

SOME REMARKS ON NONEXISTENCE IN THE GAMOW LIQUID DROP MODEL

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ABSTRACT. In this paper we deal with the celebrated Gamow liquid drop model, namely, for $m > 0$, $N \geq 2$ and $\alpha \in (0, N)$, we study the problem

$$\min \left\{ P(\Omega) + V_\alpha(\Omega) : \Omega \subset \mathbb{R}^N, |\Omega| = m \right\},$$

and focus mostly on the nonexistence of minimizers for large masses. First we show that any possible minimizer in the asymptotic regime must satisfy an optimal thickness property. Eventually, in the simplified setting of sets made up as a tiling of cubes, we show that nonexistence occurs for large masses for the whole range of parameters $N \geq 2$ and $\alpha \in (0, N)$.

1. INTRODUCTION

The celebrated Gamow liquid drop model was introduced by Gamow in [10] to quantitatively describe the binding energy curve for atomic nuclei. After a rescaling of the physical constants, this consists in solving

$$\min \left\{ P(\Omega) + V_\alpha(\Omega) : |\Omega| = m \right\}. \quad (1.1)$$

Here Ω denotes a measurable set in $\mathbb{R}^N$, $N \geq 2$, of measure $|\Omega| = m$, $P(\Omega)$ is its De Giorgi Perimeter and

$$V_\alpha(\Omega) = V_\alpha(\Omega, \Omega) = \int_\Omega \int_\Omega \frac{1}{|y - x|^{N-\alpha}} dx dy, \quad \text{with} \quad V_\alpha(E, F) = \int_E \int_F \frac{1}{|y - x|^{N-\alpha}} dx dy, \quad (1.2)$$

is the Riesz (interaction) energy, for $\alpha \in (0, N)$. By a rescaling argument, problem (1.1) is equivalent to

$$\inf \left\{ \mathcal{F}_{K,\alpha}(\Omega) := P(\Omega) + K V_\alpha(\Omega) : |\Omega| = |B_1| = \omega_N \right\}, \quad (1.3)$$

where B_1 is a ball of radius 1 and $K = (\frac{m}{\omega_N})^{\frac{1+\alpha}{N}}$. Intuitively, since for small values of m (or of K) the perimeter is the leading term, one expects that minimizers should exist and look more or less like balls. And in fact, this is well-known to be exact (see [13, 14, 7, 12]); actually much more is true, namely, for m small enough the unique minimizer is *exactly* the ball. Conversely, for large values of the mass the leading term is the repulsive term V_α , so one easily expects fragmentation of the mass and then nonexistence; one might even imagine much more, that is, that the minimizers exist only if they are a ball. Equivalently, one can say that “generalized minimizers” are unions of equal balls so that any two different balls have infinite distance (and so, no interaction). Concerning this second guess, the situation is much less clear. The fact that minimizers are equal balls at infinite distance has been established in [2] only in a very restricted range of exponents; instead, the nonexistence of minimizers for large mass (which is weaker than the “balls at infinite property”) is now known for every dimension $N \in \mathbb{N}$ but only for exponents $\alpha \in [N - 2, N)$, see [13, 14, 15, 8, 9]. A full understanding of the problem in the physical case $N = 3$, $\alpha = 2$ has appeared only recently [4]: there is a physical, explicit threshold K^* such that balls are the unique minimizer if $K \leq K^*$, while no minimizer exists for $K > K^*$. We also point out the recent papers [6, 5] that deal with sets which are critical points of the Gamow energy for any value of K (in the physical case $N = 3$ and $\alpha = 2$).

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1.1. Scope of the paper and main results. The aim of this note is to shed some light on the behavior of the problem in the regime $K \gg 1$, with particular focus on the nonexistence issue; since as said above the nonexistence is known only for $\alpha \in [N-2, N)$, the unresolved range motivating the paper is $\alpha \in (0, N-2)$.

In order to state our first result, we consider for a moment the “full conjecture”, that is, that a generalized minimizer must be a union of equal balls with infinite mutual distances. Assuming that this conjecture is true, it is immediate to calculate the corresponding radius of these balls; namely, up to a multiplicative constant, the radius of the optimal balls for $\mathcal{F}_{K,\alpha}$ must be of order

$$r_{K,\alpha} = K^{-\frac{1}{1+\alpha}}, \quad (1.4)$$

see the discussion in Section 2. Note that the volume constraint then requires the number of balls to be of order $r_{K,\alpha}^{-N}$. It is easy to check that, on this union of balls, there is *equipartition of the energy*, namely,

$$P(\cup_i B_{r_{K,\alpha}}(x_i)) \sim KV_\alpha(\cup_i B_{r_{K,\alpha}}(x_i)) \sim K^{\frac{1}{1+\alpha}}.$$

This may seem to be a peculiar property, but in fact it holds for all minimizers of $\mathcal{F}_{K,\alpha}$, as it was proved in [14] with interpolation inequalities, but may be shown also with elementary geometric arguments. Precisely, if Ω_K is an optimal set for (1.3), then

$$P(\Omega_K) \sim KV_\alpha(\Omega_K) \sim K^{\frac{1}{1+\alpha}}, \quad (1.5)$$

up to constants depending only on N, α . This fact suggests that the quantity $r_{K,\alpha}$ may have a special role in the problem.

