

ON CYLINDRICAL SINGULARITIES OF MINIMAL HYPERSURFACES

ARIA HALAVATI AND LUCA SPOLAOR

ABSTRACT. In a neighborhood of the origin in $\mathbb{R}^9$, we construct a minimal hypersurface with respect to a smooth conformally Euclidean metric, whose singular set consists of the origin and a sequence of isolated singular points converging to it. At each of the isolated singular points the hypersurface has the unique tangent cone $C_{4,3}$, the cone over $S^4(\sqrt{4/7}) \times S^3(\sqrt{3/7})$, whereas at the origin its unique tangent cone is the cone $C_{3,3} \times \mathbb{R}$. The metric is Euclidean near each isolated singular point and agrees with the Euclidean metric to infinite order at the origin. A key step, of independent interest, is the construction of an entire stationary hypersurface in Euclidean $\mathbb{R}^9$ with exactly one singular point at the origin, at which its unique tangent cone is $C_{4,3}$, and whose unique tangent cone at infinity is $C_{3,3} \times \mathbb{R}$.

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1. INTRODUCTION

For an area-minimizing hypersurface of dimension n , Federer's dimension-reduction theorem bounds the Hausdorff dimension of the interior singular set by $n - 7$ [14]. Such hypersurfaces are therefore smooth in ambient dimensions at most seven. Singularities first occur in ambient dimension eight, as exemplified by the Simons cone, proved to be area-minimizing by Bombieri–De Giorgi–Giusti [5]. Its Euclidean products give higher-dimensional singular sets, whose possible geometry becomes increasingly complicated as the dimension grows. In general, the strongest available structural conclusions are rectifiability and quantitative measure bounds: for $n \geq 7$, Simon [29] established countable $(n - 7)$ -rectifiability of the singular set, and Naber–Valtorta [24] proved local finiteness and quantitative bounds for its $(n - 7)$ -dimensional Hausdorff measure. A finer local geometric description remains unavailable in general.

The geometry of a minimal hypersurface near a singular point is described to first order by its tangent cones. If one tangent cone has multiplicity one and is regular, meaning that it is smooth away from its vertex, the situation is well understood. Building on the integrable case treated by Allard–Almgren [1], Simon's asymptotic Łojasiewicz theory [25, 26] proves uniqueness of the tangent cone and represents the hypersurface as a normal graph over it in a punctured neighborhood of the singular point: that is in a neighborhood of the singular point the minimal hypersurface is homeomorphic to its tangent model. See also Adams–Simon [2] for convergence rates and Engelstein–Spolaor–Velichkov [11] for the corresponding regularity theory for almost area-minimizing currents.

Much less is understood at points with cylindrical tangent cones, that is, cones of the form $C \times \mathbb{R}^\ell$, where C has an isolated vertex singularity and $\ell \geq 1$. The singular set $\{0\} \times \mathbb{R}^\ell$ of such a cone, called spine, is not isolated, and convergence to the cone away from the spine does not by itself describe the hypersurface near the spine itself. For quadratic cylindrical cones with a one-dimensional spine, a major breakthrough was Székelyhidi's uniqueness theorem for the Simons case $C_{3,3} \times \mathbb{R}$ [34], beyond the integrable cases covered by Simon [28]. Firester–Tsiamis–Wang [17] subsequently treated $C_{2,4} \times \mathbb{R}$. Together these works establish uniqueness for multiplicity-one tangent cones $C_{p,q} \times \mathbb{R}$ of area-minimizing hypersurfaces.

Uniqueness of the tangent cone leaves open the structure of the singular set near the point. In a striking construction, Simon [31] proved the existence of a stable minimal hypersurface in a smooth non euclidean metric in $\mathbb{R}^9$ whose singular set is an arbitrary closed subset of the singular axis of the Simons cone times $\mathbb{R}$. In Euclidean space, Székelyhidi [35] constructed hypersurfaces with an isolated singular point whose unique tangent cone is nevertheless cylindrical; his construction gives area-minimizing examples when the cross-section is strictly minimizing and strictly stable. Thus the nearby singular set need not resemble the spine of its tangent cone. Beyond such constructions, general structural results such as [27], and classifications under additional symmetry [35], a detailed local description remains largely open, even for quadratic cylindrical tangent cones.

Motivated by the question of better understanding the behavior of area minimizing hypersurfaces close to a cylindrical tangent cone, in this note we prove the following theorem. In what follows we work in $\mathbb{R}^9 = \mathbb{R}_x^4 \times \mathbb{R}_z^4 \times \mathbb{R}_y$ and, for integers $p, q \geq 1$ we use the convention

$$C_{p,q} = \{(x, z) \in \mathbb{R}^{p+1} \times \mathbb{R}^{q+1} : q|x|^2 = p|z|^2\}.$$

The two cones relevant to this paper are

$$V = C_{3,3} \times \mathbb{R} = \{|z| = |x|\}, \quad C_{4,3} = \{4|z|^2 = 3(|x|^2 + y^2)\}.$$

The first has a singular axis; the second is singular only at its vertex.

Theorem 1.1 (Accumulating singularities). *There exist an open neighborhood $\mathcal{U}$ of zero in $\mathbb{R}^9$, a conformally Euclidean C^∞ metric g on $\mathcal{U}$, and an embedded multiplicity-one hypersurface Σ , stationary with respect to g , such that, for some k_0 ,*

$$\text{Sing } \Sigma = \{0\} \cup \{p_k : k \geq k_0\}, \quad p_k = (0, 0, 2^{-k}),$$

and

$$\text{Tan}(\Sigma, 0) = \{C_{3,3} \times \mathbb{R}\}, \quad \text{Tan}(\Sigma, p_k) = \{C_{4,3}\}.$$

All these tangent cones have multiplicity one.

The metric is Euclidean near each p_k and agrees with the Euclidean metric to infinite order at zero:

$$D^j(g - g_{\text{euc}})(0) = 0 \quad \text{for every integer } j \geq 0.$$

More precisely, for every pair of nonnegative integers m, N , there is a constant $C_{m,N}$ such that

$$\|g - g_{\text{euc}}\|_{C^m(B_\rho(0))} \leq C_{m,N} \rho^N$$

for all sufficiently small $\rho > 0$. In particular, the metric can be made arbitrarily close to Euclidean in any prescribed finite collection of C^m norms by shrinking $\mathcal{U}$.

Here and below, $\text{Tan}(\Sigma, p)$ denotes the set of tangent cones at p of the varifold associated with Σ . To our knowledge, Theorem 1.1 provides the first example of a minimal hypersurface, with respect to a smooth metric, in which singular points whose tangent cones have an isolated singularity accumulate at a point with a cylindrical tangent cone.

Remark. We do not address here whether Σ is stable, or area-minimizing, with respect to g . In view of the recent work of Liu [21] on Simon's examples, we expect that this question can be settled by similar methods, but we do not pursue it here.

As is well known (see, for instance, the work of Edelen–Spolaor [9], which builds on Simon–Solomon [32]), local regularity questions near a cone are closely tied to Liouville-type theorems, that is, to the classification of complete minimal hypersurfaces with a prescribed tangent cone at infinity. An entire model interpolating between the two cones of Theorem 1.1 is therefore an essentially necessary ingredient of the construction; its existence is the content of the following result.

Theorem 1.2 (Existence of the connector). *There exists an entire Lipschitz function $U : \mathbb{R}_x^4 \times \mathbb{R}_y \rightarrow [0, \infty)$ whose symmetric graph $M = \{|z| = U(x, y)\}$ is a complete multiplicity-one stationary eight-dimensional hypersurface in Euclidean $\mathbb{R}^9$, with*

$$\text{Sing } M = \{0\}, \quad \text{Tan}(M, 0) = \{C_{4,3}\}, \quad \lim_{R \rightarrow \infty} R^{-1}M = C_{3,3} \times \mathbb{R}.$$

The last limit holds in the sense of varifolds, so that the tangent cone at infinity is unique, and the convergence is smooth on compact subsets of $\mathbb{R}^9 \setminus \{x = z = 0\}$. The function U is $O(4)_x$ -invariant, even in y , smooth away from zero, and $U_y > 0$ for $y > 0$.

We call M a *connector*, since it connects a singularity modeled on $C_{4,3}$ at the origin with the cylindrical cone $C_{3,3} \times \mathbb{R}$ at infinity. Besides its role in the proof of Theorem 1.1, its existence is of interest in connection with the Bernstein-type problem for minimal hypersurfaces asymptotic to $C_{3,3} \times \mathbb{R}$ studied in [10]. A special case of [10, Theorem 1.1], together with its stationary-varifold extension in Remark 1.2, asserts that if a nonzero stationary integral eight-varifold in all of $\mathbb{R}^9$ lies entirely on one side of $V = C_{3,3} \times \mathbb{R}$ and has density at infinity less than $2\Theta(V)$, then it is a leaf product $H(\lambda) \times \mathbb{R}$, where $H(\lambda)$ is a leaf of the Simons foliation. The connector in Theorem 1.2 satisfies the density condition, $\Theta_M(\infty) = \Theta(V)$, but meets both sides of V . Indeed, the two sides are distinguished by $q(x, z, y) = |z|^2 - |x|^2$, and on its tangent cone

$$C_{4,3} : \quad q = \frac{3}{4}(|x|^2 + y^2) - |x|^2 = \frac{3y^2 - |x|^2}{4}.$$

This is negative at $y = 0$, $x \neq 0$, and positive at $x = 0$, $y \neq 0$. Containment in either closed side would persist under dilation and passage to a tangent varifold. Since $C_{4,3}$ is contained in neither side, M meets both open sides in every neighborhood of zero. Thus the one-sidedness hypothesis of [10, Theorem 1.1 and Remark 1.2] fails.

Write $U(x, y) = u_\infty(t, y)$, where $t = |x|$, for the radial profile of the connector. At a fixed height $y \neq 0$, its orbit at $x = 0$ is the sphere $\{|z| = U(0, y)\} \cong S^3$. We call its radius $a_\infty(y) = U(0, y)$ the *smoothing radius*: it sets the scale on which the transverse cone vertex is rounded off. The *smoothing region* is the part of the slice where $|x|$ is comparable to or smaller than this radius. Figure 1 illustrates these slices and the growth of their smoothing radii away from the singular origin.

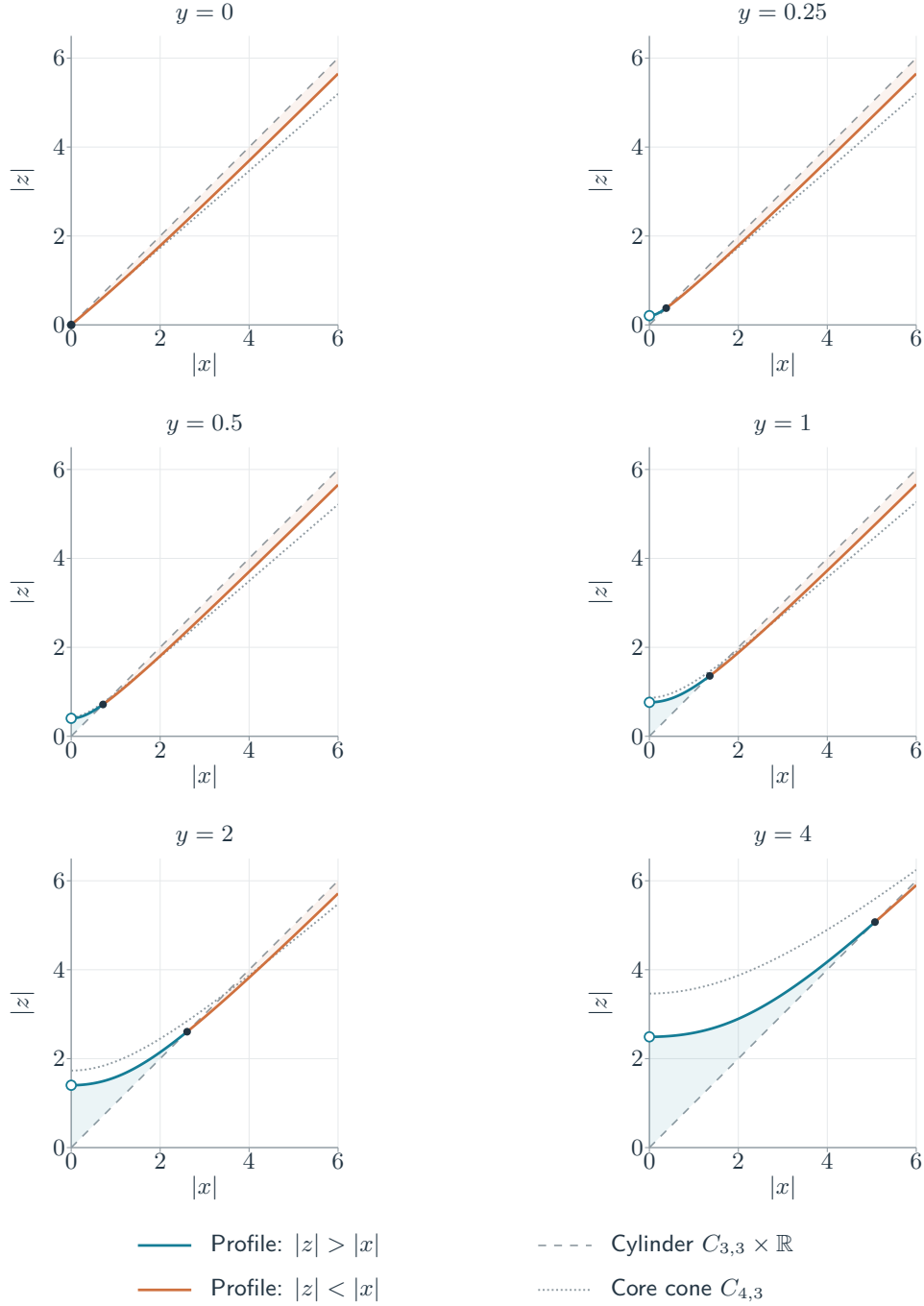


FIGURE 1. Transverse slices at $y = 0, \frac{1}{4}, \frac{1}{2}, 1, 2, 4$, read from left to right and top to bottom, with common scales. The dashed and dotted curves are the corresponding sections of $V = C_{3,3} \times \mathbb{R}$ and $C_{4,3}$. Blue and orange distinguish the regions $|z| > |x|$ and $|z| < |x|$. Open circles mark the smooth S^3 orbits at $x = 0$; filled circles mark crossings with V . The origin in the $y = 0$ slice is singular. Negative y gives the same slices by symmetry. The colored curves use an illustrative algebraic profile, not a numerical solution of the minimal surface equation.

1.1. Relation to earlier constructions and rigidity results. Hardt–Simon [20] associate smooth area-minimizing leaves to an area-minimizing hypercone with isolated singularity; their dilations foliate each complementary component. These leaves describe how a conical singularity can be smoothed on either side. Caffarelli–Hardt–Simon [6] constructed local nonconical minimal hypersurfaces with prescribed isolated conical singularities. Smale [33] subsequently constructed minimal hypersurfaces with many isolated conical singularities. Simon–Solomon [32] proved a global rigidity theorem for minimizing hypersurfaces asymptotic to suitable quadratic cones. Together with the local regularity theorem of Edelen–Spolaor [9] discussed above, these results concern cones with smooth links. The asymptotic cone in Theorem 1.2 instead, has a singular link, and the connector meets both sides of that cone.

The cylindrical analysis requires control near the axis without assuming that the singular set fills it. Székelyhidi’s barriers and nonconcentration estimates [34] address this difficulty and are essential ingredients here. The connector has a different geometry from the local examples in [35]: its tangent cone at the singular point is $C_{4,3}$, whereas its cone at infinity is V .

One-sidedness imposes a further restriction on entire hypersurfaces. Simon [30] proved a cylindrical Liouville theorem under strict stability and a finite-density hypothesis. Edelen–Székelyhidi [10] obtained the corresponding rigidity for one-sided hypersurfaces under a density bound, with an extension to stationary integral varifolds. The calculation above verifies directly why that theorem does not exclude the present connector: the density bound is satisfied, but the tangent cone forces the surface to cross V (See the orange and blue sides in Figure 1).

The prescribed-singular-set construction of Simon [31] already allows accumulation in smooth metrics. In a recent work, Liu [21] proves area-minimizing properties for these examples; his theorem covers the Simons-cone cross-section when the construction parameters are sufficiently small. The new feature in Theorem 1.1 is the change of tangent-cone type: the isolated singular points have the tangent cone $C_{4,3}$, which has an isolated singularity, while their accumulation point has the cylindrical tangent cone V . The metric is conformally Euclidean, is exactly Euclidean near each inserted singularity, and has the full Euclidean jet at the accumulation point, that is the origin. Since g is not Euclidean on any neighborhood of the origin, the last property implies that g is not real analytic near the origin. In fact we conjecture that such an example cannot be constructed with an analytic metric (in particular the euclidean metric). For related strong unique continuation results in the Euclidean setting and a discussion of real-analytic ambient metrics, see [35, Theorem 1.2 and the subsequent discussion].

1.2. The construction: main ingredients and use of AI. In this subsection we describe the main ideas of the proof, what AI contributed and where they come from. More detailed explanations of each main step are given in the corresponding sections. We first construct the entire Euclidean connector, then establish the estimates needed to insert copies of it at infinitely many scales and obtain a smooth ambient metric.

Constructing the connector. In Section 3 we begin with Dirichlet problems on a fixed rounded cylinder $\Omega \subset \mathbb{R}_x^4 \times \mathbb{R}_y$ (see Figure 3). On this domain we will use Leray–Schauder degree argument from Fouladgar–Simon [18], so that given $a > 0$ we can find boundary data $g_{\varepsilon, b_{\varepsilon, a}}$ and solutions $u_{\varepsilon, a}$ of the symmetric minimal surface equation (SME) such that the minimum of $u_{\varepsilon, a}$ is a for every $\varepsilon > 0$. The parameter $b_{\varepsilon, a}$ is chosen to guarantee such condition by representing a dip in the boundary data. Keeping ε fixed, we let $a \downarrow 0$ and pass to a subsequence; this gives a singular solution and a limiting parameter b_{ε} . Only afterward do we let $\varepsilon \downarrow 0$, obtaining convergence to $V = C_{3,3} \times \mathbb{R}$. The analysis also determines the limiting choice of b . Using leaf comparison and moving planes, we also show axial monotonicity of solutions throughout the construction. The boundary data are chosen to give more than convergence to V : the leading normalized perturbation near the origin is a positive multiple of the degree-zero Jacobi field

$$J_2(r, y) = \frac{y^2}{r^2} - \frac{1}{3}, \quad r = (|x|^2 + |z|^2)^{1/2}.$$

Section 4 uses this field to obtain a strict density drop below $\Theta(V)$. A separate density comparison excludes zeros away from the axis. The strict drop then excludes an interval of axial

zeros, and the cone classification identifies the unique remaining singularity as $C_{4,3}$. Thus each member of the family has the desired local singularity, although the family converges at fixed scale to the cylindrical cone.

Section 2 extracts an entire surface from these two facts. We choose a radius at which the density ratio has a fixed value strictly between $\Theta(C_{4,3})$ and $\Theta(V)$, and rescale that radius to one. The chosen radii tend to zero, so the rescaled domains exhaust the whole space. The density normalization prevents the limit from being either cone. Compactness and cone classification then give the connector with tangent cone $C_{4,3}$ at zero and unique blow-down V .

Quantitative control of the end. For gluing, convergence to V away from its spine is not sufficient: we need uniform control of higher order derivatives through the smoothing regions. Section 5 first proves quadratic growth of the signed Hardt–Simon phase and positivity of its leading coefficient. Székelyhidi’s barriers and nonconcentration estimates [34] transfer this information to the spine and determine the smoothing radius $a_\infty(Y) = u_\infty(0, Y) \sim \kappa_H |Y|^{2/3}$. We then differentiate the equation for u_∞ directly. Normalizing by the positive dilation Jacobi field of a comparison leaf leads to a uniformly controlled Muckenhoupt A_2 equation. The weighted estimates of [13, 12] give bounds for all axial derivatives, including those of a_∞ . These bounds control the forcing in the equation for the difference between the connector and the leaf with the same smoothing radius. Appendix A supplies the detailed derivative and remainder estimates.

Gluing the singularities. Section 6 starts from V , keeping the Simons-cone slice at $y = 0$. At other heights we replace a small neighborhood of the spine by a positive Hardt–Simon leaf and cut back to the cone outside a narrow tube around the spine. Into this background we insert translated and dilated connectors in disjoint balls whose centers approach zero. The background radius function $a(y)$ is chosen to agree with the smoothing radius of the corresponding dilated connector on the axial intervals belonging to the near-axis portions of each attachment annulus.

Matching the axis values alone does not suffice. Direct interpolation leaves a mean-curvature error that tends to zero but varies across the much smaller smoothing scale; the resulting conformal correction cannot have bounded second derivatives. We therefore improve the background before interpolating. Start with $V_0[a](t, y) = a(y)h(t/a(y))$, where $t = |x|$. If a were constant, this would be an exact minimal hypersurface. When a varies with y , each transverse slice is still a foliation leaf, but the full hypersurface has a mean-curvature error: its leading terms involve a'' and $(a')^2/a$.

To cancel these terms, fix y , linearize the radial equation about the leaf, and solve this ordinary differential equation with the negative of the leading error as forcing. The correction has zero value and zero radial derivative at $t = 0$, so it preserves the smoothing radius $a(y)$ and smoothness at the axis. Its dependence on y enters through the factors $a(a')^2$ and $a^2 a''$; the coefficients of these factors are fixed functions of $t/a(y)$, determined by that radial equation.

Adding this first correction gives $V_1[a]$. We then compute its remaining error, cancel the next terms by another radial solve, and continue to a finite approximation $V_J[a]$; see Figure 2. Section 6 proves two estimates for this procedure: the corrected background has a smaller mean-curvature residual, and, wherever $a(y)$ agrees with the connector’s axis value, it approximates the connector to the corresponding higher order. Thus both errors introduced by interpolation are controlled. At each stage we use a finite sum with a remainder estimate; convergence as $J \rightarrow \infty$ at a fixed point is not required.

For each fixed correction order, a finite-accuracy gluing lemma controls the remaining curvature and the derivatives of a conformal factor F satisfying $H + 8\nu F = 0$ on the glued surface. We first fix the insertion centers and scales and construct the matching radius function. We then choose successively smaller radii at which the correction order increases, placing the transitions away from the connector cores. This diagonal choice makes every fixed derivative of F vanish to every prescribed order at zero. The resulting metric is smooth and flat relative to the Euclidean metric there, and is exactly Euclidean near each inserted singularity. Finally, mass estimates show that the modifications preserve V as the unique tangent cone at the accumulation point and allow stationarity to extend across it.

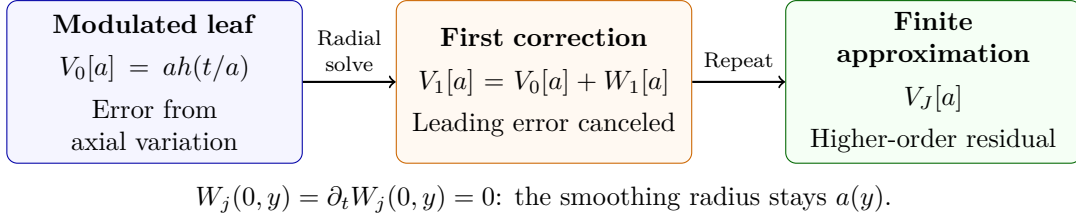


FIGURE 2. Successive radial corrections. At each step, solve the linearized radial equation to cancel the leading remaining error, then estimate the new residual. When the axis data agree with the connector's, a separate matching estimate controls the difference between $V_J[a]$ and the connector.

Use of LLMs. The higher order estimates of Section 5, in particular the details in Appendix A were produced with the help of ChatGPT-6 ASTRA, in particular the idea of using A_2 weights is entirely LLM generated, and it was surprising to us. We also used ASTRA to complete the details of the gluing process in Appendix B. Finally we used ASTRA for editing the final version of this manuscript. We take full responsibility for the correctness of the manuscript.

1.3. Forthcoming work and questions. We conclude the introduction with some natural open questions, which we expect to follow from an extension of the present construction.

Follow-up question 1 (Connectors for quadratic cones). *Let $p, q \geq 1$, and suppose that $C_{p,q}$ is area-minimizing. There exists a complete, connected, embedded hypersurface*

$$M \subset \mathbb{R}_x^{p+1} \times \mathbb{R}_z^{q+1} \times \mathbb{R}_y$$

whose associated multiplicity-one varifold is stationary, such that

$$\text{Sing } M = \{0\}, \quad \lim_{\rho \downarrow 0} \rho^{-1} M = C_{p+1,q}, \quad \lim_{R \rightarrow \infty} R^{-1} M = C_{p,q} \times \mathbb{R}.$$

Both limits are unique, have multiplicity one, and are smooth locally away from the singular sets of the limiting cones. Moreover, M is invariant under $O(p+1) \times O(q+1)$ and under $y \mapsto -y$.

Here the tangent cone is understood in the form

$$C_{p+1,q} = \{q(|x|^2 + y^2) = (p+1)|z|^2\}.$$

Applying the same conjecture with p and q interchanged gives the other attachment, whose tangent cone is

$$C_{p,q+1} = \{(q+1)|x|^2 = p(|z|^2 + y^2)\}.$$

The conjecture concerns stationary hypersurfaces.

Follow-up question 2 (Accumulation of quadratic singularities). *Let $p, q \geq 1$, and suppose that $C_{p,q}$ is area-minimizing. There exist an open neighborhood $\mathcal{U}$ of zero in*

$$\mathbb{R}_x^{p+1} \times \mathbb{R}_z^{q+1} \times \mathbb{R}_y,$$

a smooth conformally Euclidean metric g on $\mathcal{U}$, and an embedded hypersurface $\Sigma \subset \mathcal{U}$ whose associated multiplicity-one integral varifold is stationary with respect to g , such that, for some k_0 ,

$$\text{Sing } \Sigma = \{0\} \cup \{p_k : k \geq k_0\}, \quad p_k = (0, 0, 2^{-k}).$$

The tangent cones are unique and satisfy

$$\text{Tan}(\Sigma, 0) = \{C_{p,q} \times \mathbb{R}\}, \quad \text{Tan}(\Sigma, p_k) = \{C_{p+1,q}\},$$

all with multiplicity one.

The metric is Euclidean near each p_k and agrees with the Euclidean metric to infinite order at zero. More precisely, for every pair of nonnegative integers m, N , there is a constant $C_{m,N}$ such that

$$\|g - g_{\text{euc}}\|_{C^m(B_\rho(0))} \leq C_{m,N} \rho^N$$

for all sufficiently small $\rho > 0$.

Interchanging the two factors gives the analogous conjecture with tangent cone $C_{p,q+1}$ at every p_k . Analogous questions can be stated replacing $\mathbb{R}$ with $\mathbb{R}^k$, but, a part for the obvious product constructions, new techniques might be required to handle the general case $k \geq 2$.

A roadmap towards the solution of this conjecture using the results of this paper is given in Appendix C.

1.4. Organization. Section 2 reduces Theorem 1.2 to the existence of a family of singular solutions of Dirichlet problems on a bounded domain and to the classification of their singularities. Sections 3 and 4 address these two ingredients. Section 5 studies the connector at infinity, and Section 6 carries out the matching and conformal correction, thereby proving Theorem 1.1. Appendices A and B contain the detailed weighted derivative and gluing estimates. Appendix C outlines a strategy for the extension to other minimizing quadratic cones formulated in Conjectures 1 and 2.

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2. CONSTRUCTION OF THE CONNECTOR

We prove Theorem 1.2 by rescaling singular solutions to a suitable family of Dirichlet problems with boundary data parametrized by ε on a fixed bounded domain. The two propositions below give the required family and classify its singularities; their proofs occupy Sections 3 and 4. Each surface has a $C_{4,3}$ singularity at zero, whereas the family converges, as $\varepsilon \rightarrow 0$, with multiplicity one to V .

The rescaling must retain the transition between these two models. We fix an intermediate density and choose the radius where the density ratio attains that value. These radii shrink to zero, so compactness gives an entire limit after dilation. The same density normalization excludes both conical limits. The classification of symmetric, axially monotone cones then identifies the tangent cone at zero and the cone at infinity. This explains the role of the density gaps recorded below and the order in which the two propositions are used.

2.1. Notation and preliminary results. Write

$$t = |x|, \quad r = (|x|^2 + |z|^2)^{1/2}, \quad \rho = (r^2 + y^2)^{1/2}, \quad J_2(r, y) = \frac{y^2}{r^2} - \frac{1}{3}.$$

Thus $r = \sqrt{2}t$ on V . For an eight-dimensional stationary varifold N in Euclidean space,

$$\Theta(N, p, s) = \frac{\|N\|(B_s(p))}{\omega_8 s^8}, \quad \omega_8 = \frac{\pi^4}{24}, \quad \Theta(N, p) = \lim_{s \downarrow 0} \Theta(N, p, s).$$

For a cone centered at zero we abbreviate $\Theta(N) = \Theta(N, 0)$. The main logic of the existence theorem is the following density gaps:

$$(1) \quad \theta_{43} := \Theta(C_{4,3}) = \frac{768\sqrt{21}}{2401} < \theta_V := \Theta(V) = \frac{15\pi}{32} < \theta_{13} := \Theta(C_{1,3}) = \frac{9\sqrt{3}\pi}{32}.$$

The first gap provides the interval in which we choose the normalizing density. The second excludes off-axis zeros, whose density would be at least θ_{13} . To use the first gap, we must also show that the solutions of the bounded-domain Dirichlet problems have density below θ_V at zero. Their leading Jacobi field J_2 produces this strict deficit through the spherical link-area calculation in Section 4.2 dictated by the second variation along such Jacobi field, which is a strictly negative direction.

For a nonnegative function u on a domain $\Omega \subset \mathbb{R}_x^4 \times \mathbb{R}_y$ we write

$$S(u) = \{(x, z, y) \in \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R} : (x, y) \in \Omega, |z| = u(x, y)\}.$$

If $u \in C^2(\Omega)$ is positive, then $S(u)$ is a smooth hypersurface, and it is minimal if and only if u solves the symmetric minimal surface equation (SME)

$$(2) \quad \left(\delta_{ij} - \frac{u_i u_j}{1 + |Du|^2} \right) u_{ij} = \frac{3}{u}.$$

Here indices refer to the five coordinates (x, y) , and repeated indices are summed. As in [18], we work with locally uniform limits of positive solutions: a nonnegative function $u \in C^0(\Omega)$ is called an *SME limit* on Ω if it is the locally uniform limit on Ω of positive solutions of (2) of class $C^2(\Omega)$, and an SME limit with nonempty zero set is called a *singular SME solution*. An *SME cone* is a set $S(u)$, where u is an SME limit on $\mathbb{R}^5$ which is positively homogeneous of degree one. By [18, Corollary 2.5], if u is a singular SME solution, then $S(u)$, with multiplicity one, is a stationary integral varifold in $\Omega \times \mathbb{R}^4$, which is smooth away from the points $(x, 0, y)$ with $u(x, y) = 0$ and singular at each of them. Identifying (x, y) with $(x, 0, y)$, we write this as

$$\{u = 0\} = \text{Sing}(S(u)).$$

We also use the Hardt–Simon phase. The cone $C_{3,3}$ has two complementary components in $\mathbb{R}^8$, and each of them is foliated by the dilations of a smooth, $O(4) \times O(4)$ -invariant, area-minimizing Hardt–Simon leaf [20]. Outside a compact set, every leaf is a normal graph over the regular part of $C_{3,3}$, with normal height $\lambda r^{-2} + O(|\lambda|^{4/3} r^{-3})$ for some $\lambda \neq 0$, where the normal of $C_{3,3}$ points towards $\{|z| > |x|\}$ (see [34, Proposition 3.3]). We denote this leaf by H_λ , so that $H_\lambda \subset \{|z| > |x|\}$ for $\lambda > 0$ and $H_\lambda \subset \{|z| < |x|\}$ for $\lambda < 0$, and we set $H_0 = C_{3,3}$.

The *Hardt–Simon phase* $P : \mathbb{R}_x^4 \times \mathbb{R}_z^4 \times \mathbb{R}_y \rightarrow \mathbb{R}$ is defined by $P = \lambda$ on $H_\lambda \times \mathbb{R}$ and $P = 0$ on the axis $\{x = z = 0\}$. It is continuous, independent of y , positively homogeneous of degree three in (x, z) , odd under the exchange of x and z , and smooth on each of the two open sides of V . In particular, there is a constant $c_H > 0$ such that

$$P(0, z, y) = c_H |z|^3, \quad P(x, 0, y) = -c_H |x|^3.$$

2.2. Proof of Theorem 1.2. The proof relies on two propositions, which are proved in the next two sections.

The first one provides singular SME solutions on a fixed bounded domain whose normalized deviation from V is governed, as $\varepsilon \rightarrow 0$, by the Jacobi field $J_2 = y^2/r^2 - 1/3$. Its proof uses the degree argument of [18] on a mean-convex domain, with a family of boundary data depending on ε , which is designed to produce this Jacobi field in the limit $\varepsilon \rightarrow 0$.

Fix $R > 0$ and $0 < Y_0 < Y$, and choose a smooth bounded convex rounded cylinder $\Omega \subset \mathbb{R}_x^4 \times \mathbb{R}_y$, invariant under $O(4)_x$ and reflection in y , with vertical side $\{|x| = R, |y| \leq Y_0\}$ and strictly convex caps ending at the poles $(0, \pm Y)$, as in Section 3. The domain, including the cap shape, is fixed throughout the Dirichlet construction and is independent of ε ; see Figure 3.

Proposition 2.1 (Existence of SME solution). *For the domain Ω fixed above, choose $r_* > 0$ such that $B_{4r_*}^5 \subset \Omega$. There exists $\varepsilon_0 > 0$ such that, for every $0 < \varepsilon < \varepsilon_0$, there exists a singular SME solution u_ε on Ω , continuous on $\overline{\Omega}$, with the following properties:*

- (1) u_ε is $O(4)_x$ -invariant, even in y , and nondecreasing in $y \geq 0$; the varifold $M_\varepsilon = S(u_\varepsilon)$, with multiplicity one, is stationary in $\Omega \times \mathbb{R}^4$.
- (2) $\{u_\varepsilon = 0\} = \text{Sing}(S(u_\varepsilon)) \neq \emptyset$.
- (3) There is a constant $K > 0$, independent of ε , such that the uniform bound

$$|P| \leq K \varepsilon \quad \text{holds on } S(u_\varepsilon),$$

and $\sup_{0 < \varepsilon < \varepsilon_0} \text{Lip}(u_\varepsilon; U) < \infty$ for every $U \Subset \Omega$.

