV(A; Y)$,

$$\limsup_{\varepsilon \rightarrow 0} G_\varepsilon(u, A) \leq \int_{S_u \cap A} |u^+ - u^-| \int_{S^{d-1}} |\langle \nu_u, \sigma \rangle| d\mu(\sigma) d\mathcal{H}^{d-1}. \quad (4.11)$$

Let then $u \in BV(A; Y)$ and consider an extension $\bar{u} \in BV(\mathbb{R}^d; \mathbb{R}^m)$ such that $|D\bar{u}|(\partial A) = 0$ (see [9, Proposition 3.21]). Consider the function $\Phi : \sigma \in S^{d-1} \mapsto |Du \sigma|(A)$. Since $u \in BV(A; Y)$,

$$\Phi(\sigma) = \int_{S_u \cap A} |u^+ - u^-| |\langle \nu_u, \sigma \rangle| d\mathcal{H}^{d-1}. \quad (4.12)$$

Note that Φ is continuous on S^{d-1} . For $q > 0$ set

$$W_q := \{y \in \mathbb{R}^d : \text{dist}(y, A) < q\}, \quad \omega(q) := |D\bar{u}|(\overline{W}_{2q} \setminus A).$$

The function ω is nondecreasing and, since $\bigcap_{q>0} (\overline{W}_{2q} \setminus A) = \overline{A} \setminus A = \partial A$ and $|D\bar{u}|(\mathbb{R}^d) < +\infty$,

$$\lim_{q \rightarrow 0^+} \omega(q) = |D\bar{u}|(\partial A) = 0. \quad (4.13)$$

By the change of variables $z = y - x$, we have

$$\begin{aligned} G_\varepsilon(u, A) &= \int_A \int_A \rho_\varepsilon(y - x) \frac{|u(y) - u(x)|}{|y - x|} dx dy \\ &= \int_{\mathbb{R}^d} \int_{A \cap (A - z)} \rho_\varepsilon(z) \frac{|u(x + z) - u(x)|}{|z|} dx dz. \end{aligned}$$

For $z \neq 0$ we define

$$g(z) := \int_{A \cap (A - z)} \frac{|u(x + z) - u(x)|}{|z|} dx,$$

and we set $g(0) = 0$. Then, we may rewrite the energy as

$$G_\varepsilon(u, A) = \int_{\mathbb{R}^d} \rho_\varepsilon(z) g(z) dz, \quad (4.14)$$

and we claim that

$$g(z) \leq \Phi\left(\frac{z}{|z|}\right) + \omega(|z|), \quad \forall z \neq 0. \quad (4.15)$$

Indeed, fix $z \neq 0$ and set $A_z := A \cap (A - z)$. For every $x \in A_z$ and $t \in [0, 1]$ it holds

$$\text{dist}(x + tz, A) \leq t|z| < 2|z|,$$

that is

$$A_z + tz \subset W_{2|z|}. \quad (4.16)$$

Since $\bar{u} = u$ in A and $x, x + z \in A$ for every $x \in A_z$, it holds

$$|z|g(z) = \int_{A_z} |\bar{u}(x + z) - \bar{u}(x)| dx \leq |z| \left| D\bar{u} \frac{z}{|z|} \right|(\bar{W}_{2|z|}). \quad (4.17)$$

Although this estimate is standard, we include a proof for completeness. Set $\bar{u}_\delta := \vartheta_\delta * \bar{u}$, where ϑ_δ are the mollifiers introduced in (4.8). By (4.16), the change of variable $y = x + tz$ and Fubini's theorem it holds

$$\int_{A_z} |\bar{u}_\delta(x + z) - \bar{u}_\delta(x)| dx \leq |z| \int_0^1 \int_{A_z} \left| \nabla \bar{u}_\delta(x + tz) \frac{z}{|z|} \right| dx dt \leq |z| \int_{W_{2|z|}} \left| \nabla \bar{u}_\delta(y) \frac{z}{|z|} \right| dy. \quad (4.18)$$

Since $\nabla \bar{u}_\delta = \vartheta_\delta * D\bar{u}$ and $\vartheta_\delta \geq 0$, we have

$$\left| \nabla \bar{u}_\delta(y) \frac{z}{|z|} \right| \leq \int_{\mathbb{R}^d} \vartheta_\delta(y - w) d \left| D\bar{u} \frac{z}{|z|} \right|(w)$$

for every y ; integrating over $W_{2|z|}$ and using Fubini's theorem,

$$\int_{W_{2|z|}} \left| \nabla \bar{u}_\delta \frac{z}{|z|} \right| dy \leq \int_{\mathbb{R}^d} \left(\int_{W_{2|z|}} \vartheta_\delta(y - w) dy \right) d \left| D\bar{u} \frac{z}{|z|} \right|(w) \leq \left| D\bar{u} \frac{z}{|z|} \right|(W_{2|z|} + B_\delta). \quad (4.19)$$

Hence, combining (4.18) and (4.19)

$$\int_{A_z} |\bar{u}_\delta(x + z) - \bar{u}_\delta(x)| dx \leq |z| \left| D\bar{u} \frac{z}{|z|} \right|(W_{2|z|} + B_\delta).$$

We now let $\delta \rightarrow 0$. Since $\bar{u} \in L^1(\mathbb{R}^d; \mathbb{R}^m)$, we have $\bar{u}_\delta \rightarrow \bar{u}$ in $L^1(\mathbb{R}^d; \mathbb{R}^m)$, and therefore the left-hand side converges to

$$\int_{A_z} |\bar{u}(x + z) - \bar{u}(x)| dx.$$

On the other hand, the sets $W_{2|z|} + B_\delta$ decrease to

$$\bigcap_{\delta > 0} (W_{2|z|} + B_\delta) = \bar{W}_{2|z|}$$

and $|D\bar{u}|(\mathbb{R}^d) < +\infty$, so that

$$\left| D\bar{u} \frac{z}{|z|} \right|(W_{2|z|} + B_\delta) \rightarrow \left| D\bar{u} \frac{z}{|z|} \right|(\bar{W}_{2|z|}).$$

This proves (4.17). Moreover, since $D\bar{u}|_A = Du$,

$$\left| D\bar{u} \frac{z}{|z|} \right| (\overline{W}_{2|z|}) = \left| Du \frac{z}{|z|} \right| (A) + \left| D\bar{u} \frac{z}{|z|} \right| (\overline{W}_{2|z|} \setminus A) \leq \Phi \left(\frac{z}{|z|} \right) + \omega(|z|). \quad (4.20)$$

Combining (4.17) and (4.20) and dividing by $|z|$, we get (4.15). We also have

$$g(z) \leq \frac{C|A|}{|z|} \quad \text{for every } z \neq 0. \quad (4.21)$$

By Remark 4.2 there exist $C > 0$ and $\varepsilon_0 > 0$ such that

$$\int_{\mathbb{R}^d} \rho_\varepsilon(z) dz \leq C,$$

for every $\varepsilon < \varepsilon_0$. Fix $\eta > 0$. By (4.14), (4.15), the monotonicity of ω and (4.21), for $\varepsilon < \varepsilon_0$ we get

$$G_\varepsilon(u, A) \leq \int_{\mathbb{R}^d} \rho_\varepsilon(z) \Phi \left(\frac{z}{|z|} \right) dz + C \omega(\eta) + \frac{C|A|}{\eta} \int_{|z| > \eta} \rho_\varepsilon(z) dz. \quad (4.22)$$

Passing to polar coordinates and using (4.1),

$$\int_{\mathbb{R}^d} \rho_\varepsilon(z) \Phi \left(\frac{z}{|z|} \right) dz = \int_{S^{d-1}} \Phi(\sigma) \int_0^{+\infty} \rho_\varepsilon(t\sigma) t^{d-1} dt d\mathcal{H}^{d-1}(\sigma) = \int_{S^{d-1}} \Phi(\sigma) d\mu_\varepsilon. \quad (4.23)$$

Since Φ is continuous and $\mu_\varepsilon \xrightarrow{*} \mu$, combining (3.4), (4.22), (4.23) we get

$$\limsup_{\varepsilon \rightarrow 0} G_\varepsilon(u, A) \leq \int_{S^{d-1}} \Phi(\sigma) d\mu + C \omega(\eta).$$

Letting $\eta \rightarrow 0^+$, by (4.12), (4.13), together with Fubini's theorem,

$$\limsup_{\varepsilon \rightarrow 0} G_\varepsilon(u, A) \leq \int_{S^{d-1}} \Phi(\sigma) d\mu = \int_{S_u \cap A} |u^+ - u^-| \int_{S^{d-1}} |\langle \nu_u, \sigma \rangle| d\mu(\sigma) d\mathcal{H}^{d-1},$$

which is (4.11). □

By Theorem 4.1 and Remarks 4.3 and 4.4, we immediately deduce the following corollary.

Corollary 4.6. *Let $G_\varepsilon(u, A)$ be defined by (3.6) with $\rho_\varepsilon : \mathbb{R}^d \rightarrow [0, +\infty)$ satisfying (3.2), (3.3) and (3.4). If $\Gamma\text{-}\liminf_{\varepsilon \rightarrow 0} G_\varepsilon(u, A)$ is finite, then $u \in BV(A; Y)$. Moreover, there exist two positive constants α and β such that for every $A \in \mathcal{A}^{reg}(\Omega)$ and $u \in BV(A; Y)$*

$$\alpha \mathcal{H}^{d-1}(S_u \cap A) \leq \Gamma(L^1)\text{-}\liminf_{\varepsilon \rightarrow 0} G_\varepsilon(u, A) \leq \Gamma(L^1)\text{-}\limsup_{\varepsilon \rightarrow 0} G_\varepsilon(u, A) \leq \beta \mathcal{H}^{d-1}(S_u \cap A).$$

We conclude this section with four explicit examples that show that the limiting surface energy density in Theorem 4.1 may be either isotropic or anisotropic. The first two examples involve radial kernels, which yield isotropic densities, while the last two show how anisotropic densities can arise.

Example 4.7 (Scaled fractional perimeters). We first recall a notion of fractional perimeter appropriately defined in order to deal with nonlocal contributions only inside the target set Ω .

Let $E \subset \Omega$ be a measurable set and $0 < s < 1$. The fractional s -perimeter of E inside Ω is denoted by $P_s(E, \Omega)$ and it is defined as follows:

$$P_s(E, \Omega) := \int_E \int_{E^c \cap \Omega} \frac{1}{|y - x|^{d+s}} dx dy.$$

Note that $P_s(E, \Omega)$ is half of the fractional Sobolev space seminorm $|\chi_E|_{W^{s,1}(\Omega)}$, where χ_E denotes the characteristic function of E . We recall that if E is a set of finite perimeter in Ω , then $P_s(E, \Omega)$ is also finite, since in general $BV(\Omega) \subset W^{s,1}(\Omega)$ for any $s \in (0, 1)$.

