

SHARP CONVERGENCE RATE FOR STATIONARY STATES OF THE POROUS MEDIUM EQUATION

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ABSTRACT. In this paper we prove the sharp rate of convergence for stationary states of the porous medium equation with an external drift in the p -Wasserstein metric under very mild assumptions on the potential and for any $1 \leq p \leq +\infty$.

Keywords: Porous Medium Equation, Wasserstein metric, Optimal Transport, Convergence Rate.

1. INTRODUCTION

We consider the porous medium equation with drift

$$\partial_t \rho = \Delta \rho^m + \nabla \cdot (\rho \nabla V), \quad m > 1,$$

where $V : \mathbb{R}^d \rightarrow \mathbb{R}$ is a prescribed confining potential. This equation describes the evolution of a density under nonlinear diffusion and the drift induced by $-\nabla V$, and is related to models of crowd motion and tissue growth. Under suitable assumptions on V , it can be viewed as the gradient flow in the 2-Wasserstein distance of the energy

$$F_m(\rho) := \frac{1}{m-1} \int_{\mathbb{R}^d} \rho^m \, dx + \int_{\mathbb{R}^d} V \rho \, dx.$$

This interpretation belongs to the framework initiated by Jordan, Kinderlehrer and Otto [10] and developed by Ambrosio, Gigli and Savaré [2].

As $m \rightarrow +\infty$, the internal energy increasingly penalizes densities larger than one, leading to the constrained functional

$$F_\infty(\rho) := \begin{cases} \int_{\mathbb{R}^d} V \rho \, dx, & \text{if } 0 \leq \rho \leq 1 \text{ a.e.,} \\ +\infty, & \text{otherwise.} \end{cases}$$

Here the energies are defined on probability densities and extended by $+\infty$ to measures without a density. The gradient flow of F_∞ has been used in modeling congested crowd motion, where the preferred velocity $-\nabla V$ is modified to preserve the constraint $\rho \leq 1$ [12, 13]. The passage from the porous medium equation to this constrained evolution is known as the incompressible or Hele-Shaw limit.

The connection between these models was made rigorous by Alexander, Kim and Yao [1], who established convergence in W_2 and its relation to

a Hele–Shaw free boundary problem. Quantitative estimates were also studied for related aggregation models in [6] and in the homogeneous H^{-1} -norm in [7]. We refer the reader to the introduction of [8] for a detailed account of these developments and further references.

David, Mészáros and the second author [8, Theorem 2.4] obtained an improved convergence rate of order $m^{-1/2}$ in W_2 . For the equation considered here, when V has bounded Hessian and is uniformly convex, their estimate holds globally in time:

$$\sup_{t \geq 0} W_2(\rho_m(t), \rho_\infty(t)) \leq \frac{C}{\sqrt{m}},$$

for solutions starting from a common initial datum $\rho_0 \in \mathcal{P}_2(\mathbb{R}^d)$ with $0 \leq \rho_0 \leq 1$. The uniformity in time follows from the contractivity of the gradient flows in the 2-Wasserstein distance.

The same paper also examines stationary states, which are the minimizers of F_m and F_∞ . Besides proving an L^1 convergence rate of order m^{-1} , the authors obtain faster stationary W_2 rates under the additional assumption that $\text{spt}(\rho_m) \subseteq \text{spt}(\rho_\infty)$ for sufficiently large m [8, Theorems 2.6 and 2.7]. Our interest is in determining the sharp convergence rate for these stationary states.

The following Theorem is our main result.

Theorem 1.1. *Let $V \in C^2(\mathbb{R}^d)$ be a coercive function, $V(0) = 0$ and 0 be the unique critical point of V . Define, for $m \geq 2$, $\rho_m(x) = \left(\frac{m-1}{m}(C_m - V(x))_+\right)^{\frac{1}{m-1}}$ and $\rho_\infty(x) = \chi_{\{V \leq C_\infty\}}$ where C_m and C_∞ are chosen so that $\rho_m dx$ and $\rho_\infty dx$ are probability measures. Then, for sufficiently large m and any $p \in [1, +\infty]$, the following holds*

$$\frac{1}{m-1} \lesssim W_p(\rho_m, \rho_\infty) \lesssim \frac{1}{m-1}. \quad (1)$$

Let us first discuss the implication of our result on the literature. Our upper bound on (1) shows, as it was initially conjectured in [8], that the rate of convergence for stationary states solutions in W_2 can indeed be improved to be of polynomial order m^{-1} . This does not, however, guarantee that the rate of convergence for non-stationary can be improved to m^{-1} . It could happen that, locally in time, only a slower rate is attainable. What our sharpness statement guarantees is that, globally in time, the rate of convergence in W_2 cannot be improved further than m^{-1} . Thus, it remains open to establish whether this rate is sharp in the 2-Wasserstein metric for non-stationary state solutions both locally and globally in time.

Let us now discuss the assumptions on the potential V . As we discussed before, in [8] the authors examined the same question of convergence under the assumption that $\alpha I \leq D^2V \leq AI$ for some $0 < \alpha \leq A$ where I denotes the identity matrix. Our assumptions that $V \in C^2(\mathbb{R}^d)$, V coercive and V has a unique critical point are weaker than theirs.

Our extra assumption that 0 is such unique global minimizer is chosen cosmetically and of course nothing changes if the minimum point or the minimum value are different. A fine analysis of the proof shows that it would work with almost no modifications also in the case where 0 is the only *critical level* (allowing thus for many global minimizers), provided the Lebesgue measure of such a level (if positive) is strictly smaller than 1. Actually the only important point is that C_∞ itself is not a critical level, but we prefer to express our assumptions differently, since C_∞ is not necessarily explicitly known. Also, the global regularity assumptions we impose on V (namely, V of class C^2 on $\mathbb{R}^d$ and with a unique critical point) need not to hold on the entire domain $\mathbb{R}^d$, but it would suffice to assume them only in a bounded region containing the relevant sublevel sets of V . This would allow for potentials V with much wilder behavior. However, we do not see any real gains in going to this level of generality, except making the language more cumbersome. Similarly, the coercivity assumption on V is what guarantees that the sublevel sets of V , and hence the support of ρ_m and ρ_∞ , are compact sets. This could again be weakened to assuming only that the sublevel sets of V in a neighborhood of C_∞ are compact. Of course, one would need to assume separately then that V admits a unique global minimizer.

