Sobolev maps into the projective line with bounded total variation

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Abstract. *Variational problems for Sobolev maps with bounded total variation that take values into the 1-dimensional projective space are studied. We focus on the different features from the case of Sobolev maps with bounded conformal p-energy that take values into the p-dimensional projective space, for $p \geq 2$ integer, recently studied in [19].*

In the last decades there has been a growing interest in the study of variational problems for maps defined between manifolds. The most relevant problem is perhaps the one concerned with harmonic maps defined in three dimensional domains Ω that are constrained to take values into the two-dimensional unit sphere $\mathbb{S}^2$.

In this framework, one considers the Dirichlet energy

$$\mathbf{D}(u,\Omega) := \frac{1}{2}\int_\Omega |Du|^2\,dx$$

of Sobolev maps into $\mathbb{S}^2$, i.e., in the class

$$W^{1,2}(\Omega,\mathbb{S}^2) := \{u \in W^{1,2}(\Omega,\mathbb{R}^3) \,:\, |u(x)| = 1 \quad \text{for a.e. } x \in \Omega\}\,.$$

According to the continuum description in the Ericksen-Leslie theory, the unitary vector field $u(x)$ describes mathematically the configuration of a liquid crystal which occupies the domain Ω.

The general form of the energy density of a liquid crystal was derived independently by Oseen and Frank, compare e.g. [13, Vol. II, Sec. 5.1] and the references therein. For a particular choice of the physical constants, the energy of a nematic liquid crystal reduces to the Dirichlet energy above.

It is well-known that in the classical Sobolev approach to the theory of harmonic maps, the weak limit process destroys energy concentration, the so called bubbling-off phenomenon, and does not preserve geometric properties such as the degree, showing e.g. creation of cavitations.

For this reason, using tools from *Geometric measure theory*, variational problems concerning harmonic maps into the sphere have been tackled in a satisfactory way in any dimension n by means of the theory of *Cartesian currents* of Giaquinta-Modica-Souček [13], see also [15].

In a similar way, an exhaustive variational theory of liquid crystals has been developed in [10].

The same authors in [11] considered the *conformal* $\mathfrak{p}$*-energy*

$$\mathbf{D}_\mathfrak{p}(u,B^n) := \frac{1}{\mathfrak{p}^{\mathfrak{p}/2}}\int_{B^n} |Du|^\mathfrak{p}\,dx$$

of $W^{1,\mathfrak{p}}$-mappings from the unit ball B^n with values into the unit $\mathfrak{p}$-sphere $\mathbb{S}^\mathfrak{p}$, for any integer exponent $\mathfrak{p} \geq 2$, i.e., in the class

$$W^{1,\mathfrak{p}}(B^n,\mathbb{S}^\mathfrak{p}) := \{u \in W^{1,\mathfrak{p}}(B^n,\mathbb{R}^{\mathfrak{p}+1}) \,:\, |u(x)| = 1 \quad \text{for a.e. } x \in B^n\}\,.$$

Physical evidence shows that in general *the ends of the molecules of a nematic liquid cannot be distinguished*. This means that the vector field u should actually take values into the *projective plane* $\mathbb{RP}^2$.

The *Dipole problem* for harmonic maps with values into $\mathbb{RP}^2$ was studied in 1986 by Brezis-Coron-Lieb [6]. However, the *lack of orientability* of $\mathbb{RP}^2$ causes a lot of trouble in the analysis of a variational theory.

In [19], we considered the $\mathfrak{p}$-energy of mappings that take values into the $\mathfrak{p}$*-dimensional projective space* $\mathbb{RP}^\mathfrak{p}$, obtained by identification of antipodal points in $\mathbb{S}^\mathfrak{p}$. For this reason, we saw the projective $\mathfrak{p}$-space $\mathbb{RP}^\mathfrak{p}$ as an embedded submanifold $\mathrm{RP}^\mathfrak{p}$ of some Euclidean space

$$\mathrm{RP}^\mathfrak{p} := g_\mathfrak{p}(\mathbb{S}^\mathfrak{p})\,, \qquad g_\mathfrak{p} : \mathbb{S}^\mathfrak{p} \to \mathbb{R}^{N(\mathfrak{p})}\,, \qquad N(\mathfrak{p}) := \frac{(\mathfrak{p}+1)(\mathfrak{p}+2)}{2} \tag{0.1}$$

and we correspondingly worked with the Sobolev class

$$W^{1,\mathfrak{p}}(B^n, \mathrm{RP}^{\mathfrak{p}}) := \{u \in W^{1,\mathfrak{p}}(B^n, \mathbb{R}^{N(\mathfrak{p})}) \mid u(x) \in \mathrm{RP}^{\mathfrak{p}} \quad \text{for a.e. } x \in B^n\}.$$

Notice that $\mathrm{RP}^{\mathfrak{p}}$ is a smooth, compact, connected submanifold of $\mathbb{R}^{N(\mathfrak{p})}$ without boundary. Moreover, $\mathrm{RP}^{\mathfrak{p}}$ is orientable if and only if $\mathfrak{p}$ is odd. We also have $g_{\mathfrak{p}}(-y) = g_{\mathfrak{p}}(y)$, whereas

$$|Du| = |Dv| \qquad \text{if } u = g_{\mathfrak{p}} \circ v \text{ for some } v \in W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}}).$$

Our key result in [19] is the following property, that holds true in any dimension n, see also [5].

Theorem 0.1 *Let $\mathfrak{p} \geq 2$ integer. For every $u \in W^{1,\mathfrak{p}}(B^n, \mathrm{RP}^{\mathfrak{p}})$ there exist exactly two Sobolev maps $v_1, v_2 \in W^{1,\mathfrak{p}}(B^n, \mathbb{S}^{\mathfrak{p}})$ such that $g_{\mathfrak{p}} \circ v_i = u$ a.e. in B^n. Moreover, $v_2 = -v_1$ and $\mathbf{D}_{\mathfrak{p}}(v_i, B^n) = \mathbf{D}_{\mathfrak{p}}(u, B^n)$.*

Using this property, we extended some of the results from [6]. More precisely, we dealt with the concepts of *singularity, degree, D-fields, flat norm, and minimal connections* for $W^{1,\mathfrak{p}}$-maps with values in $\mathrm{RP}^{\mathfrak{p}}$. We then analyzed the relaxed $\mathfrak{p}$-energy and proved a strong density property. We also introduced a notion of optimally connecting measure for the singularity of maps in $W^{1,\mathfrak{p}}(B^n, \mathrm{RP}^{\mathfrak{p}})$. Moreover, for $\mathfrak{p} = 2$, in [19] we similarly considered the analogous problems concerning the liquid crystal energy of maps in $W^{1,2}(B^3, \mathrm{RP}^2)$.

In this paper we focus on the class of $W^{1,1}$-maps into the projective line RP^1. The function $g_{\mathfrak{p}}$ in (0.1), in the case $\mathfrak{p} = 1$ reduces to the mapping $g_1 : \mathbb{S}^1 \to \mathbb{R}^3$

$$g_1(y_1, y_2) := \Big(\frac{\sqrt{2}}{2}{y_1}^2, \, \frac{\sqrt{2}}{2}{y_2}^2, \, y_1 y_2\Big). \tag{0.2}$$

Theorem 0.1 is false in the case $\mathfrak{p} = 1$, see Example 1.2 below. In fact, its proof relies on the *lifting theorem* [22, p. 34], and on the simply-connectedness of the unit $\mathfrak{p}$-sphere $\Sigma^{\mathfrak{p}} = \mathbb{S}^{\mathfrak{p}}$, for $\mathfrak{p} \geq 2$.

For this reason, we now give the following

Definition 0.2 *For $\Omega = B^n$ or $\Sigma^{\mathfrak{p}}$, we denote by $\widetilde{W}^{1,\mathfrak{p}}(\Omega, \mathrm{RP}^{\mathfrak{p}})$ the subclass of maps $u \in W^{1,\mathfrak{p}}(\Omega, \mathrm{RP}^{\mathfrak{p}})$ for which there exists a Sobolev map $v \in W^{1,\mathfrak{p}}(\Omega, \mathbb{S}^{\mathfrak{p}})$ such that $g_{\mathfrak{p}} \circ v = u$.*

Theorem 0.1 yields that $\widetilde{W}^{1,\mathfrak{p}} = W^{1,\mathfrak{p}}$ for every $\mathfrak{p} \geq 2$, whereas for $\mathfrak{p} = 1$ the strict inclusion $\widetilde{W}^{1,1} \subsetneq W^{1,1}$ holds, a part from the case $\Omega = B^1$. As a consequence, the properties proved in [19] that are based on Theorem 0.1 fail to hold in the case $\mathfrak{p} = 1$.

For example, if $\mathfrak{p}$ is odd, and $\Sigma^{\mathfrak{p}}$ is a copy of $\mathbb{S}^{\mathfrak{p}}$, the degree of a continuous $W^{1,\mathfrak{p}}$-map u from $\Sigma^{\mathfrak{p}}$ into the oriented submanifold $\mathrm{RP}^{\mathfrak{p}}$ is defined by

$$\deg_{\mathrm{RP}^{\mathfrak{p}}}(u) := \frac{1}{2} \int_{\Sigma^{\mathfrak{p}}} u^{\#} \omega_{\mathrm{RP}^{\mathfrak{p}}}$$

where $\omega_{\mathrm{RP}^{\mathfrak{p}}}$ is a normalized volume $\mathfrak{p}$-form on $\mathrm{RP}^{\mathfrak{p}}$, so that $\int_{\mathrm{RP}^{\mathfrak{p}}} \omega_{\mathrm{RP}^{\mathfrak{p}}} = 1$. Therefore, the *double* of the degree tells the time the image of $\Sigma^{\mathfrak{p}}$ by u winds around $\mathrm{RP}^{\mathfrak{p}}$, with orientation prescribed by the sign.

According to the statements from [6, Sec. VIII-B-a], Theorem 0.1 yields that $\deg_{\mathrm{RP}^{\mathfrak{p}}}(u) \in \mathbb{Z}$ in the case $\mathfrak{p} \geq 3$ odd. However, for $\mathfrak{p} = 1$, in general we have $\deg_{\mathrm{RP}^1}(u) \in \frac{1}{2}\mathbb{Z}$, compare [6, Sec. VIII-B-b].

Main Results. In this paper we shall prove that *for every Sobolev map $u \in W^{1,1}(B^n, \mathrm{RP}^1)$ there exists a function $v \in BV(B^n, \mathbb{S}^1)$ such that $g_1 \circ v = u$*. Moreover, *v is a special function of bounded variation in $SBV(B^n, \mathbb{S}^1)$, with jump set of finite size, $\mathcal{H}^{n-1}(J_v) < \infty$*, see [3].

As to maps u in $W^{1,1}(\Sigma^1, \mathrm{RP}^1)$, for which in general $\deg_{\mathrm{RP}^1}(u) \in \frac{1}{2}\mathbb{Z}$, we shall prove that

$$\deg_{\mathrm{RP}^1}(u) \in \mathbb{Z} \iff u \in \widetilde{W}^{1,1}(\Sigma^1, \mathrm{RP}^1), \quad \text{see Definition 0.2}.$$

Similarly, for maps u in $W^{1,1}(B^2, \mathrm{RP}^1)$ that are smooth outside a discrete set of points $\Sigma(u)$, we shall prove that u *belongs to $\widetilde{W}^{1,1}(B^2, \mathrm{RP}^1)$ if and only if the degree of u around each singular point in $\Sigma(u)$ is integer.* This last property about the degree means that small circles around each point of $\Sigma(u)$ are wrapped by u around the target manifold RP^1 an *even* number of times.

More generally, in higher dimension $n \geq 2$, the singularities of Sobolev maps $u \in W^{1,1}(B^n, \mathrm{RP}^1)$ are identified by the current $\mathbf{P}(u) \in \mathcal{D}_{n-2}(B^n)$ acting on compactly supported smooth forms φ as

$$\langle \mathbf{P}(u), \varphi \rangle := \int_{B^n} d\varphi \wedge u^{\#}\omega_{\mathrm{RP}^1} , \qquad \varphi \in \mathcal{D}^{n-2}(B^n) .$$

We recall that a current $\Gamma \in \mathcal{D}_{n-2}(B^n)$ is said to be an *integral flat chain* if there exists an integer multiplicity (say i.m.) rectifiable current $L \in \mathcal{R}_{n-1}(B^n)$ such that $(\partial L) \llcorner B^n = \Gamma$.

By the coarea formula, it turns out that $\mathbf{P}(u)$ *is an integral flat chain*, i.e., we can always find an i.m. rectifiable current $L \in \mathcal{R}_{n-1}(B^n)$ that encloses the singularity of u, see Proposition 3.2 below.

According to Definition 0.2 we shall prove, Theorem 9.1, that *for every* $u \in W^{1,1}(B^n, \mathrm{RP}^1)$

$$u \in \widetilde{W}^{1,1}(B^n, \mathrm{RP}^1) \iff \textit{the current } \frac{1}{2}\,\mathbf{P}(u) \textit{ is an integral flat chain, too} .$$

Plan of the paper. In Sec. 1, we collect some preliminary facts and a counterexample to the validity of Theorem 0.1 for $\mathfrak{p} = 1$. In Sec. 2, we deal with D-fields, degree, and singularities of $W^{1,1}$-maps with values into RP^1, whereas in Sec. 3 we study a related Dipole problem. In Sec. 4, we shall introduce a class of *Cartesian currents* in $B^n \times \mathrm{RP}^1$, proving some basic properties. In Sec. 5, we discuss a notion of optimally connecting measure of the singular set of $W^{1,1}$-maps with values into RP^1, and we find an explicit formula for the relaxed total variation energy.

In order to prove the main results, in Sec. 6 we start by showing *the existence of liftings* of Cartesian currents in $B^n \times \mathrm{RP}^1$, extending a result proved in [12] for the case $B^n \times \mathbb{S}^1$, compare also [7] for the case $n = 2$. In Sec. 7, we recall the structure properties of the class of Cartesian currents in $B^n \times \mathbb{S}^1$, and we introduce a suitable current integration on the jump set of functions of bounded variation $v \in BV(B^n, \mathbb{S}^1)$. In Sec. 8, we then analyze some properties of the currents G_v carried by the graph of BV-maps in $BV(B^n, \mathbb{S}^1)$ that satisfy $g_1 \circ v = u \in W^{1,1}(B^n, \mathrm{RP}^1)$. Finally, in Sec. 9 we prove the main results stated above.

1 Maps into the projective line

For $\mathfrak{p} \geq 1$ integer, the *real projective space* $\mathbb{RP}^{\mathfrak{p}}$ is defined by the quotient space $\mathbb{RP}^{\mathfrak{p}} = \mathbb{S}^{\mathfrak{p}} / \sim_{\mathfrak{p}}$, where $\mathbb{S}^{\mathfrak{p}}$ is the unit sphere in $\mathbb{R}^{\mathfrak{p}+1}$

$$\mathbb{S}^{\mathfrak{p}} := \{ y \in \mathbb{R}^{\mathfrak{p}+1} \,:\, |y| = 1 \}$$

the equivalence relation being $y \sim_{\mathfrak{p}} \widetilde{y} \iff y = \widetilde{y}$ or $y = -\widetilde{y}$. We equip $\mathbb{RP}^{\mathfrak{p}}$ with the natural metric induced on equivalence classes. We also denote by $[y]_{\mathfrak{p}}$ the elements of $\mathbb{RP}^{\mathfrak{p}}$ and by $P_{\mathfrak{p}} : \mathbb{S}^{\mathfrak{p}} \to \mathbb{RP}^{\mathfrak{p}}$ the canonical projection $P_{\mathfrak{p}}(y) := [y]_{\mathfrak{p}}$. Recall that $\mathbb{RP}^{\mathfrak{p}}$ is orientable if and only if $\mathfrak{p}$ is odd.

Let $\Sigma^{\mathfrak{p}} = \mathbb{S}^{\mathfrak{p}} \subset \mathbb{R}^{\mathfrak{p}+1}$. The main feature that distinguishes the case $\mathfrak{p} = 1$ is related to the fact that $\Sigma^{\mathfrak{p}}$ is simply connected if and only if $\mathfrak{p} \geq 2$. In fact, the lifting theorem [22, p. 34] gives:

Proposition 1.1 (Lifting theorem). *If* $\mathfrak{p} \geq 2$, *for every continuous function* $U : \Sigma^{\mathfrak{p}} \to \mathbb{RP}^{\mathfrak{p}}$ *there exists a continuous function* $v : \Sigma^{\mathfrak{p}} \to \mathbb{S}^{\mathfrak{p}}$ *such that* $P_{\mathfrak{p}} \circ v = U$.

This property is clearly false for $\mathfrak{p} = 1$, see Example 1.2 below.