In fact, our first result says that if a minimizer exists, its “thickness” must indeed be of the same order as $r_{K,\alpha}$. More precisely, we have the following.

Theorem 1.1 (Optimal thickness). *Let $N \geq 2$. There exists a constant $c = c(N, \alpha)$ such that for every $K \geq 1$, if Ω_K is an optimal set for (1.3), then the following facts hold.*

- Ω_K is not too thick: Ω_K does not contain any ball of radius $cr_{K,\alpha}$.
- Ω_K is not too thin: For every ball $B_r(x)$ with radius $r < c^{-1}r_{K,\alpha}$, we have

$$\mathcal{H}^{N-1}(\Omega_K \cap \partial B_r(x)) \geq \frac{1}{2} \mathcal{H}^{N-1}(\partial^* \Omega_K \cap B_r(x)). \quad (1.6)$$

The above claim deserves a quick comment: while the fact that Ω_K does not contain any ball of radius $cr_{K,\alpha}$ clearly says in a sense that “ Ω_K is not too thick”, it might be at first glance less clear why (1.6) should mean that “ Ω_K is not too thin”. But in fact, that estimate is precisely saying that the “size” of Ω_K must be at least of order $r_{K,\alpha}$; indeed, (1.6) prevents Ω_K from having connected components compactly contained in a ball of radius $r_{K,\alpha}/c$, and more in general Ω_K cannot have tentacles with radius much smaller than $r_{K,\alpha}$ or similar structures. It is to be underlined that the constant $1/2$ in (1.6) can be replaced by any other constant smaller than 1. In light of the thickness result of Theorem 1.1, one can qualitatively guess the shape of possible minimizers: this is compatible, for instance, with cylinder-like or slab-like structures with thickness of order $r_{K,\alpha}$.

Our next aim is to investigate a simplified setting in which the mechanism leading to fragmentation can be analyzed explicitly. To this end, we restrict the admissible configurations to finite unions of congruent cubes on a lattice; although this model does not resolve the full problem, it allows us to isolate and quantify the competition between the perimeter gain produced by contacts and the repulsive interaction between the components.

In analogy with the conjectured picture of generalized minimizers as collections of components whose mutual distances diverge, we first consider isolated cubes and determine the side length which minimizes the energy per unit volume. Namely, setting

$$g_{K,\alpha}(r) := \frac{\mathcal{F}_{K,\alpha}(rQ)}{|rQ|} = \frac{2N}{r} + KV_\alpha(Q)r^\alpha, \quad (1.7)$$

we find the optimal side length

$$\rho_{K,\alpha} := \left(\frac{2N}{\alpha KV_\alpha(Q)} \right)^{\frac{1}{1+\alpha}}, \quad (1.8)$$

where $Q = (0, 1)^N$. Notice that

$$\rho_{K,\alpha} \sim r_{K,\alpha} \sim K^{-\frac{1}{1+\alpha}},$$

as suggested by the scaling of the original problem.

If one were to prescribe exactly the side length $\rho_{K,\alpha}$, however, the volume constraint $|E| = \omega_N$ would in general not be compatible with an integer number of congruent cubes. This does not affect the argument below, since the geometric nonattainment mechanism applies more generally whenever the common side length ℓ satisfies

$$\ell \geq \rho_{K,\alpha}.$$

We therefore change the point of view while keeping the total volume fixed. We fix an integer $M \geq 2$ and consider unions of exactly M congruent axis-parallel cubes. Precisely, we set

$$\mathcal{A}_M := \left\{ \bigcup_{i=1}^M \ell_M(Q + a_i) : a_1, \dots, a_M \in \mathbb{Z}^N \text{ distinct} \right\}. \quad (1.9)$$

The condition $|E| = \omega_N$ then determines their side length uniquely as

$$\ell_M := \left(\frac{\omega_N}{M} \right)^{1/N}. \quad (1.10)$$

For this fixed side length, the condition $\ell_M \geq \rho_{K,\alpha}$ can equivalently be written as a lower bound on K . Indeed,

$$\ell_M \geq \rho_{K,\alpha} \quad \text{if and only if} \quad \ell_M^{1+\alpha} \geq \frac{2N}{\alpha K V_\alpha(Q)},$$

which is equivalent to

$$K \geq \frac{2N}{\alpha V_\alpha(Q)} \ell_M^{-(1+\alpha)}, \quad \text{that is,} \quad K \geq \frac{2N}{\alpha V_\alpha(Q)} \left(\frac{M}{\omega_N} \right)^{\frac{1+\alpha}{N}}.$$

This motivates the definition

$$K_0(N, \alpha, M) := \frac{2N}{\alpha V_\alpha(Q)} \left(\frac{M}{\omega_N} \right)^{\frac{1+\alpha}{N}}, \quad (1.11)$$

so that the condition $\ell_M \geq \rho_{K,\alpha}$ is exactly equivalent to $K \geq K_0(N, \alpha, M)$.