Our proof strategy consists in first building a flow Ψ_m that moves $\text{spt}(\rho_\infty)$ to $\text{spt}(\rho_m)$ with unit speed and, among other things, satisfies $\|\Psi_m - \text{Id}\|_{L^\infty} \lesssim |C_m - C_\infty|$ (Corollary 2.4). Then, defining $\tilde{\rho}_m = (\Psi_m^{-1})_\# \rho_m$, we use the Dacorogna–Moser construction to obtain the estimate $W_p(\tilde{\rho}_m, \rho_\infty) \lesssim \frac{1}{m-1} + |C_m - C_\infty|$ for $1 \leq p \leq m/2$, with an implicit constant independent of m and p (Proposition 3.3). By establishing that $|C_m - C_\infty| \lesssim \frac{1}{m-1}$ (Lemma 4.1) and applying the estimate of Jylhä and Rajala [11] (which is a refined version of the result by Bouchitté, Jimenez and Rajesh [3]) with $p = m/2$, we obtain $W_\infty(\tilde{\rho}_m, \rho_\infty) \lesssim \frac{1}{m-1}$ (Lemma 4.3). Moreover, we also show that, due to the good properties of Ψ_m , $W_\infty(\rho_m, \tilde{\rho}_m) \lesssim |C_m - C_\infty|$. Finally, we prove the upper bound by the triangle inequality for the metric W_∞ and the monotonicity of the Wasserstein distances. As for the lower bound, we show that $W_1(\rho_m, \rho_\infty) \gtrsim \frac{1}{m-1}$, which gives the lower bound for every $1 \leq p \leq +\infty$ by monotonicity.

2. PRELIMINARY GEOMETRIC RESULTS

In this section we establish the results that will be used repeated times in the proofs to come. We also establish construction of the flow that will perform the geometric modification of ρ_m to a measure on the support of ρ_∞ in Corollary 2.4. This will be the key geometrical step that will allow us to employ the Dacorogna–Moser construction in the next section.

The proofs of the next two results do not contribute to the clarity of the main argument, therefore we postpone them to the Appendix A.

Proposition 2.1. *Let V be as in Theorem 1.1. For every $C > 0$, there exist $\delta > 0$, a compactly supported vector field $b \in C_c^1(\mathbb{R}^d; \mathbb{R}^d)$ and a constant $K = K(V, C)$ such that the flow generated by b , namely*

$$\begin{cases} \partial_s \Psi_s(x) = b(\Psi_s(x)) \\ \Psi_0(x) = x \end{cases} \quad (2)$$

is a $C^1(\mathbb{R}^d)$ diffeomorphism for every $s \in \mathbb{R}$, with inverse Ψ_{-s} . Moreover, for every $|s| \leq \delta$, we have $\Psi_s(\{V \leq C\}) = \{V \leq C+s\}$, and the following estimate holds:

$$\|\Psi_s - \text{id}\|_{L^\infty(\mathbb{R}^d)} + \|\det D\Psi_s - 1\|_{L^\infty(\mathbb{R}^d)} \leq K|s|. \quad (3)$$

Proposition 2.2. *Consider V as in Theorem 1.1. Define $A(t) = |\{V \leq t\}|$, then $A \in C^0([0, +\infty)) \cap C^1((0, +\infty))$. For $t \in (0, +\infty)$ we have*

$$A'(t) = \int_{V=t} \frac{1}{|\nabla V|} d\mathcal{H}^{d-1}.$$

Moreover, A is strictly increasing on $[0, +\infty)$.

Lemma 2.3. *Let V , ρ_m and ρ_∞ be as in Theorem 1.1. Then, $C_m \rightarrow C_\infty$.*

Proof. It suffices to show that $A(C_m) \rightarrow A(C_\infty) = 1$, as $t \mapsto A(t)$ is strictly increasing and continuous by Proposition 2.2. Choose constants C_0 and C_1 such that $0 < C_0 < C_\infty < C_1$ and define $I_m : \mathbb{R} \rightarrow \mathbb{R}_+$ by

$$I_m(C) = \int_{\mathbb{R}^d} \left(\frac{m-1}{m} (C - V)_+ \right)^{\frac{1}{m-1}} dx.$$

For ease of notation, let $K_m := \left(\frac{m-1}{m} \right)^{\frac{1}{m-1}}$. Since V is coercive, it admits a global minimizer. Any global minimizer is a critical point of V , and since 0 is the unique critical point by assumption, it follows that 0 is the unique minimizer of V . As $V(0) = 0$, we have $V(x) > 0$ for every $x \neq 0$. By the coarea formula and Proposition 2.2, we have

$$\begin{aligned} I_m(C) &= K_m \int_0^C (C-s)^{\frac{1}{m-1}} \left(\int_{\{V=s\}} \frac{1}{|\nabla V|} d\mathcal{H}^{d-1}(x) \right) ds \\ &= K_m \int_0^C (C-s)^{\frac{1}{m-1}} A'(s) ds. \end{aligned} \quad (4)$$

Now we estimate, for $C \in [C_0, C_1]$,

$$\begin{aligned}
|I_m(C) - A(C)| &= \left| K_m \int_0^C (C-s)^{\frac{1}{m-1}} A'(s) ds - \int_0^C A'(s) ds \right| \\
&\leq K_m \int_0^C \left| (C-s)^{\frac{1}{m-1}} - 1 \right| A'(s) ds + |K_m - 1| A(C) \\
&\leq \int_0^C \left| (C-s)^{\frac{1}{m-1}} - 1 \right| A'(s) ds + |K_m - 1| A(C) \\
&\lesssim \frac{1}{m-1} \int_0^C |\ln(C-s)| A'(s) ds + |K_m - 1| A(C).
\end{aligned} \tag{5}$$

In passing to the third line we used $0 < K_m \leq 1$. For the last inequality, for every $0 \leq s < C$, the mean value theorem applied to $r \mapsto (C-s)^r$ gives some $\Theta = \Theta(s, m) \in (0, \frac{1}{m-1})$ such that

$$\left| (C-s)^{\frac{1}{m-1}} - 1 \right| \leq \frac{1}{m-1} |\ln(C-s)| (C-s)^\Theta.$$

Since $m \geq 2$, we have $0 \leq \Theta \leq 1$, and since $0 < C-s \leq C_1$, it follows that $(C-s)^\Theta \leq \max\{1, C_1\}$. Hence, the implicit constant in (5) is independent of m . The logarithmic integral in (5) is bounded uniformly for $C \in [C_0, C_1]$. Indeed, the singularity at $s = C$ remains uniformly separated from $s = 0$, and A' is bounded on $[C_0/2, C_1]$. The claim therefore follows from $A' \in L^1(0, C_1)$ and the integrability of $|\ln u|$ near zero.