The asymptotic analysis as $s \rightarrow 1^-$ of $(1-s)P_s(E, \Omega)$ has been performed in [8], where the authors prove that the Γ -limit is the functional $\omega_{d-1}P(E, \Omega)$, with $P(E, \Omega)$ denoting the classical De Giorgi's perimeter. This family of scaled perimeters can be recast in our notation and the result in [8] is covered by Theorem 4.1. Indeed, let $s = 1 - \varepsilon$ and set

$$G_\varepsilon(u) = \varepsilon P_{1-\varepsilon}(\{u = 1\}, \Omega) = \frac{\varepsilon}{2} \int_\Omega \int_\Omega \frac{|u(y) - u(x)|}{|y - x|^{d+1-\varepsilon}} dx dy.$$

Note that, setting $D = \text{diam}(\Omega)$, $G_\varepsilon(u)$ can be written as

$$G_\varepsilon(u) = \frac{\varepsilon}{2} \int_\Omega \int_\Omega \frac{|u(y) - u(x)|}{|y - x|^{d+1-\varepsilon}} \chi_{[0,D]}(|y - x|) dx dy.$$

which is of the form (3.6) with

$$\rho_\varepsilon(z) = \frac{\varepsilon}{2} \frac{1}{|z|^{d-\varepsilon}} \chi_{[0,D]}(|z|).$$

Note that, for every $\eta > 0$,

$$\int_{|z|>\eta} \rho_\varepsilon(z) dz = \frac{\varepsilon \sigma_{d-1}}{2} \int_\eta^D t^{\varepsilon-1} dt = \frac{\sigma_{d-1}}{2} t^\varepsilon \Big|_\eta^D = \frac{\sigma_{d-1}}{2} (D^\varepsilon - \eta^\varepsilon) \rightarrow 0,$$

hence (3.4) is satisfied. Moreover,

$$\mu_\varepsilon(S^{d-1}) = \int_{\mathbb{R}^d} \rho_\varepsilon(z) dz = \frac{\sigma_{d-1}}{2} t^\varepsilon \Big|_0^D = \frac{\sigma_{d-1}}{2} D^\varepsilon.$$

Hence,

$$\mu_\varepsilon = \frac{D^\varepsilon}{2} \mathcal{H}^{d-1} \llcorner_{S^{d-1}} \stackrel{*}{\rightharpoonup} \mu = \frac{1}{2} \mathcal{H}^{d-1} \llcorner_{S^{d-1}}$$

and, by (4.3),

$$\varphi(\nu) = \frac{1}{2} \int_{S^{d-1}} |\langle \nu, \sigma \rangle| d\mathcal{H}^{d-1}(\sigma) = \omega_{d-1}.$$

Example 4.8 (Scaled volume of an ε -neighborhood of the interface). Set

$$G_\varepsilon(u) = \frac{1}{\varepsilon^{d+1}} \int_{\{(x,y) \in \Omega \times \Omega: |y-x| \leq \varepsilon\}} |u(y) - u(x)| dx dy.$$

Note that G_ε is of the form (3.6) with

$$\rho_\varepsilon(z) = \frac{|z|}{\varepsilon^{d+1}} \chi_{[0,\varepsilon]}(|z|).$$

In this case (3.4) is trivially satisfied and

$$\mu_\varepsilon(S^{d-1}) = \int_{\mathbb{R}^d} \rho_\varepsilon(z) dz = \frac{\sigma_{d-1}}{\varepsilon^{d+1}} \int_0^\varepsilon t^d dt = \frac{\sigma_{d-1}}{d+1}$$

Hence,

$$\mu_\varepsilon = \frac{1}{d+1} \mathcal{H}^{d-1}|_{S^{d-1}} = \mu$$

and, by (4.3),

$$\varphi(\nu) = \frac{1}{d+1} \int_{S^{d-1}} |\langle \nu, \sigma \rangle| d\mathcal{H}^{d-1}(\sigma) = \frac{2\omega_{d-1}}{d+1}.$$

Example 4.9 (Anisotropic finite-range interactions). Let $d \geq 2$, $C > 0$, and consider $\rho_\varepsilon(z) = \frac{1}{\varepsilon^d} \chi_{B_\varepsilon}(z) a\left(\frac{z}{|z|}\right)$, where B_ε is the ball centered at the origin with radius ε and

$$a(\sigma) := 1 + C|\langle \sigma, e_1 \rangle|.$$

In this case

$$\mu_\varepsilon = \mu = \frac{1}{d} a(\cdot) \mathcal{H}^{d-1}|_{S^{d-1}},$$

thus

$$\varphi(\nu) = \frac{1}{d} \int_{S^{d-1}} (1 + C|\langle \sigma, e_1 \rangle|) |\langle \sigma, \nu \rangle| d\mathcal{H}^{d-1}(\sigma).$$

A straightforward computation shows that $\varphi(e_1) \neq \varphi(e_2)$, and hence the resulting surface energy is anisotropic.

Example 4.10 (Anisotropic fractional perimeters). For $s \in (0, 1)$, let $k_s: \mathbb{R}^d \setminus \{0\} \rightarrow (0, +\infty)$ be a measurable function satisfying, for some constant $c \geq 1$ independent of s ,

$$k_s(-z) = k_s(z) \quad \text{for every } z \neq 0, \quad (4.24)$$

$$k_s(\lambda z) = \lambda^{-d-s} k_s(z) \quad \text{for every } \lambda > 0, z \neq 0, \quad (4.25)$$

$$\frac{1}{c} \frac{1}{|z|^{d+s}} \leq k_s(z) \leq \frac{c}{|z|^{d+s}} \quad \text{for every } z \neq 0, \quad (4.26)$$

and assume that $k_s \rightarrow k$ pointwise in $\mathbb{R}^d \setminus \{0\}$ as $s \rightarrow 1^-$, for some measurable function $k: \mathbb{R}^d \setminus \{0\} \rightarrow (0, +\infty)$. For $u = \chi_E \in L^1(\Omega; \{0, 1\})$, the anisotropic s -perimeter of E associated with k_s , restricted to the interactions inside Ω , is

$$P_{k_s}(u) := \int_{E \cap \Omega} \int_{E^c \cap \Omega} k_s(y-x) dx dy = \frac{1}{2} \int_{\Omega} \int_{\Omega} k_s(y-x) |u(y) - u(x)| dx dy,$$

where the equality follows from (4.24). These energies have been studied in [29, 25]. In this example we show that they fall within our framework.

Set $\varepsilon := 1 - s$ and $D := \text{diam}(\Omega)$. Since $|y - x| \leq D$ for every $x, y \in \Omega$, the rescaled energies $\varepsilon P_{k_{1-\varepsilon}}$ are of the form (3.1), i.e. $\varepsilon P_{k_{1-\varepsilon}} = G_\varepsilon$, with

$$\rho_\varepsilon(z) = \frac{\varepsilon}{2} k_{1-\varepsilon}(z) |z| \chi_{[0, D]}(|z|),$$

which is not radial in general. By (4.25), for every $\sigma \in S^{d-1}$ we have

$$a_\varepsilon(\sigma) = \int_0^{+\infty} \rho_\varepsilon(t\sigma) t^{d-1} dt = \frac{\varepsilon}{2} k_{1-\varepsilon}(\sigma) \int_0^D t^{\varepsilon-1} dt = \frac{D^\varepsilon}{2} k_{1-\varepsilon}(\sigma),$$

so that, by (4.1),

$$\mu_\varepsilon = \frac{D^\varepsilon}{2} k_{1-\varepsilon}(\cdot) \mathcal{H}^{d-1} \llcorner_{S^{d-1}}.$$

In the same way, for every $\eta \in (0, D)$, by (4.26) we get

$$\int_{\{|z|>\eta\}} \rho_\varepsilon(z) dz = \frac{D^\varepsilon - \eta^\varepsilon}{2} \int_{S^{d-1}} k_{1-\varepsilon}(\sigma) d\mathcal{H}^{d-1}(\sigma) \leq \frac{c\sigma_{d-1}}{2} (D^\varepsilon - \eta^\varepsilon) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0,$$

hence (3.4) is satisfied. Moreover, by (4.26) we have $1/c \leq k_{1-\varepsilon} \leq c$ on S^{d-1} for every ε ; since $k_{1-\varepsilon} \rightarrow k$ pointwise and $D^\varepsilon \rightarrow 1$, the Dominated Convergence Theorem yields

$$\mu_\varepsilon \xrightarrow{*} \mu = \frac{1}{2} k \mathcal{H}^{d-1} \llcorner_{S^{d-1}} \quad \text{as } \varepsilon \rightarrow 0.$$

Since $k \geq 1/c$ on S^{d-1} , $\text{supp } \mu = S^{d-1}$ and the spanning condition of Theorem 4.1 is satisfied.

Applying Theorem 4.1 with $A = \Omega$, and computing (4.3) with the measure μ above, we conclude that the Γ -limit of $\varepsilon P_{k_{1-\varepsilon}}$ is of the form (4.2) with

$$\varphi(\nu) = \frac{1}{2} \int_{S^{d-1}} |\langle \nu, \sigma \rangle| k(\sigma) d\mathcal{H}^{d-1}(\sigma). \quad (4.27)$$

Note that we recover the Γ -convergence result stated in [25, Theorem 1.3]. Indeed, in that theorem the energy density of the limit energy is given by

$$\frac{d+1}{2} \int_{B_k} |\langle x, \nu \rangle| dx,$$

with $B_k := \{x \in \mathbb{R}^d \setminus \{0\} : k(x) > 1\} \cup \{0\}$. By (4.25) the limit kernel satisfies $k(\lambda z) = \lambda^{-d-1} k(z)$, so that

$$B_k = \{t\sigma : \sigma \in S^{d-1}, 0 \leq t < k(\sigma)^{\frac{1}{d+1}}\};$$

hence, passing to polar coordinates in the previous formula, we get (4.27).

Finally, for $k_s(z) = |z|^{-d-s}$ we have $k \equiv 1$ on S^{d-1} , and we recover Example 4.7.

5 Proof of Theorem 3.1

The next two propositions establish that, under hypotheses (H1) and (H2), for every $A \in \mathcal{A}^{\text{reg}}(\Omega)$, $F'(u, A)$ and $F''(u, A)$ are finite only on $BV(A; Y)$ and satisfy hypothesis (iv) of Theorem 2.4. We omit the proofs, since they are a straightforward consequence of (3.7) and Corollary 4.6.

Proposition 5.1. *Let $K_\varepsilon(x, y, u, v)$ satisfy (H1) and let $A \in \mathcal{A}(\Omega)$. If $u \in L^1(\Omega; Y)$ is such that $F'(u, A)$ is finite, then $u \in BV(A; Y)$ and*

$$F'(u, A) \geq c_1 \mathcal{H}^{d-1}(S_u \cap A),$$

for some positive constant c_1 independent of u and A .

Proposition 5.2. *Let $K_\varepsilon(x, y, u, v)$ satisfy (H2) and let $A \in \mathcal{A}^{\text{reg}}(\Omega)$. Then, for every $u \in BV(A; Y)$, it holds*

$$F''(u, A) \leq c_2 \mathcal{H}^{d-1}(S_u \cap A),$$

for some positive constant c_2 independent of u and A .

In the following propositions, we show that $F''(\cdot, \cdot)$ satisfies hypotheses (i) and (ii) of Theorem 2.4.

Proposition 5.3. *Let $K_\varepsilon(x, y, u, v)$ satisfy (H1)-(H2) and let $A, B, A' \in \mathcal{A}^{\text{reg}}(\Omega)$ be given with $A' \Subset A$. Then for any $u \in BV(\Omega; Y)$*

$$F''(u, A' \cup B) \leq F''(u, A) + F''(u, B).$$

Proof. Without loss of generality we may assume $F''(u, A)$ and $F''(u, B)$ are finite. Let $u_\varepsilon, v_\varepsilon$ be such that $u_\varepsilon \rightarrow u$, $v_\varepsilon \rightarrow u$ in $L^1(\Omega; \mathbb{R}^m)$ and

$$\limsup_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon, A) = F''(u, A), \quad \limsup_{\varepsilon \rightarrow 0} F_\varepsilon(v_\varepsilon, B) = F''(u, B).$$

Recall that the distance function $\text{dist}(x, A')$ is 1-Lipschitz. For $t \in [0, \bar{d}]$, we define

$$A^t := \{x \in A : \text{dist}(x, A') \leq t\},$$

and the family of functions

$$\hat{w}_{\varepsilon, t} = \chi_{A^t} u_\varepsilon + (1 - \chi_{A^t}) v_\varepsilon.$$

Note that

$$|D_z^1 \hat{w}_{\varepsilon, t}(x)| \leq |D_z^1 u_\varepsilon(x)| + |D_z^1 v_\varepsilon(x)| + |D_z^1 \chi_{A^t}(x)| |u_\varepsilon(x) - v_\varepsilon(x)|, \quad (5.1)$$

where for a generic function $w : \Omega \rightarrow \mathbb{R}^m$ and $z \in \mathbb{R}^d \setminus \{0\}$, we adopt the notation

$$D_z^1 w(x) := \frac{w(x+z) - w(x)}{|z|}.$$

Let $\bar{d} := \text{dist}(A', A^c)$, $\eta \in (0, \bar{d})$ and set $N_\eta := \left\lfloor \frac{\bar{d}}{\eta} \right\rfloor$. Given $\varphi : \mathbb{R} \rightarrow [0, 1]$ defined by

$$\varphi(t) := \begin{cases} 0 & \text{if } t \leq 0 \\ t & \text{if } 0 < t < 1 \\ 1 & \text{if } t \geq 1 \end{cases}$$

and $k \in \{1, \dots, N_\eta - 1\}$, let $\varphi_\eta^k(x)$ be the cut-off function between $A^{k\eta}$ and $A^{(k+1)\eta}$ defined as

$$\varphi_\eta^k(x) := \varphi\left(\frac{\text{dist}(x, A')}{\eta} - k\right).$$

Note that $\varphi_\eta^k \equiv 0$ on $A^{k\eta}$, $\varphi_\eta^k \equiv 1$ on $A \setminus A^{(k+1)\eta}$ and

$$\|\nabla \varphi_\eta^k\|_\infty \leq \frac{\|\nabla \text{dist}(x, A')\|_\infty}{\eta} = \frac{1}{\eta},$$

which yields that for every $z \in \mathbb{R}^d \setminus \{0\}$

$$|D_z^1 \varphi_\eta^k| \leq \|\nabla \varphi_\eta^k\|_\infty \leq \frac{1}{\eta}. \quad (5.2)$$

