

A HOMOGENIZED MAZ'YA–SHAPOSHNIKOVA FORMULA

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ABSTRACT. We study the asymptotic behavior of fractional Gagliardo-type energies with a periodic, bounded, and bounded away from zero coefficient depending on both variables, in the joint limit of the vanishing fractional exponent $s \rightarrow 0^+$ and the vanishing period $\varepsilon \rightarrow 0^+$. Assuming that ε^{-2s} converges to some $\lambda \in [1, +\infty]$, which holds up to subsequences, we prove that the energies, rescaled by s , Γ -converge in the weak topology of $L^2(\mathbb{R}^d)$ to a multiple of the square of the L^2 norm, whose constant is given by an effective coefficient a_λ defined through a cell problem. The coefficient a_λ results from the competition between the averaging of long-range interactions, which drives the Maz'ya–Shaposhnikova limit, and the nonlocal energy of microscopic oscillations at finite range. It interpolates between the harmonic mean ($\lambda = 1$) and the arithmetic mean ($\lambda = +\infty$) of a coefficient obtained by averaging interactions “at infinity”. The proof relies on Riemann–Lebesgue-type lemmas for singular kernels, on a blow-up argument for the lower bound, and on the logarithmic-scaling properties of the energies. The case $1 < p < \infty$ is also discussed.

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1. INTRODUCTION

The Maz'ya–Shaposhnikova formula describes the asymptotic behaviour of the Gagliardo seminorm as the fractional exponent s tends to zero. More precisely, if for $s \in (0, 1)$ the

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Gagliardo seminorm of $u: \mathbb{R}^d \rightarrow \mathbb{R}$ is given by

$$[u]_{H^s(\mathbb{R}^d)}^2 := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^2}{|x - y|^{d+2s}} dx dy,$$

then, for every $u \in H^{s_0}(\mathbb{R}^d)$ for some $s_0 \in (0, 1)$,

$$\lim_{s \rightarrow 0^+} s [u]_{H^s(\mathbb{R}^d)}^2 = \sigma_{d-1} \|u\|_{L^2(\mathbb{R}^d)}^2, \quad (1.1)$$

where σ_{d-1} is the $(d-1)$ -dimensional Hausdorff measure of the unit sphere $\mathbb{S}^{d-1}$ [15]. Hence, although the seminorm is defined through nonlocal interactions between pairs of points, its asymptotic behaviour as $s \rightarrow 0^+$ is a zero-order functional.

In recent years, the Maz'ya–Shaposhnikova formula has been extended to several settings beyond its original Euclidean formulation. In metric measure spaces, volume growth at infinity (and, in particular, the asymptotic volume ratio) plays a central role in the construction of suitable convolution kernels [12, 13]. Such conditions are guaranteed, for example, by generalized Bishop–Gromov inequalities [14]. Moreover, necessary and sufficient conditions for the validity of a Maz'ya–Shaposhnikova limit for general convolution kernels are established in [9]: such conditions amount to a control on the escape of kernel mass to infinity and the attenuation of short-range interactions. Related asymptotics have been obtained for fractional perimeters [8, 10], for fractional energies associated with the Ornstein–Uhlenbeck semigroup [5], and for heat-kernel-based fractional perimeters on Riemannian manifolds [6].

The Γ -convergence counterpart of (1.1) has been established in two regimes. In [7], functions are extended by zero outside a bounded domain, so that their value at infinity is zero. In [8], localized Gagliardo seminorms, such as those defining fractional perimeters, are considered for functions that need not be globally square-integrable: the limit functional describes the interaction between the values of u in Ω and its values at infinity, thus connecting (1.1) with the asymptotics of fractional perimeters. In the Euclidean setting, the same work establishes Γ -convergence in the weak- $L^2(\Omega)$ topology for prescribed exterior data that need not vanish at infinity. In both cases the behaviour of u at infinity is fixed a priori, and the Γ -limit only acts on its values in a bounded region.

In this paper, we investigate how (1.1) is affected by periodic heterogeneities in the interaction kernel. We consider fractional energies whose interactions are weighted by a periodic coefficient depending on the spatial variables and bounded from above and below by positive constants. The resulting problem combines two limiting processes, the vanishing of the fractional exponent and the homogenization of the oscillating coefficient. Interactions at distances larger than the macroscopic scale see the behaviour of u at infinity, as in (1.1). Interactions at distances between the period ε and the macroscopic scale see oscillations as periodic perturbations with zero mean of the frozen macroscopic profile of u , with an effect reminiscent of [16], where a Maz'ya–Shaposhnikova formula is given for interactions in the torus. Since, as $s \rightarrow 0^+$, the kernel distributes its mass uniformly over logarithmic scales, the relative weight of these two ranges is governed by $\varepsilon^{-2s} = e^{2s|\log \varepsilon|}$.

Our main result characterizes this interplay through a family of effective coefficients a_λ , defined by a cell-minimization problem, and establishes the corresponding Γ -convergence of the energies in the weak topology of $L^2(\mathbb{R}^d)$. The parameter $\lambda := \lim_{\varepsilon \rightarrow 0} \varepsilon^{-2s} \in [1, +\infty]$ measures the balance between the two ranges of interactions: for $\lambda = 1$ only the behavior at infinity contributes, whereas as λ increases, the periodic interactions penalize the oscillations of the corrector more and more. The case $\lambda = +\infty$ corresponds to a *separation of scales*: the limit is obtained by first letting $\varepsilon \rightarrow 0$ and then taking the Maz'ya–Shaposhnikova limit $s \rightarrow 0$. The precise statement is the following.

Theorem 1.1 (A homogenized Maz'ya-Shaposhnikova theorem). *Let $a: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a Borel function such that there exist $0 < \alpha \leq \beta$ for which $\alpha \leq a(x, y) \leq \beta$ for all $x, y \in \mathbb{R}^d$, and 1-periodic; that is, $a(x + k, y + j) = a(x, y)$ for all $k, j \in \mathbb{Z}^d$. Let $\varepsilon > 0$ and $s := s(\varepsilon) \in (0, 1)$. Let*

$$F_\varepsilon(u) := s \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) \frac{|u(x) - u(y)|^2}{|x - y|^{d+2s}} dx dy \quad (1.2)$$

be defined in $L^2(\mathbb{R}^d)$. If $\lim_{\varepsilon \rightarrow 0} s = 0$ and

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-2s} = \lambda \in [1, +\infty],$$

then

$$\Gamma\text{-}\lim_{\varepsilon \rightarrow 0} F_\varepsilon(u) = \frac{\sigma_{d-1}}{2} a_\lambda \|u\|_{L^2(\mathbb{R}^d)}^2,$$

for all $u \in L^2(\mathbb{R}^d)$, where the Γ -limit is computed with respect to the sequential weak topology of $L^2(\mathbb{R}^d)$, a_λ is given by

$$a_\lambda := \inf \left\{ \int_Q \int_Q (a(x, y) + a(y, x)) |1 + \varphi(x)|^2 dx dy \right. \\ \left. + (\lambda - 1) \int_Q \int_Q a(x, y) |\varphi(x) - \varphi(y)|^2 dx dy : \varphi \in L^2(Q) \text{ and } \int_Q \varphi(x) dx = 0 \right\} \quad (1.3)$$

if $\lambda < +\infty$, while

$$a_{+\infty} := 2 \int_Q \int_Q a(x, y) dx dy, \quad (1.4)$$

where $Q = (0, 1)^d$ is the unit cube in $\mathbb{R}^d$, and σ_{d-1} is the $(d-1)$ -Hausdorff measure of the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$.

The counterpart of Γ -convergence of the Maz'ya-Shaposhnikova formula is recovered as a particular case of our theorem, when we take the coefficient a equal to the constant 1, in which case the minimizer for the problem in (1.3) is $\varphi = 0$ independently of λ , and we have $a_\lambda = 2$.

Theorem 1.1 can be proved for Gagliardo seminorms with an arbitrary $p > 1$. We restrict ourselves to the case $p = 2$ to focus on the novelties of this work and not to carry another parameter through the proofs. The asymptotic results hold unchanged as in the p version of the Maz'ya-Shaposhnikova formula, upon defining λ as the limit of ε^{-sp} and using the power p in the definition of a_λ (see Theorem 5.1).

For a non-constant, the form of the coefficient a_λ highlights a new nonlocal homogenization phenomenon in competition with the effect of the interactions “with points at infinity” that drive the Maz'ya-Shaposhnikova limit. This can be explained by comparing Theorem 1.1 with the Maz'ya-Shaposhnikova limit for a (smooth) function u , whose support is contained in a ball B_{R_0} . In the latter case, the limit of $s[u]_{H^s(\mathbb{R}^d)}^2$ as $s \rightarrow 0$, using the symmetry between integration in x and y and noting that all interactions in B_R can be neglected, is the same as that of

$$\lim_{s \rightarrow 0} 2s \int_{B_{R_0}} \int_{\mathbb{R}^d \setminus B_R} \frac{|u(x) - u(y)|^2}{|x - y|^{d+2s}} dx dy = \left(\lim_{s \rightarrow 0} 2s \int_{\mathbb{R}^d \setminus B_R} \frac{1}{|\xi|^{d+2s}} d\xi \right) \int_{B_{R_0}} |u(x)|^2 dx$$

for any $R > R_0$. Since the limit is independent of R , a direct computation gives the result.

In the presence of an oscillating coefficient, the Γ -limit can be different from the pointwise limit of $F_\varepsilon(u)$ at u fixed. Examining a sequence u_ε weakly converging to u , we have two issues:

1) the term given by “interactions going to infinity” is now of the form

$$s \int_{B_{R_0}} \int_{\mathbb{R}^d \setminus B_R} \left(a\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) + a\left(\frac{y}{\varepsilon}, \frac{x}{\varepsilon}\right) \right) \frac{|u_\varepsilon(x)|^2}{|x - y|^{d+2s}} dx dy.$$

By first integrating in y , we see that this double integral is asymptotically equivalent as $s \rightarrow 0$ to an oscillating integral with a 1-periodic coefficient. A Riemann–Lebesgue type lemma allows one to see that such an integral is indeed of the form

$$\frac{\sigma_{d-1}}{2} \int_{B_{R_0}} \left(\int_Q \left(a\left(\frac{x}{\varepsilon}, \xi\right) + a\left(\xi, \frac{x}{\varepsilon}\right) \right) d\xi \right) |u_\varepsilon|^2 dx. \quad (1.5)$$

A clarifying example is when $a(x, y) = a_0(x)$ (or its symmetric equivalent version $\frac{1}{2}(a_0(x) + a_0(y))$), for which this integral is

$$\frac{\sigma_{d-1}}{2} \int_{B_{R_0}} \left(a_0\left(\frac{x}{\varepsilon}\right) + \bar{a}_0 \right) |u_\varepsilon|^2 dx,$$

with $\bar{a}_0 = \int_Q a_0(\xi) d\xi$ representing the contribution “averaged at infinity”;

2) due to the presence of oscillations, the interactions at finite distance are not negligible any more. We have an additional term giving a non-vanishing contribution in the limit, which is of the form

$$s \iint_{\{\varepsilon \leq |x-y| \leq 1\}} a\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^2}{|x - y|^{d+2s}} dx dy. \quad (1.6)$$

The conditions on s and ε defining λ , and a use of a second Riemann–Lebesgue type lemma give the double-integral contribution to the definition of a_λ .

