

Homogenization of some viscous Hamilton–Jacobi equations using variational arguments on probability spaces

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Abstract

This work studies the asymptotic behavior of a class of viscous Hamilton–Jacobi equations with oscillatory periodic potentials, already studied by other authors in the PDE framework, and described in terms of an effective Hamiltonian. A novel variational approach is introduced, based on the classical stochastic control representation involving a Brownian motion, which allows the problem to be reformulated in terms of functionals defined on probability spaces. The core result is the identification of a homogenized Lagrangian obtained via a Γ -convergence procedure adapted to random processes, where filtrations and Brownian motions are treated as variables. The effective Hamiltonian is then recovered as the Legendre transform of this homogenized Lagrangian. The analysis provides both lower and upper bounds by means of blow-up arguments and the construction of stochastic correctors, yielding a complete characterization of the homogenized limit. This approach allows for a stability result with respect to the addition of some localized defects embedded in the periodic background.

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1 Introduction

In this work we consider the asymptotic behaviour of the solutions to the oscillatory Hamilton–Jacobi equation

$$\lambda U_\varepsilon(x) - \frac{\varepsilon}{2} \Delta U_\varepsilon(x) + H_{\text{per}}\left(\frac{x}{\varepsilon}, \nabla U_\varepsilon(x)\right) = \psi(x), \quad x \in \mathbb{R}^d. \quad (1.1)$$

with $\lambda > 0$, and the quadratic Hamiltonian

$$H_{\text{per}}(y, p) = \frac{1}{2}|p|^2 - V_{\text{per}}(y),$$

where V_{per} is periodic. The functions V_{per} and ψ are supposed sufficiently regular, say Lipschitz continuous, so that, for each $\varepsilon > 0$, the solution exists and is unique in the sense of viscosity solutions.

As is well known, this asymptotic behaviour can be characterized by an effective *homogenized Hamiltonian* H_{hom} , for which the solutions U_ε converge to the viscosity solution of

$$\lambda U(x) + H_{\text{hom}}(\nabla U(x)) = \psi(x), \quad x \in \mathbb{R}^d. \quad (1.2)$$

as $\varepsilon \rightarrow 0^+$. In the PDE approach to homogenization, the value of the Hamiltonian H_{hom} at p is characterized by the correctors equation

$$-\frac{1}{2}\Delta w_p(x) + H_{\text{per}}(x, p + \nabla w_p(x)) = H_{\text{hom}}(p), \quad x \in \mathbb{R}^d; \quad (1.3)$$

that is, the value $H_{\text{hom}}(p)$ is the unique constant such that there is a periodic function w_p , unique up to addition of a constant, such that (1.3) holds in the sense of viscosity solutions (see [9, 12], and related work by many authors [13]).

On the other hand, given a probability space $(\Omega, \mathcal{F}, P)$ and a d -dimensional Brownian motion B_t on it, the solution U_ε can also be characterized as the value function

$$U_\varepsilon(x) = \inf \left\{ \int_0^{+\infty} \mathbb{E} \left(\frac{1}{2} |\dot{v}(t)|^2 + V_{\text{per}} \left(\frac{\sqrt{\varepsilon} B_t + v(t)}{\varepsilon} \right) + \psi(\sqrt{\varepsilon} B_t + v(t)) \right) e^{-\lambda t} dt : v(0) = x \right\},$$

where $\mathbb{E}$ denotes the expectation with respect to $(\Omega, \mathcal{F}, P)$. Here, the infimum is taken over all random functions $v: (0, +\infty) \rightarrow L^2(\Omega; \mathbb{R}^d)$ in the corresponding weighted Sobolev space, satisfying the initial condition and adapted to the filtration corresponding to the Brownian motion (see e.g. [3, Section I.8] and [4, 5, 10]). It is interesting to remark that the minimum value is independent of the choice of the Brownian motion, since the solution U_ε to equation (1.1) is unique.

With this characterization of solutions in mind, we propose a new approach, through a variational homogenization result for the functionals

$$\int_0^{+\infty} \mathbb{E} \left(\frac{1}{2} |\dot{v}(t)|^2 + V_{\text{per}} \left(\frac{\sqrt{\varepsilon} B_t + v(t)}{\varepsilon} \right) \right) e^{-\lambda t} dt. \quad (1.4)$$

It is worth noting that a homogenization result for such functionals alone, which is expressed in terms of Γ -convergence, is not sufficient to study the behaviour of the minimum problems defining $U_\varepsilon(x)$, since the additional term

$$\int_0^{+\infty} \mathbb{E}(\psi(\sqrt{\varepsilon} B_t + v(t))) e^{-\lambda t} dt$$

cannot easily be treated as a continuous perturbation of the main functional (1.4). We adopt a different approach. Using the results established here for (1.4), we characterize

H_{hom} as the Legendre transform of a homogenized Lagrangian, for which it is sufficient to consider the case $\psi = 0$. From this we characterize the asymptotic behaviour of $U_\varepsilon(x)$ for $\psi \neq 0$ relying on the PDE results in [9].

We now describe our new type of Γ -convergence, in which the main conceptual novelty is that filtrations and Brownian motions are treated as optimization variables rather than fixed probabilistic objects.

In this limit analysis, two main issues arise. The first is the definition of a proper convergence of random functions to deterministic functions. The second is the development of technical tools of nonlinear homogenization type that allow one to treat the Brownian motion as a variable to be optimized.

In order to properly study the homogenization process, we precisely define our functionals as

$$F_\varepsilon^\lambda(v, \mathcal{F}_t, B_t) = \int_0^{+\infty} \mathbb{E} \left(\frac{1}{2} |\dot{v}(t)|^2 + V_{\text{per}} \left(\frac{\sqrt{\varepsilon} B_t + v(t)}{\varepsilon} \right) \right) e^{-\lambda t} dt,$$

explicitly depending also on a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ such that $v(t)$ is measurable with respect to $\mathcal{F}_t$ for every t , and $\{B_t\}_{t \geq 0}$ a d -dimensional Brownian motion adapted to $\{\mathcal{F}_t\}_{t \geq 0}$. The convergence of random variables v_ε to a deterministic function v is then defined as the weak convergence of v_ε towards a random function $\hat{v}$, where $\mathbb{E}(\hat{v}) = v$, and the Γ -limit is defined with respect to this convergence as $\varepsilon \rightarrow 0$. The main result is that this limit can be written in terms of a *deterministic homogenized Lagrangian* f_{hom} as

$$F_{\text{hom}}^\lambda(v) = \int_0^{+\infty} f_{\text{hom}}(\dot{v}(t)) e^{-\lambda t} dt.$$

The effective Lagrangian is defined by a variant of the usual asymptotic homogenization formula,

$$f_{\text{hom}}(\xi) = \liminf_{T \rightarrow +\infty} \frac{1}{T} \min \left\{ \int_0^T \mathbb{E} \left(\frac{1}{2} |\dot{w}(t)|^2 + V_{\text{per}}(B_t + w(t)) \right) dt : \mathbb{E}(w(T, \cdot) - w(0, \cdot)) = T\xi \right\}, \quad (1.5)$$

where the infimum is taken over all filtrations $\mathcal{F}_t$, Brownian motions B_t adapted to $\mathcal{F}_t$, and functions $w \in H^1((0, T); L^2(\Omega; \mathbb{R}^d))$, adapted to $\mathcal{F}_t$. This weak formulation is used in Stochastic control (see [14] and [15, Chapter 2, Section 4]). The solutions (functions, filtrations, and Brownian motions) to the minimum problems in the definition of f_{hom} can then be used to construct random (almost) correctors, by concatenating not only the functions v as usual, but also the corresponding filtrations and Brownian motions. This concatenation is allowed by the possibility of choosing independent minimizing Brownian motions for the same minimum problem.

The variational approach also allows us to prove that, when the periodic potential V_{per} is perturbed by adding some continuous non-negative function V_0 vanishing at infinity,

the same homogenization limit is achieved. Similar questions have been addressed for the inviscid first-order Hamilton–Jacobi equations both from the PDE theoretic viewpoint (see [12, 1, 2, 8]) and the variational viewpoint [6]. Finally, we remark that our approach extends to other Hamilton–Jacobi equations with more general Hamiltonians and non-constant viscosity (see Section 8).

Our article is organized as follows. Section 2 contains the definition (Definition 2.1) of our notion of variational convergence for random functions and the statement of our main result, namely Theorem 2.2, of variational convergence on a finite time interval (a, b) . Indeed, since the exponential $e^{-\lambda t}$ is integrable on $(0, +\infty)$, the convergence results for the energies above can be obtained from the analogous result for energies localized on bounded time intervals. We address the latter setting until Section 6. First, Section 3 yields the lower limit. We construct a sequence of suitable correctors in Section 4 and use it for the proof of the upper limit in Section 5. We then address the case of a suitable perturbation of the periodic potential in Section 6. We adapt our arguments to the infinite time interval; that is, to $(0, +\infty)$, in Section 7 and conclude on the homogenization of the original problem (1.1)–(1.2). In Section 8 we comment on possible application of the convergence scheme to other settings. Our final Section 9 contains two classical properties of the Brownian motion that we include here for the convenience of the reader.

2 Statement of the convergence theorem

Throughout the text, $(\Omega, \mathcal{F}, P)$ is a fixed probability space. Given $\lambda > 0$, $\mathcal{F}_t$ and B_t , we consider energies of the form

$$F_\varepsilon^\lambda(v, \mathcal{F}_t, B_t) = \int_0^{+\infty} \mathbb{E} \left(\frac{1}{2} |\dot{v}(t)|^2 + V_{\text{per}} \left(\frac{\sqrt{\varepsilon} B_t + v(t)}{\varepsilon} \right) \right) e^{-\lambda t} dt \quad (2.1)$$

defined on functions $v = v(t, \omega)$ with $v(0, \omega) = x$. All our Brownian motions are standard, valued in $\mathbb{R}^d$, and start at $x = 0$ for $t = 0$.

The convergence results for the energies above can be obtained from the analogous result for energies localized on a bounded interval. With fixed a, b we consider the functional

$$F_\varepsilon(v, \mathcal{F}_t, B_t) = \int_a^b \mathbb{E} \left(\frac{1}{2} |\dot{v}(t)|^2 + V_{\text{per}} \left(\frac{\sqrt{\varepsilon} B_t + v(t)}{\varepsilon} \right) \right) dt \quad (2.2)$$

defined for $\mathcal{F}_t$ a filtration of $\mathcal{F}$, B_t a Brownian motion adapted to $\mathcal{F}_t$, and $v \in H^1((a, b); L^2(\Omega))$ such that for every t the function $v(t)$ is $\mathcal{F}_t$ -measurable. In this notation, $\dot{v}$ denotes the time derivative, defined for almost all t as the limit of the difference quotients in the strong topology in $L^2(\Omega)$.

We remark that the spaces $L^2((a, b); L^2(\Omega))$ and $H^1((a, b); L^2(\Omega))$ can be characterized equivalently by requiring that $\omega \mapsto \int_a^b |u(t, \omega)|^2 dt$ and $\omega \mapsto \int_a^b |\dot{u}(t, \omega)|^2 dt$ be integrable in Ω .

We will consider the following notion of convergence.

Definition 2.1 (convergence to a deterministic variable). *Given a sequence v_k with $v_k \in H^1((a, b); L^2(\Omega; \mathbb{R}^d))$ and $v \in H^1((a, b); \mathbb{R}^d)$ we write $v_k \rightarrow v$ if for all subsequences v_{k_j} , there exist a subsequence, which we still denote by v_{k_j} , and $\hat{v} \in H^1((a, b); L^2(\Omega; \mathbb{R}^d))$, possibly depending on the subsequence, such that*

$$v_{k_j} \rightharpoonup \hat{v} \text{ weakly in } H^1((a, b); L^2(\Omega; \mathbb{R}^d)), \quad \mathbb{E}(\hat{v}(t)) = v(t) \text{ for every } t.$$

We define $f_{\text{hom}}: \mathbb{R}^d \rightarrow \mathbb{R}$ as

$$f_{\text{hom}}(\xi) = \liminf_{T \rightarrow +\infty} \frac{1}{T} \inf_{\mathcal{F}_t, B_t} \left\{ \int_0^T \mathbb{E} \left(\frac{1}{2} |\dot{w}(t)|^2 + V_{\text{per}}(B_t + w(t)) \right) dt : w \in X_0^T(\xi) \right\}, \quad (2.3)$$

where $X_0^T(\xi)$ is the set of all $w \in H^1((a, b); L^2(\Omega; \mathbb{R}^d))$ such that $w(t)$ is $\mathcal{F}_t$ -measurable for every t and $\mathbb{E}(w(T, \cdot) - w(0, \cdot)) = T\xi$, and the minimum is also taken over all filtrations $\mathcal{F}_t$ and all d -dimensional Brownian motions B_t adapted to $\mathcal{F}_t$. Note that $X_0^T(\xi)$ depends on the filtration $\mathcal{F}_t$. Note moreover that, using $w(t) = \xi t$ as a test function,

$$f_{\text{hom}}(\xi) \leq \frac{1}{2} |\xi|^2 + \max V_{\text{per}}. \quad (2.4)$$

Accordingly, we define

$$F_{\text{hom}}(v) = \int_a^b f_{\text{hom}}(\dot{v}(t)) dt, \quad (2.5)$$

for $v \in H^1((a, b); \mathbb{R}^d)$. Given $\varepsilon_k \rightarrow 0$, our goal is to prove the Γ -convergence of F_{ε_k} to F_{hom} in the following sense.