Moreover, by the definition of $\varphi_\eta^k(x)$, for every $t \in (0, 1)$ we have

$$\begin{aligned} A^{\eta(t+k)} &= \{x \in A : \text{dist}(x, A') \leq \eta(t+k)\} = \{x \in A : \varphi_\eta^k(x) \leq t\}, \\ \partial A^{\eta(t+k)} &= \{x \in A : \text{dist}(x, A') = \eta(t+k)\} = \{x \in A : \varphi_\eta^k(x) = t\}. \end{aligned}$$

By the Coarea formula, the fact that $|\nabla \text{dist}(x, A')| = 1$ and the change of variable $t = \frac{s-k\eta}{\eta}$, we get

$$\begin{aligned} |A^{(k+1)\eta} \setminus A^{k\eta}| &= \int_{\{k\eta \leq \text{dist}(x, A') \leq (k+1)\eta\}} |\nabla \text{dist}(x, A')| dx \\ &= \int_{k\eta}^{(k+1)\eta} \mathcal{H}^{d-1}(\partial A^s) ds \\ &= \eta \int_0^1 \mathcal{H}^{d-1}(\partial A^{\eta(t+k)}) dt. \end{aligned} \tag{5.3}$$

We set $g_\eta^k(t) := \mathcal{H}^{d-1}(\partial A^{\eta(t+k)})$ and

$$V_\eta^k := \left\{ t \in (0, 1) : g_\eta^k(t) > 2 \int_0^1 g_\eta^k(s) ds \right\}, \quad E_\eta^k := (0, 1) \setminus V_\eta^k.$$

We note that

$$\int_0^1 g_\eta^k(s) ds \geq \int_{V_\eta^k} g_\eta^k(s) ds \geq |V_\eta^k| 2 \int_0^1 g_\eta^k(s) ds.$$

Hence, we have that $|V_\eta^k| \leq 1/2$ and $|E_\eta^k| \geq 1/2$. By (5.3), $\forall t \in E_\eta^k$ we get

$$\mathcal{H}^{d-1}(\partial A^{\eta(t+k)}) \leq \frac{2}{\eta} |A^{(k+1)\eta} \setminus A^{k\eta}|. \tag{5.4}$$

We define

$$C_\eta^k := (A^{(k+1)\eta} \setminus A^{k\eta}) \cap B \quad \text{and} \quad S_\eta^k := (C_\eta^k + B_\eta) \cap B = \{x \in B : \text{dist}(x, C_\eta^k) < \eta\},$$

where B_η is the open ball centered at the origin with radius $\eta > 0$. Note that

$$\int_0^1 |D_z^1 \chi_{\{\varphi_\eta^k \leq t\}}(x)| dt = |D_z^1 \varphi_\eta^k(x)|.$$

Hence we have

$$\int_{|z| \leq \eta} \rho_\varepsilon(z) \int_{S_\eta^k} |D_z^1 \varphi_\eta^k(x)| |u_\varepsilon(x) - v_\varepsilon(x)| dx dz = \int_0^1 h_{\varepsilon, \eta}^k(t) dt, \tag{5.5}$$

where

$$h_{\varepsilon, \eta}^k(t) := \int_{|z| \leq \eta} \rho_\varepsilon(z) \int_{S_\eta^k} |D_z^1 \chi_{\{\varphi_\eta^k \leq t\}}(x)| |u_\varepsilon(x) - v_\varepsilon(x)| dx dz.$$

Since $h_{\varepsilon, \eta}^k \geq 0$ and $|E_\eta^k| \geq 1/2$, there exists $t_{\varepsilon, \eta}^k \in E_\eta^k$ such that

$$h_{\varepsilon, \eta}^k(t_{\varepsilon, \eta}^k) \leq \frac{1}{|E_\eta^k|} \int_{E_\eta^k} h_{\varepsilon, \eta}^k(t) dt \leq 2 \int_0^1 h_{\varepsilon, \eta}^k(t) dt. \tag{5.6}$$

Setting

$$\overline{A}_{\varepsilon,\eta}^k := A^{\eta(t_{\varepsilon,\eta}^k + k)} = \{x \in A : \varphi_\eta^k(x) \leq t_{\varepsilon,\eta}^k\},$$

by (5.5) and (5.6) we obtain

$$\begin{aligned} \int_{|z| \leq \eta} \rho_\varepsilon(z) \int_{S_\eta^k} |D_z^1 \chi_{\overline{A}_{\varepsilon,\eta}^k}(x)| |u_\varepsilon(x) - v_\varepsilon(x)| dx dz &\leq \\ &\leq 2 \int_{|z| \leq \eta} \rho_\varepsilon(z) \int_{S_\eta^k} |D_z^1 \varphi_\eta^k(x)| |u_\varepsilon(x) - v_\varepsilon(x)| dx dz. \end{aligned} \quad (5.7)$$

Moreover, since $t_{\varepsilon,\eta}^k \in E_\eta^k$, by (5.4) we also have

$$\mathcal{H}^{d-1}(\partial \overline{A}_{\varepsilon,\eta}^k) \leq \frac{2}{\eta} |A^{(k+1)\eta} \setminus A^{k\eta}|. \quad (5.8)$$

Finally, we define the family of functions $w_{\varepsilon,\eta}^k$ as

$$w_{\varepsilon,\eta}^k := \hat{w}_{\varepsilon,\eta}(t_{\varepsilon,\eta}^k + k).$$

Not that for every fixed $\eta > 0$

$$w_{\varepsilon,\eta}^k \rightarrow u \quad \text{as } \varepsilon \rightarrow 0 \quad \text{in } L^1(\Omega; \mathbb{R}^m)$$

uniformly with respect to the choice of k .

We define the set of interactions that cross $\partial \overline{A}_{\varepsilon,\eta}^k$ as follows

$$S_{z,\eta}^k := \left\{ x \in (A' \cup B) \cap ((A' \cup B) - z) : \chi_{\overline{A}_{\varepsilon,\eta}^k}(x) \neq \chi_{\overline{A}_{\varepsilon,\eta}^k}(x+z) \right\}.$$

On the complement of $S_{z,\eta}^k$ the function $w_{\varepsilon,\eta}^k$ coincides either with u_ε or with v_ε . Since $A' \subset \overline{A}_{\varepsilon,\eta}^k \subset A$, the corresponding energetic contribution is bounded by $F_\varepsilon(u_\varepsilon, A) + F_\varepsilon(v_\varepsilon, B)$. Hence,

$$\begin{aligned} F_\varepsilon(w_{\varepsilon,\eta}^k, A' \cup B) &\leq F_\varepsilon(u_\varepsilon, A) + F_\varepsilon(v_\varepsilon, B) \\ &\quad + \int_{\mathbb{R}^d} \int_{S_{z,\eta}^k} K_\varepsilon(x, x+z, w_{\varepsilon,\eta}^k(x), w_{\varepsilon,\eta}^k(x+z)) |D_z^1 w_{\varepsilon,\eta}^k(x)| dx dz. \end{aligned} \quad (5.9)$$

It is straightforward to check that, for every $z \in \mathbb{R}^d$ such that $|z| < \eta$,

$$S_{z,\eta}^k \subset S_\eta^k \cap (S_\eta^k - z). \quad (5.10)$$

Since $\overline{A}_{\varepsilon,\eta}^k$ has finite perimeter, we have

$$|S_{z,\eta}^k| \leq \int_{\mathbb{R}^d} |\chi_{\overline{A}_{\varepsilon,\eta}^k}(x+z) - \chi_{\overline{A}_{\varepsilon,\eta}^k}(x)| dx \leq |z| \mathcal{H}^{d-1}(\partial \overline{A}_{\varepsilon,\eta}^k). \quad (5.11)$$

Then,

$$\begin{aligned} &\int_{\mathbb{R}^d} \int_{S_{z,\eta}^k} K_\varepsilon(x, x+z, w_{\varepsilon,\eta}^k(x), w_{\varepsilon,\eta}^k(x+z)) |D_z^1 w_{\varepsilon,\eta}^k(x)| dx dz \\ &= \int_{|z| \leq \eta} \int_{S_{z,\eta}^k} K_\varepsilon(x, x+z, w_{\varepsilon,\eta}^k(x), w_{\varepsilon,\eta}^k(x+z)) |D_z^1 w_{\varepsilon,\eta}^k(x)| dx dz \\ &\quad + \int_{|z| > \eta} \int_{S_{z,\eta}^k} K_\varepsilon(x, x+z, w_{\varepsilon,\eta}^k(x), w_{\varepsilon,\eta}^k(x+z)) |D_z^1 w_{\varepsilon,\eta}^k(x)| dx dz \\ &= I_\varepsilon^{|z| \leq \eta} + I_\varepsilon^{|z| > \eta} \end{aligned}$$

By (H1)-(H2), (5.2), (5.7), (5.1), (5.10) and the uniform bound (3.3) we estimate $I_\varepsilon^{|z|\leq\eta}$ as follows

$$\begin{aligned}
I_\varepsilon^{|z|\leq\eta} &\leq \int_{|z|\leq\eta} \int_{S_\eta^k \cap (S_\eta^k - z)} K_\varepsilon(x, x+z, w_{\varepsilon,\eta}^k(x), w_{\varepsilon,\eta}^k(x+z)) |D_z^1 w_{\varepsilon,\eta}^k(x)| dx dz \\
&\leq \frac{C_2}{C_1} [F_\varepsilon(u_\varepsilon, S_\eta^k) + F_\varepsilon(v_\varepsilon, S_\eta^k)] + C_2 \int_{|z|\leq\eta} \rho_\varepsilon(z) \int_{S_\eta^k} |D_z^1 \chi_{\overline{A}_{\varepsilon,\eta}^k}(x)| |u_\varepsilon(x) - v_\varepsilon(x)| dx dz \\
&\leq \frac{C_2}{C_1} [F_\varepsilon(u_\varepsilon, S_\eta^k) + F_\varepsilon(v_\varepsilon, S_\eta^k)] + 2C_2 \int_{|z|\leq\eta} \rho_\varepsilon(z) \int_{S_\eta^k} |D_z^1 \varphi_\eta^k(x)| |u_\varepsilon(x) - v_\varepsilon(x)| dx dz \\
&\leq \frac{C_2}{C_1} [F_\varepsilon(u_\varepsilon, S_\eta^k) + F_\varepsilon(v_\varepsilon, S_\eta^k)] + \frac{C}{\eta} \int_{|z|\leq\eta} \rho_\varepsilon(z) \int_{S_\eta^k} |u_\varepsilon(x) - v_\varepsilon(x)| dx dz \\
&\leq \frac{C_2}{C_1} [F_\varepsilon(u_\varepsilon, S_\eta^k) + F_\varepsilon(v_\varepsilon, S_\eta^k)] + \frac{C}{\eta} \int_{S_\eta^k} |u_\varepsilon(x) - v_\varepsilon(x)| dx.
\end{aligned} \tag{5.12}$$

On the other hand, by (H2), (5.8), and (5.11) we get

$$\begin{aligned}
I_\varepsilon^{|z|>\eta} &= \int_{|z|>\eta} \int_{S_{z,\eta}^k} K_\varepsilon(x, x+z, w_{\varepsilon,\eta}^k(x), w_{\varepsilon,\eta}^k(x+z)) |D_z^1 w_{\varepsilon,\eta}^k(x)| dx dz \\
&\leq C \int_{|z|>\eta} \rho_\varepsilon(z) \frac{1}{|z|} \mathcal{H}^{d-1}(\partial \overline{A}_{\varepsilon,\eta}^k) |z| dz \\
&\leq \frac{C}{\eta} |A^{(k+1)\eta} \setminus A^{k\eta}| \int_{|z|>\eta} \rho_\varepsilon(z) dz,
\end{aligned} \tag{5.13}$$

where we also used the boundedness of $w_{\varepsilon,\eta}^k$. Now we note that

$$\bigcup_{k=0}^{N_\eta-1} (A^{(k+1)\eta} \setminus A^{k\eta}) \subset A \setminus A' \tag{5.14}$$

and

$$\bigcup_{k=1}^{N_\eta-1} S_\eta^k \subset A \cap B. \tag{5.15}$$

Since, for every $x \in A \cap B$ we have $\#\{k : x \in S_\eta^k\} \leq 3$, by (5.9), (5.12), (5.13), (5.14) and (5.15) we get that for η sufficiently small

$$\begin{aligned}
I_{\varepsilon,\eta} &:= \frac{1}{N_\eta-1} \sum_{k=1}^{N_\eta-1} F_\varepsilon(w_{\varepsilon,\eta}^k, A' \cup B) \\
&\leq F_\varepsilon(u_\varepsilon, A) + F_\varepsilon(v_\varepsilon, B) + \frac{3}{N_\eta-1} \left[\frac{C_2}{C_1} (F_\varepsilon(u_\varepsilon, A \cap B) + F_\varepsilon(v_\varepsilon, A \cap B)) \right. \\
&\quad \left. + \frac{C}{\eta} \int_{A \setminus A'} |v_\varepsilon(x) - u_\varepsilon(x)| dx + \frac{C}{\eta} |A \setminus A'| \int_{|z|>\eta} \rho_\varepsilon(z) dz \right] \\
&\leq F_\varepsilon(u_\varepsilon, A) + F_\varepsilon(v_\varepsilon, B) + C\eta + C\|v_\varepsilon - u_\varepsilon\|_{L^1(A \setminus A')} + C|A \setminus A'| \int_{|z|>\eta} \rho_\varepsilon(z) dz.
\end{aligned} \tag{5.16}$$