It is interesting to compare Theorem 1.1 with the limit taken as $s \rightarrow 1^-$ in the spirit of the Bourgain–Brezis–Mironescu formula [1]. The asymptotic behaviour of the corresponding F_ε , with the scaling factor s replaced by $1 - s$ in (1.2), has been described in [3] (see also [2]), showing a behaviour dual to ours: the relevant oscillations in that case are those at scale smaller than ε , which are negligible in Theorem 1.1. These small-scale oscillations lead to a quadratic functional in ∇u , obtained by homogenizing a quadratic functional with coefficient $a(x, x)$, concentrated on the diagonal; that is, due to interactions at distance less than ε , as opposed to the coefficient $\bar{a}^c$ (see Definition 2.1) that derives from averaging interactions at infinity. An additional contribution in the BBM limit is given by interactions at scales between ε and 1 (as in item (2) above). In contrast to the case $s \rightarrow 0$, in these interactions only the average of the oscillations matters, and they do not influence the homogenization on a lower scale. The overall limit is a convex combination of the two contributions, with the small-scale contribution now weighted by the limit of $\varepsilon^{2(1-s)}$.

We remark that in this paper we do not prove a compactness statement, which amounts to a uniform bound in L^2 . This, on a bounded domain Ω , is proved through a fractional Hardy inequality [7, 8], but it fails on the whole $\mathbb{R}^d$. On the other hand, if we restrict our attention to functions belonging to $L^2(\Omega)$ and extended to zero outside of Ω , such a compactness statement is available because $a(x, y) \geq \alpha > 0$. The proof of the theorem remains unchanged in this constrained case, so that it may be applied to the convergence of minimum problems as in [7].

Structure of the paper and core technical ideas. The whole paper is devoted to the proof of Theorem 1.1. The key observation is that, due to the oscillations forced by the periodic coefficient, in the limit $s \rightarrow 0$ we may only have weak convergence in L^2 . This led us to study the contribution of term (1.6), which is usually negligible in the Maz’ya–Shaposhnikova regime. Indeed, it works precisely as a penalization of the oscillations, with a coefficient due to the mass of the kernel at the scales between ε and 1, forcing strong convergence in the case $\lambda = +\infty$. This is in accordance with the separation-of-scale analysis for functionals F_ε , for which we have strong L^2 compactness when we let $\varepsilon \rightarrow 0$ first. The techniques used here are similar but complementary to those in [3]. In particular, certain Riemann–Lebesgue type lemmas suit very well the framework of singular kernels of the kind $|x - y|^{-d-2s}$. Indeed, these kernels are smooth

away from the diagonal and their contributions at medium range interactions can be precisely quantified.

Section 2 is devoted to the collection of several preparatory results on the average behaviour of double integrals of general interest. It should be noted that, since the kernel spreads its mass uniformly over logarithmic scales, most of the lemmas can be applied in both the Γ -liminf and the Γ -limsup inequalities. Section 3 contains the details of the Γ -liminf portion of Theorem 1.1. To prove it, we employ a Fonseca–Müller type of blow up. What is new is that the contributions to reconstruct the cell formula come from two different regions of the space. Therefore, in order to recognize a competitor for the infimization problem defining a_λ , one has to perform the same modifications both at the medium- and long-range interactions level. Remarkably, also in the case of a constant, this allows us to prove the Γ -liminf inequality without relying on a fractional Hardy inequality (see [7]). Section 4 contains the Γ -limsup argument. The techniques are similar to those of the previous section, and one can see in a more transparent way the role of each contribution (even of those that are negligible). More precisely, in the delicate term (that is, the contribution coming from (1.6)), we make use of the fact that $(d-1)$ -dimensional sets have zero fractional capacity for $s < \frac{1}{2}$ to change the boundary data and periodize the oscillations. Finally, in Section 5 we briefly illustrate the changes for the case of general $p > 1$.

2. NOTATION AND PRELIMINARIES

Throughout the paper $d \geq 1$ is a positive integer. The norm of a point $x \in \mathbb{R}^d$ is denoted by $|x|$; for $R > 0$ we set $B_R := \{y \in \mathbb{R}^d : |y| < R\}$. The Lebesgue measure of a set $A \subset \mathbb{R}^d$ is denoted by $|A|$. We use the standard notation for Lebesgue and Sobolev spaces.

Let $a : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a Borel function that is 1-periodic; i.e.,

$$a(x+k, y+j) = a(x, y) \text{ for all } k, j \in \mathbb{Z}^d, x, y \in \mathbb{R}^d,$$

and such that there exist α, β with $0 < \alpha \leq \beta < +\infty$ such that

$$\alpha \leq a(x, y) \leq \beta \text{ for all } x, y \in \mathbb{R}^d.$$

We let $Q := (0, 1)^d$ denote the unit cell of periodicity and define the *contracted function*

$$\bar{a}^c(x) := \int_Q (a(x, y) + a(y, x)) dy. \quad (2.1)$$

Let $\varepsilon > 0$ and let $s = s(\varepsilon) \in (0, 1)$ be such that $s \rightarrow 0$ as $\varepsilon \rightarrow 0$; in all formulas we drop the explicit dependence of s on ε . We assume that the limit

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-2s} = \lambda \in [1, +\infty] \quad (2.2)$$

exists, which is not restrictive up to extraction of a subsequence.

For every $\varepsilon > 0$, we define $F_\varepsilon : L^2(\mathbb{R}^d) \rightarrow [0, +\infty]$ as

$$F_\varepsilon(u) := s \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) \frac{|u(x) - u(y)|^2}{|x - y|^{d+2s}} dx dy, \quad (2.3)$$

with domain the fractional Sobolev space $H^s(\mathbb{R}^d)$.

Remark 2.1. In logarithmic scale $t = \log |x - y|$, the measure $s|h|^{-d-2s} dh$ becomes $\sigma_{d-1} s e^{-2st} dt$, which is asymptotically uniform as $s \rightarrow 0^+$. Hence, the scales $t > 0$, responsible for the interactions with infinity, carry mass $\frac{\sigma_{d-1}}{2}$, while the scales $\log \varepsilon < t < 0$, corresponding to the region in (1.6), carry mass

$$\frac{\sigma_{d-1}}{2} (\varepsilon^{-2s} - 1) \rightarrow \frac{\sigma_{d-1}}{2} (\lambda - 1).$$

On the latter scales, the macroscopic profile is essentially frozen, $u(y) \approx u(x)$, whereas y/ε is equidistributed in Q . Testing F_ε on $u_\varepsilon(x) = u(x)(1 + \varphi(x/\varepsilon))$, the corresponding energy is therefore the periodic functional of [16], weighted by a and evaluated on the corrector φ . For a periodic function, as in [16], the two ranges of scales play the same role, since its values at infinity are distributed as its values on the periodicity cell. The separation between the scale ε and the macroscopic scale splits them. Hence, the coefficient a_λ in Theorem 1.1 interpolates between homogenization at infinity and homogenization on the torus.

2.1. The effective coefficients a_λ . For every $\lambda \in [1, +\infty]$, every $z \in \mathbb{R}$ and every $\varphi \in L^2(Q)$, we set

$$A_\lambda(z, \varphi) := \int_Q \bar{a}^c(x) |z + \varphi(x)|^2 dx + (\lambda - 1) \int_Q \int_Q a(x, y) |\varphi(x) - \varphi(y)|^2 dx dy, \quad (2.4)$$

where $\bar{a}^c$ is defined in (2.1). For $\lambda = +\infty$, we use the convention $(+\infty) \cdot 0 = 0$, so that $A_{+\infty}(z, \varphi)$ is finite only if φ is constant since $a \geq \alpha > 0$. We then define

$$a_\lambda := \inf \left\{ A_\lambda(1, \varphi) : \varphi \in L^2(Q) \text{ and } \int_Q \varphi(x) dx = 0 \right\}; \quad (2.5)$$

that is, formula (1.3) or (1.4) in the case $\lambda = +\infty$. The test functions for the problem defining a_λ will be used to construct periodic recovery sequences. The function $\varphi \in L^2(Q)$ will then be identified with its 1-periodic extension. Note that

$$\inf \left\{ A_\lambda(z, \varphi) : \varphi \in L^2(Q) \text{ and } \int_Q \varphi(x) dx = 0 \right\} = a_\lambda z^2$$

by positive 2-homogeneity. The definition of a_λ allows us to define $F_\lambda: L^2(\mathbb{R}^d) \rightarrow [0, +\infty)$ as

$$F_\lambda(u) := \frac{\sigma_{d-1}}{2} a_\lambda \|u\|_{L^2(\mathbb{R}^d)}^2, \quad (2.6)$$

where $\sigma_{d-1} := \mathcal{H}^{d-1}(\mathbb{S}^{d-1})$. We note a slight abuse of notation with respect to F_ε .

Remark 2.2 (dependence on λ). The map $\lambda \mapsto a_\lambda$ is continuous, increasing and concave (an infimum of affine functions of λ), and interpolates between the harmonic and the arithmetic mean of $\bar{a}^c$.

Indeed, if $\lambda = 1$, then it is well known that the minimization gives the harmonic mean of $\bar{a}^c$; that is

$$a_1 = \left(\int_Q \frac{1}{\bar{a}^c(x)} dx \right)^{-1}.$$

If $\lambda = +\infty$, then the only test function $\varphi \in L^2(Q)$ with $A_\infty(1, \varphi) < +\infty$ is $\varphi = 0$. Therefore,

$$a_{+\infty} = \int_Q \int_Q (a(x, y) + a(y, x)) dx dy = \int_Q \bar{a}^c(x) dx = 2 \int_Q \int_Q a(x, y) dx dy; \quad (2.7)$$

that is, the average of $\bar{a}^c$ (or twice the average of a).

Remark 2.3 (symmetric coefficients). The replacement of $a(x, y)$ by its symmetric part $\frac{1}{2}(a(x, y) + a(y, x))$, which satisfies the same bounds and periodicity assumptions, does not change F_ε nor a_λ . Hence, from now on we assume, without loss of generality, that a is symmetric. In such a case, we define

$$\bar{a}^y(x) := \int_Q a(x, y) dy.$$

Then we have $\bar{a}^c = 2\bar{a}^y$, and we may use $2\bar{a}^y$ for the contracted function instead of $\bar{a}^c$. In this case, (2.4) reads

$$A_\lambda(z, \varphi) = 2 \int_Q \bar{a}^y(x) |z + \varphi(x)|^2 dx + (\lambda - 1) \int_Q \int_Q a(x, y) |\varphi(x) - \varphi(y)|^2 dx dy.$$

Remark 2.4 (coefficient depending on one variable). We consider $a(x, y)$ of the form $\frac{1}{2}(a_0(x) + a_0(y))$, with $a_0: \mathbb{R}^d \rightarrow \mathbb{R}$ a 1-periodic, bounded Borel function and satisfies $a_0 \geq \alpha > 0$ a.e. This corresponds (in the non-symmetric setting) to considering the coefficient $a_0(x/\varepsilon)$ alone in F_ε . Then the coefficient a_λ can be explicitly computed for every $\lambda \in [1, +\infty]$. Note that, by Remark 2.3, this is the symmetric kernel equivalent to the choice $a(x, y) = a_0(x)$ discussed in Section 1. By (2.1), the contracted function is

$$\bar{a}^c(x) = a_0(x) + \bar{a}_0, \quad \text{where } \bar{a}_0 := \int_Q a_0(\xi) d\xi, \quad (2.8)$$

while, expanding the square and using that $\int_Q \varphi(x) dx = 0$, for every test function φ in (2.5) we have

$$\int_Q \int_Q a(x, y) |\varphi(x) - \varphi(y)|^2 dx dy = \int_Q \bar{a}^c(x) |\varphi(x)|^2 dx; \quad (2.9)$$

that is, the nonlocal term in (2.4) is local. Hence, if $\lambda < +\infty$, by (2.9) and after completing the square, (2.4) reads

$$\begin{aligned} A_\lambda(1, \varphi) &= \int_Q \bar{a}^c(x) \left(|1 + \varphi(x)|^2 + (\lambda - 1) |\varphi(x)|^2 \right) dx \\ &= \lambda \int_Q \bar{a}^c(x) \left| \varphi(x) + \frac{1}{\lambda} \right|^2 dx + \left(1 - \frac{1}{\lambda} \right) \int_Q \bar{a}^c(x) dx, \end{aligned}$$

so that (2.5) reduces to minimizing the first term under the constraint $\int_Q (\varphi(x) + \frac{1}{\lambda}) dx = \frac{1}{\lambda}$. This amounts to saturating a Cauchy–Schwarz inequality, which produces the unique minimizer

$$\varphi_\lambda(x) = \frac{1}{\lambda} \left(\frac{a_1}{\bar{a}^c(x)} - 1 \right), \quad \text{where } a_1 = \left(\int_Q \frac{dx}{\bar{a}^c(x)} \right)^{-1} \quad (2.10)$$

is the harmonic mean of $\bar{a}^c$, and

$$a_\lambda = \left(1 - \frac{1}{\lambda} \right) \int_Q \bar{a}^c(x) dx + \frac{1}{\lambda} a_1 = \left(1 - \frac{1}{\lambda} \right) a_{+\infty} + \frac{1}{\lambda} a_1, \quad (2.11)$$

where we used (2.7) and (2.8). With the convention $\frac{1}{\lambda} = 0$ for $\lambda = +\infty$, both (2.10) and (2.11) hold for $\lambda = +\infty$ also, in accordance with Remark 2.2.