Theorem 2.2. *Let F_ε be given by (2.2). Given $\varepsilon_k \rightarrow 0$, we have*

(i) *For every $v \in H^1(a, b)$ and for every sequences $\mathcal{F}_t^k$, B_t^k , and $v_k \rightarrow v$ such that $v_k(t)$ is $\mathcal{F}_t^k$ -measurable, we have*

$$F_{\text{hom}}(v) \leq \liminf_{k \rightarrow +\infty} F_{\varepsilon_k}(v_k, \mathcal{F}_t^k, B_t^k);$$

(ii) *for every $v \in H^1(a, b)$ there exists sequences $\mathcal{F}_t^k$, B_t^k , and $v_k \rightarrow v$, such that $v_k(t)$ is $\mathcal{F}_t^k$ -measurable, such that*

$$F_{\text{hom}}(v) \geq \limsup_{k \rightarrow +\infty} F_{\varepsilon_k}(v_k, \mathcal{F}_t^k, B_t^k).$$

From this theorem, we will deduce the analogous result for the functionals (2.1). We briefly explain how that convergence result allows us to give an alternative characterization of H_{hom} as the Legendre transform of the homogenized Lagrangian f_{hom} . The details will be given in Section 7.

Given $\lambda > 0$ and $\varepsilon > 0$, for all $p \in \mathbb{R}^d$ we consider the unique viscosity solution $U_{p,\varepsilon}$ of the Hamilton-Jacobi equation

$$\lambda U_{p,\varepsilon}(x) - \frac{\varepsilon}{2} \Delta U_{p,\varepsilon}(x) + H_{\text{per}}\left(\frac{x}{\varepsilon}, p + \nabla U_{p,\varepsilon}(x)\right) = 0, \quad x \in \mathbb{R}^d. \quad (2.6)$$

From homogenization theory we know that $U_{p,\varepsilon}(x)$ converge as $\varepsilon \rightarrow 0$ to the (constant) solution U_p of

$$\lambda U_p(x) + H_{\text{hom}}(p + \nabla U_p(x)) = 0, \quad x \in \mathbb{R}^d. \quad (2.7)$$

On the other hand, given a probability space $(\Omega, \mathcal{F}, P)$ and a d -dimensional Brownian motion B_t on it, the solution $U_{p,\varepsilon}$ can also be characterized as

$$U_{p,\varepsilon}(x) = \inf \left\{ \int_0^{+\infty} \mathbb{E} \left(\frac{1}{2} |\dot{v}(t)|^2 - \langle p, \dot{v}(t) \rangle + V_{\text{per}} \left(\frac{\sqrt{\varepsilon} B_t + v(t)}{\varepsilon} \right) \right) e^{-\lambda t} dt : v(0) = x \right\},$$

where $\mathbb{E}$ denotes the expectation with respect to $(\Omega, \mathcal{F}, P)$. Here, the infimum is taken over all functions $v: (0, +\infty) \rightarrow L^2(\Omega; \mathbb{R}^d)$ in the corresponding weighted Sobolev space, satisfying the initial condition and adapted to the filtration corresponding to the Brownian motion (see e.g. [3, Section I.8] and [10]).

Using the functionals F_ε^λ in (2.1), the solutions $U_{p,\varepsilon}(x)$ can be rewritten as

$$U_{p,\varepsilon}(x) = \inf \left\{ F_\varepsilon^\lambda(v, \mathcal{F}_t, B_t) - \int_0^{+\infty} \mathbb{E}(\langle p, \dot{v}(t) \rangle) e^{-\lambda t} dt : v(0) = x \right\}, \quad (2.8)$$

where the integral will be treated separately as a continuous perturbation. In Theorem 7.3 we will show that the limit of F_ε^λ in the appropriate weighted Sobolev space can be written as

$$F_{\text{hom}}^\lambda(v) = \int_0^{+\infty} f_{\text{hom}}(\dot{v}(t)) e^{-\lambda t} dt.$$

Since in this approach the integral in (2.8) converges continuously to the corresponding deterministic term $\int_0^{+\infty} \langle p, \dot{v}(t) \rangle e^{-\lambda t} dt$, the behaviour of the solutions $U_{p,\varepsilon}(x)$ can be characterized as converging to

$$U_p(x) = \min \left\{ \int_0^{+\infty} (f_{\text{hom}}(\dot{v}(t)) - \langle p, \dot{v}(t) \rangle) e^{-\lambda t} dt : v(0) = x \right\}. \quad (2.9)$$

This minimum value is independent of x , is equal to $\frac{1}{\lambda} \min\{f_{\text{hom}}(\xi) - \langle p, \xi \rangle\}$, and it is achieved by $v(t) = x + \xi t$, where $\xi \in \mathbb{R}^d$ minimizes $f_{\text{hom}}(\xi) - \langle p, \xi \rangle$. This gives

$$\lambda U_p = \min\{f_{\text{hom}}(\xi) - \langle p, \xi \rangle\} = -f_{\text{hom}}^*(p),$$

where f_{hom}^* denotes the Legendre transform of f_{hom} . Since U_p is constant independent of x , from the previous equality it trivially solves

$$\lambda U_p + f_{\text{hom}}^*(p + \nabla U_p) = \lambda U_p + f_{\text{hom}}^*(p) = 0.$$

Since U_p is also characterized as the solution of

$$\lambda U_p + H_{\text{hom}}(p + \nabla U_p) = \lambda U_p + H_{\text{hom}}(p) = 0,$$

this shows that $f_{\text{hom}}^* = H_{\text{hom}}$ by the arbitrariness of p .

3 Lower bound

This section is devoted to the proof of claim (i) in Theorem 2.2. We will need the following lemma, which will be used to modify boundary values. Note that the functions implicitly depend on ω .

Lemma 3.1 (variation of boundary data). *Let $V_k: (0, 1) \times \mathbb{R}^d \rightarrow \mathbb{R}$ be Borel functions such that there exists $M > 0$ with $|V_k(t, x)| \leq M$ for all $k \in \mathbb{N}$, $t \in (0, 1)$, and $x \in \mathbb{R}^d$. We define*

$$G_k(z) = \int_0^1 \mathbb{E} \left(\frac{1}{2} |\dot{z}(t)|^2 dt + V_k(t, z(t)) \right) dt \quad (3.1)$$

for $z \in H^1((0, 1), L^2(\Omega; \mathbb{R}^d))$. Let $\mathcal{G}_t^k$ be a sequence of filtrations and let $z_k \in H^1((0, 1), L^2(\Omega; \mathbb{R}^d))$ be such that $z_k(t)$ is $\mathcal{G}_t^k$ -measurable for every $t \in (0, 1)$. Assume that $z_k \rightharpoonup \hat{z}$ weakly in $H^1((0, 1), L^2(\Omega; \mathbb{R}^d))$. Then, given $\eta > 0$ there exists a sequence $\tilde{z}_k$ in $H^1((0, 1), L^2(\Omega; \mathbb{R}^d))$ such that

- (i) $\tilde{z}_k \rightharpoonup \hat{z}$ weakly in $H^1((0, 1), L^2(\Omega; \mathbb{R}^d))$;
- (ii) $\tilde{z}_k(t)$ is $\mathcal{G}_t^k$ -measurable for every $t \in (0, 1)$;
- (iii) $G_k(\tilde{z}_k) \leq G_k(z_k) + \eta$ for k large enough;
- (iv) $\mathbb{E}(\tilde{z}_k(0)) = \mathbb{E}(\hat{z}(0))$ and $\mathbb{E}(\tilde{z}_k(1)) = \mathbb{E}(\hat{z}(1))$.

Proof. We first note that $\mathbb{E}(z_k(0)) \rightarrow \mathbb{E}(\hat{z}(0))$ (and similarly $\mathbb{E}(z_k(1)) \rightarrow \mathbb{E}(\hat{z}(1))$). We set $\alpha_k = \mathbb{E}(z_k(0)) - \mathbb{E}(\hat{z}(0))$ and $\beta_k = \mathbb{E}(z_k(1)) - \mathbb{E}(\hat{z}(1))$. Let $\delta > 0$ and let $\varphi \in C_c^\infty(0, 1)$ with $0 \leq \varphi \leq 1$ and $\varphi = 1$ in $(\delta, 1 - \delta)$. We set

$$\tilde{z}_k(t) = \begin{cases} z_k(t) - (1 - \varphi(t))\alpha_k & \text{if } t \in (0, \delta) \\ z_k(t) & \text{if } t \in [\delta, 1 - \delta] \\ z_k(t) - (1 - \varphi(t))\beta_k & \text{if } t \in (1 - \delta, 1), \end{cases}$$

so that

$$\dot{\tilde{z}}_k(t) = \begin{cases} \dot{z}_k(t) + \dot{\varphi}(t)\alpha_k & \text{if } t \in (0, \delta) \\ \dot{z}_k(t) & \text{if } t \in [\delta, 1 - \delta] \\ \dot{z}_k(t) + \dot{\varphi}(t)\beta_k & \text{if } t \in (1 - \delta, 1) \end{cases}$$

Properties (ii) and (iv) are true by construction, and (i) follows since $\alpha_k, \beta_k \rightarrow 0$. To prove (iii) we compute

$$\begin{aligned} G_k(\tilde{z}_k) &= \int_0^1 \mathbb{E} \left(\frac{1}{2} |\dot{\tilde{z}}_k(t)|^2 + V_k(t, \tilde{z}_k(t)) \right) dt \\ &\leq \int_\delta^{1-\delta} \mathbb{E} \left(\frac{1}{2} |\dot{z}_k(t)|^2 + V_k(t, z_k(t)) \right) dt \\ &\quad + \int_0^\delta \mathbb{E} \left(\frac{1}{2} |\dot{z}_k(t)|^2 + \alpha_k \dot{z}_k(t) \dot{\varphi}(t) + \frac{1}{2} \alpha_k^2 |\dot{\varphi}(t)|^2 \right) dt \\ &\quad + \int_{1-\delta}^1 \mathbb{E} \left(\frac{1}{2} |\dot{z}_k(t)|^2 + \beta_k \dot{z}_k(t) \dot{\varphi}(t) + \frac{1}{2} \beta_k^2 |\dot{\varphi}(t)|^2 \right) dt + 2\delta \max |V_k| \\ &\leq G_k(z_k) + \int_0^\delta \mathbb{E} \left(\alpha_k \dot{z}_k(t) \dot{\varphi}(t) + \frac{1}{2} \alpha_k^2 |\dot{\varphi}(t)|^2 \right) dt \\ &\quad + \int_{1-\delta}^1 \mathbb{E} \left(\beta_k \dot{z}_k(t) \dot{\varphi}(t) + \frac{1}{2} \beta_k^2 |\dot{\varphi}(t)|^2 \right) dt + 4\delta M. \end{aligned}$$

Since α_k and $\beta_k \rightarrow 0$, claim (iii) follows by the arbitrariness of δ . $\square$

Proof of the lower bound (claim (i) of Theorem 2.2). We will use the blow-up method [11, 7], applied to the measures $\mu_k = g_k \mathcal{L}$, where

$$g_k(t) = \mathbb{E} \left(\frac{1}{2} |\dot{v}_k(t)|^2 + V_{\text{per}} \left(\frac{\sqrt{\varepsilon_k} B_t^k + v_k(t)}{\varepsilon_k} \right) \right).$$

It is not restrictive to suppose that there exists the limit

$$\lim_{k \rightarrow +\infty} \mu_k((a, b)) = \lim_{k \rightarrow +\infty} F_{\varepsilon_k}(v_k, \mathcal{F}_t^k, B_t^k) < +\infty.$$

Hence, we can suppose that μ_k converge weakly* to a bounded Radon measure μ on (a, b) . We let $\frac{d\mu}{dt}$ denote the density of its absolutely continuous part with respect to the Lebesgue measure $\mathcal{L}$.

We fix $t_0 \in (a, b)$ a Lebesgue point for $\dot{v}$ such that

$$\frac{d\mu}{dt}(t_0) = \lim_{\rho \rightarrow 0} \frac{1}{\rho} \mu((t_0, t_0 + \rho)),$$

and note that $\mathcal{L}$ -almost every $t_0 \in (a, b)$ satisfies this property. Since $\mu_k((t_0, t_0 + \rho)) \rightarrow \mu((t_0, t_0 + \rho))$ except for at most a countable set of ρ , we can find $\rho_k \rightarrow 0$ such that

$$\frac{d\mu}{dt}(t_0) = \lim_{k \rightarrow +\infty} \frac{1}{\rho_k} \mu_k((t_0, t_0 + \rho_k));$$

that is,

$$\frac{d\mu}{dt}(t_0) = \lim_{k \rightarrow +\infty} \frac{1}{\rho_k} \int_{t_0}^{t_0 + \rho_k} \mathbb{E} \left(\frac{1}{2} |\dot{v}_k(t)|^2 + V_{\text{per}} \left(\frac{\sqrt{\varepsilon_k} B_t^k + v_k(t)}{\varepsilon_k} \right) \right) dt. \quad (3.2)$$

With $\hat{v}$ as in Definition 2.1, we can also assume that t_0 is such that $(\hat{v}(t_0 + \rho) - \hat{v}(t_0))/\rho$ converges strongly in $L^2(\Omega)$ as $\rho \rightarrow 0^+$ to $\hat{\xi} \in L^2(\Omega; \mathbb{R}^d)$, the derivative of $\hat{v}$ in t_0 . We can then assume in addition that the parameter $\rho_k > 0$ is also such that the scaled functions $z_k(t) = \frac{1}{\rho_k} (v_k(t_0 + \rho_k t) - v_k(t_0))$ converge strongly in $L^2((0, 1), L^2(\Omega; \mathbb{R}^d))$ to $\hat{z}(t) = \hat{\xi}t$.