Therefore,

$$\sup_{C \in [C_0, C_1]} \int_0^C |\ln(C-s)| A'(s) ds < +\infty. \tag{6}$$

It remains to estimate $K_m - 1$. We have

$$|K_m - 1| \leq \frac{1}{m-1} \left| \ln \left(\frac{m-1}{m} \right) \right| \leq \frac{1}{(m-1)^2}. \tag{7}$$

Since $A(C) \leq A(C_1)$ for every $C \in [C_0, C_1]$, it follows, by substituting (7) and (6) into (5) that

$$\sup_{C \in [C_0, C_1]} |I_m(C) - A(C)| \lesssim \frac{1}{m-1}. \tag{8}$$

We argue that $C_m \in [C_0, C_1]$ for sufficiently large m . Since $C_\infty \in (C_0, C_1)$ and A is strictly increasing, we have $A(C_0) < 1 < A(C_1)$. By (8), for sufficiently large m we also have $I_m(C_0) < 1 < I_m(C_1)$. Since I_m is continuous and strictly increasing on $[C_0, C_1]$ and, by the normalization of ρ_m , $I_m(C_m) = 1$, it follows that $C_m \in [C_0, C_1]$ for sufficiently large m . Substituting $C = C_m$ into (8), we obtain

$$|A(C_m) - 1| = |A(C_m) - I_m(C_m)| \lesssim \frac{1}{m-1}. \tag{9}$$

Since $1 = A(C_\infty)$, we conclude that $A(C_m) \rightarrow A(C_\infty)$ and, since A is continuous and strictly increasing, this implies that $C_m \rightarrow C_\infty$. $\square$

Corollary 2.4. *Let V , ρ_m and ρ_∞ be as in Theorem 1.1. For m sufficiently large, there exists a diffeomorphism $\Psi_m : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and constants $\delta, K > 0$ such that $\Psi_m(\{V \leq C_\infty\}) = \{V \leq C_m\}$*

$$\|\Psi_m - \text{Id}\|_{L^\infty(\{V \leq C_\infty\})} + \|\det D\Psi_m - 1\|_{L^\infty(\{V \leq C_\infty\})} \leq K|C_m - C_\infty|. \quad (10)$$

And for $x \in \{C_\infty - \frac{\delta}{3} \leq V \leq C_\infty + \frac{\delta}{3}\}$, we have

$$C_m - V(\Psi_m(x)) = C_\infty - V(x) \quad (11)$$

Proof. Let $C_\infty > 0$ play the role of C in Proposition 2.1, then there exists $\delta > 0$ and a $C^1(\mathbb{R}^d)$ diffeomorphism, Ψ_s , for $|s| \leq \delta$ we have $\Psi_s(\{V \leq C_\infty\}) = \{V \leq C_\infty + s\}$. Choosing $s = C_m - C_\infty$ and denoting $\Psi_m := \Psi_{C_m - C_\infty}$ we obtain $\Psi_m(\{V \leq C_\infty\}) = \{V \leq C_m\}$. Moreover, for m sufficiently large we have that $|C_m - C_\infty| < \delta$, since $C_m \rightarrow C_\infty$ by Lemma 2.3. Thus, by (3), estimate (10) holds.

It remains to prove (11). Recall that $\chi = 1$ on $\{C_\infty - \delta \leq V \leq C_\infty + \delta\}$ (see Appendix A). Since $C_m \rightarrow C_\infty$, for m sufficiently large we have $|C_m - C_\infty| \leq \delta/3$. Let $s_m := C_m - C_\infty$ and fix $x \in \{C_\infty - \delta/3 \leq V \leq C_\infty + \delta/3\}$. We claim that the trajectory $t \mapsto \Psi_{ts_m}(x)$ remains in the region where $\chi = 1$ for every $t \in [0, 1]$. Indeed, otherwise let t_0 be its first exit time from $\{C_\infty - \delta < V < C_\infty + \delta\}$. For every $t < t_0$ we have $\chi(\Psi_{ts_m}(x)) = 1$, and therefore

$$\frac{d}{dt} V(\Psi_{ts_m}(x)) = s_m \nabla V(\Psi_{ts_m}(x)) \cdot b(\Psi_{ts_m}(x)) = s_m. \quad (12)$$

Hence $V(\Psi_{ts_m}(x)) = V(x) + ts_m$ for every $t < t_0$, and consequently

$$|V(\Psi_{ts_m}(x)) - C_\infty| \leq |V(x) - C_\infty| + t|s_m| \leq \frac{2\delta}{3} < \delta,$$

which contradicts the definition of t_0 . Thus the trajectory remains entirely in the region where $\chi = 1$, and integrating (12) on $[0, 1]$ and recalling $s_m = C_m - C_\infty$ gives

$$V(\Psi_m(x)) = V(x) + C_m - C_\infty.$$

Rearranging, we obtain

$$C_m - V(\Psi_m(x)) = C_\infty - V(x)$$

for every $x \in \{C_\infty - \delta/3 \leq V \leq C_\infty + \delta/3\}$. $\square$

3. THE CASE OF FIXED SUPPORT.

In this section we establish the estimate $W_p((\Psi_m)_\#^{-1} \rho_m, \rho_\infty) \lesssim \frac{1}{m-1} + |C_m - C_\infty|$ for every $1 \leq p \leq m/2$. As both $(\Psi_m)_\#^{-1} \rho_m$ and ρ_∞ have support $\{V \leq C_\infty\}$, we may construct a map transporting one to the other via the Dacorogna-Moser construction. Moreover, we can estimate

from above the quadratic cost of such a map by $\|(\Psi_m)_\#^{-1} \rho_m - \rho_\infty\|_q$ with a constant depending only on the support, and as the support is fixed, such a constant will not depend on m . Finally, we estimate $\|(\Psi_m)_\#^{-1} \rho_m - \rho_\infty\|_q$. This is the content of the next Proposition.