Thus, for such kernels, a_λ is the convex combination, with parameter $\frac{1}{\lambda}$, of the harmonic mean a_1 and the arithmetic mean $a_{+\infty} = 2\bar{a}_0$ of the contracted function.

2.2. Asymptotics of double integrals. The analysis of the asymptotic behaviour of F_ε will be based on the study of double integrals defined in sets of different types. To this end, in this section we gather some asymptotic estimates that could be of independent interest for variational problems on fractional Sobolev spaces.

The first result gives an estimate of the contribution when interactions are restricted between two scales.

Lemma 2.5. *For every $s \in (0, 1)$, every measurable set $A \subseteq \mathbb{R}^d$ with $|A| < +\infty$, every $0 < r_1 < r_2$ and every $u \in L^\infty(\mathbb{R}^d)$, it holds*

$$s \int_A \int_{\{r_1 < |x-y| < r_2\}} \frac{|u(x) - u(y)|^2}{|x - y|^{d+2s}} dy dx \leq 2\sigma_{d-1} \|u\|_{L^\infty(\mathbb{R}^d)}^2 |A| (r_1^{-2s} - r_2^{-2s}), \quad (2.12)$$

where σ_{d-1} is as in (2.6). In particular, if $(u_\varepsilon) \subseteq L^\infty(\mathbb{R}^d)$ is a bounded sequence and $r_1 := r_1(\varepsilon)$, $r_2 := r_2(\varepsilon)$ satisfy $0 < r_1 < r_2$ and

$$\lim_{\varepsilon \rightarrow 0} r_1^{-2s} = \lim_{\varepsilon \rightarrow 0} r_2^{-2s} \in [0, +\infty),$$

then

$$\lim_{\varepsilon \rightarrow 0} s \int_A \int_{\{r_1 < |x-y| < r_2\}} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^2}{|x-y|^{d+2s}} dy dx = 0. \quad (2.13)$$

Proof. Using $|u(x) - u(y)|^2 \leq 4\|u\|_{L^\infty(\mathbb{R}^d)}^2$ and integrating in polar coordinates, we obtain

$$s \int_A \int_{\{r_1 < |x-y| < r_2\}} \frac{|u(x) - u(y)|^2}{|x-y|^{d+2s}} dy dx \leq 4s\|u\|_{L^\infty(\mathbb{R}^d)}^2 |A| \sigma_{d-1} \int_{r_1}^{r_2} t^{-1-2s} dt,$$

which gives (2.12).

Applying (2.12), with $u = u_\varepsilon$, we see that the quantity in the limit in (2.13) is bounded by $2\sigma_{d-1} \sup_\varepsilon \|u_\varepsilon\|_{L^\infty(\mathbb{R}^d)}^2 |A| (r_1^{-2s} - r_2^{-2s})$, which vanishes as $\varepsilon \rightarrow 0$. $\square$

The following lemma regards the contribution of interactions in a neighbourhood of a set Σ of finite $(d-1)$ -dimensional Minkowski content.

Lemma 2.6. *Let $\Sigma \subseteq \mathbb{R}^d$ be a closed set such that there exist $C, \ell > 0$ such that*

$$|\{x \in \mathbb{R}^d : \text{dist}(x, \Sigma) < \tau\}| \leq C\tau \quad \text{for every } \tau \in (0, \ell]. \quad (2.14)$$

Then, for every $s \in (0, 1/2)$ and every $u \in L^\infty(\mathbb{R}^d)$, it holds

$$s \iint_{\{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : \text{dist}(x, \Sigma) \leq |x-y| < \ell\}} \frac{|u(x) - u(y)|^2}{|x-y|^{d+2s}} dy dx \leq 4\sigma_{d-1} C \|u\|_{L^\infty(\mathbb{R}^d)}^2 \frac{s \ell^{1-2s}}{1-2s}. \quad (2.15)$$

Proof. Set $\tau(x) := \text{dist}(x, \Sigma)$ and $f(t) := (t^{-2s} - \ell^{-2s})_+$ for $t > 0$. Note that $|\Sigma| = 0$ by (2.14). For every $x \notin \Sigma$, using $|u(x) - u(y)|^2 \leq 4\|u\|_{L^\infty(\mathbb{R}^d)}^2$ and integrating in polar coordinates, we get

$$\begin{aligned} s \int_{\{\tau(x) \leq |x-y| < \ell\}} \frac{|u(x) - u(y)|^2}{|x-y|^{d+2s}} dy &\leq 4\|u\|_{L^\infty(\mathbb{R}^d)}^2 \sigma_{d-1} s \int_{\tau(x)}^{\ell} t^{-1-2s} dt \\ &= 2\sigma_{d-1} \|u\|_{L^\infty(\mathbb{R}^d)}^2 f(\tau(x)), \end{aligned}$$

where the integral is understood to be 0 if $\tau(x) \geq \ell$. Since f is decreasing on $(0, \ell]$ and vanishes on $[\ell, +\infty)$, by Cavalieri's principle and (2.14) we have

$$\begin{aligned} \int_{\mathbb{R}^d} f(\tau(x)) dx &= \int_0^{+\infty} |\{x \in \mathbb{R}^d : \tau(x) < f^{-1}(\xi)\}| d\xi \\ &\leq C \int_0^{+\infty} f^{-1}(\xi) d\xi \\ &= C \int_0^{\ell} f(t) dt = 2C \frac{s \ell^{1-2s}}{1-2s}, \end{aligned}$$

where f^{-1} denotes the inverse of f restricted to $(0, \ell)$. Integrating the previous estimate in x gives (2.15). $\square$

2.3. Averaging interactions for singular kernels. This section is devoted to the proof of two Riemann–Lebesgue type lemmas. In some sense, they are dual to [3, Lemma 2.5], where the case $s \rightarrow 1$ is treated, and can be proven in a similar way.

Given a Borel function $b: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ which is 1-periodic, we set

$$\bar{b} := \int_Q \int_Q b(x, y) dx dy.$$

For $\varepsilon > 0$ and $k \in \mathbb{Z}^d$ we let $Q_{\varepsilon, k} := \varepsilon(k + [0, 1)^d)$ denote the cubes of the partition of $\mathbb{R}^d$ induced by the lattice $\varepsilon\mathbb{Z}^d$ and, for $A \subseteq \mathbb{R}^d$, we set $\partial_\varepsilon A := \{x \in \mathbb{R}^d : \text{dist}(x, \partial A) \leq \sqrt{d}\varepsilon\}$.

Lemma 2.7 (Averaging interactions at scales larger than ε). *Let $b \in L^1_{\text{loc}}(\mathbb{R}^d \times \mathbb{R}^d)$ be 1-periodic. Then there exists $C = C_d > 0$ such that*

$$\begin{aligned} & \left| s \int_A \int_{\{c\varepsilon < |x-y| < \ell\}} \frac{b\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right)}{|x-y|^{d+2s}} dy dx - \bar{b} \frac{\sigma_{d-1}}{2} ((c\varepsilon)^{-2s} - \ell^{-2s}) |A| \right| \\ & \leq C \|b\|_{L^1(Q^2)} (c\varepsilon)^{-2s} \left(s(1+c^{-1})^{d+4} |A| + (1+c^{-1})^d |\partial_{2\varepsilon} A| \right) \end{aligned} \quad (2.16)$$

for every $s \in (0, 1)$, every $c, \ell > 0$, every $\varepsilon > 0$ such that $c\varepsilon < \ell$ and $4\sqrt{d}\varepsilon \leq \ell$, and every Borel set $A \subseteq \mathbb{R}^d$ with $|A| < +\infty$.

Proof. In the following, $C = C_d > 0$ may change from line to line. Set

$$\kappa(x, y) := \frac{s}{|x-y|^{d+2s}}, \quad \Lambda := (A \times \mathbb{R}^d) \cap \{c\varepsilon < |x-y| < \ell\}.$$

Integrating in polar coordinates, we get

$$\int_\Lambda \kappa(x, y) dy dx = \frac{\sigma_{d-1}}{2} ((c\varepsilon)^{-2s} - \ell^{-2s}) |A|, \quad (2.17)$$

so that the left-hand side of (2.16) equals the absolute value of

$$\int_\Lambda \left(b\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) - \bar{b} \right) \kappa(x, y) dy dx.$$

We cover Λ by the sets

$$\begin{aligned} \Lambda_{1,\varepsilon} &:= \bigcup \{Q_{\varepsilon,j} \times Q_{\varepsilon,k} : Q_{\varepsilon,j} \times Q_{\varepsilon,k} \subseteq \Lambda, Q_{\varepsilon,j} \subseteq A\}, \\ \Lambda_{2,\varepsilon} &:= \Lambda \cap (\partial_\varepsilon A \times \mathbb{R}^d), \\ \Lambda_{3,\varepsilon} &:= \Lambda \setminus (\Lambda_{1,\varepsilon} \cup \Lambda_{2,\varepsilon}). \end{aligned}$$

By the 1-periodicity of b , for all $j, k \in \mathbb{Z}^d$ we have

$$\int_{Q_{\varepsilon,j}} \int_{Q_{\varepsilon,k}} \left| b\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) - \bar{b} \right| dy dx = \varepsilon^{2d} \|b - \bar{b}\|_{L^1(Q^2)} \leq 2\varepsilon^{2d} \|b\|_{L^1(Q^2)}. \quad (2.18)$$

Since the sets $\Lambda_{2,\varepsilon}$ and $\Lambda_{3,\varepsilon}$ are not unions of products of cubes, we estimate the corresponding contributions by covering them with products of cubes, and using (2.18) on each of them.