Embedding of $\mathbb{RP}^1$. The function $g_{\mathfrak{p}}$ in (0.1), in the case $\mathfrak{p} = 1$ reduces to the mapping $g_1 : \mathbb{S}^1 \to \mathbb{R}^3$ defined by (0.2), that clearly induces an embedding

$$\widetilde{g}_1 : \mathbb{RP}^1 \to \mathrm{RP}^1 , \qquad \mathrm{RP}^1 := g_1(\mathbb{S}^1) \subset \mathbb{R}^3 , \qquad \widetilde{g}_1([y]_1) := g_1(y) .$$

Therefore, RP^1 is the closed arc

$$\mathrm{RP}^1 = \Big\{ z = (z_1, z_2, z_3) \in \mathbb{R}^3 \mid z_1 + z_2 = \frac{\sqrt{2}}{2} \,,\ |z - C| = \frac{1}{2} \Big\}$$

where $C := (\sqrt{2}/4, \sqrt{2}/4, 0)$, and $|z| = \sqrt{2}/2$ for every $z \in \mathrm{RP}^1$, so that

$$\mathcal{H}^1(\mathrm{RP}^1) = \pi = \frac{1}{2}\, \mathcal{H}^1(\mathbb{S}^1) .$$

Moreover, we equip RP^1 with the induced orientation, in such a way that corresponding current $[\![\mathrm{RP}^1]\!]$ satisfies

$$g_{1\#}[\![\mathbb{S}^1]\!] = 2\,[\![\mathrm{RP}^1]\!]\,.$$

Let $B^n(x,r)$ denote the n-ball in $\mathbb{R}^n$ centered at x and with radius $r>0$, and denote $B^n_r := B^n(0,r)$ and $B^n := B^n(0,1)$. For $X = C^\infty$, C^0, $W^{1,1}$, BV, L^1, and for $B\subset\mathbb{R}^n$ a Borel set, we define the classes

$$\begin{aligned} X(B,\mathbb{S}^1) &:= \{v\in X(B,\mathbb{R}^2) \,:\, |v(x)|=1 \text{ for a.e. } x\in B\}\,,\\ X(B,\mathrm{RP}^1) &:= \{u\in X(B,\mathbb{R}^3) \,:\, u(x)\in\mathrm{RP}^1 \text{ for a.e. } x\in B\}\,, \end{aligned}$$

where RP^1 is equipped with the induced metric from $\mathbb{R}^3$. We also denote by

$$\mathbf{D}_1(w,B) := \int_B |Dw(x)|\,dx$$

the *total variation* of a map w in $W^{1,1}(B,\mathbb{S}^1)$ or in $W^{1,1}(B,\mathrm{RP}^1)$. For $B=B^n$, we finally set

$$\mathbf{D}_1(w) := \mathbf{D}_1(w,B^n)\,.$$

Notice that if $u:B\to\mathrm{RP}^1$ is given by $u=g_1\circ v$ for some map $v\in W^{1,1}(B,\mathbb{S}^1)$, we have $u\in W^{1,1}(B,\mathrm{RP}^1)$ and $|Du|=|Dv|$. In particular, for every $v\in W^{1,1}(B,\mathbb{S}^1)$ we infer that

$$\mathbf{D}_1(g_1\circ v,B) = \mathbf{D}_1(v,B)\,.$$

Let now $\mathcal{Y}=\mathbb{S}^1$ or RP^1. By Schoen-Uhlenbeck density theorem [20], the class of smooth maps in $W^{1,1}(B^1,\mathcal{Y})$ is strongly dense in $W^{1,1}(B^1,\mathcal{Y})$. This is false in the case of higher dimension $n\geq 2$. For this reason, Bethuel [4] introduced the classes $R^\infty_1(B^n,\mathcal{Y})$ and $R^0_1(B^n,\mathcal{Y})$ of maps $w\in W^{1,1}(B^n,\mathcal{Y})$ that are smooth, respectively continuous, outside a smooth closed singular subset $\Sigma(w)$ of B^n of dimension $(n-2)$, e.g., a discrete set for $n=2$. He also proved that *for any $n\geq 2$, the classes $R^\infty_1(B^n,\mathcal{Y})$ and $R^0_1(B^n,\mathcal{Y})$ are strongly dense in $W^{1,1}(B^n,\mathcal{Y})$.*

Example 1.2 Let $\Sigma^1=\mathbb{S}^1$ and consider the function $\widetilde{v}:\Sigma^1\to\mathbb{R}^2$

$$\widetilde{v}(x_1,x_2) = \begin{cases} \Big(\dfrac{\sqrt{2}}{2}\sqrt{1+x_1},\ \dfrac{\sqrt{2}}{2}\dfrac{x_2}{\sqrt{1+x_1}}\Big) & \text{if } \ x_1\neq -1\\ (0,1) & \text{if } \ x_1=-1\,. \end{cases}$$

Clearly $\widetilde{v}$ is a function of *bounded variation* in $BV(\Sigma^1,\mathbb{S}^1)$, see Sec. 7 below; however, $\widetilde{v}$ is not a Sobolev function in $W^{1,1}(\Sigma^1,\mathbb{S}^1)$, due to the discontinuity at the point $(-1,0)$.

Since $x_2^2 = 1-x_1^2$ for $(x_1,x_2)\in\Sigma^1$, the corresponding function $\widetilde{u}:=g_1\circ\widetilde{v}:\Sigma^1\to\mathbb{R}^3$, see (0.2), satisfies

$$\widetilde{u}(x_1,x_2) = \Big(\frac{\sqrt{2}}{4}(1+x_1),\ \frac{\sqrt{2}}{4}(1-x_1),\ \frac{x_2}{2}\Big) \qquad \forall\,(x_1,x_2)\in\Sigma^1\,.$$

Therefore, $\widetilde{u}$ belongs to the Sobolev class $W^{1,1}(\Sigma^1,\mathrm{RP}^1)$. Moreover, $\widetilde{u}$ is continuous and winds around the embedded manifold RP^1 once.

Correspondingly, the continuous function $U:\Sigma^1\to\mathbb{RP}^1$ given by $U:=\widetilde{g}_1^{-1}\circ\widetilde{u}$ winds around $\mathbb{RP}^1$ once, hence both $\widetilde{u}$ and U are homotopically non-trivial. This also gives that Proposition 1.1 is false, for $\mathfrak{p}=1$.

Consider now the homogeneous extensions

$$\overline{u}(x) := \widetilde{u}\Big(\frac{x}{|x|}\Big)\,, \qquad \overline{v}(x) := \widetilde{v}\Big(\frac{x}{|x|}\Big)\,, \qquad x=(x_1,x_2)\in B^2\setminus\{0\}\,.$$

Clearly, $\overline{v}$ belongs to the class $BV(B^2,\mathbb{S}^1)$ but not to $W^{1,1}(B^2,\mathbb{S}^1)$, and $g_1\circ\overline{v}=\overline{u}$. Moreover, $\overline{u}$ is a Sobolev map in $W^{1,1}(B^2,\mathrm{RP}^1)$, but it does not belong to the class $\widetilde{W}^{1,1}(B^2,\mathrm{RP}^1)$, see Definition 0.2. Therefore, Theorem 0.1 is false, too, for $\mathfrak{p}=1$.

Remark 1.3 Since Sobolev maps in $W^{1,1}(B^1,\mathrm{RP}^1)$ are continuous, by the lifting theorem, and arguing as in [19], we readily check that in the case $\mathfrak{p}=1$, Theorem 0.1 holds true in low dimension $n=1$. As we have seen, it is false in higher dimension $n\geq 2$.

In [19] we also introduced the class

$$\mathcal{F}_{\mathfrak{p}} := \{u \in W^{1,\mathfrak{p}}(B^{\mathfrak{p}}, \mathrm{RP}^{\mathfrak{p}}) \cap C^0 \quad | \quad u \text{ is constant on } \partial B^{\mathfrak{p}} \text{ and homotopically non-trivial } \},$$

the homotopy to be intended with fixed boundary datum on $\partial B^{\mathfrak{p}}$, and we proved that

$$\inf\{\mathbf{D}_{\mathfrak{p}}(u) \mid u \in \mathcal{F}_{\mathfrak{p}}\} = 2\,\mathcal{H}^{\mathfrak{p}}(\mathrm{RP}^{\mathfrak{p}})$$

for every $\mathfrak{p} \geq 2$ integer. According to Example 1.2, it is readily checked that for $\mathfrak{p} = 1$ we instead have:

$$\inf\{\mathbf{D}_1(u) \mid u \in \mathcal{F}_1\} = \mathcal{H}^1(\mathrm{RP}^1).$$

2 D-fields, degree, and singularities

In this section we discuss the notions of D-field, degree, and singularities of $W^{1,1}$-maps that take values into the projective line RP^1. We first recall some notation concerning maps into the unit circle $\mathbb{S}^1$.

Maps into $\mathbb{S}^1$. Let $\omega_{\mathbb{S}^1}$ denote the *volume* 1*-form on* $\mathbb{S}^1$

$$\omega_{\mathbb{S}^1} := y^1 dy^2 - y^2 dy^1$$

so that $[\![\mathbb{S}^1]\!](\omega_{\mathbb{S}^1}) := \int_{\mathbb{S}^1} \omega_{\mathbb{S}^1} = 2\pi$. Following [13], to every Sobolev function $v \in W^{1,1}(B^n, \mathbb{S}^1)$, where $n \geq 2$, we associate the $(n-2)$-dimensional current $\mathbb{P}(v) \in \mathcal{D}_{n-2}(B^n)$ acting on compactly supported smooth $(n-2)$-forms $\varphi \in \mathcal{D}^{n-2}(B^n)$ as

$$\langle \mathbb{P}(v), \varphi \rangle := \frac{1}{2\pi} \int_{B^n} d\varphi \wedge v^{\#}\omega_{\mathbb{S}^1}. \tag{2.1}$$

We also define the $(n-1)$-current $\mathbb{D}(v) \in \mathcal{D}_{n-1}(B^n)$ by

$$\langle \mathbb{D}(v), \gamma \rangle := \frac{1}{2\pi} \int_{B^n} \gamma \wedge v^{\#}\omega_{\mathbb{S}^1}$$

for every $\gamma \in \mathcal{D}^{n-1}(B^n)$, so that clearly

$$\mathbb{P}(v) = \partial\,\mathbb{D}(v) \qquad \text{on} \quad \mathcal{D}^{n-2}(B^n). \tag{2.2}$$

The above can be stated in terms of the so called *D-field* of Brezis-Coron-Lieb [6]. In fact, for every $v \in W^{1,1}(B^n, \mathbb{S}^1)$ we have

$$v^{\#}\omega_{\mathbb{S}^1} = \sum_{i=1}^{n} v \times v_{x_i}\, dx^i \tag{2.3}$$

where

$$v \times v_{x_i} := \det\begin{pmatrix} v^1 & v^2 \\ v^1_{x_i} & v^2_{x_i} \end{pmatrix}, \qquad v = (v^1, v^2), \qquad v^j_{x_i} := \frac{\partial v^j}{\partial x_i}.$$

In dimension $n = 2$, the D-field of $v \in W^{1,1}(B^2, \mathbb{S}^1)$ is defined by

$$D(v) := (v \times v_{x_2}, -v \times v_{x_1}) \in L^1(B^2, \mathbb{R}^2).$$

Remark 2.1 In higher dimension $n \geq 3$, the $(n-1)$-vector field $D(v)$ can be defined as the dual to $v^{\#}\omega_{\mathbb{S}^1}$,

$$\langle \eta, D(v)(x) \rangle\, dx := \eta \wedge v^{\#}\omega_{\mathbb{S}^1}(x) \qquad \forall\, \eta \in \Lambda^{n-1}(\mathbb{R}^n),$$

where $dx := dx^1 \wedge \cdots \wedge dx^n$. More precisely, $D(v)$ may be identified with $*\,v^{\#}\omega_{\mathbb{S}^1}$, where $*$ is the *Hodge operator*.

If $v \in W^{1,1}(B^n, \mathbb{S}^1)$ is smooth, for a.e. $x \in B^n$ the $(n-1)$-vector $D(v)(x) \in \Lambda_{n-1}\mathbb{R}^n$ is tangent to the naturally oriented level hypersurfaces $\{z \in B^n \mid v(z) = v(x)\}$. More precisely, when normalized, the $(n-1)$-vector $D(v)(x)$ orients the slices of the current $[\![B^n]\!]$ by the map v at $v(x) \in \mathbb{S}^1$.

For maps $v \in W^{1,1}(B^n, \mathbb{S}^1)$ we thus have

$$\langle \mathbb{D}(v), \gamma \rangle = \frac{1}{2\pi} \int_{B^n} \langle \gamma, D(v) \rangle \, dx \qquad \forall \, \gamma \in \mathcal{D}^{n-1}(B^n) \, .$$

In particular, in dimension $n = 2$, formula (2.2) yields to:

$$\mathbb{P}(v) = 0 \quad \iff \quad \mathrm{Div} D(v) = 0 \quad \text{on } B^2 \, ,$$

where Div denotes the *distributional divergence.*

THE VOLUME FORM. In [19] we introduced for $\mathfrak{p} \geq 3$ odd a (normalized) volume $\mathfrak{p}$-form $\omega_{\mathrm{RP}^\mathfrak{p}}$ on $\mathrm{RP}^\mathfrak{p}$. For $\mathfrak{p} = 1$, it reads as

$$\omega_{\mathrm{RP}^1} := \frac{1}{\pi} \, (\widehat{g}_1^{-1})^{\#} \omega_{\mathbb{S}^1} \, ,$$

where $\widehat{g}_1$ is the one-to-one map given by the restriction of g_1 to the semi-circle $\mathbb{S}^1_+ := \{y \in \mathbb{S}^1 \mid y^2 > 0\}$. We then compute:

$$\omega_{\mathrm{RP}^1} = \frac{\sqrt{2}}{\pi} \left(-z^3 dz^1 + z^3 dz^2 + (z^1 - z^2) \, dz^3 \right) \qquad \forall \, z = (z^1, z^2, z^3) \in \mathrm{RP}^1 \, . \tag{2.4}$$

Denote by $j : \mathbb{R} \to \mathbb{S}^1$ and $\widehat{j} : \mathbb{R} \to \mathrm{RP}^1$ the *lifting maps*

$$j(t) := (\cos t, \sin t) \, , \qquad \widehat{j}(t) := \Big(\frac{\sqrt{2}}{2} \cos^2 t, \frac{\sqrt{2}}{2} \sin^2 t, \cos t \sin t \Big) \, , \tag{2.5}$$

so that

$$\widehat{j} = g_1 \circ j \, , \qquad j_{\#}[\![(0, 2\pi)]\!] = [\![\mathbb{S}^1]\!] \, , \qquad \widehat{j}_{\#}[\![(0, \pi)]\!] = [\![\mathrm{RP}^1]\!] \, .$$

By (2.4) we readily obtain:

$$g_1^{\#} \omega_{\mathrm{RP}^1} = \frac{1}{\pi} \, \omega_{\mathbb{S}^1} \, , \qquad \widehat{j}^{\#} \omega_{\mathrm{RP}^1} = \frac{1}{\pi} \, dt \, , \qquad j^{\#} \omega_{\mathbb{S}^1} = dt \, , \tag{2.6}$$

so that

$$[\![\mathrm{RP}^1]\!](\omega_{\mathrm{RP}^1}) = \widehat{j}_{\#}[\![(0, \pi)]\!](\omega_{\mathrm{RP}^1}) = \int_0^\pi \widehat{j}^{\#} \omega_{\mathrm{RP}^1} = 1 \, . \tag{2.7}$$

D-FIELDS. For any $u \in W^{1,1}(B^n, \mathrm{RP}^1)$, and for $i = 1, \ldots, n$, we denote

$$\mathfrak{D}_i(u) := \sqrt{2} \, \det \begin{pmatrix} (u^1 - u^2) & u^3 \\ (u^1 - u^2)_{x_i} & u^3_{x_i} \end{pmatrix} \, , \qquad u = (u^1, u^2, u^3) \, , \qquad u^j_{x_i} := \frac{\partial u^j}{\partial x_i} \, . \tag{2.8}$$

Proposition 2.2 *Let $u \in W^{1,1}(B^n, \mathrm{RP}^1)$ be such that $u = g_1 \circ v$ for some $v \in W^{1,1}(B^n, \mathbb{S}^1)$. Then*

$$\mathfrak{D}_i(u) = v \times v_{x_i} \qquad \forall \, i = 1, \ldots, n \, .$$

PROOF: By (0.2), we have:

$$\begin{aligned} (u^1 - u^2) = \frac{\sqrt{2}}{2} ((v^1)^2 - (v^2)^2) \qquad &\Longrightarrow \qquad (u^1 - u^2)_{x_i} = \sqrt{2} \, (v^1 v^1_{x_i} - v^2 v^2_{x_i}) \\ u^3 = v^1 v^2 \qquad &\Longrightarrow \qquad u^3_{x_i} = v^1 v^2_{x_i} + v^2 v^1_{x_i} \, . \end{aligned}$$

This gives

$$\det \begin{pmatrix} (u^1 - u^2) & u^3 \\ (u^1 - u^2)_{x_i} & u^3_{x_i} \end{pmatrix} = \frac{\sqrt{2}}{2} \, |v|^2 \, v \times v_{x_i} \, .$$

Since $|v| = 1$, the claim follows. □

Recall that the assumption in Proposition 2.2 is not satisfied in general. However, we check:

Proposition 2.3 $u^\# \omega_{\mathrm{RP}^1} = \dfrac{1}{\pi}\Big(\sum_{i=1}^n \mathfrak{D}_i(u)\, dx^i\Big)$ *for every* $u \in W^{1,1}(B^n, \mathrm{RP}^1)$.

PROOF: By (2.4), we compute

$$\begin{aligned} u^\# \omega_{\mathrm{RP}^1} &= \frac{\sqrt{2}}{\pi}\big(-u^3\, d(u^1-u^2) + (u^1-u^2)\, du^3\big) \\ &= \frac{\sqrt{2}}{\pi}\sum_{i=1}^n \big(-u^3\,(u^1-u^2)_{x_i} + (u^1-u^2)\, u^3_{x_i}\big)\, dx^i \\ &= \frac{1}{\pi}\sum_{i=1}^n \sqrt{2}\,\det\begin{pmatrix} (u^1-u^2) & u^3 \\ (u^1-u^2)_{x_i} & u^3_{x_i}\end{pmatrix} dx^i\,. \end{aligned}$$

This gives the claim, by (2.8). □

Definition 2.4 *The* D-field *of a Sobolev map* $u \in W^{1,1}(B^2, \mathrm{RP}^1)$ *is the vector field* $\mathfrak{D}(u) \in L^1(B^2, \mathbb{R}^2)$ *defined in components by* $\mathfrak{D}(u) = (\mathfrak{D}_2(u), -\mathfrak{D}_1(u))$*, according to* (2.8).