We prove the following nonexistence result for the Gamow problem in the cubic setting in the large- K regime.

Theorem 1.2 (Cubical nonexistence). *Let $N \geq 2$, $\alpha \in (0, N)$, and let $M \geq 2$ be an integer. Set $\ell = \ell_M$ as in (1.10). Then for every*

$$K \geq K_0(N, \alpha, M), \quad (1.12)$$

one has

$$\inf_{E \in \mathcal{A}_M} \mathcal{F}_{K,\alpha}(E) = M(2N\ell^{N-1} + K\ell^{N+\alpha}V_\alpha(Q)), \quad (1.13)$$

and the infimum is not attained. It is approached by configurations in which all mutual distances between the M cubes tend to infinity.

The main analytic ingredient is an exact comparison between the self-interaction of a cube and the interaction of two cubes sharing a face. Writing $e_1 = (1, 0, \dots, 0)$, we show in Proposition 4.2 that

$$V_\alpha(Q, Q + e_1) = \frac{\alpha}{2N} V_\alpha(Q) + R_{N,\alpha}, \quad R_{N,\alpha} > 0. \quad (1.14)$$

The proof uses the product structure of Q and integration by parts with the vector field $z|z|^{\alpha-N}$.

At the fixed side length $\ell = \ell_M$, the definition of K_0 gives

$$K_0 \ell^{1+\alpha} V_\alpha(Q) = \frac{2N}{\alpha}. \quad (1.15)$$

Combining (1.15) with (1.14) yields

$$K \ell^{1+\alpha} V_\alpha(Q, Q + e_1) \geq \frac{2N V_\alpha(Q, Q + e_1)}{\alpha V_\alpha(Q)} > 1 \quad \text{for every } K \geq K_0. \quad (1.16)$$

A shared face reduces the perimeter by $2\ell^{N-1}$, whereas the mutual interaction of the two touching cubes contributes $2K\ell^{N+\alpha}V_\alpha(Q, Q + e_1)$. Hence every face contact is strictly unfavourable for $K \geq K_0$. All

other pairs give a strictly positive interaction and no perimeter reduction. This pairwise comparison is the mechanism behind Theorem 1.2.

The plan of the paper is very simple. First, in Section 2 we fix the notation and collect some simple estimates. Then, Section 3 is devoted to prove Theorem 1.1. Finally, in Section 4 we show Theorem 1.2.

2. NOTATIONS AND PRELIMINARIES

This section is devoted to fix the notation and to present some preliminary observations and estimates. Through all the paper we will denote by $N \geq 2$ the dimension of the ambient space, and by α a given constant in $(0, N)$ that we use to define the Riesz energy as in (1.2); for any $K \geq 1$, we consider the energy $\mathcal{F}_{K,\alpha}$ as in (1.3). The elements of the canonical basis of $\mathbb{R}^N$ will be denoted as e_i for $1 \leq i \leq N$. Given any set $E \subseteq \mathbb{R}^N$, we will denote by $|E| = \mathcal{H}^N(E)$ its volume and by $P(E) = \mathcal{H}^{N-1}(\partial^* E)$ its perimeter (in the De Giorgi sense, see for instance [1]); for any open set $A \subseteq \mathbb{R}^N$, the relative perimeter of E in A will be denoted as $P(E; A) = \mathcal{H}^{N-1}(\partial^* E \cap A)$.

For any $r > 0$ and any $x \in \mathbb{R}^N$, the ball with radius r and center x will be denoted as $B_r(x)$, or just B_r if the center is the origin, or B if the ball has unit volume. By rescaling, for every $\lambda > 0$ and every $E, F \subset \mathbb{R}^N$ we have

$$P(\lambda E) = \lambda^{N-1} P(E), \quad V_\alpha(\lambda E, \lambda F) = \lambda^{N+\alpha} V_\alpha(E, F). \quad (2.1)$$

As described in the Introduction, it is conjectured that for K large the problem (1.3) has no solution, and that a minimizing sequence is given by equal balls of some optimal radius having exploding mutual distances; equivalently, we can say that a “generalized solution” is given by equal balls at infinite distance (for a formal description of these generalized solutions see for instance [16]). Let us then restrict our attention, for a moment, to a set Ω made by H equal balls, having very large mutual distances. Then, neglecting the mutual interaction between different balls (which is arbitrarily small up to take the balls distant enough) we have

$$\begin{aligned} \mathcal{F}_{K,\alpha}(\Omega) &= P(\Omega) + K V_\alpha(\Omega) = H \left(P(B_{H^{-1/N}}) + K V_\alpha(B_{H^{-1/N}}) \right) = H \left(N \omega_N H^{-\frac{N-1}{N}} + K H^{-\frac{N+\alpha}{N}} V_\alpha(B) \right) \\ &= N \omega_N H^{1/N} + K H^{-\frac{\alpha}{N}} V_\alpha(B). \end{aligned}$$

As a consequence, among the union of equal balls, the best one has a number of balls which is $H \sim K^{\frac{N}{1+\alpha}}$; the corresponding “best radius” is then of order $K^{-\frac{1}{1+\alpha}}$, so it corresponds (up to multiplicative constants) to the radius $r_{K,\alpha}$ defined in (1.4).