Let $Q_{\varepsilon,j} \times Q_{\varepsilon,k} \subseteq \Lambda_{1,\varepsilon}$, fix $(\bar{x}, \bar{y}) \in Q_{\varepsilon,j} \times Q_{\varepsilon,k}$, and set

$$m_{jk} := \inf\{|x-y|^{-d-1-2s} : (x, y) \in Q_{\varepsilon,j} \times Q_{\varepsilon,k}\}.$$

Since b is 1-periodic, we have

$$\int_{Q_{\varepsilon,j}} \int_{Q_{\varepsilon,k}} \left(b\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) - \bar{b} \right) \kappa(x, y) dy dx = \int_{Q_{\varepsilon,j}} \int_{Q_{\varepsilon,k}} \left(b\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) - \bar{b} \right) (\kappa(x, y) - \kappa(\bar{x}, \bar{y})) dy dx.$$

For $x \neq y$ we have $|\nabla_{(x,y)}\kappa(x,y)| \leq \sqrt{2}(d+2)s|x-y|^{-d-1-2s}$. Moreover, any two points $(x,y), (x',y') \in Q_{\varepsilon,j} \times Q_{\varepsilon,k}$ satisfy $|x'-y'| \geq c\varepsilon$ and

$$|x-y| \leq |x'-y'| + 2\sqrt{d}\varepsilon \leq (1+2\sqrt{d}c^{-1})|x'-y'| \leq 2\sqrt{d}(1+c^{-1})|x'-y'|,$$

so that, by the mean-value theorem,

$$|\kappa(x,y) - \kappa(\bar{x},\bar{y})| \leq C(1+c^{-1})^{d+3}\varepsilon s m_{jk} \quad \text{for } (x,y) \in Q_{\varepsilon,j} \times Q_{\varepsilon,k},$$

where we used that $d+1+2s < d+3$ since $s < 1$. Hence, by (2.18) and using $\varepsilon^{2d}m_{jk} \leq \int_{Q_{\varepsilon,j}} \int_{Q_{\varepsilon,k}} |x-y|^{-d-1-2s} dy dx$, summing over the cubes of $\Lambda_{1,\varepsilon}$ and integrating in polar coordinates we obtain

$$\begin{aligned} \left| \int_{\Lambda_{1,\varepsilon}} \left(b\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) - \bar{b} \right) \kappa(x,y) dy dx \right| &\leq C(1+c^{-1})^{d+3}\varepsilon s \|b\|_{L^1(Q^2)} |A| \sigma_{d-1} \int_{c\varepsilon}^{+\infty} t^{-2-2s} dt \\ &\leq C(1+c^{-1})^{d+4} \|b\|_{L^1(Q^2)} s (c\varepsilon)^{-2s} |A|. \end{aligned}$$

Let now $Q_{\varepsilon,j} \times Q_{\varepsilon,k}$ intersect $\Lambda_{2,\varepsilon} \cup \Lambda_{3,\varepsilon}$. Since $\Lambda \subseteq \{|x-y| > c\varepsilon\}$ and $||x-y| - |x'-y'|| \leq 2\sqrt{d}\varepsilon$ for $(x,y), (x',y') \in Q_{\varepsilon,j} \times Q_{\varepsilon,k}$, we have

$$\sup_{(Q_{\varepsilon,j} \times Q_{\varepsilon,k}) \cap \Lambda} \kappa \leq s \max \{c\varepsilon, |x-y| - 2\sqrt{d}\varepsilon\}^{-d-2s} \quad \text{for every } (x,y) \in Q_{\varepsilon,j} \times Q_{\varepsilon,k}. \quad (2.19)$$

If $Q_{\varepsilon,j} \times Q_{\varepsilon,k}$ intersects $\Lambda_{2,\varepsilon}$, then $Q_{\varepsilon,j}$ intersects $\partial_\varepsilon A$, so that $Q_{\varepsilon,j} \subseteq \partial_{2\varepsilon} A$; moreover, $|x-y| < \ell + 2\sqrt{d}\varepsilon$ on $Q_{\varepsilon,j} \times Q_{\varepsilon,k}$. Hence, by (2.18) and (2.19), the contribution of $\Lambda_{2,\varepsilon}$ is at most

$$2\|b\|_{L^1(Q^2)} \int_{\partial_{2\varepsilon} A} \int_{\{|x-y| < \ell + 2\sqrt{d}\varepsilon\}} s \max \{c\varepsilon, |x-y| - 2\sqrt{d}\varepsilon\}^{-d-2s} dy dx.$$

Integrating in polar coordinates, and using that $t \leq (1+2\sqrt{d}c^{-1})(t-2\sqrt{d}\varepsilon)$ for $t \geq (c+2\sqrt{d})\varepsilon$, the inner integral is bounded by

$$\sigma_{d-1}s(c\varepsilon)^{-d-2s} \frac{((c+2\sqrt{d})\varepsilon)^d}{d} + (1+2\sqrt{d}c^{-1})^{d-1} \sigma_{d-1}s \int_{c\varepsilon}^{+\infty} t^{-1-2s} dt \leq C(1+c^{-1})^d (c\varepsilon)^{-2s},$$

so that the contribution of $\Lambda_{2,\varepsilon}$ is bounded by $C(1+c^{-1})^d \|b\|_{L^1(Q^2)} (c\varepsilon)^{-2s} |\partial_{2\varepsilon} A|$.

If $Q_{\varepsilon,j} \times Q_{\varepsilon,k}$ intersects $\Lambda_{3,\varepsilon}$ at a point (x,y) , then $x \in A$ and $\text{dist}(x, \partial A) > \sqrt{d}\varepsilon$, so that $Q_{\varepsilon,j} \subseteq A$. Since $(x,y) \notin \Lambda_{1,\varepsilon}$, the set $Q_{\varepsilon,j} \times Q_{\varepsilon,k}$ contains a pair (x',y') with $|x'-y'| \leq c\varepsilon$ or $|x'-y'| \geq \ell$. As $||x-y| - |x'-y'|| \leq 2\sqrt{d}\varepsilon$ and $|x-y| < \ell$, we get that either $|x-y| \leq (c+2\sqrt{d})\varepsilon$ on $Q_{\varepsilon,j} \times Q_{\varepsilon,k}$, or $\ell - 2\sqrt{d}\varepsilon \leq |x-y| \leq \ell + 2\sqrt{d}\varepsilon$ on $Q_{\varepsilon,j} \times Q_{\varepsilon,k}$. In the first case, the supremum in (2.19) is at most $s(c\varepsilon)^{-d-2s}$; in the second case, since $\ell - 2\sqrt{d}\varepsilon \geq \ell/2$, it is at most $s(\ell/2)^{-d-2s}$. Hence, by (2.18), the contribution of $\Lambda_{3,\varepsilon}$ is at most

$$\begin{aligned} 2\|b\|_{L^1(Q^2)} |A| &\left(s(c\varepsilon)^{-d-2s} \omega_d ((c+2\sqrt{d})\varepsilon)^d + s(\ell/2)^{-d-2s} \sigma_{d-1} 4\sqrt{d}\varepsilon (2\ell)^{d-1} \right) \\ &\leq C\|b\|_{L^1(Q^2)} s \left((1+c^{-1})^d (c\varepsilon)^{-2s} + \frac{\varepsilon}{\ell} \ell^{-2s} \right) |A|, \end{aligned}$$

where ω_d denotes the volume of the unit ball. Since $\varepsilon \leq \ell$ and $\ell^{-2s} \leq (c\varepsilon)^{-2s}$, the right-hand side is bounded by $C(1+c^{-1})^{d+4} \|b\|_{L^1(Q^2)} s (c\varepsilon)^{-2s} |A|$, and (2.16) follows. $\square$

Observation 2.8. Note that, if $b \in L^\infty(Q^2)$, since $\|b\|_{L^1(Q^2)} \leq \|b\|_{L^\infty(Q^2)}$, estimate (2.16) holds a fortiori with $\|b\|_{L^\infty(Q^2)}$ in place of $\|b\|_{L^1(Q^2)}$.

Lemma 2.9 (Averaging long-range interactions). *Let $s \in (0, 1)$ and $\varepsilon > 0$. Let $\bar{a}^y$ be as in Remark 2.3. Then there exists $C = C_d > 0$ such that*

$$\left| s \int_A \int_{\mathbb{R}^d \setminus B_R} a\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) \frac{|u(x)|^2}{|x-y|^{d+2s}} dy dx - \frac{\sigma_{d-1}}{2} R^{-2s} \int_A \bar{a}^y\left(\frac{x}{\varepsilon}\right) |u(x)|^2 dx \right| \leq C \beta s \frac{\rho + \varepsilon}{R} R^{-2s} \|u\|_{L^2(A)}^2 \quad (2.20)$$

for every $\rho > 0$, every Borel set $A \subseteq B_\rho$, every $u \in L^2(A)$ and every $R \geq \max\{2\rho, \sqrt{d}\varepsilon\}$.

Proof. In the following, $C = C_d > 0$ may change from line to line. Set $\kappa(y) := s|y|^{-d-2s}$ and

$$J := s \int_A |u(x)|^2 \int_{\mathbb{R}^d \setminus B_R} a\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) \frac{1}{|y|^{d+2s}} dy dx.$$

We first replace $|x-y|$ with $|y|$. Let $x \in A$ and $|y| \geq R$. For every $\theta \in [0, 1]$ we have $|y - \theta x| \geq |y| - \rho \geq |y|/2$, since $R \geq 2\rho$. Hence, by the mean-value theorem applied to $z \mapsto |z|^{-d-2s}$ on the segment joining $y-x$ and y ,

$$\left| |x-y|^{-d-2s} - |y|^{-d-2s} \right| \leq (d+2) \rho (|y|/2)^{-d-1-2s} \leq C \rho |y|^{-d-1-2s},$$

where we used again that $d+1+2s < d+3$ since $s < 1$. Using $a \leq \beta$ and integrating in polar coordinates, we get

$$\left| s \int_A \int_{\mathbb{R}^d \setminus B_R} a\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) \frac{|u(x)|^2}{|x-y|^{d+2s}} dy dx - J \right| \leq C \beta s \rho \|u\|_{L^2(A)}^2 \sigma_{d-1} \int_R^{+\infty} t^{-2-2s} dt \leq C \beta s \frac{\rho}{R} R^{-2s} \|u\|_{L^2(A)}^2. \quad (2.21)$$

We now average in y . Fix $x \in A$ and set $g(y) := a(x/\varepsilon, y/\varepsilon)$. Integrating in polar coordinates, we have

$$\int_{\mathbb{R}^d \setminus B_R} \kappa(y) dy = \frac{\sigma_{d-1}}{2} R^{-2s},$$

so that

$$\int_{\mathbb{R}^d \setminus B_R} g(y) \kappa(y) dy - \frac{\sigma_{d-1}}{2} R^{-2s} \bar{a}^y\left(\frac{x}{\varepsilon}\right) = \int_{\mathbb{R}^d \setminus B_R} \left(g(y) - \bar{a}^y\left(\frac{x}{\varepsilon}\right) \right) \kappa(y) dy.$$

Note that $|g - \bar{a}^y(x/\varepsilon)| \leq \beta$. We partition $\mathbb{R}^d \setminus B_R$ into the sets

$$\Gamma_{1,\varepsilon} := \bigcup \{Q_{\varepsilon,k} : Q_{\varepsilon,k} \subseteq \mathbb{R}^d \setminus B_R\}, \quad \Gamma_{2,\varepsilon} := (\mathbb{R}^d \setminus B_R) \setminus \Gamma_{1,\varepsilon}.$$

Let $Q_{\varepsilon,k} \subseteq \Gamma_{1,\varepsilon}$ and fix $\bar{y} \in Q_{\varepsilon,k}$. Since a is 1-periodic in the second variable, we have

$$\int_{Q_{\varepsilon,k}} g(y) dy = \varepsilon^d \bar{a}^y\left(\frac{x}{\varepsilon}\right),$$

so that

$$\int_{Q_{\varepsilon,k}} \left(g(y) - \bar{a}^y\left(\frac{x}{\varepsilon}\right) \right) \kappa(y) dy = \int_{Q_{\varepsilon,k}} \left(g(y) - \bar{a}^y\left(\frac{x}{\varepsilon}\right) \right) (\kappa(y) - \kappa(\bar{y})) dy.$$

For $y \neq 0$ we have $|\nabla \kappa(y)| \leq (d+2)s|y|^{-d-1-2s}$. Moreover, any two points $y, y' \in Q_{\varepsilon,k}$ satisfy $|y'| \geq R \geq \sqrt{d}\varepsilon$ and $|y| \leq |y'| + \sqrt{d}\varepsilon \leq 2|y'|$. Hence, by the mean-value theorem,

$$|\kappa(y) - \kappa(\bar{y})| \leq C \varepsilon s |y|^{-d-1-2s} \quad \text{for } y \in Q_{\varepsilon,k},$$

where we used again that $d + 1 + 2s < d + 3$ since $s < 1$. Summing over the cubes of $\Gamma_{1,\varepsilon}$ and integrating in polar coordinates, we obtain

$$\left| \int_{\Gamma_{1,\varepsilon}} \left(g(y) - \bar{a}^y\left(\frac{x}{\varepsilon}\right) \right) \kappa(y) dy \right| \leq C\beta\varepsilon s \sigma_{d-1} \int_R^{+\infty} t^{-2-2s} dt \leq C\beta s \frac{\varepsilon}{R} R^{-2s}.$$