- (4) $u_\varepsilon \rightarrow |x|$ locally uniformly in Ω , as $\varepsilon \rightarrow 0$, and consequently $S(u_\varepsilon) \rightarrow V$ as varifolds in $\Omega \times \mathbb{R}^4$, with multiplicity one.
- (5) There are a constant $C_* > 0$ and smooth functions $f_\varepsilon : V \cap B_{2r_*}^9 \cap \{r \geq C_* \varepsilon^{1/3}\} \rightarrow \mathbb{R}$ such that the part of $S(u_\varepsilon)$ lying over this region is the normal graph $\{X + f_\varepsilon(X) \nu_V(X)\}$, where ν_V is the unit normal of V pointing towards $\{|z| > |x|\}$, and

$$(3) \quad |D^k f_\varepsilon| \leq C_k \varepsilon r^{-2-k}, \quad k = 0, 1, 2,$$

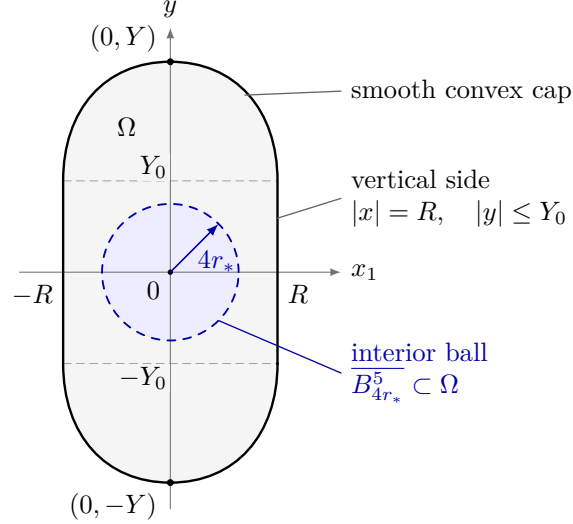


FIGURE 3. A schematic meridian section of the fixed domain Ω , in the plane $x_2 = x_3 = x_4 = 0$. The full five-dimensional domain is invariant under rotations of $x \in \mathbb{R}^4$; in this section $|x| = |x_1|$. The dashed circle is the section of the interior ball used in Proposition 2.1.

with constants independent of ε .

- (6) Let $\mathcal{D} \subset \mathbb{R}_\xi^3 \times \mathbb{R}_y$ be obtained from Ω by replacing $|x|$ with $|\xi|/\sqrt{2}$, and put $r = |\xi|$. There is a harmonic, $O(3)_\xi$ -invariant, reflection-even function W on $\mathcal{D}$ such that, with $m_0 = W(0, 0)$,

$$W_y(0, y) > 0 \quad (y > 0, (0, y) \in \mathcal{D}),$$

near the origin

$$(4) \quad W(r, y) - m_0 = c \left(y^2 - \frac{r^2}{3} \right) + E(r, y), \quad c > 0,$$

where the analytic remainder satisfies

$$|D_{\xi, y}^k E| \leq C_k \rho^{4-k}, \quad k = 0, 1, 2, 3, 4,$$

for sufficiently small $\rho > 0$, and

$$\frac{f_\varepsilon}{\varepsilon} \longrightarrow \frac{W - m_0}{r^2} \quad \text{smoothly on regular compact subsets of } V \text{ over } \Omega \text{ as } \varepsilon \rightarrow 0.$$

The second proposition classifies the singular set of the solutions u_ε produced by Proposition 2.1. Its proof combines a classification of symmetric tangent cones with a second-variation computation for the area of the link of V in the direction of J_2 ; the latter yields a strict drop of the density below $\Theta(V)$.

Proposition 2.2 (Classification of the singularities). *There is $\varepsilon_1 \in (0, \varepsilon_0]$ such that, for every $0 < \varepsilon < \varepsilon_1$, the singular SME solution u_ε of Proposition 2.1 has the following properties:*

- (1) $\{u_\varepsilon = 0\} = \text{Sing}(S(u_\varepsilon)) = \{(0, 0)\}$.
- (2) The density at the origin satisfies $\Theta(M_\varepsilon, 0) = \theta_{43} = \Theta(C_{4,3})$. Moreover, $C_{4,3}$ is the unique tangent cone of M_ε at the origin.

Using Propositions 2.1 and 2.2 together with the compactness theory of [18], we now prove Theorem 1.2. The idea is to zoom in at the scale at which $S(u_\varepsilon)$ passes from one cone to the other. Choose $\varepsilon_j \rightarrow 0$ and put $u_j = u_{\varepsilon_j}$. Each $M_j = S(u_j)$ has tangent cone $C_{4,3}$ at the origin, whereas $M_j \rightarrow V$ at fixed scales. Fixing $\theta_{43} < d < \theta_V$ and choosing ρ_j with

$$\Theta(M_j, 0, \rho_j) = d,$$

we obtain $\rho_j \rightarrow 0$. Rescaling by ρ_j^{-1} places this transition at unit scale, and compactness gives an entire limit M with

$$\Theta(M, 0, 1) = d;$$

in particular, M is neither of the two cones. Monotonicity and the classification of cones then identify $C_{4,3}$ as the tangent cone at the origin and V as the tangent cone at infinity, while the density bound and axial monotonicity exclude further singularities.

Proof of Theorem 1.2. Let u_ε , $0 < \varepsilon < \varepsilon_1$, be the singular SME solutions given by Propositions 2.1 and 2.2. Choose a sequence $\varepsilon_j \downarrow 0$, with each ε_j small enough for both propositions to apply, and set

$$u_j := u_{\varepsilon_j}, \quad M_j := S(u_j), \quad X = (x, y) \in \mathbb{R}^5.$$

Proposition 2.1(4) gives

$$u_j \rightarrow |x| \quad \text{locally uniformly in } \Omega, \quad M_j \rightarrow V \quad \text{with multiplicity one over } \Omega.$$

Step 1. Choose a shrinking radius at density d . Proposition 2.1(3) supplies a constant L , independent of j , such that $\text{Lip}(u_j; B_{r_*}^5) \leq L$. Proposition 2.2(1) gives $u_j(0) = 0$, so

$$0 \leq u_j(X) = u_j(X) - u_j(0) \leq L|X| \quad (X \in B_{r_*}^5).$$

Moreover, Proposition 2.2(2) identifies the tangent cone and density of each M_j :

$$\text{Tan}(M_j, 0) = \{C_{4,3}\}, \quad \Theta_{M_j}(0) = \theta_{43}.$$

To choose a radius with density ratio exactly d , and later pass that normalization to the limit, we use the fact that spheres carry no mass. Indeed, on the regular part of M_j , a smooth minimal hypersurface of dimension eight, the position vector $p \in \mathbb{R}^9$ satisfies

$$\Delta_M |p|^2 = 16,$$

so $|p|^2$ is not constant on any open subset of it. Analyticity implies that every spherical level has zero eight-dimensional measure on the regular part. The singular set also has zero eight-dimensional measure. Hence $\|M_j\|(\partial B_r) = 0$ for every $r > 0$; the same argument will apply to the limiting SME graph below. Consequently, the density ratio is continuous in $r > 0$, and varifold convergence gives convergence of the mass of each fixed ball whose boundary has zero mass for the limit.

Fix $d \in (\theta_{43}, \theta_V)$ and a sufficiently small fixed $r_0 < r_*$. Multiplicity-one convergence to V gives

$$\Theta_{M_j}(0) = \theta_{43} < d < \Theta_{M_j}(0, r_0) \rightarrow \theta_V$$

for all large j . Continuity and monotonicity therefore give a radius $\rho_j \in (0, r_0)$ with $\Theta_{M_j}(0, \rho_j) = d$. For every fixed $0 < r \leq r_0$,

$$\Theta_{M_j}(0, r) \rightarrow \theta_V > d \implies \rho_j < r \text{ eventually.}$$

Consequently $\rho_j \rightarrow 0$.

Step 2. Pass the normalization to an entire limit. Set $U_j(X) = \rho_j^{-1} u_j(\rho_j X)$. For each fixed R , once $\rho_j R < r_*$,

$$U_j(0) = 0, \quad \|U_j\|_{L^\infty(B_R^5)} \leq LR, \quad \text{Lip}(U_j; B_R^5) \leq L.$$

Arzelà–Ascoli and a diagonal choice of positive SME approximants give an entire SME limit U with $U(0) = 0$ and global Lipschitz constant at most L . The compactness statements in [18] yield

$$U_j \rightarrow U \quad \text{locally uniformly,} \quad S(U_j) \rightarrow M = S(U) \quad \text{with multiplicity one.}$$

The convergence is smooth where $U > 0$, and the symmetries and axial order pass to U . The varifold convergence follows from the area formula and dominated convergence, as in the proof of Proposition 2.1(4), since the zero set of U has vanishing five-dimensional measure by [18, Theorem 9.1].

For each fixed R and all large j , monotonicity gives

$$\Theta_{S(U_j)}(0, R) = \Theta_{M_j}(0, \rho_j R) \leq \Theta_{M_j}(0, r_0).$$

The zero spherical mass established in Step 1 allows passage at the specified radii, including $R = 1$. Thus

$$\Theta_M(0, 1) = d, \quad \Theta_M(0, R) \leq \theta_V \quad (R > 0), \quad \Theta_M(\infty) \leq \theta_V.$$

Step 3. Identify the singular point and its tangent. For $s > 0$, the rescaled profile $s^{-1}U(sX)$ is an SME limit on $\mathbb{R}^5$ and has the same Lipschitz bound, rotational and reflection symmetries, and axial monotonicity as U . Since $U(0) = 0$, these profiles are uniformly bounded on each fixed ball. SME compactness therefore represents every tangent cone of M at zero as an SME cone with those same properties. Its density is at most $\Theta_M(0, 1) = d < \theta_V$, so Lemma 4.8 gives

$$\text{Tan}(M, 0) = \{C_{4,3}\}.$$

Lemma 4.1 excludes off-axis zeros using the global density bound. If $U(0, y_0) = 0$ for $y_0 \neq 0$, reflection and order would give

$$0 \leq U(0, y) \leq U(0, |y_0|) = 0 \quad (|y| \leq |y_0|).$$

For any $s_i \downarrow 0$, the profiles $s_i^{-1}U(s_i X)$ would then vanish at $(0, 1)$ for all large i . Locally uniform convergence would pass this axial zero to a tangent cone, contradicting $C_{4,3}$. Therefore $U > 0$ away from zero, and SME regularity and [18, Corollary 2.5] give

$$\text{Sing } M = \{0\}.$$

Step 4. Identify the cone at infinity and obtain strict order. The same global Lipschitz bound gives compactness of $R^{-1}M$ as $R \rightarrow \infty$ in the SME class. Let K_∞ be a subsequential limit along $R_i \rightarrow \infty$. For $0 < a < b < \infty$,

$$\Theta_M(0, R_i b) - \Theta_M(0, R_i a) \rightarrow 0.$$

The vanishing monotonicity defect makes K_∞ a stationary cone, and

$$d \leq \Theta_{K_\infty}(0) = \Theta_M(\infty) \leq \theta_V.$$

It inherits the symmetries and axial order. Lemma 4.8 now gives

$$K_\infty \in \{C_{4,3}, V\}, \quad \Theta_{K_\infty}(0) \geq d > \theta_{43} \implies K_\infty = V.$$

Compactness and this identification of every subsequential limit show that the blow-down is unique and $\Theta_M(\infty) = \theta_V$. Positive-height SME regularity gives smooth convergence on regular compact subsets of V away from its axis.

Finally, $U_y \geq 0$ solves the differentiated SME on the connected region $y > 0$, where $U > 0$. If U_y were identically zero there, reflection and continuity would make U independent of y and give an entire zero axis, already excluded. The local Harnack inequality for this nonnegative solution therefore yields $U_y > 0$. $\square$

3. PROPOSITION 2.1: THE DIRICHLET PROBLEM ON A BOUNDED DOMAIN

In this section we construct the singular solutions in Proposition 2.1 by means of the following existence theorem of Fouladgar–Simon.

Theorem 3.1 (Fouladgar–Simon [18, Theorem 4.4]). *Fix integers $n, m \geq 2$ and $0 < \alpha < 1$. On a uniformly convex $C^{2,\alpha}$ domain $\Omega \subset \mathbb{R}^n$, consider*

$$\sum_{i,j=1}^n \left(\delta_{ij} - \frac{u_i u_j}{1 + |Du|^2} \right) u_{ij} = \frac{m-1}{u}.$$

Let $\varphi_\tau \in C^{2,\alpha}(\overline{\Omega})$, $\tau \in [0, 1]$, with $(X, \tau) \mapsto \varphi_\tau(X)$ jointly continuous. Assume the following conditions (cf. [18, Definition 4.1]).

- (1) *No positive $C^2(\Omega) \cap C^0(\overline{\Omega})$ solution has trace φ_0 .*
- (2) *On $\partial\Omega$, $\varphi_1 > \psi$ for some $\psi \in C^0(\partial\Omega)$ that is strongly positive: there exists $\eta > 0$ such that every positive $C^2(\Omega) \cap C^0(\overline{\Omega})$ solution v with $v|_{\partial\Omega} \geq \psi$ satisfies $\inf_\Omega v \geq \eta$.*

Then, for some $\tau_* \in (0, 1)$, there is a singular solution u with trace φ_{τ_*} and

$$\emptyset \neq \{u = 0\} \subset \Omega, \quad \text{dist}(\{u = 0\}, \partial\Omega) > 0.$$

Here a singular solution is a nonnegative function $u \in C^0(\overline{\Omega})$ with a zero in Ω which is the locally uniform limit in Ω of positive classical solutions; for $n = 5$ and $m = 4$ this is a singular SME solution in the sense of Section 2.

The remark following [18, Theorem 4.4] states that mean convexity of $\partial\Omega$ suffices. We use that extension for our rounded cylinder, whose straight side is not uniformly convex. We also use the proof in the $O(4)_x$ -invariant, reflection-even space; the preservation of this space is checked in Section 3.3. For our boundary paths, which have a positive lower bound at each fixed ε , the [18] construction supplies positive approximating solutions whose traces belong to the same path. We retain this property to pass our a priori estimates to the singular solution.

We now prove all the conclusions of Proposition 2.1. Section 3.1 chooses the domain and boundary path. Section 3.2 establishes estimates for any positive symmetric solution with these data, without assuming existence. In Section 3.3, we verify the [18] endpoint conditions, obtain the singular solution for each fixed ε , and prove its symmetries, monotonicity, phase and derivative bounds, and limiting behavior as $\varepsilon \rightarrow 0$.

The harmonic function W introduced in Section 3.1 plays two roles. Before applying [18], the negative value at the center of the normalized limit $W + b$ rules out positive solutions at the lower endpoint of the boundary path. After applying [18], the existence of a zero gives the opposite coordinate contact and forces the limiting normalized phase to vanish at the center. This identifies the leading Jacobi field needed in the next section.

3.1. The domain and boundary path. We first fix the domain and the boundary data. Fix $R > 0$ and $0 < Y_1 < Y_0 < Y$, and let $\Omega \subset \mathbb{R}_x^4 \times \mathbb{R}_y$ be a bounded convex domain with smooth boundary, invariant under $O(4)_x$ and under $y \mapsto -y$, whose boundary consists of the lateral side $\{|x| = R, |y| \leq Y_0\}$ and two strictly convex caps (the rounded ends) with poles $(0, \pm Y)$; see Figure 3. In particular, $\partial\Omega$ is mean convex. Choose a smooth even function $D : [-Y, Y] \rightarrow \mathbb{R}$ with

$$(5) \quad D \leq 0, \quad D'(y) > 0 \quad (0 < y < Y_1), \quad D = 0 \quad (|y| \geq Y_1).$$

Fix $B_0 > 0$ and define

$$(6) \quad q(y) = B_0 + \sqrt{2}R^2 D(y).$$

For $\lambda > 0$ we write $H_\lambda = \{|z| = h_\lambda(|x|)\}$ for the Hardt–Simon leaf of phase λ (see Section 2); thus $h_\lambda(0) = (\lambda/c_H)^{1/3}$, and h_λ is increasing (cf. Section 5). For $\varepsilon > 0$ and b with $B_0 + b > 0$ we define the boundary data

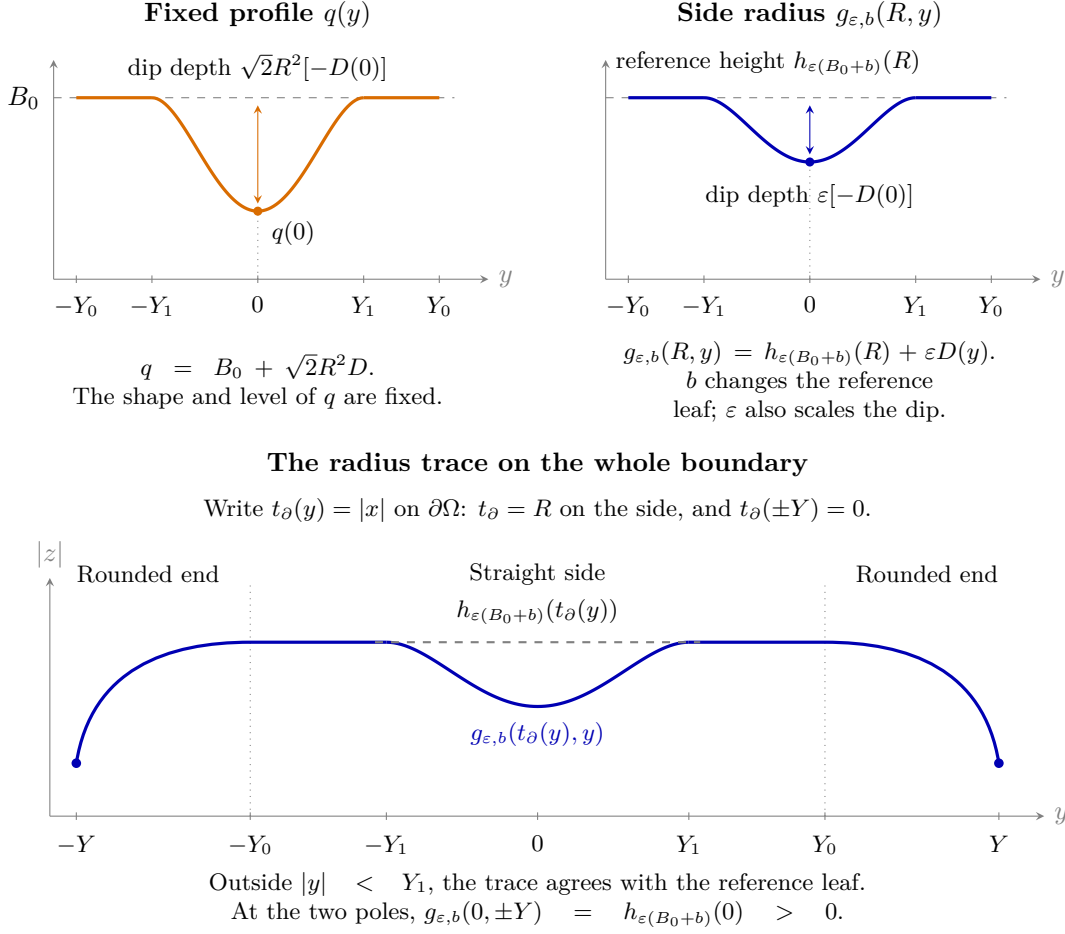
$$(7) \quad g_{\varepsilon,b}(x, y) = \begin{cases} h_{\varepsilon(B_0+b)}(R) + \varepsilon D(y), & |x| = R, |y| \leq Y_0, \\ h_{\varepsilon(B_0+b)}(|x|), & \text{on the rounded ends.} \end{cases}$$

As $\varepsilon \rightarrow 0$, these boundary data converge to $|x|$, the boundary values of V , since the phase of the leaf tends to zero. Figure 4 compares the fixed profile q with the prescribed radius $g_{\varepsilon,b}$. They are related as follows:

$$(8) \quad \frac{g_{\varepsilon,b}(R, y) - h_{\varepsilon(B_0+b)}(R)}{\varepsilon} = D(y) = \frac{q(y) - B_0}{\sqrt{2}R^2}.$$

Thus, to first order in ε , the lateral boundary data have normalized phase $q(y) + b$ (see the proof of Proposition 3.3); this is why the normalized limit of u_ε is described by the harmonic function W below. On the rounded ends the radius varies with $|x|$ along that same leaf; its phase, rather than its radius, is constant there. Since $D \leq 0$, we have $g_{\varepsilon,b}(x, y) \leq h_{\varepsilon(B_0+b)}(|x|)$ on all of $\partial\Omega$, with equality outside the perturbed part of the side.

Let $\mathcal{D} \subset \mathbb{R}_\xi^3 \times \mathbb{R}_y$ be the auxiliary domain obtained from Ω by replacing $|x|$ with $|\xi|/\sqrt{2}$. Let W be the unique bounded harmonic function on $\mathcal{D}$ with boundary values $q(y)$ on the vertical

FIGURE 4. The fixed profile q and the actual boundary radius $g_{\varepsilon,b}$.

side and B_0 on the rounded ends:

$$(9) \quad \begin{cases} \Delta_{\xi,y} W = 0 & \text{in } \mathcal{D}, \\ W(\xi, y) = q(y) & \text{for } |\xi| = \sqrt{2}R, \\ W(\xi, y) = B_0 & \text{on } \partial\mathcal{D} \setminus \{|\xi| = \sqrt{2}R\}. \end{cases}$$

These data match smoothly. By uniqueness, W is $O(3)_{\xi}$ -invariant and even in y . Write

$$m_0 = W(0, 0).$$

Since $q \leq B_0$ and $q \not\equiv B_0$, the strong maximum principle gives

$$\min q < m_0 < B_0.$$

Fix $\delta > 0$ so small that $\delta < (B_0 - m_0)/2$, and choose the parameter interval

$$(10) \quad b_- = -m_0 - \delta, \quad b_+ = -\min q + \delta.$$

The interval $[b_-, b_+]$ is fixed independently of ε , and $B_0 + b \geq B_0 - m_0 - \delta > 0$ throughout it. For each fixed sufficiently small $\varepsilon > 0$, set

$$\varphi_{\tau} = g_{\varepsilon, b_- + \tau(b_+ - b_-)}, \quad 0 \leq \tau \leq 1.$$

We will apply Theorem 3.1 to this path in the symmetric space. It selects a pair $(b_{\varepsilon}, u_{\varepsilon})$ with $\min u_{\varepsilon} = 0$. The selected parameter may depend on ε ; only its allowed interval is fixed independently of ε .

3.2. A priori estimates for symmetric solutions. We now consider any positive, $O(4)_x$ -invariant, reflection-even SME solution with boundary trace $g_{\varepsilon,b}$, where $b \in [b_-, b_+]$. The estimates in this subsection hold whenever such a solution exists. In particular, they apply to a hypothetical sequence at the lower endpoint and will allow us to rule it out in Section 3.3. We first control the phase and axial monotonicity, then identify the harmonic limit, and finally relate its sign to coordinate contacts. In the linearized equations below we write

$$A^{ij}(p) = \delta_{ij} - \frac{p_i p_j}{1 + |p|^2}.$$

Proposition 3.2 (Moving plane estimates). *There are constants $\varepsilon_2 > 0$ and $K_0 > 0$, depending only on Ω , B_0 , D and $b_{\pm}$, with the following property. Let $0 < \varepsilon < \varepsilon_2$ and $b \in [b_-, b_+]$, let $u \in C^2(\overline{\Omega})$ be a positive solution of (2) with $u|_{\partial\Omega} = g_{\varepsilon,b}$, and put $M = S(u)$. Then*

- (1) $|P| \leq K_0 \varepsilon$ on M ,
- (2) $u(x, y) \leq h_{\varepsilon(B_0+b)}(|x|)$ in Ω ,
- (3) u is nondecreasing in $y \geq 0$: $u(x, y_1) \leq u(x, y_2)$ whenever $0 \leq y_1 \leq y_2$ and $(x, y_2) \in \Omega$.

Proof. By elliptic regularity, u is smooth in Ω . On the caps of $\partial\Omega$ the boundary data lie on the leaf $H_{\varepsilon(B_0+b)}$, so that $P = \varepsilon(B_0 + b)$ there. On the lateral side, $|x| = R$ and $g_{\varepsilon,b} = R + O(\varepsilon)$ uniformly in $b \in [b_-, b_+]$; since P is Lipschitz near the point $(|x|, |z|) = (R, R)$, where it vanishes, we obtain $|P| \leq K_0 \varepsilon$ on ∂M for a suitable constant K_0 and all small ε . If $P|_M$ attained its maximum λ at an interior point, then M would touch the smooth minimal hypersurface $H_\lambda \times \mathbb{R}$ from the side $\{P \leq \lambda\}$ at that point. By the strong maximum principle and the connectedness of M , this would give $M \subset H_\lambda \times \mathbb{R}$, which is impossible because $P|_{\partial M}$ is not constant, as $D \neq 0$. The same argument applies to the minimum, and hence $|P| \leq K_0 \varepsilon$ on M . Since $D \leq 0$, moreover, $P \leq \varepsilon(B_0 + b)$ on ∂M , and therefore on M ; that is,

$$u(x, y) \leq h_{\varepsilon(B_0+b)}(|x|).$$

The inequality is strict in Ω , because the lateral data are not identically those of this leaf.

The monotonicity in y follows from the moving plane method [19, 4]. Fix a plane $y = s$, $0 < s < Y$, and

reflect the upper portion $\Omega_s = \Omega \cap \{y > s\}$ by $y \mapsto 2s - y$. Convexity and reflection symmetry of the domain imply that the reflected portion is contained in Ω . Put

$$v_s(x, y) = u(x, y) - u(x, 2s - y).$$

It vanishes on the reflecting plane. If $(x, y) \in \partial\Omega_s$ lies on a cap, then $u(x, y) = h_{\varepsilon(B_0+b)}(|x|) \geq u(x, 2s - y)$. If it lies on the lateral side, then $|x| = R$ and $|2s - y| < y \leq Y_0$, so that $(x, 2s - y)$ lies on the lateral side as well, and $D(2s - y) \leq D(y)$ because D is even and nondecreasing on $[0, Y]$. Consequently

$$\begin{aligned} y = s : \quad v_s &= 0, \\ \partial\Omega \text{ on a rounded end} : \quad v_s &= h_{\varepsilon(B_0+b)}(|x|) - u(x, 2s - y) \geq 0, \\ \partial\Omega \text{ on the side} : \quad v_s &= \varepsilon(D(y) - D(2s - y)) \geq 0. \end{aligned}$$

For the fixed positive smooth solution put $\hat{u}_s(x, y) = u(x, 2s - y)$. Subtraction of the equations gives

$$A^{ij}(Du)(v_s)_{ij} + b_s^i(v_s)_i + c_s v_s = 0, \quad c_s = \frac{3}{u\hat{u}_s} > 0,$$

where

$$b_s^k = \int_0^1 \partial_{p_k} A^{ij}(D\hat{u}_s + \tau Dv_s) d\tau (\hat{u}_s)_{ij}.$$

Ellipticity and all coefficient bounds are uniform in s for this fixed solution. The small-domain maximum principle, a consequence of the Aleksandrov–Bakelman–Pucci estimate, takes the form

$$\|(v_s)^-\|_{L^\infty(E)} \leq C_{\text{MP}} \|c_s\|_\infty |E|^{1/5} \|(v_s)^-\|_{L^\infty(E)}, \quad v_s|_{\partial E} \geq 0,$$

for every open set $E \subset \Omega_s$, where C_{MP} depends only on $\text{diam } \Omega$, the ellipticity constants and $\sup_s \|b_s\|_\infty$. For sufficiently small $|E|$ it forces $v_s \geq 0$ in E , despite the positive zero-order coefficient.

We now run the continuation argument of [19]. Let s_0 be the infimum of those $s \in (0, Y)$ such that $v_{s'} \geq 0$ in $\Omega_{s'}$ for all $s' \in [s, Y)$. Since $|\Omega_s| \rightarrow 0$ as $s \uparrow Y$, the small-domain maximum principle gives $s_0 < Y$. Suppose that $s_0 > 0$. By continuity, $v_{s_0} \geq 0$ in Ω_{s_0} . Moreover, $v_{s_0} \not\equiv 0$, since for points (x, y) of the upper cap the reflected point $(x, 2s_0 - y)$ lies in Ω , where $u < h_{\varepsilon(B_0+b)}(|x|)$. As $v_{s_0} \geq 0$ and $A^{ij}(Du)(v_{s_0})_{ij} + b_{s_0}^i(v_{s_0})_i = -c_{s_0}v_{s_0} \leq 0$, the strong maximum principle gives $v_{s_0} > 0$ in Ω_{s_0} . Choose a compact set $K \subset \Omega_{s_0}$ such that the small-domain maximum principle applies on every open set of measure at most $2|\Omega_{s_0} \setminus K|$. For $s < s_0$ sufficiently close to s_0 , uniform continuity gives $v_s > 0$ on K and $|\Omega_s \setminus K| \leq 2|\Omega_{s_0} \setminus K|$; the small-domain maximum principle on $E = \Omega_s \setminus K$, on whose boundary $v_s \geq 0$, then gives $v_s \geq 0$ in Ω_s . This contradicts the definition of s_0 , and hence $s_0 = 0$. Consequently,

$$0 \leq y_1 < y_2 \implies u(x, y_2) - u(x, y_1) = v_{(y_1+y_2)/2}(x, y_2) \geq 0.$$

These order inequalities pass to locally uniform singular limits. Note that the moving-plane argument uses reflection symmetry of the domain and of the boundary data, but does not require any symmetry of the solution. $\square$

The next proposition identifies the first-order effect of the boundary data. A direct computation, recorded in the proof below, shows that for an invariant Jacobi field f on V , the conjugate $r^2 f$ satisfies the ordinary harmonic equation in $\mathbb{R}_\xi^3 \times \mathbb{R}_y$. Boundedness removes the auxiliary axis, so the boundary data determine the limit uniquely. Reflection symmetry and axial monotonicity then force its first nonconstant term at the origin to be a positive multiple of $y^2 - r^2/3$. The proof makes this reduction and the positivity of the coefficient explicit.

Proposition 3.3 (Vanishing ε asymptotics). *Let $\varepsilon_i \downarrow 0$ and $b_i \in [b_-, b_+]$ with $b_i \rightarrow b_*$. For each i , let u_i be either an $O(4)_x$ -invariant positive solution of (2) in $C^2(\bar{\Omega})$ with $u_i|_{\partial\Omega} = g_{\varepsilon_i, b_i}$, or a uniform limit on $\bar{\Omega}$ of such solutions with the same ε_i and with parameters converging to b_i . Then $M_i = S(u_i) \rightarrow V$, and on every compact subset of the regular part of V over Ω , for all large i , M_i is the normal graph of a function f_i , with*

$$(11) \quad \frac{f_i}{\varepsilon_i} \rightarrow \frac{W(r, y) + b_*}{r^2} \quad \text{in } C_{\text{loc}}^\infty(\text{Reg}(C_{3,3} \times \mathbb{R})),$$

where $W(r, y)$ is the harmonic function defined in (9) and satisfies:

$$(12) \quad W(r, y) = m_0 + c \left(y^2 - \frac{r^2}{3} \right) + O((r^2 + y^2)^2),$$

with $m_0 = W(0, 0)$ and $c > 0$.

Proof. By Proposition 3.2, $|P| \leq K_0 \varepsilon_i$ on M_i ; this bound passes to uniform limits. Hence $M_i \rightarrow V$ in the Hausdorff sense, and the argument in the proof of Proposition 3.4 below shows that, on every compact subset of the regular part of V over Ω , M_i is eventually the normal graph of a function f_i with $|D^k f_i| \leq C_k \varepsilon_i$ for every k . Near $\partial\Omega$, away from the poles $(0, \pm Y)$, the functions u_i are bounded below by a positive constant, and $(u_i - |x|)/\varepsilon_i$ solves a uniformly elliptic linear equation whose boundary values $(g_{\varepsilon_i, b_i} - |x|)/\varepsilon_i$ are bounded in $C^{2, \alpha}$. Boundary Schauder estimates therefore give bounds for f_i/ε_i which are uniform up to $\partial\Omega \setminus \{(0, \pm Y)\}$. Consequently, every subsequence of f_i/ε_i has a further subsequence which converges smoothly on compact subsets of the regular part of V over Ω , and continuously up to the boundary away from the poles.

On the side $t = R$, the conversion from vertical height to normal height gives

$$\begin{aligned} f_i &= \frac{g_{\varepsilon_i, b_i} - R}{\sqrt{2}} + o(\varepsilon_i), \\ \frac{r^2 f_i}{\varepsilon_i} &\longrightarrow B_0 + b_* + \sqrt{2} R^2 D(y) = q(y) + b_*. \end{aligned}$$

On the rounded ends the phase is exactly $\varepsilon_i(B_0 + b_i)$. The polar points are removable for the bounded limiting harmonic Dirichlet problem. Any subsequential limit $f = \lim_i f_i/\varepsilon_i$ and its bounded conjugate $w = r^2 f$ satisfy the explicit linearization identities

$$\begin{aligned} L_V f &= f_{rr} + \frac{6}{r} f_r + f_{yy} + \frac{6}{r^2} f = 0, \\ r^2 L_V(r^{-2} w) &= w_{rr} + \frac{2}{r} w_r + w_{yy} = \Delta_{\mathbb{R}_\xi^3 \times \mathbb{R}_y} w = 0, \\ w|_{\partial \mathcal{D}} &= \begin{cases} q(y) + b_*, & \text{on the side,} \\ B_0 + b_*, & \text{on the rounded ends.} \end{cases} \end{aligned}$$

Removing the auxiliary axis and using uniqueness of the bounded Dirichlet problem gives

$$(13) \quad w = W + b_*.$$

The uniqueness shows that the entire sequence has this normalized limit whenever $b_i \rightarrow b_*$. These estimates apply to singular limits as well: on a fixed compact subset of the regular part of V , the phase estimate keeps u_ε uniformly positive for sufficiently small ε . Thus elliptic regularity still applies there, and the harmonic-limit argument goes through.