Now, we can show that minimizers of $\mathcal{F}_{K,\alpha}$ for K large, if they exist, must have very large diameter. The result is the following.

Lemma 2.1. *Let $K \geq 1$ and let Ω_K be such that $\mathcal{F}_{K,\alpha}(\Omega_K) < \zeta \inf(1.3)$, being $\zeta > 1$ given. Then, for any unit cube Q we have*

$$|\Omega_K \cap Q| \lesssim K^{-\frac{\alpha}{2(1+\alpha)}}. \quad (2.2)$$

As a consequence,

$$\text{diam}(\Omega_K) \gtrsim K^{\frac{\alpha}{(1+\alpha)(N-\alpha)}}. \quad (2.3)$$

Proof. We have already noticed that a union of balls of radius of order $K^{\frac{1}{1+\alpha}}$ and very far from each other have energy of order $K^{\frac{1}{1+\alpha}}$, and being Ω_K optimal (up to a factor ζ) it must clearly be $K V_\alpha(\Omega_K) \leq \mathcal{F}_{K,\alpha}(\Omega_K) \lesssim K^{\frac{1}{1+\alpha}}$, which means $V_\alpha(\Omega_K) \lesssim K^{-\frac{\alpha}{1+\alpha}}$. We have then

$$K^{-\frac{\alpha}{1+\alpha}} \gtrsim V_\alpha(\Omega_K) \geq \int_{\Omega_K \cap Q} \int_{\Omega_K \cap Q} \frac{1}{|y-x|^{N-\alpha}} dx dy \gtrsim |\Omega_K \cap Q|^2,$$

and this is (2.2). The same argument, substituting the unit cube Q with a cube $\tilde{Q}$ of diameter d , gives

$$K^{-\frac{\alpha}{1+\alpha}} \gtrsim |\Omega_K \cap \tilde{Q}|^2 d^{\alpha-N},$$

and this clearly implies (2.3). $\square$

3. PROOF OF THEOREM 1.1

This section is devoted to the proof of Theorems 1.1.

Proof (of Theorem 1.1). We start with the first part, namely, that if c is large enough then for every $K \geq 1$ an optimal set Ω_K cannot contain a ball with radius $cr_{K,\alpha}$. Notice that, up to increase the value of c , we are allowed to assume that $cr_{K,\alpha}$ is quite small. More precisely suppose that we have found c_0 such that for every $K \geq 1$ and every optimal set Ω_K we have that Ω_K cannot contain a ball of radius $c_0 r_{K,\alpha}$ provided that $c_0 r_{K,\alpha} \leq \ell$ for some given small $\ell \leq 1$. But then, the theorem is true with $c = c_0/\ell$; in fact, if $c_0 r_{K,\alpha} \leq \ell$ then Ω_K cannot contain a ball of radius $c_0 r_{K,\alpha}$, and then it cannot contain a ball of radius $cr_{K,\alpha}$. Instead, if $c_0 r_{K,\alpha} > \ell$, then $cr_{K,\alpha} > 1$ and then for sure Ω_K cannot contain a ball of radius $cr_{K,\alpha}$, simply because such a ball has volume strictly larger than $\omega_N = |\Omega_K|$.

Now, suppose by contradiction that for some $K \geq 1$ an optimal set Ω_K contains the ball $B_{cr_{K,\alpha}}$; then, we define the competitor

$$\tilde{\Omega} = (1 - (cr_{K,\alpha})^N)^{-\frac{1}{N}} (\Omega_K \setminus B_{cr_{K,\alpha}}),$$

which satisfies $|\tilde{\Omega}| = \omega_N$. Since $cr_{K,\alpha}$ is small, we have

$$\begin{aligned} P(\tilde{\Omega}) &= (1 - (cr_{K,\alpha})^N)^{-\frac{N-1}{N}} (P(\Omega_K) + N\omega_N(cr_{K,\alpha})^{N-1}) \leq (1 + (cr_{K,\alpha})^N) (P(\Omega_K) + N\omega_N(cr_{K,\alpha})^{N-1}) \\ &\leq P(\Omega_K) + (cr_{K,\alpha})^N P(\Omega_K) + 2N\omega_N(cr_{K,\alpha})^{N-1}, \end{aligned}$$

while

$$\begin{aligned} V_\alpha(\tilde{\Omega}) &= (1 - (cr_{K,\alpha})^N)^{-\frac{N+\alpha}{N}} V_\alpha(\Omega_K \setminus B_{cr_{K,\alpha}}) \leq (1 + 2(cr_{K,\alpha})^N) (V_\alpha(\Omega_K) - V_\alpha(B_{cr_{K,\alpha}})) \\ &\leq V_\alpha(\Omega_K) + 2(cr_{K,\alpha})^N V_\alpha(\Omega_K) - (cr_{K,\alpha})^{N+\alpha} V_\alpha(B). \end{aligned}$$