In order to apply Lemma 3.1 we now write

$$\begin{aligned} & \frac{1}{\rho_k} \int_{t_0}^{t_0 + \rho_k} \mathbb{E} \left(\frac{1}{2} |\dot{v}_k(t)|^2 + V_{\text{per}} \left(\frac{\sqrt{\varepsilon_k} B_t^k + v_k(t)}{\varepsilon_k} \right) \right) dt \\ &= \int_0^1 \mathbb{E} \left(\frac{1}{2} |\dot{v}_k(t_0 + \rho_k t)|^2 + V_{\text{per}} \left(\frac{\sqrt{\varepsilon_k} B_{t_0 + \rho_k t}^k + v_k(t_0 + \rho_k t)}{\varepsilon_k} \right) \right) dt \\ &= \int_0^1 \mathbb{E} \left(\frac{1}{2} |\dot{z}_k(t)|^2 + V_{\text{per}} \left(\frac{\sqrt{\varepsilon_k} B_{t_0 + \rho_k t}^k + v_k(t_0) + \rho_k z_k(t)}{\varepsilon_k} \right) \right) dt \\ &= \int_0^1 \mathbb{E} \left(\frac{1}{2} |\dot{z}_k(t)|^2 + V_k(t, z_k(t)) \right) dt, \end{aligned}$$

where

$$V_k(t, x) = V_{\text{per}} \left(\frac{\sqrt{\varepsilon_k} B_{t_0 + \rho_k t}^k + v_k(t_0) + \rho_k x}{\varepsilon_k} \right).$$

By applying Lemma 3.1 with the filtration $\mathcal{G}_t^k := \mathcal{F}_{t_0 + \rho_k t}^k$, we obtain $\tilde{z}_k$ satisfying claims (i)–(iv) of that lemma, with $\hat{z}(t) = \hat{\xi}t$. As a consequence, by (3.2) we have

$$\liminf_{k \rightarrow +\infty} \int_0^1 \mathbb{E} \left(\frac{1}{2} |\dot{z}_k(t)|^2 + V_{\text{per}} \left(\frac{\sqrt{\varepsilon_k} B_{t_0 + \rho_k t}^k + v_k(t_0) + \rho_k \tilde{z}_k(t)}{\varepsilon_k} \right) \right) dt \leq \frac{d\mu}{dt}(t_0) + \eta.$$

Changing variables

$$s = \frac{\rho_k}{\varepsilon_k} t, \quad T_k = \frac{\rho_k}{\varepsilon_k}, \quad w_k(s) = \frac{\rho_k}{\varepsilon_k} \tilde{z}_k \left(\frac{\varepsilon_k}{\rho_k} s \right) + \frac{v_k(t_0)}{\varepsilon_k},$$

we can rewrite the integral as

$$\begin{aligned} & \frac{1}{T_k} \int_0^{T_k} \mathbb{E} \left(\frac{1}{2} |\dot{w}_k(s)|^2 + V_{\text{per}} \left(\frac{B_{t_0 + \varepsilon_k s}^k}{\sqrt{\varepsilon_k}} + w_k(s) \right) \right) ds \\ &= \frac{1}{T_k} \int_0^{T_k} \mathbb{E} \left(\frac{1}{2} |\dot{w}_k(s)|^2 + V_{\text{per}} \left(\frac{B_{t_0 + \varepsilon_k s}^k - B_{t_0}^k}{\sqrt{\varepsilon_k}} + \frac{B_{t_0}^k}{\sqrt{\varepsilon_k}} + w_k(s) \right) \right) ds. \end{aligned} \quad (3.3)$$

We then define

$$\hat{w}_k(s) := w_k(s) + \frac{B_{t_0}^k}{\sqrt{\varepsilon_k}}, \quad \hat{B}_s^k := \frac{B_{t_0+\varepsilon_k s}^k - B_{t_0}^k}{\sqrt{\varepsilon_k}},$$

and rewrite (3.3) as

$$\frac{1}{T_k} \int_0^{T_k} \mathbb{E} \left(\frac{1}{2} |\dot{\hat{w}}_k(s)|^2 + V_{\text{per}}(\hat{B}_s^k + \hat{w}_k(s)) \right) ds. \quad (3.4)$$

If we define the filtration $\hat{\mathcal{F}}_s^k := \mathcal{F}_{t_0+\varepsilon_k s}^k$, to which the Brownian motion $\hat{B}_s^k$ is adapted, then $(\hat{\mathcal{F}}_s^k, \hat{B}_s^k, \hat{w}_k)$ is a test triplet for the problem in (2.3) for $T = T_k$, since $\mathbb{E}(\hat{w}_k(T_k)) - \mathbb{E}(\hat{w}_k(0)) = \mathbb{E}(w_k(T_k) - w_k(0)) = \mathbb{E}(\hat{\xi})T_k = \dot{v}(t_0)T_k$. We then obtain

$$\frac{d\mu}{dt}(t_0) + \eta \geq f_{\text{hom}}(\dot{v}(t_0)),$$

which then holds for $\mathcal{L}$ -almost all $t_0 \in (a, b)$, and finally,

$$\begin{aligned} \lim_{k \rightarrow +\infty} F_{\varepsilon_k}(v_k, \mathcal{F}_t^k, B_t^k) &= \lim_{k \rightarrow +\infty} \mu_k(a, b) \geq \mu(a, b) \\ &\geq \int_a^b \frac{d\mu}{dt}(t) dt \geq \int_a^b f_{\text{hom}}(\dot{v}(t)) dt - \eta(b - a), \end{aligned}$$

which gives the lower bound by the arbitrariness of $\eta > 0$. $\square$

4 Construction of correctors

In order to prove an upper bound it is convenient, for a given ξ , to define B_t and w on the whole half-line $[0, +\infty)$ such that $\sqrt{\varepsilon}B_{t/\varepsilon} + \varepsilon w(t/\varepsilon)$ is an (almost) corrector for ξt . This is done by concatenating independent copies of an almost minimizer for f_{hom} .

Proposition 4.1 (correctors). *With fixed $\xi \in \mathbb{R}^d$, $\eta > 0$, $K > 0$, $T_0 > 0$ there exist $T > T_0$, $r > 0$, a filtration $\mathcal{F}_t^\xi$, a function $w^\xi \in H^1((0, R), L^2(\Omega; \mathbb{R}^d))$ for all $R > 0$, and a Brownian motion B_t^ξ , with both w^ξ and B_t^ξ adapted to $\mathcal{F}_t^\xi$, such that for all $0 \leq s_1 < s_2 < +\infty$*

$$\begin{aligned} &\mathbb{E} \left(\int_{s_1}^{s_2} \left(\frac{1}{2} |\dot{w}^\xi(t)|^2 + V_{\text{per}}(B_t^\xi + w^\xi(t)) \right) dt \right) \\ &\leq (s_2 - s_1) \left(f_{\text{hom}}(\xi) + \eta + \left(3\frac{K}{T} + 2\eta_K \right) \max |V_{\text{per}}| \right) + C(\xi, T, K, \eta), \end{aligned} \quad (4.1)$$

where

$$C(\xi, T, K, \eta) := 2T(f_{\text{hom}}(\xi) + \eta) + 2(4K + T(1 + 2\eta_K)) \max |V_{\text{per}}|,$$

and $\eta_K \rightarrow 0$ as $K \rightarrow +\infty$. Moreover, for all $j \in \mathbb{N}$, we have

$$\mathbb{E}(w^\xi(jT + (j-1)K)) - \mathbb{E}(w^\xi(0)) = jT\xi. \quad (4.2)$$

Proof. We choose $T > T_0$ such that

$$\frac{1}{T} \min \left\{ \int_0^T \mathbb{E} \left(\frac{1}{2} |\dot{w}(t)|^2 + V_{\text{per}}(B_t + w(t)) \right) dt : \mathcal{F}_t, w \in X_0^T(\xi), B_t \right\} < f_{\text{hom}}(\xi) + \frac{\eta}{2}.$$

Recall that $X_0^T(\xi)$ depends on the filtration $\mathcal{F}_t$. Therefore, there exist a filtration $\mathcal{F}_t$, a function $w \in X_0^T(\xi)$, and a Brownian motion B_t adapted to $\mathcal{F}_t$, such that

$$\frac{1}{T} \int_0^T \mathbb{E} \left(\frac{1}{2} |\dot{w}(t)|^2 + V_{\text{per}}(B_t + w(t)) \right) dt < f_{\text{hom}}(\xi) + \frac{\eta}{2}. \quad (4.3)$$

The function w is extended as a constant $w(t)(\omega) = w(T)(\omega)$ for $t > T$.

In order to proceed inductively in the following, we will need independent copies of B_t and w . To that end, we note that, up to enlarging Ω , we can find a sequence of mutually independent filtrations $\mathcal{F}_t^j$, a sequence of Brownian motions B_t^j , adapted to $\mathcal{F}_t^j$, and corresponding w^j defined on the same $(0, T)$ belonging to the corresponding $X_0^{j,T}(\xi)$, equivalent to B_t and w ; that is, such that

$$\int_0^T \mathbb{E}(|\dot{w}^j(t)|^2) dt = \int_0^T \mathbb{E}(|\dot{w}(t)|^2) dt, \quad (4.4)$$

$$\int_0^T \mathbb{E}(V_{\text{per}}(B_t^j + w^j(t))) dt = \int_0^T \mathbb{E}(V_{\text{per}}(B_t + w(t))) dt. \quad (4.5)$$

Indeed, we can replace Ω with $\Omega^{\mathbb{N}}$ with the product probability. After denoting π_j the projection on the j -th factor, we define the filtrations

$$\mathcal{F}_t^j = \{\pi_j^{-1}(A) : A \in \mathcal{F}_t\}.$$

Note that, given distinct indices $j_1, \dots, j_m \in \mathbb{N}$, arbitrary times $t_1, \dots, t_m \in [0, T]$, and a family $E_i \in \mathcal{F}_{t_i}^{j_i}$ we have

$$P(E_1 \cap \dots \cap E_m) = \prod_{i=1}^m P(E_i);$$

that is, $\mathcal{F}_t^j$ are independent. We then set

$$B_t^j = B_t \circ \pi_j, \quad w^j(t) = w(t) \circ \pi_j,$$

both of which are adapted to $\mathcal{F}_t^j$. Note that

$$w^j(t) = w^j(T) \text{ for all } j \text{ and for all } t > T. \quad (4.6)$$

Note that for almost every $t \geq 0$ the laws of $\dot{w}^j(t)$ and $B_t^j + w^j(t)$ coincide with the corresponding laws of $\dot{w}(t)$ and $B_t + w(t)$, from which we obtain equalities (4.4) and (4.5).

For all $j \in \mathbb{N} \cup \infty$ we also introduce the filtration

$$\tilde{\mathcal{F}}_t^j := \sigma\left(\bigcup_{i=1}^j \mathcal{F}_t^i\right).$$

To define B_t^ξ and w^ξ on the half line $(0, +\infty)$ we proceed by defining inductively Brownian motions $\tilde{B}_t^j$ and functions $\tilde{w}_j$, and then setting $B_t^\xi = \tilde{B}_t^j$ and $w^\xi = \tilde{w}_j$ on intervals $(\tau_{j-1}^\xi, \tau_j^\xi)$ between two suitable stopping times.

Let $\tilde{\tau}_0 = 0$. We fix $r > 0$ such that the modulus of continuity λ for V_{per} satisfies $\lambda(r) < \eta/2$. We consider the *stopping time* τ_1 with respect to the filtration $\tilde{\mathcal{F}}_t^2$ defined as the minimal $t \geq T$ such that

$$\text{dist}(B_t^1 + w^1(T) - w^2(0), \mathbb{Z}^d) \leq r. \quad (4.7)$$

Note that in the scalar case $d = 1$ this stopping time can be replaced by the one defined by the condition

$$B_t^1 + w^1(T) - w^2(0) \in \mathbb{Z}, \quad (4.8)$$

avoiding the error term r . The same remark holds for the stopping times τ_j defined in the following.

For fixed $K > 0$ there exists $\eta_K > 0$, with $\eta_K \rightarrow 0$ as $K \rightarrow +\infty$, such that, having set $\Omega_K^1 = \{\tau_1 > T + K\}$, we have $P(\Omega_K^1) < \eta_K$ (see Proposition 9.2). In other words we have $\tau_1 \leq T + K$ on $\Omega \setminus \Omega_K^1$.

