

Rearrangements and the Maz'ya–Shaposhnikova limit: Γ -convergence for truncated interactions

Andrea Braides¹, Anneliese Defranceschi², and Enrico Vitali³

¹Department of Mathematics, University of Rome Tor Vergata, via della Ricerca Scientifica 1, 00133 Rome, Italy braides@mat.uniroma2.it

²Department of Mathematics, University of Trento, via Sommarive 14, 38123 Povo (Trento), Italy anneliese.defranceschi@unitn.it

³Department of Mathematics, University of Pavia, via A. Ferrata 5, Pavia, Italy enrico.vitali@unipv.it

Abstract

We give a proof of the Γ -liminf inequality for the Maz'ya–Shaposhnikova limit, as $s \rightarrow 0^+$, of the fractional Sobolev seminorm, based on spherical rearrangements. The key point is that the rearrangements of a weakly convergent sequence are monotone radial profiles, which converge strongly by Helly's theorem; since the energy does not increase under rearrangement (in the limit), this reduces the liminf inequality to a pointwise convergence statement for radially decreasing functions. As an application, we treat truncated energies, in which the interaction is restricted to a ball of radius $r_s \rightarrow \infty$. Assuming $r_s^{-2s} \rightarrow \lambda \in [0, 1]$, we prove that the scaled energies Γ -converge, with respect to weak L^2 -convergence, to

$$(1 - \lambda)N\omega_N\|u\|_{L^2}^2,$$

where ω_N is the measure of the unit ball of $\mathbb{R}^N$. This identifies the contribution of the long-range interactions and gives a continuous interpolation between the classical Maz'ya–Shaposhnikova limit ($\lambda = 0$) and the vanishing-energy regime ($\lambda = 1$).

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1 Introduction

The asymptotic behaviour of fractional Sobolev seminorms as the fractional exponent approaches the endpoints 0 and 1 has attracted considerable attention during the last two decades. While the limit $s \rightarrow 1^-$ is described by the

Bourgain–Brezis–Mironescu formula [2], the opposite regime was characterized by Maz’ya and Shaposhnikova [10]. In particular, for every $1 < p < \infty$, the appropriately normalized p -th power of the fractional p -seminorm converges, as $s \rightarrow 0^+$, to a multiple of the L^p -norm, provided that u belongs to $W^{s,p}(\mathbb{R}^N)$ for some $s \in (0, 1)$. In the case $p = 2$ considered here, this gives

$$\lim_{s \rightarrow 0} s \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy = N\omega_N \|u\|_{L^2(\mathbb{R}^N)}^2,$$

where ω_N denotes the measure of the N -dimensional unit ball.

Several extensions of the Maz’ya–Shaposhnikova formula have been obtained subsequently, including results for fractional Orlicz–Sobolev spaces [1], magnetic fractional Sobolev spaces [11], and more general geometric and interaction settings [7, 8, 4]. These results show that the Maz’ya–Shaposhnikova limit is sensitive both to the geometry of the underlying space and to the behaviour of the interaction at large distances.

Besides its intrinsic interest, the Maz’ya–Shaposhnikova formula has also become a fundamental tool in the analysis of nonlocal variational problems. In particular, Crismale, De Luca, Kubin, Ninno and Ponsiglione [3] showed that the classical limit admits a variational interpretation in terms of Γ -convergence with respect to weak $L^2(\Omega)$ -convergence for functions extended to 0 outside the bounded set $\Omega \subset \mathbb{R}^N$. More recently, Davoli, Fanizza and Picerni [5] obtained a generalized Maz’ya–Shaposhnikova formula for functions that need not belong to $L^2(\mathbb{R}^N)$, showing that the limit is corrected by a suitable notion of mass at infinity and establishing a close connection with the asymptotic behaviour of fractional perimeters.

In this paper we propose a different route to the Γ -liminf inequality for the Maz’ya–Shaposhnikova limit, based on spherical rearrangements. The difficulty is that the energies are not weakly continuous in L^2 , so some compactness mechanism is needed to pass from a weakly convergent sequence (u_s) to its limit. Here this is provided by rearrangement: the symmetric decreasing rearrangements $u_s^\#$ are monotone radial profiles, so by Helly’s theorem they converge strongly, and the energy does not increase under rearrangement (in the limit) by the Pólya–Szegő inequality. Since $u_s^\#$ and u_s have the same L^2 norm, the liminf inequality is thus reduced to the pointwise convergence of the energy on radially decreasing functions, which we prove by approximation with nested step functions. The argument is qualitative: it uses a rearrangement inequality and compactness of monotone profiles, and no explicit integration of the kernel.

As an application we study how the Maz’ya–Shaposhnikova limit is affected when the interaction is truncated. We consider an interaction radius $r_s \rightarrow +\infty$ and assume that $r_s^{-2s} \rightarrow \lambda \in [0, 1]$. Then the scaled energies Γ -converge, with respect to the weak $L^2(\Omega)$ -topology (for functions extended to 0 outside a bounded set Ω), to

$$(1 - \lambda)N\omega_N \int_{\Omega} |u(x)|^2 dx.$$

The parameter λ measures the asymptotic amount of interaction that is ‘lost at infinity’. When $r_s \equiv +\infty$ one has $\lambda = 0$ and the classical formula is recovered.

At the opposite extreme, if $\log r_s = o(1/s)$, then $\lambda = 1$ and the limiting energy vanishes; every intermediate regime ($\log r_s \sim \frac{1}{2s} |\log \lambda|$) produces a continuous interpolation between these two situations. The upper bound follows from the classical Maz'ya–Shaposhnikova formula and approximation arguments. For sequences supported in a fixed ball the lower bound can also be obtained by integrating the kernel directly over the region where one of the two points lies outside the support; we regard the rearrangement argument as interesting for its structure, since it handles the lack of weak continuity by a compactness property rather than by an explicit computation.

From a broader perspective, the present result complements [5]. While the latter shows how the limit changes when the admissible functions possess a nontrivial mass at infinity, here we show how it changes when the interaction with infinity is progressively removed. The two phenomena describe different mechanisms through which the behaviour at infinity governs the asymptotics of fractional Sobolev energies.

We conclude with some possible variations and extensions of our asymptotic result. The statements should carry over to the p -th power of the Gagliardo seminorm, $1 < p < \infty$, with the weak L^p -topology. In this case, the assumption on the radii becomes $r_s^{-sp} \rightarrow \lambda \in [0, 1]$, and the limit energy is $(1 - \lambda) \frac{2}{p} N \omega_N \|u\|_{L^p}^p$, in accordance with [10]. Indeed, the decomposition in Remark 4, the Pólya–Szegő inequality, and Helly's theorem do not rely on the Hilbert structure. Moreover, since our argument is based on the monotonicity of the energy under spherical rearrangement, it is natural to expect that it applies to radially symmetric decreasing kernels in the spirit of [4]. For kernels without these properties, a different approach to the Γ -liminf inequality would be needed.

An alternative to restricting interactions to $B_{r_s} \times B_{r_s}$, which cannot be written as a convolution-type kernel, is to truncate the kernel itself, replacing it with $\mathbf{1}_{\{|x-y| < r_s\}} |x-y|^{-N-2s}$. For compactly supported functions, we expect that the same value of λ governs the limit; whether this fits the framework of [7] is a natural question. It would also be interesting to study the corresponding truncated s -perimeters, in connection with [6], and to combine the truncation with functions having nontrivial mass at infinity, as in [5]. In both cases, the interplay between the radius of the truncation r_s and the behaviour at infinity is the main issue.

The paper is organized as follows. In Section 2 we state the main results. In Section 3 we establish a pointwise convergence result for spherically symmetric radially decreasing functions with a C^1 profile. In Section 4 we give details of the symmetrization argument to reduce the proof of the liminf inequality to the pointwise convergence proved in Section 3, concluding the proof of Theorem 1. Section 5 gives the pointwise convergence for Lipschitz functions with compact support and completes the proof of the recovery sequence; that is, Theorem 3.

2 Main results

Let $u: \mathbb{R}^N \rightarrow \mathbb{R}$ be a Lebesgue-measurable function. Let $s \in (0, 1)$ and define the Gagliardo s -seminorm of a function u on the set A as

$$|u|_{2,s}(A) = \left(\iint_{A \times A} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{1/2}.$$

If $A = \mathbb{R}^N$ we simply write $|u|_{2,s}$.