We get a contradiction if $\mathcal{F}_{K,\alpha}(\tilde{\Omega}) < \mathcal{F}_{K,\alpha}(\Omega_K)$, and putting together the last two estimates this is true if

$$(cr_{K,\alpha})^N P(\Omega_K) + 2N\omega_N(cr_{K,\alpha})^{N-1} + 2(cr_{K,\alpha})^N K V_\alpha(\Omega_K) \leq K(cr_{K,\alpha})^{N+\alpha} V_\alpha(B).$$

Since we can choose a large c and the highest power of c in the above inequality is on the right, and keeping in mind that $V_\alpha(B)$ is a constant, then it is enough to show that

$$r_{K,\alpha}^N \mathcal{F}_{K,\alpha}(\Omega_K) \lesssim K r_{K,\alpha}^{N+\alpha}, \quad r_{K,\alpha}^{N-1} \lesssim K r_{K,\alpha}^{N+\alpha}.$$

The definition (1.4) of $r_{K,\alpha}$ gives $K r_{K,\alpha}^{N+\alpha} = r_{K,\alpha}^{N-1}$, so the second estimate is empty true, and also the first estimate is true since $\mathcal{F}_{K,\alpha}(\Omega_K) \lesssim r_{K,\alpha}^{-1} = K^{\frac{1}{1+\alpha}}$, as already noticed.

Let us now pass to consider the second part. Let Ω_K be an optimal set for (1.3), and let us consider a ball with radius $r < c^{-1} r_{K,\alpha}$, that for simplicity of notations we assume to be the ball B_r centered at the origin. We can define the competitor

$$\tilde{\Omega} = \lambda(\Omega_K \setminus B_r),$$

where λ is the constant (slightly larger than 1) such that $|\tilde{\Omega}| = |\Omega_K| = \omega_N$, thus

$$\lambda = \left(\frac{\omega_N}{\omega_N - v} \right)^{1/N},$$

where we denote for brevity $v = |\Omega_K \cap B_r|$. If (1.6) is false, then

$$\mathcal{F}_{K,\alpha}(\Omega_K \setminus B_r) < \mathcal{F}_{K,\alpha}(\Omega_K) + (P(\Omega_K \setminus B_r) - P(\Omega_K)) \leq \mathcal{F}_{K,\alpha}(\Omega_K) - \frac{1}{3} P(\Omega_K \cap B_r) \leq \mathcal{F}_{K,\alpha}(\Omega_K) - \frac{N\omega_N^{1/N}}{3} v^{\frac{N-1}{N}}.$$

Keeping in mind the rescaling formulas (2.1) and the fact that $\lambda \geq 1$, we get

$$\begin{aligned} \mathcal{F}_{K,\alpha}(\tilde{\Omega}) &\leq \lambda^{N+\alpha} \mathcal{F}_{K,\alpha}(\Omega_K \setminus B_r) \leq \left(1 + \frac{4v}{\omega_N} \right) \left(\mathcal{F}_{K,\alpha}(\Omega_K) - \frac{N\omega_N^{1/N}}{3} v^{\frac{N-1}{N}} \right) \\ &\leq \mathcal{F}_{K,\alpha}(\Omega_K) + \frac{4v}{\omega_N} \mathcal{F}_{K,\alpha}(\Omega_K) - \frac{N\omega_N^{1/N}}{3} v^{\frac{N-1}{N}}. \end{aligned}$$

Recalling once again that $\mathcal{F}_{K,\alpha}(\Omega_K) \lesssim r_{K,\alpha}^{-1}$, this estimate provides us the required contradiction as soon as $r_{K,\alpha} \gtrsim v^{\frac{1}{N}}$, and since by construction $v \leq \omega_N (c^{-1} r_{K,\alpha})^N$ we obtain the thesis as soon as c has been chosen large enough. $\square$

Notice that in the above proof the constant $c_{N,\alpha}$ can be explicitly traced. We now notice a simple consequence of Lemma 2.1, which will not be needed in the rest of the paper but it is an interesting and elementary “cutting property”.