Let $\tilde{\tau}_1 = \tau_1 \wedge (T + K)$, so that $T \leq \tilde{\tau}_1 \leq T + K$. We set

$$\tilde{B}_t^1 = B_t^1, \quad \tilde{w}^1(t) = w^1(t),$$

recalling that $w^1(t) = w^1(T)$ for $t \geq T$, so that

$$\mathbb{E}(\tilde{w}^1(T)) = \xi T + \mathbb{E}(w^1(0)). \quad (4.9)$$

Note that

$$\begin{aligned} \int_0^{\tilde{\tau}_1} \left(\frac{1}{2} |\dot{\tilde{w}}^1(t)|^2 + V_{\text{per}}(\tilde{B}_t^1 + \tilde{w}^1(t)) \right) dt &= \int_0^{\tilde{\tau}_1} \left(\frac{1}{2} |\dot{w}^1(t)|^2 + V_{\text{per}}(B_t^1 + w^1(t)) \right) dt \\ &= \int_0^T \left(\frac{1}{2} |\dot{w}^1(t)|^2 + V_{\text{per}}(B_t^1 + w^1(t)) \right) dt + \int_T^{\tilde{\tau}_1} \left(\frac{1}{2} |\dot{w}^1(t)|^2 + V_{\text{per}}(B_t^1 + w^1(t)) \right) dt \end{aligned}$$

Hence, by (4.3), (4.4) and (4.6), we have

$$\begin{aligned}
& \mathbb{E} \left(\int_0^{\tilde{\tau}_1} \left(\frac{1}{2} |\dot{\tilde{w}}^1(t)|^2 + V_{\text{per}}(\tilde{B}_t^1 + \tilde{w}^1(t)) \right) dt \right) \\
&= \int_0^T \mathbb{E} \left(\frac{1}{2} |\dot{w}(t)|^2 + V_{\text{per}}(B_t + w(t)) \right) dt + \mathbb{E} \left(\int_T^{\tilde{\tau}_1} V_{\text{per}}(B_t^1 + w^1(T)) dt \right) \\
&\leq \int_0^T \mathbb{E} \left(\frac{1}{2} |\dot{w}(t)|^2 + V_{\text{per}}(B_t + w(t)) \right) dt + \mathbb{E} \left(\int_T^{\tilde{\tau}_1} |V_{\text{per}}(B_t^1 + w^1(T))| dt \right) \\
&\leq T \left(f_{\text{hom}}(\xi) + \frac{\eta}{2} \right) + K \max V_{\text{per}}.
\end{aligned}$$

In order to illustrate the induction argument we preliminarily deal with $j = 2$. We define $\tilde{w}^2(t)$ and $\tilde{B}_t^2$ as follows. We set

$$\tilde{w}^2(t) = \begin{cases} \tilde{w}^1(t) & \text{for } 0 \leq t \leq \tilde{\tau}_1 \\ \tilde{w}^1(\tilde{\tau}_1) + w^2(t - \tilde{\tau}_1) - w^2(0) & \text{for } t > \tilde{\tau}_1. \end{cases}$$

Observe that $\tilde{w}^2 \in H^1((0, R); L^2(\Omega; \mathbb{R}^d))$ for every $R > 0$, and that for every $t > 0$ the function $w^2(t)$ is $\tilde{\mathcal{F}}_t^2$ -measurable. Note moreover that, since $\tilde{w}^1(\tilde{\tau}_1) = w^1(T)$,

$$\tilde{w}^2(t) = \tilde{w}^2(T + \tilde{\tau}_1) = w^1(T) + w^2(T) - w^2(0) \quad \text{if } t \geq T + \tilde{\tau}_1; \quad (4.10)$$

in particular,

$$\tilde{w}^2(t) = \tilde{w}^2(2T + K) = w^1(T) + w^2(T) - w^2(0) \quad \text{if } t \geq 2T + K \quad (4.11)$$

since $\tilde{w}^1(\tilde{\tau}_1) = w^1(T)$ and $w^2(2T + K - \tilde{\tau}_1) = w^2(T)$, recalling that $2T + K - \tilde{\tau}_1 \geq T$.

We also set

$$\tilde{B}_t^2 = \begin{cases} \tilde{B}_t^1 & \text{for } 0 \leq t \leq \tilde{\tau}_1 \\ B_{t-\tilde{\tau}_1}^2 + \tilde{B}_{\tilde{\tau}_1}^1 & \text{for } t > \tilde{\tau}_1. \end{cases}$$

By the strong Markov property this formula defines a Brownian motion on $[0, +\infty)$ adapted to the filtration $\tilde{\mathcal{F}}_t^2$ (see Proposition 9.1). This argument will be used iteratively in the following.

We define the stopping time τ_2 with respect to the filtration $\tilde{\mathcal{F}}_t^3$ as the minimal $t \geq 2T + K$ such that

$$\text{dist}(\tilde{B}_t^2 + \tilde{w}^2(t) - w^3(0), \mathbb{Z}^d) \leq r,$$

or, equivalently, such that

$$\text{dist}(\tilde{B}_t^2 + \tilde{w}^2(\tilde{\tau}_1 + T) - w^3(0), \mathbb{Z}^d) \leq r,$$

recalling that $w^2(t) = w^2(T)$ for $t \geq T$. We then set $\tilde{\tau}_2 = \tau_2 \wedge (2T + 2K)$, so that $2T + K \leq \tilde{\tau}_2 \leq 2T + 2K$. Note that, having set $\Omega_K^2 = \{\tilde{\tau}_2 \neq \tau_2\}$, we still have $P(\Omega_K^2) \leq \eta_K$.

Using the definitions of $\tilde{B}_t^2$ and $\tilde{w}^2$ we have

$$\begin{aligned}
& \int_0^{\tilde{\tau}_2} \left(\frac{1}{2} |\dot{\tilde{w}}^2(t)|^2 + V_{\text{per}}(\tilde{B}_t^2 + \tilde{w}^2(t)) \right) dt \\
&= \int_0^{\tilde{\tau}_1} \left(\frac{1}{2} |\dot{\tilde{w}}^2(t)|^2 + V_{\text{per}}(\tilde{B}_t^2 + \tilde{w}^2(t)) \right) dt + \int_{\tilde{\tau}_1}^{\tilde{\tau}_2} \left(\frac{1}{2} |\dot{\tilde{w}}^2(t)|^2 + V_{\text{per}}(\tilde{B}_t^2 + \tilde{w}^2(t)) \right) dt \\
&= \int_0^{\tilde{\tau}_1} \left(\frac{1}{2} |\dot{\tilde{w}}^1(t)|^2 + V_{\text{per}}(\tilde{B}_t^1 + \tilde{w}^1(t)) \right) dt \\
&\quad + \int_{\tilde{\tau}_1}^{\tilde{\tau}_1+T} \left(\frac{1}{2} |\dot{w}^2(t - \tilde{\tau}_1)|^2 + V_{\text{per}}(B_{t-\tilde{\tau}_1}^2 + \tilde{B}_{\tilde{\tau}_1}^1 + \tilde{w}^1(\tilde{\tau}_1) + w^2(t - \tilde{\tau}_1) - w^2(0)) \right) dt \\
&\quad + \int_{\tilde{\tau}_1+T}^{\tilde{\tau}_2} \left(\frac{1}{2} |\dot{\tilde{w}}^2(t)|^2 + V_{\text{per}}(\tilde{B}_t^2 + \tilde{w}^2(t)) \right) dt \\
&= \int_0^{\tilde{\tau}_1} \left(\frac{1}{2} |\dot{\tilde{w}}^1(t)|^2 + V_{\text{per}}(\tilde{B}_t^1 + \tilde{w}^1(t)) \right) dt \\
&\quad + \int_0^T \left(\frac{1}{2} |\dot{w}^2(t)|^2 + V_{\text{per}}(B_t^2 + \tilde{B}_{\tilde{\tau}_1}^1 + \tilde{w}^1(\tilde{\tau}_1) + w^2(t) - w^2(0)) \right) dt \\
&\quad + \int_{\tilde{\tau}_1+T}^{\tilde{\tau}_2} V_{\text{per}}(\tilde{B}_t^2 + \tilde{w}^2(t)) dt.
\end{aligned}$$

On $\Omega \setminus \Omega_K^1$ we then have

$$\begin{aligned}
& \int_0^T \left(\frac{1}{2} |\dot{w}^2(t)|^2 + V_{\text{per}}(B_t^2 + \tilde{B}_{\tilde{\tau}_1}^1 + \tilde{w}^1(\tilde{\tau}_1) + w^2(t) - w^2(0)) \right) dt \\
&\leq \int_0^T \left(\frac{1}{2} |\dot{w}^2(t)|^2 + V_{\text{per}}(B_t^2 + w^2(t)) \right) dt + \lambda(r)T.
\end{aligned}$$

Note that in dimension 1 we can omit this error term.

On Ω_K^1

$$\begin{aligned}
& \int_0^T \left(\frac{1}{2} |\dot{w}^2(t)|^2 + V_{\text{per}}(B_t^2 + \tilde{B}_{\tilde{\tau}_1}^1 + \tilde{w}^1(\tilde{\tau}_1) + w^2(t) - w^2(0)) \right) dt \\
&= \int_0^T \left(\frac{1}{2} |\dot{w}^2(t)|^2 + V_{\text{per}}(B_t^2 + w^2(t)) \right) dt \\
&\quad + \int_0^T \left(V_{\text{per}}(B_t^2 + \tilde{B}_{\tilde{\tau}_1}^1 + \tilde{w}^1(\tilde{\tau}_1) + w^2(t) - w^2(0)) - V_{\text{per}}(B_t^2 + w^2(t)) \right) dt \\
&\leq \int_0^T \left(\frac{1}{2} |\dot{w}^2(t)|^2 + V_{\text{per}}(B_t^2 + w^2(t)) \right) dt + 2T \max |V_{\text{per}}|.
\end{aligned}$$

Hence,

$$\begin{aligned} & \mathbb{E} \left(\int_0^{\tilde{\tau}^2} \left(\frac{1}{2} |\dot{\tilde{w}}^2(t)|^2 + V_{\text{per}}(\tilde{B}_t^2 + \tilde{w}^2(t)) \right) dt \right) \\ & \leq 2T(f_{\text{hom}}(\xi) + \eta) + 4K \max V_{\text{per}} + 2T\eta_K \max |V_{\text{per}}|, \end{aligned}$$

recalling that $\lambda(r) < \eta/2$.

We now prove by induction that for all $j \geq 2$, there exist $\tilde{w}^j$, $\tilde{B}_t^j$, and $\tilde{\tau}^j$ with the following properties:

(i) $\tilde{w}^j \in H^1((0, R); L^2(\Omega; \mathbb{R}^d))$ for every $R > 0$, $w^j(t)$ is $\tilde{\mathcal{F}}_t^j$ -measurable for every $t > 0$,

$$\tilde{w}^j(t) = \tilde{w}^j(jT + (j-1)K) \text{ for } t \geq jT + (j-1)K, \quad (4.12)$$

and

$$\tilde{w}^j(jT + (j-1)K) = \sum_{\ell=1}^j (w^\ell(T) - w^\ell(0)) + w^1(0); \quad (4.13)$$

(ii) $\tilde{B}_t^j$ is Brownian motion adapted to the filtration $\tilde{\mathcal{F}}_t^j$;

(iii) $\tilde{\tau}^j$ is a stopping time with respect to the filtration $\tilde{\mathcal{F}}_t^{j+1}$ such that

$$jT + (j-1)K \leq \tilde{\tau}^j \leq jT + jK, \quad (4.14)$$

and

$$\text{dist}(\tilde{B}_{\tilde{\tau}^j}^j + \tilde{w}^j(\tilde{\tau}^j) - w^{j+1}(0), \mathbb{Z}^d) \leq r \quad (4.15)$$

except for a set Ω_K^j with $P(\Omega_K^j) < \eta_K$;

(iv) we have

$$\begin{aligned} & \mathbb{E} \left(\int_{\tilde{\tau}^{j-1}}^{\tilde{\tau}^j} \left(\frac{1}{2} |\dot{\tilde{w}}^j(t)|^2 + V_{\text{per}}(\tilde{B}_t^j + \tilde{w}^j(t)) \right) dt \right) \\ & \leq T(f_{\text{hom}}(\xi) + \eta) + 2K \max V_{\text{per}} + 2T\eta_K \max |V_{\text{per}}|. \end{aligned} \quad (4.16)$$

The construction above guarantees that these properties hold for $j = 2$. Suppose that we have constructed $\tilde{w}^j$, $\tilde{B}_t^j$ and $\tilde{\tau}^j$ satisfying (i)–(iv) above. We now define $\tilde{w}^{j+1}$, $\tilde{B}_t^{j+1}$ and $\tilde{\tau}^{j+1}$ satisfying (i)–(iv) above with $j+1$ in the place of j .

(i) For every $t > 0$ we define $\tilde{w}^{j+1}(t)$ as

$$\tilde{w}^{j+1}(t) = \begin{cases} \tilde{w}^j(t) & \text{for } 0 \leq t \leq \tilde{\tau}^j \\ \tilde{w}^j(\tilde{\tau}^j) + w^{j+1}(t - \tilde{\tau}^j) - w^{j+1}(0) & \text{for } t > \tilde{\tau}^j. \end{cases}$$

We observe that $\tilde{w}^{j+1} \in H^1((0, R); L^2(\Omega; \mathbb{R}^d))$ for every $R > 0$, and that for every $t > 0$ the function $\omega \mapsto w^{j+1}(t)$ is $\tilde{\mathcal{F}}_t^{j+1}$ -measurable. Moreover, by (4.14) and (4.12),

$$\begin{aligned}\tilde{w}^{j+1}((j+1)T + jK) &= \tilde{w}^j(\tilde{\tau}_j) + w^{j+1}((j+1)T + jK - \tilde{\tau}_j) - w^{j+1}(0) \\ &= \tilde{w}^j(jT + (j-1)K) + w^{j+1}(T) - w^{j+1}(0) \\ &= \sum_{\ell=1}^j (w^\ell(T) - w^\ell(0)) + w^1(0) + w^{j+1}(T) - w^{j+1}(0),\end{aligned}$$

and claim (i) is proved.