We fix a family of radii $r_s \rightarrow +\infty$ as $s \rightarrow 0$, with

$$\lim_{s \rightarrow 0} r_s^{-2s} = \lambda \in [0, 1]. \quad (1)$$

This condition is equivalent to $\log r_s \gg 1/s$ if $\lambda = 0$, $\log r_s \sim |\log \lambda|/2s$ if $\lambda \in (0, 1)$, and $\log r_s \ll 1/s$ if $\lambda = 1$.

Using the notation $B_r = B_r(0)$ for the ball centered at 0 and with radius r , we consider the truncated energies

$$\mathcal{F}_s(u) := s|u|_{2,s}^2(B_{r_s}) = s \iint_{B_{r_s} \times B_{r_s}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy. \quad (2)$$

We allow $r_s \equiv +\infty$, in which case $\lambda = 0$ and for which B_{r_s} is replaced by $\mathbb{R}^N$. We introduce a separate notation for this case and define $\mathcal{F}_s^0(u) = s|u|_{2,s}^2(\mathbb{R}^N)$.

The core of the paper is the following liminf theorem, which is proved by spherical rearrangement in Sections 3 and 4. The case $\lambda = 0$ is the Γ -liminf inequality for the classical Maz'ya–Shaposhnikova limit (for $r_s \equiv +\infty$, and more generally for r_s growing so fast that $r_s^{-2s} \rightarrow 0$), so that the theorem contains a new proof of that inequality; the truncation enters through the parameter λ .

Theorem 1. *Let $\mathcal{F}_s$ be given by (2) and let λ be defined by (1), let (u_n) be a sequence in $L^2(\mathbb{R}^N)$ weakly converging to a function u , and let (s_n) be an infinitesimal sequence in $(0, 1)$. Assume that the functions u_n vanish outside a common compact set. Then*

$$\liminf_{n \rightarrow +\infty} \mathcal{F}_{s_n}(u_n) \geq (1 - \lambda)N\omega_N \liminf_{n \rightarrow +\infty} \|u_n\|_{L^2(\mathbb{R}^N)}^2.$$

Remark 2. When the functions u_n vanish outside a common ball B_{R_0} , the inequality in Theorem 1 can also be obtained by integrating the kernel directly. If $x \in B_{R_0}$ and $2R_0 < |z| < r_s - R_0$, then $x + z \in B_{r_s} \setminus B_{R_0}$, where u_n vanishes, so that

$$\begin{aligned} \mathcal{F}_s(u_n) &\geq 2s \int_{B_{R_0}} |u_n(x)|^2 \int_{\{2R_0 < |z| < r_s - R_0\}} \frac{dz}{|z|^{N+2s}} dx \\ &= N\omega_N \left[(2R_0)^{-2s} - (r_s - R_0)^{-2s} \right] \|u_n\|_{L^2}^2, \end{aligned}$$

and the bracket tends to $1 - \lambda$. The argument below is different in nature: it is qualitative, and it handles the lack of weak continuity of the energies by a compactness property of monotone radial profiles (the rearrangements converge strongly), rather than by an explicit computation with the kernel.

This result is complemented by the next theorem, which is proved following an argument in [10, Section 4].

Theorem 3. *Let u be a Lipschitz function on $\mathbb{R}^N$ with compact support. Then*

$$\limsup_{s \rightarrow 0} \mathcal{F}_s(u) \leq (1 - \lambda)N\omega_N \|u\|_{L^2(\mathbb{R}^N)}^2.$$

Note that, in particular, the two results above imply the convergence of $\mathcal{F}_s(u)$ to $(1 - \lambda)N\omega_N \|u\|_{L^2}^2$ for every $u \in C_c^1(\mathbb{R}^N)$.

Remark 4. It is useful to point out the special role of the case $r_s \equiv +\infty$. Let u be a function with support in a ball B_{R_0} , and let $r_s > R_0$. The domain of integration $\mathbb{R}^N \times \mathbb{R}^N$ in the definition of $\mathcal{F}_s^0$ can be split taking into account the disappearance of u outside B_{r_s} . Thus, we can decompose $\mathcal{F}_s(u)$ as follows:

$$\begin{aligned} \mathcal{F}_s(u) &= \mathcal{F}_s^0(u) - \delta_s(u^2), \\ \text{with } \delta_s(f) &= 2s \iint_{B_{R_0} \times (\mathbb{R}^N \setminus B_{r_s})} \frac{f(x)}{|x - y|^{N+2s}} dx dy. \end{aligned} \quad (3)$$

This decomposition will be applied appropriately in the sequel to recover the general case from the asymptotics of $\mathcal{F}_s^0$. For example, Proposition 7 in Section 3 implies that if $u \in C_c^1(\mathbb{R}^N)$ then $\delta_s(u^2)$ tends to $\lambda N\omega_N \|u\|_{L^2}^2$ as $s \rightarrow 0$. As a consequence, it is enough to prove Theorem 3 for $\mathcal{F}_s^0$. Note that Theorem 1 cannot be derived from the case $r_s \equiv +\infty$ through this decomposition, since the convergence of $\delta_s(u^2)$ fails if u is replaced by a weakly convergent sequence; this is one reason why a compactness mechanism, here provided by rearrangements, is needed.

We can derive a Γ -convergence result as a corollary of the previous theorems. For $\lambda = 0$ it recovers the Γ -convergence of [3], and for $\lambda > 0$ it describes the effect of truncation. Let Ω be a bounded open subset of $\mathbb{R}^N$ with Lipschitz boundary. For every $u \in L^2(\Omega)$ we denote by $\tilde{u}: \mathbb{R}^N \rightarrow \mathbb{R}$ the extension of u to the whole $\mathbb{R}^N$ with value zero outside Ω . For every $s \in (0, 1)$ we define

$$F_s: L^2(\Omega) \rightarrow [0, +\infty], \quad F_s(u) = \mathcal{F}_s(\tilde{u}),$$

and $F_0(u) = (1 - \lambda)N\omega_N \|u\|_{L^2(\Omega)}^2$.

Corollary 5. *Let (s_n) be an infinitesimal sequence in $(0, 1)$. Then the functionals F_{s_n} Γ -converge to F_0 with respect to the weak $L^2(\Omega)$ -convergence as $n \rightarrow +\infty$.*

Proof. (Γ -lim inf) Let (u_n) be a sequence in $L^2(\Omega)$ weakly converging to a function u . Let $\tilde{u}_n$ and $\tilde{u}$ denote the extension of u_n and u , respectively, to the whole $\mathbb{R}^N$ with zero value outside Ω . Since Ω is bounded, and $(\tilde{u}_n)$ weakly converges to $\tilde{u}$ in $L^2(\mathbb{R}^N)$, we can apply Theorem 1, and deduce that

$$\begin{aligned} \liminf_{n \rightarrow +\infty} F_{s_n}(u_n) &= \liminf_{n \rightarrow +\infty} \mathcal{F}_{s_n}(\tilde{u}_n) \\ &\geq (1 - \lambda)N\omega_N \|\tilde{u}\|_{L^2(\mathbb{R}^N)}^2 = (1 - \lambda)N\omega_N \|u\|_{L^2(\Omega)}^2. \end{aligned}$$

(Γ -lim sup) Theorem 3 guarantees the Γ -lim sup inequality for $C_c^\infty(\Omega)$ functions. The general case follows by a standard approximation and diagonalization argument. $\square$

In the next two sections we prove Theorem 1 by rearrangements: Section 3 establishes the pointwise convergence on radially decreasing functions, and Section 4 reduces the general case to it. Theorem 3 is proved in Section 5.

In order to simplify the notation, the sequences $(u_n)_n$ in the previous statements will be denoted by $(u_s)_s$ (instead of $(u_{s_n})_n$), where s is intended to vary along a fixed infinitesimal positive sequence.

3 Convergence on radially decreasing functions

We first prove a pointwise convergence result for spherically symmetric radially decreasing functions with a C^1 profile and compact support. In the next section we reduce to this case by a rearrangement argument.

Proposition 6. *Let $\phi: [0, +\infty) \rightarrow [0, M]$ be a C^1 decreasing function, with $\phi(r) = 0$ if $r \geq R_0$ for some $R_0 > 0$. Let $w(x) = \phi(|x|)$. Then*

$$\lim_{s \rightarrow 0} \mathcal{F}_s(w) = (1 - \lambda)N\omega_N \|w\|_{L^2(\mathbb{R}^N)}^2. \quad (4)$$

We preliminarily prove a result about the kernel in the definition of δ_s (see (3)) needed in the proof of Proposition 6.