Remark 2.5 In higher dimension $n \geq 3$, the $(n-1)$-vector field $\mathfrak{D}(u)$ of maps $u \in W^{1,1}(B^n, \mathrm{RP}^1)$ can be defined by the dual to $\pi\, u^\# \omega_{\mathrm{RP}^1}$,

$$\langle \eta, \mathfrak{D}(u)(x)\rangle\, dx := \eta \wedge \pi\, u^\# \omega_{\mathrm{RP}^1}(x) \qquad \forall\, \eta \in \Lambda^{n-1}(\mathbb{R}^n)\,,$$

i.e., by $*\,\pi\, u^\# \omega_{\mathrm{RP}^1}$.

According to [6], this property justifies our definition.

Proposition 2.6 *Let* $u \in W^{1,1}(B^n, \mathrm{RP}^1)$ *be such that* $u = g_1 \circ v$ *for some* $v \in W^{1,1}(B^2, \mathbb{S}^1)$*. Then*

$$u^\# \omega_{\mathrm{RP}^1} = \frac{1}{\pi}\, v^\# \omega_{\mathbb{S}^1} \qquad \textit{and} \qquad \mathfrak{D}(u) = D(v)\,. \tag{2.9}$$

PROOF: By (2.6) we obtain

$$u^\# \omega_{\mathrm{RP}^1} = v^\# (g_1^\# (\omega_{\mathrm{RP}^1})) = \frac{1}{\pi}\, v^\# \omega_{\mathbb{S}^1}\,.$$

In dimension $n = 2$, the claim follows from Proposition 2.2, see (2.3). In higher dimension, it is a consequence of our definitions, see Remarks 2.1 and 2.5. □

DEGREE. The degree of a continuous map $U : \Sigma^1 \to \mathbb{RP}^1$, where Σ^1 is a copy of $\mathbb{S}^1$, is well-defined by identifying $\mathbb{S}^1$ with the unit circle in $\mathbb{C}$ and using the function $z \mapsto z^2$, compare [6, Sec. VIII-B-b)]. Therefore, differently to what happens in the case $\mathfrak{p} \geq 3$ odd, the degree of maps into $\mathbb{RP}^1$ in general belongs to $\frac{1}{2}\,\mathbb{Z}$.

We define the *degree* of a map $u \in W^{1,1}(\Sigma^1, \mathrm{RP}^1)$ by

$$\deg_{\mathrm{RP}^1}(u) := \frac{1}{2\pi}\int_{\Sigma^1} \mathfrak{D}(u)\cdot \nu\, d\mathcal{H}^1\,,$$

where $\mathfrak{D}(u)$ is the D-field of any smooth extension in $W^{1,1}(\Omega, \mathrm{RP}^1)$ of u to a neighborhood of Σ^1 in $\mathbb{R}^2$, see Definition 2.4, and ν is the outward unit normal to Σ^1. By Proposition 2.3, in fact, we deduce that

$$\deg_{\mathrm{RP}^1}(u) = \frac{1}{2}\int_{\Sigma^1} u^\# \omega_{\mathrm{RP}^1} \in \frac{1}{2}\,\mathbb{Z}\,. \tag{2.10}$$

Example 2.7 Taking $u = \widetilde{u}$, see Example 1.2, we compute

$$\widetilde{u}^\# \omega_{\mathrm{RP}^1} = \frac{1}{2\pi}(x^1 dx^2 - x^2 dx^1)$$

so that

$$\deg_{\mathrm{RP}^1}(\widetilde{u}) = \frac{1}{2}\int_{\Sigma^1} \widetilde{u}^\# \omega_{\mathrm{RP}^1} = \frac{1}{2}\cdot\frac{1}{2\pi}\int_{\Sigma^1}(x^1 dx^2 - x^2 dx^1) = \frac{1}{2}\,. \tag{2.11}$$

Therefore, the *double* of the degree, $2\,\deg_{\mathrm{RP}^1}(u) \in \mathbb{Z}$, tells the times the function $u : \Sigma^1 \to \mathrm{RP}^1$ winds around RP^1, with orientation prescribed by the sign.

In a similar way, if u belongs to $R^0_1(B^2, \mathrm{RP}^1)$, and $\Sigma(u) = \{a_j \mid j = 1, \dots, m\}$ is the discrete set of its singularities, the degree of u at a singular point a_j is well-defined by

$$\deg_{\mathrm{RP}^1}(u, a_j) := \frac{1}{2\pi} \int_{\partial B^2(a,r)} \mathfrak{D}(u) \cdot \nu_{a,r} \, d\mathcal{H}^1$$

for $r > 0$ small, where $\nu_{a,r}$ is the outward unit normal to $\partial B^2(a, r)$. By Proposition 2.3 we have:

$$\deg_{\mathrm{RP}^1}(u, a_j) = \frac{1}{2} \int_{\partial B^2(a,r)} u^{\#}\omega_{\mathrm{RP}^1} \in \frac{1}{2}\mathbb{Z}\,. \tag{2.12}$$

Again, the double of the degree, $2\deg_{\mathrm{RP}^1}(u, a_i) \in \mathbb{Z}$, tells the times the function $u_{|\partial B^2(a_j,r)}$, for r small, winds around RP^1, with orientation prescribed by the sign, and in general $\deg_{\mathrm{RP}^1}(u, a_i)$ belongs to $\frac{1}{2}\mathbb{Z}$.

Singularity. According to (2.1), to any map $u \in W^{1,1}(B^n, \mathrm{RP}^1)$, where $n \geq 2$, we associate the current $\mathbf{P}(u) \in \mathcal{D}_{n-2}(B^n)$ acting on forms $\varphi \in \mathcal{D}^{n-2}(B^n)$ as

$$\langle \mathbf{P}(u), \varphi \rangle := \int_{B^n} d\varphi \wedge u^{\#}\omega_{\mathrm{RP}^1}\,, \tag{2.13}$$

and the $(n-1)$-current $\widetilde{\mathbb{D}}(u) \in \mathcal{D}_{n-1}(B^n)$ given by

$$\langle \widetilde{\mathbb{D}}(u), \gamma \rangle := \int_{B^n} \gamma \wedge u^{\#}\omega_{\mathrm{RP}^1}$$

for every $\gamma \in \mathcal{D}^{n-1}(B^n)$, so that again we have

$$\mathbf{P}(u) = \partial \widetilde{\mathbb{D}}(u) \qquad \text{on} \quad \mathcal{D}^{n-2}(B^n)\,. \tag{2.14}$$

Notice that by (2.9) and the definitions (2.1) and (2.13), we readily infer:

Proposition 2.8 *Let $u \in W^{1,1}(B^n, \mathrm{RP}^1)$, where $n \geq 2$. Assume that there exists a Sobolev function $v \in W^{1,1}(B^n, \mathbb{S}^1)$ such that $g_1 \circ v = u$. Then $\frac{1}{2}\mathbf{P}(u) = \mathbb{P}(v)$.*

In dimension $n = 2$, by Proposition 2.3 and Definition 2.4 we deduce that for any $u \in W^{1,1}(B^n, \mathrm{RP}^1)$

$$\langle \mathbf{P}(u), \varphi \rangle = \frac{1}{\pi} \int_{B^2} D\varphi \cdot \mathfrak{D}(u)\, dx \qquad \forall \varphi \in C^\infty_c(B^2)\,. \tag{2.15}$$

Therefore, for every open set $\Omega \subset B^2$ we have

$$\mathbf{P}(u) \llcorner \Omega = 0 \qquad \Longleftrightarrow \qquad \mathrm{Div}(\mathfrak{D}(u) \llcorner \Omega) = 0\,. \tag{2.16}$$

In higher dimension $n \geq 3$, the D-field $\mathfrak{D}(u) \in L^1(B^n, \Lambda_{n-1}\mathbb{R}^n)$ of $u \in W^{1,1}(B^n, \mathrm{RP}^1)$ being defined as in Remark 2.5, we deduce that

$$\langle \widetilde{\mathbb{D}}(u), \gamma \rangle = \frac{1}{\pi} \int_{B^n} \langle \gamma, \mathfrak{D}(u) \rangle \, dx \qquad \forall \gamma \in \mathcal{D}^{n-1}(B^n)\,. \tag{2.17}$$

Example 2.9 Taking e.g. $u = \overline{u}$, see Example 1.2, we have $\Sigma(\overline{u}) = \{0\}$ and

$$\overline{u}^{\#}\omega_{\mathrm{RP}^1} = \frac{1}{2\pi}\Big(\frac{x^1}{\rho^2}\, dx^2 - \frac{x^2}{\rho^2}\, dx^1\Big)\,, \qquad \rho := |(x^1, x^2)|\,.$$

By (2.13) we then obtain

$$\langle \mathbf{P}(\overline{u}), \varphi \rangle = \frac{1}{2\pi} \int_{B^2} \frac{1}{\rho^2}\, (D\varphi \cdot x)\, dx = -\varphi(0)$$

for every $\varphi \in C^\infty_c(B^2)$, whereas

$$\int_{\partial B^2_r} \overline{u}^{\#}\omega_{\mathrm{RP}^1} = \frac{1}{2\pi} \int_{\partial B^2_r} \Big(\frac{x^1}{r^2}\, dx^2 - \frac{x^2}{r^2}\, dx^1\Big) = 1$$

for every $0 < r < 1$, so that

$$\mathbf{P}(\overline{u}) = -\delta_0\,, \qquad \deg_{\mathrm{RP}^1}(\overline{u}, 0) = \frac{1}{2}\,. \tag{2.18}$$

For maps u in $R^0_1(B^2,\mathrm{RP}^1)$ as above, the degrees of u at the singular points a_j are related to the current $\mathbf{P}(u)\in\mathcal{D}_0(B^2)$ as follows:

Proposition 2.10 *Let $u\in R^0_1(B^2,\mathrm{RP}^1)$ and $\Sigma(u)=\{a_j\mid j=1,\dots,m\}$ the singular set of u. Then*

$$\mathbf{P}(u)=-\sum_{j=1}^m 2\,\widetilde{\Delta}_j\,\delta_{a_j} \iff \deg_{\mathrm{RP}^1}(u,a_j)=\widetilde{\Delta}_j\in\frac12\mathbb{Z}\quad\forall\, j\,. \tag{2.19}$$

Proof: Since the argument is local, we may and do assume that u has only one singular point at the origin. In this case, we have to show that

$$\mathbf{P}(u)=-2\,\deg_{\mathrm{RP}^1}(u,0)\,\delta_0\,. \tag{2.20}$$

By (2.15), for any $\varphi\in C^\infty_c(B^2)$ we compute

$$\langle\mathbf{P}(u),\varphi\rangle=\frac{1}{\pi}\lim_{\varepsilon\to0^+}\int_{A_\varepsilon}D\varphi\cdot\mathfrak{D}(u)\,dx\,,$$

where $A_\varepsilon:=B^2\setminus B^2_\varepsilon$. Integrating by parts, since u is smooth on A_ε, for $0<\varepsilon<1$, we obtain

$$\int_{A_\varepsilon}D\varphi\cdot\mathfrak{D}(u)\,dx=\int_{\partial^+A_\varepsilon}\varphi\,(\mathfrak{D}_1(u)\,dx^1+\mathfrak{D}_2(u)\,dx^2)-\int_{A_\varepsilon}\varphi\,\mathrm{div}\,\mathfrak{D}(u)\,dx\,,$$

where $\mathrm{div}\,\mathfrak{D}(u)$ is the divergence of $\mathfrak{D}(u)$. The test function φ being compactly supported in B^2, we have

$$\int_{\partial^+A_\varepsilon}\varphi\,(\mathfrak{D}_1(u)\,dx^1+\mathfrak{D}_2(u)\,dx^2)=-\int_{\partial B^2_\varepsilon}\varphi\,(\mathfrak{D}_1(u)\,dx^1+\mathfrak{D}_2(u)\,dx^2)\,.$$

Moreover, since $\mathbf{P}(u)\llcorner A_\varepsilon=0$, by (2.16) we deduce that

$$\int_{A_\varepsilon}\varphi\,\mathrm{div}\,\mathfrak{D}(u)\,dx=0\,.$$

By the smoothness of φ, using Proposition 2.3 and (2.12) we then obtain

$$\begin{aligned}-\langle\mathbf{P}(u),\varphi\rangle &= \frac1\pi\lim_{\varepsilon\to0^+}\int_{\partial B^2_\varepsilon}\varphi\,(\mathfrak{D}_1(u)\,dx^1+\mathfrak{D}_2(u)\,dx^2)\\ &= \varphi(0)\cdot\lim_{\varepsilon\to0^+}\frac1\pi\int_{\partial B^2_\varepsilon}(\mathfrak{D}_1(u)\,dx^1+\mathfrak{D}_2(u)\,dx^2)\\ &= \varphi(0)\cdot\lim_{\varepsilon\to0^+}\int_{\partial B^2_\varepsilon}u^\#\omega_{\mathrm{RP}^1}=\varphi(0)\cdot2\,\deg_{\mathrm{RP}^1}(u,0)\end{aligned}$$

and hence (2.20), as required. □

Example 2.11 If $\overline{u}(x):=u\Big(\dfrac{x}{|x|}\Big)$ for some $u\in W^{1,1}(\Sigma^1,\mathrm{RP}^1)$, then $\overline{u}\in W^{1,1}(B^2,\mathrm{RP}^1)$. By (2.16) and (2.20) we then obtain:

$$\mathbf{P}(\overline{u})=-2\,\deg_{\mathrm{RP}^1}(\overline{u},0)\,\delta_0\,,\qquad \deg_{\mathrm{RP}^1}(\overline{u},0)=\deg_{\mathrm{RP}^1}(u)\,. \tag{2.21}$$

3 Minimal connections and dipoles

In this section we discuss the Dipole problem of $W^{1,1}$-maps u with values in RP^1. For this reason, we first recall some notation about minimal connections.

Integral flat chains and minimal connections. Let $n\geq2$ and $\Omega\subset\mathbb{R}^n$ open.

Definition 3.1 *A current* $\Gamma \in \mathcal{D}_{n-2}(\Omega)$ *is an* integral flat chain *if there exists an i.m. rectifiable current* $L \in \mathcal{R}_{n-1}(\Omega)$ *such that* $(\partial L) \llcorner \Omega = \Gamma$.

For any current $\Gamma \in \mathcal{D}_{n-2}(\Omega)$ we also denote by

$$\begin{array}{rcl} m_{r,\Omega}(\Gamma) & := & \inf\{\mathbf{M}(D) \mid D \in \mathcal{D}_{n-1}(\Omega), \quad (\partial D) \llcorner \Omega = \Gamma\} \\ m_{i,\Omega}(\Gamma) & := & \inf\{\mathbf{M}(L) \mid L \in \mathcal{R}_{n-1}(\Omega), \quad (\partial L) \llcorner \Omega = \Gamma\} \end{array} \tag{3.1}$$

the *real* and *integral mass* of Γ *relative to* Ω, respectively. Therefore, $m_{i,\Omega}(\Gamma) < \infty$ if and only if Γ is an integral flat chain. In this case, moreover, Federer-Fleming's closure theorem [9] yields that the minimum in (3.1) is always attained, and an i.m. rectifiable current $L \in \mathcal{R}_{n-1}(\Omega)$ is an *integral minimal connection* of Γ *allowing connections to the boundary* of Ω if $(\partial L) \llcorner \Omega = \Gamma$ and $\mathbf{M}(L) = m_{i,\Omega}(\Gamma)$, see [13, Vol. II, Sec. 4.2.6].

For example, the current $\mathbf{P}(u) \in \mathcal{D}_{n-2}(B^n)$ of the singularities of a Sobolev map u in $\in W^{1,1}(B^n, \mathrm{RP}^1)$, see (2.13), is an integral flat chain.

Proposition 3.2 *Let* $u \in W^{1,1}(B^n, \mathrm{RP}^1)$*, where* $n \geq 2$*. Then*

$$\pi \cdot m_{i,B^n}(\mathbf{P}(u)) \leq \mathbf{D}_1(u, B^n) < \infty\,.$$

PROOF: By the coarea formula [2], we have

$$\mathbf{D}_1(u, B^n) = \int_{\mathrm{RP}^1} \mathcal{H}^{n-1}(u^{-1}(z))\, d\mathcal{H}^1(z)\,.$$

We then find $z \in \mathrm{RP}^1$ such that the i.m. rectifiable current $L_z \in \mathcal{R}_{n-1}(B^n)$

$$L_z := \tau(u^{-1}(z), 1, \overrightarrow{\xi}\,)\,, \qquad \overrightarrow{\xi}(x) := \frac{\mathfrak{D}(u(x))}{|\mathfrak{D}(u(x))|}\,, \quad x \in u^{-1}(z)\,,$$

acting on forms $\gamma \in \mathcal{D}^{n-1}(B^n)$ as

$$\langle L_z, \gamma \rangle = \int_{u^{-1}(z)} \langle \gamma(x), \overrightarrow{\xi}(x) \rangle\, d\mathcal{H}^{n-1}(x)\,,$$

has finite mass

$$\mathbf{M}(L_z) = \mathcal{H}^{n-1}(u^{-1}(z)) \leq \frac{1}{\pi}\, \mathbf{D}_1(u, B^n) < \infty\,.$$

Finally, by (2.14) and (2.17), or by (2.15) for $n = 2$, we deduce that $(\partial L_z) \llcorner B^n = \mathbf{P}(u)$. □

THE DIPOLE PROBLEM. We let $n = 2$ and fix $a_i \in \mathbb{R}^2$, for $i = 1, \ldots, m$. As in [6, Sec. VIII-B-b)], the dipole problem involves the class

$$\begin{array}{rcl} \widetilde{\mathcal{F}}_1 := \{u \in L^1_{\mathrm{loc}}(\mathbb{R}^2, \mathrm{RP}^1) & : & |Du| \in L^1(\mathbb{R}^2)\,,\ u \in C^\infty(\mathbb{R}^2 \setminus \{a_i \mid i = 1, \ldots, m\})\,, \\ & & u \text{ is constant at infinity},\ \deg_{\mathrm{RP}^1}(u, a_i) = \widetilde{\Delta}_i \quad \forall\, i\} \end{array}$$

where to each point a_i we assign a non-zero number $\widetilde{\Delta}_i \in \frac{1}{2}\mathbb{Z}$, and we set $\widetilde{\Gamma}_0 := -\sum_{i=1}^m \widetilde{\Delta}_i\, \delta_{a_i}$, so that $2\,\widetilde{\Gamma}_0$ is an i.m. rectifiable current in $\mathcal{R}_0(\mathbb{R}^2)$.