If $y \in \Gamma_{2,\varepsilon}$, the cube $Q_{\varepsilon,k}$ containing y intersects B_R , so that $|y| < R + \sqrt{d}\varepsilon$. Hence, integrating in polar coordinates,

$$\int_{\Gamma_{2,\varepsilon}} \kappa(y) dy \leq \sigma_{d-1}s \int_R^{R+\sqrt{d}\varepsilon} t^{-1-2s} dt \leq \sqrt{d}\sigma_{d-1}s\varepsilon R^{-1-2s} = Cs \frac{\varepsilon}{R} R^{-2s}.$$

Using $|g - \bar{a}^y(x/\varepsilon)| \leq \beta$ once more, we conclude that

$$\left| \int_{\mathbb{R}^d \setminus B_R} a\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) \kappa(y) dy - \frac{\sigma_{d-1}}{2} R^{-2s} \bar{a}^y\left(\frac{x}{\varepsilon}\right) \right| \leq C\beta s \frac{\varepsilon}{R} R^{-2s}$$

for every $x \in A$. Multiplying by $|u(x)|^2$ and integrating over A , we get

$$\left| J - \frac{\sigma_{d-1}}{2} R^{-2s} \int_A \bar{a}^y\left(\frac{x}{\varepsilon}\right) |u(x)|^2 dx \right| \leq C\beta s \frac{\varepsilon}{R} R^{-2s} \|u\|_{L^2(A)}^2,$$

which, together with (2.21), gives (2.20). $\square$

3. LIMINF INEQUALITY

In this section we prove that for all $u \in L^2(\mathbb{R}^d)$ and for all sequences $u_\varepsilon \in L^2(\mathbb{R}^d)$ weakly converging to u in $L^2(\mathbb{R}^d)$ we have

$$\liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) \geq F_\lambda(u).$$

To this end, we use a variant of the Fonseca–Müller blow-up method [4, 11]. This allows us to reduce to competitors defined on small cubes and converging to a constant value. From these, a periodization-by-convexity argument gives a lower bound in terms of $F_\lambda(u)$ by comparison with the problems defining a_λ .

3.1. Localization, truncation and blow-up. Let $u \in L^2(\mathbb{R}^d)$ and let $(u_\varepsilon) \subseteq L^2(\mathbb{R}^d)$ be such that u_ε converges weakly to u in $L^2(\mathbb{R}^d)$, and $F_\varepsilon(u_\varepsilon) \leq E$ for every $\varepsilon > 0$ and some fixed $E > 0$. Up to subsequences, we assume that the limit $\lim_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon)$ exists and that $|u_\varepsilon|$ converges weakly to v in $L^2(\mathbb{R}^d)$ for some $v \in L^2(\mathbb{R}^d)$. We also fix a ball $B \subseteq \mathbb{R}^d$ and a parameter $R := R(\varepsilon)$ such that

$$\lim_{\varepsilon \rightarrow 0} R = +\infty, \quad \lim_{\varepsilon \rightarrow 0} R^{-2s} = 1. \quad (3.1)$$

For every $M \in \mathbb{N}$ we set

$$u_\varepsilon^M := \max \{ \min \{ u_\varepsilon, M \}, -M \}, \quad (3.2)$$

so that, pointwise,

$$|u_\varepsilon^M| \leq M, \quad |u_\varepsilon^M(x) - u_\varepsilon^M(y)| \leq |u_\varepsilon(x) - u_\varepsilon(y)|, \quad |u_\varepsilon^M - u_\varepsilon| \leq \frac{|u_\varepsilon|^2}{M}. \quad (3.3)$$

Up to subsequences, we assume that for every $M \in \mathbb{N}$ there exists $\xi_M \in L^2(\mathbb{R}^d)$, such that u_ε^M converges weakly to ξ_M in $L^2(\mathbb{R}^d)$. By the last inequality in (3.3) and by the lower semicontinuity of the $L^1(B)$ norm with respect to the weak convergence in $L^2(B)$,

$$\|\xi_M - u\|_{L^1(B)} \leq \frac{1}{M} \sup_{\varepsilon > 0} \|u_\varepsilon\|_{L^2(\mathbb{R}^d)}^2,$$

so that $\xi_M \rightarrow u$ in $L^1(B)$ and, up to a subsequence, almost everywhere in B , as $M \rightarrow +\infty$.

For ε small enough we have $B \subseteq B_R$, so that, by the symmetry of a and by the nonnegativity of the integrand, $F_\varepsilon(u_\varepsilon) \geq \mu_{B,\varepsilon}(B)$, where $\mu_{B,\varepsilon}$ is the measure defined, for every Borel set $A \subseteq \mathbb{R}^d$, by

$$\begin{aligned} \mu_{B,\varepsilon}(A) := & 2s \int_{B \cap A} \int_{\mathbb{R}^d \setminus B_R} a\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^2}{|x - y|^{d+2s}} dy dx \\ & + s \int_{B \cap A} \int_B a\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^2}{|x - y|^{d+2s}} dy dx. \end{aligned} \quad (3.4)$$

The sequence $(\mu_{B,\varepsilon})$ is equibounded, so that, up to subsequences, $\mu_{B,\varepsilon} \rightharpoonup \mu_B$ in duality with $\mathcal{C}_b(\mathbb{R}^d)$, and, since B is open,

$$\liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) \geq \liminf_{\varepsilon \rightarrow 0} \mu_{B,\varepsilon}(B) \geq \mu_B(B) \geq \int_B \frac{d\mu_B}{d\mathcal{L}^d}(x) dx. \quad (3.5)$$

Let $Q_\rho(x)$ denote the closed hypercube of side ρ centred at x . By the Lebesgue differentiation theorem and by a diagonal argument, for $\mathcal{L}^d$ -almost every $x_0 \in B$ we may choose $\rho := \rho(\varepsilon) \rightarrow 0$ so that

$$\frac{d\mu_B}{d\mathcal{L}^d}(x_0) = \lim_{\varepsilon \rightarrow 0} \frac{\mu_{B,\varepsilon}(Q_\rho(x_0))}{\rho^d} < +\infty. \quad (3.6)$$

Since ρ is free to tend to 0 as slowly as desired, we fix x_0 to be, in addition, a Lebesgue point of v and of ξ_M for every $M \in \mathbb{N}$, and a point at which $\xi_M \rightarrow u$ as $M \rightarrow +\infty$, and we further require that

$$\lim_{\varepsilon \rightarrow 0} \rho^{-2s} = 1, \quad \lim_{\varepsilon \rightarrow 0} s\rho^{-d} = 0, \quad N := \frac{\rho}{\varepsilon} \in \mathbb{N}, \quad (3.7)$$

and that

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\rho^d} \int_{Q_\rho(x_0)} |u_\varepsilon| dx = v(x_0), \quad \lim_{\varepsilon \rightarrow 0} \frac{1}{\rho^d} \int_{Q_\rho(x_0)} u_\varepsilon^M dx = \xi_M(x_0) \quad \text{for every } M \in \mathbb{N}. \quad (3.8)$$

From now on we drop the dependence on x_0 and simply write Q_ρ ; note that $Q_\rho \subseteq B$ for ε small enough, since B is open. By (3.5) and (3.6), the Γ -liminf inequality follows once we produce a lower bound for

$$\begin{aligned} \frac{\mu_{B,\varepsilon}(Q_\rho)}{\rho^d} = & \frac{2s}{\rho^d} \int_{Q_\rho} \int_{\mathbb{R}^d \setminus B_R} a\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^2}{|x - y|^{d+2s}} dy dx \\ & + \frac{s}{\rho^d} \int_{Q_\rho} \int_B a\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^2}{|x - y|^{d+2s}} dy dx =: I_{1,\varepsilon} + I_{2,\varepsilon}. \end{aligned} \quad (3.9)$$

3.2. The periodic competitor. We fix $M \in \mathbb{N}$ and a parameter $\delta := \delta(\varepsilon) > 0$ such that

$$\lim_{\varepsilon \rightarrow 0} \frac{\delta}{\rho} = 0, \quad \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon}{\delta} = 0, \quad \lim_{\varepsilon \rightarrow 0} \delta^{-2s} = 1, \quad (3.10)$$

and, for $w \in L^2(Q_\rho)$, we set

$$G_\varepsilon(w) := \frac{s}{\rho^d} \int_{Q_\rho} \int_{Q_\rho \cap \{|x-y| > \varepsilon\}} a\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) \frac{|w(x) - w(y)|^2}{|x - y|^{d+2s}} dy dx, \quad (3.11)$$

so that, discarding nonnegative contributions and using (3.3),

$$I_{2,\varepsilon} \geq G_\varepsilon(u_\varepsilon^M). \quad (3.12)$$

We now replace u_ε^M by an ε -periodic function, in three steps: we flatten u_ε^M near ∂Q_ρ , we extend it by ρ -periodicity, and we average its translates by $\varepsilon\mathbb{Z}^d$.

Lemma 3.1 (Periodic competitor). *Let $S := Q_\rho \setminus Q_{\rho-\delta}$, where $Q_{\rho-\delta}$ denotes the closed hypercube of side $\rho - \delta$ centred at x_0 , and let*

$$c_\varepsilon := \frac{1}{|S|} \int_S u_\varepsilon^M(x) dx, \quad v_\varepsilon^M := u_\varepsilon^M \chi_{Q_{\rho-\delta}} + c_\varepsilon \chi_S,$$

extended to $\mathbb{R}^d$ by ρ -periodicity. Let N be as in (3.7) and

$$w_\varepsilon^M := \frac{1}{Nd} \sum_{k \in \{0, \dots, N-1\}^d} v_{\varepsilon, k}^M, \quad v_{\varepsilon, k}^M := v_\varepsilon^M(\cdot + \varepsilon k).$$

Then w_ε^M is ε -periodic, $\|w_\varepsilon^M\|_{L^\infty(\mathbb{R}^d)} \leq M$,

$$\frac{1}{\rho^d} \int_{Q_\rho} w_\varepsilon^M(x) dx = \frac{1}{\rho^d} \int_{Q_\rho} u_\varepsilon^M(x) dx, \quad (3.13)$$

$$\begin{aligned} G_\varepsilon(w_\varepsilon^M) &\leq G_\varepsilon(u_\varepsilon^M) + 4d \sigma_{d-1} \beta M^2 (\varepsilon^{-2s} - (\sqrt{d} \rho)^{-2s}) \frac{\delta}{\rho} \\ &\quad + 2\sigma_{d-1} \beta M^2 \left(\left(\frac{\delta}{2} \right)^{-2s} - (\sqrt{d} \rho)^{-2s} \right), \end{aligned} \quad (3.14)$$

and

$$\int_{Q_\rho} \bar{a}^y\left(\frac{x}{\varepsilon}\right) |w_\varepsilon^M(x)|^2 dx \leq \int_{Q_\rho} \bar{a}^y\left(\frac{x}{\varepsilon}\right) |u_\varepsilon^M(x)|^2 dx + d \beta M^2 \rho^{d-1} \delta. \quad (3.15)$$

Proof. By construction w_ε^M is ε -periodic and, since $|u_\varepsilon^M| \leq M$ and $|c_\varepsilon| \leq M$, also $|v_\varepsilon^M| \leq M$ and $|w_\varepsilon^M| \leq M$. By the definition of c_ε we have $\int_{Q_\rho} v_\varepsilon^M dx = \int_{Q_\rho} u_\varepsilon^M dx$, while the ρ -periodicity of v_ε^M gives $\int_{Q_\rho} v_{\varepsilon, k}^M = \int_{Q_\rho} v_\varepsilon^M$ for every $k \in \mathbb{Z}^d$; averaging over k gives (3.13).