Proposition 7. *For any $x \in \mathbb{R}^N$, define*

$$\eta_s(x) = 2s \int_{\mathbb{R}^N \setminus B_{r_s}} \frac{1}{|x - y|^{N+2s}} dy.$$

Then $\eta_s \rightarrow \lambda N\omega_N$ uniformly on compact sets.

Proof. We set

$$I_s = 2s \int_{\mathbb{R}^N \setminus B_{r_s}} \frac{1}{|y|^{N+2s}} dy.$$

If $|x| \leq R < r_s$ then we have

$$|\eta_s(x) - I_s| \leq 2s \int_{\mathbb{R}^N \setminus B_{r_s}} \frac{1}{|y|^{N+2s}} \left| \left(\frac{|y|}{|x - y|} \right)^{N+2s} - 1 \right| dy.$$

For every $y \in \mathbb{R}^N \setminus B_{r_s}$, from the inequality $||x - y| - |y|| \leq |x| \leq R$ we deduce that $|x - y| \geq r_s - R$, and

$$\left| \frac{|y|}{|x - y|} - 1 \right| \leq \frac{|x|}{|x - y|} \leq \frac{R}{r_s - R}.$$

Therefore, to estimate the integral above, for r_s sufficiently large we consider the function $t \mapsto t^{N+2s}$ in a neighbourhood of $t = 1$, where the function is Lipschitz. It follows that there exists a constant $L > 0$ such that for every s

$$\left| \left(\frac{|y|}{|x-y|} \right)^{N+2s} - 1 \right| \leq L \left| \frac{|y|}{|x-y|} - 1 \right| \leq L \frac{R}{r_s - R}.$$

Hence,

$$|\eta_s(x) - I_s| \leq L \frac{R}{r_s - R} I_s.$$

A direct computation shows that I_s tends to $\lambda N \omega_N$ as $s \rightarrow 0$, thus concluding the proof. $\square$

Proof of Proposition 6. It is enough to consider the case $r_s \equiv +\infty$; i.e., $\mathcal{F}_s^0$. Indeed, by the decomposition in (3), we have $\mathcal{F}_s(w) = \mathcal{F}_s^0(w) - \delta_s(w^2)$, and $\delta_s(w^2) \rightarrow \lambda N \omega_N \|w\|_{L^2}^2$ as $s \rightarrow 0$ by Proposition 7.

The proof will proceed by approximation of w by radially symmetric step functions.

Fix $h > 0$ and let $n = \lfloor M/h \rfloor$. For every $i = 0, \dots, n$, let

$$B_i = \{x \in \mathbb{R}^N : w(x) > ih\} = \{x \in \mathbb{R}^N : \phi(|x|) > ih\}.$$

Note that the radial symmetry of w implies that B_i is a ball centered at the origin. We denote by r_i the radius. We can assume that $r_0 = R_0$ and $B_n \neq \emptyset$. Thus, since $\phi(r_i) = ih$ and ϕ is a C^1 -function, we have $0 < r_{i+1} < r_i \leq R_0$.

Define

$$w^h = \sum_{i=0}^n h 1_{B_i}. \quad (5)$$

In Step 1 we prove equality (4) for the characteristic function of a ball; then (in Step 2 and Step 3) we consider the general case of simple functions. In Step 4 we show that $\mathcal{F}_s^0(w)$ can be approximated by $\mathcal{F}_s^0(w^h)$ as $h \rightarrow 0$, *uniformly with respect to s* . This allows us to conclude. Indeed, from Step 3 and 4 we deduce that

$$\begin{aligned} \lim_{s \rightarrow 0} \mathcal{F}_s^0(w) &= \lim_{s \rightarrow 0} \lim_{h \rightarrow 0} \mathcal{F}_s^0(w^h) = \lim_{h \rightarrow 0} \lim_{s \rightarrow 0} \mathcal{F}_s^0(w^h) \\ &= N \omega_N \lim_{h \rightarrow 0} \|w^h\|_{L^2(\mathbb{R}^N)}^2 = N \omega_N \|w\|_{L^2(\mathbb{R}^N)}^2. \end{aligned}$$

Step 1 We first prove (4) in the case where $w = 1_{B_r(0)}$, in which

$$\lim_{s \rightarrow 0} \mathcal{F}_s^0(w) = N \omega_N \|w\|_{L^2(\mathbb{R}^N)}^2. \quad (6)$$

To show this, we note that, setting

$$\sigma(x) = \int_{B_r(0) \setminus B_{r-|x|}(x)} \frac{1}{|x-y|^{N+2s}} dy,$$

we have

$$\begin{aligned}
\mathcal{F}_s^0(w) &= s \int \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|w(x) - w(y)|^2}{|x - y|^{N+2s}} dx dy \\
&= 2s \int_{B_r(0)} dx \int_{\mathbb{R}^N \setminus B_r(0)} \frac{1}{|x - y|^{N+2s}} dy \\
&= 2s \int_{B_r(0)} \left(\int_{\mathbb{R}^N \setminus B_{r-|x|}(x)} \frac{1}{|x - y|^{N+2s}} dy - \sigma(x) \right) dx.
\end{aligned}$$

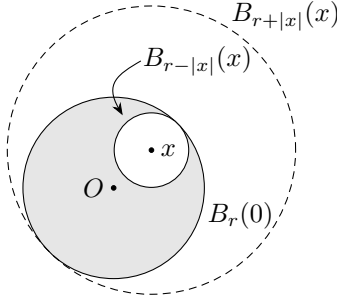


Figure 1: in the estimate of $\sigma(x)$, the domain of integration (shaded part) is controlled by $B_{r+|x|}(x) \setminus B_{r-|x|}(x)$.

Since $B_r(0) \subseteq B_{r+|x|}(x)$ (see Figure 1), we get

$$\begin{aligned}
\sigma(x) &\leq \int_{B_{r+|x|}(x) \setminus B_{r-|x|}(x)} \frac{1}{|x - y|^{N+2s}} dy = \int_{B_{r+|x|}(0) \setminus B_{r-|x|}(0)} \frac{1}{|\xi|^{N+2s}} d\xi \\
&= N\omega_N \int_{r-|x|}^{r+|x|} \frac{1}{\varrho^{N+2s}} \varrho^{N-1} d\varrho = \frac{N\omega_N}{2s} \left[\frac{1}{(r-|x|)^{2s}} - \frac{1}{(r+|x|)^{2s}} \right].
\end{aligned}$$

Hence,

$$2s \int_{B_r(0)} \sigma(x) dx \leq (N\omega_N)^2 \int_0^r \left[\frac{1}{(r-\varrho)^{2s}} - \frac{1}{(r+\varrho)^{2s}} \right] \varrho^{N-1} d\varrho.$$

The integral on the right-hand side is majorized by

$$r^{N-1} \int_0^r \left[\frac{1}{(r-\varrho)^{2s}} - \frac{1}{(r+\varrho)^{2s}} \right] d\varrho = \frac{r^{N-1}}{1-2s} [2r^{1-2s} - (2r)^{1-2s}] \xrightarrow{s \rightarrow 0} 0.$$

Therefore,

$$\mathcal{F}_s^0(w) = 2s \int_{B_r(0)} dx \int_{\mathbb{R}^N \setminus B_{r-|x|}(x)} \frac{1}{|x - y|^{N+2s}} dy + o_s(1).$$

A straightforward computation of the inner integral yields $\frac{N\omega_N}{2s}(r - |x|)^{-2s}$. Therefore,

$$\mathcal{F}_s^0(w) = N\omega_N \int_{B_r(0)} (r - |x|)^{-2s} dx + o_s(1) = (N\omega_N)^2 \int_0^r \frac{\varrho^{N-1}}{(r - \varrho)^{2s}} d\varrho + o_s(1).$$

By applying Lebesgue's Dominated Convergence Theorem, the limit of the right-hand side as $s \rightarrow 0$ gives

$$(N\omega_N)^2 \int_0^r \varrho^{N-1} d\varrho = N(\omega_N)^2 r^N = N\omega_N \|w\|_{L^2}^2,$$

which proves (6).

Step 2 Let $0 < r < R$ and $h, H > 0$. Then $w = H1_{B_R(0)} + h1_{B_r(0)}$ satisfies (4).