The reflection argument in the proof of Proposition 3.2 applies to W , with the ordinary maximum principle in place of the small-domain one. Indeed, for $0 < s < Y$ the function $W(\xi, y) - W(\xi, 2s - y)$ is harmonic in $\mathcal{D} \cap \{y > s\}$, vanishes on $\{y = s\}$, and is nonnegative on $\partial \mathcal{D} \cap \{y > s\}$, since $W \leq B_0$ with equality on the rounded ends, and the lateral data q are even and nondecreasing in $|y|$. Hence $W_y \geq 0$ for $y > 0$. It is not identically zero, since the boundary data vary. The strong maximum principle for its harmonic derivative gives $W_y > 0$ in the upper half-domain. On the central plane, $W_y = 0$, and Hopf's lemma at the origin of the auxiliary half-domain gives

$$W_{yy}(0, 0) > 0.$$

Analyticity and the two symmetries yield

$$W(r, y) = m_0 + a_2 r^2 + c y^2 + O((r^2 + y^2)^2), \quad 6a_2 + 2c = 0, \quad c = \frac{1}{2} W_{yy}(0, 0) > 0.$$

This proves the estimate, including the positivity of c and the analytic, $O(3)$ -invariant, reflection-even remainder. In particular, $W(0, y)$ has its unique minimum m_0 at $y = 0$. $\square$

Proposition 3.4. *Let $0 < \varepsilon < \varepsilon_2$, and let u_ε be a positive solution as in Proposition 3.2, or a uniform limit on $\bar{\Omega}$ of such solutions with the same ε . There is a constant C_* , independent of ε and b , such that the part of $S(u_\varepsilon)$ lying over $V \cap B_{2r_*}^9 \cap \{r \geq C_* \varepsilon^{1/3}\}$ is the normal graph over V of a function f_ε satisfying*

$$(14) \quad |D^k f_\varepsilon| \leq C_k \varepsilon r^{-2-k}, \quad k = 0, 1, 2,$$

with constants independent of ε , and the same bound holds for every $k \geq 0$, with C_k depending on k .

Proof. By Proposition 3.2, $|P| \leq K_0 \varepsilon$ on $S(u_\varepsilon)$, and this bound passes to uniform limits. Given this bound, the leaf expansion and separation estimates [34, Proposition 3.3 and Lemma 5.1] give

$$r \geq C_* \varepsilon^{1/3} \implies t \approx u_\varepsilon \approx r, \quad |u_\varepsilon - t| \leq C \varepsilon r^{-2}.$$

Fix a point of this region, and work in a base ball of radius cr about it, contained in the interior, and dilate by r^{-1} . The rescaled functions $\hat{u} = r^{-1} u_\varepsilon$ and $\hat{t} = r^{-1} t$ have uniformly positive height and bounded derivatives, by [18, Theorem 8.1] and positive-height elliptic regularity. Their difference $z = \hat{u} - \hat{t}$ satisfies

$$a^{ij} z_{ij} + b^i z_i + \frac{3}{\hat{u} \hat{t}} z = 0, \quad \|z\|_{C^k(B_{c/2})} \leq C_k \|z\|_{C^0(B_c)} \leq C_k \varepsilon r^{-3},$$

with uniform ellipticity and coefficient bounds. Here and below a C^k estimate means any fixed finite order. Scaling back gives the same bounds for $u_\varepsilon - t$. For C_* sufficiently large, the C^1 error makes normal projection onto V a diffeomorphism. Its normal height therefore satisfies

$$(15) \quad |D^k f_\varepsilon| \leq C_k \varepsilon r^{-2-k}, \quad r \geq C_* \varepsilon^{1/3}.$$

The constants are uniform on each fixed interior region. $\square$

The preceding normal-graph convergence holds away from the spine $\{x = z = 0\}$. Although the limiting function $w = r^2 f$ extends continuously to the spine, we cannot evaluate the normal heights f_i there: the reference cylinder is singular at those points. The next lemma supplies the missing link between this regular convergence and intersections of the hypersurfaces with the coordinate subspaces $\{x = 0\}$ and $\{z = 0\}$. The phase is nonnegative on $\{x = 0\}$ and nonpositive on $\{z = 0\}$. The lemma transfers these signs to the limiting value $w(0, y_*)$, using barriers near the spine. The lemma is needed both before existence, to check the lower endpoint in [18], and after existence, to select the limiting phase.

Lemma 3.5 (Signed transmission at the spine). *Let $I \subset \mathbb{R}$ be an open interval, and let $\mathcal{U} \subset \mathbb{R}^9$ be a fixed open neighborhood of the spine segment*

$$\{(0, 0, y) : y \in I\}.$$

Let M_i be multiplicity-one stationary integral varifolds in $\mathcal{U}$ of the form $S(u_i)$, where the u_i are $O(4)_x$ -invariant SME limits (so that M_i is invariant under $O(4)_x \times O(4)_z$), and let $\varepsilon_i \downarrow 0$. Assume that their signed Hardt–Simon phase satisfies

$$|P| \leq K\varepsilon_i \quad \text{on } \text{spt } M_i \cap \mathcal{U}$$

for a constant K independent of i . Suppose that, near each compact subset of $V \cap \mathcal{U} \cap \{r > 0\}$, the whole support of M_i is, for all sufficiently large i , a single normal graph over V . Write f_i for its signed Euclidean normal height, and assume

$$\frac{r^2 f_i}{\varepsilon_i} \longrightarrow w(r, y) \quad \text{smoothly on compact subsets of } V \cap \mathcal{U} \cap \{r > 0\},$$

where w extends continuously to $r = 0$ for $y \in I$.

For any $y_ \in I$, the following conclusions hold:*

- (i) *If there are points $(0, z_i, y_i) \in \text{spt } M_i \cap \mathcal{U}$ with $y_i \rightarrow y_*$, then $w(0, y_*) \geq 0$.*
- (ii) *If there are points $(x_i, 0, y_i) \in \text{spt } M_i \cap \mathcal{U}$ with $y_i \rightarrow y_*$, then $w(0, y_*) \leq 0$.*

In each case the phase bound forces the contact points to converge to $(0, 0, y_)$. If both types of contact approach the same spine point, then $w(0, y_*) = 0$.*

Proof. We recall precisely the signed barrier input. Let $H = H_1$ be the Hardt–Simon leaf with $P = 1$. For a uniformly C^2 -bounded family of real functions q on a fixed axial interval, [34, Propositions 5.8 and 5.10] give barriers X_q modeled on the slicewise dilated leaves $\varepsilon^{1/3} q(y)^3 H$; when $q(y) < 0$, the dilation by the negative factor $\varepsilon^{1/3} q(y)^3$ is understood as the dilation by $\varepsilon^{1/3} |q(y)|^3$ followed by the exchange of the factors x and z , so that in all cases the leaf has phase $\varepsilon q(y)^9$. Given $\eta > 0$, there is a lateral radius $r_0 > 0$, depending only on η and on the family, such that for all sufficiently small ε the barriers exist in $\{r < r_0\}$ and satisfy

$$(16) \quad \varepsilon(q^9 - \eta) \leq P|_{X_q} \leq \varepsilon(q^9 + \eta).$$

The constants are uniform for the family. Each barrier separates the comparison region into a negative and a positive side, lies between the two comparison leaves of phases $\varepsilon(q^9 \pm \eta)$, and cannot be touched from its negative side by a stationary hypersurface of the type considered here. The cited result includes contact at their singular spine points when $q = 0$; ordinary smooth comparison alone would not suffice there. Exchanging x and z reverses both the phase and the comparison direction.

We prove (i). Suppose that contact points as in (i) exist, but $w(0, y_*) < 0$. Choose $c > 0$ and $I_0 \Subset I$ containing y_* so that $w(r, y) < -3c$ for small r and $y \in I_0$. Choose $\psi \in C_c^\infty(I_0)$ with $0 \leq \psi \leq 1$ and $\psi = 1$ near y_* . For A sufficiently large, use

$$q_s = A - s\psi, \quad 0 \leq s \leq A + c^{1/9}, \quad A^9 - c/4 > K.$$

This is a uniformly C^2 -bounded signed family, with $q_s^9 \geq -c$, $q_s = A$ near the axial ends, and final value $q_s^9 = -c$ near y_* . Choose one lateral radius r_0 for the whole family in (16), with

$\eta = c/4$. The fixed-leaf expansion converts regular graphical convergence to $P/\varepsilon_i \rightarrow w$ on $r = r_0$. Thus, for large i ,

$$\begin{aligned} r = r_0 : \quad & P|_{M_i} < -2c\varepsilon_i < -\frac{5}{4}c\varepsilon_i \leq P|_{X_{q_s}}, \\ y \in \partial I_0 \text{ or } s = 0 : \quad & P|_{M_i} \leq K\varepsilon_i < (A^9 - c/4)\varepsilon_i \leq P|_{X_{q_s}}. \end{aligned}$$

The cylinder is chosen compactly inside the common domain. These inequalities order its entire comparison boundary. Starting with $s = 0$, a first loss of containment cannot occur at that boundary or in its interior, by the signed barrier comparison just recalled. At the final parameter the barrier near y_* has phase at most $-3c\varepsilon_i/4$; its negative side contains no point with $x = 0$. But $P(0, z_i, y_i) = c_H|z_i|^3$ and the phase bound give $|z_i| = O(\varepsilon_i^{1/3})$, so the assumed contact is in this comparison region for large i . This is a contradiction. Exchanging the two four-dimensional factors proves the other assertion. $\square$

3.3. Proof of Proposition 2.1.

We divide the proof into four steps.
Step 1: verify the boundary-path and endpoint conditions. Use the convex domain and r_* chosen in Section 3.1. For fixed sufficiently small $\varepsilon > 0$, the traces $\varphi_\tau = g_{\varepsilon, b_- + \tau(b_+ - b_-)}$ form a continuous path in $C^{2,\alpha}(\partial\Omega)$ with the required symmetries. Indeed $D = 0$ near the junctions with the rounded ends, so the two formulas in (7) match smoothly. The leaves depend smoothly on the positive phase $\varepsilon(B_0 + b)$. On the side, $g_{\varepsilon, b} = R + O(\varepsilon) > 0$ uniformly in b , and on the rounded ends it is the positive leaf radius. Compactness gives

$$\min_{b \in [b_-, b_+]} \min_{\partial\Omega} g_{\varepsilon, b} > 0.$$

These traces admit symmetric $C^{2,\alpha}(\overline{\Omega})$ extensions depending continuously on τ . The positive lower bound may depend on the fixed ε .

At the lower endpoint, no positive solution in the imposed symmetry class (such solutions are understood to be of class $C^{2,\alpha}(\overline{\Omega})$, as are the fixed points of the homotopy in [18]) exists for $b \in [b_-, b_- + \delta/2]$ when ε is sufficiently small. Otherwise take such solutions with $\varepsilon_i \rightarrow 0$ and $b_i \rightarrow b$ in this interval. Proposition 3.3 gives $r^2 f_i/\varepsilon_i \rightarrow W + b$ on regular compact subsets. Every radius graph contains $(0, u_i(0, 0)\omega, 0)$, and the phase bound makes these points approach the origin. Lemma 3.5 implies $m_0 + b \geq 0$, contradicting $m_0 + b \leq -\delta/2$. The remaining hypotheses of that lemma follow from the phase bound and Proposition 3.4.

At the upper endpoint, and throughout $b \in [b_+ - \delta/4, b_+]$, we have $q + b \geq 3\delta/4$. The side phase is $\varepsilon(q + b) + o(\varepsilon)$ uniformly in this interval; on the rounded ends it is exactly $\varepsilon(B_0 + b) \geq 3\delta\varepsilon/4$. Thus $\min_{S(g_{\varepsilon, b})} P \geq \delta\varepsilon/2$ for small ε , where $S(g_{\varepsilon, b}) = \{(x, z, y) : (x, y) \in \partial\Omega, |z| = g_{\varepsilon, b}(x, y)\}$, and the trace strictly dominates that of the positive leaf $h_{\delta\varepsilon/4}$. The phase minimum principle (see the proof of Proposition 3.2), together with the monotonicity of $h_{\delta\varepsilon/4}$, gives

$$(17) \quad u|_{\partial\Omega} \geq h_{\delta\varepsilon/4}(|x|) \implies \inf_{\Omega} u \geq h_{\delta\varepsilon/4}(0) =: \mu_\varepsilon > 0$$

for every positive SME solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, symmetric or not, with these dominating boundary values. Hence this leaf trace is strongly positive, and φ_1 lies strictly above it, as required.

Step 2: apply [18] and obtain items (1) and (2). We use Theorem 3.1 with $n = 5$ and $m = 4$, the mean-convex extension stated after [18, Theorem 4.4], and its proof in the closed space

$$\mathcal{B} = \{v \in C^{1,\alpha}(\overline{\Omega}) : v(Qx, y) = v(x, y) = v(x, -y), Q \in O(4)\}.$$

The domain, the boundary path, and the terminal constant in the FS homotopy have these symmetries. For invariant input, each linear Dirichlet problem in that homotopy has invariant coefficients and data; uniqueness makes its solution invariant. Thus the homotopy preserves $\mathcal{B}$, and the nonsolvability established in this space is precisely the lower-endpoint condition needed for the degree argument.

Theorem 3.1 now gives a parameter b_ε and a nonnegative singular solution u_ε , with

$$(18) \quad u_\varepsilon|_{\partial\Omega} = g_{\varepsilon, b_\varepsilon}, \quad \min_{\overline{\Omega}} u_\varepsilon = 0, \quad \text{dist}(\{u_\varepsilon = 0\}, \partial\Omega) > 0.$$

We also retain the approximation supplied by their proof: there are positive solutions $u_{\varepsilon,j} \in \mathcal{B}$ and parameters $b_{\varepsilon,j} \in (b_-, b_+)$ such that

$$(19) \quad \begin{aligned} u_{\varepsilon,j}|_{\partial\Omega} &= g_{\varepsilon,b_{\varepsilon,j}}, & b_{\varepsilon,j} &\longrightarrow b_\varepsilon, \\ u_{\varepsilon,j} &\longrightarrow u_\varepsilon & \text{uniformly on } \overline{\Omega}. \end{aligned}$$

In particular $\min u_{\varepsilon,j} \rightarrow 0$. Step 1 excludes the lower endpoint interval for every approximant and the upper endpoint interval for all sufficiently large j , since there the minimum is at least μ_ε . Consequently

$$b_\varepsilon \in [b_- + \delta/2, b_+ - \delta/4] \Subset (b_-, b_+).$$

The symmetries pass to the limit. Applying Proposition 3.2 to the approximants and then using (19) gives

$$0 \leq y_1 \leq y_2 \implies u_\varepsilon(x, y_1) \leq u_\varepsilon(x, y_2)$$

whenever the two points belong to Ω . By [18, Corollary 2.5], $M_\varepsilon = S(u_\varepsilon)$ is stationary with multiplicity one, is smooth where $u_\varepsilon > 0$, and has a singularity at each point corresponding to a zero of u_ε . This proves items (1) and (2).

Step 3: phase, compactness, and derivative bounds, items (3)–(5). The phase estimate for the same approximants passes to the limit:

$$|P| \leq K\varepsilon \quad \text{on } M_\varepsilon,$$

with K independent of ε . The boundary traces have a common upper bound for small ε and $b \in [b_-, b_+]$. The maximum principle (at an interior maximum the left-hand side of (2) would be nonpositive) bounds the positive approximants by the same constant, and hence also bounds u_ε . The interior gradient estimate [18, Theorem 8.1], applied on balls compactly contained in Ω , therefore gives

$$\sup_{0 < \varepsilon < \varepsilon_0} \|u_\varepsilon\|_{C^{0,1}(U)} < \infty \quad \text{for every } U \Subset \Omega.$$

Every sequence $\varepsilon_i \rightarrow 0$ has a locally uniform subsequential limit. By continuity of P , its radius graph lies in $\{P = 0\} = V$, so the limit must be $|x|$. Thus $u_\varepsilon \rightarrow |x|$ locally uniformly throughout Ω . On compact subsets with $|x| > 0$, the height stays uniformly positive, and elliptic regularity upgrades this to smooth convergence.

The area density of the radius parametrization is $u_\varepsilon^3 \sqrt{1 + |Du_\varepsilon|^2}$, apart from integration over S^3 . The uniform local Lipschitz bound dominates this density; the parametrizations and their tangent planes converge away from $\{x = 0\}$, a set of zero base measure. Dominated convergence in the area formula gives $M_\varepsilon \rightarrow V$ as varifolds with multiplicity one, with no additional mass on the spine. This proves item (4).

Proposition 3.4, applied to these singular limits of solutions with the prescribed boundary data, gives a normal graph on $V \cap B_{2r_*}^9 \cap \{r \geq C_* \varepsilon^{1/3}\}$ and

$$|D^k f_\varepsilon| \leq C_k \varepsilon r^{-2-k}, \quad k = 0, 1, 2,$$

for fixed sufficiently large C_* , with constants independent of ε . This proves item (5).

Step 4: identify the boundary parameter and prove item (6). The existence of an interior zero does not by itself identify b_ε . Along any sequence $\varepsilon_i \rightarrow 0$, take a subsequence with $b_{\varepsilon_i} \rightarrow b$. Proposition 3.3 gives

$$\frac{r^2 f_{\varepsilon_i}}{\varepsilon_i} \longrightarrow W + b$$

on regular compact subsets. If $u_{\varepsilon_i}(x_i, y_i) = 0$, reflection and axial monotonicity imply $u_{\varepsilon_i}(x_i, 0) = 0$. The signed phase identity at $z = 0$ gives

$$c_H |x_i|^3 \leq K \varepsilon_i,$$

so these central contacts approach the origin. Lemma 3.5 then implies $m_0 + b \leq 0$. On the other hand, $(0, u_{\varepsilon_i}(0, 0)\omega, 0)$ always belongs to the symmetric graph. Its radius is also $O(\varepsilon_i^{1/3})$,

and the other conclusion of the same lemma gives $m_0 + b \geq 0$. Therefore $b = -m_0$. Every subsequential limit has this value, so

$$b_\varepsilon \longrightarrow -m_0, \quad \frac{f_\varepsilon}{\varepsilon} \longrightarrow \frac{W - m_0}{r^2}$$

smoothly on regular compact subsets, with normal $\nu = (-x, z, 0)/r$. The same proposition gives the symmetries of W , $W_y > 0$ for $y > 0$, and

$$W(r, y) - m_0 = c \left(y^2 - \frac{r^2}{3} \right) + O((r^2 + y^2)^2), \quad c > 0.$$

Analyticity supplies the differentiated Taylor bounds. This proves item (6), and all conclusions of Proposition 2.1 follow. $\square$

4. PROPOSITION 2.2: THE CLASSIFICATION OF SINGULARITIES

The previous section produces a singular solution with symmetry, axial monotonicity, and a controlled leading Jacobi field. It does not yet locate all of its zeros or identify its tangent cones at such zeros. We first show that an off-axis zero would have density at least $\theta_{13} > \theta_V$. Monotonicity at a fixed interior radius and convergence of the family to V exclude such zeros. Symmetry and axial monotonicity then make the zero set a nonempty closed interval on the axis.

The next step is a strict density drop at the origin. The field J_2 has negative second variation on the spherical link of V , but that link has singular poles. We estimate the area of the omitted polar neighborhoods using the uniform Lipschitz bound and expand the regular part of the link to second order. On a sufficiently small fixed sphere, the leading negative term dominates the errors as $\varepsilon \rightarrow 0$. Stationarity converts this link-area deficit into a density strictly below θ_V . A nontrivial interval of axial zeros would force density at least θ_V in a tangent cone, so the interval reduces to the origin. Only then do we use the minimal-bigraph classification of [16, Corollary 3.18] to identify every tangent cone as $C_{4,3}$.

4.1. Ruling out off-axis zeros. In this subsection we rule out zeros away from the axis $\{x = 0\}$. The main ingredients are the density comparison between $C_{1,3}$ and V and the dimension bound for the zero set of SME limits in [18, Theorem 9.1]. Recall from (1) that

$$\theta_V = \Theta(V) = \frac{15\pi}{32} < \theta_{13} = \Theta(C_{1,3}) = \frac{9\sqrt{3}\pi}{32}.$$

Lemma 4.1 (Density at an off-axis zero). *Let u be an $O(4)_x$ -invariant SME limit on a neighborhood of a point $(x_0, y_0) \in \mathbb{R}^5$ with $x_0 \neq 0$ and $u(x_0, y_0) = 0$, and put $N = S(u)$ and $p = (x_0, 0, y_0)$. Then*

$$\Theta_N(p) \geq \Theta(C_{1,3}) > \theta_V.$$

Consequently, if u is an $O(4)_x$ -invariant SME limit on all of $\mathbb{R}^5$ with $\Theta_N(\infty) \leq \theta_V$ (in particular, if N is an SME cone with $\Theta(N) \leq \theta_V$), then u has no zeros off the axis $\{x = 0\}$.

Proof. Let T be a tangent cone of N at p , along scales $s_i \rightarrow 0$. We first show that the tangent directions of the symmetry orbit through p are directions of translation invariance of T . The orbit of p under rotations of x is

$$\{(x, 0, y_0) : |x| = |x_0|\},$$

a three-dimensional sphere. When we blow-up around p , this sphere becomes flatter and converges to its tangent plane

$$E = \{(w, 0, 0) : w \cdot x_0 = 0\} \cong \mathbb{R}^3.$$

Given $e \in E$, choose $Q_i \in O(4)$, acting on x , with $Q_i x_0 = x_0 + s_i e + O(s_i^2)$ and $|Q_i - I| = O(s_i)$. The maps $Z \mapsto s_i^{-1}(Q_i(p + s_i Z) - p)$ of $\mathbb{R}^9$ preserve $s_i^{-1}(N - p)$ and converge locally uniformly to the translation $Z \mapsto Z + e$. Hence the tangent cone satisfies

$$T + e = T \quad \text{for every } e \in E.$$

Identifying E with $\mathbb{R}^3$ and writing $X \in \mathbb{R}^2$ for the remaining base coordinates (the x direction parallel to x_0 and the axial direction y), and noticing that the local SME Lipschitz bound [18, Theorem 8.1] passes to the rescaled profiles of u at p along the same scales, we deduce that T is a homogeneous radius graph U and

$$U(e, X) = v(X), \quad T = \mathbb{R}^3 \times K, \quad K = S(v) = \{(X, z) \in \mathbb{R}^2 \times \mathbb{R}^4 : |z| = v(X)\},$$

where $v(\lambda X) = \lambda v(X)$ for $\lambda > 0$. The factor K is therefore a five-dimensional stationary cone in $\mathbb{R}^6$.

If $v(X_0) = 0$ at $X_0 \neq 0$, the zero set of the full tangent profile contains $\mathbb{R}^3 \times \{\tau X_0 : \tau \geq 0\}$, of dimension four. This contradicts [18, Theorem 9.1], according to which the zero set of an SME limit on a five-dimensional base has Hausdorff dimension at most $5 - 2 = 3$. Hence $v > 0$ away from zero.

The link $\Sigma = K \cap S^5$ is therefore a smooth compact, embedded minimal hypersurface of S^5 which is rotational, that is, invariant under the action of $O(4)$ on the z -factor. It cannot be a great sphere: an $O(4)_z$ -invariant hyperplane has its normal in $\mathbb{R}^2$ and cannot project onto the whole two-dimensional base as $S(v)$ does. By [7, Theorem 1.1], its area is either the great-sphere area, the area of $S^1(1/2) \times S^3(\sqrt{3}/2)$, or greater than $2(1 - 1/\pi)$ times the latter. The first alternative would make K a density-one cone; monotonicity centered at a regular point would then force K to be a plane, already excluded. Therefore

$$\Theta_N(p) = \Theta(T) = \Theta(K) \geq \Theta(C_{1,3}) = \theta_{13}.$$

For an entire N , monotonicity centered at p gives $\Theta_N(p) \leq \Theta_N(\infty)$; the limit at infinity is independent of the center by comparison of concentric balls of radii $R - |p|$ and $R + |p|$. A cone has the same density at its vertex and at infinity. Both stated consequences follow. $\square$

Corollary 4.2 (The zero set of the Dirichlet solution lies on the axis). *For all sufficiently small ε , every zero of u_ε lies on $x = 0$. Its zero set is a nonempty closed axial interval $\{(0, y) : |y| \leq \ell_\varepsilon\}$; in particular $u_\varepsilon(0, 0) = 0$.*

Proof. If off-axis zeros existed for a sequence $\varepsilon_i \rightarrow 0$, axial monotonicity would give central off-axis zeros $p_i = (x_i, 0, 0)$ of M_{ε_i} . The phase bound implies $|x_i| = O(\varepsilon_i^{1/3})$, so $p_i \rightarrow 0$. Choose a fixed $\rho > 0$ so that $B_{2\rho}(0)$ lies in the common interior domain. Lemma 4.1 and monotonicity give

$$\theta_{13} \leq \Theta(M_{\varepsilon_i}, p_i) \leq \frac{\|M_{\varepsilon_i}\|(B_\rho(p_i))}{\omega_8 \rho^8} \rightarrow \theta_V.$$

For completeness, the limit follows by enclosing the moving ball between $B_{\rho-\eta}(0)$ and $B_{\rho+\eta}(0)$, using multiplicity-one convergence to V , and then sending $\eta \downarrow 0$. The limiting cone gives zero mass to the bounding spheres. This contradicts $\theta_{13} > \theta_V$. The zero set is nonempty by construction. Reflection, axial monotonicity, continuity, and the positivity of u_ε near $\partial\Omega$ (the zero set has positive distance from $\partial\Omega$) now give the asserted interval description. Points where the profile is positive are smooth by elliptic regularity. $\square$

4.2. Decreasing area from the degree-zero Jacobi field. We turn the leading Jacobi field into a quantitative density deficit. The argument has three parts: remove small neighborhoods of the two singular poles and bound their area; compute the second-order area change on the remaining smooth link; then use stationarity to compare link area with the density ratio. The cutoff radius is chosen so that the omitted polar neighborhoods and cutoff errors are smaller than the negative quadratic contribution of J_2 . The required inputs are the uniform Lipschitz bound, phase trapping, and the normalized expansion from the preceding section. Write $L = V \cap S^8$. The link L has two singular poles. Its regular part is minimal in S^8 , has dimension seven, and satisfies

$$|A_L|^2 = \frac{6}{r^2}, \quad d\mu_L \approx r^6 dr d\omega \quad \text{near either pole.}$$

Here $d\omega$ denotes the fixed angular measure on $S^3 \times S^3$. Define its quadratic form by

$$(20) \quad \mathcal{Q}_L(\phi) = \int_L (|\nabla_L \phi|^2 - (|A_L|^2 + 7)\phi^2) d\mu_L.$$

The next lemma shows that the part of a spherical slice of $S(v)$, for an SME limit v , lying near the spine has small area; this will allow us to expand the area of the remaining part.

Lemma 4.3. *Let $M = \{|z| = v(|x|, y)\}$ be the multiplicity-one symmetric graph of a nonnegative function with Lipschitz constant at most Λ , defined on a neighborhood of the two points $(x, y) = (0, \pm 1)$. Then, for sufficiently small s , depending only on Λ ,*

$$(21) \quad \mathcal{H}^7(M \cap S^8 \cap \{r < s\}) \leq C_\Lambda s^7.$$

The estimate includes graphs which are singular where $v = 0$.

Proof. For small s , the sphere equation $|x|^2 + v(|x|, y)^2 + y^2 = 1$ is uniformly transverse to the y -direction near either pole: on the north polar region, $y + vv_y \geq \sqrt{1 - s^2} - \Lambda s \geq 1/2$, and the south pole has the opposite sign. Each polar part therefore admits a Lipschitz parametrization

$$(x, \omega) \mapsto (x, \hat{v}(x)\omega, y(x)), \quad x \in D \subset B_s^4, \quad \omega \in S^3,$$

where $\hat{v}(x) = v(|x|, y(x))$, $0 \leq \hat{v} \leq s$, and the Lipschitz bounds for y and $\hat{v}$ depend only on Λ . Its area Jacobian is bounded by $C_\Lambda \hat{v}^3$. Hence the area formula gives

$$\mathcal{H}^7(\text{polar region}) \leq C_\Lambda \int_D \hat{v}^3 dx \leq C_\Lambda s^3 |B_s^4| \leq C_\Lambda s^7.$$

Collapsed fibers at $\hat{v} = 0$ contribute no seven-dimensional area. Summing over the two poles proves the claim. $\square$

In the next proposition we expand the area up to second order and prove the relevant estimates.

Proposition 4.4 (Second-order link expansion). *Let $\varepsilon_i \downarrow 0$, and let $M_i = S(u_i)$, where the u_i are $O(4)_x$ -invariant SME limits on a fixed neighborhood of the closed unit ball of $\mathbb{R}^5$, with a uniform Lipschitz bound. Suppose that, near S^8 and away from the spine, the M_i are Euclidean normal graphs over V of functions f_i , and that*

$$(22) \quad \varepsilon_i^{-1} f_i \longrightarrow f \quad \text{smoothly on compact subsets of } V \setminus \{r = 0\}.$$

Suppose in addition that for $r \geq C_ \varepsilon_i^{1/3}$,*

$$(23) \quad |D^k f_i| \leq C_k \varepsilon_i r^{-2-k}, \quad 0 \leq k \leq 2,$$

on a fixed annular neighborhood of the unit sphere. Then

$$(24) \quad \mathcal{H}^7(M_i \cap S^8) = \mathcal{H}^7(L) + \frac{\varepsilon_i^2}{2} \mathcal{Q}_L(f|_L) + o(\varepsilon_i^2).$$

In particular, the integral defining the displayed quadratic form is absolutely convergent.

Proof. Choose $s_i = \varepsilon_i^\beta$ with $2/7 < \beta < 1/3$. Since $\varepsilon_i s_i^{-3} \rightarrow 0$, the graph bounds give a spherical normal height ϕ_i over $L \cap \{r \geq s_i\}$ satisfying

$$(25) \quad |\phi_i| \leq C \varepsilon_i r^{-2}, \quad |\nabla_L \phi_i| \leq C \varepsilon_i r^{-3}, \quad \varepsilon_i^{-1} \phi_i \longrightarrow f|_L \quad \text{in } C_{\text{loc}}^1.$$

These estimates follow by solving the sphere equation $a^2 + u_i(ap)^2 = 1$ for a near 1, with $p \in L$. The omitted part of $M_i \cap S^8$ lies in $\{r < C s_i\}$. By Lemma 4.3, it and the omitted part of L have total area $O(s_i^7) = o(\varepsilon_i^2)$.

The standard area expansion for a spherical normal graph over the minimal link L has quadratic term $\frac{1}{2}(|\nabla_L \phi_i|^2 - (|A_L|^2 + 7)\phi_i^2)$ and a pointwise remainder bounded by

$$C((1 + |A_L|)|\phi_i| + |\nabla_L \phi_i|)^3 \leq C \varepsilon_i^3 r^{-9}.$$

Since $d\mu_L \sim r^6 dr d\omega$ near the poles, the integrated remainder is $O(\varepsilon_i^3 s_i^{-2}) = o(\varepsilon_i^2)$. Thus

$$\mathcal{H}^7(M_i \cap S^8) - \mathcal{H}^7(L) = \frac{1}{2} \int_{L \cap \{r \geq s_i\}} (|\nabla_L \phi_i|^2 - (|A_L|^2 + 7)\phi_i^2) d\mu_L + o(\varepsilon_i^2).$$

Finally, after division by ε_i^2 and extension by zero on $\{r < s_i\}$, the absolute value of the quadratic integrand is bounded by $C r^{-6}$, which is integrable on L . The local convergence in

(25) and dominated convergence therefore give $\mathcal{Q}_L(f|_L)$ as its integral limit. This also proves absolute convergence of the quadratic form and yields (24). $\square$

We next show that J_2 is a direction of negative second variation for the area of L , and that the same holds for the restrictions to small spheres of Jacobi fields whose conjugate $w = r^2 f$ has leading term proportional to $y^2 - r^2/3$.

Lemma 4.5. *For $J_2 = \frac{y^2}{r^2} - \frac{1}{3}$, one has*

$$(26) \quad \mathcal{Q}_L(J_2) = -7 \int_L J_2^2 d\mu_L < 0.$$

More generally, suppose

$$f(r, y) = r^{-2} w(r, y),$$

where w is smooth across $r = 0$ as a rotationally invariant function on $\mathbb{R}^3 \times \mathbb{R}$, and near the origin

$$(27) \quad w(r, y) = c \left(y^2 - \frac{r^2}{3} \right) + O((r^2 + y^2)^{3/2}), \quad c \neq 0, .$$

Then, for all sufficiently small fixed $\rho > 0$,

$$\mathcal{Q}_L(f(\rho \cdot)) < 0.$$

Proof. A direct calculation gives $(\Delta_L + |A_L|^2)J_2 = 0$. Since $|J_2| \leq Cr^{-2}$ and $|\nabla_L J_2| \leq Cr^{-3}$, the boundary term in integration by parts over $L \cap \{r > d\}$ is $O(d)$. Letting $d \downarrow 0$ therefore gives

$$\mathcal{Q}_L(J_2) = - \int_L J_2(\Delta_L + |A_L|^2 + 7)J_2 d\mu_L = -7\|J_2\|_{L^2(L)}^2 < 0.$$

By the differentiated Taylor bounds in (27),

$$f(\rho p) = cJ_2(p) + E_\rho(p), \quad |E_\rho| \leq C\rho r^{-2}, \quad |\nabla_L E_\rho| \leq C\rho r^{-3}.$$

Since $\int_L r^{-6} d\mu_L < \infty$, expanding the quadratic form and applying these bounds yields

$$\mathcal{Q}_L(f(\rho \cdot)) = c^2 \mathcal{Q}_L(J_2) + O(\rho) < 0$$

for all sufficiently small $\rho > 0$. $\square$

We can now show that the density drops below θ_V .