(ii) We set

$$\tilde{B}_t^{j+1} = \begin{cases} \tilde{B}_t^j & \text{for } 0 \leq t \leq \tilde{\tau}_j \\ B_{t-\tilde{\tau}_j}^{j+1} + \tilde{B}_{\tilde{\tau}_j}^j & \text{for } t > \tilde{\tau}_j. \end{cases}$$

By the strong Markov property this formula defines a Brownian motion on $[0, +\infty)$ adapted to the filtration $\tilde{\mathcal{F}}_t^{j+1}$.

(iii) We define the stopping time τ_{j+1} with respect to the filtration $\tilde{\mathcal{F}}_t^{j+2}$ as the minimal $t \geq (j+1)T + jK$ such that

$$\text{dist}(\tilde{B}_t^{j+1} + \tilde{w}^{j+1}(t) - w^{j+2}(0), \mathbb{Z}^d) \leq r,$$

or, equivalently, such that

$$\text{dist}(\tilde{B}_t^{j+1} + \tilde{w}^{j+1}(\tilde{\tau}_j + T) - w^{j+2}(0), \mathbb{Z}^d) \leq r,$$

recalling that $w^{j+1}(t) = w^{j+1}(T)$ for $t \geq T$. Then, we set

$$\tilde{\tau}_{j+1} = \tau_{j+1} \wedge ((j+1)T + (j+1)K).$$

Note that, having set $\Omega_K^{j+1} = \{\tilde{\tau}_{j+1} \neq \tau_{j+1}\}$, we still have $P(\Omega_K^{j+1}) < \eta_K$.

(iv) We finally estimate

$$\mathbb{E} \left(\int_{\tilde{\tau}_j}^{\tilde{\tau}_{j+1}} \left(\frac{1}{2} |\dot{\tilde{w}}^{j+1}(t)|^2 + V_{\text{per}}(\tilde{B}_t^{j+1} + \tilde{w}^{j+1}(t)) \right) dt \right).$$

To that end, we note that

$$\begin{aligned}& \int_{\tilde{\tau}_j}^{\tilde{\tau}_{j+1}} \left(\frac{1}{2} |\dot{\tilde{w}}^{j+1}(t)|^2 + V_{\text{per}}(\tilde{B}_t^{j+1} + \tilde{w}^{j+1}(t)) \right) dt \\ &= \int_{\tilde{\tau}_j}^{\tilde{\tau}_j + T} \left(\frac{1}{2} |\dot{\tilde{w}}^{j+1}(t)|^2 + V_{\text{per}}(\tilde{B}_t^{j+1} + \tilde{w}^{j+1}(t)) \right) dt \\ &+ \int_{\tilde{\tau}_j + T}^{\tilde{\tau}_{j+1}} \left(\frac{1}{2} |\dot{\tilde{w}}^{j+1}(t)|^2 + V_{\text{per}}(\tilde{B}_t^{j+1} + \tilde{w}^{j+1}(t)) \right) dt,\end{aligned}$$

and that

$$\begin{aligned}
& \int_{\tilde{\tau}_j}^{\tilde{\tau}_j+T} \left(\frac{1}{2} |\dot{\tilde{w}}^{j+1}(t)|^2 + V_{\text{per}}(\tilde{B}_t^{j+1} + \tilde{w}^{j+1}(t)) \right) dt \\
&= \int_{\tilde{\tau}_j}^{\tilde{\tau}_j+T} \left(\frac{1}{2} |\dot{w}^j(t - \tilde{\tau}_j)|^2 \right. \\
&\quad \left. + V_{\text{per}}(B_{t-\tilde{\tau}_j}^{j+1} + \tilde{B}_{\tilde{\tau}_j}^j + \tilde{w}^j(\tilde{\tau}_j) + w^{j+1}(t - \tilde{\tau}_j) - w^{j+1}(0)) \right) dt \\
&= \int_0^T \left(\frac{1}{2} |\dot{w}^{j+1}(t)|^2 + V_{\text{per}}(B_t^{j+1} + \tilde{B}_{\tilde{\tau}_j}^j + \tilde{w}^j(\tilde{\tau}_j) + w^{j+1}(t) - w^{j+1}(0)) \right) dt.
\end{aligned}$$

On $\Omega \setminus \Omega_K^j$ by the inductive hypothesis (iii) we have

$$\begin{aligned}
& \int_0^T \left(\frac{1}{2} |\dot{w}^{j+1}(t)|^2 + V_{\text{per}}(B_t^{j+1} + \tilde{B}_{\tilde{\tau}_j}^j + \tilde{w}^j(\tilde{\tau}_j) + w^{j+1}(t) - w^{j+1}(0)) \right) dt \\
&\leq \int_0^T \left(\frac{1}{2} |\dot{w}^{j+1}(t)|^2 + V_{\text{per}}(B_t^{j+1} + w^{j+1}(t)) \right) dt + \lambda(r)T,
\end{aligned}$$

while on Ω_K^j we write

$$\begin{aligned}
& \int_0^T \left(\frac{1}{2} |\dot{w}^{j+1}(t)|^2 + V_{\text{per}}(B_t^{j+1} + \tilde{B}_{\tilde{\tau}_j}^j + \tilde{w}^j(\tilde{\tau}_j) + w^{j+1}(t) - w^{j+1}(0)) \right) dt \\
&= \int_0^T \left(\frac{1}{2} |\dot{w}^{j+1}(t)|^2 + V_{\text{per}}(B_t^{j+1} + w^{j+1}(t)) \right) dt \\
&+ \int_0^T \left(V_{\text{per}}(B_t^{j+1} + \tilde{B}_{\tilde{\tau}_j}^j + \tilde{w}^j(\tilde{\tau}_j) + w^{j+1}(t) - w^{j+1}(0)) - V_{\text{per}}(B_t^{j+1} + w^{j+1}(t)) \right) dt.
\end{aligned}$$

Hence, recalling that $\lambda(r) < \eta/2$,

$$\begin{aligned}
& \mathbb{E} \left(\int_{\tilde{\tau}_j}^{\tilde{\tau}_j+T} \left(\frac{1}{2} |\dot{\tilde{w}}^{j+1}(t)|^2 + V_{\text{per}}(\tilde{B}_t^{j+1} + \tilde{w}^{j+1}(t)) \right) dt \right) \\
&\leq T(f_{\text{hom}}(\xi) + \eta) + T2\eta_K \max |V_{\text{per}}|.
\end{aligned} \tag{4.17}$$

Furthermore, we have

$$\begin{aligned}
\int_{\tilde{\tau}_j+T}^{\tilde{\tau}_j+1} \left(\frac{1}{2} |\dot{\tilde{w}}^{j+1}(t)|^2 + V_{\text{per}}(\tilde{B}_t^{j+1} + \tilde{w}^{j+1}(t)) \right) dt &= \int_{\tilde{\tau}_j+T}^{\tilde{\tau}_j+1} V_{\text{per}}(\tilde{B}_t^{j+1} + \tilde{w}^{j+1}(t)) dt \\
&\leq 2K \max |V_{\text{per}}|.
\end{aligned} \tag{4.18}$$

From (4.17) and (4.18) we obtain claim (iv) with j replaced by $j+1$.

We finally define $\mathcal{F}_t^\xi = \tilde{\mathcal{F}}_t^\infty$, B_t^ξ and w^ξ on $(0, +\infty)$ as

$$B_t^\xi = \tilde{B}_t^j, \quad w^\xi(t) = \tilde{w}^j(t) \text{ for } t \leq jT. \quad (4.19)$$

Since $\tilde{\tau}_j \geq jT$ and we have $\tilde{B}_t^j = \tilde{B}_t^k$ and $\tilde{w}^j = \tilde{w}^k$ on $[0, \tilde{\tau}_j]$ for $k \geq j$, this is a good definition. We observe that B_t^ξ is a Brownian motion adapted to the filtration $\mathcal{F}_t^\xi$, and $w^\xi \in H^1((0, R); L^2(\Omega; \mathbb{R}^d))$ for every $R > 0$, and that for every $t > 0$ the function $w^\xi(t)$ is $\mathcal{F}_t^\xi$ -measurable. Eventually, (4.1) follows from (4.14) and (4.16) by choosing r small enough, and estimating the integral on (s_1, s_2) with the integral between $(\tilde{\tau}_{j_1}, \tilde{\tau}_{j_2})$, where $\tilde{\tau}_{j_1}$ is the largest stopping time not greater than s_1 and $\tilde{\tau}_{j_2}$ is the smallest stopping time not smaller than s_2 . $\square$

Remark 4.2 (correctors in the one-dimensional case). If $d = 1$, as remarked in the proof, taking into account condition (4.8) and its iterations at every j , stopping times $\tilde{\tau}_j$ can be assumed to satisfy

$$jT + (j-1)K \leq \tilde{\tau}_j \leq jT + jK,$$

and

$$B_{\tilde{\tau}_j}^\xi + w^\xi(\tilde{\tau}_j) - w^{j+1}(0) \in \mathbb{Z}^d$$

except for a set Ω_K^j with $P(\Omega_K^j) < \eta_K$, avoiding the error term r .

5 Upper bound

We now prove claim (ii) of Theorem 2.2. We begin with the case of an affine target function; that is, $v(t) := \xi t + x$, for $t \in (a, b)$, with $x, \xi \in \mathbb{R}^d$.

With fixed $\eta > 0$ and K we define the correctors $\mathcal{F}_t^\xi$, B_t^ξ and w^ξ as in Proposition 4.1. We then define v_ε and a new Brownian motion B_t^ε on $(0, +\infty)$ by

$$v_\varepsilon(t) = \varepsilon w^\xi\left(\frac{t}{\varepsilon}\right) + \varepsilon \left\lfloor \frac{x}{\varepsilon} \right\rfloor, \quad \text{and} \quad B_t^\varepsilon = \sqrt{\varepsilon} B_{\frac{t}{\varepsilon}}^\xi, \quad (5.1)$$

where for every $y \in \mathbb{R}^d$ the symbol $\lfloor y \rfloor$ denotes the vector whose components are the integer parts of the components of y .

We observe that B_t^ε is a Brownian motion adapted to the new filtration $\mathcal{F}_t^\varepsilon = \mathcal{F}_{t/\varepsilon}^\xi$, $v_\varepsilon \in H^1((a, b); L^2(\Omega; \mathbb{R}^d))$, and $v_\varepsilon(t)$ is $\mathcal{F}_t^\varepsilon$ -measurable for every $t \in (a, b)$.

Proposition 5.1. *There exists a constant $M > 0$, independent of K and T , such that*

$$\mathbb{E}\left(\int_a^b |\dot{v}_\varepsilon(t)|^2 dt\right) \leq M(b-a)(|\xi|^2 + 1)$$

for every $0 \leq a < b < +\infty$ and $0 < \varepsilon < (b-a)/(3T + 3K)$.

Proof. With fixed $0 \leq a < b < +\infty$, and $0 < \varepsilon < (b-a)/(3T+3K)$, let j_a be the largest integer such that $\varepsilon(j_a T + (j_a - 1)K) \leq a$, and j_b be the least integer such that $\varepsilon(j_b T + (j_b - 1)K) \geq b$. Then, after a change of variables, we have

$$\begin{aligned} \mathbb{E}\left(\int_a^b |\dot{v}_\varepsilon(t)|^2 dt\right) &\leq \varepsilon \mathbb{E}\left(\sum_{\ell=j_a}^{j_b-1} \int_{\tilde{\tau}_\ell}^{\tilde{\tau}_\ell+T} |\dot{w}^\ell(t)|^2 dt\right) = \varepsilon \mathbb{E}\left(\sum_{\ell=j_a}^{j_b-1} \int_0^T |\dot{w}^\ell(t)|^2 dt\right) \\ &= \varepsilon(j_b - j_a) \mathbb{E}\left(\int_0^T |\dot{w}(t)|^2 dt\right) \leq \left(\frac{b-a}{T+K} + 2\varepsilon\right) \mathbb{E}\left(\int_0^T |\dot{w}(t)|^2 dt\right) \\ &\leq 2\left(\frac{b-a}{T+K} + 2\varepsilon\right) \left(\int_0^T \mathbb{E}\left(\frac{1}{2}|\dot{w}(t)|^2 + V_{\text{per}}(B_t + w(t))\right) dt + T \max |V_{\text{per}}|\right) \\ &\leq 4(b-a)(|\xi|^2 + \eta + 2 \max |V_{\text{per}}|), \end{aligned}$$

using that $f_{\text{hom}}(\xi) \leq \frac{1}{2}|\xi|^2 + \max |V_{\text{per}}|$ and that $\varepsilon \leq (b-a)/(3T+3K)$. The claim is obtained by sending η to 0, and choosing e.g. $M = 4(1 + 2 \max |V_{\text{per}}|)$. $\square$

Proposition 5.2. *Let $0 \leq a < b < +\infty$. For all sequences $\varepsilon_k \rightarrow 0$, the sequence v_{ε_k} in (5.1) converges in (a, b) in the sense of Definition 2.1 to the function given by $v_{\xi, T, K}$ given by*

$$v_{\xi, T, K}(t) = x + \frac{T}{T+K} t \xi.$$

Proof. Since $\varepsilon[\frac{x}{\varepsilon}]$ converges to x in $\mathbb{R}^d$, it suffices to consider the case $x = 0$. By Proposition 5.1, up to subsequences, we can suppose that $v_{\varepsilon_k} \rightharpoonup \hat{v}$ weakly in $H^1((a, b); L^2(\Omega; \mathbb{R}^d))$. To conclude we have only to prove that $\mathbb{E}(\hat{v}(t)) = v_{\xi, T, K}(t)$.