We first note that we have

$$\begin{aligned} |w(x) - w(y)|^2 &= H^2 |1_{B_R(0)}(x) - 1_{B_R(0)}(y)|^2 + h^2 |1_{B_r(0)}(x) - 1_{B_r(0)}(y)|^2 \\ &\quad + 2Hh (1_{B_R(0)}(x) - 1_{B_R(0)}(y)) (1_{B_r(0)}(x) - 1_{B_r(0)}(y)). \end{aligned}$$

If the last term does not vanish, then one of x and y must lie outside $B_R(0)$. The other must belong to $B_r(0)$, since otherwise $1_{B_r(0)}(x) - 1_{B_r(0)}(y) = 0$ (recall that $B_r(0) \subseteq B_R(0)$). In both cases, the last term takes the value $2Hh$. Therefore,

$$\mathcal{F}_s^0(w) = H^2 \mathcal{F}_s^0(1_{B_R(0)}) + h^2 \mathcal{F}_s^0(1_{B_r(0)}) + \mathcal{R}, \quad (7)$$

where $\mathcal{R} = 4Hhs \int_{B_r(0)} dx \int_{\mathbb{R}^N \setminus B_R(0)} \frac{1}{|x - y|^{N+2s}} dy$.

We now argue as in the previous step:

$$\mathcal{R} = 4Hhs \int_{B_r(0)} \left(\int_{\mathbb{R}^N \setminus B_{R-|x|}(x)} \frac{1}{|x - y|^{N+2s}} dy - \tilde{\sigma}(x) \right) dx,$$

where $\tilde{\sigma}(x) = \int_{B_R(0) \setminus B_{R-|x|}(x)} \frac{1}{|x - y|^{N+2s}} dy$. Since $B_R(0) \subseteq B_{R+|x|}(x)$, as in Step 1 we have

$$\tilde{\sigma}(x) \leq \int_{B_{R+|x|}(x) \setminus B_{R-|x|}(x)} \frac{dy}{|x - y|^{N+2s}} = \frac{N\omega_N}{2s} \left(\frac{1}{(R - |x|)^{2s}} - \frac{1}{(R + |x|)^{2s}} \right),$$

and $s \int_{B_r(0)} \tilde{\sigma}(x) dx$ is infinitesimal as $s \rightarrow 0$.

Thus,

$$\mathcal{R} = 4Hhs \int_{B_r(0)} dx \int_{\mathbb{R}^N \setminus B_{R-|x|}(x)} \frac{1}{|x - y|^{N+2s}} dy + o_s(1).$$

As in the previous step, the inner integral is equal to $\frac{N\omega_N}{2s}(R-|x|)^{-2s}$. Hence,

$$\begin{aligned}\mathcal{R} &= 2HhN\omega_N \int_{B_r(0)} \frac{1}{(R-|x|)^{2s}} dx \\ &= 2Hh(N\omega_N)^2 \int_0^r \frac{1}{(R-\varrho)^{2s}} \varrho^{N-1} d\varrho \xrightarrow{s \rightarrow 0} 2HhN\omega_N |B_r(0)|.\end{aligned}$$

Recall now (7) and apply the result of Step 1; then

$$\lim_{s \rightarrow 0} \mathcal{F}_s^0(w) = N\omega_N(H^2|B_R(0)| + h^2|B_r(0)| + 2Hh|B_r(0)|) = N\omega_N \|w\|_{L^2(\mathbb{R}^N)}^2.$$

This concludes the proof of Step 2.

Step 3 Let $(r_i)_{i=0,\dots,n}$ and $(h_i)_{i=0,\dots,n}$ be strictly positive values with $R_0 = r_0 > r_1 > \dots > r_n > 0$. Let $w = \sum_{i=0}^n h_i 1_{B_{r_i}(0)}$. Then w satisfies (4).

We first note that we have

$$\begin{aligned}|w(x) - w(y)|^2 &= \left| \sum_i h_i (1_{B_{r_i}(0)}(x) - 1_{B_{r_i}(0)}(y)) \right|^2 \\ &= \sum_i h_i^2 |1_{B_{r_i}(0)}(x) - 1_{B_{r_i}(0)}(y)|^2 \\ &\quad + 2 \sum_{i < j} h_i h_j (1_{B_{r_i}(0)}(x) - 1_{B_{r_i}(0)}(y)) (1_{B_{r_j}(0)}(x) - 1_{B_{r_j}(0)}(y)).\end{aligned}$$

If we use this equality in the expression of $\mathcal{F}_s^0(w)$, the first sum on the right-hand side equals $\sum_i h_i^2 \mathcal{F}_s^0(1_{B_{r_i}(0)})$, which, by Step 1, tends to $N\omega_N \sum_i h_i^2 |B_{r_i}(0)|$ as $s \rightarrow 0$. For the sum $\sum_{i < j}$, we can argue as in the previous step. Finally, we conclude that

$$\lim_{s \rightarrow 0} \mathcal{F}_s^0(w) = N\omega_N \left(\sum_i h_i^2 |B_{r_i}(0)| + 2 \sum_{i < j} h_i h_j |B_{r_j}(0)| \right).$$

We show that this limit equals the square of the L^2 norm of w . Indeed, since $B_{r_j}(0) \subseteq B_{r_i}(0)$ if $i < j$, we have

$$\begin{aligned}\int_{\mathbb{R}^N} w^2(x) dx &= \sum_{i,j} h_i h_j \int_{\mathbb{R}^N} 1_{B_{r_i}(0)}(x) 1_{B_{r_j}(0)}(x) dx \\ &= \sum_i h_i^2 |B_{r_i}(0)| + 2 \sum_{i < j} h_i h_j |B_{r_j}(0)|.\end{aligned}$$

This proves Step 3.

We next derive an estimate, which will be needed in the next step, for

$$\delta := \frac{1}{2} \min_{0 \leq i \leq n-1} (r_i - r_{i+1})$$

in terms of h . Fix a closed subinterval J of $[0, R_0]$ where ϕ' does not vanish: let $|\phi'| \geq a > 0$ in J , for some a . Let h be less than half the width of the interval $\phi(J)$; thus, there exists j such that $jh, (j+1)h \in \phi(J)$. We deduce that

$$h = \phi(r_{j+1}) - \phi(r_j) \geq a(r_j - r_{j+1}) \geq 2a\delta.$$

On the other hand, if L is a Lipschitz constant of ϕ , for every i we have

$$h = \phi(r_{i+1}) - \phi(r_i) \leq L(r_i - r_{i+1}),$$

hence $h \leq 2L\delta$. We conclude that there exist constants $L, a > 0$, depending only on w , such that

$$\frac{h}{2L} \leq \delta \leq \frac{h}{2a} \quad (8)$$

for h sufficiently small.

Step 4 We now fix w as in the statement of the proposition; let w^h be defined as in (5). We show that

$$\lim_{h \rightarrow 0} \mathcal{F}_s^0(w^h) = \mathcal{F}_s^0(w) \quad \text{uniformly for } s \text{ sufficiently close to } 0. \quad (9)$$

We first prove an estimate for $|w^h - w|_{2,s}^2$. Note that the function $\nu_h := w^h - w$ vanishes outside $B_{R_0}(0)$; then, in the integral that defines $|\nu_h|_{2,s}$, we require that at least one of the points x and y lies in $B_{R_0}(0)$:

$$|\nu_h|_{2,s}^2 \leq 2 \int_{B_{R_0}(0)} \int_{\mathbb{R}^N} \frac{|\nu_h(x) - \nu_h(y)|^2}{|x - y|^{N+2s}} dy dx.$$

Let now $\delta = \frac{1}{2} \min_i (r_i - r_{i+1})$, and divide the inner y -integral into two sets, depending on $x \in B_{R_0}(0)$:

$$\begin{aligned} |\nu_h|_{2,s}^2 &= 2 \int_{B_{R_0}(0)} dx \int_{\mathbb{R}^N \setminus B_\delta(x)} \frac{|\nu_h(x) - \nu_h(y)|^2}{|x - y|^{N+2s}} dy \\ &\quad + 2 \int_{B_{R_0}(0)} dx \int_{B_\delta(x)} \frac{|\nu_h(x) - \nu_h(y)|^2}{|x - y|^{N+2s}} dy =: I_1 + I_2. \end{aligned}$$

To estimate I_1 we note that $|w^h - w| \leq h$, hence $|\nu_h(x) - \nu_h(y)| \leq 2h$. Then

$$I_1 \leq 2(2h)^2 \int_{B_{R_0}(0)} dx \int_{\mathbb{R}^N \setminus B_\delta(x)} \frac{1}{|x - y|^{N+2s}} dy = 8N\omega_N h^2 \delta^{-2s} \frac{|B_{R_0}(0)|}{2s}.$$

We now address I_2 , for which we use the following decomposition:

$$I_2 = 2 \sum_{i=0}^n \int_{S_i} dx \int_{B_\delta(x)} \frac{|\nu_h(x) - \nu_h(y)|^2}{|x - y|^{N+2s}} dy,$$

where $S_i = B_i \setminus \overline{B_{i+1}}$ if $i < n$, and $S_n = B_n$.