Indeed, for η sufficiently small, we have

$$N_\eta - 1 \geq \frac{\bar{d}}{2\eta},$$

and the energies $F_\varepsilon(u_\varepsilon, A \cap B)$ and $F_\varepsilon(v_\varepsilon, A \cap B)$ are uniformly bounded. Then, there exists $k_\eta^\varepsilon \in \{1, \dots, N_\eta - 1\}$ such that

$$F_\varepsilon(w_{\varepsilon,\eta}^{k_\eta^\varepsilon}, A' \cup B) \leq I_{\varepsilon,\eta}.$$

Setting $w_{\varepsilon,\eta} := w_{\varepsilon,\eta}^{k_\eta^\varepsilon}$, for every fixed $\eta > 0$, we have $w_{\varepsilon,\eta} \rightarrow u$ in L^1 as $\varepsilon \rightarrow 0$. Then, passing to the limit in (5.16) and using the decay property (3.4), we obtain

$$F''(u, A' \cup B) \leq \limsup_{\varepsilon \rightarrow 0} F_\varepsilon(w_{\varepsilon,\eta}, A' \cup B) \leq F''(u, A) + F''(u, B) + C\eta.$$

Letting $\eta \rightarrow 0$, the desired conclusion follows. $\square$

Proof of Theorem 3.1. Taking into account Proposition 5.2 and 5.3, one can infer that for any $u \in BV(\Omega; Y)$, $F''(u, \cdot)$ is inner regular and subadditive on $\mathcal{A}^{\text{reg}}(\Omega)$ (see as a reference for this procedure [16, Proposition 11.5 and 11.6]). By the compactness property of the Γ -convergence and arguing as in [16, Theorem 10.3], noting that only the inner regularity of F'' is needed, for every sequence $\varepsilon_j \rightarrow 0$ there exists a subsequence $\varepsilon_{j_k} \rightarrow 0$ such that, for any $(u, A) \in BV(\Omega; Y) \times \mathcal{A}^{\text{reg}}(\Omega)$ there exists

$$\Gamma\text{-}\lim_{k \rightarrow \infty} F_{\varepsilon_{j_k}}(u, A) =: F(u, A).$$

Moreover, by setting

$$F(u, A) := \sup\{F(u, A') : A' \in \mathcal{A}^{\text{reg}}(\Omega), A' \Subset A\}, \quad (5.17)$$

we can extend $F(u, \cdot)$ to $\mathcal{A}(\Omega)$ and verify that all properties satisfied by $F'(u, \cdot)$ and $F''(u, \cdot)$, as stated in Propositions 5.1, 5.2 and 5.3 still hold for $F(\cdot, \cdot)$ in (5.17). Since in Proposition 5.1 we proved that the domain of the Γ -limit is $BV(\Omega; Y)$, we have

$$\Gamma(L^1)\text{-}\lim_k F_{\varepsilon_{j_k}}(u) = +\infty$$

for $u \in L^1(\Omega; Y) \setminus BV(\Omega; Y)$. So far, it suffices to check that, for every $(u, A) \in BV(\Omega; Y) \times \mathcal{A}(\Omega)$, $F(u, A)$ satisfies all the hypotheses of Theorem 2.4. Note that $F_\varepsilon(u, \cdot)$ is superadditive, and this property is preserved in the limit. Thus, we can apply Theorem 2.3 and we get hypotheses (i) in Theorem 2.4. By Propositions 5.1 and 5.2 we get (iv), while (ii) is trivially satisfied. Finally, hypothesis (iii) is a consequence of the lower semicontinuity of the Γ -limit. Therefore all the assumptions of Theorem 2.4 are satisfied, and the conclusion follows. $\square$

6 Convergence of minimum problems.

In this section, we prove the convergence of minimum problems involving energies as in (3.1) to the corresponding minimum problem involving the limiting energy. We first consider boundary value problems and subsequently extend the analysis to minimum problems with prescribed volume fraction.

Our analysis relies on two compactness results proved in [31], for $d \geq 2$ and $d = 1$, respectively, which ensure the coerciveness of our family of functionals. The statements of these results are provided below and are slightly rephrased for our purposes.

Theorem 6.1. [31, Theorem 1.2] Assume $d \geq 2$. Let $G_\varepsilon(u)$ be defined by (3.6) with $\rho_\varepsilon : \mathbb{R}^d \rightarrow [0, +\infty)$ satisfying (3.2), (3.3), and (3.4). If $(u_\varepsilon) \subset L^1(\Omega; \mathbb{R}^m)$ is a bounded sequence such that

$$\sup_\varepsilon G_\varepsilon(u_\varepsilon) < +\infty,$$

then (u_ε) is relatively compact in $L^1(\Omega; \mathbb{R}^m)$. Assume that $u_{\varepsilon_j} \rightarrow u$ in $L^1(\Omega; \mathbb{R}^m)$, then $u \in BV(\Omega; \mathbb{R}^m)$.

If $d = 1$, we need to impose an additional condition on the kernels ρ_ε in order to have compactness. More precisely, given $\rho_\varepsilon \in L^1(\mathbb{R})$, we shall assume that ρ_ε is defined for every $x \in \mathbb{R}$ as

$$\rho_\varepsilon(x) = \begin{cases} \lim_{r \rightarrow 0} \frac{1}{2r} \int_{x-r}^{x+r} \rho_\varepsilon(y) dy & \text{if } x \text{ is a Lebesgue point of } \rho_\varepsilon \\ +\infty & \text{otherwise.} \end{cases} \quad (6.1)$$

Given $\theta_0 \in (0, 1)$ and setting

$$\rho_{\varepsilon, \theta_0}(x) := \inf_{\theta_0 \leq \theta \leq 1} \rho_\varepsilon(\theta x) \quad \forall x \in \mathbb{R},$$

we will consider the following hypothesis: there exists $\alpha_0 > 0$ such that

$$\int_{\mathbb{R}} \rho_{\varepsilon, \theta_0} \geq \alpha_0 > 0 \quad \forall \varepsilon > 0. \quad (6.2)$$

Theorem 6.2. [31, Theorem 1.3] Assume $d = 1$ and $\Omega = (a, b)$. Let $G_\varepsilon(u)$ be defined by (3.6) with $\rho_\varepsilon : \mathbb{R} \rightarrow [0, +\infty)$ defined as in (6.1) and satisfying (3.2), (3.3), and (3.4). Assume there exist $\theta_0 \in (0, 1)$ and $\alpha_0 > 0$ such that (6.2) is satisfied. If $(u_\varepsilon) \subset L^1((a, b); \mathbb{R}^m)$ is a bounded sequence such that

$$\sup_\varepsilon G_\varepsilon(u_\varepsilon) < +\infty,$$

then (u_ε) is relatively compact in $L^1((a, b); \mathbb{R}^m)$; moreover, if $u_{\varepsilon_j} \rightarrow u$ in $L^1((a, b); \mathbb{R}^m)$, then $u \in BV((a, b); \mathbb{R}^m)$.

6.1 Boundary value minimum problems

Given $u_0 \in BV(\Omega; Y)$, let $A \in \mathcal{A}^{\text{reg}}(\Omega)$ and for any $\eta > 0$ set

$$A^\eta = \{x \in A : \text{dist}(x, \partial A) > \eta\}$$

and

$$m_\varepsilon^\eta(u_0, A) := \inf\{F_\varepsilon(u, A) : u \in L^1(A; Y), u = u_0 \text{ on } A \setminus A^\eta\}. \quad (6.3)$$

We observe that $m_\varepsilon^\eta(u_0, A)$ is increasing in η . Moreover $m_\varepsilon^\eta(u_0, A)$ is uniformly bounded. Indeed,

$$m_\varepsilon^\eta(u_0, A) \leq F_\varepsilon(u_0, A)$$

which is uniformly bounded by (3.7), (4.4) and Corollary 4.6.

Theorem 6.3. Let $K_\varepsilon(x, y, u, v)$ satisfy (H1)-(H2). If $d = 1$, assume there exist $\theta_0 \in (0, 1)$ and $\alpha_0 > 0$ such that (6.2) is satisfied. Given a sequence $\varepsilon_j \rightarrow 0$, let (ε_{j_k}) and $F(u, A)$ be as in Theorem 3.1. Then it holds

$$\lim_{\eta \rightarrow 0} \liminf_{k \rightarrow +\infty} m_{\varepsilon_{j_k}}^\eta(u_0, A) = \lim_{\eta \rightarrow 0} \limsup_{k \rightarrow +\infty} m_{\varepsilon_{j_k}}^\eta(u_0, A) = m(u_0, A),$$

where $m_\varepsilon^\eta(u_0, A)$ is defined by (6.3) and

$$m(u_0, A) = \inf\{F(u, A) : u \in BV(A; Y), u = u_0 \text{ in a neighborhood of } \partial A\}.$$

Proof. Let $\eta > 0$ be fixed and let $u_k \in L^1(A; Y)$ such that $u_k = u_0$ on $A \setminus A^\eta$ and

$$F_{\varepsilon_{j_k}}(u_k, A) \leq m_{\varepsilon_{j_k}}^\eta(u_0, A) + \frac{1}{k}. \quad (6.4)$$

Since $m_{\varepsilon_{j_k}}^\eta(u_0, A)$ is uniformly bounded, by Theorems 6.1 and 6.2 there exists a subsequence (not relabeled) such that u_k converges strongly in $L^1(A; \mathbb{R}^m)$ to some $u \in BV(A; Y)$. In particular, $u = u_0$ in $A \setminus A^\eta$. By Theorem 3.1 and (6.4) we get

$$m(u_0, A) \leq F(u, A) \leq \liminf_{k \rightarrow +\infty} F_{\varepsilon_{j_k}}(u_k, A) \leq \liminf_{k \rightarrow +\infty} m_{\varepsilon_{j_k}}^\eta(u_0, A),$$

and then, taking the limit as $\eta \rightarrow 0$, it follows

$$m(u_0, A) \leq \lim_{\eta \rightarrow 0} \liminf_{k \rightarrow +\infty} m_{\varepsilon_{j_k}}^\eta(u_0, A).$$

It remains to prove the reverse inequality. To this end, let $\delta > 0$ and choose $u \in BV(A; Y)$ such that

$$F(u, A) \leq m(u_0, A) + \delta \quad (6.5)$$

and $u = u_0$ in a neighborhood of ∂A . Hence, there exists $\eta_0 > 0$ such that $u = u_0$ in $A \setminus A^{\eta_0}$. Let $u_k \in L^1(A; Y)$ be such that $u_k \rightarrow u$ strongly in $L^1(A; \mathbb{R}^m)$ and

$$\lim_{k \rightarrow +\infty} F_{\varepsilon_{j_k}}(u_k, A) = F(u, A).$$

We may argue as in the proof of Proposition 5.3 in order to modify the values of u_k in $A \setminus A^\eta$ without increasing the energy too much. More precisely, given $\eta \in (0, \eta_0)$, we can select open sets $A_k \in \mathcal{A}^{\text{reg}}(\Omega)$ such that $A^\eta \Subset A_k \Subset A^{\eta/2}$ and the functions

$$v_k = \chi_{A_k} u_k + (1 - \chi_{A_k}) u_0$$

satisfy

- (i) $v_k = u_0$ on $A \setminus A^{\eta/2}$;
- (ii) $v_k \rightarrow u$ strongly in $L^1(A; \mathbb{R}^m)$;
- (iii) $F_{\varepsilon_{j_k}}(v_k, A) \leq F_{\varepsilon_{j_k}}(u_k, A) + F_{\varepsilon_{j_k}}(u_0, A \setminus A^\eta) + C\eta + C\|u_k - u_0\|_{L^1(A \setminus A^\eta)} + o(1)$.