Corollary 3.1. *Let Ω_K be a set such that $\mathcal{F}_{K,\alpha}(\Omega_K) < \zeta \inf (1.3)$, for some given $\zeta > 1$. Then, provided that K is large enough, there exists a hyperplane H such that, calling $H^\pm$ the two parts in which $\mathbb{R}^N$ is divided by H , one has*

$$\mathcal{H}^{N-1}(\Omega_K \cap H) \leq 1, \quad |\Omega_K \cap H^\pm| \in \left(\frac{\omega_N}{3N}, \omega_N - \frac{\omega_N}{3N} \right). \quad (3.1)$$

Proof. Let us define $a_1 < b_1$ as

$$a_1 = \sup \left\{ t \in \mathbb{R}, |\Omega_K \cap \{x_1 \leq t\}| \leq \frac{\omega_N}{3N} \right\}, \quad b_1 = \inf \left\{ t \in \mathbb{R}, |\Omega_K \cap \{x_1 \geq t\}| \leq \frac{\omega_N}{3N} \right\},$$

so that the part of Ω_K which is not contained in the slab $\Pi_1 = [a_1, b_1] \times \mathbb{R}^{N-1}$ has volume $2\omega_N/3N$. If the width $b_1 - a_1$ of the slab is greater than ω_N , then there is some $a_1 < t < b_1$ such that

$$\mathcal{H}^{N-1}(\Omega_K \cap \{x_1 = t\}) \leq 1,$$

and then the proof is obtained with $H = \{x_1 = t\}$. If we repeat the same argument with slabs Π_j with $2 \leq j \leq N$, we conclude the proof unless each of the slabs has width smaller than ω_N . In other words, if the claim is false then the intersection of the slabs is contained in a cube Q of side ω_N ; moreover, by construction the intersection of this cube with Ω_K has volume at least $\omega_N/3$. And this is clearly in contradiction with Lemma 2.1 as soon as K is large enough. $\square$

4. PROOF OF THEOREM 1.2

Thanks to the scaling highlighted in Section 2, for $r > 0$, we have

$$\mathcal{F}_{K,\alpha}(rQ) = 2Nr^{N-1} + Kr^{N+\alpha}V_\alpha(Q). \quad (4.1)$$

For the prescribed-volume class (1.9), the side length is $r = \ell (= \ell_M)$ and is therefore completely determined by ω_N and the integer M .

Note also that (1.11) is equivalent to

$$K_0 \ell^{1+\alpha} V_\alpha(Q) = \frac{2N}{\alpha}. \quad (4.2)$$

Recalling the energy per unit volume of a cube of side length r from (1.7), K_0 is precisely the value for which the fixed side length ℓ satisfies the stationary scaling relation for the isolated-cube specific energy.

Remark 4.1. *Let us also show that the asymptotic value of the functional among disjoint union of cubes of sidelength ℓ at reciprocal diverging distance is exactly $M\mathcal{F}_{K,\alpha}(\ell Q)$, that is, the right hand side of (1.13). For integers $n \geq 2$, we define*

$$E_n := \bigcup_{i=0}^{M-1} \ell(Q + ine_1). \quad (4.3)$$

Every E_n belongs to $\mathcal{A}_M$, and the cubes have no shared faces. For $0 \leq i < j \leq M-1$, their first coordinates are separated by at least $\ell((j-i)n-1)$. Since $N-\alpha > 0$,

$$V_\alpha(\ell(Q + ine_1), \ell(Q + jne_1)) \leq \frac{\ell^{N+\alpha}}{(n-1)^{N-\alpha}}.$$

It follows that, as $n \rightarrow +\infty$,

$$0 \leq \mathcal{F}_{K,\alpha}(E_n) - M\mathcal{F}_{K,\alpha}(\ell Q) \leq \frac{KM(M-1)\ell^{N+\alpha}}{(n-1)^{N-\alpha}} \rightarrow 0.$$

4.1. A technical identity. We develop here an exact identity for two cubes sharing a face, by a sort of calibration method. Define

$$h(t) := (1 - |t|)_+, \quad H(z) := \prod_{i=1}^N h(z_i). \quad (4.4)$$

Since $H(z) = |Q \cap (Q - z)|$, Fubini's theorem gives

$$V_\alpha(Q) = \int_{(-1,1)^N} H(z) |z|^{\alpha-N} dz. \quad (4.5)$$

The proof of the next result is based on the autocorrelation representation of the Riesz energy and on an integration by parts argument exploiting the homogeneity of the Riesz kernel. Related representations of Riesz energies in terms of overlap functions can be found in [11], while integration by parts arguments for Riesz-type energies in a geometric setting have been used, for example, recently in [3].

Proposition 4.2. *Let $N \geq 2$ and $\alpha \in (0, N)$. Then*

$$V_\alpha(Q, Q + e_1) = \frac{\alpha}{2N} V_\alpha(Q) + R_{N,\alpha}, \quad (4.6)$$

where, calling $z' = (z_2, \dots, z_N)$,

$$R_{N,\alpha} := \int_{(-1,1)^{N-1}} \prod_{j=2}^N h(z_j) \int_1^2 \frac{2-s}{(s^2 + |z'|^2)^{(N-\alpha)/2}} ds dz' > 0. \quad (4.7)$$