Proposition 3.1. *Let V , ρ_m and ρ_∞ be as in Theorem 1.1. Let $\tilde{\rho}_m = (\Psi_m)_\#^{-1} \rho_m$ where Ψ_m is the diffeomorphism from Corollary 2.4, then, for m sufficiently large and $1 \leq q < +\infty$, the following holds*

$$\|\tilde{\rho}_m - 1\|_{L^q(\{V \leq C_\infty\})} \lesssim \frac{1}{m-1} + |C_m - C_\infty| \quad (13)$$

Proof. For ease of notation let $K_m := ((m-1)/m)^{\frac{1}{m-1}}$, and rewrite the density ρ_m as $\rho_m(x) = K_m(C_m - V(x))_+^{\frac{1}{m-1}}$. The relation $\tilde{\rho}_m(x) dx = (\Psi_m^{-1})_\# \rho_m dx$, which is equivalent to

$$\tilde{\rho}_m(x) = \rho_m(\Psi_m(x)) \det(D\Psi_m(x)). \quad (14)$$

By the equation above, we have

$$\begin{aligned} |\tilde{\rho}_m(x) - 1|^q &= \left| K_m \det D\Psi_m(x) (C_m - V(\Psi_m(x)))^{\frac{1}{m-1}} - 1 \right|^q \\ &\leq \left(K_m |\det D\Psi_m(x)| \left| (C_m - V(\Psi_m(x)))^{\frac{1}{m-1}} - 1 \right| + |K_m \det D\Psi_m(x) - 1| \right)^q \\ &\leq 2^q \left((K_m)^q |\det D\Psi_m(x)|^q \left| (C_m - V(\Psi_m(x)))^{\frac{1}{m-1}} - 1 \right|^q + |K_m \det D\Psi_m(x) - 1|^q \right). \end{aligned} \quad (15)$$

and thus

$$\begin{aligned} \|\tilde{\rho}_m - 1\|_{L^q(\Omega_\infty)}^q &\leq 2^q \left[(K_m)^q \|\det D\Psi_m\|_{L^\infty(\Omega_\infty)}^q \right. \\ &\quad \times \left\| (C_m - V \circ \Psi_m)^{\frac{1}{m-1}} - 1 \right\|_{L^q(\Omega_\infty)}^q \\ &\quad \left. + \|K_m \det D\Psi_m - 1\|_{L^q(\Omega_\infty)}^q \right]. \end{aligned} \quad (16)$$

We will estimate each of the terms and factors in (16). Let

$$F_1 = (K_m)^q, F_2 = \|\det D\Psi_m(x)\|_{L^\infty}^q$$

and

$$T_1 = \|(C_m - V(\Psi_m(x)))^{\frac{1}{m-1}} - 1\|_{L^q(\{V \leq C_\infty\})}^q,$$

$$T_2 = \|K_m \det D\Psi_m(x) - 1\|_{L^q(\{V \leq C_\infty\})}^q.$$

We start with T_1 :

$$\begin{aligned}
& \| (C_m - V(\Psi_m(x)))^{\frac{1}{m-1}} - 1 \|_{L^q(\{V \leq C_\infty\})}^q \\
&= \int_{\{V \leq C_\infty\}} \left| (C_m - V(\Psi_m(x)))^{\frac{1}{m-1}} - 1 \right|^q dx \\
&= \int_{\{V \leq C_\infty\}} \left| \exp \left(\frac{1}{m-1} \ln(C_m - V(\Psi_m(x))) \right) - 1 \right|^q dx \quad (17) \\
&\lesssim \frac{1}{(m-1)^q} \int_{\{V \leq C_\infty\}} |\ln(C_m - V(\Psi_m(x)))|^q dx
\end{aligned}$$

where, in the last inequality, we used that the exponent $\frac{1}{m-1} \ln(C_m - V(\Psi_m(x)))$ is bounded above uniformly in m and x , since C_m is uniformly bounded, and that the exponential function is Lipschitz on any interval bounded above.

We must show that

$$\sup_m \int_{\{V \leq C_\infty\}} |\ln(C_m - V(\Psi_m(x)))|^q dx < +\infty. \quad (18)$$

Let $\delta > 0$ be as in Corollary 2.4. We may split the integral above into the regions $\{V < C_\infty - \frac{\delta}{3}\}$ and $\{C_\infty - \frac{\delta}{3} \leq V \leq C_\infty\}$.

$$\begin{aligned}
& \| (C_m - V(\Psi_m(x)))^{\frac{1}{m-1}} - 1 \|_{L^q(\{V \leq C_\infty\})}^q \\
&\lesssim \frac{1}{(m-1)^q} \left(\underbrace{\int_{\{V \leq C_\infty - \frac{\delta}{3}\}} |\ln(C_m - V(\Psi_m(x)))|^q dx}_{T_{1.1}} \right. \\
&\quad \left. + \underbrace{\int_{\{C_\infty - \frac{\delta}{3} \leq V \leq C_\infty\}} |\ln(C_m - V(\Psi_m(x)))|^q dx}_{T_{1.2}} \right) \quad (19)
\end{aligned}$$

By (11) and the co-area formula, if $f(s) = \int_{V=s} |\nabla V|^{-1} d\mathcal{H}^{d-1}$, then

$$\begin{aligned}
T_{1.2} &= \int_{\{C_\infty - \frac{\delta}{3} \leq V \leq C_\infty\}} |\ln(C_m - V(\Psi_m(x)))|^q dx \\
&= \int_{\{C_\infty - \frac{\delta}{3} \leq V \leq C_\infty\}} |\ln(C_\infty - V(x))|^q dx \quad (20) \\
&= \int_{C_\infty - \frac{\delta}{3}}^{C_\infty} |\ln(C_\infty - s)|^q f(s) ds
\end{aligned}$$

fix $\varepsilon > 0$ small, splitting away from the singularity we have

$$\int_{C_\infty - \frac{\delta}{3}}^{C_\infty - \varepsilon} |\ln(C_\infty - s)|^q f(s) ds + \int_{C_\infty - \varepsilon}^{C_\infty} |\ln(C_\infty - s)|^q f(s) ds < +\infty \quad (21)$$

the first integral is finite because $s \mapsto |\ln(C_\infty - s)|^q$ is bounded from above and below on $[C_\infty - \frac{\delta}{3}, C_\infty - \varepsilon]$ and $s \mapsto f(s)$ is integrable on the same interval (see Proposition 2.2) also the second integral is finite since $s \mapsto f(s)$ is bounded on $[C_\infty - \varepsilon, C_\infty]$ while $s \mapsto |\ln(C_\infty - s)|^q$ is integrable on the same region. As neither integral in (21) depends on m , $T_{1,2}$ is bounded uniformly in m .

From Lemma 2.3 we have $C_m \rightarrow C_\infty$, also from Corollary 2.4 we have that $\Psi_m \rightarrow \text{Id}$ uniformly. Since V is continuous and coercive, the set $\{V \leq C_\infty - \frac{\delta}{3}\}$ is compact and V is uniformly continuous there. We obtain that $C_m - V(\Psi_m(x)) \rightarrow C_\infty - V(x)$ uniformly in x . As on $\{V \leq C_\infty - \frac{\delta}{3}\}$ we have $C_\infty - V(x) \geq \frac{\delta}{3} > 0$, for m sufficiently large we also have $C_m - V(\Psi_m(x)) \geq \frac{\delta}{3} > 0$, thus $|\ln(C_m - V(\Psi_m(x)))|$ is bounded from above and below uniformly in x , and the bounds can be chosen independently of m , hence $T_{1,1}$ is bounded uniformly in m .