By (3.7), (4.4) and Corollary 4.6, we get

$$\begin{aligned} \limsup_{k \rightarrow +\infty} m_{\varepsilon_{j_k}}^{\eta/2}(u_0, A) &\leq \limsup_{k \rightarrow +\infty} F_{\varepsilon_{j_k}}(v_k, A) \\ &\leq F(u, A) + C\mathcal{H}^{d-1}(S_{u_0} \cap (A \setminus A^\eta)) + C\eta. \end{aligned}$$

By (6.5), we infer that

$$\lim_{\eta \rightarrow 0} \limsup_{k \rightarrow +\infty} m_{\varepsilon_{j_k}}^\eta(u_0, A) \leq F(u, A) \leq m(u_0, A) + \delta$$

and the conclusion follows by the arbitrariness of δ . □

6.2 Minimum problems with prescribed volume fraction

Given $l = (l^1, l^2, \dots, l^N) \in (0, 1)^N$ such that $\sum_{i=1}^N l^i = 1$, we set

$$\mathcal{V}^l(\Omega; Y) := \{u \in L^1(\Omega; Y) : |\{u = y_i\}| = l^i |\Omega|, i = 1, \dots, N\}$$

and define $F_\varepsilon^l : L^1(\Omega; Y) \rightarrow [0, +\infty]$ as

$$F_\varepsilon^l(u) = \begin{cases} F_\varepsilon(u) & \text{if } u \in \mathcal{V}^l(\Omega; Y) \\ +\infty & \text{otherwise,} \end{cases}$$

with F_ε as in (3.1). The following Γ -convergence theorem holds.

Theorem 6.4. *Let $K_\varepsilon(x, y, u, v)$ satisfy (H1)-(H2). If $d = 1$ and $\Omega = (a, b)$, assume in addition that*

$$K_\varepsilon(x, y, u, v) = K_\varepsilon(y - x, u, v).$$

Let $l_\varepsilon, l \in (0, 1)^N$ such that $\sum_{i=1}^N l_\varepsilon^i = \sum_{i=1}^N l^i = 1$ and $l_\varepsilon \rightarrow l$ as $\varepsilon \rightarrow 0$. Given $\varepsilon_j \rightarrow 0$, let ε_{j_k} and $F(\cdot)$ as in Theorem 3.1. Then $F_{\varepsilon_{j_k}}^{l_{\varepsilon_{j_k}}}$ Γ -converges with respect to the $L^1(\Omega; \mathbb{R}^m)$ strong topology to the functional $F^l : L^1(\Omega; Y) \rightarrow [0, +\infty]$ defined by

$$F^l(u) = \begin{cases} F(u) & \text{if } |\{u = y_i\}| = l^i |\Omega|, \forall i = 1, \dots, N \\ +\infty & \text{otherwise.} \end{cases}$$

Proof. For the sake of notation, in what follows we drop the dependence on the sequence j_k . Let $u_\varepsilon \in \mathcal{V}^{l_\varepsilon}(\Omega; Y)$ converge to $u \in BV(\Omega; Y)$ strongly in $L^1(\Omega; \mathbb{R}^m)$ such that

$$\liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) < +\infty.$$

Then, $u \in \mathcal{V}^l(\Omega; Y)$ and by Theorem 3.1 the Γ -liminf inequality follows; i.e.

$$\liminf_{\varepsilon \rightarrow 0} F_\varepsilon^{l_\varepsilon}(u_\varepsilon) \geq F^l(u).$$

We now prove the Γ -limsup inequality. Thanks to the density result stated in [4, Theorem A.1, Remark A.2], it is enough to consider $u \in BV(\Omega; Y)$ such that $|\{u = y_i\}| = l^i |\Omega|$ and whose jump set S_u is polyhedral. Let u_ε be a recovery sequence for $F(u)$. Arguing as in Proposition 5.3, we can find $\delta > 0$ and a suitable neighborhood of S_u , A_ε , such that, for ε sufficiently small, $A_\varepsilon \subset S_\delta := \{x \in \Omega : \text{dist}(x, S_u) \leq \delta\}$ and the sequence of functions

$$\tilde{u}_\varepsilon := \chi_{A_\varepsilon} u_\varepsilon + (1 - \chi_{A_\varepsilon}) u,$$

is still a recovery sequence for $F(u)$. In particular $\tilde{u}_\varepsilon \equiv u$ in $\Omega \setminus S_\delta$. To get the conclusion, it is enough to replace $\tilde{u}_\varepsilon$ with a new sequence $\bar{u}_\varepsilon \in \mathcal{V}^{l_\varepsilon}(\Omega; Y)$ such that $\bar{u}_\varepsilon \rightarrow u$ in $L^1(\Omega; \mathbb{R}^m)$ and

$$F_\varepsilon(\bar{u}_\varepsilon) \leq F_\varepsilon(\tilde{u}_\varepsilon) + o(1).$$

For simplicity, assume $N = 3$. Let us first construct $\bar{u}_\varepsilon$ in the case $d \geq 2$. We set

$$\tilde{l}_\varepsilon^i := \frac{|\{\tilde{u}_\varepsilon = y_i\}|}{|\Omega|}, \quad r_\varepsilon^i := \left(\frac{|\tilde{l}_\varepsilon^i - l_\varepsilon^i| |\Omega|}{\omega_d} \right)^{1/d}.$$

Since $\tilde{u}_\varepsilon \rightarrow u$ in $L^1(\Omega; \mathbb{R}^m)$, we have

$$r_\varepsilon^i \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0 \quad \forall i \in \{1, 2, 3\}. \quad (6.6)$$

Moreover, since S_u is polyhedral and

$$|\{u = y_i\}| = l^i |\Omega| > 0 \quad \text{for every } i = 1, 2, 3,$$

there exist $\bar{r} > 0$ and $x_i \in \Omega \setminus S_\delta$, $i = 1, 2, 3$, such that

$$B(x_i, \bar{r}) \subset \{u = y_i\} \cap (\Omega \setminus S_\delta) = \{\tilde{u}_\varepsilon = y_i\} \cap (\Omega \setminus S_\delta).$$

By (6.6), we may assume that

$$\sum_{i=1}^3 r_\varepsilon^i < \bar{r}.$$

If $\tilde{l}_\varepsilon = l_\varepsilon$, we simply set $\bar{u}_\varepsilon = \tilde{u}_\varepsilon$. Otherwise, there are two components of $l_\varepsilon - \tilde{l}_\varepsilon$ that have opposite sign. Assume, for example, that $l_\varepsilon^1 > \tilde{l}_\varepsilon^1$ and $l_\varepsilon^2 < \tilde{l}_\varepsilon^2$. If $l_\varepsilon^3 > \tilde{l}_\varepsilon^3$, we set

$$\bar{u}_\varepsilon(x) = \begin{cases} \tilde{u}_\varepsilon & \text{if } x \notin \tilde{B}_\varepsilon^1 \cup \tilde{B}_\varepsilon^3 \\ y_1 & \text{if } x \in \tilde{B}_\varepsilon^1, \\ y_3 & \text{if } x \in \tilde{B}_\varepsilon^3, \end{cases}$$

where $\tilde{B}_\varepsilon^1$ and $\tilde{B}_\varepsilon^3$ are disjoint balls of radius r_ε^1 and r_ε^3 , respectively, and $\tilde{B}_\varepsilon^1 \cup \tilde{B}_\varepsilon^3 \subset B(x_2, \bar{r})$. If $l_\varepsilon^3 < \tilde{l}_\varepsilon^3$, we set

$$\bar{u}_\varepsilon(x) = \begin{cases} \tilde{u}_\varepsilon & \text{if } x \notin B(x_2, r_\varepsilon^2) \cup B(x_3, r_\varepsilon^3) \\ y_1 & \text{if } x \in B(x_2, r_\varepsilon^2) \cup B(x_3, r_\varepsilon^3). \end{cases}$$

The case $l_\varepsilon^3 = \tilde{l}_\varepsilon^3$ is treated analogously by using only one correcting ball. In all cases, we have $\bar{u}_\varepsilon \in \mathcal{V}^{l_\varepsilon}(\Omega; Y)$ and $\bar{u}_\varepsilon \rightarrow u$ in $L^1(\Omega; \mathbb{R}^m)$. Moreover, setting $J_\varepsilon := S_{\bar{u}_\varepsilon} \cap (\Omega \setminus S_\delta)$ and $J_\varepsilon^z := \{x = y + tz : y \in J_\varepsilon, |t| \leq 1\} \cap \Omega$, by (H2), (6.6) and taking into account that, by (3.4), interactions between points at distance of order 1 are negligible, we get

$$\begin{aligned} F_\varepsilon(\bar{u}_\varepsilon) &\leq F_\varepsilon(\tilde{u}_\varepsilon) + C \int_{\mathbb{R}^d} \frac{\rho_\varepsilon(z)}{|z|} |J_\varepsilon^z| dz \\ &\leq F_\varepsilon(\tilde{u}_\varepsilon) + C \mathcal{H}^{d-1}(J_\varepsilon) \int_{\mathbb{R}^d} \rho_\varepsilon(z) dz \\ &\leq F_\varepsilon(\tilde{u}_\varepsilon) + C \sum_{i=1}^3 (r_\varepsilon^i)^{d-1} = F_\varepsilon(\tilde{u}_\varepsilon) + o(1), \end{aligned}$$

which yields the conclusion. The previous construction does not apply in the case $d = 1$ since $\mathcal{H}^{d-1}(J_\varepsilon)$ is not negligible as $\varepsilon \rightarrow 0$. In this case, note that the additional assumption $K_\varepsilon(x, y, u, v) = K_\varepsilon(y - x, u, v)$ yields that

$$F_\varepsilon(u, A) = F_\varepsilon(u(\cdot + \tau), A - \tau), \quad \forall \tau \in \mathbb{R}. \quad (6.7)$$

We then exploit this property and construct $\bar{u}_\varepsilon$ by suitably ‘locally’ shifting $\tilde{u}_\varepsilon$. Let $S_u = \{x_1, \dots, x_M\} \subset (a, b)$. We may assume that δ is sufficiently small so that $\min_{\substack{h, k \in \{1, \dots, M\} \\ h \neq k}} |x_k - x_h| >$

4δ and $\min_{k \in \{1, \dots, M\}} |x_k - a| \wedge |x_k - b| > 2\delta$. We then set

$$\bar{u}_\varepsilon(x) = \begin{cases} \tilde{u}_\varepsilon(x) & \text{if } x \notin \bigcup_{k=1}^M (x_k - 2\delta, x_k + 2\delta) \\ \tilde{u}_\varepsilon(x + \tau_\varepsilon^k) & \text{if } x \in (x_k - 2\delta, x_k + 2\delta), \quad k \in \{1, \dots, M\}, \end{cases}$$

where $\tau_\varepsilon := (\tau_\varepsilon^1, \dots, \tau_\varepsilon^M) \rightarrow 0$ as $\varepsilon \rightarrow 0$ and it is suitably chosen so that $\bar{u}_\varepsilon \in \mathcal{V}^{l_\varepsilon}(\Omega; Y)$. By (6.7) and using again that, by (3.4), interactions between points at distance of order 1 are negligible, we get

$$F_\varepsilon(\bar{u}_\varepsilon) = \sum_{k=1}^M F_\varepsilon(\bar{u}_\varepsilon, (x_k - 2\delta, x_k + 2\delta)) + o(1) = F_\varepsilon(\tilde{u}_\varepsilon) + o(1),$$

which yields the conclusion. $\square$

As a consequence of Theorem 6.4, the compactness results stated in Theorems 6.1 and 6.2, and the main properties of Γ -convergence, we get the following result.