Proposition 4.6 (Strict density drop). *Assume that the M_i in Proposition 4.4 are stationary in a neighborhood of the closed unit ball. If $\mathcal{Q}_L(f|_L) < 0$, then for all sufficiently large i ,*

$$\Theta(M_i, 0) < \Theta(V, 0).$$

The same conclusion holds if the preceding hypotheses hold, after a fixed dilation, on a fixed neighborhood of the origin, and the conjugate $w = r^2 f$ of the limit Jacobi field satisfies (27); in this case the link is evaluated at a sufficiently small fixed radius.

Proof. For a compactly supported radial cutoff η , stationarity gives

$$\begin{aligned} 0 &= \delta M_i(\eta(|X|)X) \\ &= \int_{M_i} \left(8\eta(|X|) + |X|\eta'(|X|) |\nabla_{M_i} X|^2 \right) d\|M_i\|. \end{aligned}$$

Approximating the characteristic function of B_ρ and applying coarea gives, for almost every ρ ,

$$(28) \quad 8\|M_i\|(B_\rho) = \rho \int_{M_i \cap \partial B_\rho} |\nabla_{M_i} X| d\mathcal{H}^7 \leq \rho \mathcal{H}^7(M_i \cap \partial B_\rho).$$

The same first-variation calculation with the usual radial powers gives Euclidean monotonicity in the form

$$\frac{\|M_i\|(B_\rho)}{\rho^8} - \frac{\|M_i\|(B_\sigma)}{\sigma^8} = \int_{M_i \cap (B_\rho \setminus B_\sigma)} \frac{|X^\perp|^2}{|X|^{10}} d\|M_i\| \geq 0, \quad 0 < \sigma < \rho.$$

Both calculations are varifold first-variation identities and remain valid across the singular set. Therefore

$$\Theta(M_i, 0) \leq \frac{\|M_i\|(B_\rho)}{\omega_8 \rho^8} \leq \frac{\mathcal{H}^7(M_i \cap \partial B_\rho)}{8\omega_8 \rho^7}.$$

At unit radius, whenever the flux identity holds,

$$\begin{aligned} \Theta(M_i, 0) &\leq \frac{\mathcal{H}^7(L)}{8\omega_8} + \frac{\varepsilon_i^2}{16\omega_8} \mathcal{Q}_L(f|_L) + o(\varepsilon_i^2) \\ &< \frac{\mathcal{H}^7(L)}{8\omega_8} = \Theta(V, 0). \end{aligned}$$

If necessary choose a nearby common admissible radius. Indeed, if E_i is the exceptional set for M_i , then

$$|\cup_i E_i| = 0, \quad \mathcal{Q}_L(f(\rho \cdot)) < 0 \quad \text{for all } \rho \text{ sufficiently close to 1.}$$

The second statement follows by the weighted continuity already used in the link expansion.

At a general fixed radius ρ , set $F_\rho(p) = f(\rho p)$, for $p \in L$. Rescaling Proposition 4.4 gives

$$\mathcal{H}^7(M_i \cap \partial B_\rho) = \rho^7 \mathcal{H}^7(L) + \frac{\varepsilon_i^2 \rho^5}{2} \mathcal{Q}_L(F_\rho) + o(\varepsilon_i^2).$$

Lemma 4.5 makes this coefficient negative at every sufficiently small fixed radius under (27). Choose a common admissible one and apply (28). $\square$

4.3. Density bounds and classification of cone limits. The next lemma shows that a symmetric SME cone with a zero on the axis away from its vertex has density at least θ_V , with equality only for V .

Lemma 4.7 (An axial zero of a cone forces cylindrical density). *Let $K = S(u)$ be an SME cone, with u $O(4)_x$ -invariant. If $u(0, 1) = 0$, that is, if $(0, 0, 1) \in \text{Sing } K$, then $\Theta_K(0) \geq \theta_V$. If also $\Theta_K(0) \leq \theta_V$, then $K = V$.*

Proof. Put $q = (0, 0, 1)$. A tangent cone T to a cone at a nonvertex point splits its radial direction; thus $T = T_0 \times \mathbb{R}_y$. This follows directly by conjugating the dilations fixing the vertex by the translation and dilation used to blow up at q ; in the limit they give every translation in the direction q . The SME compactness and graph property are preserved in this limit. The transverse profile is $O(4)_x$ -invariant and homogeneous of degree one, hence equals $c|x|$ for a constant $c \geq 0$. The noncollapse consequence of [18, Lemma 2.3 and Remark 2.4] excludes $c = 0$: every base ball of radius ρ contains a point where the height is at least $c_* \rho$. Substituting the radial profile $c|x|$ in the four-dimensional-base equation gives

$$\frac{d^2(ct)/dt^2}{1+c^2} + \frac{3}{t} \frac{d(ct)}{dt} - \frac{3}{ct} = \frac{3}{t}(c - c^{-1}) = 0 \implies c = 1, \quad T = V.$$

Thus monotonicity centered at q gives

$$\theta_V = \Theta_K(q) \leq \Theta_K(q, R) \longrightarrow \Theta_K(0) \quad (R \rightarrow \infty).$$

If $\Theta_K(0) = \theta_V$, then for $0 < s < R$ the exact monotonicity identity yields that K is a cone centered at q as well and it equals its tangent cone $T = V$ there. $\square$

We next classify symmetric, axially monotone SME cones with density at most θ_V .

Lemma 4.8 (Cone classification with axial monotonicity). *Let $K = S(u)$ be an SME cone with $\Theta(K) \leq \theta_V$, where $u(x, y) = u(|x|, y)$ is $O(4)_x$ -invariant. Suppose u is even in y and nondecreasing in $y \geq 0$. Then K is either $C_{4,3}$ or V .*

Proof. Lemma 4.1 excludes off-axis zeros. By homogeneity, either $u(0, 1) = 0$, in which case Lemma 4.7 gives $K = V$, or $u(0, 1) > 0$. In the latter case $u > 0$ on $\mathbb{R}^5 \setminus \{0\}$, so the cone has an isolated singularity and a smooth link.

On the connected region $y > 0$, the function u_y satisfies the linear equation obtained by differentiating the SME. It is nonnegative by hypothesis and is positive on the positive y axis,

where homogeneity gives $u(0, y) = yu(0, 1)$. The local Harnack inequality for this nonnegative solution gives

$$u_y(0, y) = u(0, 1) > 0 \quad (y > 0) \implies u_y > 0 \text{ on } \{y > 0\}.$$

For each $a = |x| > 0$, homogeneity gives

$$u(a, 0) = c_0 a, \quad \frac{u(a, y)}{y} = u(a/y, 1) \longrightarrow u(0, 1) > 0.$$

Thus $y \mapsto u(a, y)$ maps $(0, \infty)$ bijectively onto $(c_0 a, \infty)$. Its inverse $G(a, b)$ satisfies

$$\begin{aligned} K \cap \{y > 0\} &= \{y = G(|x|, |z|) : |z| > c_0 |x|\}, \\ K \cap \{y < 0\} &= \{y = -G(|x|, |z|) : |z| > c_0 |x|\}, \\ G(a, c_0 a) &= 0, \quad u_y(a, 0) = 0. \end{aligned}$$

Smoothness along $y = 0$ therefore makes the two graphs meet that hyperplane orthogonally.

These verify the precise hypotheses of [16, Corollary 3.18]: a cohomogeneity-two minimal cone, equivariant under $O(n - k) \times O(k) \times \mathbb{Z}_2$, expressible as a bigraph. Set $n = 8$, $k = 4$, and identify their two four-dimensional factors with z and x . Their conclusion is the Lawson cone

$$C(S^3 \times S^4) = \left\{ |z| = \frac{\sqrt{3}}{2} \sqrt{|x|^2 + y^2} \right\} = C_{4,3}.$$

The relevant ODE endpoint is also explicit in [16, Proposition 3.17]: the zero-height, infinite-slope endpoint occurs only at $a_{8,4} = \sqrt{4/3}$. $\square$

We can finally prove Proposition 2.2.

Proof of Proposition 2.2. Let M_ε be the family in Proposition 2.1, with $0 < \varepsilon < \varepsilon_0$. Its normalized limit is $f = r^{-2}w$, with $w = W - m_0$, where

$$w = c(y^2 - r^2/3) + O(\rho^4), \quad c > 0.$$

Lemma 4.5 gives, on a sufficiently small fixed sphere of radius $s_* > 0$,

$$\mathcal{Q}_L(f(s_* \cdot)) < 0.$$

The phase, Lipschitz and normal derivative estimates from the first proposition verify the hypotheses of Propositions 4.4 and 4.6 after this fixed dilation. Consequently, arguing by contradiction along sequences $\varepsilon_i \rightarrow 0$, we obtain $\varepsilon_1 \in (0, \varepsilon_0]$ such that, for $0 < \varepsilon < \varepsilon_1$,

$$\Theta(M_\varepsilon, 0) < \theta_V.$$

Decreasing ε_1 if necessary, we may also assume that Corollary 4.2 applies for $0 < \varepsilon < \varepsilon_1$.

For such a fixed ε , suppose that the half-length ℓ_ε of its zero interval were positive. For any sequence $s_j \downarrow 0$, the rescaled profiles $s_j^{-1}u_\varepsilon(s_j x, s_j y)$ vanish at $(x, y) = (0, 1)$ once $s_j < \ell_\varepsilon$. The local Lipschitz bound and SME compactness therefore give a homogeneous tangent profile with this same axial zero. If K is its symmetric cone, Lemma 4.7 gives

$$\theta_V \leq \Theta(K) = \Theta(M_\varepsilon, 0) < \theta_V,$$

a contradiction. Thus $\ell_\varepsilon = 0$. Note that this argument uses no classification of the isolated cone: the density obstruction alone rules out the interval.

Every tangent cone at zero inherits the symmetries and axial order and has density less than θ_V . The remaining classification, Lemma 4.8, gives

$$\text{Tan}(M_\varepsilon, 0) = \{C_{4,3}\}, \quad \Theta(M_\varepsilon, 0) = \theta_{43}.$$

Positive profile points are smooth by SME regularity, including the smooth radial axis where the profile is positive. The only zero is the origin, so it is the only singular point of the graph over Ω . This proves all assertions of the proposition. $\square$

5. QUANTITATIVE ESTIMATES FOR THE CONNECTOR

The estimates needed for the gluing construction of the next sections concern the error between the connector and the Hardt–Simon leaf with the same smoothing radius $a(y) := u(0, y)$. Write

$$u = u_\infty, \quad a(y) = u(0, y), \quad v(t, y) = a(y)h(t/a(y)), \quad e = u - v,$$

where the Hardt–Simon’s foliation h and the minimal-surface operator Q are as in the previous sections. The required bounds, on an axial band of size $R \approx |y|$, are

$$|\partial_t^m \partial_y^n e| \leq C_{mn} a^3 R^{-2-n} (a+t)^{-m}.$$

Subtracting the equations for u and v gives a linear equation for e with right-hand side $-Q(v)$. Its forcing is determined by the derivatives of a . Indeed, with $s = t/a$ and $\zeta(s) = h(s) - sh'(s)$, a straightforward computation yields

$$v_y = a' \zeta(s), \quad v_{yy} = a'' \zeta(s) + \frac{(a')^2}{a} s^2 h''(s),$$

which combined with the leaf equation for h gives

$$(29) \quad |Q(v)| \leq C \left(|a''| + \frac{(a')^2}{a} \right) (1 + t/a)^{-2}.$$

Thus the desired forcing scale is available once the bound $|a^{(j)}| \leq C_j a R^{-j}$ is known. Differentiating the forcing requires the corresponding higher derivatives of a . At the axis, the exact identity

$$4e_{tt}(0, y) + \frac{a''(y)}{1 + (a'(y))^2} = 0$$

already shows that the remainder estimate and the smoothing-radius derivative estimate cannot be separated by treating a'' as known data.

We obtain those derivatives by differentiating the equation for u itself. The function u_y solves the Jacobi equation, whose inverse-square potential prevents ordinary estimates from being uniform between the smoothing scale and the axial scale. Dividing by the positive dilation Jacobi field of the local leaf absorbs the fixed-leaf potential. The principal coefficients of the resulting equation are comparable to

$$\omega_\beta(x, y) = (\beta^2 + |x|^2)^{-1/2}, \quad \beta = u(0, y_0),$$

where β is held fixed on a band about y_0 . This is a Muckenhoupt A_2 weight with constant independent of β . Weighted local estimates then control the normalized axial derivatives at scale R .

To apply that theory, we must first identify the smoothing scale and compare the actual Jacobi coefficients with the displayed weight uniformly throughout the band. The quadratic growth estimate, together with the positive quadratic coefficient and comparison near the axis, gives $\beta^3 \approx |y_0|^2$, a positive range of comparable leaves, and a quantitative approximation by those leaves. The approximation also controls the residual Jacobi potential. The A_2 property itself holds for every β ; the growth and comparison estimates are what place this particular surface in a uniform weighted coefficient class. We therefore prove the quadratic estimate first, then estimate the axial derivatives of u , and finally return to the forced radial equation for e .

Throughout the section we consider the invariant hypersurfaces

$$\Sigma_v = \{|z| = v(t, y)\}.$$

Recall that the Euclidean minimal surface equation for a positive v is

$$(30) \quad Q(v) := \left(\delta_{ij} - \frac{v_i v_j}{1 + |Dv|^2} \right) v_{ij} + \frac{3}{t} v_t - \frac{3}{v} = 0, \quad i, j \in \{t, y\}.$$

For the normal pointing towards increasing $|z|$, the scalar mean curvature, with convention $H = \operatorname{div}_{\Sigma_v} \nu$, is

$$H = -Q(v)(1 + |Dv|^2)^{-1/2}.$$

Let $|z| = h(|x|)$ be the invariant Hardt-Simon leaf on the $|z| > |x|$ side, normalized by $h(0) = 1$. The external fixed-leaf input is the global parametrization in [34, §3.1, proof of Proposition 3.3] and its asymptotic expansion [34, Proposition 3.3]. In our vertical graph coordinates these give

$$(31) \quad h'(0) = 0, \quad \frac{h''}{1+h'^2} + \frac{3}{t}h' - \frac{3}{h} = 0, \quad h(t) = t + b_H t^{-2} + O(t^{-3}), \quad b_H > 0.$$

together with the corresponding expansions of the derivatives of h . In the notation of Section 2, $\{|z| = h(|x|)\} = H_{c_H}$, and $0 < h' < 1$ on $(0, \infty)$; see the proof of Lemma B.1. By rotational invariance, write $u_\infty(t, y) = U(x, y)$ when $|x| = t$; within this section we also abbreviate this profile by u . Our goal is the following:

Proposition 5.1 (The sharp connector end). *Let $M = S(u_\infty)$ be the connector of Theorem 1.2, and put*

$$a_\infty(Y) = u_\infty(0, Y) \quad (Y \neq 0).$$

There are constants $\kappa_H, c > 0$ and $Y_ > 0$ such that*

$$a_\infty(Y) \sim \kappa_H |Y|^{2/3} \quad (|Y| \rightarrow \infty), \quad |a_\infty^{(j)}(Y)| \leq C_j a_\infty(Y) |Y|^{-j} \quad (j \geq 1, |Y| \geq Y_*).$$

For $|Y| \geq Y_$ and $0 \leq T \leq c|Y|$,*

$$(32) \quad \begin{aligned} u_\infty(T, Y) &= a_\infty(Y) h(T/a_\infty(Y)) + e_\infty(T, Y), \\ |\partial_T^m \partial_Y^n e_\infty(T, Y)| &\leq C_{mn} a_\infty(Y)^3 |Y|^{-2-n} (a_\infty(Y) + T)^{-m}, \quad m, n \geq 0. \end{aligned}$$

On every fixed regular angular sector $\{r \geq \theta\rho\}$ of V , where $r = \sqrt{2}t$ and $\rho = (r^2 + Y^2)^{1/2}$, the normal graph height f_∞ of M satisfies

$$|D^j f_\infty| \leq C_{\theta,j} \rho^{-j} \quad (j \geq 0).$$

Every derivative order is fixed but arbitrary. The constants $C_j, C_{mn}, C_{\theta,j}$ may depend on those orders, while c and Y_ can be chosen independently of them.*

The first subsection proves the phase bound and the positivity of its leading coefficient. The second assembles the comparison in the smoothing region, direct derivative estimates, and radial remainder bound from Appendix A to prove the proposition.

5.1. The quadratic phase bound. The cylindrical blow-down gives only subcubic phase growth. We first improve this to growth of order $R^{2+\eta}$ for every $\eta > 0$, using compactness and the harmonic description of Jacobi fields. Harmonic replacements on annuli then isolate the quadratic mode, while nonconcentration controls the part of the surface near the spine. We must also prove that its coefficient is positive: if it vanished, iterating the improved growth estimate would give bounded phase. Transferring that bound and its asymptotic value through the smoothing regions would force an interior contact with a foliation leaf, contradicting the connector's change of side near zero. The positive quadratic mode is what determines the smoothing scale used in the derivative estimates. Define:

$$(33) \quad D(R) = \sup_{M \cap B_R} |P|.$$

Here P is the Hardt-Simon phase of Section 2; it is independent of y and homogeneous of degree three:

$$(34) \quad P(sx, sz, y) = s^3 P(x, z, y) \quad s > 0.$$

Note that $D(R) > 0$ for every $R > 0$, since M meets both sides of V near the origin. We aim to prove:

Proposition 5.2 (Quadratic bound). *For the connector $S(u) = M$ of Theorem 1.2, we have:*

$$(35) \quad D(R) \leq CR^2,$$

for some $C > 0$ and all $R \geq 1$. Moreover there is $A > 0$ such that, on every fixed regular sector $r \geq \theta\rho$, the normal height f of M over V satisfies

$$|D^k(f - AJ_2)| \leq C_{k,\theta}\rho^{-7/33-k} \quad (k \geq 0)$$

for sufficiently large ρ .

We first establish the following preliminary bound:

Lemma 5.3 (Super quadratic bound). *For any $\eta > 0$, there is $C_\eta > 0$ such that:*

$$(36) \quad D(R) \leq C_\eta R^{2+\eta} \quad \text{for } R \geq 1.$$

Proof. For the proof we use the nonconcentration estimates of [34, Proposition 5.6] and a linearization argument, which produces Jacobi fields in the limit.

Step 1: contradiction sequence. In the notation of [34, Definition 5.2(a)], for a fixed value of its parameter $\beta > 0$, the distance $D_{\beta,V_0}(M; B_R)$ of M from V in B_R is comparable to $D(R)$, the maximum of the Hardt–Simon phase. Since $R^{-1}M \rightarrow V$ as $R \rightarrow \infty$, we have $R^{-3}D(R) = \sup_{R^{-1}M \cap B_1} |P| \rightarrow 0$. It suffices to prove the lemma for $0 < \eta < 1$, and since D is nondecreasing it suffices to consider the radii $R_k = 16^k$, $k \geq 0$. Put $E_k = R_k^{-(2+\eta)}D(R_k)$, and suppose that $\sup_k E_k = \infty$. We claim that there are indices $k_1 < k_2 < \dots$ such that

$$(37) \quad \begin{cases} E_{k_j} = R_{k_j}^{-(2+\eta)}D(R_{k_j}) \rightarrow \infty, \\ D(16R_{k_j}) \leq 16^{3+\delta}D(R_{k_j}), & \text{for all } j \in \mathbb{N} \\ E_{k_j} \geq E_k, & \text{for all } k \leq k_j. \end{cases}$$

To see this, call k a *record index* if $E_k \geq E_{k'}$ for all $k' \leq k$. Since (E_k) is unbounded, there are infinitely many record indices, and $E_k \rightarrow \infty$ along them. If k is a record index at which the second condition fails, then

$$E_{k+1} = 16^{-(2+\eta)}R_k^{-(2+\eta)}D(16R_k) > 16^1 E_k,$$

so that $k+1$ is again a record index. The second condition cannot fail at all the indices $k, k+1, \dots, k+\ell$ for every ℓ : otherwise $D(16^\ell R_k) > 16^{(3+\eta)\ell}D(R_k)$ for every ℓ , which, since $D(R_k) > 0$, contradicts $R^{-3}D(R) \rightarrow 0$. Replacing each record index by the first subsequent index at which the second condition holds, which is again a record index with a larger value of E , and passing to a subsequence, we obtain the claim.

The first and third conditions now give

$$(38) \quad D(sR_{k_j}) \leq 16^{2+\eta}s^{2+\eta}D(R_{k_j}) \quad \text{for } 0 < s \leq 1 \text{ with } sR_{k_j} \geq 1.$$

Indeed, if $k \leq k_j$ is the least index with $sR_{k_j} \leq R_k$, then $R_k \leq 16sR_{k_j}$ and $D(sR_{k_j}) \leq D(R_k) = E_k R_k^{2+\eta} \leq E_{k_j} R_k^{2+\eta} = (R_k/R_{k_j})^{2+\eta}D(R_{k_j})$. Relabel the selected radii as R_j .

Step 2: Jacobi field. Now let f_j be the normal height of $R_j^{-1}M$, and set

$$\varepsilon_j = \frac{D(R_j)}{R_j^3}, \quad v_j = \varepsilon_j^{-1}f_j.$$

By the second condition in (37), $|P| \leq 16^{3+\eta}\varepsilon_j$ on $R_j^{-1}M \cap B_{16}$. The outer bound and interior graph estimates yield

$$v_j \longrightarrow v \quad \text{in } C_{\text{loc}}^\infty(\text{Reg } C_{3,3} \times \mathbb{R} \cap B_8), \quad L_{C_{3,3} \times \mathbb{R}} v = 0,$$

where

$$\begin{aligned} L_{C_{3,3} \times \mathbb{R}} &= \partial_r^2 + \frac{6}{r}\partial_r + \partial_y^2 + \frac{6}{r^2}, \\ L_{C_{3,3} \times \mathbb{R}}(r^{-2}w) &= r^{-2} \left(w_{rr} + \frac{2}{r}w_r + w_{yy} \right) = r^{-2} \Delta_{\mathbb{R}_\xi^3 \times \mathbb{R}_y} w. \end{aligned}$$

Thus $w = r^2v$ is harmonic off the auxiliary axis and bounded near it.

Nonconcentration prevents the normalized limit w from vanishing. Indeed, the nonconcentration estimate [34, Proposition 5.6], with its parameter δ equal to zero, together with the maximum principle for $|P|$, gives for every $\gamma > 0$

$$\Delta w = 0 \text{ in } B_8, \quad |w(X)| \leq C|X|^{2+\eta} \text{ near } 0.$$

We aim to show that non-concentration prevents the normalized limit from vanishing. Notice the the non-concentration estimates of [34, Proposition 5.6] with the choice $\delta = 0$ and the maximum principle for $|P|$ asserts that for every $\gamma > 0$ we have:

$$(39) \quad D(R) \leq C_\gamma \sup_{\substack{M \text{ represented over } C_{3,3} \times \mathbb{R} \\ R/2 < \rho < 2R, \ r > \theta_\gamma R}} |r^2 f| + C_0 \gamma D(4R),$$

where f is the normal graph of M over $C_{3,3} \times \mathbb{R} = V$. Here R is sufficiently large, $\theta_\gamma > 0$ and the threshold may depend on γ , and C_0 is independent of γ . Choose $\gamma > 0$ so that

$$C_0 \gamma 16^{3+\eta} < \frac{1}{2}.$$

Apply (39) at R_j and divide by $D(R_j)$. Since $D(4R_j) \leq D(16R_j)$, this with the second condition in (37) gives

$$1 \leq C_\gamma \sup_{\substack{1/2 < \rho < 2 \\ r > \theta_\gamma}} |r^2 v_j| + \frac{1}{2}, \quad \sup |r^2 v| \geq (2C_\gamma)^{-1} > 0.$$

Finally, $w_y \geq 0$ for $y > 0$. To check this in normal coordinates, write $t = (r - f)/\sqrt{2}$ and $u(t, y) = (r + f)/\sqrt{2}$. Then

$$f_y = \frac{\sqrt{2}u_y}{1 + u_t} \geq 0$$

where the normal graph is small; the inequality passes to the limit.

Step 3: final contradiction. Evenness and the strong maximum principle now give the alternatives

$$\begin{aligned} w_y \not\equiv 0 &\implies w_y > 0 \text{ in } B_8 \cap \{y > 0\}, \ w_y = 0 \text{ on } \{y = 0\} \\ &\implies w_{yy}(0) > 0 \quad (\text{Hopf lemma}), \\ w_y \equiv 0 &\implies w = c_1 + c_2/r \implies w \equiv 0 \quad (\text{boundedness and } w(0) = 0). \end{aligned}$$

The first contradicts $w = O(\rho^{2+\eta})$; the second contradicts nontriviality and (36) follows. $\square$

We next improve this bound to a quadratic one by linearization. The main difficulty is that one cannot linearize near the spine; the missing ingredient is the nonconcentration estimate [34, Proposition 5.6], which allows the linearization to go through. We also need the following transfer lemma, which relies on the barriers recalled in the proof of Lemma 3.5.

Lemma 5.4 (Uniform phase convergence up to the spine). *Let M be the connector and P its signed Hardt–Simon phase. Assume that $|P| \leq K$ on $M \setminus B_{R_0}$ for some $K, R_0 > 0$. Suppose there is a constant $b \in \mathbb{R}$ such that, for every fixed $0 < \theta < 1$,*

$$\lim_{R \rightarrow \infty} \sup_{M \cap \{\rho \geq R, \ r \geq \theta \rho\}} |P - b| = 0.$$

Then

$$\lim_{R \rightarrow \infty} \sup_{M \cap \{\rho \geq R\}} |P - b| = 0.$$

Thus the same convergence holds uniformly without any restriction on the distance to the spine.

Proof. Fix $0 < \delta < 1$, $I = (2/3, 4/3)$, and $I' = [3/4, 5/4]$. Choose $\chi \in C^\infty(I; [0, 1])$ equal to zero on I' and one near both ends of I . With real ninth roots of the constants, put

$$q_+ = (b + \delta)^{1/9} + B\chi, \quad q_- = (b - \delta)^{1/9} - B\chi.$$

Choose B so large that $q_+^9 > K + 2$ and $q_-^9 < -K - 2$ near the ends. Choose s_0 so large that $(q_+ + s_0)^9 > K + 2$ and $(q_- - s_0)^9 < -K - 2$ throughout I . These choices give uniformly C^2 -bounded signed families for $0 \leq s \leq s_0$; they do not depend on the dilation below.

Set $M_R = R^{-1}M$ and $\varepsilon = R^{-3}$. Cubic homogeneity gives $|P| \leq K\varepsilon$ on M_R in this slab for large R . Let $X_\varepsilon(q)$ denote the negative-side comparison barrier in the proof of Lemma 3.5, and let $\mathcal{S}(x, z, y) = (z, x, y)$. Use the upper and lower barriers

$$X_\varepsilon(q_+ + s), \quad \mathcal{S}X_\varepsilon(-q_- + s), \quad 0 \leq s \leq s_0.$$

The sign in the second expression is necessary: after exchanging the factors, its phase is within $\varepsilon\delta/4$ of $\varepsilon(q_- - s)^9$, and it excludes contact from its positive side. By (16), first choose a common fixed lateral radius $r_0 > 0$ giving phase error at most $\varepsilon\delta/4$ for both families. Only then take R large enough that the barriers exist and

$$|P/\varepsilon - b| < \delta/4 \quad \text{on } M_R \cap \{r = r_0, y \in I\}.$$

The latter follows from the assumed regular-sector convergence, since r_0 is fixed and the rescaled axial coordinate stays away from zero.

At $s = s_0$, phase trapping puts the whole surface between the barriers. The phase bound preserves strict ordering at the axial ends for every s . At the lateral boundary, the upper barrier has phase at least $\varepsilon(b + 3\delta/4)$ and the lower at most $\varepsilon(b - 3\delta/4)$, whereas the surface has phase within $\varepsilon\delta/4$ of $b\varepsilon$. Slide s down to zero. Boundary contact is excluded by these inequalities; interior contact, including at a barrier spine point, is excluded by [34, Proposition 5.8]. At $s = 0$ and $y \in I'$, trapping between the two barriers gives

$$|P/\varepsilon - b| \leq 5\delta/4 \quad \text{on } M_R \cap \{r < r_0, y \in I'\}.$$

For any physical point with large $y > 0$ and $r < r_0 y$, apply this estimate with $R = y$; its rescaled axial coordinate is one. Reflection handles $y < 0$. The complement lies in a fixed regular angular sector, where the original convergence applies. Letting $\delta \downarrow 0$ proves uniform convergence on the end. This argument uses neither a positive quadratic coefficient nor a positive lower bound for the phase. $\square$

We can now prove the quadratic decay. The combination of growth bounds and interior elliptic estimates used in the proof is also familiar from optimal regularity, see for instance [15, §4, Lemma 4.2 and Theorem 4.1].

Proof of Proposition 5.2. The idea is to replace the regular normal graph by harmonic functions on full auxiliary annuli and control the error caused by the removed tube. Their expansion identifies the quadratic term. If its coefficient vanished, iteration of the replacement and nonconcentration estimates would give bounded phase. Lemma 5.4 would then give constant phase at infinity, contradicting the two-sided geometry of the connector by the strong maximum principle.

Step 1: harmonic replacement in annuli. Let f be the normal height of M over V on the regular graph region in $\{\rho < 8R\}$, and define

$$(40) \quad d = D(8R), \quad \varepsilon_R = d/R^3, \quad f_R(X) = R^{-1}f(RX),$$

Then f_R is the normal height of $R^{-1}M$ over V , and $|P| \leq \varepsilon_R$ on $R^{-1}M \cap B_8$. The proof of Proposition 3.4, which uses only this phase bound and the Lipschitz bound, shows that in B_5 the function f_R is defined on $\{r > \sigma\}$, $\sigma = C_*\varepsilon_R^{1/3}$, for a fixed large $C_* > 0$, and satisfies $|D^k f_R| \leq C_k \varepsilon_R r^{-2-k}$ there. We set

$$(41) \quad w_R = r^2 f_R / \varepsilon_R.$$

These estimates, together with the quadratic structure of the nonlinear terms in the minimal surface equation for normal graphs over V , give

$$(42) \quad \begin{cases} w_R : B_5 \setminus \{r \leq C_* \varepsilon_R^{1/3}\} \rightarrow \mathbb{R}, \\ w_R(x, y) = w_R(r, y) \\ |w_R| \leq C, \\ |\Delta_{\xi, y} w_R| \leq C \varepsilon_R r^{-5}. \end{cases}$$

Regard w_R as an $O(3)$ -invariant function in the auxiliary space $\mathbb{R}_\xi^3 \times \mathbb{R}_y$. Put $\mathcal{A} = B_4 \setminus \overline{B}_{1/4}$. Prescribe $g_R = w_R$ on $\partial\mathcal{A} \cap \{r > \sigma\}$ and extend it to a bounded continuous function over the remaining spherical polar neighborhoods, preserving the symmetries. Let H_R be its harmonic extension to the full annulus $\mathcal{A}$; then $|H_R| \leq C$.

Step 2: error estimates. We now construct harmonic replacements on dyadic annuli and compare them on their overlaps. This uses the approximation principle underlying harmonic-replacement proofs of Schauder estimates; see, for example, [36, §1]. Here we must also control the error across the removed tube before patching the replacements together.

On the region $r > \sigma$, define

$$\mathcal{B}(r) = C \frac{\sigma}{r} + C \varepsilon_R \left(\frac{\sigma^{-2}}{r} - \frac{1}{r^3} \right).$$

Fixing $\sigma = C_* \varepsilon_R^{1/3}$, the boundary values and Laplacian satisfy

$$\begin{aligned} \mathcal{B} &\geq 0, & \mathcal{B}(\sigma) &= C \geq |w_R - H_R|_{\{r=\sigma\}}, \\ (w_R - H_R)|_{\{\rho=1/4, 4\} \cap \{r>\sigma\}} &= 0, \\ \Delta(r^{-1}) &= 0, & \Delta(r^{-3}) &= 6r^{-5}, \\ -\Delta \mathcal{B} &= 6C \varepsilon_R r^{-5} \geq |\Delta(w_R - H_R)| = |\Delta(H_R - w_R)|. \end{aligned}$$

The maximum principle in $\mathcal{A} \cap \{r > \sigma\}$ therefore gives

$$(43) \quad |w_R - H_R| \leq \mathcal{B} \leq C \frac{\sigma + \varepsilon_R \sigma^{-2}}{r} \leq C \frac{\varepsilon_R^{1/3}}{r}.$$

Interior estimates for the Poisson equation give the corresponding scaled derivative bounds away from $\{r = \sigma\}$. We now perform this construction at dyadic scales and patch the results together. Let $R_j = 2^j R_0$, with $R_0 > 1$ large, let $H^{(j)}$ be the harmonic replacement constructed above at scale $R = R_j$, and put $H_j(X) = d_j H^{(j)}(X/R_j)$, where $d_j = D(8R_j)$; thus H_j approximates $r^2 f$ at scale R_j . Let $\{\chi_j\}_{j \geq 1}$ be radial functions with $\text{supp } \chi_j \subset B_{2R_j} \setminus B_{R_j/2}$, $|D^k \chi_j| \leq C_k R_j^{-k}$ and $\sum_j \chi_j = 1$ on $\{\rho \geq R_1\}$, and define

$$(44) \quad W = \sum_{j=1}^{\infty} \chi_j H_j.$$

For the rescaled functions, on a slightly smaller auxiliary annulus, the replacement bound gives

$$\begin{aligned} \|w_{R_j} - H^{(j)}\|_{L^2(\mathcal{A} \cap \{r > \sigma_{R_j}\})}^2 &\leq C \varepsilon_{R_j}^{2/3} \int_{\sigma_{R_j}}^4 r^{-2} r^2 dr \leq C \varepsilon_{R_j}^{2/3}, \\ |\mathcal{A} \cap \{r < \sigma_{R_j}\}|^{1/2} &\leq C \sigma_{R_j}^{3/2} = C \varepsilon_{R_j}^{1/2} \leq C \varepsilon_{R_j}^{1/3}. \end{aligned}$$

The L^2 norms here are taken on fixed rescaled annuli. Taking the square root in the first estimate gives an error of order $\varepsilon_{R_j}^{1/3}$ outside the tube, whereas the omitted tube contributes only $\varepsilon_{R_j}^{1/2}$. Since $\varepsilon_{R_j} \rightarrow 0$, the first contribution dominates. Restoring the physical amplitude d_j , these two contributions are

$$d_j \varepsilon_{R_j}^{1/3} = \frac{d_j^{4/3}}{R_j}, \quad d_j \varepsilon_{R_j}^{1/2} = \left(\frac{d_j}{R_j} \right)^{3/2}.$$

Thus the overlap estimate must retain the larger first term.