We note that for all $j \in \mathbb{N}$, setting $t_j^k := \varepsilon_k(jT + (j-1)K)$, by (4.13) we have

$$v_{\varepsilon_k}(t_j^k) = \varepsilon_k w^1(0) + \varepsilon_k \sum_{\ell=1}^j (w^\ell(T) - w^\ell(0)),$$

so that, since $w^\ell \in X_0^{\ell, T}(\xi)$,

$$\begin{aligned} \mathbb{E}(v_{\varepsilon_k}(t_j^k)) &= \varepsilon_k \mathbb{E}(w^1(0)) + \varepsilon_k \sum_{\ell=1}^j \mathbb{E}(w^\ell(T) - w^\ell(0)) \\ &= \varepsilon_k \mathbb{E}(w(0)) + \varepsilon_k j \mathbb{E}(w(T) - w(0)) \end{aligned}$$

and, recalling that $w \in X_0^T(\xi)$, we have

$$\mathbb{E}(v_{\varepsilon_k}(t_j^k)) = \varepsilon_k \mathbb{E}(w^1(0)) + \varepsilon_k j T \xi = \varepsilon_k \mathbb{E}(w^1(0)) + v_{\xi, T, K}(t_j^k) + \varepsilon_k \frac{K}{T+K}.$$

Since for all $t \in (a, b)$ and for all k there exists t_j^k such that $|t - t_j^k| \leq \varepsilon_k(T + K)$, by the equi-Hölder continuity of $t \mapsto \mathbb{E}(v_{\varepsilon_k}(t))$ (see Proposition 5.1) we have

$$|\mathbb{E}(v_{\varepsilon_k}(t)) - \mathbb{E}(v_{\varepsilon_k}(t_j^k))| \leq \sqrt{M\varepsilon_k(T + K)(|\xi|^2 + 1)}$$

We then obtain the claim as $k \rightarrow +\infty$. $\square$

Proposition 5.3. *With fixed $0 \leq a < b$, we have*

$$\limsup_{\varepsilon \rightarrow 0} F_\varepsilon(v_\varepsilon, \mathcal{F}_t^\varepsilon, B_t^\varepsilon; (a, b)) \leq (b - a) \left(f_{\text{hom}}(\xi) + \eta + \left(2\frac{K}{T} + 2\eta_K \right) \max |V_{\text{per}}| \right).$$

Proof. Using a change of variables and the periodicity of V_{per} , by (4.1) with $s_1 = a/\varepsilon$ and $s_2 = b/\varepsilon$ we obtain

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} F_\varepsilon(v_\varepsilon, \mathcal{F}_t^\varepsilon, B_t^\varepsilon; (a, b)) &= \limsup_{\varepsilon \rightarrow 0} \varepsilon \mathbb{E} \left(\int_{a/\varepsilon}^{b/\varepsilon} \left(\frac{1}{2} |\dot{w}^\xi(t)|^2 + V_{\text{per}}(B_t^\xi + w^\xi(t)) \right) dt \right) \\ &\leq (b - a) \left(f_{\text{hom}}(\xi) + \eta + \left(3\frac{K}{T} + 2\eta_K \right) \max |V_{\text{per}}| \right), \end{aligned}$$

and the claim $\square$

Corollary 5.4 (recovery sequence for an affine target function). *For all ξ and x and for all $\delta > 0$ there exists a sequence $\bar{v}_\varepsilon \rightarrow x + \xi t$ and such that*

$$\limsup_{\varepsilon \rightarrow 0} F_\varepsilon(\bar{v}_\varepsilon, \mathcal{F}_t^\varepsilon, B_t^\varepsilon; (a, b)) \leq f_{\text{hom}}(\xi)(b - a).$$

Proof. Let $\eta = \delta/4$, and let K be large enough so that $2\eta_K \max |V_{\text{per}}| \leq \delta/4$. Let T_j be a diverging sequence such that

$$f_{\text{hom}}(\xi) = \lim_{j \rightarrow +\infty} \frac{1}{T_j} \inf \left\{ \int_0^{T_j} \mathbb{E} \left(\frac{1}{2} |\dot{w}(t)|^2 + V_{\text{per}}(B_t^j + w(t)) \right) dt : w \in X_0^{T_j}(\xi), B_t^j, \mathcal{F}_t \right\}.$$

By Proposition 5.2 for all fixed j , the sequence v_ε^j defined therein with $T = T_j$ converges in the sense of Definition 2.1 to the function given by $v_{\xi, T, K}$ given by

$$v_{\xi, K}^j(t) = x + \frac{T_j}{T_j + K} t \xi.$$

Moreover, letting $\mathcal{F}_t^{j, \varepsilon}$ and $B_t^{j, \varepsilon}$ be the corresponding scaled filtration and Brownian motion, by Proposition 5.3 we have

$$\limsup_{\varepsilon \rightarrow 0} F_\varepsilon(v_\varepsilon^j, \mathcal{F}_t^{j, \varepsilon}, B_t^{j, \varepsilon}; (a, b)) \leq \left(f_{\text{hom}}(\xi) + \frac{\delta}{2} + 2\frac{K}{T_j} \max |V_{\text{per}}| \right) (b - a).$$

The conclusion follows by letting $j \rightarrow +\infty$ and using a diagonal argument. $\square$

Proposition 5.5. *With fixed $0 \leq a < b$, and v piecewise affine, there exists v_ε converging to v in the sense of Definition 2.1, filtrations $\mathcal{F}_t^\varepsilon$, and Brownian motions B_t^ε such that*

$$\limsup_{\varepsilon \rightarrow 0} F_\varepsilon(v_\varepsilon, \mathcal{F}_t^\varepsilon, B_t^\varepsilon; (a, b)) \leq \int_a^b f_{\text{hom}}(v'(t)) dt.$$

Proof. We define $a = a_0 < a_1 < \dots < a_m = b$ and $\xi_1, \dots, \xi_m$ such that $v'(t) = \xi_i$ in (a_{i-1}, a_i) . It is not restrictive to suppose that $\frac{a_i}{\varepsilon} > \frac{a_{i-1}}{\varepsilon} + K$ for all $i \in \{1, \dots, m\}$. Note that, up to a translation, it is not restrictive to suppose that $v(t) = \xi_1 t$ in the first interval.

With fixed $\eta > 0$, T_0 and K we define $T_i > T_0$ and the correctors $\mathcal{F}_t^{\xi_i}$, $B_t^{\xi_i}$ and w^{ξ_i} as in Proposition 4.1 for $i \in \{1, \dots, m\}$. It is not restrictive to suppose that the filtrations $\mathcal{F}_t^{\xi_i}$ are mutually independent. For all j we define

$$\hat{\mathcal{F}}_t^j = \sigma\left(\bigcup_{i=1}^j \mathcal{F}_t^{\xi_i}\right).$$

With fixed $\varepsilon > 0$ we define w^ε , $\mathcal{F}_t^\varepsilon$, and B_t^ε by a finite induction argument. With fixed $r > 0$ let τ_1^ε be the stopping time with respect to $\hat{\mathcal{F}}_t^2$ defined as the smallest $t \geq a_1/\varepsilon$ such that

$$\text{dist}(B_t^{\xi_1} + w^{\xi_1}(\frac{a_1}{\varepsilon}) - w^{\xi_2}(0), \mathbb{Z}^d) \leq r$$

and set $\tilde{\tau}_1^\varepsilon = \tau_1^\varepsilon \wedge (\frac{a_1}{\varepsilon} + K)$. For $t \geq 0$ we set

$$w_1^\varepsilon(t) = w^{\xi_1}(t) \text{ and } B_t^{\varepsilon,1} = B_t^{\xi_1}.$$

For $t \geq 0$ we then let

$$w_2^\varepsilon(t) = w^{\xi_2}((t - \tilde{\tau}_1^\varepsilon)^+) - w^{\xi_2}(0) + w^{\varepsilon,1}(t \wedge \tilde{\tau}_1^\varepsilon), \text{ and } B_t^{\varepsilon,2} = B_{(t - \tilde{\tau}_1^\varepsilon)^+}^{\xi_2} + B_{t \wedge \tilde{\tau}_1^\varepsilon}^{\varepsilon,1}.$$

and let τ_2^ε be the stopping time with respect to $\hat{\mathcal{F}}_t^3$ defined as the smallest $t \geq a_2/\varepsilon$ such that

$$\text{dist}(B_t^{\varepsilon,2} + w_2^\varepsilon(\frac{a_2}{\varepsilon}) - w^{\xi_3}(0), \mathbb{Z}^d) \leq r$$

and set $\tilde{\tau}_2^\varepsilon = \tau_2^\varepsilon \wedge (\frac{a_2}{\varepsilon} + K)$

This construction can then be iterated, by setting for $t \geq 0$

$$w_i^\varepsilon(t) = w^{\xi_i}((t - \tilde{\tau}_{i-1}^\varepsilon)^+) - w^{\xi_i}(0) + w^{\varepsilon,i-1}(t \wedge \tilde{\tau}_{i-1}^\varepsilon) \text{ and } B_t^{\varepsilon,i} = B_{(t - \tilde{\tau}_{i-1}^\varepsilon)^+}^{\xi_i} + B_{t \wedge \tilde{\tau}_{i-1}^\varepsilon}^{\varepsilon,i-1}.$$

and letting τ_i^ε be the stopping time with respect to $\hat{\mathcal{F}}_t^{i+1}$ defined as the smallest $t \geq a_i/\varepsilon$ such that

$$\text{dist}(B_t^{\varepsilon,i} + w_i^\varepsilon(\frac{a_i}{\varepsilon}) - w^{\xi_{i+1}}(0), \mathbb{Z}^d) \leq r$$

and set $\tilde{\tau}_i^\varepsilon = \tau_i^\varepsilon \wedge (\frac{a_i}{\varepsilon} + K)$. We finally set

$$w^\varepsilon(t) := w_m^\varepsilon(t), \quad B_t^\varepsilon = B_t^{\varepsilon, m}, \quad \text{and } \mathcal{F}_t^\varepsilon = \hat{\mathcal{F}}_t^{i+1}.$$

If we consider $v_\varepsilon(t) = \varepsilon w^\varepsilon(t/\varepsilon)$, then $v_\varepsilon \rightarrow v_{K, \{T_i\}}$, the piecewise affine function with

$$v'_{K, \{T_i\}} = \frac{T_i}{T_i + K} \xi_i$$

on (a_{i-1}, a_i) . After a diagonalization, the same computation as in Proposition 5.3 gives the conclusion. $\square$

Corollary 5.6. *The function $f_{\text{hom}}: \mathbb{R}^d \rightarrow \mathbb{R}$ is convex.*

Proof. With fixed $\xi_1, \xi_2 \in \mathbb{R}^d$, $\lambda \in (0, 1)$, for all $k \in \mathbb{N}$ we consider the piecewise-affine function $u_k: (0, 1) \rightarrow \mathbb{R}^d$, with $u_k(0) = 0$ and whose derivative is the $\frac{1}{k}$ -periodic function equal to ξ_1 on $(0, \lambda/k)$ and ξ_2 on $(\lambda/k, 1/k)$. By Proposition 5.5 for all $k \in \mathbb{N}$ there exists $u_{k, \varepsilon}$ converging to u_k in the sense of Definition 2.1, filtrations $\mathcal{F}_t^{k, \varepsilon}$, and Brownian motions $B_t^{k, \varepsilon}$ such that

$$\limsup_{\varepsilon \rightarrow 0} F_\varepsilon(u_{k, \varepsilon}, \mathcal{F}_t^{k, \varepsilon}, B_t^{k, \varepsilon}; (0, 1)) \leq \lambda f_{\text{hom}}(\xi_1) + (1 - \lambda) f_{\text{hom}}(\xi_2).$$

Note that the right-hand side is independent of $k \in \mathbb{N}$. Since u_k converge weakly in $H^1((0, 1); \mathbb{R}^d)$ to the function $u(t) = \xi t$, with $\xi = \lambda \xi_1 + (1 - \lambda) \xi_2$, by a diagonal argument we can select a sequence $\varepsilon_k \rightarrow 0$ such that the corresponding u_{k, ε_k} converge to u in the sense of Definition 2.1, and

$$\limsup_{k \rightarrow +\infty} F_\varepsilon(u_{k, \varepsilon_k}, \mathcal{F}_t^{k, \varepsilon_k}, B_t^{k, \varepsilon_k}; (0, 1)) \leq \lambda f_{\text{hom}}(\xi_1) + (1 - \lambda) f_{\text{hom}}(\xi_2).$$

Since by Theorem 2.2(i) we have

$$\liminf_{k \rightarrow +\infty} F_\varepsilon(u_{k, \varepsilon_k}, \mathcal{F}_t^{k, \varepsilon_k}, B_t^{k, \varepsilon_k}; (0, 1)) \geq f_{\text{hom}}(\xi)$$

we conclude that $f_{\text{hom}}(\xi) \leq \lambda f_{\text{hom}}(\xi_1) + (1 - \lambda) f_{\text{hom}}(\xi_2)$; that is, the claim. $\square$

Proof of Theorem 2.2(ii). By the convexity and the upper bound of f_{hom} , proved in Corollary 5.6 and (2.4), the functional $u \mapsto \int_a^b f_{\text{hom}}(u') dt$ is strongly continuous in $H^1((a, b); \mathbb{R}^d)$. Hence, the conclusion follows from Proposition 5.5 and the strong density of piecewise-affine functions, using the same diagonal argument as in the proof of Corollary 5.6. $\square$

6 A stability result for Lagrangians with defects

We consider a bounded Borel function $V_0: \mathbb{R}^d \rightarrow [0, +\infty)$ such that $V_0(x) \rightarrow 0$ as $|x| \rightarrow +\infty$, and the perturbed functional

$$\tilde{F}_\varepsilon(v, \mathcal{F}_t, B_t) = \int_a^b \mathbb{E} \left(\frac{1}{2} |\dot{v}(t)|^2 + V_{\text{per}} \left(\frac{\sqrt{\varepsilon} B_t + v(t)}{\varepsilon} \right) + V_0 \left(\frac{\sqrt{\varepsilon} B_t + v(t)}{\varepsilon} \right) \right) dt \quad (6.1)$$

Then the Γ -limit of the functionals $\tilde{F}_\varepsilon$ is the same as that obtained for the periodic case in Theorem 2.2. More precisely, we have the following result.