We prove (3.14). We first compare v_ε^M with u_ε^M . Splitting the domain of integration in $G_\varepsilon(v_\varepsilon^M)$ according to whether x and y belong to $Q_{\rho-\delta}$ or to S , the contribution of $Q_{\rho-\delta} \times Q_{\rho-\delta}$ is at most $G_\varepsilon(u_\varepsilon^M)$, the contribution of $S \times S$ vanishes because v_ε^M is constant on S , and, using $|v_\varepsilon^M(x) - v_\varepsilon^M(y)| \leq 2M$, $a \leq \beta$ and $|S| \leq d \rho^{d-1} \delta$, the contribution of the remaining pairs is at most

$$\frac{8\beta M^2 s}{\rho^d} |S| \sigma_{d-1} \int_\varepsilon^{\sqrt{d}\rho} t^{-1-2s} dt \leq 4d \sigma_{d-1} \beta M^2 (\varepsilon^{-2s} - (\sqrt{d} \rho)^{-2s}) \frac{\delta}{\rho}.$$

We then compare w_ε^M with v_ε^M . Since a is 1-periodic, the integrand of G_ε is invariant under the simultaneous translation $(x, y) \mapsto (x + \varepsilon k, y + \varepsilon k)$; hence, setting $Q' := Q_\rho + \varepsilon k$ and assuming without loss of generality that $\varepsilon k \in [0, \rho]^d$, we obtain

$$G_\varepsilon(v_{\varepsilon, k}^M) = \frac{s}{\rho^d} \int_{Q'} \int_{Q' \cap \{|x-y| > \varepsilon\}} a\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) \frac{|v_\varepsilon^M(x) - v_\varepsilon^M(y)|^2}{|x-y|^{d+2s}} dy dx.$$

We write $Q' = \bigcup_{\sigma \in \{0,1\}^d} A_\sigma$ with $A_\sigma := Q' \cap (Q_\rho + \rho\sigma)$; the sets $A_\sigma - \rho\sigma$ are then pairwise disjoint subsets of Q_ρ covering it, and we split the double integral accordingly. Since $\rho/\varepsilon \in \mathbb{N}$ (recall (3.7)) and v_ε^M is ρ -periodic, the integrand is invariant under $(x, y) \mapsto (x + \rho\sigma, y + \rho\sigma)$, so that the contributions of the sets $A_\sigma \times A_\sigma$ add up to at most $G_\varepsilon(v_\varepsilon^M)$. If instead $\sigma \neq \tau$, $x \in A_\sigma$ and $y \in A_\tau$, the segment joining x and y meets $\bigcup_{j \in \mathbb{Z}^d} \partial(Q_\rho + \rho j)$, on whose $\delta/2$ -neighbourhood v_ε^M is constant; hence $v_\varepsilon^M(x) = v_\varepsilon^M(y)$ whenever $|x-y| < \delta/2$, and the contributions of the sets $A_\sigma \times A_\tau$ with $\sigma \neq \tau$ add up to at most

$$\frac{4\beta M^2 s}{\rho^d} \int_{Q'} \int_{\{\delta/2 \leq |x-y| \leq \sqrt{d}\rho\}} \frac{dy dx}{|x-y|^{d+2s}} = 2\sigma_{d-1} \beta M^2 \left(\left(\frac{\delta}{2} \right)^{-2s} - (\sqrt{d} \rho)^{-2s} \right).$$

Since G_ε is convex, averaging over $k \in \{0, \dots, N-1\}^d$ gives (3.14).

Finally we prove (3.15). Since $|u_\varepsilon^M| \leq |u_\varepsilon|$ pointwise, it is enough to prove it with u_ε^M in place of u_ε . As $v_\varepsilon^M = u_\varepsilon^M$ on $Q_{\rho-\delta}$ and $|v_\varepsilon^M| \leq M$, using $\bar{a}^y \leq \beta$ and $|S| \leq d\rho^{d-1}\delta$ we get

$$\int_{Q_\rho} \bar{a}^y\left(\frac{x}{\varepsilon}\right) |v_\varepsilon^M(x)|^2 dx \leq \int_{Q_\rho} \bar{a}^y\left(\frac{x}{\varepsilon}\right) |u_\varepsilon^M(x)|^2 dx + d\beta M^2 \rho^{d-1} \delta.$$

Moreover $\bar{a}^y$ is 1-periodic, so that, arguing as above but without the contributions of the sets $A_\sigma \times A_\tau$ with $\sigma \neq \tau$, we obtain $\int_{Q_\rho} \bar{a}^y(x/\varepsilon) |v_{\varepsilon,k}^M|^2 dx = \int_{Q_\rho} \bar{a}^y(x/\varepsilon) |v_\varepsilon^M|^2 dx$ for every $k \in \mathbb{Z}^d$; the convexity of $t \mapsto t^2$ concludes the proof. $\square$

3.3. Lower bounds in the cell. We set

$$\nu_\varepsilon := \frac{1}{\rho^d} \int_{Q_\rho} w_\varepsilon^M(x) dx, \quad \varphi_\varepsilon(z) := w_\varepsilon^M(\varepsilon z) - \nu_\varepsilon, \quad (3.16)$$

so that φ_ε is 1-periodic with zero mean on Q , $|\nu_\varepsilon| \leq M$ and $\|\varphi_\varepsilon\|_{L^\infty(\mathbb{R}^d)} \leq 2M$. By (3.13) and (3.8),

$$\lim_{\varepsilon \rightarrow 0} \nu_\varepsilon = \xi_M(x_0). \quad (3.17)$$

Moreover, since $x \mapsto \bar{a}^y(x/\varepsilon) |w_\varepsilon^M(x)|^2$ is ε -periodic and Q_ρ is the union of N^d periodicity cells,

$$\frac{1}{\rho^d} \int_{Q_\rho} \bar{a}^y\left(\frac{x}{\varepsilon}\right) |w_\varepsilon^M(x)|^2 dx = \int_Q \bar{a}^y(z) |\nu_\varepsilon + \varphi_\varepsilon(z)|^2 dz. \quad (3.18)$$

Long-range interactions. Expanding the square in $I_{1,\varepsilon}$, see (3.9), and dropping the nonnegative term containing $|u_\varepsilon(y)|^2$, we get

$$I_{1,\varepsilon} \geq \frac{2s}{\rho^d} \int_{Q_\rho} \int_{\mathbb{R}^d \setminus B_R} a\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) \frac{|u_\varepsilon(x)|^2}{|x-y|^{d+2s}} dy dx - J_\varepsilon,$$

where

$$J_\varepsilon := \frac{4s}{\rho^d} \int_{Q_\rho} \int_{\mathbb{R}^d \setminus B_R} a\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) \frac{u_\varepsilon(x)u_\varepsilon(y)}{|x-y|^{d+2s}} dy dx. \quad (3.19)$$

Let $r > 0$ be a fixed radius such that $B \subseteq B_r$. For ε small enough we have $R \geq 2r$, so that $|x-y| \geq |y|/2$ for every $x \in B$ and $y \in \mathbb{R}^d \setminus B_R$ and, integrating in polar coordinates, one obtains

$$\int_{\mathbb{R}^d \setminus B_R} \frac{dy}{|x-y|^{2d+4s}} \leq C_d R^{-d-4s} \quad \text{for every } x \in B.$$

Indeed, unlike its L^1 norm, the L^2 norm of the kernel vanishes as $R \rightarrow +\infty$. Hence, by $a \leq \beta$, applying the Cauchy-Schwarz inequality in y , and since the resulting bound is uniform in x , it follows that

$$|J_\varepsilon| \leq C_d \beta s R^{-\frac{d}{2}-2s} \|u_\varepsilon\|_{L^2(\mathbb{R}^d)} \frac{1}{\rho^d} \int_{Q_\rho} |u_\varepsilon(x)| dx,$$

which vanishes as $\varepsilon \rightarrow 0$ by (3.8), since $s \rightarrow 0$ and $R \rightarrow +\infty$. Applying Lemma 2.9 with $A = Q_\rho \subseteq B_r$ and $u = u_\varepsilon$, which is admissible for ε small enough, and then (3.15) and (3.18), we conclude that

$$I_{1,\varepsilon} \geq \sigma_{d-1} R^{-2s} \int_Q \bar{a}^y(z) |\nu_\varepsilon + \varphi_\varepsilon(z)|^2 dz - \sigma_{d-1} \beta d M^2 \frac{\delta}{\rho} - J_\varepsilon - K_\varepsilon, \quad (3.20)$$

where $K_\varepsilon := \frac{2C_d \beta}{\rho^d} s \frac{r+\varepsilon}{R} R^{-2s} \|u_\varepsilon\|_{L^2(\mathbb{R}^d)}^2$ vanishes as $\varepsilon \rightarrow 0$ by (3.7).

Short-range interactions. We set $b_\varepsilon(x, y) := a(x, y)|\varphi_\varepsilon(x) - \varphi_\varepsilon(y)|^2$, which is a 1-periodic function with $0 \leq b_\varepsilon \leq 16\beta M^2$ and

$$\|b_\varepsilon\|_{L^1(Q^2)} = \overline{b_\varepsilon} = \int_Q \int_Q a(x, y)|\varphi_\varepsilon(x) - \varphi_\varepsilon(y)|^2 dx dy \geq 2\alpha \|\varphi_\varepsilon\|_{L^2(Q)}^2, \quad (3.21)$$

where the last inequality follows from $a \geq \alpha$ and from $\int_Q \varphi_\varepsilon = 0$. Since $w_\varepsilon^M(x) - w_\varepsilon^M(y) = \varphi_\varepsilon(x/\varepsilon) - \varphi_\varepsilon(y/\varepsilon)$, and since $y \in Q_\rho$ whenever $x \in Q_{\rho-2\delta}$ and $|x - y| < \delta$, discarding nonnegative contributions in $G_\varepsilon(w_\varepsilon^M)$ and applying Lemma 2.7 with $b = b_\varepsilon$, $A = Q_{\rho-2\delta}$, $c = 1$ and outer radius δ , we get

$$G_\varepsilon(w_\varepsilon^M) \geq \frac{\sigma_{d-1}}{2} (\varepsilon^{-2s} - \delta^{-2s}) \overline{b_\varepsilon} \frac{|Q_{\rho-2\delta}|}{\rho^d} - C_d \overline{b_\varepsilon} \varepsilon^{-2s} \left(2^{d+4} s + 2^d \frac{|\partial_{2\varepsilon} Q_{\rho-2\delta}|}{\rho^d} \right). \quad (3.22)$$

Since $|Q_{\rho-2\delta}|/\rho^d \rightarrow 1$, $|\partial_{2\varepsilon} Q_{\rho-2\delta}| \leq C_d \rho^{d-1} \varepsilon$ and $\overline{b_\varepsilon} \leq 16\beta M^2$, combining (3.12), (3.14) and (3.22) we obtain

$$\begin{aligned} I_{2,\varepsilon} &\geq \frac{\sigma_{d-1}}{2} (\varepsilon^{-2s} - \delta^{-2s}) \overline{b_\varepsilon} (1 + o(1)) \\ &\quad - C M^2 \left(\varepsilon^{-2s} \left(s + \frac{\varepsilon}{\rho} + \frac{\delta}{\rho} \right) + \left(\frac{\delta}{2} \right)^{-2s} - (\sqrt{d} \rho)^{-2s} \right), \end{aligned} \quad (3.23)$$

where $C = C(d, \alpha, \beta)$.