For adjacent indices j and $j+1$, the function $H_j - H_{j+1}$ is harmonic on the overlap of the corresponding annuli. Combining the first bound in the preceding display, the volume bound together with $|H_j| + |H_{j+1}| \leq C(d_j + d_{j+1})$, and interior estimates for harmonic functions on $\mathbb{R}_\xi^3 \times \mathbb{R}_y$, we obtain

$$(45) \quad \|H_j - H_{j+1}\|_{C_{\text{scaled}}^k(B_{R_{j+1}} \setminus B_{R_j})} \leq C_k \frac{d_j^{4/3}}{R_j} + C_k \frac{d_{j+1}^{4/3}}{R_{j+1}},$$

where $\|g\|_{C_{\text{scaled}}^k} := \sum_{i=0}^k R^i \|D^i g\|_\infty$. By Lemma 5.3, $d_j \leq C_\eta R_j^p$ with $p = 2 + \eta$, and hence

$$(46) \quad \|H_j - H_{j+1}\|_{C_{\text{scaled}}^k(B_{R_{j+1}} \setminus B_{R_j})} \leq C_{k,\eta} R_j^{\frac{4}{3}p-1}.$$

Accordingly, the exponent used below is $p' = \frac{4}{3}p - 1$. Since $\sum_j D\chi_j = 0$ and $\sum_j \Delta\chi_j = 0$ on $\{\rho \geq R_1\}$, on each overlap, subtracting a fixed neighbor H_* gives

$$\Delta_{\xi,y} W = \sum_j ((\Delta\chi_j)(H_j - H_*) + 2D\chi_j \cdot D(H_j - H_*)).$$

Thus

$$(47) \quad |W| \leq C\rho^p, \quad |D^k \Delta_{\xi,y} W| \leq C_k \rho^{\frac{4}{3}p-3-k}.$$

On each fixed regular angular sector $\{r \geq \theta\rho, \rho \approx R\}$, the replacement estimate and interior elliptic estimates give:

$$(48) \quad \|W - r^2 f\|_{C_{\text{scaled}}^k} \leq C_{k,\theta,p} R^{\frac{4}{3}p-1}.$$

Step 3: leading order of the harmonic part of W . We next subtract a particular solution of the Poisson equation and expand the remaining harmonic function at infinity. For the general method of modified Newton potentials and asymptotic harmonic expansions used here, see [3, Theorem 1.7 and Theorem 1.17 with Remark 1]. We give the estimates needed here explicitly. For this we just integrate the Newtonian potential as follows. We will use (47) with $-2 < p' < 2$ and $p' \neq 0$. Extend its source smoothly across a fixed compact set, preserving $O(3)$ invariance and reflection in y . If $p' > 0$, solve it with the modified Newton potential

$$v(X) = \int_{\mathbb{R}^4} [\Gamma(X - Y) - \chi(Y)\{\Gamma(-Y) + D\Gamma(-Y) \cdot X\}](\Delta W)(Y) dY,$$

where $\Gamma(X) = -(4\pi^2|X|^2)^{-1}$ is the fundamental solution of the Laplacian on $\mathbb{R}^4$, and χ is radial, zero near zero and one near infinity. The kernel at $|Y| > 2|X|$ satisfies

$$|\Gamma(X - Y) - \Gamma(-Y) - D\Gamma(-Y) \cdot X| \leq C|X|^2|Y|^{-4},$$

$$\int_{2|X|}^\infty |X|^2 s^{p'-3} ds \leq C|X|^{p'} \quad (p' < 2).$$

On every centered truncated ball the linear subtraction integrates to zero by symmetry. Splitting the remaining integral into $|Y| < |X|/2$ and $|X|/2 < |Y| < 2|X|$ gives

$$|X|^{-2} \int_1^{|X|/2} s^{p'+1} ds + \int_1^{|X|/2} s^{p'-1} ds \leq C|X|^{p'} \quad (p' > 0),$$

$$\int_{\{|X|/2 < |Y| < 2|X|\}} (|\Gamma(X - Y)| + |\Gamma(-Y)|) |Y|^{p'-2} dY \leq C|X|^{p'}.$$

The combined integral is therefore absolutely convergent and $v = O(\rho^{p'})$. For $-2 < p' < 0$, use the unmodified Newton kernel:

$$\rho^{-2} \int_1^\rho s^{p'+1} ds + \int_\rho^\infty s^{p'-1} ds \leq C\rho^{p'}.$$

Interior Poisson estimates yield $|D^k v| \leq C_k \rho^{p'-k}$. Hence

$$\Delta(W - v) = 0$$

on the full exterior auxiliary domain. The invariant even expansion has the form

$$(49) \quad W - v = \sum_{\ell \in 2\mathbb{N}_0} \left(a_\ell \rho^\ell + b_\ell \rho^{-\ell-2} \right) Y_\ell(r/\rho, y/\rho),$$

where Y_ℓ is the $O(3)$ invariant spherical harmonic of degree ℓ on S^3 .

We first take $\eta = 1/11$, so that

$$(50) \quad p = \frac{23}{11}, \quad p' = \frac{59}{33} = 2 - \kappa, \quad \kappa = \frac{7}{33},$$

and using (49), (48) and the growth, we see that:

$$\begin{aligned} W - v &= O(\rho^{2+\eta}), \quad 2 + \eta < 3, \\ \Rightarrow W - v &= A \left(y^2 - \frac{r^2}{3} \right) + a_0 + O(\rho^{-2}), \\ f &= AJ_2 + O(\rho^{-\kappa}) \quad \text{on } \{r > \theta\rho\}. \end{aligned}$$

Axial monotonicity passes to the normal coordinates by the exact identity $f_y = \sqrt{2}u_y/(1+u_t) \geq 0$. The differentiated expansion on fixed regular sectors then gives $A \geq 0$.

Step 4: non degeneracy of the quadratic harmonic part. Suppose now that $A = 0$. Then $|r^2 f| \leq C(1 + R^{p'})$ on the regular sectors at scale R , and the nonconcentration estimate (39) gives

$$(51) \quad D(R) \leq C_\gamma(1 + R^{p'}) + C_0\gamma D(4R).$$

Now name $\alpha_0 = C_0\gamma$ and choose γ small such that $\alpha_0 4^p < 1$ and iterate the above to see that for every N ,

$$\begin{aligned} D(R) &\leq C_\gamma \sum_{j=0}^{N-1} \alpha_0^j (1 + (4^j R)^{p'}) + \alpha_0^N D(4^N R), \\ \alpha_0^N D(4^N R) &\leq CR^p (\alpha_0 4^p)^N \rightarrow 0. \end{aligned}$$

Since $\alpha_0 4^{p'} < \alpha_0 4^p < 1$, letting $N \rightarrow \infty$ improves the bound to

$$(52) \quad D(R) \leq C(1 + R^{\max(p', 0)}) = C(1 + R^{\max(\frac{4}{3}p-1, 0)}).$$

We now repeat the argument with p replaced by p' . Since $p' < 2$, the quadratic term is then absent from the expansion of $W - v$, and we obtain the same conclusion with the exponent $\frac{4}{3}p' - 1$. Iterating the exponent map $p \mapsto \frac{4}{3}p - 1$ gives

$$\frac{23}{11} \mapsto \frac{59}{33} \mapsto \frac{137}{99} \mapsto \frac{251}{297} \mapsto \frac{113}{891} \mapsto -\frac{2221}{2673}.$$

None of these exponents is zero, and the last is negative, so the preceding bound gives $D(R) \leq C$.

Repeat the replacement with this bounded D and retain the unweakened errors. In particular, $d_j \varepsilon_{R_j}^{1/3} = d_j^{4/3}/R_j = O(R_j^{-1})$. The overlap and regular-sector estimates give

$$\Delta W = O(\rho^{-3}), \quad W - r^2 f = O_\theta(\rho^{-1}) \quad \text{on } r \geq \theta\rho.$$

The unmodified Newton potential is $O(\rho^{-1})$, and the remaining bounded exterior harmonic function is $b_\infty + O(\rho^{-2})$. The fixed-leaf expansion [34, Proposition 3.3], with the phase bounded, therefore yields

$$P = b_\infty + O_\theta(\rho^{-1}) \quad \text{on } r \geq \theta\rho.$$

Lemma 5.4 now supplies the missing uniformity through the smoothing regions:

$$P = b_\infty + o(1) \quad \text{on the entire end of } M.$$

If $b_\infty \geq 0$, the $C_{4,3}$ tangent at zero supplies a regular point of negative phase. Uniform convergence at infinity and $P(0) = 0$ imply that the negative global minimum of $P|_M$ is attained at a regular interior point. At that point M touches the smooth negative Hardt–Simon leaf from one side. The strong maximum principle gives local equality. The set of points of equality is then open and closed in the connected regular hypersurface, by the same comparison at each

of its limit points. This would make P constant there, contradicting $P(0) = 0$ and the negative minimum. If $b_\infty \leq 0$, a positive-phase point gives the analogous contradiction at a positive maximum. Hence $A > 0$.

Step 5: conclusion of the proof. Finally, the expansion $f = AJ_2 + O(\rho^{-\kappa})$ on regular sectors gives

$$\sup_{\{r > \theta R, \rho \approx R\}} |r^2 f| \leq CR^2 \xrightarrow{(39)} D(R) \leq CR^2 + qD(4R) \implies D(R) \leq CR^2,$$

using the same forward iteration (note that $16\alpha_0 < \alpha_0 4^p < 1$) and the previously known growth bound. $\square$

5.2. Higher derivative estimates. We prove Proposition 5.1 by highlighting the geometric properties coming the problem, and deferring to the appendix the standard PDE techniques. The ideas of this section were provided to us by ChatGPT Astra.

Proof of Proposition 5.1. First we apply Lemma A.1 (which is a somewhat standard application of the estimates in [34]), to push the quadratic decay of Proposition 5.2 to the spine to bound the smoothing radius by

$$a_\infty(Y) = \left(\frac{A}{c_H}\right)^{1/3} |Y|^{2/3} + O(|Y|^{2/3-1/6}), \quad A > 0.$$

The details are given for the readers convenience in the appendix for the reader's convenience.

Fix a large $|y_0|$, put $\beta = u_\infty(0, y_0)$ and $R = \theta_0|y_0|$, and keep β fixed throughout the band. Here $\theta_0 > 0$ is a sufficiently small geometric constant. Then $\beta^3 \approx R^2$ and $\beta/R \rightarrow 0$. For the fixed leaf family, write

$$H(t, \lambda) = \lambda^{1/3} h(t/\lambda^{1/3}), \quad P(t, H(t, \lambda)) = c_H \lambda.$$

We now identify the weighted equation. All derivatives and divergences in the following formulas are in the base variables $X = (x, y) \in \mathbb{R}^4 \times \mathbb{R}$. Put

$$W = \sqrt{1 + |Du|^2}, \quad B = \frac{I}{W} - \frac{Du \otimes Du}{W^3}, \quad d(X) = H_\lambda \left(t, c_H^{-1} P(t, u(X)) \right),$$

where H_λ denotes the derivative of H in the second variable. Differentiating the Euler–Lagrange equation for $u^3 W$ in y gives the exact equation

$$\mathcal{J}u_y = 0, \quad \mathcal{J}\phi = -\operatorname{div}(u^3 B D\phi) - \frac{3u}{W} \phi.$$

For $q_1 = u_y/(\beta^3 d)$, the identity $d\mathcal{J}(dq) = -\operatorname{div}(u^3 d^2 B Dq) + d(\mathcal{J}d)q$ therefore gives

$$(53) \quad -\operatorname{div}(\mathcal{A} Dq_1) + \mathcal{V}q_1 = 0, \quad \mathcal{A} = u^3 d^2 B, \quad \mathcal{V} = d\mathcal{J}d.$$

The field d is evaluated at the leaf through each point of the connector. Because that leaf varies with X , the residual potential $\mathcal{V}$ need not vanish. On a fixed leaf, however, the positive dilation field $d_* = H_\lambda(\cdot, \lambda_*)$ satisfies $\mathcal{J}_* d_* = 0$. The normalization therefore cancels the potential of the model leaf, leaving only the error from comparison with that leaf.

Here is where the positive quadratic asymptotics enter. On the chosen band the quadratic estimate of Proposition 5.2, in the form of Lemma A.1, gives

$$(54) \quad c\beta^3 \leq c_H^{-1} P(t, u(t, y)) \leq C\beta^3.$$

(see Lemma A.2). The fixed-leaf bounds consequently imply $u = H(t, \lambda) \approx s$ and $d = H_\lambda(t, \lambda) \approx s^{-2}$, where $s = \beta + t$. Since $|Du| \leq \Lambda$, the matrix B is uniformly elliptic, and hence

$$c\omega_\beta I \leq \mathcal{A} \leq C\omega_\beta I, \quad \omega_\beta(X) = (\beta^2 + |x|^2)^{-1/2}.$$

The model weights satisfy $[\omega_\beta]_{A_2} \leq C$ uniformly for every $\beta > 0$, as verified in Appendix A.3. Thus the scalar factor $u^3 d^2 \approx \omega_\beta$ also has a uniform local A_2 bound. The role of the quadratic asymptotics is to obtain this comparison for the actual solution throughout the band; the A_2 property of the model weight itself is independent of those asymptotics.

The quantitative comparison with a fixed leaf also gives

$$|\mathcal{V}| \leq C\omega_\beta s^{-2}(s/R)^{1/2}.$$

The factor $(s/R)^{1/2}$ comes from the phase oscillation estimate (54), using $\beta^3 \approx R^2$. Relative to its principal weight, the original Jacobi potential has size $3/(Wu^2) \approx s^{-2}$. The new residual potential gains the factor $(s/R)^{1/2}$, as it is measuring the error of the foliation after modulating the dilation, although it need not satisfy a pointwise bound by $C\omega_\beta R^{-2}$ near the axis. The useful smallness is instead measured by the Dirichlet Green kernel G_D of $-\operatorname{div}(\mathcal{A}D)$ on an interior ball $D = B_\rho$. Lemma A.3, using [12, Theorem 3.3], proves

$$\sup_{X \in D} \int_D G_D(X, Z) |\mathcal{V}(Z)| dZ \leq C(\rho/R)^{1/2},$$

with a constant independent of β/R .

The summability behind this estimate can be seen on dyadic shells centered on the axis. At radius $r \gg \beta$, their weighted volume is comparable to r^4 , the Green kernel is bounded by Cr^{-2} , and $|\mathcal{V}|/\omega_\beta \leq Cr^{-2}(r/R)^{1/2}$. The contribution of each shell is consequently bounded by

$$Cr^{-2} r^4 r^{-2} (r/R)^{1/2} = C(r/R)^{1/2}.$$

These contributions form a summable geometric series. Without the additional factor $(r/R)^{1/2}$, the same calculation would give a constant on each shell and a bound growing with the number of scales between β and ρ .

Choose $\rho = \delta_0 R$, with $\delta_0 > 0$ fixed sufficiently small that the Green norm above is at most $1/4$. Lemma A.3 then solves

$$\vartheta(X) = 1 - \int_D G_D(X, Z) \mathcal{V}(Z) \vartheta(Z) dZ, \quad \frac{1}{2} \leq \vartheta \leq 2.$$

Thus $\mathcal{L}\vartheta = 0$, where $\mathcal{L} = -\operatorname{div}(\mathcal{A}D \cdot) + \mathcal{V}$. Equation (53) then becomes

$$-\operatorname{div}(\vartheta^2 \mathcal{A}D(q_1/\vartheta)) = 0.$$

Writing $z = q_1/\vartheta$ and testing with $\eta^2 z$ gives the weighted Caccioppoli, or reverse Poincaré, inequality

$$\int_D \eta^2 |Dz|^2 \omega_\beta dX \leq C \int_D |D\eta|^2 z^2 \omega_\beta dX.$$

This energy estimate uses only weighted ellipticity. The uniform A_2 bound supplies the accompanying weighted Sobolev inequality, which gives local boundedness by iteration. This is the equation to which [13, Theorem 2.3.1] applies: its symmetric coefficient matrix is comparable to the same A_2 weight, with constants independent of β/R . Appendix A.3 states the weighted estimates and treats the potential and the higher-order forcing.

Using higher order analogue of the estimate (54), obtained in Lemma A.2 by scaling of standard elliptic estimates, Proposition A.4 proves, for every fixed finite derivative order,

$$(55) \quad |D^k \partial_y^j u| \leq C_{j,k} \beta^3 R^{-j} (\beta + t)^{-2-k}, \quad j \geq 1, \quad k \geq 0,$$

on $B_{R/4}(0, y_0)$. Its proof uses the same normalization $q_j = \partial_y^j u / (\beta^3 d)$ at every order. The additional factor $(\beta/(\beta + t))^3$ in the higher-order forcing makes its Green potential uniformly bounded. Energy estimates and the weighted local boundedness estimate then give one factor R^{-1} for each axial derivative.

At $t = 0$, (55) immediately gives

$$|a_\infty^{(j)}(y)| \leq C_j \beta R^{-j} \leq C_j a_\infty(y) R^{-j}.$$

Proposition A.5 applies these direct derivative estimates to the radial equation. For $0 \leq t \leq cR$ and $|y - y_0| \leq cR$ it gives

$$u_\infty(t, y) = a_\infty(y) h(t/a_\infty(y)) + e_\infty(t, y),$$

with

$$|\partial_t^m \partial_y^n e_\infty| \leq C_{mn} a_\infty(y)^3 R^{-2-n} (a_\infty(y) + t)^{-m}.$$

For any specified m, n , use the direct estimates through axial order $n + 2$. The geometric band and lower height threshold are independent of the chosen finite order; the constants may depend on it. Taking $y_0 = Y$ proves (32), and the smoothing-radius asymptotic identifies $\kappa_H = (A/c_H)^{1/3}$.

On a fixed regular angular sector, the quadratic phase bound and the fixed-leaf expansion give $|f_\infty| \leq C_\theta$. After rescaling an annulus of radius ρ to unit size, the normal height is at most $C_\theta \rho^{-1}$. The interior normal-graph estimates of Proposition 3.4, on a slightly larger regular sector, give the same bound for every fixed derivative order in rescaled coordinates. Scaling back yields $|D^j f_\infty| \leq C_{\theta,j} \rho^{-j}$, as required.

Set

$$d(X) = H_\lambda(t, c_H^{-1} P(t, u(t, y))) \approx (\beta + t)^{-2}.$$

This is the fixed leaf's positive Jacobi field evaluated at the solution. We differentiate the equation for u ; P is used only to select this normalizing field. Proposition A.4 proves, for every fixed finite derivative order,

$$(56) \quad |D^k \partial_y^j u| \leq C_{j,k} \beta^3 R^{-j} (\beta + t)^{-2-k}, \quad j \geq 1, \quad k \geq 0,$$

on $B_{R/4}(0, y_0)$. Its proof starts with the exact Jacobi equation for u_y . Normalizing each $\partial_y^j u$ by $\beta^3 d$ gives a common weighted operator. The comparison above makes its potential small in the Green-kernel norm. The nonlinear forcing at higher orders decays by an additional factor $(\beta/(\beta + t))^3$, which makes its Green potential uniformly bounded. Energy estimates and weighted local boundedness then give one factor R^{-1} for each axial derivative.

At $t = 0$, (56) immediately gives

$$|a_\infty^{(j)}(y)| \leq C_j \beta R^{-j} \leq C_j a_\infty(y) R^{-j}.$$

Proposition A.5 applies these direct derivative estimates to the radial equation. For $0 \leq t \leq cR$ and $|y - y_0| \leq cR$ it gives

$$u_\infty(t, y) = a_\infty(y) h(t/a_\infty(y)) + e_\infty(t, y),$$

with

$$|\partial_t^m \partial_y^n e_\infty| \leq C_{mn} a_\infty(y)^3 R^{-2-n} (a_\infty(y) + t)^{-m}.$$

For any specified m, n , use the direct estimates through axial order $n + 2$. The geometric band and lower height threshold are independent of the chosen finite order; the constants may depend on it. Taking $y_0 = Y$ proves (32), and the smoothing-radius asymptotic identifies $\kappa_H = (A/c_H)^{1/3}$.

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6. ACCUMULATING SINGULARITIES

This section proves Theorem 1.1 using the quantitative connector estimates and a suitable gluing construction.

We first describe the background model and show why direct interpolation with a modulation of the foliation fails.

We then prove Theorem 1.1 by preparing a background adapted to the connector insertions. The construction is organized around one transverse correction $f(t, y)$. We first specify its required estimates, then use them to prepare the background and its metric, insert the connectors, and update the metric locally. Finally we choose f from the finite-order profiles constructed in Section 6.4, where we prove the residual and matching estimates.

The purpose of f is to control both the background mean curvature and the error on the attachment annuli. These two requirements permit a smooth conformal metric with the full Euclidean jet at the accumulation point. The parameter-selection order is similar to that in [8]:

choose a background modification to fit the intended insertions. Here its transverse dependence is essential.

Preparing the pieces before joining them also appears in the constant-mean-curvature construction of Mazzeo–Pacard–Pollack [23], where Green-function corrections for the Jacobi operator adapt the summands to the inserted neck. Here we prepare the background by successive radial corrections about Hardt–Simon leaves and then correct the ambient metric conformally.

6.1. The base model and the interpolation obstruction. We first keep only the zeroth-order approximation by foliation leaves. On a transition annulus it has exactly the connector's axis value $a(y)$, so any error comes from how the profiles open away from the axis, that is how far the connector is from the foliation modulated by $a(y)$. The calculation below isolates that error at the axis and shows why decreasing the smoothing radius does not by itself give a regular conformal correction. This identifies the terms that the higher-order profiles must cancel.

The unmodified model is $V = C_{3,3} \times \mathbb{R}$. Let $a(y) > 0$ for $y \neq 0$, with $a(0) = 0$, and let $\mathcal{R}(y) \downarrow 0$ be a transverse cutoff scale, with $a(y) \ll \mathcal{R}(y) \ll |y|$. Near the spine, replace the cone in the slice of height y by the positive Hardt–Simon leaf

$$v_0(t, y) = a(y)h(t/a(y)), \quad t = |x|.$$

Cut this profile off to t for $t \gtrsim \mathcal{R}(y)$ and set the slice at $y = 0$ equal to the Simons cone. This is the first candidate background. Later we take $a(y) \approx e^{-1/y^2}$ and $\mathcal{R}(y) \approx |y|^4$: other choices are possible but $a(y)$ must vanish at infinite order at 0 to obtain a C^∞ metric whose jet at zero coincide with the euclidean metric.

At $p_k = (0, 0, y_k)$, $y_k \downarrow 0$, place a translated and dilated connector

$$u_k(t, y) = \lambda_k u_\infty(t/\lambda_k, (y - y_k)/\lambda_k)$$

in a ball of radius comparable to $R_k \ll y_k$ (to keep the transition regions disjoint), with $\lambda_k/R_k \rightarrow 0$ (so that the connector is blown down to resemble V at infinity, that is the model). The parts of its attachment annulus near $x = 0$, above and below $y = y_k$, will be called the *polar regions*.

On the corresponding axial matching intervals choose $a(y) = u_k(0, y)$, so that the connector and the background leaf have exactly the same smoothing radius. One might then try the direct interpolation

$$v_\chi = v_0 + \chi \left(\frac{\sqrt{2t^2 + (y - y_k)^2}}{R_k} \right) (u_k - v_0),$$

where $\chi = 1$ on $[0, 1]$ and $\chi = 0$ on $[2, \infty)$. The transverse cutoff of the background is one near the axis, so it does not affect the following calculation.

Write Q for the radial SME operator, as in (30):

$$(57) \quad Q(v) = \left(\delta_{ij} - \frac{v_i v_j}{1 + |Dv|^2} \right) v_{ij} + \frac{3}{t} v_t - \frac{3}{v}, \quad i, j \in \{t, y\}.$$

At $t = 0$ the smooth radial interpretation gives $Q(v) = 4v_{tt} + v_{yy}/(1 + v_y^2) - 3/v$. Since $h''(0) = 3/4$ and u_k solves $Q(u_k) = 0$, as already observed in Section 5, the difference $e = u_k - v_0$ satisfies

$$e(0, y) = e_t(0, y) = 0, \quad 4e_{tt}(0, y) = -\frac{a''}{1 + (a')^2}.$$

Writing $\chi_0(y) = \chi(|y - y_k|/R_k)$, all derivatives of the interpolation cutoff multiply data that vanish at $t = 0$. Consequently the scalar mean curvature, with the normal pointing towards increasing $|z|$, satisfies the exact identity

$$(58) \quad H_\chi(0, y) = -\frac{Q(v_\chi)(0, y)}{\sqrt{1 + (a')^2}} = -(1 - \chi_0(y)) \frac{a''(y)}{(1 + (a'(y))^2)^{3/2}}.$$

Now suppose there is a C^2 metric $g = e^{2F} g_{\text{euc}}$, then the equation for the mean curvature is $H_\chi + 8\nu F = H_\chi^g = 0$, if we require minimality. Writing $z = a(y)\omega$, with $|\omega| = 1$, the normal

pointing toward increasing $|z|$ is

$$\nu = \frac{(0, \omega, -a'(y))}{\sqrt{1 + (a'(y))^2}}.$$

Since the averaged F depends on z through $s = |z|$, this gives

$$\nu F = \frac{F_s(0, a(y), y) - a'(y)F_y(0, a(y), y)}{\sqrt{1 + (a'(y))^2}}.$$

Substitute this into $H_\chi + 8\nu F = 0$:

$$-\frac{(1 - \chi_0)a''}{(1 + (a')^2)^{3/2}} + 8\frac{F_s - a'F_y}{\sqrt{1 + (a')^2}} = 0.$$

Multiplying by $\sqrt{1 + (a')^2}$ and dividing by $8a$ yields exactly

$$\frac{(1 - \chi_0)a''}{8a(1 + (a')^2)} = \frac{F_s(0, a, y)}{a} - \frac{a'}{a}F_y(0, a, y).$$

Fix $\tau \in (1, 2)$ such that $1 - \chi(\tau) > 0$, and evaluate at $y = y_k + \tau R_k$. By Proposition 5.1,

$$a(y) \sim \kappa_H \lambda_k^{1/3} (y - y_k)^{2/3},$$

with the asymptotic valid through two derivatives on the scale R_k . Since $\lambda_k/R_k \rightarrow 0$, it follows that

$$a'(y) \rightarrow 0, \quad \left| \frac{a'}{a} \right| \leq CR_k^{-1}, \quad -\frac{a''}{a} \geq cR_k^{-2}.$$

On the other hand, rotational invariance and C^2 regularity give $F_s(0, 0, y) = 0$, and hence

$$\left| \frac{F_s(0, a(y), y)}{a(y)} \right| \leq \|D^2 F\|_{L^\infty(U)},$$

where U is a fixed neighborhood of the origin containing these points for all sufficiently large k . Taking absolute values in the preceding identity therefore yields

$$cR_k^{-2} \leq \|D^2 F\|_{L^\infty(U)} + CR_k^{-1} \|DF\|_{L^\infty(U)}.$$

Multiplying by R_k^2 and letting $k \rightarrow \infty$ gives a contradiction. Thus the interpolated hypersurface cannot be minimal in a C^2 conformally Euclidean metric near the origin.

6.2. The foliation and the prescribed insertions. Write $(x, z, y) \in \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}$ and $t = |x|$, and let h be the positive normalized Hardt–Simon profile from (31). Thus

$$(59) \quad \frac{h''}{1 + h'^2} + \frac{3h'}{\rho} - \frac{3}{h} = 0, \quad h(0) = 1, \quad h'(0) = 0, \quad h''(0) = \frac{3}{4}.$$

We use its differentiated end expansion

$$(60) \quad h(\rho) = \rho + b_H \rho^{-2} + O(\rho^{-3}), \quad b_H > 0.$$

Write $S(v) = \{|z| = v(|x|, y)\}$. For a positive radius function a define the uncorrected back-ground

$$(61) \quad V_0[a](t, y) = a(y)h(t/a(y)).$$

It suffices to consider this surface in a tube $0 \leq t < \mathcal{R}(y)$ with $\mathcal{R}(y)/a(y) \rightarrow \infty$ as $y \downarrow 0$.

For a rotational profile v , put

$$(62) \quad Q(v) = \frac{(1 + v_y^2)v_{tt} - 2v_tv_yv_{ty} + (1 + v_t^2)v_{yy}}{1 + v_t^2 + v_y^2} + \frac{3v_t}{t} - \frac{3}{v}.$$

With the normal pointing towards increasing $|z|$, its scalar Euclidean mean curvature is $H = -Q(v)/\sqrt{1 + v_t^2 + v_y^2}$.

Let $u_\infty(t, Y)$ be the connector of Theorem 1.2. Its properties and Proposition 5.1 give the following properties. The Cartesian profile $u_\infty(|x|, Y)$ is globally Lipschitz and vanishes only at $(x, Y) = (0, 0)$. The hypersurface $S(u_\infty)$ is stationary with multiplicity one. Its only singular

point is the ambient origin, with unique tangent cone $C_{4,3}$; at infinity it has finite density and unique tangent cone $C_{3,3} \times \mathbb{R}$. Its cap satisfies

$$u_\infty(0, Y) \sim \kappa_H |Y|^{2/3}, \quad |\partial_Y^n u_\infty(0, Y)| \leq C_n |Y|^{2/3-n} \quad (|Y| \text{ large}).$$

We also use its regular-sector derivative estimates and the matching estimates recalled in Section 6.4.

Fix a small $c_* > 0$. For all sufficiently large k , choose

$$(63) \quad \begin{aligned} y_k &= 2^{-k}, & p_k &= (0, 0, y_k), & R_k &= c_* y_k^4, \\ \lambda_k &= \frac{e^{-3/y_k^2}}{\kappa_H^3 R_k^2}, & u_k(t, y) &= \lambda_k u_\infty(t/\lambda_k, (y - y_k)/\lambda_k). \end{aligned}$$

Thus $u_k(0, y) \approx e^{-1/y_k^2}$ when $|y - y_k| \approx R_k$, and $\lambda_k \ll e^{-1/y_k^2} \ll R_k \ll y_k$. The insertion neighborhoods are disjoint.

The cutoff coordinate $\rho_k = \sqrt{2t^2 + (y - y_k)^2}$ separates an exact core $\rho_k \leq R_k$ from the attachment annulus $R_k < \rho_k < 2R_k$. The parts of this annulus near $t = 0$ are its *polar regions*; the remaining part is the *regular region*. Figure 5 shows these regions and the corrected background to be chosen.

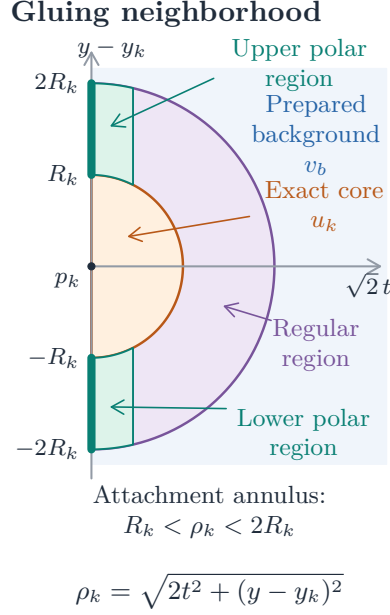


FIGURE 5. The gluing neighborhood in the parameter half-plane $(\sqrt{2}t, y - y_k)$ (schematic; scales are exaggerated). The semicircles are level sets of ρ_k , rather than ambient distance. The orange region contains the exact connector core. The attachment annulus consists of the green upper and lower polar regions near $t = 0$ and the purple regular region away from the axis. The thick green segments on $t = 0$ are the axial matching intervals. Outside the outer semicircle, the construction retains the prepared background v_b . It agrees with $\hat{v} = V_0[a] + f$ in the inner transverse tube and is cut back to the cone outside that tube; these profiles are defined below.

Choose η_k supported in $R_k/4 < |y - y_k| < 4R_k$, equal to one on $R_k/2 \leq |y - y_k| \leq 3R_k$, with $|\eta_k^{(n)}| \leq C_n R_k^{-n}$. For $y \neq 0$, set

$$(64) \quad a(y) = e^{-1/y^2} + \sum_k \eta_k(y) (u_k(0, y) - e^{-1/y^2}), \quad \mathcal{R}(y) = c_* |y|^4.$$

Each summand is extended by zero outside its support. Since $R_k = O(y_k^4)$, the functions e^{-1/y^2} and e^{-1/y_k^2} are comparable on the cutoff supports. The cap estimates therefore imply

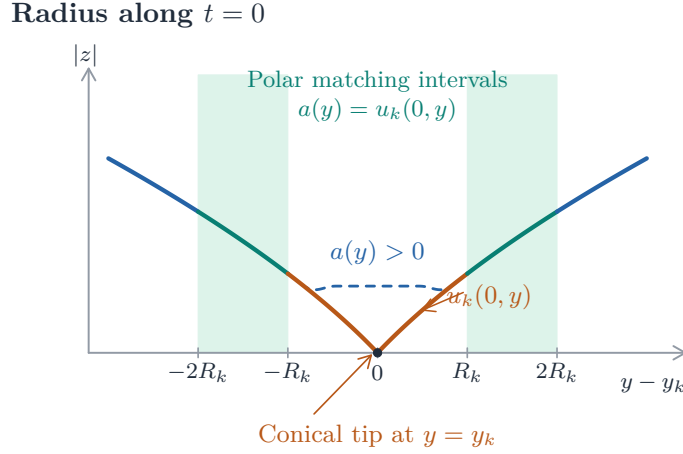
$$(65) \quad a(y) \approx e^{-1/y^2}, \quad |a^{(n)}(y)| \leq C_n a(y) \mathcal{R}(y)^{-n}.$$

In particular a extends smoothly and flatly by $a(0) = 0$. On the polar matching intervals it equals the cap radius of the corresponding connector. We have chosen the insertion data before the background: the function f will be adapted to these data.