Fix $i < n$. For every $x \in S_i$ and $y \in B_\delta(x)$ we have:

$$|\nu_h(x) - \nu_h(y)|^2 \leq 2(|w^h(x) - w^h(y)|^2 + |w(x) - w(y)|^2). \quad (10)$$

We have $w^h(x) - w^h(y) = 0$ unless $d(x, \partial S_i) < \delta$, and $y \in B_\delta(x) \setminus S_i$. In this case, since δ is at most half the width of every S_j , we have $|w^h(x) - w^h(y)| = h$. Then

$$\begin{aligned} \int_{S_i} dx \int_{B_\delta(x)} \frac{|w^h(x) - w^h(y)|^2}{|x - y|^{N+2s}} dy \\ = \int_{\{x \in S_i: d(x, \partial S_i) < \delta\}} dx \int_{B_\delta(x) \setminus S_i} \frac{h^2}{|x - y|^{N+2s}} dy. \end{aligned} \quad (11)$$

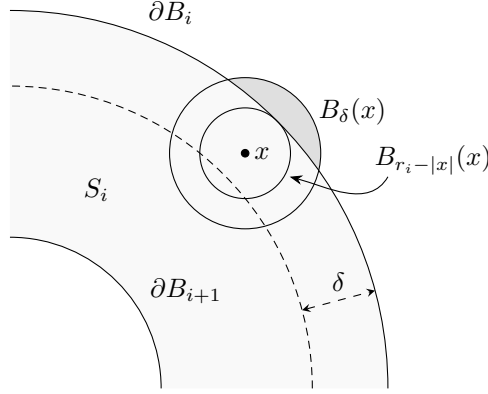


Figure 2: when x is δ -close to ∂S_i , the darker shaded part of the ball $B_\delta(x)$ (where the difference $w^h(x) - w^h(y)$ does not vanish) is controlled by the annular region.

Assume now, for example, that $r_i - \delta < |x| < r_i$ (see Figure 2), which implies that $y \in S_{i-1}$ (the case $r_{i+1} < |x| < r_{i+1} + \delta$ being analogous). Then

$$\begin{aligned} \int_{B_\delta(x) \setminus S_i} \frac{1}{|x - y|^{N+2s}} dy &\leq \int_{\{y: r_i - |x| < |y - x| < \delta\}} \frac{1}{|x - y|^{N+2s}} dy \\ &= \frac{N\omega_N}{2s} [(r_i - |x|)^{-2s} - \delta^{-2s}]. \end{aligned}$$

We now integrate the term in square brackets with respect to x in the outer

strip $r_i - \delta < |x| < r_i$:

$$\begin{aligned}
& \int_{\{x: r_i - \delta < |x| < r_i\}} [(r_i - |x|)^{-2s} - \delta^{-2s}] dx \\
&= N\omega_N \int_{r_i - \delta}^{r_i} [(r_i - \varrho)^{-2s} - \delta^{-2s}] \varrho^{N-1} d\varrho \\
&\leq N\omega_N r_i^{N-1} \int_{r_i - \delta}^{r_i} [(r_i - \varrho)^{-2s} - \delta^{-2s}] d\varrho = \frac{2sN\omega_N}{1-2s} r_i^{N-1} \delta^{1-2s}.
\end{aligned}$$

An analogous estimate holds for the inner strip.

Returning to (11), we get the following:

$$\int_{S_i} dx \int_{B_\delta(x)} \frac{|w^h(x) - w^h(y)|^2}{|x - y|^{N+2s}} dy \leq 2 \frac{(N\omega_N)^2}{1-2s} R_0^{N-1} h^2 \delta^{1-2s}.$$

We now consider the integral of the term $|w(x) - w(y)|^2$ in (10). Let L be a Lipschitz constant for w . Then

$$\begin{aligned}
\int_{S_i} dx \int_{B_\delta(x)} \frac{|w(x) - w(y)|^2}{|x - y|^{N+2s}} dy &\leq L^2 \int_{S_i} dx \int_{B_\delta(x)} \frac{1}{|x - y|^{N-2+2s}} dy \\
&= L^2 |S_i| \frac{N\omega_N}{2-2s} \delta^{2-2s}.
\end{aligned}$$

We conclude that for s sufficiently small (say $s < 1/3$) and $i = 0, \dots, n-1$:

$$\int_{S_i} dx \int_{B_\delta(x)} \frac{|\nu_h(x) - \nu_h(y)|^2}{|x - y|^{N+2s}} dy \leq C(N, w) (h^2 \delta^{1-2s} + |S_i| \delta^{2-2s}),$$

where $C(N, w)$ is a constant depending only on N and w . The same inequality holds for S_n (only the outer boundary has to be considered). Finally, we estimate I_2 using the fact that $n \leq M/h$, so that we can assume $(n+1)h \leq 2M$:

$$I_2 \leq 2C(N, w) (2Mh \delta^{1-2s} + |B_{R_0}(0)| \delta^{2-2s}).$$

Now, we gather the estimate for I_1 and I_2 , and take into account the inequalities (8). Then

$$|\nu_h|_{2,s}^2 \leq 2C(w) h^{2-2s}, \tag{12}$$

where $C(w)$ denotes a constant that depends only on w .

Since $\mathcal{F}_s^0(f)^{1/2} = \sqrt{s} |f|_{2,s}$ is a seminorm,

$$|\mathcal{F}_s^0(w^h)^{1/2} - \mathcal{F}_s^0(w)^{1/2}| \leq \mathcal{F}_s^0(\nu_h)^{1/2} \leq Ch^{1-s},$$

and $\mathcal{F}_s^0(w)$ is bounded for small s by Theorem 3. Therefore $|\mathcal{F}_s^0(w^h) - \mathcal{F}_s^0(w)| \leq Ch^{1-s} \leq Ch^{1-s_0}$ for $s \leq s_0$. This concludes Step 4, and the proof of the proposition. $\square$

4 Reduction to pointwise convergence via rearrangements

The key idea of the proof of Theorem 1 is to replace functions u_s by their spherically symmetric rearrangements. This is possible since the energy decreases (in the limit) by rearrangements (Step 1 in this section). Symmetric functions enjoy better convergence properties and allow passing from a weakly convergent u_s to a strongly convergent sequence (Step 2). In turn, the strong convergence makes it possible to replace the sequence with its limit (Step 3), thus reducing the problem to a pointwise convergence of the energy, which is proved in the previous section.

We note two technical points. First, the property of strong convergence of rearrangements demands an equiboundedness assumption, which will be subsequently removed (Step 4). Second, since the reduction to a pointwise-convergence problem can be carried out more easily for smooth functions, we make use of a standard regularizing procedure in Step 3.

As we will see, the case $r_s \equiv +\infty$ makes the structure of the proof more transparent, and it represents an essential part of the analysis. In the following steps we now show how the general case reduces to this one by relying on the decomposition in (3).

Consider $u_s \in L^2(\mathbb{R}^N)$, with $s > 0$, and such that

$$u_s \rightharpoonup u \text{ weakly in } L^2(\mathbb{R}^N) \text{ as } s \rightarrow 0.$$

We assume that there exists $R_0 > 0$ such that the functions u_s have support in a common compact subset of the open ball $B_{R_0}(0)$. Moreover, up to a subsequence we can suppose that $\liminf$ in the statement of Theorem 1 is a limit, so that it does not change by considering further subsequences.

As mentioned above, we first argue under the additional assumption that

$$\text{there exists } M > 0 \text{ such that } |u_s| \leq M. \quad (13)$$

If $w: \mathbb{R}^N \rightarrow \mathbb{R}$ is a Lebesgue-measurable function, we denote by $w^\#$ the *spherically symmetric rearrangement* of w [9, Definitions 10.16 and 10.33]. If w vanishes at infinity, then (see [9, Theorem 10.34]) $|w^\#|_{2,s} \leq |w|_{2,s}$. This inequality yields $\mathcal{F}_s^0(u_s^\#) \leq \mathcal{F}_s^0(u_s)$. The next step shows that an analogous inequality holds for $\mathcal{F}_s$ as $s \rightarrow 0$.