3.4. Conclusion. Now, we divide the two cases $\lambda < +\infty$ and $\lambda = +\infty$, which are treated separately.

The case $\lambda < +\infty$. Here $\varepsilon^{-2s} \rightarrow \lambda$ is bounded and $\varepsilon^{-2s} - \delta^{-2s} \rightarrow \lambda - 1$, so that the error term in (3.23) vanishes by (3.7) and (3.10), and

$$I_{2,\varepsilon} \geq \frac{\sigma_{d-1}}{2} (\lambda - 1) \int_Q \int_Q a(x, y)|\varphi_\varepsilon(x) - \varphi_\varepsilon(y)|^2 dx dy + o(1) \quad \text{as } \varepsilon \rightarrow 0.$$

Gathering this estimate with (3.9) and (3.20), and recalling (2.4) and that $R^{-2s} \rightarrow 1$, we get

$$\frac{\mu_{B,\varepsilon}(Q_\rho)}{\rho^d} \geq \frac{\sigma_{d-1}}{2} A_\lambda(\nu_\varepsilon, \varphi_\varepsilon) + o(1) \quad \text{as } \varepsilon \rightarrow 0. \quad (3.24)$$

The case $\lambda = +\infty$. Here we discard $I_{2,\varepsilon} \geq 0$ in (3.9), and we only use (3.23) to rule out the oscillations of w_ε^M . Indeed, $\varepsilon^{2s} \delta^{-2s} \rightarrow 0$, so that $\varepsilon^{2s} (\varepsilon^{-2s} - \delta^{-2s}) \rightarrow 1$; multiplying (3.23) by ε^{2s} , and using that $\varepsilon^{2s} (\varepsilon^{-2s} - \delta^{-2s}) \leq 1$ and that $I_{2,\varepsilon} \leq \mu_{B,\varepsilon}(Q_\rho)/\rho^d$ is bounded by (3.6), we obtain

$$\frac{\sigma_{d-1}}{2} \overline{b_\varepsilon} (1 + o(1)) \leq \varepsilon^{2s} I_{2,\varepsilon} + C M^2 \left(s + \frac{\varepsilon}{\rho} + \frac{\delta}{\rho} + \varepsilon^{2s} \right) \rightarrow 0,$$

therefore $\overline{b_\varepsilon} \rightarrow 0$ and, by (3.21), $\varphi_\varepsilon \rightarrow 0$ in $L^2(Q)$. Since $|\nu_\varepsilon| \leq M$ and $\overline{a^y} \leq \beta$, this gives

$$\int_Q \overline{a^y}(z) |\nu_\varepsilon + \varphi_\varepsilon(z)|^2 dz = |\nu_\varepsilon|^2 \int_Q \overline{a^y}(z) dz + o(1) = \overline{a} |\nu_\varepsilon|^2 + o(1),$$

so that, by (3.9) and (3.20),

$$\frac{\mu_{B,\varepsilon}(Q_\rho)}{\rho^d} \geq \sigma_{d-1} \overline{a} |\nu_\varepsilon|^2 + o(1) = \frac{\sigma_{d-1}}{2} a_{+\infty} |\nu_\varepsilon|^2 + o(1) \quad \text{as } \varepsilon \rightarrow 0,$$

where we used (2.7).

Conclusion. In both cases, since $A_\lambda(z, z\varphi) = z^2 A_\lambda(1, \varphi)$, we have $\inf_\varphi A_\lambda(z, \varphi) = a_\lambda z^2$, so that

$$\frac{\mu_{B,\varepsilon}(Q_\rho)}{\rho^d} \geq \frac{\sigma_{d-1}}{2} a_\lambda |\nu_\varepsilon|^2 + o(1) \quad \text{as } \varepsilon \rightarrow 0.$$

Hence, by (3.6) and (3.17), and letting $M \rightarrow +\infty$,

$$\frac{d\mu_B}{d\mathcal{L}^d}(x_0) \geq \frac{\sigma_{d-1}}{2} a_\lambda |u(x_0)|^2 \quad \text{for } \mathcal{L}^d\text{-almost every } x_0 \in B. \quad (3.25)$$

Integrating (3.25) over B and recalling (3.5), we obtain

$$\liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) \geq \frac{\sigma_{d-1}}{2} a_\lambda \int_B |u(x)|^2 dx.$$

Since B is an arbitrary ball, taking an increasing sequence of balls invading $\mathbb{R}^d$, by monotone convergence we conclude that

$$\liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) \geq \frac{\sigma_{d-1}}{2} a_\lambda \|u\|_{L^2(\mathbb{R}^d)}^2 = F_\lambda(u),$$

which is the Γ -liminf inequality.

4. LIMSUP INEQUALITY

The limit functional F_λ is continuous with respect to the strong topology of $L^2(\mathbb{R}^d)$; therefore, by a diagonal argument it suffices to prove the existence of a recovery sequence, for every u in a strongly dense subset of $L^2(\mathbb{R}^d)$. In our case, we consider the functions $u \in L^2(\mathbb{R}^d)$ whose support is essentially contained in a cube and which are piecewise constant on dyadic cubes of arbitrarily small side length. Without loss of generality, we may assume that u is supported in the reference cube $[0, 1]^d$ and that there exists $n \in \mathbb{N}$ such that

$$u = \sum_{k \in K_n} u_k \chi_{Q_k}, \quad (4.1)$$

where $u_k \in \mathbb{R}$, $Q_k := 2^{-n}(k + [0, 1]^d)$ and $K_n := \{k \in \mathbb{Z}^d : Q_k \subseteq [0, 1]^d\} = \{0, \dots, 2^n - 1\}^d$.

We fix $\delta > 0$, and let $\varphi \in L^2(Q)$ with zero mean be such that

$$A_\lambda(1, \varphi) \leq a_\lambda + \delta. \quad (4.2)$$

The function φ is extended to $\mathbb{R}^d$ by 1-periodicity. Note that, by a density argument, we may assume $\varphi \in C^\infty(\mathbb{R}^d)$ and that if $\lambda = +\infty$, then $\varphi = 0$ (see Remark 2.2). We then set $L := \text{Lip}(\varphi) \max_{k \in K_n} |u_k| < +\infty$.

We define the recovery sequence as

$$u_\varepsilon(x) := u(x) \left(1 + \varphi\left(\frac{x}{\varepsilon}\right)\right), \quad x \in \mathbb{R}^d. \quad (4.3)$$

First, note that $u_\varepsilon \xrightarrow{L^2} u$, and there exists $C > 0$, independent of ε , such that $\|u_\varepsilon\|_{L^\infty(\mathbb{R}^d)} \leq C$. Moreover, for every $k \in K_n$, the restriction of $u_\varepsilon - u$ to Q_k is smooth and

$$\|\nabla(u_\varepsilon - u)\|_{L^\infty(Q_k)} = \frac{|u_k|}{\varepsilon} \|\nabla \varphi\|_{L^\infty(\mathbb{R}^d)} \leq \frac{L}{\varepsilon}. \quad (4.4)$$

We also point out that $u_\varepsilon \in H^t(\mathbb{R}^d)$ for every $t \in (0, 1/2)$.

We fix a parameter $R := R(\varepsilon) > 0$ such that

$$\lim_{\varepsilon \rightarrow 0} R = +\infty, \quad \lim_{\varepsilon \rightarrow 0} R^{-2s} = 1.$$

We also set

$$\Sigma := \bigcup_{k \in K_n} \partial Q_k.$$

Note that Σ depends only on u , and that it satisfies

$$|\{x \in \mathbb{R}^d : \text{dist}(x, \Sigma) < \tau\}| \leq C_\Sigma \tau \quad \text{for every } \tau \in (0, 1], \quad (4.5)$$

where $C_\Sigma > 0$ depends only on d and u . We decompose the interactions into several regions and discuss separately the contributions. Given a measurable set $\Lambda \subseteq \mathbb{R}^d \times \mathbb{R}^d$, we consider the localized functional

$$F_\varepsilon(v, \Lambda) := s \iint_{\Lambda} a\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) \frac{|v(x) - v(y)|^2}{|x - y|^{d+2s}} dx dy. \quad (4.6)$$

For ε small enough so that $[0, 1]^d \subseteq B_R$, we divide $\mathbb{R}^d \times \mathbb{R}^d$ into the following sets

$$\begin{aligned} \Lambda_{1,\varepsilon} &:= \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : x \notin B_R \text{ or } y \notin B_R\}, \\ \Lambda_{2,\varepsilon} &:= \bigcup_{k \in K_n} \{(x, y) \in Q_k \times Q_k : |x - y| < 1\}, \\ \Lambda_{3,\varepsilon} &:= \{(x, y) \in B_R \times B_R : |x - y| \geq 1\}, \\ \Lambda_{4,\varepsilon} &:= \{(x, y) \in B_R \times B_R : |x - y| < 1\} \setminus \Lambda_{2,\varepsilon}. \end{aligned}$$

We note that $\Lambda_{2,\varepsilon}$ does not depend on ε , but we include the index to simplify the notation throughout. For every $\varepsilon > 0$ small enough and every $v \in L^2(\mathbb{R}^d)$, it holds

$$F_\varepsilon(v) = \sum_{k=1}^4 F_\varepsilon(v, \Lambda_{k,\varepsilon}). \quad (4.7)$$

We show that the contributions coming from sets $\Lambda_{3,\varepsilon}$ and $\Lambda_{4,\varepsilon}$ are asymptotically negligible.

We first consider $\Lambda_{3,\varepsilon}$. Since u_ε vanishes outside $[0, 1]^d$, only pairs $(x, y) \in \Lambda_{3,\varepsilon}$ with $x \in [0, 1]^d$ or $y \in [0, 1]^d$ contribute, and for such pairs we have $1 \leq |x - y| < 2R$. Hence, using the symmetry of a , $a \leq \beta$ and Lemma 2.5 with $A = [0, 1]^d$, $r_1 = 1$ and $r_2 = 2R$, we obtain

$$\begin{aligned} F_\varepsilon(u_\varepsilon, \Lambda_{3,\varepsilon}) &\leq 2\beta s \int_{[0,1]^d} \int_{\{1 \leq |x-y| < 2R\}} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^2}{|x - y|^{d+2s}} dy dx \\ &\leq 4\beta \sigma_{d-1} \sup_{\varepsilon} \|u_\varepsilon\|_{L^\infty(\mathbb{R}^d)}^2 (1 - (2R)^{-2s}). \end{aligned}$$

Since $R^{-2s} \rightarrow 1$ and (u_ε) is bounded in $L^\infty(\mathbb{R}^d)$, it follows that

$$\limsup_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon, \Lambda_{3,\varepsilon}) = 0. \quad (4.8)$$

We now consider $\Lambda_{4,\varepsilon}$. Again, only pairs $(x, y) \in \Lambda_{4,\varepsilon}$ with $x \in [0, 1]^d$ or $y \in [0, 1]^d$ contribute. For such pairs, x and y do not belong to the same cube Q_k , so that the segment that joins them intersects Σ and $\text{dist}(x, \Sigma) \leq |x - y| < 1$. Therefore, by (4.5) and Lemma 2.6, for ε small enough so that $s < 1/2$, we get

$$\begin{aligned} F_\varepsilon(u_\varepsilon, \Lambda_{4,\varepsilon}) &\leq \beta s \iint_{\{\text{dist}(x, \Sigma) \leq |x-y| < 1\}} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^2}{|x - y|^{d+2s}} dy dx \\ &\leq 4\beta \sigma_{d-1} C_\Sigma \sup_{\varepsilon} \|u_\varepsilon\|_{L^\infty(\mathbb{R}^d)}^2 \frac{s}{1 - 2s}. \end{aligned}$$

Since $s \rightarrow 0$ and (u_ε) is bounded in $L^\infty(\mathbb{R}^d)$, it follows that

$$\limsup_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon, \Lambda_{4,\varepsilon}) = 0. \quad (4.9)$$

We now consider $\Lambda_{1,\varepsilon}$. Since u_ε vanishes outside $[0, 1]^d$ and $[0, 1]^d \subseteq B_R$, by the symmetry of a we have

$$F_\varepsilon(u_\varepsilon, \Lambda_{1,\varepsilon}) = 2 \sum_{k \in K_n} s |u_k|^2 \int_{Q_k} \int_{\mathbb{R}^d \setminus B_R} a\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) \frac{|1 + \varphi(x/\varepsilon)|^2}{|x - y|^{d+2s}} dy dx.$$