6.3. The transverse correction. We now leave f unspecified and write

$$(66) \quad \widehat{v}(t, y) = V_0[a](t, y) + f(t, y), \quad s = a(y) + t.$$

We impose $f(0, y) = f_t(0, y) = 0$ to preserve the prescribed cap radius and the radial first-derivative condition. The correction changes how the profile opens away from the axis. Here f denotes a correction to the SME radius profile, distinct from the normal graph heights used earlier in the paper. Figure 6 shows the prescribed axis data: the background stays positive at y_k , while the inserted connector vanishes there. Agreement on the matching intervals fixes the axis value; f must also control the transverse error.



$$\widehat{v}(0, y) = a(y): f \text{ preserves the axis value.}$$

It improves the transverse profile and mean curvature.

FIGURE 6. Axis radii before and after inserting the connector. The horizontal coordinate is $y - y_k$ and the vertical coordinate is the radius $|z|$ at $t = 0$. The dashed blue curve is the positive background radius $a(y)$; the orange curve is $u_k(0, y)$ in the exact core and vanishes conically at y_k . On the green matching intervals $R_k \leq |y - y_k| \leq 2R_k$, the two radii agree. The solid blue portions represent the retained background. Matching these values does not match the full transverse profiles. The correction f preserves the axis value and improves the transverse profile and mean curvature. All curves are schematic.

The two essential requirements are small mean curvature and accurate matching to the connectors. The following formulation records them without mentioning a correction order.

Definition 6.1 (An admissible transverse correction). On a sufficiently small punctured neighborhood of $y = 0$, the function f is smooth, with a smooth radial extension in x , and satisfies $f(0, y) = f_t(0, y) = 0$. On a tube $0 \leq t < C\mathcal{R}(y)$, the profile $\widehat{v}$ satisfies

$$(67) \quad \widehat{v} \approx s, \quad |D^m \widehat{v}| \leq C_m s^{1-m} \quad (m \geq 1).$$

Here and below D denotes Cartesian base derivatives in (x, y) ; radial formulas at $x = 0$ are interpreted by their smooth extension. For every $L, m, N \geq 0$, its residual obeys

$$(68) \quad |D^m Q(\hat{v})| \leq C_{L,m,N} |y|^N s^{L-m}.$$

Fix small constants $0 < c < c_1$ so that $t \leq cR_k$ lies inside $t < c_1\mathcal{R}(y)$ on each polar attachment region. On

$$R_k \leq \sqrt{2t^2 + (y - y_k)^2} \leq 2R_k, \quad t \leq cR_k,$$

we require, for every $L, m, N \geq 0$,

$$(69) \quad |D^m(u_k - \hat{v})| \leq C_{L,m,N} |y|^N s^{L+2-m}.$$

The constants are independent of k . Finally, throughout each fixed transverse collar $\theta\mathcal{R} \leq t \leq C\mathcal{R}$, $\theta > 0$, we require

$$(70) \quad |D^m(\hat{v} - t)| \leq C_{\theta,m,N} |y|^N \quad (m, N \geq 0).$$

The quantifiers in (68) and (69) are useful: every fixed power lost by differentiating at the small scale s can be absorbed by choosing L larger. The annular estimate has no such small-scale issue because $s \approx \mathcal{R}$ there and $\mathcal{R}$ is a fixed power of $|y|$.

Proposition 6.2 (Choice of the correction). *For the radius function a and scaled connector profiles u_k fixed in (63)–(64), there exists an admissible transverse correction f .*

We prove this proposition after the geometric construction. No smooth extension of f to $(x, y) = 0$ is asserted. Its high transverse derivatives may grow in the shrinking caps. The smooth flat object obtained below is the ambient metric exponent. Given such a correction we can conclude the proof of Theorem 1.1.

Proof of Theorem 1.1. Take an admissible correction given by Proposition 6.2. We first make its background minimal in a smooth conformal metric. We then insert the connectors and update that same metric formula locally. The estimates in Definition 6.1 control both stages.

Step 1: Prepare the background and its metric. Choose a smooth cutoff χ equal to one on $[0, 1]$ and zero on $[2, \infty)$. Given an admissible f , set

$$(71) \quad v_b(t, y) = t + \chi\left(\frac{t}{c_1\mathcal{R}(y)}\right) (\hat{v}(t, y) - t) \quad (y \neq 0), \quad v_b(t, 0) = t.$$

The background $S(v_b)$ retains the prescribed cap $a(y)$ and agrees with the cone outside the narrow tube. Its slope is uniformly bounded. It is smooth away from the origin.

We use the explicit conformal extension from Lemma B.3 for both the background and the surface after insertion. For any positive profile v , let H_v be its Euclidean scalar mean curvature. Fix $\vartheta \in C_c^\infty((-1, 1))$ equal to one near zero and define

$$(72) \quad F[v](x, z, y) = -\frac{H_v(t, y)}{8\sqrt{1 + v_t^2 + v_y^2}} (|z| - v(t, y))\vartheta\left(\frac{4(|z| - v(t, y))}{v(t, y)}\right).$$

A common outer cutoff, equal to one on the working neighborhood, may be included. The radial support avoids $z = 0$, and the smooth even dependence in x makes the expression smooth at $x = 0$. On $S(v)$,

$$(73) \quad F[v] = 0, \quad \nu_v F[v] = -H_v/8, \quad H_{S(v), e^{2F[v]}g_{\text{euc}}} = 0.$$

The formula is zero wherever $H_v = 0$.

Here is the derivative estimate that explains the choice of conditions on f . On a region with geometric scale $s = a + t$, the profile bounds (67) and differentiation of (72) give

$$(74) \quad |D^m F[v]| \leq C_m \sum_{j=0}^m s^{1-m+j} |D^j H_v|.$$

Derivatives on the left are ambient Cartesian derivatives; those on the right are base derivatives of the scalar curvature. The factor s comes from $|z| - v$ on the support, and each derivative of a profile coefficient or cutoff costs s^{-1} .

Inside the tube, (68) gives $|D^j H_v| \leq C|y|^N s^{L-j}$. Taking $L \geq m-1$ in (74) controls the m th metric derivative by $C|y|^N$. On the transverse cutoff collar, (70) gives the same conclusion: cutoff derivatives cost only powers of $\mathcal{R}^{-1}$, which are absorbed by increasing N in that estimate. Consequently

$$(75) \quad F_b := F[v_b], \quad |D^m F_b(x, z, y)| \leq C_{m,N} |y|^N \quad (m, N \geq 0).$$

Extend F_b by zero at the origin and wherever the background is exactly the cone. It is smooth and flat at zero, and $g_b = e^{2F_b} g_{\text{euc}}$ makes the prepared background minimal.

Step 2: Insert the connectors and update the metric. With the same cutoff χ , define

$$(76) \quad v = v_b + \sum_k \chi \left(\frac{\sqrt{2t^2 + (y - y_k)^2}}{R_k} \right) (u_k - v_b).$$

The supports are disjoint. The surface $\Sigma = S(v)$ agrees with the exact connector in each inner region and with the prepared background outside the insertion regions.

On a polar attachment region, the transverse cutoff in (71) is one. The matching estimate (69) loses two derivatives when the curvature operator is applied. Its remaining bound is therefore of the form (68). Differentiating the interpolation cutoff costs $R_k^{-1} \leq Cs^{-1}$, so these terms obey the same bound. The positive profiles and their first derivatives stay in the controlled range in (67), after restricting to sufficiently large k .

On the rest of the attachment annulus, $t \approx R_k$. The connector's regular-sector estimates give

$$|D^m(u_k - t)| \leq C_m \lambda_k R_k^{-m} \leq C_m e^{-3/y_k^2} R_k^{-2-m}.$$

Together with (70), this controls the interpolation and every curvature derivative by every prescribed power of $|y|$. Since $s \approx R_k$ here, the metric extension again has the bounds (75). Inside the exact connector regions its curvature is zero, so the extension vanishes identically; there is no derivative cost at scale λ_k .

It follows that $F = F[v]$, defined by (72) off the singular points and extended by zero on the exact cores, is smooth away from zero and satisfies

$$(77) \quad |D^m F(x, z, y)| \leq C_{m,N} |y|^N \quad (m, N \geq 0).$$

It extends smoothly and flatly at zero. The final metric is $g = e^{2F} g_{\text{euc}}$.

The background-first metric construction can be stated literally:

$$(78) \quad g = e^{2\Delta F} g_b, \quad \Delta F = F[v] - F[v_b].$$

Use the same cutoffs in the two extensions. Locality of (72) gives $\Delta F = 0$ where the profiles agree on a base neighborhood. Thus the metric update is supported over the insertion regions. On an exact connector core, $F[v] = 0$ and $\Delta F = -F_b$: the final metric is Euclidean there, although the preliminary metric need not be. The estimates for ΔF follow from (75) and (77). No independent patching of metric pieces is needed.

Step 3: Mass bounds and stationarity. For completeness, the varifold conclusions follow from the same localization. The modified background has mass in B_r bounded by

$$C \int_{|y| < r} \int_0^{C|y|^4} t^3 (t + a(y))^3 dt dy \leq Cr^{29}.$$

Each inserted region has mass at most CR_k^8 , using the bounded slopes on its annulus and the connector's finite density at infinity on its core. The total contribution near zero is $C \sum_{y_k \leq 2r} R_k^8 \leq Cr^{32}$. The removed cone pieces have the same negligible bounds. Hence the unique tangent at zero is $C_{3,3} \times \mathbb{R}$ with multiplicity one, and $\mu_\Sigma(B_r) \leq Cr^8$. Exact agreement on smaller ambient balls about p_k gives the unique $C_{4,3}$ tangent there. There are no other singularities.

Stationarity holds on the regular part by (73), and at each p_k by the exact Euclidean connector. Removing $B_r(0)$ with a cutoff whose derivative is $O(r^{-1})$ gives a first-variation error $O(r^8 + r^7) \rightarrow 0$. This proves stationarity also at zero. Finally, (77) implies $D^m(g - g_{\text{euc}})(0) = 0$

for all m . This proves the theorem after restricting to a sufficiently small open ball. Indeed, the same bounds give $\|g - g_{\text{euc}}\|_{C^m(B_\rho)} \leq C_{m,N} \rho^N$ for every m, N . $\square$

6.4. Successive corrections around the foliation and the choice of f . We establish two estimates with different roles. Proposition 6.3 summarizes the control of the corrected background and its remaining mean curvature for a slowly varying radius function. Corollary 6.4 shows that, when the radius function comes from the connector, the same profiles approximate that connector to the corresponding order. The first estimate controls the region between insertions; the second controls the attachment annuli. In both, the axis value is kept fixed.

We construct the finite-order profiles used below to choose the transverse correction f . At each fixed y , we retain the prescribed axis value $a(y)$ and solve a radial linearized equation around the leaf $a(y)h(t/a(y))$. Successive solutions give the approximations $V_J[a]$ and cancel their mean-curvature errors through increasing even orders. When a is the connector radius, these profiles also approximate the connector to the corresponding order. The construction is a finite expansion in the normalized axial jets $q_j = a^{j-1}a^{(j)}$, with smooth radial coefficients in $\rho = t/a(y)$ determined by the linearized Hardt–Simon equation. No convergence of an infinite series is required.

A related scheme for successively improving approximate constant-mean-curvature hypersurfaces by solving model linearized equations appears in Mahmoudi–Mazzeo–Pacard [22].

At order J , the estimates stated below and proved in Appendix B.2, together with the conformal correction lemma, yield

$$(79) \quad |D^m F| \leq C_{J,m} \left(a^3 \mathcal{R}^{-m-3} + a^{2J+2-m} \mathcal{R}^{-2J-2} \right).$$

The second term measures the cost near the axis. In the uncorrected interpolation that cost has the scaling $a^{2-m} R_k^{-2}$; successive corrections supply the additional positive powers of $a/\mathcal{R}$. For each fixed m , taking $2J + 2 > m$ makes both terms flat in y . The diagonal choice below increases J as y tends to zero, while retaining exact connector cores. This is the mechanism that gives one smooth metric with the full Euclidean jet at zero.

We now define the correctors and state the estimates used in this construction. The scalar mean curvature is $H = -Q(v)/\sqrt{1 + |Dv|^2}$, with Q as in (62). Set $\rho = t/a(y)$, $q_j = a^{j-1}a^{(j)}$, and

$$\mathcal{I}_d = \left\{ \gamma \in \mathbb{N}_0^d : \sum_{j=1}^d j\gamma_j = d \right\}, \quad q^\gamma = \prod_{j=1}^d q_j^{\gamma_j}.$$

The initial approximation is

$$V_0[a](t, y) = a(y)h\left(\frac{t}{a(y)}\right).$$

Suppose $V_{J-1}[a]$ has been constructed, and write $v = V_{J-1}[a]$. For a vertical perturbation $v_\tau = v + \tau\varphi$, its mean curvature, viewed as a function of (t, y) , is

$$H_{v_\tau} = -\frac{Q(v_\tau)}{\sqrt{1 + |Dv_\tau|^2}}.$$

Consequently,

$$\frac{d}{d\tau} H_{v+\tau\varphi} \Big|_{\tau=0} = -\frac{DQ_v[\varphi]}{\sqrt{1 + |Dv|^2}} + \frac{Q(v) Dv \cdot D\varphi}{(1 + |Dv|^2)^{3/2}}, \quad DQ_v[\varphi] := \frac{d}{d\tau} Q(v + \tau\varphi) \Big|_{\tau=0}.$$

Express the normalized residual $aQ(v)$ in the variables $(\rho, q_1, q_2, \dots)$, and define $S_{2J,\gamma}$ by

$$[aQ(v)]_{2J} = \sum_{\gamma \in \mathcal{I}_{2J}} q^\gamma S_{2J,\gamma}(\rho).$$

Here $[\cdot]_{2J}$ denotes the coefficient of ε^{2J} after the substitution $q_j \mapsto \varepsilon^j q_j$. By induction, all residual terms of weight less than $2J$ vanish.

Consider a correction of the form

$$\varphi(t, y) = a(y) \sum_{\gamma \in \mathcal{I}_{2J}} q^\gamma(y) P_{2J,\gamma}(\rho).$$

At weight $2J$, its contribution to $aDQ_v[\varphi]$ is

$$[aDQ_v[\varphi]]_{2J} = \sum_{\gamma \in \mathcal{I}_{2J}} q^\gamma L_h P_{2J,\gamma},$$

where

$$L_h w = \frac{w''}{1+h^2} + \left(\frac{3}{\rho} - \frac{2h'h''}{(1+h^2)^2} \right) w' + \frac{3}{h^2} w.$$

Indeed, replacing the linearization at v/a by that at h only changes higher-weight terms, as do the axial and nonlinear contributions of the new correction. Thus the perturbed mean curvature satisfies

$$[aH_{v+\varphi}]_{2J} = -\frac{1}{\sqrt{1+h^2}} \sum_{\gamma \in \mathcal{I}_{2J}} q^\gamma (S_{2J,\gamma} + L_h P_{2J,\gamma}).$$

We therefore define $P_{2J,\gamma}$ as the unique regular even solution of

$$L_h P_{2J,\gamma} = -S_{2J,\gamma}, \quad P_{2J,\gamma}(0) = P'_{2J,\gamma}(0) = 0.$$

Set

$$(80) \quad W_J[a](t, y) = a(y) \sum_{\gamma \in \mathcal{I}_{2J}} q^\gamma(y) P_{2J,\gamma} \left(\frac{t}{a(y)} \right), \quad V_J[a] = V_{J-1}[a] + W_J[a].$$

This cancels the weight- $2J$ residual without changing the lower-weight cancellations. Since only even weights occur, the residual $aQ(V_J[a])$ has no terms of weight less than $2J+2$.

In Proposition B.2 we prove precise estimates for the profiles constructed above, for arbitrary given slowly varying profile a . We state the result informally here and collect the precise estimates in the Appendix B.

Proposition 6.3 (Informal version; see Proposition B.2 in Appendix B). *Let $a(y) > 0$ be any prescribed smooth radius varying on an axial scale R , with $|a^{(n)}| \leq C_n a R^{-n}$. For each fixed J , if a/R is sufficiently small, the universal corrections in (80) give a positive smooth profile $V_J[a] \approx a + t$ with the same axis data as $V_0[a]$. They cancel successive even-weight curvature errors, leaving*

$$|Q(V_J[a])| \leq C_J a^3 R^{-2J-2} (a+t)^{2J-2}$$

on the interior of the band, with corresponding weighted estimates for all derivatives. The construction uses only a and its derivatives; it does not require a to come from the connector.

For the connector radius, the same profiles also approximate u_∞ to arbitrary finite order. The proof of the following corollary is given in Appendix B.2.

Corollary 6.4 (Matching to arbitrary order). *Take $a(y) = u_\infty(0, y)$ and let $V_J[a]$ be the profiles supplied by Proposition B.2. For each $J > 0$ and $|y_0|$ large enough depending on J and $R = \theta_0 |y_0|$ with $\theta_0 > 0$ small and fixed, we have:*

$$(81) \quad |\partial_t^m \partial_y^n (u_\infty - V_J[a])| \leq C_{J,m,n} a(y)^3 R^{-2J-2-n} (a(y) + t)^{2J-m} \quad (m, n \geq 0),$$

for all $m, n \in \mathbb{N}_0$ and $(t, y) \in \{t^2 + |y - y_0|^2 \leq R^2\}$.

We now use these finite-order estimates to choose the single correction f required by Proposition 6.2.

Proof of Proposition 6.2. The preceding construction used only the properties in Definition 6.1. We choose f using the finite-order profiles of Proposition B.2 and Corollary 6.4, whose proofs are given in Appendix B.2.

Step 1: The finite-order estimates. The complete statement of Proposition B.2 applies on an enlarged band of scale R where $a(y) \approx a(y_0)$, $|a^{(n)}| \leq C_n a R^{-n}$, and a/R is sufficiently small. It provides profiles satisfying

$$V_J[a] = V_0[a] + \sum_{j=1}^J W_j[a], \quad W_j(0, y) = \partial_t W_j(0, y) = 0.$$

For every fixed finite collection of derivative orders,

$$(82) \quad |\partial_t^m \partial_y^n W_j[a]| \leq C_{j,m,n} a^3 R^{-2j-n} (a+t)^{2j-2-m},$$

$$(83) \quad |\partial_t^m \partial_y^n Q(V_J[a])| \leq C_{J,m,n} a^3 R^{-2J-2-n} (a+t)^{2J-2-m}.$$

The profiles are comparable to $a+t$ and have the corresponding scaled derivative bounds. For matching, translate and dilate the connector and cover each polar attachment region by finitely many bands whose radii are comparable to $\theta_0 R_k$. The enlarged bands lie where $a(y) = u_k(0, y)$; choose the fixed polar width c small enough for their transverse estimates. Corollary 6.4 then gives

$$(84) \quad |\partial_t^m \partial_y^n (u_k - V_J[a])| \leq C_{J,m,n} a^3 R_k^{-2J-2-n} (a+t)^{2J-m}.$$

These are precisely the model and matching bounds (119)–(120) and (81). Constants and thinness thresholds may depend on J and on the prescribed derivative orders. No uniformity in J or convergence of an infinite series is assumed. Even radial smoothness gives the same weighted bounds in Cartesian variables.

Step 2: Choice of f and conclusion. Choose radii $r_j \downarrow 0$, $j \geq 2$, and smooth cutoffs ζ_j equal to one on $|y| \leq r_j$, zero on $|y| \geq 5r_j/4$, with $|\zeta_j^{(n)}| \leq C_n r_j^{-n}$. Take $r_{j+1} < r_j/4$ and arrange their transition intervals between consecutive insertion neighborhoods. For example, use $r_j = \frac{3}{4} 2^{-k_j}$ with sufficiently separated large integers k_j . Set

$$(85) \quad f = W_1[a] + \sum_{j=2}^{\infty} \zeta_j(y) W_j[a].$$

For each $y \neq 0$ the sum is locally finite. The profile $V_0 + f$ is either a completed V_J or a transition $V_J + \zeta_{J+1} W_{J+1}$. Its order is constant throughout each insertion neighborhood. Since $\mathcal{R}(y) \ll r_j$ on the j th transition, derivatives of its cutoff cost at most the corresponding powers of $\mathcal{R}^{-1}$.

The transition obeys the residual estimate at the lower order J . Indeed, (82) bounds its added term by $C a^3 \mathcal{R}^{-2J-2} s^{2J}$, with differentiated bounds, and the curvature operator costs two derivatives at scale s . Thus (83) remains valid during the transition.

Here is the diagonal choice needed for admissibility. Fix L, m, N . Convert the axial factors $\mathcal{R}^{-n}$ in the input estimates to $C s^{-n}$, since $s \leq C \mathcal{R}$. After dividing the residual by s^{L-m} , or the matching error by s^{L+2-m} , both estimates reduce to

$$(86) \quad C_J a^3 \mathcal{R}^{-2J-2} s^{2J-2-L} \leq C_J \left(a^{2J+1-L} \mathcal{R}^{-2J-2} + a^3 \mathcal{R}^{-L-4} \right).$$

The two terms on the right are the endpoint bounds for $a \leq s \leq C \mathcal{R}$. If $2J+1 > L$, both are positive powers of a times fixed powers of $\mathcal{R}^{-1}$. By (65), they are bounded by $|y|^N$ sufficiently near zero.

Choose r_j successively so small that these bounds hold throughout $0 < |y| \leq 2r_j$ for $J = j-1, j$ and all $L, m, N \leq j$, including the finite number of additional derivatives needed for the curvature estimates. All relevant constants and thinness thresholds are included in this choice. Here $2(j-1)+1-L \geq j-1 > 0$ for $j \geq 2$. On a transverse collar, the leaf expansion and (82) give $|D^m(V_J - t)| \leq C_{J,m} a^3 \mathcal{R}^{-2-m}$; include the corresponding flatness requirements as well.

The same choice can ensure that each activated W_j , with its cutoff, has scaled C^m norm at most 2^{-j} for $m \leq j$. To see that this is available, divide (82) by s^{1-m-n} . For $j \geq 2$ the result is bounded by $C_{j,m,n} (a/\mathcal{R})^3$ on the tube. The fixed first correction is bounded in the same norm by $C(a/\mathcal{R})^2$. These quantities tend to zero. This gives the uniform geometric bounds (67) and preserves positivity after the working neighborhood is reduced.

For each fixed L, m, N , sufficiently small nonzero heights lie in bands whose completed orders exceed all these indices. The finitely many outer bands only change the constants. Equations (86) and (84) therefore give (68)–(70). The zero cap data and radial smoothness follow term by term. $\square$

APPENDIX A. HIGHER DERIVATIVE ESTIMATES FOR THE CONNECTOR

This appendix proves the analytic implications used in Proposition 5.1. The first step transfers the regular-sector quadratic asymptotics to the spine, determining the smoothing radius without assuming bounds for its derivatives. Local comparison with fixed leaves then identifies a uniform weighted class for the differentiated equation. We prove the axial derivative estimates by induction and only afterward use the radial equation to estimate the difference from the leaf with the same smoothing radius. This order avoids using derivatives of the smoothing radius before they have been controlled.

A.1. Quadratic phase bound near the spine. Here we improve the quadratic estimates of Proposition 5.2 to hold near the spine:

Lemma A.1 (Quadratic phase asymptotics near the spine). *Let $M = S(u_\infty)$ be the connector of Theorem 1.2, and assume the quadratic bound of Proposition 5.2. Let $A > 0$ be the quadratic coefficient in that proposition, and let $c_H > 0$ be determined by*

$$P(0, z, Y) = c_H |z|^3.$$

Then, as $|Y| \rightarrow \infty$,

$$a_\infty(Y) = \left(\frac{A}{c_H}\right)^{1/3} |Y|^{2/3} + O(|Y|^{2/3-\kappa_2}), \quad a_\infty(Y) = u_\infty(0, Y),$$

where one may take $\kappa_2 = 1/6$. Moreover, there is a fixed $\theta_0 > 0$ such that, for $T = |y_0|$ large and $R = \theta_0 T$, throughout the base ball $B_{4R}(0, y_0)$,

$$(87) \quad \left| P(t, u_\infty(t, y)) - A \left(y^2 - \frac{t^2 + u_\infty(t, y)^2}{3} \right) \right| \leq CT^{11/6}, \quad cT^2 \leq P(t, u_\infty(t, y)) \leq CT^2.$$

These constants are independent of y_0 .

Proof. By reflection symmetry, it suffices to consider $Y \rightarrow +\infty$. For R large, set

$$M_R = R^{-1}M, \quad \varepsilon = R^{-1}, \quad \Pi_R(X) = R^{-2}P(RX) \quad (X \in M_R).$$

The cubic homogeneity of P gives

$$\Pi_R(X) = R P(X) = \varepsilon^{-1} P(X).$$

Thus Π_R is the physical phase divided by R^2 , pulled back to the rescaled surface; the actual phase on M_R is $\varepsilon \Pi_R$. In what follows, (r, y) are rescaled coordinates, and the annulus under consideration has fixed size.

We first establish the regular-region estimate

$$(88) \quad \left| \Pi_R - A \left(y^2 - \frac{r^2}{3} \right) \right| \leq C \frac{R^{-\kappa}}{r}, \quad \frac{1}{2} < \sqrt{r^2 + y^2} < 2, \quad r > C_* R^{-1/3},$$

where we take $\kappa = 1/4$ and C_* is sufficiently large.

To see this, put

$$d_R = D(8R), \quad \varepsilon_R = d_R/R^3.$$

The quadratic bound gives $d_R \leq CR^2$. Multiplying (43) by d_R , the error between the physical conjugate $s^2 f(s, Y)$ and its harmonic replacement, at $(s, Y) = (Rr, Ry)$, is bounded by

$$C \frac{d_R \varepsilon_R^{1/3}}{r} = C \frac{d_R^{4/3}}{Rr} \leq C \frac{R^{5/3}}{r}.$$

Repeat the dyadic patching construction in the proof of Proposition 5.2, now using this quadratic bound. The overlap errors are $O(R^{5/3})$, so the resulting function $\widehat{W}$ on the exterior auxiliary space satisfies

$$|\widehat{W}| \leq C\rho^2, \quad |\Delta_{\xi, Y} \widehat{W}| \leq C\rho^{-1/3}, \quad \rho = (s^2 + Y^2)^{1/2}.$$

The same Newton-potential construction gives a correction $v = O(\rho^{5/3})$ for which $\widehat{W} - v$ is harmonic. Its invariant, reflection-even harmonic expansion therefore gives

$$\widehat{W}(s, Y) = A \left(Y^2 - \frac{s^2}{3} \right) + O(\rho^{5/3}).$$

The coefficient agrees with the previously identified A by comparison on any fixed regular angular sector.

The fixed-leaf expansion [34, Proposition 3.3] gives

$$f_{\text{leaf}}(s) = \lambda s^{-2} + O(|\lambda|^{4/3} s^{-3}).$$

Consequently, at points of the normal graph with $s > Cd_R^{1/3}$,

$$|P - s^2 f| \leq C \frac{d_R^{4/3}}{s}.$$

Combining these estimates and dividing by R^2 yields

$$\Pi_R = A \left(y^2 - \frac{r^2}{3} \right) + O \left(\frac{R^{-1/3}}{r} \right),$$

which implies (88). Here the computation initially uses the cone radius. It also holds with $r = |(x, z)|$ on M_R : the squared ambient and cone radii differ by f_R^2 , where $|f_R| \leq CR^{-1}r^{-2}$, and this difference is absorbed by the displayed error.

We next transfer this estimate near the spine. Fix

$$I = (2/3, 4/3), \quad J = [3/4, 5/4], \quad r_0 = R^{-\kappa/3}.$$

On the lateral boundary $r = r_0$, (88) gives

$$(89) \quad |\Pi_R - Ay^2| \leq C \left(r_0^2 + \frac{R^{-\kappa}}{r_0} \right).$$

Moreover, the quadratic bound implies $|\Pi_R| \leq K$ on $M_R \cap \{r < r_0, y \in I\}$, for a fixed K .

Let H_{ph} be the positive Hardt-Simon leaf normalized by $P = 1$. For a uniformly bounded C^2 family of functions $q : I \rightarrow \mathbb{R}$, the barriers of [34, Proposition 5.8] are normal graphs over the slice-wise leaves

$$\varepsilon^{1/3} q(y)^3 H_{\text{ph}}, \quad P|_{\varepsilon^{1/3} q(y)^3 H_{\text{ph}}} = \varepsilon q(y)^9.$$

Negative dilations here mean positive dilations with the x and z factors interchanged. These barriers have negative mean curvature; interchanging the factors also provides the barriers with the opposite sign.

We record the quantitative phase error. The normal correction ψ_q in that construction satisfies

$$|\psi_q| \leq C\varepsilon(|q|^3 + r), \quad r \geq c\varepsilon^{1/3}|q|^3.$$

Hence $|\psi_q|/r \leq C(\varepsilon^{2/3} + \varepsilon)$. Using $|DP| \leq Cr^2$, we obtain, on $r < r_0$,

$$(90) \quad \varepsilon^{-1}|P - \varepsilon q^9| \leq Cr_0^2.$$

This is the quantitative form of the phase comparison used in [34, Proposition 5.10]; it follows directly from the correction estimates in its proof.

Choose

$$\delta = L \left(r_0^2 + \frac{R^{-\kappa}}{r_0} \right),$$

where L is a sufficiently large fixed constant. Let $\chi \in C^\infty(I; [0, 1])$ vanish on J and equal one near the endpoints of I , and define

$$q_+(y) = (Ay^2 + \delta)^{1/9} + B\chi(y), \quad q_-(y) = (Ay^2 - \delta)^{1/9} - B\chi(y).$$

For large R , the expressions inside the roots are uniformly positive. Choose the fixed constant B so large that, near the endpoints of I ,

$$q_+^9 > K + 2, \quad q_-^9 < -K - 2.$$

These functions have uniformly bounded C^2 norms and satisfy

$$q_+^9 \geq Ay^2 + \delta, \quad q_-^9 \leq Ay^2 - \delta \quad \text{on } I,$$

with equality on J .

Apply the upper barriers to $q_+ + s$ and the lower barriers to $q_- - s$. Start with a sufficiently large fixed s , so that the barriers lie beyond the global phase range, and decrease s to zero. The axial boundary comparisons follow from the choice of B . The lateral comparisons follow from (89), (90), and the choice of L . Interior first contact is excluded by [34, Proposition 5.8], including at singular points of the barriers. Thus the comparison persists to $s = 0$.

On the smaller band J , the resulting phase bounds are

$$|\Pi_R - Ay^2| \leq C\delta \quad \text{on } M_R \cap \{r < r_0, y \in J\}.$$

The axis orbit at height $y = 1$ belongs to this region for large R : indeed, the quadratic phase bound gives $a_\infty(R)/R \leq CR^{-1/3} \ll r_0$. Evaluating there gives

$$\left| \frac{c_H a_\infty(R)^3}{R^2} - A \right| \leq C\delta.$$

Finally, since $\kappa = 1/4$,

$$r_0 = R^{-1/12}, \quad \delta = O(R^{-1/6}).$$

Therefore

$$c_H a_\infty(R)^3 = AR^2 + O(R^{2-1/6}).$$

Taking cube roots, using $A > 0$, proves the claimed expansion with $\kappa_2 = 1/6$.

We also record the uniform band conclusion. Apply the preceding rescaling at $T = y_0$. Choose θ_0 so small that the graph over $B_{4\theta_0 T}(0, y_0)$ lies in the rescaled annulus and has $y \in J$. This follows from $|Du_\infty| \leq \Lambda$ and $a_\infty(y_0)/T = O(T^{-1/3})$. In the shrinking tube $r < r_0 = T^{-1/12}$, the barrier comparison gives $|\Pi_T - Ay^2| \leq CT^{-1/6}$, and the omitted term $Ar^2/3$ has the same bound. Outside that tube, (88) gives an error at most $CT^{-1/4}/r_0 = CT^{-1/6}$. Here $r^2 = T^{-2}(t^2 + u_\infty^2)$ in the physical graph coordinates. Multiplying by T^2 proves the first assertion of (87). The quadratic polynomial is uniformly positive on the chosen band, so the second assertion follows for large T . Reflection gives all conclusions for negative y_0 . $\square$

A.2. Local comparison with fixed leaves. We now work directly with the minimal graph $u = u_\infty$. The quadratic asymptotics first give local comparison estimates; these will control the coefficients of its differentiated equation. Throughout this subsection and the next, set

$$T = |y_0|, \quad R = \theta_0 T, \quad \beta = u(0, y_0), \quad X_0 = (0, y_0), \quad s(X) = \beta + |x|.$$

Choose the fixed $\theta_0 > 0$ sufficiently small and then T sufficiently large. In particular $\beta^3 \approx T^2$ and $\beta < R/100$. Cartesian derivatives are denoted by D .

Lemma A.2 (Fixed-leaf comparison and Jacobi normalization). *On $B_{4R}(X_0)$ the leaf labels satisfy*

$$c\beta^3 \leq c_H^{-1}P(t, u(t, y)) \leq C\beta^3.$$

For $X_* = (x_*, y_*) \in B_{2R}(X_0)$ put $s_* = \beta + |x_*|$, $\lambda_* = c_H^{-1}P(t_*, u(X_*))$, and $u_*(t) = H(t, \lambda_*)$. There is a fixed $c > 0$ such that

$$(91) \quad |D^k(u - u_*)| \leq C_k \beta^3 s_*^{-2-k} (s_*/R)^{1/2} \quad \text{on } B_{cs_*}(X_*), \quad k \geq 0.$$

Define, by evaluation of fixed functions at u ,

$$(92) \quad d(X) = H_\lambda(t, c_H^{-1}P(t, u(X))), \quad W = \sqrt{1 + |Du|^2}, \quad B = \frac{I}{W} - \frac{Du \otimes Du}{W^3}.$$

Then $d \approx s^{-2}$ and $|D^k d| \leq C_k s^{-2-k}$. For the vertical Jacobi operator and its conjugate, set

$$(93) \quad \begin{aligned} \mathcal{J}\phi &= -\operatorname{div}(u^3 B D\phi) - \frac{3u}{W}\phi, \\ \mathcal{A} &= u^3 d^2 B, \quad \mathcal{V} = d\mathcal{J}d, \quad \mathcal{L}q = -\operatorname{div}(\mathcal{A}Dq) + \mathcal{V}q. \end{aligned}$$

With $\omega_\beta = (\beta^2 + t^2)^{-1/2}$, these coefficients satisfy

$$(94) \quad \begin{aligned} c\omega_\beta I &\leq \mathcal{A} \leq C\omega_\beta I, \quad |D^k \mathcal{A}| \leq C_k \omega_\beta s^{-k}, \\ |D^k \mathcal{V}| &\leq C_k \omega_\beta s^{-2-k} (s/R)^{1/2} \quad \text{on } B_{2R}(X_0). \end{aligned}$$

All constants are independent of β/R .