Therefore, from (17) we have

$$T_1 = \|(C_m - V(\Psi_m(x)))^{\frac{1}{m-1}} - 1\|_{L^q(\{V \leq C_\infty\})}^q \lesssim \frac{1}{(m-1)^q}. \quad (22)$$

We have already estimated F_1 in (7) to be

$$|K_m - 1| \lesssim \frac{1}{(m-1)^2}. \quad (23)$$

In particular, K_m , and hence F_1 , is bounded.

By (10), we have $\det D\Psi_m = 1 + O(|C_m - C_\infty|)$ uniformly on $\{V \leq C_\infty\}$. Since $C_m \rightarrow C_\infty$ by Lemma 2.3, it follows that F_2 is uniformly bounded in m . Moreover, (23) gives $K_m = 1 + O(m^{-2})$, and hence, using $|\{V \leq C_\infty\}| = 1$,

$$T_2 = \|K_m \det D\Psi_m - 1\|_{L^q(\{V \leq C_\infty\})}^q \lesssim |C_m - C_\infty|^q + m^{-2q}. \quad (24)$$

Returning to (16) we have, since $(K_m)^q$ and $\|\det D\Psi_m(x)\|_{L^\infty(\{V \leq C_\infty\})}^q$ are bounded uniformly in m , (22) and (24)

$$\|\tilde{\rho}_m - 1\|_{L^q(\{V \leq C_\infty\})}^q \lesssim |C_m - C_\infty|^q + \frac{1}{(m-1)^q}$$

and thus

$$\|\tilde{\rho}_m - 1\|_{L^q(\{V \leq C_\infty\})} \lesssim |C_m - C_\infty| + \frac{1}{(m-1)}.$$

□

Define u_m as a solution to the following Neumann problem

$$\begin{cases} -\Delta u_m = \tilde{\rho}_m - \rho_\infty & \text{in } \{V < C_\infty\} \\ \partial_n u_m = 0 & \text{on } \{V = C_\infty\}. \end{cases} \quad (25)$$

As $\tilde{\rho}_m dx$ and $\rho_\infty dx$ are probability measures we have $\int (\tilde{\rho}_m - \rho_\infty) dx = 0$, which guarantees a solution exists (see [9, Chapter 6, Exercise 3]).

Lemma 3.2. *For $q > d$, a solution u_m of (25) satisfies the estimate*

$$\|\nabla u_m\|_{L^\infty(\{V \leq C_\infty\})} \lesssim \|\tilde{\rho}_m - 1\|_{L^q(\{V \leq C_\infty\})} \quad (26)$$

Proof. The set $\{V < C_\infty\}$ is a bounded connected domain with C^2 boundary. By elliptic regularity for the Neumann problem (see [4, Theorem B.2]) and the Sobolev embedding for $q > d$, we obtain

$$\|\nabla u_m\|_{L^\infty(\{V < C_\infty\})} \lesssim \|\nabla u_m\|_{W^{1,q}(\{V < C_\infty\})} \lesssim \|\tilde{\rho}_m - 1\|_{L^q(\{V < C_\infty\})}.$$

The implicit constants depend only on the domain and q , and are therefore independent of m . $\square$

Proposition 3.3. *Let V , ρ_m , $\tilde{\rho}_m$ and ρ_∞ be as in Proposition 3.1. Then, for sufficiently large m and every $1 \leq p \leq m/2$*

$$W_p(\tilde{\rho}_m, \rho_\infty) \lesssim \frac{1}{m-1} + |C_m - C_\infty|.$$

with the implicit constant independent of m and p .

Proof. It suffices to consider $3 \leq p \leq m/2$, since the case $1 \leq p < 3$ follows by taking $p = 3$ and using the monotonicity of the Wasserstein distances. We do the Dacorogna–Moser interpolation (see [14, Box 4.3]) with the flux $w_m := \nabla u_m$. Set $\rho_t := (1-t)\rho_\infty + t\tilde{\rho}_m$ for $t \in [0, 1]$. On $\{V \leq C_\infty\}$, this is simply $\rho_t = (1-t) + t\tilde{\rho}_m$, and the equation $-\Delta u_m = \tilde{\rho}_m - 1$ gives $\partial_t \rho_t + \operatorname{div} w_m = 0$, with no boundary flux because $\partial_n u_m = 0$ on $\{V = C_\infty\}$. Therefore (ρ_t, w_m) is an admissible pair in the Benamou–Brenier formula, and hence

$$W_p^p(\tilde{\rho}_m, \rho_\infty) \leq \int_0^1 \int_{\{V \leq C_\infty\}} \frac{|w_m|^p}{\rho_t^{p-1}} dx dt.$$

Swapping the order of integration and using that w_m is independent of time gives

$$W_p^p(\tilde{\rho}_m, \rho_\infty) \leq \int_{\{V \leq C_\infty\}} |w_m|^p \int_0^1 \frac{1}{\rho_t^{p-1}} dt dx. \quad (27)$$

By Corollary 2.4, Ψ_m maps the interior of $\{V \leq C_\infty\}$ diffeomorphically into the interior of $\{V \leq C_m\}$, thus $\tilde{\rho}_m > 0$ on $\{V < C_\infty\}$ and we may compute the integral in time above to obtain

$$\begin{aligned} W_p^p(\tilde{\rho}_m, \rho_\infty) &\leq \int_{\{V \leq C_\infty\}} |w_m|^p \frac{\tilde{\rho}_m(x)^{2-p} - 1}{(p-2)(1 - \tilde{\rho}_m(x))} dx \\ &\leq \|w_m\|_\infty^p J_m \end{aligned} \quad (28)$$

where

$$J_m := \int_{\{V < C_\infty\}} \frac{\tilde{\rho}_m^{2-p} - 1}{(p-2)(1 - \tilde{\rho}_m)} dx.$$

It remains to prove that $J_m^{1/p}$ is bounded uniformly in m and p . Since $p \geq 3$,

$$0 < \frac{r^{2-p} - 1}{(p-2)(1-r)} \leq 1 + r^{2-p}, \quad r > 0,$$

and $|\{V < C_\infty\}| = 1$, we have

$$J_m \leq 1 + \int_{\{V < C_\infty\}} \tilde{\rho}_m^{2-p} dx.$$

Recall that $\tilde{\rho}_m(x) = K_m(C_m - V(\Psi_m(x)))^{\frac{1}{m-1}} \det D\Psi_m(x)$ where we have denoted $K_m := (\frac{m-1}{m})^{\frac{1}{m-1}}$. Since K_m and $\det D\Psi_m$ are uniformly bounded away from zero for sufficiently large m , there exists $0 < c \leq 1$, independent of m , such that $K_m \det D\Psi_m \geq c$ on $\{V < C_\infty\}$. Therefore,

$$\int_{\{V < C_\infty\}} \tilde{\rho}_m^{2-p} dx \leq c^{2-p} \int_{\{V < C_\infty\}} (C_m - V(\Psi_m(x)))^{\frac{2-p}{m-1}} dx.$$