For ε small enough, we have $R \geq \max\{2\sqrt{d}, \sqrt{d}\varepsilon\}$. Hence, applying Lemma 2.9 with $A = Q_k \subseteq B_{\sqrt{d}}$, $\rho = \sqrt{d}$ and $u = u_\varepsilon$, we get

$$\left| F_\varepsilon(u_\varepsilon, \Lambda_{1,\varepsilon}) - \sigma_{d-1} R^{-2s} \sum_{k \in K_n} |u_k|^2 \int_{Q_k} \bar{a}^y\left(\frac{x}{\varepsilon}\right) \left| 1 + \varphi\left(\frac{x}{\varepsilon}\right) \right|^2 dx \right| \leq 2C_d \beta s \frac{\sqrt{d} + \varepsilon}{R} R^{-2s} \|u_\varepsilon\|_{L^2(\mathbb{R}^d)}^2,$$

where C_d is the constant given by Lemma 2.9. Since $s \rightarrow 0$, $R \rightarrow +\infty$, $R^{-2s} \rightarrow 1$ and (u_ε) is bounded in $L^2(\mathbb{R}^d)$, it follows that

$$\lim_{\varepsilon \rightarrow 0} \left(F_\varepsilon(u_\varepsilon, \Lambda_{1,\varepsilon}) - \sigma_{d-1} \sum_{k \in K_n} |u_k|^2 \int_{Q_k} \bar{a}^y\left(\frac{x}{\varepsilon}\right) \left| 1 + \varphi\left(\frac{x}{\varepsilon}\right) \right|^2 dx \right) = 0.$$

Finally, since the function $x \mapsto \bar{a}^y(x) |1 + \varphi(x)|^2$ is 1-periodic and bounded, by the Riemann–Lebesgue lemma, we obtain

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon, \Lambda_{1,\varepsilon}) &= \frac{\sigma_{d-1}}{2} \sum_{k \in K_n} |Q_k| |u_k|^2 2 \int_Q \bar{a}^y(x) |1 + \varphi(x)|^2 dx \\ &= \frac{\sigma_{d-1}}{2} \left(2 \int_Q \bar{a}^y(x) |1 + \varphi(x)|^2 dx \right) \|u\|_{L^2(\mathbb{R}^d)}^2. \end{aligned} \quad (4.10)$$

We finally consider $\Lambda_{2,\varepsilon}$. We first assume that $\lambda < +\infty$. For $x, y \in Q_k$ we have $u_\varepsilon(x) - u_\varepsilon(y) = u_k(\varphi(x/\varepsilon) - \varphi(y/\varepsilon))$; therefore, enlarging the domain of integration in y , we get

$$F_\varepsilon(u_\varepsilon, \Lambda_{2,\varepsilon}) \leq \sum_{k \in K_n} s |u_k|^2 \int_{Q_k} \int_{\{|x-y| < 1\}} a\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) \frac{|\varphi(x/\varepsilon) - \varphi(y/\varepsilon)|^2}{|x-y|^{d+2s}} dy dx.$$

We split the inner integral into the contributions of $\{|x-y| \leq \varepsilon\}$ and $\{\varepsilon < |x-y| < 1\}$.

Since $|u_k| |\varphi(x/\varepsilon) - \varphi(y/\varepsilon)| \leq L|x-y|/\varepsilon$, integrating in polar coordinates we get

$$\begin{aligned} \sum_{k \in K_n} s |u_k|^2 \int_{Q_k} \int_{\{|x-y| \leq \varepsilon\}} a\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) \frac{|\varphi(x/\varepsilon) - \varphi(y/\varepsilon)|^2}{|x-y|^{d+2s}} dy dx &\leq \beta L^2 \varepsilon^{-2s} \sigma_{d-1} \int_0^\varepsilon t^{1-2s} dt \\ &= \frac{\beta \sigma_{d-1} L^2}{2-2s} s \varepsilon^{-2s}, \end{aligned}$$

which vanishes as $\varepsilon \rightarrow 0$, since $s \rightarrow 0$ and $\varepsilon^{-2s} \rightarrow \lambda < +\infty$.

Now, we set $b(x, y) := a(x, y) |\varphi(x) - \varphi(y)|^2$, which is a 1-periodic Borel function with $0 \leq b \leq 4\beta \|\varphi\|_{L^\infty(\mathbb{R}^d)}^2$ and

$$\bar{b} = \int_Q \int_Q a(x, y) |\varphi(x) - \varphi(y)|^2 dx dy.$$

Applying Lemma 2.7 with $A = Q_k$, $c = 1$ and outer radius 1, together with Observation 2.8, for ε small enough we get

$$\begin{aligned} s \int_{Q_k} \int_{\{\varepsilon < |x-y| < 1\}} a\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) \frac{|\varphi(x/\varepsilon) - \varphi(y/\varepsilon)|^2}{|x-y|^{d+2s}} dy dx &\leq \frac{\sigma_{d-1}}{2} (\varepsilon^{-2s} - 1) \bar{b} |Q_k| \\ &\quad + 4C_d \beta \|\varphi\|_{L^\infty(\mathbb{R}^d)}^2 \varepsilon^{-2s} (2^{d+4} s |Q_k| + 2^d |\partial_{2\varepsilon} Q_k|), \end{aligned}$$

where C_d is the constant given by Lemma 2.7. Since $s \rightarrow 0$, $\varepsilon^{-2s} \rightarrow \lambda$, and $|\partial_{2\varepsilon} Q_k| \rightarrow 0$, it follows that

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon, \Lambda_{2,\varepsilon}) &\leq \frac{\sigma_{d-1}}{2} \sum_{k \in K_n} |u_k|^2 |Q_k| (\lambda - 1) \int_Q \int_Q a(x, y) |\varphi(x) - \varphi(y)|^2 dx dy \\ &= \frac{\sigma_{d-1}}{2} (\lambda - 1) \int_Q \int_Q a(x, y) |\varphi(x) - \varphi(y)|^2 dx dy \|u\|_{L^2(\mathbb{R}^d)}^2. \end{aligned} \quad (4.11)$$

If $\lambda = +\infty$, then $\varphi = 0$, so that $F_\varepsilon(u_\varepsilon, \Lambda_{2,\varepsilon}) = 0$ and (4.11) holds with the convention $(+\infty) \cdot 0 = 0$.

Gathering (4.7), (4.8), (4.9), (4.10) and (4.11), and recalling (2.4), we obtain

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) &\leq \limsup_{\varepsilon \rightarrow 0} (F_\varepsilon(u_\varepsilon, \Lambda_{1,\varepsilon}) + F_\varepsilon(u_\varepsilon, \Lambda_{2,\varepsilon})) + \sum_{k=3}^4 \limsup_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon, \Lambda_{k,\varepsilon}) \\ &\leq \frac{\sigma_{d-1}}{2} A_\lambda(1, \varphi) \|u\|_{L^2(\mathbb{R}^d)}^2. \end{aligned}$$

By (4.2), we conclude that

$$\limsup_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) \leq F_\lambda(u) + \frac{\sigma_{d-1}}{2} \delta \|u\|_{L^2(\mathbb{R}^d)}^2. \quad (4.12)$$

By the arbitrariness of $\delta \rightarrow 0$ and a diagonal argument, this gives the Γ -limsup inequality.

5. THE CASE OF A GENERAL $1 < p < \infty$

In this section, we briefly comment on the modifications needed to prove the analogue of Theorem 1.1 in the case of $1 < p < \infty$. The preparatory results of Section 2.2 hold for general p , *mutatis mutandis*, and similarly for the strategies carried out in the Γ -liminf/limsup inequalities. The only role of $p = 2$ is played in the Γ -liminf inequality, for expanding $|u(x) - u(y)|^2$ and consequently proving an estimate of the form

$$2s \int_B \int_{\mathbb{R}^d \setminus B_R} a\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^2}{|y|^{d+2s}} dy dx \geq 2s \int_B \int_{\mathbb{R}^d \setminus B_R} a\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) \frac{|u_\varepsilon(x)|^2}{|y|^{d+2s}} dy dx + o(1), \quad (5.1)$$

where $B \subset \mathbb{R}^d$ is a fixed ball and $R \gg 1$. Indeed, expanding the square and throwing away the contribution from $|u(y)|^2$, one obtains an error where $u(y)$ appears to be raised to a lower power. This allows us to bound it by using that the L^2 -norm of the singular kernels $s|y|^{-d-2s}$ in the region $\mathbb{R}^d \setminus B_R$ goes to zero as $R \rightarrow +\infty$ and $s \rightarrow 0$.

In the case of a general p , there is a natural replacement of this procedure. Indeed, for every $t, \tau \in \mathbb{R}$ it holds

$$||t - \tau|^p - |t|^p| \leq C_p (|t|^{p-1} |\tau| + |\tau|^p), \quad C_p := p \max\{1, 2^{p-2}\},$$

so that, integrating $||u_\varepsilon(x) - u_\varepsilon(y)|^p - |u_\varepsilon(x)|^p|$ against $s|y|^{-d-sp}$, the error in the analog of (5.1) is bounded by

$$2\beta C_p s \int_B \int_{\mathbb{R}^d \setminus B_R} \frac{|u_\varepsilon(x)|^{p-1} |u_\varepsilon(y)|}{|y|^{d+sp}} dy dx + 2\beta C_p s \int_B \int_{\mathbb{R}^d \setminus B_R} \frac{|u_\varepsilon(y)|^p}{|y|^{d+sp}} dy dx.$$

In the formula above, the second term is bounded by $C_{B,d,p} s R^{-d-sp} \sup_{\varepsilon > 0} \|u_\varepsilon\|_{L^p(\mathbb{R}^d)}^p$, which is infinitesimal, while it can be shown that the first term goes to zero by improving the integrability of the kernels as in the case $p = 2$, using a bound in some $L^{1+\delta}$ with $\delta > 0$ instead of L^2 .

Finally, the complete statement in the general case is as follows.

Theorem 5.1 (Homogenized Maz'ya–Shaposhnikova). *Let $a: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a Borel 1-periodic function such that there exist $0 < \alpha \leq \beta$ for which $\alpha \leq a(x, y) \leq \beta$ for all $x, y \in \mathbb{R}^d$. Let $1 < p < \infty$, $\varepsilon > 0$ and $s := s(\varepsilon) \in (0, 1)$. Let*

$$F_{\varepsilon,p}(u) := s \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) \frac{|u(x) - u(y)|^p}{|x - y|^{d+sp}} dx dy \quad (5.2)$$

be defined in $L^p(\mathbb{R}^d)$. If $\lim_{\varepsilon \rightarrow 0} s = 0$ and

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-sp} = \lambda \in [1, +\infty],$$

then

$$\Gamma\text{-}\lim_{\varepsilon \rightarrow 0} F_{\varepsilon,p}(u) = \frac{\sigma_{d-1}}{p} a_{\lambda,p} \|u\|_{L^p(\mathbb{R}^d)}^p,$$

for all $u \in L^p(\mathbb{R}^d)$, where the Γ -limit is computed with respect to the sequential weak topology of $L^p(\mathbb{R}^d)$, and $a_{\lambda,p}$ is given by

$$\begin{aligned} a_{\lambda,p} := \inf \left\{ \int_Q \int_Q (a(x, y) + a(y, x)) |1 + \varphi(x)|^p dx dy \right. \\ \left. + (\lambda - 1) \int_Q \int_Q a(x, y) |\varphi(x) - \varphi(y)|^p dx dy : \varphi \in L^p(Q) \text{ and } \int_Q \varphi(x) dx = 0 \right\} \end{aligned} \quad (5.3)$$

if $\lambda < +\infty$, while

$$a_{+\infty,p} := 2 \int_Q \int_Q a(x, y) dx dy. \quad (5.4)$$

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