Theorem 6.5. *Let the hypotheses of Theorem 6.4 be satisfied. If $d = 1$, assume there exist $\theta_0 \in (0, 1)$ and $\alpha_0 > 0$ such that (6.2) holds true. Then, it holds*

$$\begin{aligned} \lim_{k \rightarrow +\infty} \min\{F_{\varepsilon_{j_k}}(u) : |\{u = y_i\}| = l_{\varepsilon_{j_k}}^i |\Omega|, \quad \forall i \in \{1, \dots, N\}\} \\ = \min\{F(u) : |\{u = y_i\}| = l^i |\Omega|, \quad \forall i \in \{1, \dots, N\}\}. \end{aligned}$$

Moreover, if $u_{\varepsilon_{j_k}}$ is a converging sequence such that $|\{u_{\varepsilon_{j_k}} = y_i\}| = l_{\varepsilon_{j_k}}^i |\Omega|$ for every $i \in \{1, \dots, N\}$ and

$$\lim_{k \rightarrow +\infty} F_{\varepsilon_{j_k}}(u_{\varepsilon_{j_k}}) = \lim_{k \rightarrow +\infty} \min\{F_{\varepsilon_{j_k}}(u) : |\{u = y_i\}| = l_{\varepsilon_{j_k}}^i |\Omega|, \quad \forall i \in \{1, \dots, N\}\},$$

then its limit is a minimizer for $\min\{F(u) : |\{u = y_i\}| = l^i |\Omega|, \quad \forall i \in \{1, \dots, N\}\}$.

7 Homogenization.

In this section, we study the homogenization of the functionals F_ε defined in (3.1) under a periodicity assumption in the space variable of the kernels K_ε . Assume that $K_\varepsilon(x, y, u, v) = \frac{1}{\varepsilon^d} K\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, u, v\right)$, where $K(\cdot, \cdot, u, v) : \mathbb{R}^d \times \mathbb{R}^d \rightarrow [0, +\infty)$ is 1-periodic with respect to simultaneous translations of the two spatial variables, namely

$$K(x + h, y + h, u, v) = K(x, y, u, v) \quad \text{for every } x, y \in \mathbb{R}^d, \quad u, v \in Y, \quad \text{and } h \in \mathbb{Z}^d. \quad (7.1)$$

The analog of assumptions (H1) and (H2) is given by the following set of hypotheses on the function K : there exists a radial Borel function $\rho : \mathbb{R}^d \rightarrow [0, +\infty)$ such that

$$\int_{\mathbb{R}^d} \rho(z) dz = 1 \quad (7.2)$$

and for a.e. $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ and for every $(u, v) \in Y \times Y$ it holds:

(H1') (Coerciveness) $\exists C_1 > 0$ such that

$$K(x, y, u, v) \geq C_1 \rho(y - x)$$

(H2') (*Growth*) $\exists C_2 > 0$ such that

$$K(x, y, u, v) \leq C_2 \rho(y - x).$$

Setting

$$\rho_\varepsilon(z) = \frac{1}{\varepsilon^d} \rho\left(\frac{z}{\varepsilon}\right),$$

it is straightforward to verify that if (H1') and (H2') hold, then K_ε satisfies assumptions (H1) and (H2).

Throughout the section we will use the change of variable $z = y - x$. For any open set $A \subset \mathbb{R}^d$ and $z \in \mathbb{R}^d$ we set

$$A_z = A \cap (A - z) = \{x \in A : x + z \in A\}.$$

Then, the localized energy in (3.1) can be written as

$$F_\varepsilon(u, A) = \frac{1}{\varepsilon^d} \int_{\mathbb{R}^d} \int_{A_z} K\left(\frac{x}{\varepsilon}, \frac{x+z}{\varepsilon}, u(x), u(x+z)\right) \frac{|u(x+z) - u(x)|}{|z|} dx dz. \quad (7.3)$$

Remark 7.1. Set

$$\widehat{K}(x, z, u, v) := K(x, x+z, u, v) \quad \widehat{K}_\varepsilon(x, z, u, v) := K_\varepsilon(x, x+z, u, v).$$

In particular,

$$\widehat{K}_\varepsilon(x, z, u, v) = \frac{1}{\varepsilon^d} \widehat{K}\left(\frac{x}{\varepsilon}, \frac{z}{\varepsilon}, u, v\right).$$

By the periodicity assumption on K , the function $\widehat{K}$ is 1-periodic with respect to the first variable and then $\widehat{K}_\varepsilon$ satisfies

$$\widehat{K}_\varepsilon(x + \varepsilon h, z, u, v) = \widehat{K}_\varepsilon(x, z, u, v), \quad \text{for every } h \in \mathbb{Z}^d. \quad (7.4)$$

Moreover, by (H1') and (H2'),

$$C_1 \rho(z) \leq \widehat{K}(x, z, u, v) \leq C_2 \rho(z), \quad C_1 \rho_\varepsilon(z) \leq \widehat{K}_\varepsilon(x, z, u, v) \leq C_2 \rho_\varepsilon(z). \quad (7.5)$$

With this notation, (7.3) can be written as

$$F_\varepsilon(u, A) = \int_{\mathbb{R}^d} \int_{A_z} \widehat{K}_\varepsilon(x, z, u(x), u(x+z)) \frac{|u(x+z) - u(x)|}{|z|} dx dz.$$

The following theorem holds.

Theorem 7.2. *Let F_ε be defined by (3.1) and let $K_\varepsilon(x, y, u, v) = \frac{1}{\varepsilon^d} K\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, u, v\right)$, where $K(\cdot, \cdot, u, v)$ satisfies (7.1). If (H1') and (H2') hold, then the functionals F_ε Γ -converge with respect to the $L^1(\Omega; \mathbb{R}^m)$ strong topology, as $\varepsilon \rightarrow 0^+$, to the homogenized functional $F_{\text{hom}} : L^1(\Omega; Y) \rightarrow [0, +\infty]$ defined by*

$$F_{\text{hom}}(u) = \begin{cases} \int_{S_u} \varphi_{\text{hom}}(u^+, u^-, \nu_u) d\mathcal{H}^{d-1} & \text{if } u \in BV(\Omega; Y) \\ +\infty & \text{otherwise.} \end{cases}$$

where

$$\varphi_{\text{hom}}(a, b, \nu) = \inf_{0 < \eta < 1} \limsup_{T \rightarrow +\infty} \frac{1}{T^{d-1}} \inf \left\{ F_1(u, TQ_\nu) : \begin{array}{l} u \in L^1(TQ_\nu; Y), \\ u = u_{a,b,\nu} \text{ on } TQ_\nu \setminus (1-\eta)TQ_\nu \end{array} \right\}, \quad (7.6)$$

with $u_{a,b,\nu}$ defined as in (2.2).

Proof. By Theorem 3.1, given a sequence $\varepsilon_j \rightarrow 0^+$, we can extract a subsequence ε_{j_k} such that the functionals $F_{\varepsilon_{j_k}}$ Γ -converge to a functional F defined as in (3.5) and for any $u \in BV(\Omega; Y)$ and $A \in \mathcal{A}^{\text{reg}}(\Omega)$

$$\Gamma(L^1)\text{-}\lim_k F_{\varepsilon_{j_k}}(u, A) = \int_{S_u \cap A} \varphi(x, u^+, u^-, \nu_u) d\mathcal{H}^{d-1}.$$

For simplicity we write ε instead of ε_{j_k} , and all the limits as $\varepsilon \rightarrow 0$ are understood along this subsequence. In order to prove Theorem 7.2 we first show that φ does not depend on x . More precisely, for every fixed $x_0, z_0 \in \Omega$, $a, b \in Y$ and $\nu \in S^{d-1}$, we have

$$\varphi(x_0, a, b, \nu) = \varphi(z_0, a, b, \nu).$$

By (2.1), to get the claim it is enough to prove that, for every $r > 0$ such that $Q_\nu(x_0, r) \cup Q_\nu(z_0, r) \subset \Omega$, it holds

$$m(u_{x_0, a, b, \nu}, Q_\nu(x_0, r)) = m(u_{z_0, a, b, \nu}, Q_\nu(z_0, r)).$$

Given $\delta > 0$, let $w \in BV(Q_\nu(x_0, r); Y)$ such that $w = u_{x_0, a, b, \nu}$ in $Q_\nu(x_0, r) \setminus Q_\nu(x_0, (1-\eta)r)$ for some $\eta \in (0, 1)$ and

$$F(w, Q_\nu(x_0, r)) \leq m(u_{x_0, a, b, \nu}, Q_\nu(x_0, r)) + \delta. \quad (7.7)$$

For $x \in Q_\nu(z_0, r)$, we consider the function $\tilde{w}(x) := w(x + x_0 - z_0)$. We note that $\tilde{w} = u_{z_0, a, b, \nu}$ in $Q_\nu(z_0, r) \setminus Q_\nu(z_0, (1-\eta)r)$. Let u_ε be a recovery sequence for $F(w, Q_\nu(x_0, r))$. We define $\tau_\varepsilon \in \mathbb{R}^d$ such that, for each $i = 1, \dots, d$ the i -th component of τ_ε is given by

$$\tau_\varepsilon^i = \left\lfloor \frac{x_0^i - z_0^i}{\varepsilon} \right\rfloor \varepsilon$$

and, for $x \in Q_\nu(z_0, r)$, $\tilde{u}_\varepsilon(x) := u_\varepsilon(x + \tau_\varepsilon)$. Let $r' < r$. For ε sufficiently small $Q_\nu(z_0, r') + \tau_\varepsilon \subset Q_\nu(x_0, r)$. Since $u_\varepsilon \rightarrow w$ in $L^1(Q_\nu(x_0, r))$, it holds that $\tilde{u}_\varepsilon \rightarrow \tilde{w}$ in $L^1(Q_\nu(z_0, r'))$. Since $(Q_\nu(z_0, r'))_z + \tau_\varepsilon = (Q_\nu(z_0, r') + \tau_\varepsilon)_z$, by the change of variables $x' = x + \tau_\varepsilon$ and (7.4) we obtain

$$\begin{aligned} F_\varepsilon(\tilde{u}_\varepsilon, Q_\nu(z_0, r')) &= \int_{\mathbb{R}^d} \int_{(Q_\nu(z_0, r'))_z} \hat{K}_\varepsilon(x, z, \tilde{u}_\varepsilon(x), \tilde{u}_\varepsilon(x+z)) \frac{|\tilde{u}_\varepsilon(x+z) - \tilde{u}_\varepsilon(x)|}{|z|} dx dz \\ &= \int_{\mathbb{R}^d} \int_{(Q_\nu(z_0, r') + \tau_\varepsilon)_z} \hat{K}_\varepsilon(x' - \tau_\varepsilon, z, u_\varepsilon(x'), u_\varepsilon(x' + z)) \frac{|u_\varepsilon(x' + z) - u_\varepsilon(x')|}{|z|} dx' dz \\ &= F_\varepsilon(u_\varepsilon, Q_\nu(z_0, r') + \tau_\varepsilon) \leq F_\varepsilon(u_\varepsilon, Q_\nu(x_0, r)) \end{aligned}$$

Then,

$$F(\tilde{w}, Q_\nu(z_0, r')) \leq \liminf_{\varepsilon \rightarrow 0} F_\varepsilon(\tilde{u}_\varepsilon, Q_\nu(z_0, r')) \leq \lim_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon, Q_\nu(x_0, r)) = F(w, Q_\nu(x_0, r)).$$

Therefore, using the inner regularity

$$F(\tilde{w}, Q_\nu(z_0, r)) = \sup_{r' < r} F(\tilde{w}, Q_\nu(z_0, r')) \leq F(w, Q_\nu(x_0, r)). \quad (7.8)$$

Since $\tilde{w}$ is an admissible function for the minimum problem $m(u_{z_0, a, b, \nu}, Q_\nu(z_0, r))$, using (7.7) and (7.8), the following chain of inequalities holds

$$m(u_{z_0, a, b, \nu}, Q_\nu(z_0, r)) \leq F(\tilde{w}, Q_\nu(z_0, r)) \leq F(w, Q_\nu(x_0, r)) \leq m(u_{x_0, a, b, \nu}, Q_\nu(x_0, r)) + \delta.$$

By the arbitrariness of δ , it follows

$$m(u_{z_0, a, b, \nu}, Q_\nu(z_0, r)) \leq m(u_{x_0, a, b, \nu}, Q_\nu(x_0, r)).$$

Reversing the role of x_0 and z_0 we get the equality.