Theorem 6.1. *Let $\tilde{F}_\varepsilon$ be given by (6.1). Given $\varepsilon_k \rightarrow 0$, we have*

(i) *For every $v \in H^1(a, b)$ and for every sequences $\mathcal{F}_t^k$, B_t^k , and $v_k \rightarrow v$ such that $v_k(t)$ is $\mathcal{F}_t^k$ -measurable, we have*

$$F_{\text{hom}}(v) \leq \liminf_{k \rightarrow +\infty} \tilde{F}_{\varepsilon_k}(v_k, \mathcal{F}_t^k, B_t^k);$$

(ii) *for every $v \in H^1(a, b)$ there exists sequences $\mathcal{F}_t^k$, B_t^k , and $v_k \rightarrow v$, such that $v_k(t)$ is $\mathcal{F}_t^k$ -measurable, such that*

$$F_{\text{hom}}(v) \geq \limsup_{k \rightarrow +\infty} \tilde{F}_{\varepsilon_k}(v_k, \mathcal{F}_t^k, B_t^k).$$

Proof. Since $V_0 \geq 0$ (i) follows from the analogous property in Theorem 2.2.

On order to prove (ii), we fix ε_k , v_k , B_t^k and $\mathcal{F}_t^k$ with v_k converging to some v . For every $s \geq 0$, let μ_s^k be the probability law of $(\sqrt{\varepsilon_k} B_t^k + v_k(s))/\varepsilon_k$; that is, for every Borel set A in $\mathbb{R}$

$$\mu_t^k(A) := P \left(\left\{ \frac{\sqrt{\varepsilon_k} B_t^k + v_k(s)}{\varepsilon_k} \in A \right\} \right).$$

We also introduce the probability measure defined by

$$\mu^k(A) := \frac{1}{b-a} \int_a^b \mu_t^k(A) dt.$$

We now fix $\eta > 0$, and $M \in \mathbb{N}$ such that $M^{-d} \leq \eta$. Since $\mu^k([-ML, ML]^d) \leq 1$, there exists $z_k \in \mathbb{Z}^d \cap [-ML, ML]^d$ such that

$$\mu^k(z_k + [-L, L]^d) \leq \frac{1}{M^d} \leq \eta.$$

We can then use as recovery sequence the same B_t^k and $\mathcal{F}_t^k$, and the function $\tilde{v}_k(t) = v_k(t) - \varepsilon_k z_k$, obtained by translating the function v_k . We note that $\tilde{v}_k$ still converges to v as $k \rightarrow +\infty$ since z_k is bounded, and that

$$V_{\text{per}} \left(\frac{\sqrt{\varepsilon_k} B_t^k + v_k(t)}{\varepsilon_k} - z_k \right) = V_{\text{per}} \left(\frac{\sqrt{\varepsilon_k} B_t^k + v_k(t)}{\varepsilon} \right)$$

by the periodicity of V_{per} . Hence,

$$\tilde{F}_{\varepsilon_k}(\tilde{v}_k, \mathcal{F}_t^k, B_t^k) = F_{\varepsilon_k}(\tilde{v}_k, \mathcal{F}_t^k, B_t^k) + \int_a^b \mathbb{E} \left(V_0 \left(\frac{\sqrt{\varepsilon_k} B_t^k + v_k(t)}{\varepsilon_k} - z_k \right) \right) dt$$

We now observe that

$$\int_a^b \mathbb{E} \left(V_0 \left(\frac{\sqrt{\varepsilon_k} B_t^k + v_k(t)}{\varepsilon_k} - z_k \right) \right) dt \leq (b-a) \eta \sup V_0.$$

Hence, we have

$$F_{\varepsilon_k}(v_k, \mathcal{F}_t^k, B_t^k) + (b-a) \eta \sup V_0 \geq \tilde{F}_{\varepsilon_k}(\tilde{v}_k, \mathcal{F}_t^k, B_t^k),$$

which proves (ii) by choosing as v_k as in the corresponding claim in Theorem 2.2 by the arbitrariness of η . $\square$

7 Convergence in $(0, +\infty)$

We now consider the extension of the Γ -convergence result to some functionals defined on functions on the half line, in view of the applications to steady-state Hamilton-Jacobi equations.

Definition 7.1. *Let $\lambda > 0$ be a given parameter. The space $H_\lambda^1((0, +\infty); \mathbb{R}^d)$ is defined as the space of all functions $u: (0, +\infty) \rightarrow \mathbb{R}^d$ such that $u \in H^1((0, t_0); \mathbb{R}^d)$ for all $t_0 > 0$, and*

$$\int_0^{+\infty} |u'(t)|^2 e^{-\lambda t} dt < +\infty,$$

endowed with the norm

$$\left(|u(0)|^2 + \int_0^{+\infty} |u'(t)|^2 e^{-\lambda t} dt \right)^{1/2}.$$

Similarly, the space $H_\lambda^1((0, +\infty); L^2(\Omega; \mathbb{R}^d))$ is defined as the space of all functions $u: (0, +\infty) \rightarrow L^2(\Omega; \mathbb{R}^d)$, such that $u \in H^1((0, t_0); L^2(\Omega; \mathbb{R}^d))$ for all $t_0 > 0$, and

$$\int_0^{+\infty} \mathbb{E}(|u'(t)|^2) e^{-\lambda t} dt < +\infty,$$

endowed with the norm

$$\left(\mathbb{E}(|u(0)|^2) + \int_0^{+\infty} \mathbb{E}(|u'(t)|^2) e^{-\lambda t} dt \right)^{1/2}.$$

We observe that $H_\lambda^1((0, +\infty); \mathbb{R}^d)$ and $H_\lambda^1((0, +\infty); L^2(\Omega; \mathbb{R}^d))$ are Hilbert spaces.

Definition 7.2 (convergence to a deterministic variable on $(0, +\infty)$). *Let $\lambda > 0$ be fixed. Given a sequence $v_k \in H_\lambda^1((0, +\infty); L^2(\Omega; \mathbb{R}^d))$ and $v \in H_\lambda^1((0, +\infty); \mathbb{R}^d)$ we write $v_k \rightarrow v$ if for all subsequences v_{k_j} , there exist a subsequence, that we still denote by v_{k_j} , and $\hat{v} \in H_\lambda^1((0, +\infty); L^2(\Omega; \mathbb{R}^d))$, possibly depending on the subsequence, such that*

$$v_{k_j} \rightharpoonup \hat{v} \text{ weakly in } H_\lambda^1((0, +\infty); L^2(\Omega; \mathbb{R}^d)) \quad \text{and} \quad \mathbb{E}(\hat{v}(t)) = v(t) \text{ for every } t.$$

7.1 Homogenization on the half line

Given $\lambda > 0$, for all $\varepsilon > 0$, for all $v \in H_\lambda^1((0, +\infty); L^2(\Omega; \mathbb{R}^d))$, $\{\mathcal{F}_t\}_{t \geq 0}$ filtrations such that $v(t)$ is measurable with respect to $\mathcal{F}_t$ for every t , and $\{B_t\}_{t \geq 0}$ Brownian motions adapted to $\{\mathcal{F}_t\}_{t \geq 0}$, we set

$$F_\varepsilon^\lambda(v, \mathcal{F}_t, B_t) = \int_0^{+\infty} \mathbb{E} \left(\frac{1}{2} |\dot{v}(t)|^2 + V_{\text{per}} \left(\frac{\sqrt{\varepsilon} B_t + v(t)}{\varepsilon} \right) \right) e^{-\lambda t} dt. \quad (7.1)$$

For $v \in H_\lambda^1((0, +\infty); \mathbb{R}^d)$, we also define

$$F_{\text{hom}}^\lambda(v) = \int_0^{+\infty} f_{\text{hom}}(\dot{v}(t)) e^{-\lambda t} dt. \quad (7.2)$$

We will prove the following homogenization result, whose stability with respect to perturbations can be proved using the assumptions and the results in Section 6.

Theorem 7.3. *Given $\lambda > 0$, let F_ε^λ and F_{hom}^λ be given by (7.1) and (7.2), respectively. Given $\varepsilon_k \rightarrow 0$, for every $v \in H_\lambda^1((0, +\infty); \mathbb{R}^d)$ we have*

(i) *for every sequence of functions $v_k \in H_\lambda^1((0, +\infty); L^2(\Omega; \mathbb{R}^d))$ with $v_k \rightarrow v$ in the sense of Definition 7.2, for every sequence $\{\mathcal{F}_t^k\}_{t \geq 0}$ of filtrations such that each $v_k(t)$ is $\mathcal{F}_t^k$ -measurable, and for every sequence $\{B_t^k\}_{t \geq 0}$ of Brownian motions adapted to $\{\mathcal{F}_t^k\}_{t \geq 0}$, we have*

$$F_{\text{hom}}^\lambda(v) \leq \liminf_{k \rightarrow +\infty} F_{\varepsilon_k}^\lambda(v_k, \mathcal{F}_t^k, B_t^k);$$

(ii) *there exist a sequence of functions $v_k \in H_\lambda^1((0, +\infty); L^2(\Omega; \mathbb{R}^d))$ with $v_k \rightarrow v$ in the sense of Definition 7.2, a sequence $\{\mathcal{F}_t^k\}_{t \geq 0}$ of filtrations such that each $v_k(t)$ is $\mathcal{F}_t^k$ -measurable, and a sequence $\{B_t^k\}_{t \geq 0}$ of Brownian motions adapted to $\{\mathcal{F}_t^k\}_{t \geq 0}$, such that*

$$F_{\text{hom}}^\lambda(v) \geq \limsup_{k \rightarrow +\infty} F_{\varepsilon_k}^\lambda(v_k, \mathcal{F}_t^k, B_t^k).$$

Proof. Claim (i). Let $\{a_j\}_{j \geq 0}$ be an increasing sequence in $[0, +\infty)$ with $a_0 = 0$ and $a_j \rightarrow +\infty$ as $j \rightarrow +\infty$. Using the notation

$$F_\varepsilon(v, \mathcal{F}_t, B_t; (a, b)) = \int_a^b \mathbb{E} \left(\frac{1}{2} |\dot{v}(t)|^2 + V_{\text{per}} \left(\frac{\sqrt{\varepsilon} B_t + v(t)}{\varepsilon} \right) \right) dt \quad (7.3)$$

and Theorem 2.2, we have

$$\begin{aligned} \liminf_{k \rightarrow +\infty} F_{\varepsilon_k}^\lambda(v_k, \mathcal{F}_t^k, B_t^k) &\geq \sum_{j \geq 1} \liminf_{k \rightarrow +\infty} F_{\varepsilon_k}(v_k, \mathcal{F}_t^k, B_t^k; (a_{j-1}, a_j)) e^{-\lambda a_j} \\ &\geq \sum_{j \geq 1} \int_{a_{j-1}}^{a_j} f_{\text{hom}}(\dot{v}(t)) e^{-\lambda a_j} dt \geq \sum_{j \geq 1} e^{-\lambda(a_j - a_{j-1})} \int_{a_{j-1}}^{a_j} f_{\text{hom}}(\dot{v}(t)) e^{-\lambda t} dt \\ &\geq \sup_{j \geq 1} e^{-\lambda(a_j - a_{j-1})} \int_0^{+\infty} f_{\text{hom}}(\dot{v}(t)) e^{-\lambda t} dt. \end{aligned}$$

Since $\sup_{j \geq 1} e^{-\lambda(a_j - a_{j-1})}$ can be taken arbitrarily close to 1, we have proved the claim.