Let $\delta > 0$ be as in Corollary 2.4. As shown in the proof of Proposition 3.1, $C_m - V(\Psi_m(x))$ is bounded above and below by positive constants uniformly in m on $\{V \leq C_\infty - \frac{\delta}{3}\}$. On the remaining region, (11) and the coarea formula show that the integral is bounded uniformly in m and p , since A' is bounded near C_∞ and

$$-\frac{1}{2} < \frac{2-p}{m-1} < 0$$

for $3 \leq p \leq m/2$. Therefore, there exists $C > 0$, independent of m and p , such that

$$J_m \leq 1 + Cc^{2-p} \leq (1+C)c^{-p},$$

and hence

$$J_m^{1/p} \leq c^{-1}(1+C)^{1/p} \leq c^{-1}(1+C)^{1/3}.$$

By (28), Lemma 3.2 and Proposition 3.1, both with exponent $d+1$, we obtain

$$W_p(\tilde{\rho}_m, \rho_\infty) \leq \|w_m\|_\infty J_m^{1/p} \lesssim \frac{1}{m-1} + |C_m - C_\infty|.$$

□

4. MAIN RESULT

Lemma 4.1. *Let V , ρ_m and ρ_∞ be as in Theorem 1.1. Then, for m sufficiently large,*

$$|C_m - C_\infty| \lesssim \frac{1}{m-1}. \quad (29)$$

Proof. By Proposition 2.2 we have $A \in C^1((0, +\infty))$ and from $A'(C_\infty) = \int_{V=C_\infty} \frac{1}{|\nabla V|} d\mathcal{H}^{d-1} > 0$, there exists constants $c > 0$ and $0 < \eta < C_\infty$ such that

$$A'(t) \geq c > 0 \text{ for } [C_\infty - \eta, C_\infty + \eta].$$

From Lemma 2.3 we have that, for sufficiently large m , $C_m \in [C_\infty - \eta, C_\infty + \eta]$. By the mean value Theorem there exists $\xi_m \in [C_\infty - \eta, C_\infty + \eta]$ such that

$$|A(C_m) - A(C_\infty)| = A'(\xi_m)|C_m - C_\infty| \geq c|C_m - C_\infty|. \quad (30)$$

In the proof of Lemma 2.3 we already obtained that $|A(C_m) - A(C_\infty)| \lesssim \frac{1}{(m-1)}$, thus by (30) we conclude

$$|C_m - C_\infty| \lesssim \frac{1}{m-1}.$$

□

For $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$, the ∞ -Wasserstein metric (see [14, Section 3.2]) is defined by

$$W_\infty(\mu, \nu) := \inf_{\lambda \in \Pi(\mu, \nu)} \|x - y\|_{L^\infty(\lambda)}.$$

We recall the following consequence of [11, Theorem 1.1] for p -power costs.

Theorem 4.2 (Theorem 1.1 in [11]). *Let $\mu \in \mathcal{P}(\mathbb{R}^d)$ have compact and connected support. Define $m(t) := \inf_{x \in \text{spt}(\mu)} \mu(B(x, t))$. Then, for every $1 \leq p < +\infty$*

$$W_p^p(\mu, \nu) \geq \frac{1}{2} m \left(\frac{W_\infty(\mu, \nu)}{17} \right) \left(\frac{W_\infty(\mu, \nu)}{17} \right)^p, \quad (31)$$

for every $\nu \in \mathcal{P}(\text{spt}(\mu))$

Lemma 4.3. *Let V , $\tilde{\rho}_m$ and ρ_∞ be as in Proposition 3.1. Then, for sufficiently large m ,*

$$W_\infty(\rho_\infty, \tilde{\rho}_m) \lesssim \frac{1}{m-1} \quad (32)$$

Proof. Since $\{V < C_\infty\}$ is a bounded connected domain with C^2 boundary, there exists $c > 0$ such that $m(t) \geq ct^d$ for every $0 < t < \text{diam}(\{V < C_\infty\})$. Since $\tilde{\rho}_m \in \mathcal{P}(\text{spt}(\rho_\infty))$, Theorem 4.2 gives us that, for every $1 \leq p < +\infty$

$$W_\infty(\rho_\infty, \tilde{\rho}_m) \leq 17 \left(\frac{2}{c} \right)^{\frac{1}{p+d}} W_p^{\frac{p}{p+d}}(\rho_\infty, \tilde{\rho}_m). \quad (33)$$

The factor $17 \left(\frac{2}{c} \right)^{\frac{1}{p+d}}$ is uniformly bounded in p , therefore the inequality above can be written with a constant $C > 0$ depending only on d and the domain $\{V < C_\infty\}$.

Choose $p = m/2$, by Proposition 3.3 and Lemma 4.1 and Equation (33)

$$W_\infty(\rho_\infty, \tilde{\rho}_m) \lesssim \frac{1}{(m-1)^{\frac{m}{m+2d}}} \lesssim \frac{1}{m-1}$$

since $(m-1)^{\frac{2d}{m+2d}} \rightarrow 1$ as $m \rightarrow \infty$, and thus it is uniformly bounded in m . □

Finally, we are ready to prove the main theorem.

Proof of Theorem 1.1. Let $\tilde{\rho}_m = (\Psi_m^{-1})_{\#} \rho_m$ where Ψ_m is the diffeomorphism given by corollary 2.4. The triangular inequality gives

$$W_{\infty}(\rho_m, \rho_{\infty}) \leq W_{\infty}(\rho_m, \tilde{\rho}_m) + W_{\infty}(\tilde{\rho}_m, \rho_{\infty}). \quad (34)$$

Observe that Ψ_m^{-1} is a candidate transport map from ρ_m to $\tilde{\rho}_m$, therefore, by Corollary 2.4 equation (3) we have

$$W_{\infty}(\rho_m, \tilde{\rho}_m) \leq \|\Psi_m - \text{Id}\|_{L^{\infty}(\mathbb{R}^d)} \lesssim |C_m - C_{\infty}|. \quad (35)$$