Step 1 We prove that for every $s > 0$

$$\mathcal{F}_s(u_s^\#) \leq \mathcal{F}_s(u_s) + o_s(1). \quad (14)$$

Apply (3) to $\mathcal{F}_s(u_s)$ and $\mathcal{F}_s(u_s^\#)$; we have

$$\mathcal{F}_s(u_s^\#) - \mathcal{F}_s(u_s) = \mathcal{F}_s^0(u_s^\#) - \mathcal{F}_s^0(u_s) + \sigma(s),$$

where

$$\sigma(s) = \delta_s(u_s^2 - (u_s^\#)^2) = \int_{B_{R_0}} \eta_s(x) (u_s^2(x) - (u_s^\#)^2(x)) dx.$$

Since $\mathcal{F}_s^0(u_s^\#) \leq \mathcal{F}_s^0(u_s)$, it is enough to show that $\sigma(s) \rightarrow 0$ as $s \rightarrow 0$. Recalling that u_s and $u_s^\#$ have the same L^2 norm, we can write

$$\sigma(s) = \int_{B_{R_0}} (\eta_s(x) - \lambda N \omega_N) (u_s^2(x) - (u_s^\#)^2(x)) dx.$$

We now get (14) by the L^2 equiboundedness of u_s and $u_s^\#$ and by Proposition 7.

Step 2 Since $u_s^\#$ is radially symmetric and radially decreasing (see [9, Theorem 10.17]) there exists a decreasing function $\varphi_s: [0, +\infty) \rightarrow [0, +\infty)$ such that

$$u_s^\#(x) = \varphi_s(|x|), \quad x \in \mathbb{R}^N.$$

From (13) we also have

$$0 \leq \varphi_s \leq M; \quad \varphi_s(r) = 0 \quad \text{if } r > R_0.$$

By Helly's Theorem there exists a subsequence of $(\varphi_s)_s$ that converges pointwise in $[0, R_0]$. We can assume, for simplicity, that the whole sequence $(\varphi_s)_s$ converges; if we denote by φ the limit, we deduce that

$$u_s^\# \rightarrow v \quad \text{strongly in } L^2(\mathbb{R}^N), \text{ where } v(x) = \varphi(|x|). \quad (15)$$

It remains to prove Theorem 1 for the sequence $(u_s^\#)_s$, which strongly converges to v ; i.e., we have to prove that

$$(1 - \lambda)N\omega_N \|v\|_{L^2}^2 \leq \liminf_{s \rightarrow 0} \mathcal{F}_s(u_s^\#). \quad (16)$$

Indeed, $\mathcal{F}_s(u_s^\#) \leq \mathcal{F}_s(u_s) + o_s(1)$ by (14), and $\|v\|_{L^2(\mathbb{R}^N)} \geq \|u\|_{L^2(\mathbb{R}^N)}$. This last inequality follows from $\|u_s^\#\|_{L^2(\mathbb{R}^N)} = \|u_s\|_{L^2(\mathbb{R}^N)}$, since

$$\|v\|_{L^2(\mathbb{R}^N)} = \lim_{s \rightarrow 0} \|u_s^\#\|_{L^2(\mathbb{R}^N)} = \lim_{s \rightarrow 0} \|u_s\|_{L^2(\mathbb{R}^N)} \geq \|u\|_{L^2(\mathbb{R}^N)}. \quad (17)$$

Remark 8. For future reference we explicitly note that the last inequality in (17) is the only point where the weak convergence of (u_s) is used. In particular, this will be used in the final step of this section, when we will remove the additional assumption (13).

Thanks to the previous steps we have reduced the problem to the proof of (16).

Step 3 (*Reduction to pointwise convergence; regularization*) In order to prove (16):

- a) we replace $\liminf_{s \rightarrow 0} \mathcal{F}_s(u_s^\#)$ with $\liminf_{s \rightarrow 0} \mathcal{F}_s(v)$, by exploiting the strong L^2 convergence of $(u_s^\#)$ to v ;
- b) we apply the pointwise convergence of $\mathcal{F}_s$ for spherically symmetric and radially decreasing functions obtained in the previous section.

It is convenient to first prove claim (a) for the regularized functions; we will then obtain the general case by letting the smoothing parameter ε tend to 0.

Let $(\varrho_\varepsilon)_\varepsilon$ be standard symmetric regularizing kernels. Define

$$\bar{u}_s^\varepsilon = \varrho_\varepsilon * u_s^\#, \quad v^\varepsilon = \varrho_\varepsilon * v. \quad (18)$$

For every $w \in L^2_{\text{loc}}(\mathbb{R}^N)$ with $|w|_{2,s} < +\infty$, we know that $|\varrho_\varepsilon * w|_{2,s} \leq |w|_{2,s}$ ([9], Theorem 6.62). This implies that $\mathcal{F}_s^0(\bar{u}_s^\varepsilon) \leq \mathcal{F}_s^0(u_s^\#)$. We next extend this inequality to $\mathcal{F}_s$. More precisely, we claim that

$$\liminf_{s \rightarrow 0} \mathcal{F}_s(\bar{u}_s^\varepsilon) \leq \liminf_{s \rightarrow 0} \mathcal{F}_s(u_s^\#) + o_\varepsilon(1). \quad (19)$$

To prove this claim, we apply (3) to $\mathcal{F}_s(\bar{u}_s^\varepsilon)$ and $\mathcal{F}_s(u_s^\#)$, and use the inequality $\mathcal{F}_s^0(\bar{u}_s^\varepsilon) \leq \mathcal{F}_s^0(u_s^\#)$. Then we have

$$\mathcal{F}_s(\bar{u}_s^\varepsilon) - \mathcal{F}_s(u_s^\#) = \mathcal{F}_s^0(\bar{u}_s^\varepsilon) - \mathcal{F}_s^0(u_s^\#) + \sigma^\varepsilon(s) \leq \sigma^\varepsilon(s),$$

where $\sigma^\varepsilon(s) = \delta_s((u_s^\#)^2 - (\bar{u}_s^\varepsilon)^2)$, so that

$$\begin{aligned} |\sigma^\varepsilon(s)| &\leq \int_{B_{R_0}} \eta_s |(u_s^\#)^2 - (\bar{u}_s^\varepsilon)^2| dx \\ &\leq \int_{B_{R_0}} |\eta_s - \lambda N \omega_N| |(u_s^\#)^2 - (\bar{u}_s^\varepsilon)^2| dx + \lambda N \omega_N \int_{B_{R_0}} |(u_s^\#)^2 - (\bar{u}_s^\varepsilon)^2| dx. \end{aligned}$$

By Proposition 7 and the L^2 boundedness of u_s (and, hence, of $u_s^\#$ and $\bar{u}_s^\varepsilon$), the first integral in the right-hand side of the last inequality tends to zero as $s \rightarrow 0$ (uniformly with respect to ε). The same L^2 boundedness applied to the other integral yields the following estimate (C a positive constant independent of s and ε):

$$\begin{aligned} \int_{B_{R_0}} |u_s^\# + \bar{u}_s^\varepsilon| |u_s^\# - \bar{u}_s^\varepsilon| dx &\leq C \|u_s^\# - \bar{u}_s^\varepsilon\|_{L^2(\mathbb{R}^N)} \\ &\leq C (\|u_s^\# - v\|_{L^2} + \|v - v^\varepsilon\|_{L^2} + \|v^\varepsilon - \bar{u}_s^\varepsilon\|_{L^2}) \\ &\leq C (2\|u_s^\# - v\|_{L^2} + \|v - v^\varepsilon\|_{L^2}), \end{aligned}$$

since $\|v^\varepsilon - \bar{u}_s^\varepsilon\|_{L^2} = \|(v - u_s^\#)^\varepsilon\|_{L^2} \leq \|v - u_s^\#\|_{L^2}$. Therefore, we get

$$\limsup_{s \rightarrow 0} |\sigma^\varepsilon(s)| \leq C \|v - v^\varepsilon\|_{L^2(\mathbb{R}^N)},$$

concluding the proof. This proves claim (19).