Proposition 3.3 *The class* $\widetilde{\mathcal{F}}_1$ *is non-empty if and only if the compatibility condition*

$$\sum_{i=1}^m \widetilde{\Delta}_i = 0\,, \qquad \widetilde{\Delta}_i \in \frac{1}{2}\mathbb{Z} \setminus \{0\} \tag{3.2}$$

is satisfied. If (3.2) *holds, moreover, we have*

$$\inf\{\mathbf{D}_1(u, \mathbb{R}^2) \mid u \in \widetilde{\mathcal{F}}_1\} = \pi \cdot m_{i,\mathbb{R}^2}(2\,\widetilde{\Gamma}_0)\,. \tag{3.3}$$

PROOF: By (2.19) it turns out that $\mathbf{P}(u) = 2\,\widetilde{\Gamma}_0$ for every $u \in \widetilde{\mathcal{F}}_1$. Therefore, the first statement follows from the fact that the maps in $\widetilde{\mathcal{F}}_1$ are constant at infinity. If (3.2) holds, we have $m_{i,\mathbb{R}^2}(2\,\widetilde{\Gamma}_0) < \infty$, see (3.1), and we can find an integral minimal connection for $2\,\widetilde{\Gamma}_0$, i.e., an i.m. rectifiable current $L_0 \in \mathcal{R}_1(\mathbb{R}^2)$ such that $\partial L_0 = 2\,\widetilde{\Gamma}_0$ and $\mathbf{M}(L_0) = m_{i,\mathbb{R}^2}(2\,\widetilde{\Gamma}_0)$. Moreover, arguing as in [19], for every $\varepsilon > 0$ we find a map $u_\varepsilon \in \widetilde{\mathcal{F}}_1$ such that $\mathbf{D}_1(u_\varepsilon, \mathbb{R}^2) \leq \mathcal{H}^1(\mathrm{RP}^1) \cdot \mathbf{M}(L_0) + \varepsilon$. This proves the inequality "$\leq$" in (3.3).

To prove the converse inequality, we follow the proof of Thm. 1 in [13, Vol. II, Sec. 4.2.10]. More precisely, let $R := [\![\, \mathbb{R}^2 \setminus \{a_i \mid i = 1, \dots, m\} \,]\!]$. For every $u \in \widetilde{\mathcal{F}}_1$, similarly to Proposition 3.2, consider the slices of the current R at points $z \in \mathrm{RP}^1$,

$$\langle R, u, z \rangle := \tau(u^{-1}(z), 1, \overrightarrow{\zeta}\,)\,,$$

$\overrightarrow{\zeta}$ being the unit $(n-1)$-vector field orienting $u^{-1}(z)$ in the natural way. Therefore, $\langle R, u, z \rangle \in \mathcal{R}_1(\mathbb{R}^2)$ and $\partial \langle R, u, z \rangle = 2\,\widetilde{\Gamma}_0$ for $\mathcal{H}^1$-a.e. $z \in \mathrm{RP}^1$, so that by the definition of L_0 we get

$$\mathcal{H}^1(u^{-1}(z)) = \mathbf{M}(\langle R, u, z \rangle) \geq \mathbf{M}(L_0)\,.$$

Moreover, by the coarea formula,

$$\int_{\mathrm{RP}^1} \mathbf{M}(\langle R, u, z \rangle)\, d\mathcal{H}^1(z) = \int_{\mathrm{RP}^1} \mathcal{H}^1(u^{-1}(z))\, d\mathcal{H}^1(z) = \int_{\mathbb{R}^2} |Du(x)|\, dx\,.$$

We have thus obtained

$$\mathbf{D}_1(u, \mathbb{R}^2) \geq \int_{\mathrm{RP}^1} \mathbf{M}(L_0)\, d\mathcal{H}^1(z) = \mathcal{H}^1(\mathrm{RP}^1) \cdot m_{i,\mathbb{R}^2}(2\,\widetilde{\Gamma}_0)$$

for every $u \in \widetilde{\mathcal{F}}_1$, as required. □

4 Cartesian currents in $\mathbf{B^n \times RP^1}$

In this section we introduce a class of Cartesian currents in $B^n \times \mathrm{RP}^1$, proving some basic properties.

GRAPHS. If $u : B^n \to \mathrm{RP}^1$ is smooth, the graph current G_u in $\mathcal{R}_n(B^n \times \mathrm{RP}^1)$ is defined by the integration of compactly supported smooth n-forms ω in $B^n \times \mathrm{RP}^1$ over the naturally oriented n-manifold given by the graph $\mathcal{G}_u$ of u, i.e.,

$$G_u(\omega) := \int_{\mathcal{G}_u} \omega\,, \qquad \omega \in \mathcal{D}^n(B^n \times \mathrm{RP}^1)\,.$$

We thus have

$$G_u(\omega) = \int_{B^n} (\mathrm{Id} \bowtie u)^{\#}\omega \qquad \forall\, \omega \in \mathcal{D}^n(B^n \times \mathrm{RP}^1)\,, \tag{4.1}$$

where $(\mathrm{Id} \bowtie u)(x) := (x, u(x))$. Following [13, Vol. I, Sec. 3.2], the i.m. rectifiable current $G_u \in \mathcal{R}_n(B^n \times \mathrm{RP}^1)$ carried by the graph of a function $u \in W^{1,1}(B^n, \mathrm{RP}^1)$ is well-defined in the a.e. approximate sense by (4.1). Therefore, the area formula yields

$$\mathbf{M}(G_u) = \int_{B^n} \sqrt{1 + |Du|^2}\, dx\,.$$

Moreover, for $n \geq 2$, the current $\mathbf{P}(u)$ of the singularity, see (2.13), satisfies

$$\langle \mathbf{P}(u), \varphi \rangle = G_u(d\varphi \wedge \omega_{\mathrm{RP}^1}) = \partial G_u(\varphi \wedge \omega_{\mathrm{RP}^1}) \tag{4.2}$$

for every $\varphi \in \mathcal{D}^{n-2}(B^n)$, as $G_u(\varphi \wedge d\omega_{\mathrm{RP}^1}) = 0$.

WEAK LIMITS. Recall that the weak convergence $T_k \rightharpoonup T$ as currents in $\mathcal{D}_n(B^n \times \mathrm{RP}^1)$ is defined in the dual sense by $T_k(\omega) \to T(\omega)$ for every $\omega \in \mathcal{D}^n(B^n \times \mathrm{RP}^1)$.

Let $\{u_k\}$ be a sequence of smooth maps in $W^{1,1}(B^n, \mathrm{RP}^1)$ satisfying $\sup_k \mathbf{D}_1(u_k) < \infty$ and converging in L^1 to a Sobolev function $u \in W^{1,1}(B^n, \mathrm{RP}^1)$. By Stoke's theorem we have

$$\partial G_{u_k}(\widetilde{\omega}) := G_{u_k}(d\widetilde{\omega}) = \int_{\mathcal{G}_{u_k}} d\widetilde{\omega} = \int_{\partial \mathcal{G}_{u_k}} \widetilde{\omega} = 0$$

for every $\widetilde{\omega} \in \mathcal{D}^{n-1}(B^n \times \mathrm{RP}^1)$. Then, by Federer-Flemings closure theorem [9], possibly passing to a subsequence the currents G_{u_k} weakly converge to an i.m. rectifiable current T in $\mathcal{R}_n(B^n \times \mathrm{RP}^1)$ satisfying the null-boundary condition

$$\partial T(\widetilde{\omega}) = 0 \qquad \forall \widetilde{\omega} \in \mathcal{D}^{n-1}(B^n \times \mathrm{RP}^1)\,. \tag{4.3}$$

Moreover, the L^1-convergence $u_k \to u$ yields that on "horizontal" forms we have

$$T(\phi(x,y)\,dx) = \int_{B^n} \phi(x, u_T(x))\,dx \qquad \forall\, \phi \in C_c^\infty(B^n \times \mathrm{RP}^1)\,, \tag{4.4}$$

where $u_T = u$. Also, the following structure property holds:

Proposition 4.1 *Let T in $\mathcal{R}_n(B^n \times \mathrm{RP}^1)$ satisfying (4.3) and (4.4), where $u_T \in W^{1,1}(B^n, \mathrm{RP}^1)$. Then there exists an i.m. rectifiable current $L_T \in \mathcal{R}_{n-1}(B^n)$ such that*

$$T = G_{u_T} + L_T \times [\![\, \mathrm{RP}^1 \,]\!]\,. \tag{4.5}$$

PROOF: Every $(n-1)$-form in $\mathcal{D}^{n-1}(B^n)$ can be written as

$$\omega_\eta := \sum_{i=1}^n (-1)^{i-1} \eta^i\, \widehat{dx^i}\,, \qquad \widehat{dx^i} := dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^n \tag{4.6}$$

for some vector field $\eta = (\eta^1, \ldots, \eta^n) \in C_c^\infty(B^n, \mathbb{R}^n)$, so that $d\omega_\eta = \operatorname{div} \eta\, dx$. Define $L_T \in \mathcal{R}_{n-1}(B^n)$ by

$$L_T(\omega_\eta) := S_T(\omega_\eta \wedge \omega_{\mathrm{RP}^1})\,, \qquad \eta \in C_c^\infty(B^n, \mathbb{R}^n)\,,$$

where

$$S_T := T - G_{u_T} \in \mathcal{R}_n(B^n \times \mathrm{RP}^1)\,.$$

Forms of the type $\phi(x,y)\,dx + \omega_\eta \wedge \alpha$, where $\phi \in C_c^\infty(B^n \times \mathrm{RP}^1)$, $\eta \in C_c^\infty(B^n, \mathbb{R}^n)$, and $\alpha \in \mathcal{D}^1(\mathrm{RP}^1)$, are dense in $\mathcal{D}^n(B^n \times \mathrm{RP}^1)$. Therefore, it suffices to show that

$$S_T(\omega) = L_T \times [\![\, \mathrm{RP}^1 \,]\!](\omega) \qquad \forall\, \omega = \phi(x,y)\,dx + \omega_\eta \wedge \alpha\,. \tag{4.7}$$

Now, by (4.4), and by definition of cartesian product of currents, we have

$$S_T(\phi(x,y)\,dx) = L_T \times [\![\, \mathrm{RP}^1 \,]\!](\phi(x,y)\,dx) = 0\,.$$

Moreover, since the de Rham cohomology group $H^1_{dR}(\mathrm{RP}^1) \simeq \mathbb{Z}$, by Hodge decomposition theorem we can write $\alpha = \lambda\, \omega_{\mathrm{RP}^1} + d\beta$ for some $\lambda \in \mathbb{R}$ and $\beta \in C^\infty(\mathrm{RP}^1)$, so that

$$\omega_\eta \wedge \alpha = \lambda\, \omega_\eta \wedge \omega_{\mathrm{RP}^1} + \omega_\eta \wedge d\beta\,.$$

Lemma 4.2 *$S_T(\omega_\eta \wedge d\beta) = 0$ for every $\eta \in C_c^\infty(B^n, \mathbb{R}^n)$ and $\beta \in C^\infty(\mathrm{RP}^1)$.*

Lemma 4.2, the proof of which is postponed, gives:

$$S_T(\omega_\eta \wedge \alpha) = \lambda\, S_T(\omega_\eta \wedge \omega_{\mathrm{RP}^1})\,.$$

Since moreover $[\![\, \mathrm{RP}^1 \,]\!](d\beta) = \partial [\![\, \mathrm{RP}^1 \,]\!](\beta) = 0$, formula (2.7) gives

$$[\![\, \mathrm{RP}^1 \,]\!](\alpha) = \lambda\, [\![\, \mathrm{RP}^1 \,]\!](\omega_{\mathrm{RP}^1}) + [\![\, \mathrm{RP}^1 \,]\!](d\beta) = \lambda \tag{4.8}$$

and hence, by definition of Cartesian product of currents,

$$L_T \times [\![\, \mathrm{RP}^1 \,]\!](\omega_\eta \wedge \alpha) = L_T(\omega_\eta) \cdot [\![\, \mathrm{RP}^1 \,]\!](\alpha) = \lambda\, L_T(\omega_\eta) = \lambda\, S_T(\omega_\eta \wedge \omega_{\mathrm{RP}^1})\,.$$

This gives (4.7), as required. □

Proof of Lemma 4.2: Since

$$d(\omega_\eta \wedge \beta) = \operatorname{div} \eta(x)\, \beta(y)\, dx + (-1)^{n-1} \omega_\eta \wedge d\beta$$

by (4.3) we have

$$T(\operatorname{div} \eta(x)\, \beta(y)\, dx) = (-1)^n T(\omega_\eta \wedge d\beta)\,,$$

so that

$$(-1)^n S_T(\omega_\eta \wedge d\beta) = T(\operatorname{div} \eta(x)\, \beta(y)\, dx) + (-1)^{n-1} G_{u_T}(\omega_\eta \wedge d\beta)\,.$$

By (4.4) we find that

$$T(\operatorname{div} \eta(x)\, \beta(y)\, dx) = \int_{B^n} \operatorname{div} \eta(x)\, \beta(u_T(x))\, dx\,.$$

Moreover, since $(-1)^{n-i} \widehat{dx^i} \wedge dx^h = \delta_i^h\, dx$, we compute

$$\begin{aligned} (-1)^{n-1} (Id \bowtie u_T)^\# (\omega_\eta \wedge d\beta) &= (-1)^{n-1} \omega_\eta \wedge u_T^\# (d\beta) \\ &= \sum_{i=1}^n (-1)^{n-i} \eta^i \widehat{dx^i} \wedge \sum_{j=1}^2 D_j \beta(u_T) \sum_{h=1}^n D_h u_T^j\, dx^h \\ &= \sum_{i=1}^n \eta^i \sum_{j=1}^2 D_j \beta(u_T)\, D_i u_T^j\, dx \\ &= \sum_{i=1}^n \eta^i D_i [\beta(u_T)]\, dx\,. \end{aligned}$$

By (4.1), and integrating by parts, this gives

$$\begin{aligned} (-1)^{n-1} G_{u_T}(\omega_\eta \wedge d\beta) &= (-1)^{n-1} \int_{B^n} (Id \bowtie u_T)^\# (\omega_\eta \wedge d\beta) \\ &= \sum_{i=1}^n \int_{B^n} \eta^i(x)\, D_i [\beta(u_T(x))]\, dx \\ &= -\int_{B^n} \operatorname{div} \eta(x)\, \beta(u_T(x))\, dx \end{aligned}$$

and finally $S_T(\omega_\eta \wedge d\beta) = 0$. □

Cartesian currents. For this reason, we introduce the following

Definition 4.3 *Denote by* $\mathrm{Cart}^{1,1}(B^n \times \mathrm{RP}^1)$ *the class of i.m. rectifiable currents* $T \in \mathcal{R}_n(B^n \times \mathrm{RP}^1)$ *satisfying the null-boundary condition* (4.3) *and the structure property* (4.5) *for some Sobolev map* $u_T \in W^{1,1}(B^n, \mathrm{RP}^1)$ *and some i.m. rectifiable current* $L_T \in \mathcal{R}_{n-1}(B^n)$*.*

Notice that each current $T \in \mathrm{Cart}^{1,1}(B^n \times \mathrm{RP}^1)$ has finite mass

$$\mathbf{M}(T) = \mathbf{M}(G_{u_T}) + \pi \cdot \mathbf{M}(L_T) < \infty\,.$$

Moreover, for future use, we point out the following property:

Proposition 4.4 *Let* $n \geq 2$ *and* $T \in \mathrm{Cart}^{1,1}(B^n \times \mathrm{RP}^1)$ *satisfying* (4.5)*. According to* (2.13)*, the null-boundary condition* (4.3) *is equivalent to the formula*

$$(\partial L_T) \llcorner B^n = -\mathbf{P}(u_T)\,. \tag{4.9}$$

Proof: In order to prove that (4.9) implies (4.3), we decompose any form $\omega \in \mathcal{D}^k(B^n \times \mathrm{RP}^1)$ as $\omega = \omega^{(0)} + \omega^{(1)}$ according to the number of differentials in the "vertical" y-directions. Moreover, we split the differential $d = d_x + d_y$.

Since $u_T \in W^{1,1}(B^n, \mathrm{RP}^1)$, arguing as e.g. in [13, Vol. II, Sec. 5.4.2], see also [15, Prop. 4.22], we get:

(i) $\partial G_{u_T}(\eta^{(0)}) = 0$ for every $\eta \in \mathcal{D}^{n-1}(B^n \times \mathrm{RP}^1)$;

(ii) $\partial G_{u_T}(d_y \gamma^{(0)}) = 0$ for every $\gamma \in \mathcal{D}^{n-2}(B^n \times \mathrm{RP}^1)$.