By Lemma 4.3

$$W_{\infty}(\tilde{\rho}_m, \rho_{\infty}) \lesssim \frac{1}{m-1}. \quad (36)$$

Therefore, plugging (35) and (36) into (34) we have, for all sufficiently large m and any $p \in [1, +\infty]$

$$W_p(\rho_m, \rho_{\infty}) \leq W_{\infty}(\rho_m, \rho_{\infty}) \lesssim \frac{1}{m-1} \quad (37)$$

It remains to show that the rate is sharp. Let φ be a nonnegative, nonzero, 1-Lipschitz function compactly supported in $\{0 < C_{\infty} - V < 1/4\}$. By Lemma 2.3, for sufficiently large m we have $(C_m - V(x))_+ \leq 1/2$ on $\text{spt}(\varphi)$. Since $K_m \leq 1$, the concavity of $z \mapsto z^{\frac{1}{m-1}}$ gives

$$\rho_m(x) \leq 1 + \frac{(C_m - V(x))_+ - 1}{m-1} \leq 1 - \frac{1}{2(m-1)}$$

on $\text{spt}(\varphi)$. Since $\rho_{\infty} = 1$ there, the Kantorovich–Rubinstein theorem gives

$$W_1(\rho_{\infty}, \rho_m) \geq \int_{\mathbb{R}^d} \varphi(x)(\rho_{\infty} - \rho_m) dx \geq \frac{1}{2(m-1)} \int_{\mathbb{R}^d} \varphi(x) dx \gtrsim \frac{1}{m-1}.$$

As φ is fixed, the implicit constant is independent of m . The lower bound for every $p \in [1, +\infty]$ follows by monotonicity of the Wasserstein distances, with a constant independent of p . $\square$

APPENDIX A.

Proof of Proposition 2.1. Consider $\delta > 0$ sufficiently small so that $C - 3\delta > 0$. Let χ be a smooth function such that $0 \leq \chi \leq 1$ with χ equal to 1 on $\{C - \delta \leq V \leq C + \delta\}$ and equal to 0 outside $\{C - 3\delta \leq V \leq C + 3\delta\}$. Since V is coercive, the set $\{V \leq C + 3\delta\}$ is compact, thus χ is compactly supported. Define the vector field

$$b(x) = \begin{cases} \chi(x) \frac{\nabla V(x)}{|\nabla V(x)|^2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases} \quad (38)$$

we have that b is compactly supported as χ is compactly supported. Since V is coercive, it admits a global minimizer. Any global minimizer is a critical point of V , and since 0 is the unique critical point by assumption,

it follows that 0 is the unique minimizer of V . As $V(0) = 0$, we have $V(x) > 0$ for every $x \neq 0$. Moreover, by uniqueness of the critical point, $\nabla V(x) \neq 0$ for every $x \neq 0$. Since $\text{spt}(\chi) \subset \{V \geq C - 3\delta\}$ and $C - 3\delta > 0$, we have $0 \notin \text{spt}(\chi)$. Hence $\nabla V \neq 0$ on $\text{spt}(\chi)$. Since $V \in C^2(\mathbb{R}^d)$ and $\nabla V \neq 0$ on $\text{spt}(\chi)$, we have $b \in C_c^1(\mathbb{R}^d; \mathbb{R}^d)$.

Consider now the flow generated by b as in (2), by the standard existence and uniqueness results for autonomous ODEs (see [5] Ch. 1) it follows that Ψ_s is a C^1 -diffeomorphism for every $s \in \mathbb{R}$. Let us show the desired properties of Ψ_s : Let us first show that $\Psi_s(\{V \leq C\}) = \{V \leq C + s\}$.

Let $x \in \{V \leq C\}$, computing $\frac{d}{ds} V(\Psi_s(x))$ gives

$$\frac{d}{ds} V(\Psi_s(x)) = \frac{\nabla V(\Psi_s(x)) \cdot \nabla V(\Psi_s(x))}{|\nabla V(\Psi_s(x))|^2} \chi(\Psi_s(x)) = \chi(\Psi_s(x)) \leq 1, \quad (39)$$

hence integrating (39) on $[0, s]$ gives

$$V(\Psi_s(x)) \leq V(x) + s \leq C + s \quad (40)$$

thus $\Psi_s(x) \in \{V \leq C + s\}$ which shows that $\Psi_s(\{V \leq C\}) \subset \{V \leq C + s\}$. To show the other inclusion, let $y \in \{V \leq C + s\}$ and assume, for the sake of contradiction, that $V(\Psi_{-s}(y)) > C$. Consider the backwards flows $t \mapsto \Psi_{-t}(y)$ for $t \in [0, s]$. A similar computation to (39) gives

$$\frac{d}{dt} V(\Psi_{-t}(y)) = -\chi(\Psi_{-t}(y)) \leq 0 \quad (41)$$

thus, the map $t \mapsto V(\Psi_{-t}(y))$ is non-increasing. Thus, from the contradicting assumption and from $y \in \{V \leq C + s\}$ we have

$$C < V(\Psi_{-s}(y)) \leq V(\Psi_{-t}(y)) \leq V(\Psi_0(y)) = V(y) \leq C + s$$

which means that the entire curve $t \mapsto \Psi_{-t}(y)$ lies in the region $\{C < V \leq C + \delta\}$ and thus $\frac{d}{dt} V(\Psi_{-t}(y)) = -1$. Integrating this equality on $[0, s]$ leads to

$$V(\Psi_{-s}(y)) - V(y) = -s \quad (42)$$

and thus, from $y \in \{V \leq C + s\}$ we get $V(\Psi_{-s}(y)) = V(y) - s \leq C$ which contradicts $V(\Psi_{-s}(y)) > C$. Hence $\Psi_s(\{V \leq C\}) = \{V \leq C + s\}$ holds for $0 < s < \delta$.

For the case $-\delta < s < 0$, suffices to note that, from what was previously shown, $\Psi_r(\{V \leq C - r\}) = \{V \leq C\}$ for $0 < r < \delta$, because $\chi = 1$ on the region $C - r \leq V < C$. From $\Psi_r^{-1} = \Psi_{-r}$ we obtain that $\{V \leq C - r\} = \Psi_{-r}(\{V \leq C\})$ and thus choosing $s = -r$ we obtain the result for $-\delta < s < 0$.