Proof. Let $X(z) := z|z|^{\alpha-N}$ for $z \neq 0$ and H be as in (4.4). A direct calculation gives

$$\operatorname{div} X(z) = \alpha |z|^{\alpha-N} \quad \text{for } z \neq 0.$$

For $0 < \varepsilon < 1$, set $D_\varepsilon := (-2, 2)^N \setminus \overline{B_\varepsilon}$. Since HX is Lipschitz on $\overline{D_\varepsilon}$, the divergence theorem, together with $H = 0$ on $\partial(-2, 2)^N$ and $\nu_{D_\varepsilon}(z) = -z/\varepsilon$ on ∂B_ε , gives

$$\alpha \int_{D_\varepsilon} H(z) |z|^{\alpha-N} dz + \int_{D_\varepsilon} X(z) \cdot \nabla H(z) dz = \int_{\partial B_\varepsilon} HX \cdot \nu_{D_\varepsilon} d\mathcal{H}^{N-1} = -\varepsilon^{\alpha-N+1} \int_{\partial B_\varepsilon} H d\mathcal{H}^{N-1}.$$

Using $0 \leq H \leq 1$ and $|\nabla H| \leq \sqrt{N}$ almost everywhere, we estimate the terms

$$\begin{aligned} \varepsilon^{\alpha-N+1} \int_{\partial B_\varepsilon} H d\mathcal{H}^{N-1} &\leq N\omega_N \varepsilon^\alpha, \\ \alpha \int_{B_\varepsilon} H(z) |z|^{\alpha-N} dz &\leq \alpha N\omega_N \int_0^\varepsilon r^{\alpha-1} dr = N\omega_N \varepsilon^\alpha, \quad \text{and} \\ \int_{B_\varepsilon} |X(z) \cdot \nabla H(z)| dz &\leq N^{3/2}\omega_N \int_0^\varepsilon r^\alpha dr = \frac{N^{3/2}\omega_N}{\alpha+1} \varepsilon^{\alpha+1}. \end{aligned}$$

All three terms tend to zero as $\varepsilon \downarrow 0$, since $\alpha > 0$. Recalling (4.5) and the support of H , we obtain

$$\alpha V_\alpha(Q) = - \int_{\mathbb{R}^N} X(z) \cdot \nabla H(z) dz. \quad (4.8)$$

Now note that, for almost every $z \in \mathbb{R}^N$,

$$\partial_i H(z) = -\operatorname{sgn}(z_i) \mathbf{1}_{\{|z_i| < 1\}} \prod_{j \neq i} h(z_j).$$

Substituting in (4.8) gives

$$\alpha V_\alpha(Q) = \sum_{i=1}^N \int_{(-1,1)^N} |z_i| \prod_{j \neq i} h(z_j) |z|^{\alpha-N} dz. \quad (4.9)$$

The N summands are equal by coordinate symmetry. Using also evenness in z_1 , we conclude that

$$\begin{aligned} \frac{\alpha}{2N} V_\alpha(Q) &= I_{\alpha,N}, \\ I_{\alpha,N} &:= \int_{(-1,1)^{N-1}} \prod_{j=2}^N h(z_j) \int_0^1 \frac{s}{(s^2 + |z'|^2)^{(N-\alpha)/2}} ds dz'. \end{aligned} \quad (4.10)$$

These integrals are finite: near zero, the integrands in (4.9) are bounded by $|z|^{\alpha-N+1}$. We now compute $V_\alpha(Q, Q + e_1)$. For $x \in Q$, $y \in Q + e_1$, set $z = y - x$. Making the change of variables $s = y_1 - x_1$, we observe that, for each fixed s , the admissible values of x_1 belong to the interval

$$(0, 1) \cap (1 - s, 2 - s),$$

whose length is

$$a(s) := \begin{cases} s, & 0 < s < 1, \\ 2 - s, & 1 \leq s < 2, \\ 0, & \text{otherwise.} \end{cases}$$

The corresponding density in every other coordinate is h . Therefore

$$\begin{aligned} V_\alpha(Q, Q + e_1) &= \int_{(-1,1)^{N-1}} \prod_{j=2}^N h(z_j) \int_0^2 \frac{a(s)}{(s^2 + |z'|^2)^{(N-\alpha)/2}} ds dz' \\ &= I_{\alpha,N} + R_{N,\alpha}. \end{aligned}$$

Together with (4.10), this proves (4.6). The remainder is finite because $s \geq 1$, and it is strictly positive because its integrand is positive for $1 < s < 2$ and $z' \in (-1, 1)^{N-1}$. $\square$

4.2. Perimeter decomposition and proof of nonexistence. Fix an integer $M \geq 2$ and abbreviate $\ell := \ell_M$. Let $E \in \mathcal{A}_M$ and write $E = \bigcup_{i=1}^M C_i$, with $C_i = \ell(Q + a_i)$. Let

$$S(E) := \{(i, j) : 1 \leq i < j \leq M, |a_i - a_j| = 1\}, \quad c(E) := \#S(E).$$

Thus $c(E)$ counts the shared faces, each counted once. Throughout this section, an empty sum is understood to be zero.