Moreover, $\partial(L_T \times [\![\mathrm{RP}^1]\!])(\eta^{(0)}) = 0$ for every $\eta \in \mathcal{D}^{n-1}(B^n \times \mathrm{RP}^1)$. Then, by (4.5) and (i) we deduce that the null-boundary condition (4.3) is equivalent to the property

$$\partial(L_T \times [\![\mathrm{RP}^1]\!])(\eta^{(1)}) = -\partial G_{u_T}(\eta^{(1)}) \qquad \forall \, \eta \in \mathcal{D}^{n-1}(B^n \times \mathrm{RP}^1)\,. \tag{4.10}$$

By a density argument we reduce to prove (4.10) when $\eta^{(1)} = \varphi \wedge \alpha$ for some $\varphi \in \mathcal{D}^{n-2}(B^n)$ and $\alpha \in \mathcal{D}^1(\mathrm{RP}^1)$. As in the proof of Proposition 4.1, we then decompose $\alpha = \lambda\, \omega_{\mathrm{RP}^1} + d\beta$, so that

$$\eta^{(1)} = \varphi \wedge \alpha = \lambda\, \varphi \wedge \omega_{\mathrm{RP}^1} + \varphi \wedge d\beta\,, \qquad \lambda \in \mathbb{R}\,, \quad \beta \in C^\infty(\mathrm{RP}^1)\,.$$

Using (4.2) and (ii), we have

$$\partial G_{u_T}(\varphi \wedge \alpha) = \partial G_{u_T}(\lambda\, \varphi \wedge \omega_{\mathrm{RP}^1}) + \partial G_{u_T}(\varphi \wedge d\beta) = \lambda\, \langle \mathbf{P}(u_T), \varphi \rangle + 0\,.$$

Since moreover $\partial [\![\mathrm{RP}^1]\!] = 0$, by (4.8) we obtain

$$\partial(L_T \times [\![\mathrm{RP}^1]\!])(\varphi \wedge \alpha) = (\partial L_T \times [\![\mathrm{RP}^1]\!])(\varphi \wedge \alpha) = \partial L_T(\varphi) \cdot [\![\mathrm{RP}^1]\!](\alpha) = \lambda\, \partial L_T(\varphi)\,,$$

so that (4.9) implies (4.10), hence (4.3). The converse implication follows from the previous computation, by taking $\eta = \eta^{(1)} = \varphi \wedge \omega_{\mathrm{RP}^1}$, i.e., $\lambda = 1$ and $\beta = 0$. □

The total variation energy. Using the *parametric lower semicontinuous extension* of the total variation energy integrand, Giaquinta-Modica-Souček defined a non-negative functional $T \mapsto \mathbf{D}_1(T)$ on the class of Cartesian currents $\mathrm{cart}(B^n \times \mathbb{S}^1)$, see Sec. 7 below, called the *total variation energy.*

It turns out that such a functional can be defined on our class of currents $\mathrm{Cart}^{1,1}(B^n \times \mathrm{RP}^1)$ in such a way that the following properties hold:

Theorem 4.5 *We have:*

(*a*) *$T \mapsto \mathbf{D}_1(T)$ is lower semicontinuous with respect to the weak $\mathcal{D}_n$-convergence in $\mathrm{Cart}^{1,1}(B^n \times \mathrm{RP}^1)$;*

(*b*) *if T satisfies* (4.5)*, then $\mathbf{D}_1(T) = \mathbf{D}_1(u_T, B^n) + \pi \cdot \mathbf{M}(L_T)$.*

(*c*) *for every $T \in \mathrm{Cart}^{1,1}(B^n \times \mathrm{RP}^1)$, there exists a sequence of* smooth *maps $\{u_k\} \subset W^{1,1}(B^n, \mathrm{RP}^1)$ such that $G_{v_k} \rightharpoonup T$ in $\mathcal{D}_n$ and $\mathbf{D}_1(u_k, B^n) \to \mathbf{D}_1(T)$ as $k \to \infty$.*

(*d*) *we also have mass convergence $\mathbf{M}(G_{u_k}) \to \mathbf{M}(T)$ as $k \to \infty$.*

Sketch of the proof: Properties (a) and (b) follow from the definition of total variation energy, compare [13, Vol. II, Sec. 1.2.4]. In order to prove the density property (c), we may argue as in [13, Vol. II, Sec. 5.4.2]. Roughly speaking, for every $T \in \mathrm{Cart}^{1,1}(B^n \times \mathrm{RP}^1)$, by Bethuel's theorem [4] we find a sequence of maps $\{u_k\} \subset R_1^\infty(B^n, \mathrm{RP}^1)$ strongly converging to u_T in $W^{1,1}$. This gives that the real mass $m_{r,B^n}(\mathbf{P}(u_k) - \mathbf{P}(u)) \to 0$ as $k \to \infty$, see (3.1). By Proposition 3.2 and by Hardt-Pitts theorem [16], we deduce that the integral mass $m_{i,B^n}(\mathbf{P}(u_k) - \mathbf{P}(u)) \to 0$ as $k \to \infty$. Therefore, we reduce to prove the density property (c) for currents in $\mathrm{Cart}^{1,1}(B^n \times \mathrm{RP}^1)$ satisfying (4.5) for some $u_T \in R_1^\infty(B^n, \mathrm{RP}^1)$ and some *integral current* $L_T \in \mathcal{R}_{n-2}(B^n)$, i.e., such that $\mathbf{M}_{B^n}(\partial L_T) < \infty$. By Federer's strong approximation theorem [8, 4.2.20], we then reduce to the case in which L_T is an $(n-2)$-dimensional *integral polyhedral chain.* Therefore, a Dipole-type construction yields the claim in (c). Finally, the mass convergence in (d) follows from the strong $W^{1,1}$-convergence $u_k \to u_T$ at the first step. □

A FEW REMARKS. For $\mathfrak{p}\geq 3$ odd, in [19] we found that the weak limits of sequences of currents carried by graphs of smooth maps $u_k\in W^{1,\mathfrak{p}}(B^n,\mathrm{RP}^{\mathfrak{p}})$ satisfying $\sup_k \mathbf{D}_{\mathfrak{p}}(u_k,B^n)<\infty$, for $n\geq \mathfrak{p}+1$, are i.m. rectifiable currents in $\mathcal{R}_n(B^n\times \mathrm{RP}^{\mathfrak{p}})$ of the type

$$T=G_{u_T}+2\,\widetilde{L}_T\times [\![\,\mathrm{RP}^{\mathfrak{p}}\,]\!]$$

for some $u_T\in W^{1,\mathfrak{p}}(B^n,\mathrm{RP}^{\mathfrak{p}})$ and some i.m. rectifiable current $\widetilde{L}_T\in\mathcal{R}_{n-\mathfrak{p}}(B^n)$. Moreover, compare (4.9), the null-boundary condition reads as $(\partial\widetilde{L}_T)\llcorner B^n=-\frac{1}{2}\,\mathbf{P}(u_T)$, where $\mathbf{P}(u_T)\in\mathcal{D}_{n-\mathfrak{p}-1}(B^n)$ is the current of the singularities of u_T. This is a consequence of Theorem 0.1, and it actually defines the class $\mathrm{cart}^{\mathfrak{p},1}(B^n\times\mathrm{RP}^{\mathfrak{p}})$, that is *closed* under the weak convergence of sequences with equibounded masses, or $\mathfrak{p}$-energies. Moreover, the current $L_T:=2\widetilde{L}_T$ has *even multiplicity.*

In the case $\mathfrak{p}=1$, taking e.g. $n=2$ and $u_T=\overline{u}$, see Example 1.2, by (2.18) we infer that in order to enclose the singularity of $\overline{u}$, one has to take an i.m. rectifiable current $L\in\mathcal{R}_1(B^2)$ such that $(\partial L)\llcorner B^2=\delta_0$, e.g., an oriented line from the boundary of B^2 to the origin. This yields that in general *the current* $L_T\in\mathcal{R}_{n-1}(B^n)$ *in* (4.5) *does not have an even multiplicity.*

Moreover, the class $\mathrm{Cart}^{1,1}(B^n\times\mathrm{RP}^1)$ is *not closed* under the weak convergence of sequences with equibounded masses, or total variation energies. In fact, if a sequence $\{u_k\}$ of smooth maps in $W^{1,1}(B^n,\mathrm{RP}^1)$ satisfies $\sup_k\mathbf{D}_1(u_k,B^n)<\infty$, possibly passing to a subsequence, in general the u_k's weakly converge in the BV-sense to a function of bounded variation in $BV(B^n,\mathrm{RP}^1)$. Therefore, the weak limits of the corresponding currents $G_{u_k}\in\mathrm{Cart}^{1,1}(B^n\times\mathrm{RP}^1)$ are i.m. rectifiable currents in $B^n\times\mathrm{RP}^1$ with a more complicated structure, as they involve the integration on the "graph" of functions in $BV(B^n,\mathrm{RP}^1)$.

5 Optimally connecting measure and relaxed energy

In this section we discuss a notion of optimally connecting measure of the singular set of u. We then analyze the relaxed total variation energy.

OPTIMALLY CONNECTING MEASURE. Proposition 3.2 yields that for every $u\in W^{1,1}(B^n,\mathrm{RP}^1)$, where $n\geq 2$, we can find an integral minimal connection of the singularity $\mathbf{P}(u)$, i.e., an i.m. rectifiable current $L_u\in\mathcal{R}_{n-1}(B^n)$ such that

$$(\partial L_u)\llcorner B^n=\mathbf{P}(u)\qquad\text{and}\qquad \mathbf{M}(L_u)=m_{i,B^n}(\mathbf{P}(u))<\infty\,.$$

We thus have

$$L_u(\gamma)=\int_{\mathcal{L}_u}\theta_u\,\langle\gamma,\overrightarrow{\mathcal{L}}_u\rangle\,d\mathcal{H}^{n-1}\qquad\forall\,\gamma\in\mathcal{D}^{n-1}(B^n)\,,$$

where $\mathcal{L}_u$ is a countably $(n-1)$-rectifiable set in B^n, the multiplicity function $\theta_u:\mathcal{L}_u\to\mathbb{N}^+$ is $\mathcal{H}^{n-1}\llcorner\mathcal{L}_u$-summable, and $\overrightarrow{\mathcal{L}}_u:\mathcal{L}_u\to\Lambda_{n-1}\mathbb{R}^n$ is an $\mathcal{H}^{n-1}\llcorner\mathcal{L}_u$-measurable unit $(n-1)$-vector field that provides an orientation to the $(n-1)$-dimensional approximate tangent space to $\mathcal{L}_u$ at $\mathcal{H}^{n-1}$-a.e. point.

We then call $\widetilde{\mu}_u:=\theta_u\,\mathcal{H}^{n-1}\llcorner\mathcal{L}_u$ an *optimally connecting measure* of the singular set of u. Notice that the total variation of $\widetilde{\mu}_u$ satisfies

$$|\widetilde{\mu}_u|(B^n)=\int_{B^n}\theta_u\,d\mathcal{H}^{n-1}=\mathbf{M}(L_u)=m_{i,B^n}(\mathbf{P}(u))\,. \tag{5.1}$$

By Proposition 4.4, it turns out that the current $T_u:=G_u-L_u\times[\![\,\mathrm{RP}^1\,]\!]$ actually belongs to the class $\mathrm{Cart}^{1,1}(B^n\times\mathrm{RP}^1)$. This clearly gives:

Proposition 5.1 *For every* $u\in W^{1,1}(B^n,\mathrm{RP}^1)$ *there exists a current* $T\in\mathrm{Cart}^{1,1}(B^n\times\mathrm{RP}^1)$ *with corresponding* $W^{1,1}$*-function* $u_T=u$ *in* (4.5).

Moreover, we have:

Theorem 5.2 *Let* $u\in W^{1,1}(B^n,\mathrm{RP}^1)$ *and let* $\widetilde{\mu}_u$ *as above. Then there exists a sequence of* smooth *maps* $\{u_k\}\subset W^{1,1}(B^n,\mathrm{RP}^1)$ *satisfying the following properties:*

i) $u_k \rightharpoonup u$ *weakly in* $W^{1,1}$ *as* $k \to \infty$;

ii) $\mathbf{D}_1(u_k, B^n) \to \mathbf{D}_1(u, B^n) + \pi \cdot |\widetilde{\mu}_u|(B^n)$ *as* $k \to \infty$;

iii) $|Du_k|\,\mathcal{L}^n \llcorner B^n \rightharpoonup |Du|\,\mathcal{L}^n \llcorner B^n + \pi\,\widetilde{\mu}_u$ *weakly as measures;*

iv) *for any open set* A *contained in* $B^n \setminus \operatorname{spt} \widetilde{\mu}_u$, *we have strong* $W^{1,1}$*-convergence of* $u_{k|A}$ *to* $u_{|A}$.

Proof: The first three assertions follow by applying the density property (c) in Theorem 4.5 to the current $T_u \in \mathrm{Cart}^{1,1}(B^n \times \mathrm{RP}^1)$. Moreover, the mass convergence (d) in Theorem 4.5 implies that

$$\lim_{k\to\infty} \int_A \sqrt{1+|Du_k|^2}\,dx = \int_A \sqrt{1+|Du_T|^2}\,dx\,.$$

for any open set A contained in $B^n \setminus \operatorname{spt} \widetilde{\mu}_u$. Therefore, the last assertion follows from a theorem due to Reshetnyak, as observed in [1]. □

Relaxed energy. In the same spirit as for Lebesgue's area, the relaxed total variation energy with respect to the L^1-convergence is defined on maps $u \in L^1(B^n, \mathrm{RP}^1)$ by

$$\widehat{\mathbf{D}}_1(u, B^n) := \inf\Big\{ \liminf_{k\to\infty} \mathbf{D}_1(u_k, B^n) \quad | \quad \{u_k\} \subset C^\infty(B^n, \mathrm{RP}^1)\,, \; u_k \to u \text{ in } L^1(B^n, \mathbb{R}^3) \Big\}\,. \tag{5.2}$$

We readily obtain:

Proposition 5.3 *The relaxed energy* $\widehat{\mathbf{D}}_1(u, B^n)$ *is finite if and only if* $u \in BV(B^n, \mathrm{RP}^1)$.

We write an explicit formula for the relaxed energy of $W^{1,1}$-maps. In dimension $n = 1$, by Schoen-Uhlenbeck density theorem [20] we clearly have:

$$\widehat{\mathbf{D}}_1(u, B^1) = \mathbf{D}_1(u, B^1) \qquad \forall\, u \in W^{1,1}(B^1, \mathrm{RP}^1)\,.$$

In higher dimension $n \geq 2$, Proposition 5.1 yields that *for every* $u \in W^{1,1}(B^n, \mathrm{RP}^1)$ *the class*

$$\mathcal{T}_u^{1,1} := \{T \in \mathrm{Cart}^{1,1}(B^n \times \mathrm{RP}^1) \mid u_T = u \text{ in } (4.5)\}$$

is non-empty, whereas by Proposition 4.4

$$\mathcal{T}_u^{1,1} = \Big\{ G_u + L \times [\![\, \mathrm{RP}^1 \,]\!] \mid L \in \mathcal{R}_{n-1}(B^n)\,, \, (\partial L) \llcorner B^n = -\mathbf{P}(u) \Big\}\,,$$

where $\mathbf{P}(u) \in \mathcal{D}_{n-2}(B^n)$ is given by (2.13). Since moreover the current $T_u := G_u - L_u \times [\![\, \mathrm{RP}^1 \,]\!]$ belongs to $\mathcal{T}_u^{1,1}$, by (5.1) and property (b) from Theorem 4.5 we obtain:

$$\inf\{\mathbf{D}_1(T) \mid T \in \mathcal{T}_u^{1,1}\} = \mathbf{D}_1(T_u) = \mathbf{D}_1(u, B^n) + \pi \cdot \mathbf{M}(L_u)\,. \tag{5.3}$$

Proposition 5.4 *For every* $u \in W^{1,1}(B^n, \mathrm{RP}^1)$ *we have*

$$\begin{aligned} \widehat{\mathbf{D}}_1(u, B^n) &= \mathbf{D}_1(u, B^n) + \pi \cdot m_{i,B^n}(\mathbf{P}(u)) \\ &= \mathbf{D}_1(u, B^n) + \pi \cdot |\widetilde{\mu}_u|(B^n)\,. \end{aligned}$$

Proof: By (5.1) and (5.3), it suffices to show that

$$\widehat{\mathbf{D}}_1(u, B^n) = \inf\{\mathbf{D}_1(T) \mid T \in \mathcal{T}_u^{1,1}\}\,. \tag{5.4}$$

Let $T \in \mathcal{T}_u^{1,1}$, and apply the density property (c) from Theorem 4.5. Since the weak convergence $G_{u_k} \rightharpoonup T$ with $\mathbf{D}_1(u_k) \to \mathbf{D}_1(T)$ yields the L^1-convergence $u_k \to u_T$, and $u_T = u$, we deduce that the inequality "$\leq$" holds in (5.4). To prove the converse inequality, let $\{u_k\} \subset C^\infty(B^n, \mathrm{RP}^1)$ such that $u_k \to u$ in L^1 and $\sup_k \mathbf{D}_1(u_k) < \infty$. Possibly passing to a subsequence, we can assume that $\liminf_k \mathbf{D}_1(u_k) = \lim_k \mathbf{D}_1(u_k)$. The argument at the beginning of Sec. 4 gives that (possibly passing again to a subsequence) the currents G_{u_k} weakly converge in $\mathcal{D}_n$ to some current $T \in \mathrm{Cart}^{1,1}(B^n \times \mathrm{RP}^1)$ such that $u_T = u$, i.e., $T \in \mathcal{T}_u^{1,1}$. Since $\mathbf{D}_1(G_{u_k}) = \mathbf{D}_1(u_k)$, the lower semicontinuity property (a) from Theorem 4.5 yields $\mathbf{D}_1(T) \leq \liminf_k \mathbf{D}_1(u_k)$, hence the inequality "$\geq$" holds in (5.4). □

Remark 5.5 Similarly to the case of maps into $\mathbb{S}^1$, compare [12] and [15, Sec. 7.8], a representation formula for the relaxed energy can be obtained on the larger class of maps $BV(B^n, \mathrm{RP}^1)$, arguing as e.g. in [18].

6 Existence of liftings

In this section we prove the *existence of liftings* of currents in $\mathrm{Cart}^{1,1}(B^n \times \mathrm{RP}^1)$. This will be used in Sec. 8 below to deduce some preliminary properties to our main result, Theorem 9.1. We write a complete proof, even if it is very similar, with minor modifications, to the analogous existence result proved in [12], see Thm. 2 in [13, Vol. II, Sec. 6.2.2], for Cartesian currents in $B^n \times \mathbb{S}^1$.