We now prove (a) with $\bar{u}_s^\varepsilon$ and v^ε in place of $u_s^\#$ and v , respectively. To this end we claim that

$$\liminf_{s \rightarrow 0} \mathcal{F}_s(\bar{u}_s^\varepsilon) = \liminf_{s \rightarrow 0} \mathcal{F}_s(v^\varepsilon). \quad (20)$$

We first address the case $r_s \equiv +\infty$. Let $\varepsilon > 0$ be fixed. Since the functions $u_s^\# - v$ are equibounded in $L^1(\mathbb{R}^N)$, we can choose a Lipschitz constant L_ε for $\bar{u}_s^\varepsilon - v^\varepsilon = (u_s^\# - v)^\varepsilon$ which is independent of s : indeed, a Lipschitz constant is given by $\|\nabla \varrho_\varepsilon\|_\infty \|u_s^\# - v\|_{L^1}$. We now show that

$$\lim_{s \rightarrow 0} s |\bar{u}_s^\varepsilon - v^\varepsilon|_{2,s}^2 = 0. \quad (21)$$

For every $R > 1$ we have that

$$\begin{aligned} I &:= |\bar{u}_s^\varepsilon - v^\varepsilon|_{2,s}^2 = \int_{\mathbb{R}^N} \frac{1}{|\xi|^{N+2s}} d\xi \int_{\mathbb{R}^N} |\Delta_\xi(\bar{u}_s^\varepsilon - v^\varepsilon)|^2(x) dx \\ &= \left(\int_{B_R(0)} \frac{1}{|\xi|^{N+2s}} d\xi + \int_{\mathbb{R}^N \setminus B_R(0)} \frac{1}{|\xi|^{N+2s}} d\xi \right) \int_{\mathbb{R}^N} |\Delta_\xi(\bar{u}_s^\varepsilon - v^\varepsilon)(x)|^2 dx \\ &=: I_1 + I_2, \end{aligned}$$

where we have set $(\Delta_\xi f)(x) = f(x + \xi) - f(x)$. We now estimate I_1 . Take into account that the support of $(\bar{u}_s^\varepsilon - v^\varepsilon)$ is contained in $B_{R_0}(0)$, and let $c = \lfloor R_0/R \rfloor + 2$; then $R + R_0 \leq cR$ and

$$I_1 \leq \int_{B_R(0)} \frac{1}{|\xi|^{N+2s}} (L_\varepsilon |\xi|)^2 |B_{cR}(0)| d\xi = CR^N \int_0^R \frac{1}{\varrho^{N+2s-2}} \varrho^{N-1} d\varrho,$$

where C is a positive constant independent of s . We conclude that, for $s < 1/2$,

$$I_1 \leq CR^{N+2-2s}. \quad (22)$$

We now address I_2 . Note that for every $\xi \in \mathbb{R}^N$

$$\begin{aligned} \int_{\mathbb{R}^N} |\Delta_\xi(\bar{u}_s^\varepsilon - v^\varepsilon)(x)|^2 dx &\leq 2 \left(\int_{\mathbb{R}^N} |(\bar{u}_s^\varepsilon - v^\varepsilon)(x + \xi)|^2 dx + \int_{\mathbb{R}^N} |(\bar{u}_s^\varepsilon - v^\varepsilon)(x)|^2 dx \right) \\ &= 4 \|\bar{u}_s^\varepsilon - v^\varepsilon\|_{L^2(\mathbb{R}^N)}^2 \leq 4 \|u_s^\# - v\|_{L^2(\mathbb{R}^N)}^2. \end{aligned}$$

Therefore

$$\begin{aligned} I_2 &= \int_{\{|\xi| \geq R\}} \frac{1}{|\xi|^{N+2s}} d\xi \int_{\mathbb{R}^N} |\Delta_\xi(\bar{u}_s^\varepsilon - v^\varepsilon)(x)|^2 dx \\ &\leq 4 \|u_s^\# - v\|_{L^2(\mathbb{R}^N)}^2 \int_{\{|\xi| \geq R\}} \frac{1}{|\xi|^{N+2s}} d\xi \leq 2N\omega_N \|u_s^\# - v\|_{L^2(\mathbb{R}^N)}^2 \frac{1}{sR^{2s}}. \end{aligned}$$

Combining the two estimates and recalling that $R > 1$, we have

$$\begin{aligned} s |\bar{u}_s^\varepsilon - v^\varepsilon|_{2,s}^2 &= sI \leq C(sR^{N+2-2s} + \|u_s^\# - v\|_{L^2(\mathbb{R}^N)}^2 R^{-2s}) \\ &\leq sCR^{N+2-2s} + 2\|u_s^\# - v\|_{L^2(\mathbb{R}^N)}^2. \end{aligned}$$

Passing to the limit as $s \rightarrow 0$ and using (15), we obtain (21).

The seminorm property $||\bar{u}_s^\varepsilon|_{2,s} - |v^\varepsilon|_{2,s}| \leq |\bar{u}_s^\varepsilon - v^\varepsilon|_{2,s}$, together with (21), now implies that

$$\lim_{s \rightarrow 0} |(\mathcal{F}_s^0(\bar{u}_s^\varepsilon))^{1/2} - (\mathcal{F}_s^0(v^\varepsilon))^{1/2}| = 0,$$

from which $\liminf_{s \rightarrow 0} (\mathcal{F}_s^0(\bar{u}_s^\varepsilon))^{1/2} = \liminf_{s \rightarrow 0} (\mathcal{F}_s^0(v^\varepsilon))^{1/2}$. Claim (20) follows in the case $r_s \equiv +\infty$.

We now consider the general case. Apply (3) to $\mathcal{F}_s(\bar{u}_s^\varepsilon)$ and $\mathcal{F}_s(v^\varepsilon)$, and consider $\delta_s((\bar{u}_s^\varepsilon)^2 - (v^\varepsilon)^2)$. By the L^2 boundedness of u_s there exists $C > 0$ such that

$$\int_{B_{R_0}} |(\bar{u}_s^\varepsilon)^2 - (v^\varepsilon)^2| dx \leq C \|\bar{u}_s^\varepsilon - v^\varepsilon\|_{L^2(\mathbb{R}^N)} \leq C \|u_s^\# - v\|_{L^2(\mathbb{R}^N)} \rightarrow 0 \quad \text{as } s \rightarrow 0.$$

Together with the uniform convergence of η_s (Proposition 7) this implies that $\delta_s((\bar{u}_s^\varepsilon)^2 - (v^\varepsilon)^2) \rightarrow 0$ as $s \rightarrow 0$. Since $\delta_s((v^\varepsilon)^2)$ has a limit (with value $\lambda N \omega_N \|v^\varepsilon\|_{L^2}^2$) as $s \rightarrow 0$, then $\delta_s((\bar{u}_s^\varepsilon)^2)$ has limit, too, with the same value. Hence, taking into account what already proved above for $\mathcal{F}_s^0$, we deduce that

$$\begin{aligned} \liminf_{s \rightarrow 0} \mathcal{F}_s(\bar{u}_s^\varepsilon) &= \liminf_{s \rightarrow 0} [\mathcal{F}_s^0(\bar{u}_s^\varepsilon) - \delta_s((\bar{u}_s^\varepsilon)^2)] \\ &= \liminf_{s \rightarrow 0} [\mathcal{F}_s^0(\bar{u}_s^\varepsilon) - \delta_s((v^\varepsilon)^2)] \\ &= \liminf_{s \rightarrow 0} [\mathcal{F}_s^0(v^\varepsilon) - \delta_s((v^\varepsilon)^2)] = \liminf_{s \rightarrow 0} \mathcal{F}_s(v^\varepsilon); \end{aligned}$$

that is, (20).

Step 4 We now collect the results above, and take the pointwise convergence proved in the previous section into account. In this way, we have proved Theorem 1 under the additional assumption (13). Next, we will remove this condition.

By (14) (Step 1) we have

$$\liminf_{s \rightarrow 0} \mathcal{F}_s(u_s) \geq \liminf_{s \rightarrow 0} \mathcal{F}_s(u_s^\#),$$

and by (19) and (20) (Step 3)

$$\liminf_{s \rightarrow 0} \mathcal{F}_s(u_s^\#) + o_\varepsilon(1) \geq \liminf_{s \rightarrow 0} \mathcal{F}_s(\bar{u}_s^\varepsilon) = \liminf_{s \rightarrow 0} \mathcal{F}_s(v^\varepsilon). \quad (23)$$

The function v^ε defined in (18) is radially symmetric, radially decreasing and has compact support. Thus, $v^\varepsilon(x) = \phi^\varepsilon(|x|)$, with ϕ^ε non-negative, decreasing C^1 function on $[0, +\infty)$ with compact support. Therefore, we can apply Proposition 6 with $w = v^\varepsilon$ and $\phi = \phi^\varepsilon$, and obtain that the liminf on the right-hand side equals $(1 - \lambda)N\omega_N \|v^\varepsilon\|_{L^2(\mathbb{R}^N)}^2$. Finally, letting ε go to zero and applying (17) (Step 2) we conclude.