Lemma 4.3. *For every $E \in \mathcal{A}_M$,*

$$P(E) = 2NM\ell^{N-1} - 2c(E)\ell^{N-1} \quad (4.11)$$

and

$$V_\alpha(E) = M\ell^{N+\alpha}V_\alpha(Q) + 2 \sum_{1 \leq i < j \leq M} V_\alpha(C_i, C_j). \quad (4.12)$$

Consequently,

$$\mathcal{F}_{K,\alpha}(E) - M\mathcal{F}_{K,\alpha}(\ell Q) = 2c(E)\ell^{N-1} (K\ell^{1+\alpha}V_\alpha(Q, Q + e_1) - 1) + 2K \sum_{\substack{1 \leq i < j \leq M \\ (i,j) \notin S(E)}} V_\alpha(C_i, C_j). \quad (4.13)$$

Proof. The sum of the individual cube perimeters is $2NM\ell^{N-1}$. At each shared face, the two copies of that face are absent from the reduced boundary of the union. Their total area is $2\ell^{N-1}$. All other intersections of cube closures have dimension at most $N - 2$ and hence zero $\mathcal{H}^{N-1}$ measure. This proves (4.11).

Splitting the double integral over the cubes gives (4.12). The diagonal terms are equal by translation invariance and scaling, and each unordered pair contributes twice its mutual interaction. If $(i, j) \in S(E)$, invariance under coordinate permutations and reflections yields

$$V_\alpha(C_i, C_j) = \ell^{N+\alpha}V_\alpha(Q, Q + e_1).$$

Separating these terms proves (4.13). $\square$

Proof of Theorem 1.2. Let $K \geq K_0(N, \alpha, M)$. By (4.2) and Proposition 4.2,

$$\begin{aligned} K\ell^{1+\alpha}V_\alpha(Q, Q + e_1) &\geq K_0\ell^{1+\alpha}V_\alpha(Q, Q + e_1) \\ &= \frac{2NV_\alpha(Q, Q + e_1)}{\alpha V_\alpha(Q)} = 1 + \frac{2NR_{N,\alpha}}{\alpha V_\alpha(Q)} > 1. \end{aligned} \quad (4.14)$$

Hence both terms on the right-hand side of (4.13) are nonnegative. If $c(E) > 0$, the first term is strictly positive. If $c(E) = 0$, the second term contains all $M(M - 1)/2$ pairs, each with strictly positive interaction. Since $M \geq 2$, in either case

$$\mathcal{F}_{K,\alpha}(E) > M\mathcal{F}_{K,\alpha}(\ell Q). \quad (4.15)$$

On the other hand, thanks to Remark 4.1, we can infer that

$$\inf_{E \in \mathcal{A}_M} \mathcal{F}_{K,\alpha}(E) = M\mathcal{F}_{K,\alpha}(\ell Q),$$

which is exactly (1.13), recalling (4.1). The strict inequality (4.15) proves that the infimum is not attained. $\square$

Remark 4.4. *Since the prescribed mass is ω_N and the integer M is fixed, the side length $\ell = (\omega_N/M)^{1/N}$ is fixed. Hence the dimensionless parameter governing a face contact is $K\ell^{1+\alpha}V_\alpha(Q, Q+e_1)$, which grows linearly with K . Theorem 1.2 therefore describes a true strong-repulsion regime. No mass is sent to infinity as K decreases, and no change in the number of cubes is hidden in the statement.*

Remark 4.5. *The contribution of a single face contact in (4.13) is positive precisely when*

$$K\ell^{1+\alpha}V_\alpha(Q, Q+e_1) > 1.$$

Recalling the definition of ℓ (1.10), the corresponding contact threshold is

$$K_c(N, \alpha, M) := \frac{1}{V_\alpha(Q, Q+e_1)} \left(\frac{M}{\omega_N} \right)^{\frac{1+\alpha}{N}}.$$

Proposition 4.2 yields

$$V_\alpha(Q, Q+e_1) > \frac{\alpha}{2N} V_\alpha(Q),$$

and therefore

$$K_c(N, \alpha, M) < K_0(N, \alpha, M).$$

In fact, the proof of Theorem 1.2 gives nonattainment for every $K > K_c$. We state the theorem with the larger threshold K_0 because it arises naturally from the equivalent condition $\ell \geq \rho_{K,\alpha}$ and depends only on the self-interaction $V_\alpha(Q)$. Moreover, the strict remainder $R_{N,\alpha} > 0$ in Proposition 4.2 ensures that the required strict inequality still holds at $K = K_0$. No optimality of K_0 as a nonexistence threshold is claimed.

NOTE ON THE USE OF AI

We started working on this paper in 2023 and we presented the ideas contained in it at some workshops. However, we were able to show the strict positivity of the remainder $R_{N,\alpha}$ in the equality (4.6) only for $\alpha \in (N-2-\delta, N)$ for a certain $\delta > 0$, thus having a weaker result than Theorem 1.2, which is valid for all $\alpha \in (0, N)$. The technical trick in Subsection 4.1 yielding to the stronger present version of the result has been obtained through interactions with AI.

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