SUBGRAPHS. We first recall that the *current subgraph* of a real valued L^1-function $\psi \in L^1(B^n)$ is the $(n+1)$-dimensional current SG_ψ in $\mathcal{D}_{n+1}(B^n \times \mathbb{R})$ defined by

$$SG_\psi(\phi(x,t)dx \wedge dt) := \int_{B^n} \left(\int_0^{\psi(x)} \phi(x,t)\,dt \right) dx\,, \qquad \phi \in C_c^\infty(B^n \times \mathbb{R})\,. \tag{6.1}$$

Moreover, see [8], the mass of the boundary current ∂SG_ψ agrees with the total variation of ψ,

$$\mathbf{M}_{B^n \times \mathbb{R}}(\partial SG_\psi) = |D\psi|(B^n)\,. \tag{6.2}$$

Therefore, by the *boundary rectifiability theorem* [21, 30.3], it turns out that ∂SG_ψ is an i.m. rectifiable current in $\mathcal{R}_n(B^n \times \mathbb{R})$ if and only if ψ is a function of bounded variation in $BV(B^n)$, see [3].

ANGLE FUNCTION. According to (2.5), denote by $\widehat{\imath} : B^n \times \mathbb{R} \to B^n \times \mathrm{RP}^1$ the lifting map

$$\widehat{\imath}(x,t) := (x, \widehat{\jmath}(t))\,, \qquad \widehat{\jmath}(t) := \Big(\frac{\sqrt{2}}{2} \cos^2 t, \frac{\sqrt{2}}{2} \sin^2 t, \cos t \sin t \Big)\,. \tag{6.3}$$

Since by (6.3) we have $(z_1 - z_2)^2 + 2z_3^2 = 1/2$ for every $z = (z_1, z_2, z_3) \in \mathrm{RP}^1$, the function $\phi : \mathrm{RP}^1 \to \mathbb{R}$

$$\phi(z) := \frac{1}{2} \arctan\Big(\frac{\sqrt{2}\, z_3}{z_1 - z_2} \Big)$$

satisfies $d\phi(z) = \widehat{\Theta}(z)$ for $\mathcal{H}^1$-a.e. $z \in \mathrm{RP}^1$, where $\widehat{\Theta}$ is the *non-normalized* volume 1-form on RP^1

$$\widehat{\Theta}(z) := \pi \cdot \omega_{\mathrm{RP}^1}(z) = \sqrt{2}\,\big(-z^3 dz^1 + z^3 dz^2 + (z^1 - z^2)\, dz^3 \big)\,,$$

see (2.4). Define the *angle function* $\widehat{\theta} : \mathrm{RP}^1 \to [0, \pi[$ by

$$\widehat{\theta}(z) := \begin{cases} \phi(z) & \text{if } z \in \widehat{\jmath}([0, \pi/4[) \\ \phi(z) + \pi/2 & \text{if } z \in \widehat{\jmath}(]\pi/4, 3\pi/4[) \\ \phi(z) + \pi & \text{if } z \in \widehat{\jmath}(]3\pi/4, \pi[)\,, \end{cases}$$

whereas $\widehat{\theta}(\sqrt{2}/4, \sqrt{2}/4, 1/2) := \pi/4$ and $\widehat{\theta}(\sqrt{2}/4, \sqrt{2}/4, -1/2) := 3\pi/4$. We thus have $\widehat{\jmath} \circ \widehat{\theta} = Id_{\mathrm{RP}^1}$ and $\widehat{\theta}(\widehat{\jmath}(t)) = t$ for every $t \in [0, \pi[$. Finally, by (2.6) we have

$$d\widehat{\theta} = \widehat{\Theta}\,, \qquad \widehat{\jmath}^{\#} \widehat{\Theta} = dt\,. \tag{6.4}$$

EXISTENCE OF LIFTINGS. Denote by G_{p_0} the current in $\mathcal{R}_n(B^n \times \mathrm{RP}^1)$ integration over the graph of the constant map $p_0(x) \equiv \widehat{\jmath}(0) = (\sqrt{2}/2, 0, 0)$.

Theorem 6.1 *Let $T \in \mathrm{Cart}^{1,1}(B^n \times \mathrm{RP}^1)$, see Definition 4.3. Then there exists a real valued BV-function $\psi_T \in BV(B^n)$ such that*

$$T - G_{p_0} = (-1)^n \widehat{\imath}_{\#} \partial SG_{\psi_T}\,. \tag{6.5}$$

Moreover, if $u_T \in W^{1,1}(B^n, \mathrm{RP}^1)$ is the corresponding $W^{1,1}$-function in (4.5), we have

$$u_T = \widehat{\jmath} \circ \psi_T \qquad \mathcal{L}^n\text{-a.e. on } B^n\,. \tag{6.6}$$

Proof: We divide it in three steps.

Step 1: Arguing as in [14], we find a current $\Sigma\in\mathcal{D}_{n+1}(B^n\times\mathrm{RP}^1)$ such that

$$T-G_{p_0}=(-1)^n\partial\Sigma\qquad\text{on}\quad\mathcal{D}^n(B^n\times\mathrm{RP}^1)\,. \tag{6.7}$$

In fact, B^n being simply-connected, both the *real relative homology groups* $H_n(B^n\times\mathrm{RP}^1,\partial B^n\times\mathrm{RP}^1;\mathbb{R})$ and $H_n(B^n,\partial B^n;\mathbb{R})$ and equal to $\mathbb{R}$, and the canonical projection of the first one into the second one is an isomorphism. Denoting by $\pi:B^n\times\mathrm{RP}^1\to B^n$ the orthogonal projection onto the first factor, by (4.5) we have $\pi_\# T=\pi_\# G_{p_0}=[\![B^n]\!]$. Therefore, T and G_{p_0} are homologous relative cycles in $H_n(B^n\times\mathrm{RP}^1,\partial B^n\times\mathrm{RP}^1;\mathbb{R})$. This gives (6.7), compare Thm. 2 in [13, Vol. II, Sec. 6.2.2].

Step 2: Proof of (6.5). Since the current Σ in (6.7) is an $(n+1)$-dimensional *normal* current in $B^n\times\mathrm{RP}^1$, by [21, 26.28] we find the existence of a function $\widetilde g\in BV_{\mathrm{loc}}(B^n\times\mathrm{RP}^1)$ such that for any $\widetilde f\in C^\infty_c(B^n\times\mathrm{RP}^1)$

$$\Sigma(\widetilde f(x,\widehat\theta)\,dx\wedge\widehat\Theta)=\int_{B^n\times\mathrm{RP}^1}\widetilde f(x,\widehat\theta)\,\widetilde g(x,\widehat\theta)\,d\mathcal{H}^{n+1}\,. \tag{6.8}$$

Setting then $f,\overline g:B^n\times\mathbb{R}\to\mathbb{R}$ by

$$f(x,t):=\widetilde f(x,\widehat j(t))\,,\qquad \overline g(x,t):=\widetilde g(x,\widehat j(t))\,,$$

clearly f and $\overline g$ are π-periodic in t and

$$\Sigma(\widetilde f(x,\widehat\theta)\,dx\wedge\widehat\Theta)=\int_{B^n}dx\int_0^\pi f(x,t)\,\overline g(x,t)\,dt\,. \tag{6.9}$$

Moreover, in the sense of measures $|D\widetilde g|=\|\partial\Sigma\|$, whereas by (6.7) we infer that $\partial\Sigma$ is i.m. rectifiable in $\mathcal{R}_n(B^n\times\mathrm{RP}^1)$. Therefore, we find that $|D\widetilde g|=\sigma\,\mathcal{H}^n\llcorner\mathcal{S}$ for some n-rectifiable set $\mathcal{S}\subset B^n\times\mathrm{RP}^1$ and some *integer-valued* $\mathcal{H}^n$-integrable function σ on $\mathcal{S}$. As a consequence, we find a real number $r_0\in\mathbb{R}$ and an *integer-valued* locally BV-function $g\in BV_{\mathrm{loc}}(B^n\times\mathbb{R},\mathbb{Z})$, actually $g\in BV_{\mathrm{loc}}(B^n\times(0,\pi))$, such that

$$\overline g(x,t)=r_0+g(x,t)\,. \tag{6.10}$$

Consider the function $\psi=\psi_T\in BV_{\mathrm{loc}}(B^n)$ defined by

$$\psi(x):=\int_0^\pi g(x,t)\,dt \tag{6.11}$$

and the $(n+1)$-dimensional current SG_ψ in (6.1). In Step 3 below we will prove the following claim:

$$\widehat\imath_\# SG_\psi+r_0\,[\![B^n\times\mathrm{RP}^1]\!]=\Sigma\qquad\text{on}\quad\mathcal{D}^{n+1}(B^n\times\mathrm{RP}^1)\,. \tag{6.12}$$

Since $\widehat\imath_\#\partial SG_\psi=\partial\widehat\imath_\# SG_\psi$ and $\partial[\![B^n\times\mathrm{RP}^1]\!]=0$ on $\mathcal{D}^n(B^n\times\mathrm{RP}^1)$, by (6.7) we readily obtain (6.5). As a consequence we infer

$$\mathbf{M}(\partial SG_\psi)\le\mathbf{M}(T)+\mathcal{L}^n(B^n)<\infty\,,$$

which yields that the total variation of ψ is finite, see (6.2). Also, a Poincaré type inequality yields that $\psi\in BV(B^n)$. Finally, formula (6.6) is an immediate consequence of (4.4), (6.5), and of the definition (6.1) of SG_ψ.

Step 3: Proof of the claim (6.12). By (4.4) and (6.7) we obtain for any $\widetilde f\in C^\infty_c(B^n\times\mathrm{RP}^1)$

$$\int_{B^n}\big[\widetilde f(x,u_T(x))-\widetilde f(x,p_0)\big]\,dx=(-1)^n\partial\Sigma(\widetilde f(x,\widehat\theta)\,dx)\,.$$

Therefore, since $(-1)^n d\widetilde f(x,\widehat\theta)\,dx=\widetilde f_{,\widehat\theta}(x,\widehat\theta)\,dx\wedge\widehat\Theta$, compare (6.4), by (6.8) we get

$$\int_{B^n}\big[\widetilde f(x,u_T(x))-\widetilde f(x,p_0)\big]\,dx=\int_{B^n\times\mathrm{RP}^1}\widetilde f_{,\widehat\theta}(x,\widehat\theta)\,\widetilde g(x,\widehat\theta)\,d\mathcal{H}^{n+1}\,.$$

Denoting for a.e. $x \in B^n$ by $l(x)$ the point in $[0,\pi)$ such that $\widehat{j}(l(x)) = u_T(x)$, and since $\widehat{j}(0) = p_0$, by (6.9) we may rewrite

$$\int_{B^n} \big[f(x,l(x)) - f(x,0)\big]\,dx = \int_{B^n} dx \int_0^\pi f_{,t}(x,t)\,\overline{g}(x,t)\,dt\,.$$

Since by the π-periodicity of f

$$\int_0^\pi f_{,t}(x,t)\ dt = 0 \qquad \forall\, x \in B^n\,,$$

by (6.10) we get

$$\int_{B^n} dx \int_0^\pi \big[g(x,t) - \chi_{[0,l(x)]}(t)\big]\,f_{,t}(x,t)\,dt = 0\,,$$

where χ_A is the characteristic function of $A \subset \mathbb{R}$. The last equality yields that

$$\int_{B^n} dx \int_0^\pi \big[g(x,t) - \chi_{[0,l(x)]}(t)\big]\,\varphi(x,t)\,dt = 0\,.$$

for all C^∞-maps φ which are π-periodic in t and such that $\int_0^\pi \varphi(x,t)\,dt = 0$ for every $x \in B^n$. Consequently, for a.e. $x \in B^n$

$$g(x,t) = c(x) + \chi_{[0,l(x)]}(t)\,,$$

in particular, $c(x)$ is integer-valued. Integrating with respect to $t \in [0,\pi]$, by (6.11) we obtain

$$\psi(x) = \pi\,c(x) + l(x) \qquad \text{for } \mathcal{L}^n\text{-a.e. } x \in B^n$$

and hence, taking account again the π-periodicity of f in t,

$$\begin{aligned} \int_{B^n} dx \int_0^\pi f(x,t)\,g(x,t)\,dt &= \int_{B^n} dx \int_0^\pi f(x,t)\,\big(c(x) + \chi_{[0,l(x)]}(t)\big)\,dt \\ &= \int_{B^n} dx \Big\{ c(x) \int_0^\pi f(x,t)\,dt + \int_0^{l(x)} f(x,t)\,dt \Big\} \\ &= \int_{B^n} dx \int_0^{\psi(x)} f(x,t)\,dt \;=:\; SG_\psi(f(x,t)\,dx \wedge dt)\,, \end{aligned} \tag{6.13}$$

see (6.1). Now, (6.3) and (6.4) yield

$$\widehat{i}^{\#}(\widetilde{f}(x,\widehat{\theta})\,dx \wedge \widehat{\Theta}) = f(x,t)\,dx \wedge \widehat{j}^{\#}\widehat{\Theta} = f(x,t)\,dx \wedge dt\,.$$

This gives

$$\widehat{i}_{\#} SG_\psi(\widetilde{f}(x,\widehat{\theta})\,dx \wedge \widehat{\Theta}) := SG_\psi(\widehat{i}^{\#}(\widetilde{f}(x,\widehat{\theta})\,dx \wedge \widehat{\Theta})) = SG_\psi(f(x,t)\,dx \wedge dt)\,.$$

Since moreover $\widehat{i}_{\#}[\![\, B^n \times (0,\pi)\,]\!] = [\![\, B^n \times \mathrm{RP}^1\,]\!]$, we also have

$$r_0\,[\![\, B^n \times \mathrm{RP}^1\,]\!](\widetilde{f}(x,\widehat{\theta})\,dx \wedge \widehat{\Theta}) = r_0\,[\![\, B^n \times (0,\pi)\,]\!](f\,dx \wedge dt) = \int_{B^n} dx \int_0^\pi r_0\,f(x,t)\,dt\,.$$

By (6.9), (6.10), and (6.13) we finally obtain the claim (6.12). □

7 Cartesian currents in $\mathbf{B^n} \times \mathbf{S^1}$

In order to prove Theorem 9.1 below, in this section we recall the structure properties of the class $\mathrm{cart}(B^n \times \mathbb{S}^1)$. We then introduce a suitable current integration on the jump set of functions in $BV(B^n, \mathbb{S}^1)$.

GRAPHS OF $W^{1,1}$-FUNCTIONS INTO $\mathbb{S}^1$. To every Sobolev map v in $W^{1,1}(B^n, \mathbb{S}^1)$ we associate an i.m. rectifiable current $G_v \in \mathcal{R}_n(B^n \times \mathbb{S}^1)$ by

$$G_v(\omega) := \int_{B^n} (\mathrm{Id} \bowtie v)^{\#}\omega \qquad \forall\, \omega \in \mathcal{D}^n(B^n \times \mathbb{S}^1)\,, \tag{7.1}$$

where $(\mathrm{Id} \bowtie v)(x) := (x, v(x))$ and the pull-back is defined in the a.e. approximate sense. If v is smooth, G_v is the current integration over the oriented graph of v.

Remark 7.1 For $n \geq 2$, the current $\mathbb{P}(v) \in \mathcal{D}_{n-2}(B^n)$ of the singularity, see (2.1), satisfies

$$2\pi \cdot \langle \mathbb{P}(v), \varphi \rangle = G_v(d\varphi \wedge \omega_{\mathbb{S}^1}) = \partial G_v(\varphi \wedge \omega_{\mathbb{S}^1}) \tag{7.2}$$

for every $\varphi \in \mathcal{D}^{n-2}(B^n)$, as $G_v(\varphi \wedge d\omega_{\mathbb{S}^1}) = 0$. Moreover, arguing as in Proposition 3.2 one infers that

$$2\pi \cdot m_{i,\Omega}(\mathbb{P}(v)) \leq \mathbf{D}_1(v, B^n) < \infty\,,$$

hence $\mathbb{P}(v)$ is an integral flat chain, see Definition 3.1.

Therefore, if $u \in W^{1,1}(B^n, \mathrm{RP}^1)$ satisfies the property $u = g_1 \circ v$ for some $v \in W^{1,1}(B^n, \mathbb{S}^1)$, by Proposition 2.8 we deduce that $\frac{1}{2}\mathbf{P}(u)$ is an integral flat chain, too.

As we have seen, in general the above condition $u = g_1 \circ v$ is not satisfied. However, in Theorem 9.1 below we shall prove that *the converse implication holds true*, too.