Let $0 < s < \delta$, the integral form of (2) gives

$$\Psi_s(x) - x = \int_0^s b(\Psi_\tau(x)) d\tau \quad (43)$$

since b is continuous and compactly supported, $\|b\|_\infty < +\infty$ and thus

$$|\Psi_s - x| \leq \|b\|_\infty s \quad (44)$$

and for $-\delta < s < 0$ the same inequality follows by observing that $\Psi_{-s} = \Psi_s^{-1}$, thus

$$\|\Psi_s - \text{Id}\|_\infty \leq \|b\|_\infty |s| \quad (45)$$

Differentiating (2) we obtain

$$\frac{d}{ds} D\Psi_s(x) = D[b(\Psi_s(x))] = Db(\Psi_s(x))D\Psi_s(x), \quad (46)$$

then, by Jacobi's formula we have, since $D\Psi_s(x)$ is invertible,

$$\begin{aligned} \frac{d}{ds} \det(D\Psi_s(x)) &= \det(D\Psi_s(x)) \operatorname{tr}([D\Psi_s(x)]^{-1} \frac{d}{ds} D\Psi_s(x)) \\ &= \det(D\Psi_s(x)) \operatorname{tr}([D\Psi_s(x)]^{-1} Db(\Psi_s(x))D\Psi_s(x)) \\ &= \det(D\Psi_s(x)) \operatorname{tr}(Db(\Psi_s(x))) \\ &= \det(D\Psi_s(x)) \nabla \cdot (b(\Psi_s(x))) \end{aligned} \quad (47)$$

thus, the function $(x, s) \mapsto \det(D\Psi_s(x))$ solves the following ODE

$$\begin{cases} \frac{d}{ds} \det(D\Psi_s(x)) = (\nabla \cdot b)(\Psi_s(x)) \det(D\Psi_s(x)) \\ \det(D\Psi_0(x)) = 1 \end{cases} \quad (48)$$

therefore, we obtain

$$\det(D\Psi_s(x)) = \exp \left(\int_0^s (\nabla \cdot b)(\Psi_\tau(x)) d\tau \right). \quad (49)$$

Using the standard estimate $|e^a - 1| \leq e^{|a|}|a|$ we obtain

$$\begin{aligned} |\det(D\Psi_s(x)) - 1| &\leq \exp \left(\int_0^s |(\nabla \cdot b)(\Psi_\tau(x))| d\tau \right) \int_0^s |(\nabla \cdot b)(\Psi_\tau(x))| d\tau \\ &\leq \exp(\|\nabla \cdot b\|_\infty |s|) \|\nabla \cdot b\|_\infty |s| \\ &\leq \exp(\|\nabla \cdot b\|_\infty \delta) \|\nabla \cdot b\|_\infty |s| \end{aligned} \quad (50)$$

Thus, choosing $K := 2 \max(\|b\|_\infty, \exp(\|\nabla \cdot b\|_\infty \delta) \|\nabla \cdot b\|_\infty)$ and combining (44) and (50) we obtain (3) $\square$

Proof of Proposition 2.2. Let $C > 0$, and let b , δ and Ψ_s be as in the proof of Proposition 2.1. Thus, we have $\Psi_s(\{V \leq C\}) = \{V \leq C + s\}$, hence $|\Psi_s(\{V \leq C\})| = A(C + s)$. By the change of variables we have

$$A(C + s) = \int_{\{V \leq C\}} \det D\Psi_s(x) dx.$$

By (49), for $|s| < \delta$

$$A(C + s) = \int_{\{V \leq C\}} \underbrace{\exp \left(\int_0^s (\nabla \cdot b)(\Psi_\tau(x)) d\tau \right)}_{g(x,s)} dx.$$

Since the function $(x, s) \mapsto g(x, s)$ is continuous, $(x, s) \mapsto \partial_s g(x, s)$ is continuous and bounded on $\{V \leq C\} \times [-\delta, \delta]$, we may differentiate under the integral sign to obtain

$$\frac{d}{ds} A(C + s) = \int_{V \leq C} (\nabla \cdot b)(\Psi_s(x)) \exp \left(\int_0^s (\nabla \cdot b)(\Psi_\tau(x)) d\tau \right) dx.$$

Now we show that $s \mapsto \frac{d}{ds} A(C + s)$ is continuous: Let $s_n \rightarrow s$, then by continuity we have

$$\begin{aligned} & (\nabla \cdot b)(\Psi_{s_n}(x)) \exp \left(\int_0^{s_n} (\nabla \cdot b)(\Psi_\tau(x)) d\tau \right) \\ & \longrightarrow (\nabla \cdot b)(\Psi_s(x)) \exp \left(\int_0^s (\nabla \cdot b)(\Psi_\tau(x)) d\tau \right). \end{aligned}$$

and

$$\left| (\nabla \cdot b)(\Psi_{s_n}(x)) \exp \left(\int_0^{s_n} (\nabla \cdot b)(\Psi_\tau(x)) d\tau \right) \right| \leq \|\nabla \cdot b\|_\infty \exp(\|\nabla \cdot b\|_\infty \delta)$$

hence, continuity of $s \mapsto \frac{d}{ds} A(C + s)$ follows by dominated convergence. We thus obtain that A is of class C^1 in a neighbourhood of C , and since $C > 0$ was arbitrarily taken, we have $A \in C^1((0, +\infty))$.

We still must show that A is continuous at 0. From the assumptions on V it follows that the unique minimizer of V is $x = 0$, thus $A(0) = |\{V \leq 0\}| = 0$. Let t_n be a decreasing sequence converging to zero, then $\lim_{t_n \rightarrow 0} A(t_n) = \lim_{t_n \rightarrow 0} |\{V \leq t_n\}| = |\cap_n \{V \leq t_n\}| = |\{V = 0\}| = 0 = A(0)$.

By (47) we obtain

$$\begin{aligned} A'(C) &= \left. \frac{d}{ds} A(C + s) \right|_{s=0} \\ &= \int_{\{V \leq C\}} \left. \frac{d}{ds} \det D\Psi_s(x) \right|_{s=0} dx \\ &= \int_{\{V \leq C\}} \nabla \cdot b \, dx, \end{aligned}$$

and using the divergence theorem, observing that the outward unit normal vector on the set $\{V = C\}$ is $\frac{\nabla V}{|\nabla V|}$, we have

$$A'(C) = \int_{\{V=C\}} b \cdot \frac{\nabla V}{|\nabla V|} d\mathcal{H}^{d-1} = \int_{\{V=C\}} \frac{1}{|\nabla V|} d\mathcal{H}^{d-1}.$$

Finally, for every $t > 0$ we have that $\{V = t\} \neq \emptyset$ and $\nabla V \neq 0$ on $\{V = t\}$, thus,

$$A'(t) = \int_{\{V=t\}} \frac{1}{|\nabla V|} d\mathcal{H}^{d-1} > 0,$$

which implies that A is strictly increasing on $(0, +\infty)$. Since $A(0) = 0 < A(t)$ for every $t > 0$, A is actually strictly increasing on the larger set $[0, +\infty)$. $\square$

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AI STATEMENT

No AI has been used in this work.

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