We now prove that φ coincides with the density in (7.6). Let $m_\varepsilon^\eta(u_{a, b, \nu}, A)$ be defined as in (6.3). Observe that, if $A = Q$ is a d -cube, Theorem 6.3 is still valid if we assume that the test functions in the definitions of $m_\varepsilon^\eta(u_{a, b, \nu}, Q)$ satisfy $u = u_{a, b, \nu}$ on $Q \setminus (1 - \eta)Q$. By Theorem 6.3 and rescaling the functionals, we obtain

$$\begin{aligned} \varphi(a, b, \nu) &= \limsup_{r \rightarrow 0^+} \frac{1}{r^{d-1}} m(u_{a, b, \nu}, rQ_\nu) \\ &= \limsup_{r \rightarrow 0^+} \frac{1}{r^{d-1}} \lim_{\eta \rightarrow 0} \limsup_{k \rightarrow +\infty} m_{\varepsilon_{j_k}}^\eta(u_{a, b, \nu}, rQ_\nu) \\ &= \limsup_{r \rightarrow 0^+} \frac{1}{r^{d-1}} \inf_{0 < \eta < 1} \limsup_{k \rightarrow +\infty} \inf \left\{ F_{\varepsilon_{j_k}}(u, rQ_\nu) : \begin{array}{l} u \in L^1(rQ_\nu; Y), \\ u = u_{a, b, \nu} \text{ on } rQ_\nu \setminus (1 - \eta)rQ_\nu \end{array} \right\} \\ &= \limsup_{r \rightarrow 0^+} \inf_{0 < \eta < 1} \limsup_{k \rightarrow +\infty} \frac{1}{T_k^{d-1}} \inf \left\{ F_1(u, T_k Q_\nu) : \begin{array}{l} u \in L^1(T_k Q_\nu; Y), \\ u = u_{a, b, \nu} \text{ on } T_k Q_\nu \setminus (1 - \eta)T_k Q_\nu \end{array} \right\} \end{aligned}$$

where $T_k = \frac{r}{\varepsilon_{j_k}}$. For any fixed $T > 0$ we set

$$m(T, \eta) := \inf \{ F_1(u, TQ_\nu) : u \in L^1(TQ_\nu; Y), u = u_{a, b, \nu} \text{ on } TQ_\nu \setminus (1 - \eta)TQ_\nu \},$$

$$m'(\eta) := \liminf_{T \rightarrow +\infty} \frac{1}{T^{d-1}} m(T, \eta), \quad m''(\eta) := \limsup_{T \rightarrow +\infty} \frac{1}{T^{d-1}} m(T, \eta).$$

Since

$$m'(\eta) \leq \liminf_{T_k \rightarrow +\infty} \frac{1}{T_k^{d-1}} m(T_k, \eta) \leq \limsup_{T_k \rightarrow +\infty} \frac{1}{T_k^{d-1}} m(T_k, \eta) \leq m''(\eta),$$

to get the conclusion it is enough to prove that

$$\inf_{0 < \eta < 1} m'(\eta) = \inf_{0 < \eta < 1} m''(\eta). \quad (7.9)$$

The equality (7.9) follows once we prove that, for every $\eta \in (0, 1)$,

$$m''(\eta) \leq m'(\eta) + C\eta. \quad (7.10)$$

In what follows we consider the case $d \geq 2$. The proof can be easily adapted to the case $d = 1$ and it is left to the reader.

Let $B_\nu := \{\nu^1, \dots, \nu^d\}$ be an orthonormal basis of $\mathbb{R}^d$ such that $\nu^d = \nu$ and

$$Q_\nu = \bigoplus_{j=1}^d \left(-\frac{1}{2}, \frac{1}{2} \right) \nu^j.$$

Let $S \gg T$ and fix $\delta > 0$. Let $\bar{u}_T \in L^1(TQ_\nu; Y)$ such that

$$\bar{u}_T = u_{a, b, \nu} \quad \text{on } TQ_\nu \setminus (1 - \eta)TQ_\nu$$

and

$$F_1(\bar{u}_T, TQ_\nu) \leq m(T, \eta) + \delta T^{d-1}. \quad (7.11)$$

We introduce the auxiliary length

$$\ell_{T,\eta} := T + \frac{\eta T}{2} + 2\sqrt{d}$$

and let $\Pi^\nu := \{x \in \mathbb{R}^d : \langle x, \nu \rangle = 0\}$ be the hyperplane orthogonal to ν . Consider the lattice

$$\Lambda_{T,\eta} := \left\{ z_k = \ell_{T,\eta} \sum_{j=1}^{d-1} k_j \nu^j : k = (k_1, \dots, k_{d-1}) \in \mathbb{Z}^{d-1} \right\} \subset \Pi^\nu.$$

For every $k \in \mathbb{Z}^{d-1}$, choose $y_k \in \mathbb{Z}^d$ such that

$$|y_k - z_k| \leq \sqrt{d}. \quad (7.12)$$

In particular, since $z_k \in \Pi^\nu$,

$$|\langle y_k, \nu \rangle| = |\langle y_k - z_k, \nu \rangle| \leq \sqrt{d}. \quad (7.13)$$

We set

$$I_{S,T} := \left\{ k \in \mathbb{Z}^{d-1} : |k_j| \ell_{T,\eta} \leq \frac{(1-\eta)S - T - 2\sqrt{d}}{2} \text{ for every } j = 1, \dots, d-1 \right\}$$

and

$$N_{S,T} := \#I_{S,T}.$$

For every $k \in I_{S,T}$, we define

$$Q_k := y_k + TQ_\nu.$$

By (7.12), (7.13), and the definition of $I_{S,T}$, we have

$$Q_k \subset (1-\eta)SQ_\nu \cap \left\{ x \in \mathbb{R}^d : |\langle x, \nu \rangle| \leq \frac{T}{2} + \sqrt{d} \right\} \quad \forall k \in I_{S,T}. \quad (7.14)$$

Moreover, the cubes Q_k are pairwise disjoint. Indeed, for every $k, h \in I_{S,T}$ such that $k \neq h$, there exists $j \in \{1, \dots, d-1\}$ such that $|k_j - h_j| \geq 1$. Hence,

$$\begin{aligned} |\langle y_k - y_h, \nu^j \rangle| &= |\langle z_k - z_h, \nu^j \rangle + \langle y_k - z_k, \nu^j \rangle - \langle y_h - z_h, \nu^j \rangle| \\ &\geq |\langle z_k - z_h, \nu^j \rangle| - |\langle y_k - z_k, \nu^j \rangle| - |\langle y_h - z_h, \nu^j \rangle| \\ &\geq \ell_{T,\eta} - 2\sqrt{d} \\ &= T + \frac{\eta T}{2}, \end{aligned}$$

from which it follows that

$$\text{dist}(Q_k, Q_h) \geq \frac{\eta T}{2}, \quad \text{for every } k \neq h. \quad (7.15)$$

Arguing analogously, we infer that

$$\text{dist} \left(Q_k, \bigcup_{\substack{h \in I_{S,T} \\ h \neq k}} Q_h \right) \leq \frac{\eta T}{2} + 4\sqrt{d} \leq \eta T \quad (7.16)$$

for T large enough. By the definition of $I_{S,T}$, we also have

$$N_{S,T} = \left(2 \left\lfloor \frac{(1-\eta)S - T - 2\sqrt{d}}{2\ell_{T,\eta}} \right\rfloor + 1 \right)^{d-1}.$$

Consequently,

$$\lim_{S \rightarrow +\infty} \frac{N_{S,T}}{S^{d-1}} = \left(\frac{1-\eta}{\ell_{T,\eta}} \right)^{d-1}$$

and therefore, since $\ell_{T,\eta} \geq T$,

$$\lim_{S \rightarrow +\infty} \frac{N_{S,T} T^{d-1}}{S^{d-1}} = (1-\eta)^{d-1} \left(\frac{T}{\ell_{T,\eta}} \right)^{d-1} \leq 1; \quad (7.17)$$

i.e.

$$N_{S,T} \leq C \frac{S^{d-1}}{T^{d-1}}. \quad (7.18)$$

We now define $u_S \in L^1(SQ_\nu; Y)$ by

$$u_S(x) := \begin{cases} \bar{u}_T(x - y_k) & \text{if } x \in Q_k \text{ for } k \in I_{S,T}, \\ u_{a,b,\nu}(x) & \text{otherwise.} \end{cases}$$

Since the cubes Q_k are pairwise disjoint, u_S is well defined. Moreover, by (7.14), the function u_S coincides with $u_{a,b,\nu}$ on

$$SQ_\nu \setminus (1-\eta)SQ_\nu.$$

Hence u_S is admissible for $m(S, \eta)$, and thus

$$m(S, \eta) \leq F_1(u_S, SQ_\nu). \quad (7.19)$$

We decompose $F_1(u_S, SQ_\nu)$ as

$$F_1(u_S, SQ_\nu) = F_{S,T}^{\text{cell}} + F_{S,T}^{\text{rem}},$$

where

$$F_{S,T}^{\text{cell}} = \sum_{k \in I_{S,T}} F_1(u_S, Q_k)$$

denotes the contribution of the cubes Q_k and $F_{S,T}^{\text{rem}}$ the remaining contribution. By the periodicity of $\widehat{K}_1$ in its first variable (see (7.4) for $\varepsilon = 1$) and the fact that $y_k \in \mathbb{Z}^d$, for every $k \in I_{S,T}$ we have

$$F_1(u_S, Q_k) = F_1(\bar{u}_T, TQ_\nu).$$

Therefore, by (7.11), we get

$$F_{S,T}^{\text{cell}} \leq N_{S,T} (m(T, \eta) + \delta T^{d-1}). \quad (7.20)$$

We now estimate $F_{S,T}^{\text{rem}}$ by distinguishing interactions of length at most $\eta T/4$ from the remaining ones. Namely,

$$F_{S,T}^{\text{rem}} = F_{S,T}^{sr} + F_{S,T}^{lr},$$

where

$$F_{S,T}^{sr} := F_1^{\frac{\eta T}{4}}(u_S, SQ_\nu) - \sum_{k \in I_{S,T}} F_1^{\frac{\eta T}{4}}(u_S, Q_k)$$

and

$$F_{S,T}^{lr} := (F_1(u_S, SQ_\nu) - F_1^{\frac{\eta T}{4}}(u_S, SQ_\nu)) - \sum_{k \in I_{S,T}} (F_1(u_S, Q_k) - F_1^{\frac{\eta T}{4}}(u_S, Q_k)).$$

We estimate $F_{S,T}^{sr}$. Set

$$U_{S,T} := \bigcup_{k \in I_{S,T}} Q_k$$

and let us first estimate the interactions between points in

$$U_{S,T}^\eta := \{x \in \mathbb{R}^d : \text{dist}(x, U_{S,T}) \leq \eta T\}.$$

Suppose that $x, x+z \in U_{S,T}^\eta$. We recall that, by (7.15), if $|z| \leq \eta T/4$, the points x and $x+z$ cannot belong to two distinct cubes. Hence, the points x and $x+z$ whose interactions are not taken into account in $F_{S,T}^{\text{cell}}$ are such that both points lie outside $U_{S,T}$ or one of them lies in $\bigcup_{k \in I_{S,T}} (Q_k \setminus (y_k + (1-\eta)TQ_\nu))$. In the first case,

$$u_S(x) = u_{a,b,\nu}(x), \quad u_S(x+z) = u_{a,b,\nu}(x+z).$$

If these two values are different, the segment $[x, x+z]$ intersects Π^ν . By (7.16) and (7.18), the energetic contribution of these interactions is bounded by

$$C \left(\frac{S}{T} \right)^{d-1} T^{d-2} T \eta \int_{|z| \leq \frac{\eta T}{4}} \rho(z) dz \leq C S^{d-1} \eta. \quad (7.21)$$

Suppose now that $x \in Q_k \setminus (y_k + (1-\eta)TQ_\nu)$ and $x+z \notin U_{S,T}$. Then

$$u_S(x) = u_{a,b,\nu}(x - y_k), \quad u_S(x+z) = u_{a,b,\nu}(x+z).$$

If these two values are different, then

$$\langle x - y_k, \nu \rangle \quad \text{and} \quad \langle x + z, \nu \rangle$$

have opposite signs, from which we easily infer that

$$|\langle x, \nu \rangle| \vee |\langle x - y_k, \nu \rangle| \leq |\langle y_k, \nu \rangle| + |\langle z, \nu \rangle| \leq \sqrt{d} + |z|. \quad (7.22)$$

Moreover, for T sufficiently large, the segment $[x, x+z]$ cannot leave Q_k through one of the two faces orthogonal to ν . Indeed, this would require

$$|\langle x - y_k, \nu \rangle| \geq \frac{T}{2} - |z|,$$

which is incompatible with (7.22) for T sufficiently large, since $|z| \leq \eta T/4$. Therefore, the segment must intersect one of the lateral faces of Q_k . Taking into account also (7.22), we infer that there exists $\bar{j} \in \{1, \dots, d-1\}$ such that, setting

$$D_k := \left\{ \xi \in \bar{Q}_k : \langle \xi - y_k, \nu^{\bar{j}} \rangle = \pm \frac{T}{2}, \quad |\langle \xi, \nu \rangle| \leq \sqrt{d} + 2|z| \right\},$$

we have

$$[x, x+z] \cap D_k \neq \emptyset.$$

Hence,

$$x \in \{x = x' + tz, x' \in D_k, |t| \leq 1\}$$

whose measure is bounded by

$$CT^{d-2}(\sqrt{d} + |z|)|z|.$$

By (7.18) and since $|z| \leq \eta T/4$, the energetic contribution of these interactions is bounded by

$$C \left(\frac{S}{T} \right)^{d-1} T^{d-2} \int_{|z| \leq \frac{\eta T}{4}} \rho(z)(\sqrt{d} + |z|) dz \leq CS^{d-1} \left(\frac{1}{T} + \eta \right). \quad (7.23)$$