Weak limits. Let $\{v_k\}$ a sequence of smooth maps from B^n into $\mathbb{S}^1$ satisfying $\sup_k \mathbf{D}_1(v_k, B^n) < \infty$. Arguing as in Sec. 4, we infer that the currents G_{v_k}, possibly passing to a subsequence, weakly converge in $\mathcal{D}_n(B^n \times \mathbb{S}^1)$ to an i.m. rectifiable current $\widetilde{T} \in \mathcal{R}_n(B^n \times \mathbb{S}^1)$ satisfying the null-boundary condition

$$\partial \widetilde{T}(\widetilde{\omega}) := \widetilde{T}(d\widetilde{\omega}) = 0 \qquad \forall\, \widetilde{\omega} \in \mathcal{D}^{n-1}(B^n \times \mathbb{S}^1) \tag{7.3}$$

and acting on "horizontal" forms as

$$\widetilde{T}(\phi(x,y)\,dx) = \int_{B^n} \phi(x, v_T(x))\,dx \qquad \forall\, \phi \in C_c^\infty(B^n \times \mathbb{S}^1) \tag{7.4}$$

for some function of bounded variation $v_T \in BV(B^n, \mathbb{S}^1)$. Therefore, the weak $\mathcal{D}_n$-limits of the G_{v_k}'s involve the currents G_v in $B^n \times \mathbb{S}^1$ integration on the "graph" of functions in $BV(B^n, \mathbb{S}^1)$, see Definition 7.4 below. We recall that a function $v \in L^1(B^n, \mathbb{S}^1)$ belongs to the class $BV(B^n, \mathbb{S}^1)$ if its distributional derivative Dv is a measure with bounded total variation. Following e.g. [3, Sec. 3.9], one decomposes

$$Dv = \nabla v\,dx + D^C v + (v^+ - v^-) \otimes \nu_v\,\mathcal{H}^{n-1} \llcorner J_v\,,$$

where ∇v is the approximate gradient of v, the countably $\mathcal{H}^{n-1}$-rectifiable subset J_v of B^n, the so called *jump set*, is given by the jump points of v, we choose $\nu_v = (\nu_v^1, \ldots, \nu_v^n)$ a unit normal to J_v, and $v^-(x)$ and $v^+(x)$ are the one-sided limits of v at $x \in J_v$ with respect to ν_v, for $\mathcal{H}^{n-1}$-a.e. $x \in J_v$. We also recall that v is a *special function of bounded variation* in $SBV(B^n, \mathbb{S}^1)$ if v belongs to $BV(B^n, \mathbb{S}^1)$ and its distributional derivative Dv has *no Cantor part*, i.e., $D^C v = 0$. Notice that in general $\mathcal{H}^{n-1}(J_v) \leq \infty$, even if v belongs to $SBV(B^n, \mathbb{S}^1)$, and the strict inclusion $W^{1,1}(B^n, \mathbb{S}^1) \subsetneq SBV(B^n, \mathbb{S}^1)$ holds.

Graphs of BV-functions into $\mathbb{S}^1$. Following the notation from [18], to every function $v \in BV(B^n, \mathbb{S}^1)$ we again associate an i.m. rectifiable current G_v in $\mathcal{R}_n(B^n \times \mathbb{S}^1)$. We decompose G_v into its absolutely continuous, Cantor, and Jump parts

$$G_v := G_v^a + G_v^C + G_v^J\,.$$

Every n-form $\omega \in \mathcal{D}^n(B^n \times \mathbb{S}^1)$ splits as $\omega^{(0)} + \omega^{(1)}$ according to the number of "vertical" differentials. Write $\omega^{(0)} = \phi(x,y)\,dx$ for some $\phi \in C_0^\infty(B^n \times \mathbb{S}^1)$,

$$\omega^{(1)} = \sum_{i=1}^{n} \sum_{j=1}^{2} (-1)^{n-i} \phi_i^j(x,y)\,\widehat{dx^i} \wedge dy^j \tag{7.5}$$

for some $\phi_i^j \in C_0^\infty(B^n \times \mathbb{S}^1)$, and denote $\phi^j := (\phi_1^j, \ldots, \phi_n^j)$. We set

$$G_v^C(\phi(x,y)\,dx) = G_v^J(\phi(x,y)\,dx) = 0\,,$$

$$G_v(\phi(x,y)\,dx) = G_v^a(\phi(x,y)\,dx) := \int_{B^n} \phi(x,v(x))\,dx\,.$$

Moreover, we define

$$\begin{aligned}
G_v^a(\omega^{(1)}) &:= \sum_{j=1}^{2}\int_{B^n}\big(\nabla v^j(x)\cdot\phi^j(x,v(x))\big)\,dx \\
&= \sum_{j=1}^{2}\sum_{i=1}^{n}\int_{B^n}\nabla_i v^j(x)\phi_i^j(x,v(x))\,dx \\
G_v^C(\omega^{(1)}) &:= \sum_{j=1}^{2}\int_{B^n}\phi^j(x,v(x))\,dD^C v^j \\
G_v^J(\omega^{(1)}) &:= \sum_{i=1}^{n}\sum_{j=1}^{2}\int_{J_v}\left(\int_{l_x}\phi_i^j(x,y)\,dy^j\right)\nu_v^i\,d\mathcal{H}^{n-1}(x)\,.
\end{aligned}$$

In this formula, for $\mathcal{H}^{n-1}$-a.e. $x\in J_v$ we denote by l_x the oriented simple arc of $\mathbb{S}^1$ from $v^-(x)$ to $v^+(x)$ and satisfying the following properties:

i) if $v^+(x)=v^-(x)$, then l_x is constantly the point $v^+(x)$;

ii) if $v^+(x)\neq -v^-(x)$, then l_x is a geodesic arc;

iii) if $v^+(x)\neq -v^-(x)$, then l_x is oriented in the counterclockwise sense in the case $\mathrm{Arg}\,(v^+(x))\in[0,\pi]$, and in the clockwise sense in the case $\mathrm{Arg}\,(v^+(x))\in]-\pi,0[$.

Here, $\mathrm{Arg}\,(\theta)\in]-\pi,\pi]$ is the *argument* of the unit complex number $\theta\in\mathbb{S}^1\subset\mathbb{C}$.

Notice that for $\mathcal{H}^{n-1}$-a.e. $x\in J_v$ we have $\partial[\![\, l_x \,]\!] = \delta_{v^+(x)}-\delta_{v^-(x)}$ and

$$\int_{l_x}\omega_{\mathbb{S}^1} = \rho(v^+(x),v^-(x))\,, \tag{7.6}$$

where $\rho:\mathbb{S}^1\times\mathbb{S}^1\to[-\pi,\pi]$ is the signed distance on $\mathbb{S}^1$, compare [17], defined by

$$\rho(\theta_1,\theta_2) := \begin{cases} \mathrm{Arg}\,\big(\theta_1/\theta_2\big) & \text{if } \theta_1/\theta_2\neq -1\,,\\ \mathrm{Arg}\,\big(\theta_1\big)-\mathrm{Arg}(\theta_2) & \text{if } \theta_1/\theta_2 = -1\,,\end{cases} \qquad \forall\,\theta_1,\,\theta_2\in\mathbb{S}^1\,. \tag{7.7}$$

Similarly to Proposition 4.1, we have:

Proposition 7.2 *Let $\widetilde{T}\in\mathcal{R}_n(B^n\times\mathbb{S}^1)$ satisfy (7.3) and (7.4) for some $v_T\in BV(B^n,\mathbb{S}^1)$. Then there exists an i.m. rectifiable current $\widetilde{L}_T$ in $\mathcal{R}_{n-1}(B^n)$ such that*

$$\widetilde{T} = G_{v_T} + \widetilde{L}_T\times[\![\,\mathbb{S}^1\,]\!]\,. \tag{7.8}$$

Proof: Let $\omega_\eta\in\mathcal{D}^{n-1}(B^n)$ given by (4.6). Define $\widetilde{L}_T\in\mathcal{R}_{n-1}(B^n)$ by

$$\widetilde{L}_T(\omega_\eta) := \frac{1}{2\pi}\,\widetilde{S}_T(\omega_\eta\wedge\omega_{\mathbb{S}^1})\,,\qquad \eta\in C_c^\infty(B^n,\mathbb{R}^n)\,,$$

where we have set

$$\widetilde{S}_T := \widetilde{T} - G_{v_T}\in\mathcal{R}_n(B^n\times\mathbb{S}^1)\,.$$

As in the proof of Proposition 4.1, it suffices to show that

$$\widetilde{S}_T(\omega) = \widetilde{L}_T\times[\![\,\mathbb{S}^1\,]\!](\omega)\qquad\forall\,\omega=\phi(x,y)\,dx+\omega_\eta\wedge\alpha\,, \tag{7.9}$$

where $\phi\in C_c^\infty(B^n\times\mathbb{S}^1)$, $\eta\in C_c^\infty(B^n,\mathbb{R}^n)$, and $\alpha\in\mathcal{D}^1(\mathbb{S}^1)$. By (7.4) we check

$$\widetilde{S}_T(\phi(x,y)\,dx) = \widetilde{L}_T\times[\![\,\mathbb{S}^1\,]\!](\phi(x,y)\,dx) = 0\,,$$

whereas by Hodge decomposition theorem, we can write $\alpha=\lambda\,\omega_{\mathbb{S}^1}+d\beta$ for some $\lambda\in\mathbb{R}$ and $\beta\in C^\infty(\mathbb{S}^1)$. We also have:

Lemma 7.3 *$S_T(\omega_\eta \wedge d\beta) = 0$ for every $\eta \in C_c^\infty(B^n, \mathbb{R}^n)$ and $\beta \in C^\infty(\mathbb{S}^1)$.*

This gives $\widetilde{S}_T(\omega_\eta \wedge \alpha) = \lambda\, \widetilde{S}_T(\omega_\eta \wedge \omega_{\mathbb{S}^1})$, whereas

$$\widetilde{L}_T \times [\![\mathbb{S}^1]\!](\omega_\eta \wedge \alpha) = \widetilde{L}_T(\omega_\eta) \cdot [\![\mathbb{S}^1]\!](\alpha) = \lambda\, 2\pi \widetilde{L}_T(\omega_\eta) = \lambda\, \widetilde{S}_T(\omega_\eta \wedge \omega_{\mathbb{S}^1})$$

and hence (7.9). □

Proof of Lemma 7.3: As in Lemma 4.2, by (7.3) we obtain

$$(-1)^n \widetilde{S}_T(\omega_\eta \wedge d\beta) = \widetilde{T}(\operatorname{div} \eta(x)\, \beta(y)\, dx) + (-1)^{n-1} G_{v_T}(\omega_\eta \wedge d\beta)\,.$$

By (7.4) we find that

$$\widetilde{T}(\operatorname{div} \eta(x)\, \beta(y)\, dx) = \int_{B^n} \operatorname{div} \eta(x)\, \beta(v_T(x))\, dx =: -\langle D(\beta \circ v_T), \eta \rangle\,.$$

Moreover, by the definition of G_{v_T}, with $\phi_i^j = \eta^i\, D_{y_j} \beta$ in (7.5), and since $\partial [\![l_x]\!] = \delta_{v_T^+(x)} - \delta_{v_T^-(x)}$, we infer

$$\begin{aligned}(-1)^{n-1} G_{v_T}(\omega_\eta \wedge d\beta) = & \sum_{j=1}^2 \int_{B^n} \frac{\partial \beta}{\partial y^j}(v_T(x)) \langle \nabla v_T^j(x), \eta(x) \rangle\, dx \\ & + \sum_{j=1}^2 \int_{B^n} \frac{\partial \beta}{\partial y^j}(v_T(x))\, \eta(x)\, dD^C v_T^j \\ & + \int_{J_{v_T}} \big(\beta(v_T^+(x)) - \beta(v_T^-(x) \big) \langle \eta(x), \nu(x) \rangle\, d\mathcal{H}^{n-1}\,.\end{aligned}$$

Finally, by the chain rule for the derivative $D(\eta \circ v_T)$, see [3, Sec. 3.10], we obtain

$$(-1)^{n-1} G_{v_T}(\omega_\eta \wedge d\beta) = \langle D(\beta \circ v_T), \eta \rangle$$

and hence $\widetilde{S}_T(\omega_\eta \wedge d\beta) = 0$. □

The above facts motivate the following

Definition 7.4 *We denote by $\operatorname{cart}(B^n \times \mathbb{S}^1)$ the class of i.m. rectifiable currents $\widetilde{T} \in \mathcal{R}_n(B^n \times \mathbb{S}^1)$ with finite mass, $\mathbf{M}(\widetilde{T}) < \infty$, satisfying the null-boundary condition (7.3), that can be decomposed as in (7.8) for some function $v_T \in BV(B^n, \mathbb{S}^1)$ and some i.m. rectifiable current $\widetilde{L}_T$ in $\mathcal{R}_{n-1}(B^n)$.*

Remark 7.5 The class $\operatorname{cart}(B^n \times \mathbb{S}^1)$ agrees with the one from [12] and [13, Vol. II, Sec. 6.2]. Moreover, for any $v \in BV(B^n, \mathbb{S}^1)$ one has

$$G_v^a(\omega) = \int_{B^n} (Id \bowtie v)^\# \omega \qquad \forall\, \omega \in \mathcal{D}^n(B^n \times \mathbb{S}^1)\,,$$

where the pull-back is defined in the approximate sense. Therefore, if v_T is a Sobolev map in $W^{1,1}(B^n, \mathbb{S}^1)$, we have $G_{v_T}^C = G_{v_T}^J = 0$, hence G_{v_T} agrees with the current in definition (7.1). This yields that the class $\operatorname{cart}(B^n \times \mathbb{S}^1)$ contains the currents of the type

$$T = G_v + \widetilde{L} \times [\![\mathbb{S}^1]\!]\,,$$

for some $v \in W^{1,1}(B^n, \mathbb{S}^1)$ and $\widetilde{L} \in \mathcal{R}_{n-1}(B^n)$. In this case, arguing as in Proposition 4.4, by (7.2) one infers that for $n \geq 2$, the null-boundary condition (7.3) is equivalent to:

$$(\partial \widetilde{L}) \llcorner B^n = -\, \mathbb{P}(v)\,.$$

Currents integration on the jump set. We denote by $[\![J_v]\!]$ the i.m. current in $\mathcal{D}_{n-1}(B^n)$ integration of $(n-1)$-forms on the jump set J_v of a function $v \in BV(B^n, \mathbb{S}^1)$. More precisely, we set

$$[\![J_v]\!] := \tau(J_v, 1, *\nu_v) \,,$$

where the tangent unit $(n-1)$-vector $*\nu_v$ is defined $\mathcal{H}^{n-1}$-a.e. on J_v by

$$*\nu_v := \sum_{i=1}^{n} (-1)^{i-1} \nu_v^i \, \widehat{e_i} \,, \qquad \widehat{e_i} := e_1 \wedge \cdots \wedge e_{i-1} \wedge e_{i+1} \wedge \cdots \wedge e_n \,.$$

Notice that $\mathbf{M}([\![J_v]\!]) = \mathcal{H}^{n-1}(J_v)$, so that $[\![J_v]\!]$ has finite mass, and hence it is an i.m. rectifiable current in $\mathcal{R}_{n-1}(B^n)$, if and only if $\mathcal{H}^{n-1}(J_v) < \infty$. Moreover, if $\omega_\eta \in \mathcal{D}^{n-1}(B^n)$ is given by (4.6), we have

$$[\![J_v]\!](\omega_\eta) = \int_{J_v} \langle \omega_\eta, *\nu_v \rangle \, d\mathcal{H}^{n-1} = \int_{J_v} \eta \cdot \nu_v \, d\mathcal{H}^{n-1} \,.$$

Assume now that $g_1 \circ v = u$ for some Sobolev map u in $W^{1,1}(B^n, \mathrm{RP}^1)$. This yields that $v^- = -v^+$ $\mathcal{H}^{n-1}$-a.e. in J_v. By (7.7), we then deduce that for $\mathcal{H}^{n-1}$-a.e. $x \in J_v$

$$\rho(v^+(x), v^-(x)) = \begin{cases} \pi & \text{if} \quad \mathrm{Arg}(v^+(x)) \in [0, \pi] \\ -\pi & \text{if} \quad \mathrm{Arg}(v^+(x)) \in]-\pi, 0[\,. \end{cases}$$

Therefore, setting $\overrightarrow{\mathfrak{J}_v}(x) := \theta_v(x) \cdot (*\nu_v(x))$, where

$$\theta_v(x) := (-1)^{n-1} \frac{1}{\pi} \, \rho(v^+(x), v^-(x)) \in \{-1, +1\} \,, \qquad x \in J_v \,,$$

we conclude that the current $\mathfrak{J}_v := \tau(J_v, 1, \overrightarrow{\mathfrak{J}_v})$ has *multiplicity one* and satisfies

$$\mathfrak{J}_v(\omega_\eta) = \int_{J_v} \langle \omega_\eta, \overrightarrow{\mathfrak{J}_v} \rangle \, d\mathcal{H}^{n-1} = \frac{(-1)^{n-1}}{\pi} \int_{J_v} \rho(v^+, v^-) \, \eta \cdot \nu_v \, d\mathcal{H}^{n-1} \,. \tag{7.10}$$

Remark 7.6 We again have $\mathbf{M}(\mathfrak{J}_v) = \mathcal{H}^{n-1}(J_v)$. Therefore, $\mathfrak{J}_v$ has finite mass, and hence $\mathfrak{J}_v \in \mathcal{R}_{n-1}(B^n)$, if and only if $\mathcal{H}^{n-1}(J_v) < \infty$.

Remark 7.7 Finally, if a function $v \in BV(B^n, \mathbb{S}^1)$ satisfies $g_1 \circ v \in W^{1,1}(B^n, \mathrm{RP}^1)$, we have $D^C v = 0$, see Remark 8.1 below. Therefore, we infer that v *belongs to* $W^{1,1}(B^n, \mathbb{S}^1)$ *if and only if* $\mathfrak{J}_v = 0$.

8 Preliminary results

In this section, using the lifting theorem 6.1, we analyze some properties of the currents G_v carried by the graph of maps in $BV(B^n, \mathbb{S}^1)$ that satisfy $g_1 \circ v = u \in W^{1,1}(B^n, \mathrm{RP}^1)$. This properties will be used to prove of our main result, Theorem 9.1 below. For the sake of clarity, we postpone the proofs to the end of the section.

According to (2.5) and (6.3), denote by $i : B^n \times \mathbb{R} \to B^n \times \mathbb{S}^1$ the map

$$i(x, t) := (x, j(t)) \,, \qquad j(t) := (\cos t, \sin t) \,.$$

Remark 8.1 We first observe that the function u_T in Theorem 6.1 belongs to $W^{1,1}(B^n, \mathrm{RP}^1)$. Therefore, by applying the chain rule to (6.6), see [3], we readily infer that the lifting map $\psi_T : B^n \to \mathbb{R}$ is a *special function of bounded variation* in $SBV(B^n)$, i.e., the *Cantor part* of the distributional derivative is zero, $D^C \psi_T = 0$. As a consequence, the corresponding function

$$v_T := j \circ \psi_T : B^n \to \mathbb{S}^1$$

is a special function of bounded variation in $SBV(B^n, \mathbb{S}^1)$, i.e., $D^C v_T = 0$.