Proof. Lemma A.1 gives the positive leaf band and the uniform estimate

$$\left| P(t, u) - A\left(y^2 - \frac{t^2 + u^2}{3}\right) \right| \leq CT^{11/6}.$$

The polynomial evaluated on the graph is Lipschitz with constant CT on $B_{4R}(X_0)$, because $|Du| \leq \Lambda$ and $u \leq C(\beta + t)$. Its variation on a ball of radius cs_* is therefore at most CTs_* . Since $\beta^3 \approx T^2$,

$$\frac{\operatorname{osc}_{B_{2cs_*}(X_*)} P(t, u)}{\beta^3} \leq C \left(\frac{s_*}{R} + T^{-1/6} \right) \leq C(s_*/R)^{1/2}.$$

The last inequality uses $s_* \geq \beta \approx T^{2/3}$ and $s_*/R \leq C$. This is a consequence of the quantitative geometric comparison, without differentiating a leaf-coordinate equation.

For $\lambda \approx \beta^3$, the fixed-leaf bounds give

$$(95) \quad H \approx \beta + t, \quad H_\lambda \approx (\beta + t)^{-2}, \quad |\partial_t^k \partial_\lambda^r H| \leq C_{k,r} \beta^{3-3r} (\beta + t)^{-2-k}, \quad r \geq 1.$$

The mean-value theorem gives the $k = 0$ case of (91). After rescaling lengths and graph heights by s_* , both u and u_* have uniformly positive height and bounded gradient. Standard interior regularity for the SME bounds their coefficient derivatives, and their difference satisfies a linear uniformly elliptic equation with uniformly controlled lower-order coefficients on this scale. Interior estimates for that difference prove (91) for all k .

For clarity, the required differentiation of the fixed functions in (92) can be checked using the implicit identity $u = H(t, c_H^{-1} P(t, u))$ on this local ball. Inverting its parameter derivative and using (95) gives, with $d_*(t) = H_\lambda(t, \lambda_*)$,

$$|D^k(d - d_*)| \leq C_k s_*^{-2-k} (s_*/R)^{1/2}.$$

This also gives the asserted bounds on d and $\mathcal{A}$. The field d_* is an exact Jacobi field on the fixed leaf u_* , so $\mathcal{J}_* d_* = 0$. Subtract this identity from $\mathcal{J}d$ on the local ball. Coefficients of the principal part of $\mathcal{J}$ have size s_*^3 , and its potential has size s_* . Their differences from the fixed-leaf coefficients retain a factor $(s_*/R)^{1/2}$. Together with the displayed bound for $d - d_*$, this gives

$$|D^k(\mathcal{J}d)| \leq C_k s_*^{-1-k} (s_*/R)^{1/2}.$$

Multiplication by $d \approx s_*^{-2}$ proves the bound on $\mathcal{V}$ in (94). $\square$

A.3. Weighted estimates for the differentiated equation. The Euler–Lagrange expression for the area density $u^3 \sqrt{1 + |Du|^2}$ is $-u^3 Q(u)/W$. Its linearization at a solution is the operator $\mathcal{J}$ in (93). Thus

$$(96) \quad \mathcal{J}u_y = 0, \quad \mathcal{J}\phi = -\frac{u^3}{W} DQ_u[\phi], \quad d\mathcal{J}(dq) = \mathcal{L}q.$$

These identities explain both the direct differentiation and the Jacobi normalization. In particular, the remaining potential $\mathcal{V}$ is controlled by the local difference from a fixed leaf.

We recall the precise weighted results used below. For a positive weight ω on $\mathbb{R}^5$, its Muckenhoupt characteristic is

$$(97) \quad [\omega]_{A_2} := \sup_{\mathbb{B}} \left(\frac{1}{|\mathbb{B}|} \int_{\mathbb{B}} \omega dX \right) \left(\frac{1}{|\mathbb{B}|} \int_{\mathbb{B}} \omega^{-1} dX \right),$$

where the supremum is over Euclidean balls. Membership in A_2 means that this quantity is finite. With $d\mu = \omega dX$, [13, Theorems 1.2 and 1.5] give, for some $\kappa > 1$,

$$\left(\frac{1}{\mu(\mathbb{B})} \int_{\mathbb{B}} |\psi|^{2\kappa} d\mu \right)^{1/(2\kappa)} \leq C\rho \left(\frac{1}{\mu(\mathbb{B})} \int_{\mathbb{B}} |D\psi|^2 d\mu \right)^{1/2}, \quad \psi \in C_c^\infty(\mathbb{B}),$$

$$\int_{\mathbb{B}} |\psi - \psi_{\mathbb{B},\mu}|^2 d\mu \leq C\rho^2 \int_{\mathbb{B}} |D\psi|^2 d\mu, \quad \psi \in C^1(\overline{\mathbb{B}}),$$

where ρ is the ball radius and $\psi_{\mathbb{B},\mu}$ is the weighted average. The constants depend only on dimension and $[\omega]_{A_2}$; the measure is also doubling [12, equation (1.2)]. For a measurable symmetric matrix $\mathcal{A}$ with $c\omega I \leq \mathcal{A} \leq C\omega I$, a weak solution $h \in H_{\text{loc}}^1(d\mu)$ of $-\text{div}(\mathcal{A}Dh) = 0$ in $\mathbb{B}$ satisfies

$$(98) \quad \|h\|_{L^\infty(\mathbb{B}/2)} \leq C \left(\frac{1}{\mu(\mathbb{B})} \int_{\mathbb{B}} |h|^2 d\mu \right)^{1/2}.$$

This is [13, Theorem 2.3.1] applied to h and $-h$; see also [12, equations (1.5)–(1.7)]. Its constant depends only on the same data and the weighted ellipticity constants. We use this local boundedness result; higher derivatives will follow by differentiating the SME and applying ordinary interior estimates on the local scale s .

For the present weight $\omega_\beta = (\beta^2 + |x|^2)^{-1/2}$, the A_2 bound is independent of $\beta > 0$. Indeed, for a ball $\mathbb{B} = B_\rho(X)$ put $\ell = (\beta^2 + |x|^2)^{1/2}$. If $\rho \leq \ell/2$, the weight is comparable to ℓ^{-1} throughout the ball. If $\rho > \ell/2$, then $|x| < 2\rho$, $\omega_\beta^{-1} \leq 3\rho$ on $\mathbb{B}$, and

$$\int_{\mathbb{B}} \omega_\beta dX \leq 2\rho \int_{|\xi| < 3\rho, \xi \in \mathbb{R}^4} |\xi|^{-1} d\xi \leq C\rho^4.$$

Since $|\mathbb{B}| \approx \rho^5$, the product in (97) is uniformly bounded in both cases. Direct integration also gives, now with $d\mu = \omega_\beta dX$,

$$(99) \quad \mu(B_\rho(X)) \approx \frac{\rho^5}{\beta + |x| + \rho}.$$

Thus all the preceding constants are uniform as $\beta/R \rightarrow 0$. Lemma A.2 puts the actual matrix $\mathcal{A} = u^3 d^2 B$ in this weighted ellipticity class, using the positive quadratic phase band to obtain $u \approx s$ and $d \approx s^{-2}$. Multiplication of the weight by a function bounded above and below by fixed positive constants preserves its uniform A_2 bound.

Equation $\mathcal{L}q = F$ still contains $\mathcal{V}q$ and F . To use (98), we remove the potential by a bounded positive solution of $\mathcal{L}\vartheta = 0$ and then subtract the Green potential of the source. The Green-function bounds and representation are supplied by [12, Theorem 3.3 and Lemma 2.5]; the next proof checks the uniform bounds needed for these two reductions.

Lemma A.3 (Uniform weighted local estimate). *Let $d\mu = \omega_\beta dX$, and let $\mathcal{L}$ be as in Lemma A.2. Suppose on an interior ball $B \subset B_{2R}(X_0)$ that*

$$\mathcal{L}q = F, \quad |F| \leq M\omega_\beta s^{-2}(\beta/s)^3.$$

If $B' \Subset B$ have radii and separation comparable to R , then

$$(100) \quad \|q\|_{L^\infty(B')} \leq C \left[\left(\frac{1}{\mu(B)} \int_B q^2 d\mu \right)^{1/2} + M \right].$$

The constant may depend on the fixed ball ratios, but is independent of β/R .

Proof. The A_2 theory applies to the principal part of $\mathcal{L}$. We first estimate the Green potentials of the zeroth-order term and the source, then reduce to the homogeneous equation in (98).

Let D be a ball of radius ρ in the coefficient domain. The Dirichlet Green kernel G_D of $-\text{div}(\mathcal{A}D)$ is normalized with respect to Lebesgue measure. Extend $\mathcal{A}$ by $\omega_\beta I$ outside D and consider its Green function on the concentric ball of radius 8ρ . Points of D lie in the inner quarter of that ball, so [12, Theorem 3.3] applies. Domain monotonicity and (99) give

$$(101) \quad G_D(X, Z) \leq C \int_{|X-Z|}^{8\rho} \frac{r}{\mu(B_r(X))} dr \leq C \frac{s(X) + |X - Z|}{|X - Z|^3}.$$

The same conclusion holds for any coefficients uniformly comparable to $\omega_\beta I$.

Two different integrability gains will be used. For the potential, (94) and the kernel bound imply

$$(102) \quad \sup_{X \in D} \int_D G_D(X, Z) |\mathcal{V}(Z)| dZ \leq C(\rho/R)^{1/2}.$$

Indeed, at distances $r \leq s(X)/2$ the contribution is bounded by $CR^{-1/2}s(X)^{-3/2}r^2 \leq C(r/R)^{1/2}$. On a dyadic annulus with radius $r \geq s(X)/2$, the kernel is at most Cr^{-2} , while

$$\int_{B_r(X)} \omega_\beta s^{-3/2} dZ \leq Cr \int_0^{Cr} t^3 (\beta + t)^{-5/2} dt \leq Cr^{5/2}.$$

Summing gives (102).

For the source, the separate radial factor gives

$$(103) \quad \sup_{X \in D} \int_D G_D(X, Z) \omega_\beta(Z) s(Z)^{-2} (\beta/s(Z))^3 dZ \leq C.$$

For $r \leq s(X)/2$, its contribution is bounded by $C\beta^3 s(X)^{-5}r^2 \leq C$. For $r \geq s(X)/2$, the source mass obeys

$$\int_{B_r(X)} \omega_\beta s^{-2} (\beta/s)^3 dZ \leq Cr\beta^3 \int_0^{Cr} \frac{t^3}{(\beta + t)^6} dt \leq C\beta r.$$

The dyadic contribution is consequently at most $C\beta/r$, whose sum is uniformly bounded because $s(X) \geq \beta$. This source need not be small at the axis.

Work first on a ball D of radius εR . Choose a fixed $\varepsilon > 0$ so that (102) is at most $1/4$. The convergent Green-operator series solves

$$\vartheta = 1 - G_D(\mathcal{V}\vartheta), \quad \frac{1}{2} \leq \vartheta \leq 2, \quad \mathcal{L}\vartheta = 0.$$

Writing $z = q/\vartheta$ gives

$$-\operatorname{div}(\vartheta^2 \mathcal{A} D z) = \vartheta F.$$

The matrix $\vartheta^2 \mathcal{A}$ is still symmetric and uniformly comparable to $\omega_\beta I$, so its Green kernel satisfies (101) with the same uniform constants. Subtract its Dirichlet Green potential of ϑF . The supremum of this potential is at most CM by (103). For each fixed $\beta > 0$, the density $\vartheta F/\omega_\beta$ is bounded; thus the representation follows from [12, Lemma 2.5]. The remaining function solves a homogeneous weighted divergence equation, and (98) yields

$$\|q\|_{L^\infty(D/2)} \leq C \left[\left(\frac{1}{\mu(D)} \int_D q^2 d\mu \right)^{1/2} + M \right].$$

A finite covering of B' by such balls, and doubling to compare their averages with the average on B , proves (100). $\square$

Proposition A.4 (Direct axial estimates for the minimal graph). *In the band of Lemma A.2, for every fixed $N \geq 1$ and every fixed $k \geq 0$,*

$$(104) \quad |D^k \partial_y^j u(X)| \leq C_{N,k} \beta^3 R^{-j} (\beta + t)^{-2-k}, \quad 1 \leq j \leq N, \quad X \in B_{R/4}(X_0).$$

In particular, $|a_\infty^{(j)}(y)| \leq C_j a_\infty(y) R^{-j}$ on the smaller axial interval. The band and the lower threshold for $|y_0|$ are independent of N .

Proof. Fix N and use finitely many nested concentric balls between $B_R(X_0)$ and $B_{R/4}(X_0)$. Their separations may depend on N . Let

$$u_j = \partial_y^j u, \quad q_j = \frac{u_j}{\beta^3 d}, \quad j \geq 1,$$

where β is fixed. These are normalized derivatives of the original graph. We prove

$$(105) \quad \|q_j\|_\infty \leq C_j R^{-j}, \quad |D^k u_j| \leq C_{j,k} \beta^3 R^{-j} s^{-2-k}, \quad k \geq 0,$$

on successively smaller balls. No axial derivative estimate for a_∞ is assumed.

For $j = 1$, (96) gives $\mathcal{L}q_1 = 0$. The local comparison (91), whose comparison leaf has zero axial derivative, gives $|q_1| \leq Cs^{-1}$. Since $\mu(B_R) \approx R^4$,

$$\int_{B_R} q_1^2 d\mu \leq CR \int_0^R \frac{t^3}{(\beta + t)^3} dt \leq CR^2 \leq CR^{-2} \mu(B_R).$$

Lemma A.3, with zero forcing, therefore gives $\|q_1\|_\infty \leq C/R$ on a smaller ball. The local elliptic estimate described below supplies the mixed derivatives and completes this first step.

We record the sources at higher orders. In Cartesian coordinates,

$$Q(u) = A^{ij}(Du)u_{ij} - 3/u, \quad A^{ij}(\xi) = \delta_{ij} - \frac{\xi_i \xi_j}{1 + |\xi|^2}.$$

Differentiating j times and separating the complete linearization gives

$$(106) \quad DQ_u[u_j] = -T_j, \quad T_1 = 0.$$

Every term of T_j contains at least two factors with positive axial order, all strictly less than j . Apart from fixed contractions and constants, its principal terms are

$$(\partial_\xi^b A^{ij})(Du) \prod_{\nu=1}^b Du_{r_\nu} D_{ij} u_q, \quad r_\nu \geq 1, \quad \sum r_\nu + q = j,$$

with the linear terms $(b, q) = (0, j)$ and $(b, r_1, q) = (1, j, 0)$ removed. Its remaining terms come from differentiating $-3/u$ and have the form

$$Cu^{-1-b} \prod_{\nu=1}^b u_{r_\nu}, \quad b \geq 2, \quad r_\nu \geq 1, \quad \sum r_\nu = j.$$

The local bounds for the undifferentiated solution are $|D^k u| \leq C_k s^{1-k}$, and $|Du| \leq \Lambda$. Assuming (105) at lower axial orders, each differentiated gradient costs $\beta^3 R^{-r} s^{-3}$, and each differentiated Hessian costs $\beta^3 R^{-q} s^{-4}$. In a principal term with $q = 0$ there are at least two gradient factors and $|D^2 u| \leq C/s$. The reciprocal terms have the same bound. Since $\beta/s \leq 1$, the product rule therefore gives

$$(107) \quad |D^k T_j| \leq C_{j,k} \beta^6 R^{-j} s^{-7-k}, \quad j \geq 2.$$

All fixed local Cartesian derivative orders are retained in the induction, so the extra derivatives in these products are already available.

By (96), the normalized equation is

$$(108) \quad \mathcal{L}q_j = F_j, \quad F_j = \frac{du^3}{\beta^3 W} T_j, \quad |D^k F_j| \leq C_{j,k} \omega_\beta R^{-j} s^{-2-k} (\beta/s)^3.$$

The factor $(\beta/s)^3$ is the gain that permits direct differentiation of u at every order. It gives a uniformly bounded source potential by (103).

To obtain the needed L^2 bound at order $j \geq 2$, test the equation for q_{j-1} with $\zeta^2 q_{j-1}$, where $|D\zeta| \leq C_N/R$. Ellipticity and Young's inequality give

$$\begin{aligned} \int \zeta^2 |Dq_{j-1}|^2 d\mu &\leq CR^{-2} \int_B q_{j-1}^2 d\mu + C \|q_{j-1}\|_\infty^2 \int_B |\mathcal{V}| dX \\ &\quad + C \|q_{j-1}\|_\infty \int_B |F_{j-1}| dX. \end{aligned}$$

The ball B is centered at X_0 and has radius comparable to R . The radial integrations in the preceding lemma give

$$\int_B |\mathcal{V}| dX \leq CR^{-2} \mu(B), \quad \int_B \omega_\beta s^{-2} (\beta/s)^3 dX \leq C\beta R \leq CR^{-2} \mu(B).$$

Together with the induction hypothesis and (108), this proves

$$\int \zeta^2 |Dq_{j-1}|^2 d\mu \leq CR^{-2j} \mu(B).$$

The normalization contributes one additional, controlled term:

$$q_j = \partial_y q_{j-1} + \frac{d_y}{d} q_{j-1}, \quad \frac{d_y}{d} = \beta^3 \frac{H_{\lambda\lambda}}{H_\lambda} q_1.$$

Here the fixed-profile derivatives are evaluated at the leaf through $u(X)$. By (95) and the first step, $|d_y/d| \leq C/R$. Thus, on the next smaller ball B_1 ,

$$\left(\frac{1}{\mu(B_1)} \int_{B_1} q_j^2 d\mu \right)^{1/2} \leq CR^{-j}.$$

Applying Lemma A.3 to (108), with $M = CR^{-j}$, gives $\|q_j\|_\infty \leq CR^{-j}$.

At each step, recover the mixed derivatives on local balls of radius comparable to $s = \beta + t$. Divide the normalized equation by ω_β and rescale by s . Its principal part is uniformly elliptic, and Lemma A.2 bounds all coefficient derivatives on that scale. Equation (108) bounds the rescaled source and its derivatives by $C_{j,k} R^{-j}$. Standard interior estimates therefore give

$$|D^k q_j| \leq C_{j,k} R^{-j} s^{-k}.$$

Multiplying by $\beta^3 d$ proves the second estimate in (105). Near the boundary of an intermediate ball, its radius can be reduced by a factor depending on N ; this changes only the constants. The induction is now complete.

Restricting to $B_{R/4}(X_0)$ proves (104). At $x = 0$, $a_\infty^{(j)}(y) = u_j(0, y)$, so the smoothing-radius estimates follow from $\beta \approx a_\infty(y)$. The geometric threshold was used only in the initial comparison lemma. Higher orders change constants and intermediate ball separations, not that threshold. $\square$

A.4. Expansion around Hardt–Simon leaves. The direct axial estimates now control both the smoothing radius $a_\infty(y) = u_\infty(0, y)$ and the axial terms in the SME. We compare each slice with the leaf having that smoothing radius:

$$u_\infty(t, y) = a_\infty(y)h(t/a_\infty(y)) + e_\infty(t, y), \quad e_\infty(0, y) = \partial_t e_\infty(0, y) = 0.$$

Moving the axial terms of the SME to the right-hand side gives a forced radial equation. Each term contains two axial derivatives or a product of two first axial derivatives, hence a factor R^{-2} . The following proposition converts that forcing estimate into the remainder bounds required for gluing.

Proposition A.5 (Expansion about Hardt–Simon leaves). *Let $N \geq 1$ be an arbitrary integer. Fix $|y_0|$ sufficiently large and set*

$$\beta := u_\infty(0, y_0), \quad R := \theta_0 |y_0|,$$

where $\theta_0 > 0$ is the fixed constant chosen in Lemma A.2. Thus β is the smoothing radius at the reference height y_0 and is held fixed throughout the band. Assume the direct estimates of Proposition A.4 through axial order $N + 2$. There is a fixed $c > 0$, independent of N , such that, for $|y - y_0| \leq cR$, the varying smoothing radius

$$a_\infty(y) := u_\infty(0, y)$$

satisfies

$$|a_\infty^{(j)}(y)| \leq C_j a_\infty(y) R^{-j}, \quad 1 \leq j \leq N + 2.$$

For $0 \leq t \leq cR$ and $|y - y_0| \leq cR$, one has

$$(109) \quad u_\infty(t, y) = a_\infty(y)h(t/a_\infty(y)) + e_\infty(t, y),$$

where

$$(110) \quad |\partial_t^m \partial_y^n e_\infty(t, y)| \leq C_{mn} a_\infty(y)^3 R^{-2-n} (a_\infty(y) + t)^{-m}, \quad m + n \leq N.$$

The constants may depend on N and the indicated derivative orders, but are independent of β/R . The geometric constant c and the lower threshold for $|y_0|$ can be chosen independently of N .

Proof. Choose $c > 0$ so that the enlarged rectangle

$$0 \leq t \leq 2cR, \quad |y - y_0| \leq 2cR$$

lies inside $B_{R/4}(X_0)$, where the direct derivative estimates hold. Abbreviate $u = u_\infty$, $a = a_\infty$, and $e = e_\infty$.

1. *Smoothing radius and the forced radial equation.* Evaluating Proposition A.4 at $t = 0$, where $a(y) = u(0, y)$, gives

$$a \approx \beta, \quad |a^{(j)}| \leq C_j \beta R^{-j}, \quad 1 \leq j \leq N + 2.$$

The local fixed-leaf comparison of Lemma A.2 and the direct derivative estimates of Proposition A.4 give

$$(111) \quad \begin{aligned} |\partial_t^m(u - t)| &\leq C_m \beta^3 (\beta + t)^{-2-m}, \\ |\partial_t^m \partial_y^j u| &\leq C_{mj} \beta^3 R^{-j} (\beta + t)^{-2-m}, \quad 1 \leq j \leq N + 2, \end{aligned}$$

for every fixed transverse derivative order $m \geq 0$. The same estimates hold for $u_a(t, y) = a(y)h(t/a(y))$. Write $D_u = 1 + u_t^2 + u_y^2$. Rearranging the SME (30) gives

$$(112) \quad \frac{u_{tt}}{1 + u_t^2} + \frac{3}{t}u_t - \frac{3}{u} = F,$$

where

$$F = -\frac{u_t^2 u_y^2}{(1 + u_t^2)D_u} u_{tt} + \frac{2u_t u_y}{D_u} u_{ty} - \frac{1 + u_t^2}{D_u} u_{yy}.$$

In particular,

$$|F| \leq C(|u_y|^2 |u_{tt}| + |u_y u_{ty}| + |u_{yy}|) \leq C\beta^3 R^{-2}(\beta + t)^{-2}.$$

Differentiating this identity and using (111) yields

$$(113) \quad |\partial_t^m \partial_y^n F| \leq C_{mn} \beta^3 R^{-2-n} (\beta + t)^{-2-m}, \quad 0 \leq n \leq N,$$

for every fixed $m \geq 0$. At most $n + 2 \leq N + 2$ axial derivatives of u are needed, because the forcing contains at most two axial derivatives of u . The direct mixed estimates already control every fixed transverse order, so increasing m requires no further axial derivatives.

2. *The equation for the remainder.* Define $e = u - u_a$. Since u_a solves the radial Hardt-Simon equation, subtraction gives

$$(114) \quad \mathbf{a} e_{tt} + \mathbf{b} e_t + \mathbf{c} e = F, \quad e(0, y) = e_t(0, y) = 0.$$

More explicitly, with $v_\theta = u_a + \theta e$,

$$\begin{aligned} \mathbf{a} &= \int_0^1 \frac{d\theta}{1 + (v_\theta)_t^2}, \\ \mathbf{b} &= \frac{3}{t} - \int_0^1 \frac{2(v_\theta)_t (v_\theta)_{tt}}{(1 + (v_\theta)_t^2)^2} d\theta, \\ \mathbf{c} &= \int_0^1 \frac{3}{v_\theta^2} d\theta. \end{aligned}$$

The bounds already established for u and u_a imply $\mathbf{a} \geq c_* > 0$ and

$$(115) \quad \begin{aligned} |\partial_t^m \partial_y^j \mathbf{a}| + (\beta + t) |\partial_t^m \partial_y^j (\mathbf{b} - 3/t)| \\ + (\beta + t)^2 |\partial_t^m \partial_y^j \mathbf{c}| \leq C_{mj} R^{-j} (\beta + t)^{-m}, \quad 0 \leq j \leq N. \end{aligned}$$

At the axis, $\mathbf{a}(0, y) = 1$, $\mathbf{b} - 3/t$ is smooth and odd in t , and $\mathbf{a}, \mathbf{c}$ are smooth and even. All equations at $t = 0$ are understood through this smooth radial extension.

For $t \geq T\beta$, with T a sufficiently large fixed constant, (111) gives

$$(116) \quad \mathbf{a} = \frac{1}{2} + O(\beta^3 t^{-3}), \quad \mathbf{b} = \frac{3}{t} + O(\beta^3 t^{-4}), \quad \mathbf{c} = \frac{3}{t^2} + O(\beta^3 t^{-5}).$$

These estimates hold with their differentiated versions. Since $\beta/R \rightarrow 0$ as $|y_0| \rightarrow \infty$, we may take $|y_0|$ sufficiently large that $2T\beta < cR$.

3. *A uniform estimate for the radial equation.* For the fixed coefficients above, consider a smooth even solution

$$\mathbf{a}z_{tt} + \mathbf{b}z_t + \mathbf{c}z = g, \quad z(0) = z_t(0) = 0,$$

at a fixed value of y . We claim that

$$(117) \quad |\partial_t^m g| \leq K_m M(\beta + t)^{-2-m} \implies |\partial_t^m z| \leq C_m M(\beta + t)^{-m}$$

for every fixed $m \geq 0$. The constants depend on finitely many K_m and coefficient bounds, but not on β/R .

First consider the smoothing region. Set $r = t/\beta$ and $Z(r) = z(\beta r)$. After division by the leading coefficient, the equation becomes

$$Z'' + \left(\frac{3}{r} + b_1(r)\right) Z' + c_1(r)Z = g_1(r), \quad Z(0) = Z'(0) = 0.$$

On $0 \leq r \leq 2T$, the coefficients have uniform smooth bounds, b_1 is odd with $|b_1(r)| \leq Cr$, and c_1, g_1 are even, with $|g_1| \leq CM$. Writing

$$J(r) = r^3 \exp\left(\int_0^r b_1(s) ds\right),$$

we obtain

$$Z'(r) = J(r)^{-1} \int_0^r J(s)(g_1(s) - c_1(s)Z(s)) ds.$$

Since $J(r) \approx r^3$ on this fixed interval, integration and Gronwall's inequality give

$$|Z| + |Z'| \leq CM.$$

The equation is a smooth radial elliptic equation in $\mathbb{R}^4$: the coefficient $b_1(r)/r$ extends smoothly across the origin. Interior elliptic estimates therefore give $|\partial_r^m Z| \leq C_m M$ on $0 \leq r \leq T$. This proves (117) in the smoothing region.

For $t \geq T\beta$, put $\tau = \log(t/\beta)$ and $\mathcal{Z} = (z, \partial_\tau z)$. By (116), the equation becomes

$$\partial_\tau \mathcal{Z} = \left[\begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix} + O(e^{-3\tau}) \right] \mathcal{Z} + \mathcal{G}, \quad |\mathcal{G}| \leq CM.$$

The constant matrix has eigenvalues -2 and -3 . Since the perturbation is integrable in τ , its evolution operator satisfies

$$|\Phi(\tau, \sigma)| \leq Ce^{-2(\tau-\sigma)}, \quad \tau \geq \sigma \geq \log T.$$

Variation of constants, together with the estimate in the smoothing region at $t = T\beta$, gives

$$|\mathcal{Z}(\tau)| \leq Ce^{-2(\tau-\log T)} |\mathcal{Z}(\log T)| + CM \int_{\log T}^\tau e^{-2(\tau-\sigma)} d\sigma \leq CM.$$

Differentiating the system in τ gives bounds for every fixed τ -derivative, because the differentiated coefficients and forcing are uniformly controlled. Converting back to t proves (117) in the remaining region.

Applying (117) to (114), with $M = \beta^3 R^{-2}$, yields

$$|\partial_t^m e| \leq C_m \beta^3 R^{-2} (\beta + t)^{-m}$$

for every fixed $m \geq 0$.

4. *Mixed derivatives.* We induct on $n = 0, \dots, N$, proving the stronger intermediate statement

$$(118) \quad |\partial_t^m \partial_y^n e| \leq C_{mn} \beta^3 R^{-2-n} (\beta + t)^{-m} \quad \text{for every fixed } m \geq 0.$$

Keeping all transverse orders at each step ensures that the second transverse derivatives in the commutators are already controlled.

Put $e_n = \partial_y^n e$. Differentiating (114) gives

$$\begin{aligned} (\mathbf{a}\partial_t^2 + \mathbf{b}\partial_t + \mathbf{c})e_n &= \partial_y^n F - \sum_{j=1}^n \binom{n}{j} [(\partial_y^j \mathbf{a})\partial_t^2 + (\partial_y^j \mathbf{b})\partial_t + \partial_y^j \mathbf{c}] e_{n-j}, \\ e_n(0, y) &= \partial_t e_n(0, y) = 0. \end{aligned}$$

For $j \geq 1$, the singular term $3/t$ has zero y -derivative. Thus (115), the induction hypothesis, and (113) show that the complete right-hand side, denoted by g_n , satisfies

$$|\partial_t^m g_n| \leq C_{mn} \beta^3 R^{-2-n} (\beta + t)^{-2-m}$$

for every fixed $m \geq 0$. It is smooth and even at the axis. Applying (117) with $M = \beta^3 R^{-2-n}$ proves (118) and closes the induction.

Finally, $a_\infty(y) \approx \beta$ throughout the band. Replacing β by this comparable quantity and restricting to $m+n \leq N$ gives (110). The choices of the smoothing region and the tail transition T use only fixed low-order coefficient bounds. Consequently the requirement $2T\beta < cR$ is independent of N ; higher derivatives change only the constants in the estimates. Together with the smoothing-radius estimates, this proves the proposition. Since $R = \theta_0|y_0|$, taking $y_0 = y$ also gives the corresponding bounds with R replaced by $|y|$, on $0 \leq t \leq c|y|$ after adjusting the fixed constant c . $\square$

APPENDIX B. GLUING ESTIMATES

The gluing argument uses two analytic operations, established here. First, we invert the radial linearized Hardt–Simon operator with zero data at the axis, obtaining the growth and derivative bounds for each corrector profile. We then prove the full corrector estimates and the connector matching statement used in Section 6.4. Finally, we extend a prescribed mean-curvature correction off the hypersurface to an ambient conformal factor. The weighted bounds in this extension record the loss from taking derivatives at the shrinking smoothing scale. They explain why the high-order cancellations in Section 6 are needed for a smooth metric at the accumulation point.

B.1. Linearized Hardt–Simon operator. In the next lemma we prove estimates for solutions of the linearized Hard–Simon ODE.

Lemma B.1 (Radial inverse for the linearized Hardt–Simon operator). *Let h be the normalized Hardt–Simon profile in (31), and let*

$$L_h w = \frac{w''}{1+h'^2} + \left(\frac{3}{\rho} - \frac{2h'h''}{(1+h'^2)^2} \right) w' + \frac{3}{h^2} w.$$

Suppose $b \geq 0$ and f is smooth and even, with

$$|f^{(m)}(\rho)| \leq C_m (1+\rho)^{b-2-m}, \quad \rho \geq 0, \quad m \geq 0.$$

Then there is a unique smooth even solution of

$$L_h w = f, \quad w(0) = w'(0) = 0.$$

It is defined on $[0, \infty)$ and satisfies

$$|w^{(m)}(\rho)| \leq C_{b,m} \left(\sum_{j=0}^m C_j \right) (1+\rho)^{b-m}, \quad \rho \geq 0, \quad m \geq 0,$$

where $C_{b,m}$ depends only on b , m , and the fixed profile h .

Proof. The dilation Jacobi field

$$d(\rho) = h(\rho) - \rho h'(\rho)$$

satisfies $L_h d = 0$, as follows by differentiating the family $\lambda h(\rho/\lambda)$ at $\lambda = 1$. Moreover, d is strictly positive. Indeed, $h(\rho) > \rho$ because the leaf lies on the $|z| > |x|$ side. Since $h'(0) = 0$, a first point at which $h' = 1$ would satisfy

$$h'' = 6 \left(\frac{1}{h} - \frac{1}{\rho} \right) < 0,$$

contradicting a first crossing from below. Hence $h' < 1$ and $d > 0$.

The differentiated asymptotics of h give

$$d(\rho) = 3b_H \rho^{-2} + O(\rho^{-3}) \quad \text{as } \rho \rightarrow \infty.$$

Together with positivity and smoothness on compact intervals, this implies

$$d(\rho) \approx (1 + \rho)^{-2}, \quad |d^{(m)}(\rho)| \leq C_m(1 + \rho)^{-2-m}.$$

Introduce the integrating factor

$$\mu(\rho) = \frac{\rho^3 h(\rho)^3}{(1 + h'(\rho)^2)^{3/2}}.$$

Using the equation for h , one checks that

$$L_h w = f \iff (\mu w')' + \frac{3\mu(1 + h'^2)}{h^2} w = \mu(1 + h'^2) f.$$

Since $L_h d = 0$, reduction of order gives

$$\left(\mu d^2 \left(\frac{w}{d} \right) \right)' = \mu(1 + h'^2) df.$$

The zero data at the axis therefore lead to the formula

$$w(\rho) = d(\rho) \int_0^\rho \frac{1}{\mu(r)d(r)^2} \left(\int_0^r \mu(s)(1 + h'(s)^2) d(s) f(s) ds \right) dr.$$

This formula defines a solution for every $\rho \geq 0$. Near zero, $\mu(\rho)/\rho^3$ and $d(\rho)$ are smooth, positive, and even. The inner integral is consequently r^4 times a smooth even function, and the outer integrand is r times a smooth even function. Thus w is smooth and even, with $w(0) = w'(0) = 0$. Conversely, every regular solution with zero data at the axis must satisfy this formula, proving uniqueness.