The remaining interactions in $F_{S,T}^{sr}$, which have not been taken into account so far, are those between points x and $x+z$ such that the segment $[x, x+z]$ intersects Π^ν at distance of order $S\eta$ from $\partial(SQ_\nu)$. Arguing as above the energetic contribution of these interactions is bounded by

$$CS^{d-1} \eta \int_{|z| \leq \frac{\eta T}{4}} \rho(z) dz \leq CS^{d-1} \eta. \quad (7.24)$$

Hence, combining (7.21) and (7.23), (7.24), we get

$$F_{S,T}^{sr} \leq CS^{d-1} \eta. \quad (7.25)$$

We now estimate $F_{S,T}^{lr}$. For every $z \in \mathbb{R}^d$,

$$|\{x \in (SQ_\nu)_z : x \in U_{S,T} \text{ or } x+z \in U_{S,T}\}| \leq 2|U_{S,T}| = 2N_{S,T}T^d.$$

Therefore, by (7.5), the energetic contribution of the pairs of points such that one of them is in $U_{S,T}$ is bounded by

$$CN_{S,T}T^d \int_{|z| > \frac{\eta T}{4}} \frac{\rho(z)}{|z|} dz. \quad (7.26)$$

If both points are outside $U_{S,T}$, then

$$u_S(x) = u_{a,b,\nu}(x), \quad u_S(x+z) = u_{a,b,\nu}(x+z).$$

For every $z \in \mathbb{R}^d$,

$$|\{x \in (SQ_\nu)_z : u_{a,b,\nu}(x) \neq u_{a,b,\nu}(x+z)\}| \leq CS^{d-1}|z|. \quad (7.27)$$

Consequently, combining (7.26) and (7.27)

$$F_{S,T}^{lr} \leq CN_{S,T}T^d \int_{|z| > \frac{\eta T}{4}} \frac{\rho(z)}{|z|} dz + CS^{d-1} \int_{|z| > \frac{\eta T}{4}} \rho(z) dz.$$

Since

$$\int_{|z| > \frac{\eta T}{4}} \frac{\rho(z)}{|z|} dz \leq \frac{4}{\eta T} \int_{|z| > \frac{\eta T}{4}} \rho(z) dz$$

and

$$N_{S,T} \leq C \left(\frac{S}{T} \right)^{d-1},$$

we obtain

$$F_{S,T}^{lr} \leq \frac{CS^{d-1}}{\eta} \int_{|z| > \frac{\eta T}{4}} \rho(z) dz + CS^{d-1} \int_{|z| > \frac{\eta T}{4}} \rho(z) dz. \quad (7.28)$$

Combining (7.19), (7.20), (7.25), and (7.28), we find

$$\begin{aligned} m(S, \eta) &\leq N_{S,T} (m(T, \eta) + \delta T^{d-1}) + CS^{d-1}\eta \\ &\quad + \frac{CS^{d-1}}{\eta} \int_{|z| > \frac{\eta T}{4}} \rho(z) dz + CS^{d-1} \int_{|z| > \frac{\eta T}{4}} \rho(z) dz. \end{aligned}$$

Dividing by S^{d-1} , we obtain

$$\begin{aligned} \frac{1}{S^{d-1}} m(S, \eta) &\leq \frac{N_{S,T} T^{d-1}}{S^{d-1}} \left(\frac{1}{T^{d-1}} m(T, \eta) + \delta \right) + C\eta \\ &\quad + \frac{C}{\eta} \int_{|z| > \frac{\eta T}{4}} \rho(z) dz + C \int_{|z| > \frac{\eta T}{4}} \rho(z) dz. \end{aligned}$$

Letting first S and then T tend to $+\infty$, by (7.2) and (7.17) we get

$$\limsup_{S \rightarrow +\infty} \frac{1}{S^{d-1}} m(S, \eta) \leq \liminf_{T \rightarrow +\infty} \frac{1}{T^{d-1}} m(T, \eta) + \delta + C\eta.$$

By the arbitrariness of δ we get (7.10). $\square$

8 Approximation of nonlocal functionals

In this section we show that if the decay condition (3.4) is not satisfied, then the Γ -limits of energies as in (3.1) may be nonlocal. We provide two examples.

Example 8.1. Fix $r \in (0, 1)$ and assume $\Omega \subset \mathbb{R}^d$ such that $\text{diam}(\Omega) > 1 - r$. Consider the family of functionals

$$\begin{aligned} G_\varepsilon(u) = G_\varepsilon^{sr}(u) + G^{nl}(u) &:= \frac{1}{\varepsilon^{d+1}} \int_{\{(x,y) \in \Omega \times \Omega: |y-x| \leq \varepsilon\}} |u(y) - u(x)| dx dy \\ &\quad + \int_{\{(x,y) \in \Omega \times \Omega: 1-r \leq |y-x| \leq 1+r\}} |u(y) - u(x)| dx dy. \end{aligned}$$

Note that, for $0 < \varepsilon < 1 - r$, G_ε is of the form (3.6) with $A = \Omega$, $\rho_\varepsilon(z) = \tilde{\rho}_\varepsilon(|z|)$ and $\tilde{\rho}_\varepsilon : [0, +\infty) \rightarrow [0, +\infty)$ defined by

$$\tilde{\rho}_\varepsilon(t) = \begin{cases} \frac{t}{\varepsilon^{d+1}} & \text{if } 0 \leq t \leq \varepsilon \\ t & \text{if } 1 - r \leq t \leq 1 + r \\ 0 & \text{otherwise.} \end{cases}$$

We find that ρ_ε is uniformly bounded in $L^1(\mathbb{R}^d)$, whereas (3.4) fails. Indeed, by passing to polar coordinates,

$$\int_{\mathbb{R}^d} \rho_\varepsilon(z) dz = \frac{\sigma_{d-1}}{d+1} [1 + (1+r)^{d+1} - (1-r)^{d+1}],$$

while, given $\varepsilon < \eta \leq 1 - r$,

$$\int_{|z| > \eta} \rho_\varepsilon(z) dz = \frac{\sigma_{d-1}}{d+1} [(1+r)^{d+1} - (1-r)^{d+1}] > 0.$$

As shown in Example 4.8, G_ε^{sr} Γ -converges with respect to the strong $L^1(\Omega; \mathbb{R}^m)$ topology to the local functional

$$G^{sr}(u) := \begin{cases} \frac{2\omega_{d-1}}{d+1} \int_{S_u} |u^+ - u^-| d\mathcal{H}^{d-1} & u \in BV(\Omega; Y) \\ +\infty & \text{otherwise.} \end{cases} \quad (8.1)$$

We note that G^{nl} is continuous with respect to the strong convergence in $L^1(\Omega; \mathbb{R}^m)$. Indeed, through the change of variable $z = y - x$, $G^{nl}(u)$ can be written as

$$G^{nl}(u) = \int_{B_{1+r} \setminus B_{1-r}} \int_{\{x \in \Omega : x+z \in \Omega\}} |u(x+z) - u(x)| dx dz.$$

Hence, given $u, v \in L^1(\Omega; \mathbb{R}^m)$, we have

$$\begin{aligned} |G^{nl}(v) - G^{nl}(u)| &\leq \int_{B_{1+r} \setminus B_{1-r}} \int_{\{x \in \Omega : x+z \in \Omega\}} |v(x+z) - u(x+z)| dx dz \\ &\quad + \int_{B_{1+r} \setminus B_{1-r}} \int_{\{x \in \Omega : x+z \in \Omega\}} |v(x) - u(x)| dx dz \\ &\leq 2 |B_{1+r} \setminus B_{1-r}| \|v - u\|_{L^1(\Omega; \mathbb{R}^m)}. \end{aligned}$$

Then, the stability of Γ -convergence under continuous perturbations yields

$$\Gamma(L^1)\text{-}\lim_{\varepsilon \rightarrow 0} G_\varepsilon(u) = G^{sr}(u) + G^{nl}(u).$$

Hence, the first term is a local, perimeter-type energy, whereas the second is genuinely nonlocal.

Example 8.2. Let $0 < s < 1$ and consider the family of functionals

$$\begin{aligned} \hat{G}_\varepsilon(u) = G_\varepsilon^{sr}(u) + \hat{G}^{nl}(u) &:= \frac{1}{\varepsilon^{d+1}} \int_{\{(x,y) \in \Omega \times \Omega : |y-x| \leq \varepsilon\}} |u(y) - u(x)| dx dy \\ &\quad + \int_\Omega \int_\Omega \frac{|u(y) - u(x)|}{|y-x|^{d+s}} dx dy. \end{aligned}$$

As in the previous example, $\hat{G}_\varepsilon$ is of the form (3.6) with $A = \Omega$, $\rho_\varepsilon(z) = \hat{\rho}_\varepsilon(|z|)$ and $\hat{\rho}_\varepsilon : [0, +\infty) \rightarrow [0, +\infty)$ defined by

$$\hat{\rho}_\varepsilon(t) := \chi_{[0,D]}(t) \frac{1}{t^{d+s-1}} + \begin{cases} \frac{t}{\varepsilon^{d+1}} & \text{if } 0 \leq t \leq \varepsilon \\ 0 & \text{if } t > \varepsilon, \end{cases}$$

where $D := \text{diam}(\Omega)$. Note that, if $Y = \{0, 1\}$, $\hat{G}^{nl}(u)$ is twice the s -fractional perimeter of $\{u = 1\}$ inside Ω . Also in this case ρ_ε is uniformly bounded in $L^1(\mathbb{R}^d)$, whereas (3.4) fails. Indeed,

$$\int_{\mathbb{R}^d} \hat{\rho}_\varepsilon(z) dz = \sigma_{d-1} \left[\frac{1}{d+1} + \frac{D^{1-s}}{1-s} \right],$$

while, if $\varepsilon < \eta < D$, we have

$$\int_{|z| > \eta} \hat{\rho}_\varepsilon(z) dz = \frac{\sigma_{d-1}}{1-s} (D^{1-s} - \eta^{1-s}) > 0.$$

We now prove that $\hat{G}_\varepsilon \xrightarrow{\Gamma} G^{sr} + \hat{G}^{nl}$ in the strong $L^1(\Omega; \mathbb{R}^m)$ topology, where G^{sr} is defined in (8.1), but we cannot argue as in the previous example. Indeed, by Fatou's Lemma, we can easily infer that $\hat{G}^{nl}(u)$ is lower semicontinuous with respect to strong $L^1(\Omega; \mathbb{R}^m)$ convergence, but it is not continuous.

From the lower semicontinuity of $\hat{G}^{nl}$ we immediately deduce that the Γ -liminf inequality holds true. Indeed, for every $u_\varepsilon \rightarrow u$ in $L^1(\Omega; \mathbb{R}^m)$, we have

$$\liminf_{\varepsilon \rightarrow 0} \hat{G}_\varepsilon(u_\varepsilon) \geq \liminf_{\varepsilon \rightarrow 0} G_\varepsilon^{sr}(u_\varepsilon) + \liminf_{\varepsilon \rightarrow 0} \hat{G}^{nl}(u_\varepsilon) \geq G^{sr}(u) + \hat{G}^{nl}(u).$$

We now prove the Γ -limsup inequality. If $u \notin BV(\Omega; Y)$ there is nothing to prove, since $G^{sr}(u) = +\infty$. If $u \in BV(\Omega; Y)$, we take $u_\varepsilon \equiv u$. By Theorem 4.1, applied with $A = \Omega$ and the kernel of Example 4.8, we have $\lim_{\varepsilon \rightarrow 0} G_\varepsilon^{sr}(u) = G^{sr}(u)$. Hence, since $\hat{G}^{nl}$ does not depend on ε ,

$$\limsup_{\varepsilon \rightarrow 0} \hat{G}_\varepsilon(u) = \lim_{\varepsilon \rightarrow 0} G_\varepsilon^{sr}(u) + \hat{G}^{nl}(u) = G^{sr}(u) + \hat{G}^{nl}(u).$$

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