Setting $h_1(x,y)=(x,g_1(x))\in B^n\times\mathrm{RP}^1$ for $(x,y)\in B^n\times\mathbb{S}^1$, we have $\widehat{j}=g_1\circ j$ and $\widehat{i}=h_1\circ i$, whence

$$i_\#[\![B^n\times(0,2\pi)]\!]=[\![B^n]\!]\times[\![\mathbb{S}^1]\!]\,,\qquad \widehat{i}_\#[\![B^n\times(0,\pi)]\!]=[\![B^n]\!]\times[\![\mathrm{RP}^1]\!]\,.$$

Proposition 8.2 *Let $u\in W^{1,1}(B^n,\mathrm{RP}^1)$. Assume that there exists a Sobolev map $v\in W^{1,1}(B^n,\mathbb{S}^1)$ such that $g_1\circ v=u$, see Definition 0.2. Then $h_{1\#}G_v=G_u$. More generally, if $u=g_1\circ v$ for some $v\in BV(B^n,\mathbb{S}^1)$, we have $h_{1\#}G^a_v=G_u$.*

Denote now by G_{q_0} the current in $\mathcal{R}_n(B^n\times\mathbb{S}^1)$ integration over the graph of the constant map $q_0(x)\equiv j(0)=(1,0)$. As a consequence of the lifting theorem 6.1, we obtain:

Proposition 8.3 *Under the hypotheses of Theorem 6.1, the image current by the lifting i satisfies*

$$(-1)^n i_\#\partial SG_{\psi_T}=\widetilde{T}-G_{q_0}\,,$$

for some Cartesian current $\widetilde{T}\in\mathrm{cart}(B^n\times\mathbb{S}^1)$ with corresponding BV-function equal to $v_T:=j\circ\psi_T$.

As a consequence, we also have:

Proposition 8.4 *For every $u\in W^{1,1}(B^n,\mathrm{RP}^1)$, there exists a function $v\in SBV(B^n,\mathbb{S}^1)$ such that*

$$u=g_1\circ v\qquad \mathcal{L}^n\text{-a.e. on } B^n\,.$$

Moreover, for every $T\in\mathrm{Cart}^{1,1}(B^n\times\mathrm{RP}^1)$, with corresponding function $u_T=u$ in (4.5), there exists a current $\widetilde{T}\in\mathrm{cart}(B^n\times\mathbb{S}^1)$, with corresponding BV-function $v_T=v$ in (7.8), such that $T=h_{1\#}\widetilde{T}$, i.e.,

$$h_{1\#}(G_v+\widetilde{L}_T\times[\![\mathbb{S}^1]\!])=G_u+L_T\times[\![\mathrm{RP}^1]\!]\,,\tag{8.1}$$

where $\widetilde{L}_T$, $L_T\in\mathcal{R}_{n-1}(B^n)$.

Recall now that the i.m. current $\mathfrak{J}_v\in\mathcal{D}_{n-1}(B^n)$ is given by (7.10). We finally obtain:

Proposition 8.5 *Under the hypotheses of Proposition 8.4, property (8.1) yields*

$$L_T=2\,\widetilde{L}_T+\mathfrak{J}_v\,.\tag{8.2}$$

In particular, the function $v\in SBV(B^n,\mathbb{S}^1)$ has jump set of finite measure, $\mathcal{H}^{n-1}(J_v)<\infty$.

Proofs. We finally collect the proofs of the results stated above.

Proof of Proposition 8.2: Assume first that $v\in W^{1,1}(B^n,\mathbb{S}^1)$. Since $h_1\circ(Id\bowtie v)=Id\bowtie(g_1\circ v)$, by (7.1) and (4.1) we get

$$\begin{aligned}\langle h_{1\#}G_v,\omega\rangle &:= \langle G_v,h_1^\#\omega\rangle=\int_{B^n}(Id\bowtie v)^\#\big(h_1^\#\omega\big)\\ &= \int_{B^n}\big(Id\bowtie(g_1\circ v)\big)^\#\omega=\int_{B^n}(Id\bowtie u)^\#\omega=:\langle G_u,\omega\rangle\end{aligned}$$

for every form $\omega\in\mathcal{D}^n(B^n\times\mathrm{RP}^1)$. If $v\in BV(B^n,\mathbb{S}^1)$, the claim follows from Remark 7.5. □

Proof of Proposition 8.3: For every $\phi\in C^\infty_c(B^n\times\mathbb{S}^1)$ we have

$$(-1)^n d\phi(x,j(t))\,dx=\frac{\partial}{\partial t}\phi(x,j(t))\,dx\wedge dt\,.$$

Therefore, using (6.1) we compute

$$\begin{aligned}&(-1)^n i_\#\partial SG_{\psi_T}(\phi(x,y)\,dx)\\ &\quad:=(-1)^n\partial SG_{\psi_T}(i^\#\phi(x,y)\,dx)=(-1)^n\partial SG_{\psi_T}(\phi(x,j(t))\,dx)\\ &\quad=(-1)^n SG_{\psi_T}(d\phi(x,j(t))\,dx)=\int_{B^n}\left(\int_0^{\psi_T(x)}\frac{\partial}{\partial t}\phi(x,j(t))\,dt\right)dx\\ &\quad=\int_{B^n}\big(\phi(x,j(\psi_T(x)))-\phi(x,j(0))\big)\,dx=(G_{v_T}-G_{q_0})(\phi(x,y)\,dx)\,.\end{aligned}$$

Moreover, the current ∂SG_{ψ_T} belongs to $\mathcal{R}_n(B^n\times\mathbb{R})$, and has null boundary inside $B^n\times\mathbb{R}$. This yields that $\widetilde{T}:=(-1)^n i_\#\partial SG_{\psi_T}+G_{q_0}$ is an i.m. rectifiable current in $\mathcal{R}_n(B^n\times\mathbb{S}^1)$ that satisfies the null-boundary condition (7.3) and agrees with G_{v_T} on horizontal forms $\phi(x,y)\,dx$, see (7.4). The assertion follows from Proposition 7.2 and Definition 7.4. $\square$

Proof of Proposition 8.4: Recall that $\widehat{i}=h_1\circ i$, whereas $p_0=g_1(q_0)$. This gives

$$(-1)^n\widehat{i}_\#\partial SG_{\psi_T}=h_{1\#}((-1)^n i_\#\partial SG_{\psi_T})\,,\qquad h_{1\#}G_{q_0}=G_{p_0}\,.$$

The claim follows from Proposition 5.1, Theorem 6.1, and Proposition 8.3. $\square$

Proof of Proposition 8.5: Let $\widetilde{T}\in\mathrm{cart}(B^n\times\mathbb{S}^1)$ given by Proposition 8.4, so that $v_T=v$. Since $g_{1\#}[\![\,\mathbb{S}^1\,]\!]=2\,[\![\,\mathrm{RP}^1\,]\!]$, we get

$$h_{1\#}\widetilde{T}=h_{1\#}(G_v+\widetilde{L}_T\times[\![\,\mathbb{S}^1\,]\!])=h_{1\#}G_v+2\,\widetilde{L}_T\times[\![\,\mathrm{RP}^1\,]\!]\,.$$

Moreover, Proposition 8.2 yields that $h_{1\#}G_v^a=G_u$, whereas $G_v=G_v^a+G_v^J$, as $G_v^C=0$, see Remark 8.1. Therefore, (8.1) is equivalent to

$$h_{1\#}G_v^J+2\,\widetilde{L}_T\times[\![\,\mathrm{RP}^1\,]\!]=L_T\times[\![\,\mathrm{RP}^1\,]\!]\,. \tag{8.3}$$

Let now $\omega_\eta\wedge\omega_{\mathrm{RP}^1}\in\mathcal{D}^n(B^n\times\mathrm{RP}^1)$, where $\omega_\eta\in\mathcal{D}^{n-1}(B^n)$ is given by (4.6). By (2.6), we have

$$h_1^\#(\omega_\eta\wedge\omega_{\mathrm{RP}^1})=\omega_\eta\wedge g_1^\#\omega_{\mathrm{RP}^1}=\frac{1}{\pi}\,\omega_\eta\wedge\omega_{\mathbb{S}^1}\,.$$

Therefore, denoting $y^{\overline{1}}:=y^2$ and $y^{\overline{2}}:=y^1$, according to the notation in (7.5) we infer that

$$\pi\,h_1^\#(\omega_\eta\wedge\omega_{\mathrm{RP}^1})=\omega^{(1)}\,,\quad\text{where}\quad\phi_i^j(x,y):=(-1)^{n-1+j}\,\eta^i(x)\,y^{\overline{j}}\,.$$

By the definition of $G_v^J(\omega^{(1)})$ from Sec. 7, we thus obtain:

$$\begin{aligned}\pi\,h_{1\#}G_v^J(\omega_\eta\wedge\omega_{\mathrm{RP}^1}) &= G_v^J(\pi\,h_1^\#(\omega_\eta\wedge\omega_{\mathrm{RP}^1}))\\ &= \sum_{i=1}^n\sum_{j=1}^2\int_{J_v}\Big(\int_{l_x}(-1)^{n-1+j}\eta^i(x)\,y^{\overline{j}}dy^j\Big)\,\nu_v^i(x)\,d\mathcal{H}^{n-1}(x)\\ &= (-1)^{n-1}\sum_{i=1}^n\int_{J_v}\eta^i(x)\Big(\int_{l_x}(y^1dy^2-y^2dy^1)\Big)\,\nu_v^i(x)\,d\mathcal{H}^{n-1}(x)\,.\end{aligned}$$

Using (7.6), we get

$$\pi\,h_{1\#}G_v^J(\omega_\eta\wedge\omega_{\mathrm{RP}^1})=(-1)^{n-1}\int_{J_v}\rho(v^+,v^-)\,\eta\cdot\nu_v\,d\mathcal{H}^{n-1}$$

and hence

$$h_{1\#}G_v^J(\omega_\eta\wedge\omega_{\mathrm{RP}^1})=\mathfrak{J}_v(\omega_\eta)\,,$$

compare (7.10), whereas by (2.7)

$$2\,\widetilde{L}_T\times[\![\,\mathrm{RP}^1\,]\!](\omega_\eta\wedge\omega_{\mathrm{RP}^1})=2\,\widetilde{L}_T(\omega_\eta)\,,\qquad L_T\times[\![\,\mathrm{RP}^1\,]\!](\omega_\eta\wedge\omega_{\mathrm{RP}^1})=L_T(\omega_\eta)\,.$$

By (8.3), we conclude that

$$\mathfrak{J}_v(\omega_\eta)+2\,\widetilde{L}_T(\omega_\eta)=L_T(\omega_\eta)$$

for every $\eta\in C_c^\infty(B^n,\mathbb{R}^n)$, that gives (8.2). Finally, the property $\mathcal{H}^{n-1}(J_v)<\infty$ follows from Remark 7.6 and (8.2), as $\mathbf{M}(L_T)+\mathbf{M}(2\,\widetilde{L}_T)<\infty$. $\square$

9 Main results

Let now $u \in W^{1,1}(B^n, \mathrm{RP}^1)$, where $n \geq 2$, and let $\mathbf{P}(u) \in \mathcal{D}_{n-2}(B^n)$ the current of the singularities of u, given by (2.13). In Proposition 3.2 we have noticed that $\mathbf{P}(u)$ is always an integral flat chain.

We now consider the following properties:

(a) *there exists a Sobolev map* $v \in W^{1,1}(B^n, \mathbb{S}^1)$ *such that* $g_1 \circ v = u$ *a.e. in* B^n, see Definition 0.2;

(b) *the current* $\frac{1}{2}\mathbf{P}(u)$ *is an integral flat chain*, see Definition 3.1.

Proposition 2.8 yields that the implication (a) $\Longrightarrow$ (b) is true, see Remark 7.1. As we have seen in Examples 1.2 and 2.9, both the above properties (a) and (b) are not verified, in general. Recall also that property (a) is always true in low dimension $n = 1$, see Remark 1.3. Moreover, property (b) means that we can find an i.m. rectifiable current $L \in \mathcal{R}_{n-1}(B^n)$, with finite mass, such that $(\partial L) \llcorner B^n = \frac{1}{2}\mathbf{P}(u)$. Finally, notice that the function $\overline{v}$ from Example 1.2 satisfies $g_1 \circ \overline{v} = \overline{u} \in W^{1,1}(B^2, \mathrm{RP}^1)$, belongs to the class $SBV(B^2, \mathbb{S}^1)$, and its jump set $J_{\overline{v}} = \{(x_1, 0) \in B^2 \mid -1 < x_1 < 0\}$ has finite size.

We now show that the converse implication (b) $\Longrightarrow$ (a) holds true, too. More precisely, from the results of the previous sections we obtain:

Theorem 9.1 *Let* $u \in W^{1,1}(B^n, \mathrm{RP}^1)$*, where* $n \geq 2$*. Then there exists a function* $v \in SBV(B^n, \mathbb{S}^1)$ *with* $\mathcal{H}^{n-1}(J_v) < \infty$ *such that* $g_1 \circ v = u$*. Moreover, the above properties* (*a*) *and* (*b*) *are equivalent.*

PROOF: By Propositions 5.1, 8.4, and 8.5, we deduce the first assertion and the corresponding formula (8.2), where the currents L_T, $\widetilde{L}_T$, and $\mathfrak{J}_v$ are i.m. rectifiable in $\mathcal{R}_{n-1}(B^n)$, and $\mathfrak{J}_v$ is given by (7.10).

As we have seen, Proposition 2.8 and Remark 7.1 yield the implication (a) $\Longrightarrow$ (b). To prove the converse implication, assume that $\frac{1}{2}\mathbf{P}(u)$ is an integral flat chain, see Definition 0.2. Then there exists an i.m. rectifiable current $\widehat{L} \in \mathcal{R}_{n-1}(B^n)$ such that

$$2(\partial \widehat{L}) \llcorner B^n = -\mathbf{P}(u)\,.$$

Proposition 4.4 yields that

$$\widehat{T} := G_u + 2\widehat{L} \times [\![\, \mathrm{RP}^1 \,]\!] \in \mathrm{Cart}^{1,1}(B^n \times \mathrm{RP}^1)\,.$$

Therefore, applying the arguments of the previous section to $T = \widehat{T}$, formula (8.2) gives

$$\mathfrak{J}_v = 2\,(\widehat{L} - \widetilde{L}_T)\,, \quad \text{where} \quad \widehat{L} - \widetilde{L}_T \in \mathcal{R}_{n-1}(B^n)\,.$$

Since the current $\mathfrak{J}_v$ has *multiplicity one*, see Sec. 7, this gives that $\mathfrak{J}_v = 0$, condition that is equivalent to the membership of v to the Sobolev class $W^{1,1}(B^n, \mathbb{S}^1)$, see Remark 7.7, as required. $\square$

In the case e.g. of maps in $R^0_1(B^2, \mathrm{RP}^1)$, the above property (b) says that the degree of u at each singular point a_j of $\Sigma(u)$ is integer, see Proposition 2.10. Therefore, if (b) holds, the image of the circle $\partial B^2(a_j, r)$ by the function u, for $r > 0$ small, covers the target space RP^1 an *even* number of times, given by the number $2\,|\deg_{\mathrm{RP}^1}(u, a_j)| \in 2\,\mathbb{N}$, with orientation prescribed by the sign of $\deg_{\mathrm{RP}^1}(u, a_j)$. Moreover, property (b) has to be compared with the formulas (2.11) and (2.18) for the functions from Example 1.2.

In fact, as a consequence of Theorem 9.1 we finally obtain:

Corollary 9.2 *If* $u \in R^0_1(B^2, \mathrm{RP}^1)$*, the degree of* u *at each singular point* a_j *of* $\Sigma(u)$ *is integer if and only if there exists a Sobolev map* $v \in W^{1,1}(B^2, \mathbb{S}^1)$ *such that* $g_1 \circ v = u$*. Similarly, if* $u \in W^{1,1}(\Sigma^1, \mathrm{RP}^1)$*, the degree* (2.10) *is integer if and only if there exists a Sobolev map* $v \in W^{1,1}(\Sigma^1, \mathbb{S}^1)$ *such that* $g_1 \circ v = u$*.*

PROOF: If $u \in R^0_1(B^2, \mathrm{RP}^1)$, by (2.19) we deduce that $\frac{1}{2}\mathbf{P}(u)$ is an integral flat chain if and only if $\deg_{\mathrm{RP}^1}(u, a_i) \in \mathbb{Z}$ for every i. The first claim then follows from Theorem 9.1. Moreover, we observe that for every $u \in W^{1,1}(\Sigma^1, \mathrm{RP}^1)$, the corresponding homogeneous extension $\overline{u}(x) := u\big(\frac{x}{|x|}\big)$ belongs to

$W^{1,1}(B^2, \mathrm{RP}^1)$. Moreover, by (2.21) we deduce that the current $\frac{1}{2}\,\mathbf{P}(\overline{u})$ is an integral flat chain if and only if the degree (2.10) of u is integer. Therefore, by Theorem 9.1, $\deg_{\mathrm{RP}^1}(u) \in \mathbb{Z}$ if and only if we find a Sobolev map $\overline{v} \in W^{1,1}(B^2, \mathbb{S}^1)$ such that $g_1 \circ \overline{v} = \overline{u}$. In this case, moreover, we have $\overline{v}(x) := v\big(\frac{x}{|x|}\big)$ for some $v \in W^{1,1}(\Sigma^1, \mathbb{S}^1)$ such that $g_1 \circ v = u$. This gives the second claim. □

Acknowledgments I thank the referee for pointing out some remarks to the first version of this paper.

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