We next estimate the solution. The leaf bounds imply

$$\mu(\rho) \approx \rho^3(1 + \rho)^3, \quad \mu(\rho)d(\rho)^2 \approx \frac{\rho^3}{1 + \rho}.$$

Hence, for $b \geq 0$,

$$\left| \int_0^r \mu(s)(1 + h'(s)^2) d(s) f(s) ds \right| \leq C_b C_0 r^4 (1 + r)^{b-1}.$$

Substituting this into the solution formula yields

$$|w(\rho)| \leq C_b C_0 (1 + \rho)^{-2} \int_0^\rho r(1 + r)^b dr \leq C_b C_0 (1 + \rho)^b.$$

Differentiating the formula and using $|d'(\rho)| \leq C(1 + \rho)^{-3}$ likewise gives

$$|w'(\rho)| \leq C_b C_0 (1 + \rho)^{b-1}.$$

On $0 \leq \rho \leq 1$, rescaling both integration variables to fixed intervals and differentiating the formula bounds $w^{(m)}$ by a constant times $\sum_{j=0}^m C_j$. On $\rho \geq 1$, divide the equation by its leading coefficient. The coefficient of w' and its k th derivative are $O(\rho^{-1-k})$, while those of w are $O(\rho^{-2-k})$. The bounds already proved for w, w' , followed by successive differentiation of the equation, therefore give

$$|w^{(m)}(\rho)| \leq C_{b,m} \left(\sum_{j=0}^m C_j \right) (1 + \rho)^{b-m}.$$

Combining the two regions proves the lemma. □

B.2. Corrector estimates and connector matching. We give the complete statement underlying Proposition 6.3 and its proof, followed by the proof of Corollary 6.4. The profiles are those defined in Section 6.4.

Proposition B.2 (Correctors for a prescribed radius function). *The recursively defined profiles $P_{2\ell,\gamma}$ depend only on the normalized Hardt–Simon profile h , and are independent of the prescribed radius function a . For every $\ell \geq 1$ and $\gamma \in \mathcal{I}_{2\ell}$, they are smooth and even, satisfy*

$$P_{2\ell,\gamma}(0) = P'_{2\ell,\gamma}(0) = 0,$$

and obey

$$|\partial_\rho^m P_{2\ell,\gamma}(\rho)| \leq C_{\ell,\gamma,m}(1 + \rho)^{2\ell-2-m}, \quad \rho \geq 0, \quad m \geq 0.$$

Fix $C_1 > 0$ and constants $M_n \geq 1$, $n \geq 1$. For each fixed $J \geq 1$, there is an $\varepsilon_J > 0$, depending only on J , C_1 , and $M_1, \dots, M_{2J+2}$, with the following property.

Let $y_0 \in \mathbb{R}$, $R > 0$, and let

$$a \in C^\infty(I_{2R}; (0, \infty)), \quad I_{2R} = (y_0 - 2R, y_0 + 2R),$$

be any prescribed function satisfying

$$|a^{(n)}(y)| \leq M_n a(y) R^{-n} \quad (n \geq 1), \quad \sup_{y \in I_{2R}} \frac{a(y)}{R} \leq \varepsilon_J.$$

Then $V_J[a]$, defined in (80), is positive on

$$0 \leq t < (C_1 + 1)R, \quad y \in I_{2R},$$

and $V_J[a](|x|, y)$ is smooth in Cartesian coordinates. Throughout the interior region

$$0 \leq t \leq C_1 R, \quad |y - y_0| \leq R,$$

one has

$$V_J[a](t, y) \approx a(y) + t, \quad |\partial_t V_J[a](t, y)| \leq C_J, \quad |\partial_y V_J[a](t, y)| \leq C_J \frac{a(y)}{R}.$$

Moreover, for every $m, n \geq 0$,

$$(119) \quad |\partial_t^m \partial_y^n W_\ell[a](t, y)| \leq C_{\ell,m,n} a(y)^3 R^{-2\ell-n} (a(y) + t)^{2\ell-2-m}, \quad 1 \leq \ell \leq J,$$

$$(120) \quad |\partial_t^m \partial_y^n Q(V_J[a])(t, y)| \leq C_{J,m,n} a(y)^3 R^{-2J-2-n} (a(y) + t)^{2J-2-m}.$$

Define

$$\Sigma_J = \left\{ (x, z, y) \in \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R} : |x| < (C_1 + 1)R, \quad y \in I_{2R}, \quad |z| = V_J[a](|x|, y) \right\}.$$

Its Euclidean scalar mean curvature satisfies, for every $j \geq 0$,

$$|\nabla_{\Sigma_J}^j H_{\Sigma_J}(X)| \leq C_{J,j} a(y)^3 R^{-2J-2} (a(y) + |x|)^{2J-2-j}$$

whenever $X = (x, z, y) \in \Sigma_J$, $|x| \leq C_1 R$, and $|y - y_0| \leq R$.

All constants are independent of a , y_0 , and R , apart from the prescribed modulation bounds M_n . Each constant depends on only finitely many of these bounds. The smallness threshold ε_J is independent of the derivative orders m, n, j .

Proof of Proposition B.2. Throughout the proof, a is an arbitrary prescribed function satisfying the hypotheses. In particular,

$$|(\log a)'| \leq M_1 R^{-1},$$

so

$$e^{-2M_1} a(y_0) \leq a(y) \leq e^{2M_1} a(y_0), \quad y \in I_{2R}.$$

We work on the enlarged region

$$0 \leq t < (C_1 + 1)R, \quad y \in I_{2R},$$

and write

$$\rho = \frac{t}{a(y)}, \quad q_j = a^{j-1} a^{(j)}, \quad s = a(y) + t.$$

We may decrease ε_J below 1 when necessary. In particular, $s \leq (C_1 + 2)R$ on this region.

We first establish the universal profile estimates. The identities

$$a\partial_y\rho = -q_1\rho, \quad a\partial_yq_j = (j-1)q_1q_j + q_{j+1}$$

show that an axial differentiation, multiplied by a , raises the weight by one. These are identities for every positive smooth function a .

The differentiated leaf estimates give

$$|\partial_\rho^m(h - \rho h')| \leq C_m(1 + \rho)^{-2-m}.$$

Thus the weight-one coefficient of

$$\partial_y(ah(t/a)) = q_1(h - \rho h')$$

has the required decay.

Suppose the profiles of weights below 2ℓ have already been constructed with the asserted bounds. A coefficient of weight k in $V_{\ell-1}/a - h$ has radial growth $(1 + \rho)^{k-2}$, with one power lost for each radial derivative. The differentiation identities therefore show that a coefficient of weight k in $(V_{\ell-1})_y$ has growth $(1 + \rho)^{k-3}$, and one in $a(V_{\ell-1})_{yy}$ has growth $(1 + \rho)^{k-4}$. These statements include all differentiated bounds.

Substituting into $aQ(V_{\ell-1})$ and expanding the denominators about their weight-zero terms h and $1 + h'^2$ gives

$$|\partial_\rho^m S_{2\ell,\gamma}(\rho)| \leq C_{\ell,\gamma,m}(1 + \rho)^{2\ell-4-m}.$$

Indeed, the second-derivative terms have this order, while additional products of positive-weight corrections have further radial decay. The coefficients are smooth and even. At the axis, terms involving ρ^{-1} have smooth even extensions because their numerators are odd.

Lemma B.1, applied with $b = 2\ell - 2$ and $f = -S_{2\ell,\gamma}$, now gives the unique smooth even solution of

$$L_h P_{2\ell,\gamma} = -S_{2\ell,\gamma}, \quad P_{2\ell,\gamma}(0) = P'_{2\ell,\gamma}(0) = 0,$$

with

$$|\partial_\rho^m P_{2\ell,\gamma}(\rho)| \leq C_{\ell,\gamma,m}(1 + \rho)^{2\ell-2-m}.$$

This closes the induction. In particular, the construction involves only the formal axial jets and the fixed radial operator L_h ; the resulting profiles are independent of the choice of a .

We next evaluate these profiles on the prescribed function. The modulation hypotheses imply

$$|\partial_y^n q_j| \leq C_{j,n} \left(\frac{a(y)}{R} \right)^j R^{-n},$$

and hence, for $\gamma \in \mathcal{I}_{2\ell}$,

$$|\partial_y^n q^\gamma| \leq C_{\ell,n} \left(\frac{a(y)}{R} \right)^{2\ell} R^{-n}.$$

Using the formula for W_ℓ , the profile estimates, and $\partial_y\rho = -(a'/a)\rho$, we obtain

$$\begin{aligned} |\partial_t^m \partial_y^n W_\ell[a]| &\leq C_{\ell,m,n} a(y)^{1-m} \left(\frac{a(y)}{R} \right)^{2\ell} R^{-n} (1 + \rho)^{2\ell-2-m} \\ &= C_{\ell,m,n} a(y)^3 R^{-2\ell-n} (a(y) + t)^{2\ell-2-m}. \end{aligned}$$

This proves (119).

Since $s \leq (C_1 + 2)R$, these estimates also yield

$$\sum_{\ell=1}^J \left(\frac{|W_\ell|}{s} + |\partial_t W_\ell| \right) \leq C_J \left(\frac{a}{R} \right)^2, \quad \sum_{\ell=1}^J |\partial_y W_\ell| \leq C_J \left(\frac{a}{R} \right)^3.$$

On the other hand,

$$ah(t/a) \approx s, \quad \partial_t(ah(t/a)) = h'(\rho), \quad \partial_y(ah(t/a)) = a'(h - \rho h').$$

Choosing ε_J sufficiently small therefore gives positivity, comparability, and the asserted first-derivative bounds for $V_J[a]$ on the enlarged region. This choice uses only finitely many modulation bounds and does not depend on the derivative orders appearing later. The axis identities

follow from $h(0) = 1$, $h'(0) = 0$, and the zero data of the correctors. Evenness gives smoothness in Cartesian x coordinates.

We now estimate the residual. The recursive construction cancels all terms of weight at most $2J$ in $aQ(V_J[a])$. Only even weights occur, so the first possible remaining weight is $2J + 2$. To turn this formal cancellation into a uniform estimate, we use a finite Taylor expansion.

At a fixed y , put $\varepsilon = a(y)/R$. Express $aQ(V_J[a])$ in $(\rho, q_1, \dots, q_{2J+2})$, and let E_δ be the resulting expression after the substitution $q_j \mapsto \delta^j q_j$, $0 \leq \delta \leq 1$. Then

$$E_1 = aQ(V_J[a]), \quad \partial_\delta^k E_\delta|_{\delta=0} = 0 \quad (0 \leq k \leq 2J+1).$$

A weight- k contribution to $V_J/a - h$ and its first two radial derivatives is bounded, respectively, by

$$(1+\rho)^{-2}(\varepsilon(1+\rho))^k, \quad (1+\rho)^{-3}(\varepsilon(1+\rho))^k, \quad (1+\rho)^{-4}(\varepsilon(1+\rho))^k.$$

The corresponding contributions to $(V_J)_y$ and $a(V_J)_{yy}$ are bounded by

$$(1+\rho)^{-3}(\varepsilon(1+\rho))^k, \quad (1+\rho)^{-4}(\varepsilon(1+\rho))^k.$$

All these estimates hold with differentiated radial bounds.

After subtracting the equation for h , the residual is therefore $(1+\rho)^{-4}$ times a rational expression in the weighted variables, with uniformly controlled coefficients. The relevant normalized quantities satisfy

$$\varepsilon(1+\rho) = \frac{s}{R} \leq C_1 + 2, \quad |\varepsilon^{-j} q_j| = \left| \frac{R^j a^{(j)}}{a} \right| \leq M_j.$$

Along the entire interpolation $0 \leq \delta \leq 1$, the normalized profile divided by h equals $1 + O_J(\varepsilon^2)$. The slope denominator is bounded below by 1. Thus all inverse denominators and their differentiated expressions are uniformly controlled.

Consequently,

$$|\partial_\rho^m \partial_\delta^{2J+2} E_\delta| \leq C_{J,m} \varepsilon^{2J+2} (1+\rho)^{2J-2-m}, \quad 0 \leq \delta \leq 1.$$

Taylor's formula with integral remainder gives

$$aQ(V_J[a]) = \frac{1}{(2J+1)!} \int_0^1 (1-\delta)^{2J+1} \partial_\delta^{2J+2} E_\delta d\delta.$$

The same argument applies after axial differentiation. Indeed, the modulation hypotheses give

$$|\partial_y^n \varepsilon| \leq C_n \varepsilon R^{-n}, \quad |\partial_y^n \rho| \leq C_n \rho R^{-n},$$

and

$$\left| \partial_y^n \left(\frac{R^j a^{(j)}}{a} \right) \right| \leq C_{j,n} R^{-n}.$$

Thus each axial differentiation contributes a factor R^{-1} to the preceding bounds. Differentiating the integral remainder and dividing by a therefore yields

$$\begin{aligned} |\partial_t^m \partial_y^n Q(V_J[a])| &\leq C_{J,m,n} a^{-1-m} R^{-n} \left(\frac{a}{R} \right)^{2J+2} (1+\rho)^{2J-2-m} \\ &= C_{J,m,n} a(y)^3 R^{-2J-2-n} (a(y) + t)^{2J-2-m}. \end{aligned}$$

This proves (120). Higher derivative orders require higher modulation bounds, but do not require a smaller ε_J .

Finally, the correction estimates and the differentiated leaf bounds imply

$$|D_{x,y}^k V_J[a](|x|, y)| \leq C_{J,k} (a(y) + |x|)^{1-k}, \quad k \geq 1.$$

Near the axis, these follow from the smooth even radial extension; away from it, they follow from the usual radial differentiation formulas.

Since $V_J[a] \approx a + |x|$, the parametrization

$$(x, y, \omega) \mapsto (x, V_J[a](|x|, y)\omega, y), \quad \omega \in S^3,$$

gives uniformly controlled local charts after rescaling by $a(y) + |x|$. On patches of a sufficiently small fixed multiple of this radius, the rescaled induced metric, its inverse, and their derivatives of every fixed order are uniformly bounded. The enlarged domain provides these patches throughout the stated interior region.

In these coordinates,

$$H_{\Sigma_J} = -\frac{Q(V_J[a])}{\sqrt{1 + |DV_J[a]|^2}}.$$

The differentiated bounds for $V_J[a]$ control the denominator. Since $a(y) + |x| \leq CR$, each axial factor R^{-1} is bounded by $C(a(y) + |x|)^{-1}$. The defect estimate therefore gives the coordinate derivative bounds

$$|D^j H_{\Sigma_J}| \leq C_{J,j} a(y)^3 R^{-2J-2} (a(y) + |x|)^{2J-2-j}.$$

The uniformly controlled rescaled metric and connection give the same bounds for covariant derivatives, proving the final assertion. $\square$

Proof of Corollary 6.4. Write $u = u_\infty$ throughout this proof.

Choose θ_0 sufficiently small that the connector estimates hold on the enlarged rectangle

$$0 \leq t \leq 2R, \quad |y - y_0| \leq 2R.$$

We prove the estimate on $0 \leq t \leq R$, $|y - y_0| \leq R$, which contains the region in the statement.

We induct on J , retaining every fixed mixed derivative order at each stage. The case $J = 0$ is Proposition 5.1. Suppose the assertion holds at order $J - 1$, and set

$$E = u - V_J[a].$$

The induction hypothesis and (119) first give the weaker estimate

$$|\partial_t^m \partial_y^n E| \leq C_{J,m,n} a(y)^3 R^{-2J-n} (a(y) + t)^{2J-2-m}.$$

Moreover, because $a(y) = u(0, y)$ and all corrector profiles have zero data at the axis,

$$E(0, y) = E_t(0, y) = 0.$$

Separate the axial terms of Q by writing

$$Q(v) = \frac{v_{tt}}{1 + v_t^2} + \frac{3}{t} v_t - \frac{3}{v} + \mathcal{T}(v),$$

where

$$\mathcal{T}(v) = \frac{v_t^2 v_y^2 v_{tt}}{(1 + v_t^2)(1 + v_t^2 + v_y^2)} - \frac{2v_t v_y v_{ty}}{1 + v_t^2 + v_y^2} + \frac{(1 + v_t^2) v_{yy}}{1 + v_t^2 + v_y^2}.$$

The established estimates for u and V_J , and hence for their interpolations, imply

$$|v_t| \leq C_J, \quad |v_{tt}| \leq \frac{C_J}{a+t}, \quad |v_y| \leq C_J \frac{a+t}{R}, \quad |v_{ty}| \leq \frac{C_J}{R}, \quad |v_{yy}| \leq C_J \frac{a+t}{R^2},$$

together with their differentiated versions. Linearizing the displayed expression for $\mathcal{T}$ along the segment between V_J and u therefore gives

$$\begin{aligned} |\mathcal{T}(u) - \mathcal{T}(V_J)| &\leq C_J \left(\frac{(a+t)^2}{R^2} |E_{tt}| + \frac{a+t}{R} |E_{ty}| + |E_{yy}| \right. \\ &\quad \left. + \frac{a+t}{R^2} |E_t| + \frac{1}{R} |E_y| \right). \end{aligned}$$

The same coefficient bounds hold after differentiation, with an additional factor $(a+t)^{-1}$ for each transverse derivative and R^{-1} for each axial derivative. Consequently, the weaker estimate for E yields

$$|\partial_t^m \partial_y^n (\mathcal{T}(u) - \mathcal{T}(V_J))| \leq C_{J,m,n} a(y)^3 R^{-2J-2-n} (a(y) + t)^{2J-2-m}.$$

Here the induction hypothesis is used with up to two additional axial derivatives.

Since $Q(u) = 0$, subtraction of the radial parts gives

$$(a\partial_t^2 + b\partial_t + c)E = -Q(V_J) - (\mathcal{T}(u) - \mathcal{T}(V_J)), \quad E(0, y) = E_t(0, y) = 0.$$

The coefficients are obtained by averaging the radial linearization between V_J and u , exactly as in (114). By (120) and the preceding estimate, the right-hand side satisfies

$$\left| \partial_t^m \partial_y^n [-Q(V_J) - \mathcal{T}(u) + \mathcal{T}(V_J)] \right| \leq C_{J,m,n} a(y)^3 R^{-2J-2-n} (a(y) + t)^{2J-2-m}.$$

We verify that the radial response argument used in Proposition A.5 applies to these averaged coefficients. The estimates for u give

$$|\partial_t^m \partial_y^n (u - t)| \leq C_{m,n} a(y)^3 R^{-n} (a(y) + t)^{-2-m}.$$

The same bound, with constants depending on J , holds for $V_J - t$: indeed, each corrector satisfies

$$|\partial_t^m \partial_y^n W_\ell| \leq C_{\ell,m,n} a(y)^3 R^{-n} (a(y) + t)^{-2-m} \left(\frac{a(y) + t}{R} \right)^{2\ell},$$

and the last factor is bounded on the region under consideration. Thus the averaged coefficients satisfy the bounds (115), with β replaced by $a(y)$ and constants depending on J . They also have the tail expansions

$$\mathbf{a} = \frac{1}{2} + O_J(a^3 t^{-3}), \quad \mathbf{b} = \frac{3}{t} + O_J(a^3 t^{-4}), \quad \mathbf{c} = \frac{3}{t^2} + O_J(a^3 t^{-5}),$$

with their differentiated versions. At the axis, $\mathbf{a}(0, y) = 1$, $\mathbf{b} - 3/t$ is smooth and odd, and $\mathbf{a}, \mathbf{c}$ are smooth and even.

The radial response estimate (117) extends to the polynomial forcing occurring here. To see this, fix y . Rescaling t by $a(y)$ in the smoothing region gives a bound of size

$$a(y)^{2J+3} R^{-2J-2}$$

for E on a fixed interval in the smoothing region. On the tail, the logarithmic variable $\tau = \log(t/a(y))$ gives the same stable system with limiting eigenvalues -2 and -3 as in that proof. The forcing now grows like $e^{2J\tau}$, and

$$\int_{\log T_J}^{\tau} e^{-2(\tau-\sigma)} e^{2J\sigma} d\sigma \leq C_J e^{2J\tau},$$

where T_J is a fixed constant separating the inner and outer regions. Since $a(y)/R \rightarrow 0$, this transition lies inside the available region once $|y_0|$ is sufficiently large. Variation of constants and differentiation of the radial equation therefore give

$$|\partial_t^m E| \leq C_{J,m} a(y)^3 R^{-2J-2} (a(y) + t)^{2J-m}$$

for every fixed $m \geq 0$.

Finally, differentiate the radial equation in y and induct on the axial derivative order. The commutators are sums of terms of the form

$$(\partial_y^r \mathbf{a}) \partial_t^2 \partial_y^{n-r} E, \quad (\partial_y^r \mathbf{b}) \partial_t \partial_y^{n-r} E, \quad (\partial_y^r \mathbf{c}) \partial_y^{n-r} E, \quad 1 \leq r \leq n.$$

The coefficient bounds give

$$\begin{aligned} & |\partial_t^m \partial_y^r \mathbf{a}| + (a + t) |\partial_t^m \partial_y^r (\mathbf{b} - 3/t)| \\ & + (a + t)^2 |\partial_t^m \partial_y^r \mathbf{c}| \leq C_{J,m,r} R^{-r} (a + t)^{-m}. \end{aligned}$$

The singular term $3/t$ has zero axial derivative. Hence the induction hypothesis in n , retaining every fixed transverse order, bounds the differentiated forcing and all commutators by

$$C_{J,m,n} a(y)^3 R^{-2J-2-n} (a(y) + t)^{2J-2-m}.$$

The differentiated data at the axis still vanish. Applying the same radial response estimate proves

$$|\partial_t^m \partial_y^n E| \leq C_{J,m,n} a(y)^3 R^{-2J-2-n} (a(y) + t)^{2J-m}.$$

This closes both inductions and proves (81). $\square$

B.3. Metric corrections. In the next lemma we collect the estimates needed for the conformal metric correction.

Lemma B.3 (Conformal correction with weighted estimates). *Use the axial band, the function a , and the scale R from Proposition B.2. Let V be a positive smooth even profile on*

$$0 \leq t < (C_1 + 1)R, \quad |y - y_0| < 2R,$$

and set

$$\Sigma = \{(x, z, y) : |z| = V(|x|, y)\}.$$

Suppose that throughout this enlarged region

$$V(t, y) \approx a(y) + t \leq CR, \quad |D_{x,y}^k V(|x|, y)| \leq C_k(a(y) + |x|)^{1-k} \quad (k \geq 1),$$

and, for some fixed $J \geq 1$,

$$|\nabla_\Sigma^j H_\Sigma| \leq C_{J,j} a(y)^3 R^{-2J-2} (a(y) + t)^{2J-2-j} \quad (j \geq 0).$$

Here $H_\Sigma = \operatorname{div}_\Sigma \nu$, with ν pointing towards increasing $|z|$.

There is a smooth compactly supported function F such that $g = e^{2F} g_{\text{euc}}$ makes Σ minimal on

$$t \leq C_1 R, \quad |y - y_0| \leq R.$$

Moreover, $F|_\Sigma = 0$, its support lies in the enlarged band and in

$$\frac{3}{4}V(t, y) < |z| < \frac{5}{4}V(t, y),$$

and, for every $m \geq 0$,

$$(121) \quad |D^m F(x, z, y)| + |D^m (g - g_{\text{euc}})(x, z, y)| \leq C_{J,m} a(y)^3 R^{-2J-2} (a(y) + t)^{2J-1-m}, \quad t = |x|.$$

The derivatives in this estimate are ambient Cartesian derivatives. The constants are independent of a and R , apart from the constants in the assumed estimates.

Proof. Choose a smooth cutoff $\eta(t, y)$, supported in the enlarged band and equal to one on the smaller region, such that $\eta(|x|, y)$ is smooth and

$$|D_{x,y}^k \eta| \leq C_k R^{-k}.$$

Choose also $\chi \in C_c^\infty((-1, 1))$ with $\chi = 1$ near zero. By rotational symmetry, write the mean curvature as $H_\Sigma(t, y)$, and define

$$F(x, z, y) = -\frac{\eta(t, y) H_\Sigma(t, y)}{8\sqrt{1 + V_t(t, y)^2 + V_y(t, y)^2}} (|z| - V(t, y)) \chi\left(\frac{4(|z| - V(t, y))}{V(t, y)}\right).$$

Extend this function by zero outside the indicated region. The cutoff vanishes near $z = 0$, while smoothness at $x = 0$ follows from the smooth even profiles. The cutoff η ensures smooth extension across the boundaries of the band.

On Σ ,

$$F = 0, \quad \nu(|z| - V(t, y)) = \sqrt{1 + V_t^2 + V_y^2}.$$

Consequently,

$$\nu F = -\frac{1}{8} \eta H_\Sigma.$$

Since $\dim \Sigma = 8$, the conformal transformation formula gives

$$H_{\Sigma,g} = e^{-F} (H_\Sigma + 8\nu F) = (1 - \eta) H_\Sigma.$$

Thus Σ is minimal wherever $\eta = 1$.

We next justify the ambient derivative estimates. In the coordinates (x, y, ω) , where $z = V(|x|, y)\omega$ and $\omega \in S^3$, the induced metric is

$$(I + D_{x,y} V \otimes D_{x,y} V) \oplus V^2 g_{S^3}.$$

The bounds on V imply that the connection coefficients in the (x, y) directions have k th derivatives bounded by $C_k(a(y) + t)^{-1-k}$. Since H_Σ is independent of ω , conversion of its covariant derivatives to Cartesian (x, y) derivatives therefore gives

$$|D_{x,y}^k H_\Sigma(t, y)| \leq C_{J,k} a(y)^3 R^{-2J-2} (a(y) + t)^{2J-2-k}.$$

On the support of the normal cutoff,

$$|z| \approx V(t, y) \approx a(y) + t.$$

Hence, for every $k \geq 0$,

$$\begin{aligned} |D^k(|z| - V)| &\leq C_k(a(y) + t)^{1-k}, \\ |D^k(1 + V_t^2 + V_y^2)^{-1/2}| &\leq C_k(a(y) + t)^{-k}, \\ \left| D^k \chi \left(\frac{4(|z| - V)}{V} \right) \right| &\leq C_k(a(y) + t)^{-k}. \end{aligned}$$

Also, $R^{-k} \leq C_k(a(y) + t)^{-k}$ on the enlarged band. Applying the product rule to the formula for F yields

$$|D^m F| \leq C_{J,m} a(y)^3 R^{-2J-2} (a(y) + t)^{2J-1-m}.$$

Finally, the undifferentiated factor satisfies

$$a(y)^3 R^{-2J-2} (a(y) + t)^{2J-1} = \left(\frac{a(y)}{R} \right)^3 \left(\frac{a(y) + t}{R} \right)^{2J-1} \leq C_J.$$

In differentiating e^{2F} , each product of derivatives of F is therefore bounded by the same weight as a single derivative of the corresponding total order. Together with $|e^{2F} - 1| \leq C_J |F|$, this proves (121) for the metric. $\square$

APPENDIX C. GENERAL QUADRATIC CONES

The proposed construction follows the proof of Theorem 1.1. We will insert successively smaller copies of the conjectured connector and increase the correction order as the insertion points approach zero. We will then remove the remaining mean curvature by a conformal change supported away from the exact connector cores. The increasing correction order should ensure that this metric change is smooth and flat at the accumulation point. Carrying this out requires the corresponding arbitrary-order end and matching estimates in addition to the existence of the connectors.

The expected behavior at infinity is determined by the same Jacobi calculation as in the present example. Put

$$m = p + q, \quad \gamma = \frac{m - 1 - \sqrt{m^2 - 6m + 1}}{2}, \quad V = C_{p,q} \times \mathbb{R},$$

and write $r = (|x|^2 + |z|^2)^{1/2}$. For the minimizing quadratic cones, $1 < \gamma \leq 2$. On rotationally invariant functions, the Jacobi operator of V is

$$L_V = \partial_{rr} + \frac{m}{r} \partial_r + \partial_{yy} + \frac{m}{r^2}.$$

A direct calculation gives

$$L_V \left[r^{-\gamma} \left(y^2 - \frac{r^2}{m + 1 - 2\gamma} \right) \right] = 0.$$

This field has homogeneity $2 - \gamma < 1$ and therefore describes a perturbation compatible with the cylindrical blow-down.

More precisely, orient the normal of V towards $p|z|^2 - q|x|^2 > 0$, and let f_R be the normal height of $R^{-1}M$ over the regular part of V . For the attachment with tip $C_{p+1,q}$, we expect

$$R^{\gamma-1} f_R \longrightarrow A r^{-\gamma} \left(y^2 - \frac{r^2}{m + 1 - 2\gamma} \right), \quad A > 0,$$

locally smoothly on $\text{reg } V$. The opposite attachment has the opposite sign. When $p = q = 3$, this reduces to

$$\gamma = 2, \quad A \left(\frac{y^2}{r^2} - \frac{1}{3} \right).$$

We expect to use the following replacements for the ingredients of the present construction.

- (1) The transverse models will be the Hardt–Simon leaves associated with $C_{p,q}$. Their dilations provide the foliations on the two sides of the cone. For a radius graph $|z| = v(t, y)$, with $t = |x|$, the equation becomes

$$\left(\delta_{ij} - \frac{v_i v_j}{1 + |Dv|^2} \right) v_{ij} + \frac{p}{t} v_t - \frac{q}{v} = 0, \quad i, j \in \{t, y\}.$$

The cylindrical profile is $\sqrt{q/p} t$. The two sides must be treated with their respective leaf models; they need not have identical expansions.

- (2) The barriers and non-concentration estimates used here have general versions in [35, Propositions 2.9 and 4.3]. These apply to strictly stable, strictly minimizing cones, including the minimizing quadratic cones considered here. They provide the replacements for the corresponding Simons-cone inputs in our barrier and end arguments.

The phase analysis must also be adjusted. If f is an invariant Jacobi field and $P = r^\gamma f$, then

$$P_{rr} + \frac{m-2\gamma}{r} P_r + P_{yy} = 0.$$

For $m = 6$, this is the radial harmonic equation used in the present paper. In general, the harmonic replacement must be carried out for this weighted equation. The quadratic growth estimate and the nonvanishing of its quadratic coefficient still need to be proved.

- (3) The existence argument for the Dirichlet problem on a bounded domain and the classification of singularities must be adapted to the new equation. The relevant density inequalities can be checked directly.

The link of $C_{p,q}$ is

$$S^p \left(\sqrt{\frac{p}{p+q}} \right) \times S^q \left(\sqrt{\frac{q}{p+q}} \right).$$

Consequently, its normalized density is

$$\Theta(C_{p,q}) = \frac{|S^p| |S^q|}{|S^{p+q}|} \left(\frac{p}{p+q} \right)^{p/2} \left(\frac{q}{p+q} \right)^{q/2}.$$

Taking a product with $\mathbb{R}$ preserves the normalized density, so

$$\Theta(C_{p,q} \times \mathbb{R}) = \Theta(C_{p,q}).$$

We claim that the displayed density strictly decreases when either index increases. To see this, set

$$\Phi(s) = \log \Gamma \left(\frac{s+1}{2} \right) - \frac{s}{2} \log s, \quad s > 0.$$

The sphere-area formula gives

$$\log \Theta(C_{p,q}) = \log(2\sqrt{\pi}) + \Phi(p+q) - \Phi(p) - \Phi(q).$$

Moreover, the integral representation of the derivatives of $\log \Gamma$ yields

$$\Phi''(s) = \frac{1}{2} \int_0^\infty e^{-su} \left(\frac{u}{\sinh u} - 1 \right) du < 0.$$

Thus Φ' is strictly decreasing, and

$$\log \frac{\Theta(C_{p+1,q})}{\Theta(C_{p,q})} = \int_0^1 [\Phi'(p+q+s) - \Phi'(p+s)] ds < 0.$$

Interchanging p and q proves the other inequality. Therefore both proposed attachments satisfy

$$\frac{\Theta(C_{p+1,q})}{\Theta(C_{p,q} \times \mathbb{R})} < 1, \quad \frac{\Theta(C_{p,q+1})}{\Theta(C_{p,q} \times \mathbb{R})} < 1.$$

These are precisely the density comparisons required by the monotonicity formula.

- (4) Once the connector and its quantitative end estimates are available, we expect to repeat the corrector and matching construction using the linearized operator of the corresponding Hardt–Simon leaf.

The exponents change with γ . A leaf dilated by a has leading normal coefficient proportional to $a^{1+\gamma}$, so the natural phase has homogeneity $1 + \gamma$. A quadratic axial phase therefore predicts

$$a_\infty(Y) \sim \kappa |Y|^{2/(1+\gamma)}, \quad \kappa > 0.$$

Thus the cubic phase and the exponent $2/3$ in the present paper are the special case $\gamma = 2$.

The corresponding conformal estimate to establish is

$$|D^j F| \leq C_{J,j} a(y)^{1+\gamma} R^{-2J-2} (a(y) + t)^{2J+1-\gamma-j}.$$

The conformal construction itself is unchanged, with the dimension factor 8 replaced by $m + 2$:

$$H_{e^{2F} g_{\text{euc}}} = e^{-F} (H + (m + 2)\nu F).$$

With these weighted estimates, the same increasing-order choice would give a smooth metric flat at the accumulation point.

The general cylindrical estimates therefore supply several of the analytic tools. The remaining work is to construct the entire connectors, identify their isolated tangent cones, and establish the weighted end and matching estimates needed for the accumulating construction.

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DEPARTMENT OF DECISION SCIENCES AND BIDSA, BOCCONI UNIVERSITY, MILANO, ITALY
Email address: aria.halavati@unibocconi.it

DEPARTMENT OF MATHEMATICS, UC SAN DIEGO, AP&M, LA JOLLA, CALIFORNIA, USA
Email address: lskolaor@ucsd.edu