We finally remove assumption (13). For every $M > 0$ and $w: \mathbb{R}^N \rightarrow \mathbb{R}$ let $w^M = (w \wedge M) \vee (-M)$. Since $t \mapsto (t \wedge M) \vee (-M)$ is a Lipschitz function with Lipschitz constant 1, we have

$$\mathcal{F}_s(u_s^M) \leq \mathcal{F}_s(u_s).$$

Taking Remark 8 into account, we obtain that

$$\liminf_{s \rightarrow 0} \mathcal{F}_s(u_s^M) \geq (1 - \lambda)N\omega_N \|v_M\|_{L^2(\mathbb{R}^N)}^2,$$

where v_M is the strong L^2 -limit of $(u_s^M)^\#$ (recall (15)). Since (compare (17))

$$\|v_M\|_{L^2(\mathbb{R}^N)} = \lim_{s \rightarrow 0} \|(u_s^M)^\#\|_{L^2(\mathbb{R}^N)} = \lim_{s \rightarrow 0} \|u_s^M\|_{L^2(\mathbb{R}^N)},$$

we conclude by choosing the truncation level M in a suitable way. This follows by a general result on weakly converging sequences, which we state and prove in the following Lemma 9. $\square$

Lemma 9. *Let $(w_n)_n$ be a weakly convergent sequence in $L^2(\mathbb{R}^N)$. Let w be the limit function. Then, for every $\varepsilon > 0$ there exists $M > 0$ such that*

$$\liminf_{n \rightarrow +\infty} \|w_n^M\|_{L^2(\mathbb{R}^N)} \geq \|w\|_{L^2(\mathbb{R}^N)} - \varepsilon,$$

where $w_n^M = (w_n \wedge M) \vee (-M)$.

Proof. We may assume that $w \not\equiv 0$. For every $M > 0$ we have

$$\int_{\mathbb{R}^N} w_n^M w \, dx = \int_{\mathbb{R}^N} w_n w \, dx - \int_{\mathbb{R}^N} (w_n - w_n^M) w \, dx.$$

We show that the second integral on the right-hand side can be made arbitrarily small for a suitable M independent of n . Indeed,

$$\begin{aligned} \left| \int_{\mathbb{R}^N} (w_n - w_n^M) w \, dx \right| &\leq \int_{\{|w_n| > M\}} (|w_n| - M) |w| \, dx \leq \int_{\{|w_n| > M\}} |w_n w| \, dx \\ &\leq \|w_n\|_{L^2(\mathbb{R}^N)} \left(\int_{\{|w_n| > M\}} w^2 \, dx \right)^{1/2}. \end{aligned}$$

The weak convergence of (w_n) implies that $\|w_n\|_{L^2(\mathbb{R}^N)} \leq C$ for a suitable C independent of n . By Chebyshev's inequality $|\{|w_n| > M\}| \leq (C/M)^2$. Thus, the integrability of w^2 implies that for every $\sigma > 0$ we can find a value M such that $(\int_{\{|w_n| > M\}} w^2 \, dx)^{1/2} \leq \sigma$ for every n . Then

$$\liminf_{n \rightarrow +\infty} \int_{\mathbb{R}^N} w_n^M w \, dx \geq \liminf_{n \rightarrow +\infty} \int_{\mathbb{R}^N} w_n w \, dx - \sigma C = \|w\|_{L^2(\mathbb{R}^N)}^2 - \sigma C.$$

The left-hand side is majorized by $\|w\|_{L^2} \liminf_n \|w_n^M\|_{L^2}$. Thus

$$\liminf_n \|w_n^M\|_{L^2} \geq \|w\|_{L^2} - \sigma C / \|w\|_{L^2}.$$

We conclude by taking $\sigma = \varepsilon \|w\|_{L^2} / C$. $\square$

5 Convergence on smooth functions

In Remark 4 we noted that it is enough to prove Theorem 3 in the case $r_s \equiv +\infty$; i.e., for $\mathcal{F}_s^0$. As we mentioned, this result is contained in Theorem 3 of [10]. For completeness, we adapt the proof here to Lipschitz functions with compact support.

Let u be a Lipschitz function on $\mathbb{R}^N$ with compact support. In the integral defining the Gagliardo s -seminorm of u , we split $\mathbb{R}^N \times \mathbb{R}^N$ into the set $\{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : |y| \geq 2|x| \text{ or } |x| \geq 2|y|\}$ and its complement. By symmetry, $\mathcal{F}_s^0(u) = s|u|_{2,s}^2 = 2s(I_1 + I_2)$, with

$$I_1 = \iint_{\{|y| \geq 2|x|\}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy,$$

and

$$I_2 = \iint_{\{|x| < |y| < 2|x|\}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy.$$

By the Minkowski inequality (with respect to the measure $1/|x - y|^{N+2s} dx dy$) we get

$$I_1^{\frac{1}{2}} \leq \left(\iint_{\{|y| \geq 2|x|\}} \frac{|u(x)|^2}{|x - y|^{N+2s}} dx dy \right)^{\frac{1}{2}} + \left(\iint_{\{|y| \geq 2|x|\}} \frac{|u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{\frac{1}{2}}.$$

Since $|y| \geq 2|x|$ implies that $|x - y| \geq |x|$, for the first integral on the right-hand side, we have

$$\begin{aligned} & \int_{\mathbb{R}^N} |u(x)|^2 dx \int_{\{|y| \geq 2|x|\}} \frac{1}{|x - y|^{N+2s}} dy \\ & \leq \int_{\mathbb{R}^N} |u(x)|^2 dx \int_{\{|x - y| \geq |x|\}} \frac{1}{|x - y|^{N+2s}} dy = \frac{N\omega_N}{2s} \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2s}} dx. \end{aligned}$$

For the second integral, we note that the condition $|y| \geq 2|x|$ implies $|x - y| \geq |y|/2$; hence,

$$\begin{aligned} & \int_{\mathbb{R}^N} |u(y)|^2 dy \int_{\{|y| \geq 2|x|\}} \frac{1}{|x - y|^{N+2s}} dx \\ & \leq \int_{\mathbb{R}^N} |u(y)|^2 dy \int_{\{|x| < \frac{1}{2}|y|\}} \frac{1}{(|y|/2)^{N+2s}} dx = 2^{2s}\omega_N \int_{\mathbb{R}^N} \frac{|u(y)|^2}{|y|^{2s}} dy. \end{aligned}$$

If we collect the above estimates, we deduce that

$$sI_1 \leq \left(\left(\frac{N\omega_N}{2} \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2s}} dx \right)^{1/2} + 2^s \left(\omega_N s \int_{\mathbb{R}^N} \frac{|u(y)|^2}{|y|^{2s}} dy \right)^{1/2} \right)^2.$$

We now note that $\int_{\mathbb{R}^N \setminus B_1(0)} |u(x)|^2 / |x|^{2s} dx \leq \int_{\mathbb{R}^N \setminus B_1(0)} |u(x)|^2 dx$, while in $B_1(0)$ the function $|u(x)|^2 / |x|^{2s}$ is integrable (since u is bounded) and decreasing as $s \rightarrow 0$; then

$$\limsup_{s \rightarrow 0} \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2s}} dx \leq \|u\|_{L^2(\mathbb{R}^N)}^2.$$

It follows that $\limsup_{s \rightarrow 0} 2sI_1 \leq N\omega_N \|u\|_{L^2(\mathbb{R}^N)}^2$. It remains to show that sI_2 vanishes as $s \rightarrow 0$.

Since u has compact support, the set of points (x, y) for which the integrand in I_2 does not vanish is contained in a bounded subset E of $\mathbb{R}^{2N}$. Hence, if we denote by L a Lipschitz constant for u , we have

$$sI_2 \leq sL^2 \int \int_E \frac{1}{|x - y|^{N-2+2s}} dx dy \rightarrow 0 \quad \text{as } s \rightarrow 0. \quad \square$$

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