Claim (ii). By a density argument (see Proposition 4.5 in [6]), it is sufficient to consider a target function $v \in H_\lambda^1((0, +\infty); \mathbb{R}^d)$ such that there exists t_0 such that $v(t) = 0$ if $t \geq t_0$. We fix $t_1 > t_0$. By Theorem 2.2 with $(a, b) = (0, t_1)$, there exist $v_k, \{\mathcal{F}_t^k\}, \{B_t^k\}$ as in Claim (ii) of that theorem. By Lemma 3.1 we can assume that $\mathbb{E}(v_k(0)) = v(0)$ and $\mathbb{E}(v_k(t_1)) = v(t_1) = 0$. The function v_k is then extended by setting $v_k(t) = v_k(t_1)$ for $t \geq t_1$, and, recalling Definition 7.1, we have that $v_k \in H_\lambda^1((0, +\infty); L^2(\Omega))$, and $v_k \rightarrow v$ in the sense of Definition 7.2. Let $N \in \mathbb{N}$, and set $a_j = jt_1/N$. We note that, using the validity of Claim (i) of Theorem 2.2 for $(a, b) = (a_{j-1}, a_j)$, and the validity of Claim (ii) of the same theorem with $(a, b) = (0, t_1)$ we can prove v_k satisfies Claim (ii) of Theorem 2.2 with $(a, b) = (a_{j-1}, a_j)$ for any $j \in \{1, \dots, N\}$. We then have

$$\begin{aligned} &\limsup_{k \rightarrow +\infty} F_{\varepsilon_k}^\lambda(v_k, \mathcal{F}_t^k, B_t^k) \\ &\leq \sum_{j=1}^N \limsup_{k \rightarrow +\infty} F_{\varepsilon_k}(v_k, \mathcal{F}_t^k, B_t^k; (a_{j-1}, a_j)) e^{-\lambda a_{j-1}} + \int_{t_1}^{+\infty} e^{-\lambda t} \max |V_{\text{per}}| dt \\ &\leq \sum_{j=1}^N \int_{a_{j-1}}^{a_j} f_{\text{hom}}(\dot{v}(t)) e^{-\lambda a_{j-1}} dt + e^{-\lambda t_1} \frac{\max |V_{\text{per}}|}{\lambda} \\ &\leq e^{t_1/N} \int_0^{t_1} f_{\text{hom}}(\dot{v}(t)) e^{-\lambda t} dt + e^{-\lambda t_1} \frac{\max |V_{\text{per}}|}{\lambda}. \end{aligned}$$

By the arbitrariness of N and $t_1 > t_0$ we conclude the proof. $\square$

7.2 Convergence of value functions

Given $\lambda > 0$, define

$$U_\varepsilon(x) = \inf \{F_\varepsilon^\lambda(v, \mathcal{F}_t, B_t) : \mathbb{E}(v(0)) = x\}, \quad (7.4)$$

where the infimum is taken over all $v \in H_\lambda^1((0, +\infty); L^2(\Omega; \mathbb{R}^d))$, satisfying the initial condition, $\{\mathcal{F}_t\}_{t \geq 0}$ filtrations such that $v(t)$ is measurable with respect to $\mathcal{F}_t$ for every t , and $\{B_t\}_{t \geq 0}$ Brownian motions adapted to $\{\mathcal{F}_t\}_{t \geq 0}$. Similarly, we define

$$U_{\text{hom}}(x) = \inf \{F_{\text{hom}}^\lambda(v) : v \in H_\lambda^1((0, +\infty); \mathbb{R}^d), v(0) = x\}, \quad (7.5)$$

Theorem 7.4. *Given λ and x we have $U_\varepsilon(x) \rightarrow U_{\text{hom}}(x)$ as $\varepsilon \rightarrow 0$.*

Proof. With fixed ε_k converging to 0, we choose all $v_k \in H_\lambda^1((0, +\infty); L^2(\Omega; \mathbb{R}^d))$, with $\mathbb{E}(v_k(0)) = x$, $\{\mathcal{F}_t^k\}_{t \geq 0}$ filtrations such that $v(t)$ is measurable with respect to $\mathcal{F}_t^k$ for every t , and $\{B_t^k\}_{t \geq 0}$ Brownian motions adapted to $\{\mathcal{F}_t^k\}_{t \geq 0}$ such that

$$F_{\varepsilon_k}^\lambda(v_k, \mathcal{F}_t^k, B_t^k) \leq U_{\varepsilon_k}(x) + \frac{1}{k}.$$

Using the constant function x as competitor in (7.4), from (7.1) we obtain that $U_{\varepsilon_k}(x)$ is bounded uniformly. Again, using (7.1) we then obtain

$$\int_0^{+\infty} \mathbb{E} \left(\frac{1}{2} |\dot{v}_k|^2 + V_{\text{per}} \left(\frac{\sqrt{\varepsilon} B_t^k + v_k}{\varepsilon} \right) \right) e^{-\lambda t} dt \leq C,$$

from which we deduce that v_k is bounded in $H_\lambda^1((0, +\infty); L^2(\Omega; \mathbb{R}^d))$, and hence we can suppose that it converges to a function $\hat{v}$ weakly in $H_\lambda^1((0, +\infty); L^2(\Omega; \mathbb{R}^d))$. This implies that $v_k \rightarrow v$ in the sense of Definition 7.2, where $v(t) := \mathbb{E}(\hat{v}(t))$ for every $t \geq 0$, and $v(0) = \mathbb{E}(\hat{v}(0)) = x$. From Theorem 7.3(i) we have that

$$U_{\text{hom}}(x) \leq F_{\text{hom}}^\lambda(v) \leq \liminf_{k \rightarrow +\infty} F_{\varepsilon_k}^\lambda(v_k, \mathcal{F}_t^k, B_t^k) \leq \liminf_{k \rightarrow +\infty} U_{\varepsilon_k}(x). \quad (7.6)$$

To prove

$$U_{\text{hom}}(x) \geq \limsup_{k \rightarrow +\infty} U_{\varepsilon_k}(x), \quad (7.7)$$

with fixed $\eta > 0$ we consider $v \in H_\lambda^1((0, +\infty); \mathbb{R}^d)$, with $v(0) = x$, such that

$$U_{\text{hom}}(x) + \eta \geq F_{\text{hom}}^\lambda(v).$$

Let v_k , $\{\mathcal{F}_t^k\}_{t \geq 0}$, and $\{B_t^k\}_{t \geq 0}$ be as in Theorem 7.3(ii). By Lemma 3.1 we can suppose that $\mathbb{E}(v_k(0)) = x$ for every k . Thus we have

$$U_{\text{hom}}(x) + \eta \geq F_{\text{hom}}^\lambda(v) \geq \limsup_{k \rightarrow +\infty} F_{\varepsilon_k}^\lambda(v_k, \mathcal{F}_t^k, B_t^k) \geq \limsup_{k \rightarrow +\infty} U_{\varepsilon_k}(x).$$

By the arbitrariness of $\eta > 0$ we get (7.7), which, together with (7.6), proves the claim. $\square$

Remark 7.5 (addition of continuous perturbations). For $v \in H_\lambda^1((0, +\infty); L^2(\Omega; \mathbb{R}^d))$ and $g \in H_\lambda^1((0, +\infty); \mathbb{R}^d)$, we set

$$G^\lambda(v) = \int_0^{+\infty} \mathbb{E}(\langle g(t), \dot{v}(t) \rangle) e^{-\lambda t} dt. \quad (7.8)$$

We note that if $v_k \rightarrow v$ then

$$\lim_{k \rightarrow +\infty} G^\lambda(v_k) = \int_0^{+\infty} \langle g(t), \dot{v}(t) \rangle e^{-\lambda t} dt.$$

Indeed, if $v_k \rightarrow \hat{v}$ in $H_\lambda^1((0, +\infty); L^2(\Omega; \mathbb{R}^d))$ then $v_k \rightarrow \hat{v}$ in $L_\lambda^2((0, +\infty); L^2(\Omega; \mathbb{R}^d))$, so that $G^\lambda(v_k) \rightarrow G^\lambda(\hat{v})$, from which we conclude since

$$G^\lambda(\hat{v}) = \int_0^{+\infty} \langle g(t), \dot{v}(t) \rangle e^{-\lambda t} dt.$$

Given $\lambda > 0$, define

$$U_\varepsilon(x) = \inf_{\mathcal{F}_t, B_t} \{F_\varepsilon^\lambda(v, \mathcal{F}_t, B_t) + G^\lambda(v) : \mathbb{E}(v(0)) = x\}, \quad (7.9)$$

where the infimum is taken over all $v \in H_\lambda^1((0, +\infty); L^2(\Omega; \mathbb{R}^d))$, such that $v(t)$ is measurable with respect to $\mathcal{F}_t$ for every t and satisfies the initial condition, as well as over all filtrations $\{\mathcal{F}_t\}_{t \geq 0}$ of $\mathcal{F}$ and over all Brownian motions $\{B_t\}_{t \geq 0}$ adapted to $\{\mathcal{F}_t\}_{t \geq 0}$. Similarly, we define

$$U_{\text{hom}}(x) = \inf \{F_{\text{hom}}^\lambda(v) + G^\lambda(v) : v \in H_\lambda^1((0, +\infty); \mathbb{R}^d), v(0) = x\}. \quad (7.10)$$

We then have $U_\varepsilon(x) \rightarrow U_{\text{hom}}(x)$.

7.3 Characterization of the limit Hamiltonian and stability

We take $g(t)$ equal to a constant $-p \in \mathbb{R}^d$ in the previous remark. In this case, $U_\varepsilon = U_{p, \varepsilon}$ is the solution to a problem of the form (7.4) with $\frac{1}{2}|\dot{v}|^2 - \langle p, \dot{v} \rangle$ in the place of $\frac{1}{2}|\dot{v}|^2$ in the energy. Correspondingly, $U_{\text{hom}}(x)$ is the solution $U_p(x)$ of the limit problem with $f_{\text{hom}}(\dot{v}) - \langle p, \dot{v} \rangle$ in the place of $f_{\text{hom}}(\dot{v})$. Note that U_p is the constant

$$U_p = \frac{1}{\lambda} \min \{f_{\text{hom}}(\xi) - \langle p, \xi \rangle\} = -\frac{1}{\lambda} f_{\text{hom}}^*(p).$$

Since, on the other hand, U_p is also characterized as solving the limit Hamilton-Jacobi equation (2.7), we deduce that $H_{\text{hom}} = f_{\text{hom}}^*$.

As a consequence of the convergence result for Hamilton-Jacobi equations in terms of viscosity solutions [9], we deduce that, for any fixed uniformly continuous and bounded ψ we also have that the value functions

$$U_\varepsilon(x) = \inf \left\{ \int_0^{+\infty} \mathbb{E} \left(\frac{1}{2} |\dot{v}(t)|^2 + V_{\text{per}} \left(\frac{\sqrt{\varepsilon} B_t + v(t)}{\varepsilon} \right) + \psi(\sqrt{\varepsilon} B_t + v(t)) \right) e^{-\lambda t} dt : v(0) = x \right\}$$

converge to the value function

$$U_{\text{hom}}(x) = \inf \left\{ \int_0^{+\infty} (f_{\text{hom}}(\dot{v}(t)) + \psi(v(t))) e^{-\lambda t} dt : v(0) = x \right\}. \quad (7.11)$$

Remark 7.6 (stability for defects). From the arguments in Section 6, we can conclude that, if V_0 is as in that section and for any fixed uniformly continuous and bounded ψ , we also have that the value functions

$$U_\varepsilon^0(x) = \inf \left\{ \int_0^{+\infty} \mathbb{E} \left(\frac{1}{2} |\dot{v}(t)|^2 + V_{\text{per}} \left(\frac{\sqrt{\varepsilon} B_t + v(t)}{\varepsilon} \right) + V_0 \left(\frac{\sqrt{\varepsilon} B_t + v(t)}{\varepsilon} \right) + \psi(\sqrt{\varepsilon} B_t + v(t)) \right) e^{-\lambda t} dt : v(0) = x \right\}$$

converge to the value function U_{hom} for the unperturbed case in (7.11).

Indeed, for a fixed sequence ε_k , we can apply the arguments in the proof of Theorem 6.1 to a sequence of almost minimizers v_k , B^k for $U_{\varepsilon_k}^0$, upon defining μ^k as

$$\mu^k(A) := \frac{1}{\lambda} \int_0^{+\infty} \mu_t^k(A) e^{-\lambda t} dt.$$

This gives a variational proof of a convergence result for U_ε^0 by P.L. Lions [12].

8 Related settings

We briefly describe how our proof could cover some more general Hamilton–Jacobi equations, and outline some possible extensions. Although definite conclusions are yet to be obtained on the various settings we mention here, we believe that bringing up possible connections of our arguments and results with these other research lines might shed further light on the mechanisms at play in our study and we hope it may motivate further research.

We first consider the case of non-constant viscosity. Throughout our work, $\sqrt{\varepsilon} B_t + v(t)$ stands for the solution to the stochastic differential equation

$$dX_t^\varepsilon = \alpha(t) dt + \sqrt{\varepsilon} dB_t, \quad (8.1)$$

with $\dot{v} = \alpha$, starting at $X_0^\varepsilon = x$ at initial time $t = 0$. Equation (8.1) is the random trajectory to be considered in the stochastic-control interpretation of the Hamilton–Jacobi equation (1.1). We believe that our analysis carries over, *mutatis mutandis*, to the case of the random trajectory solution to

$$dX_t^\varepsilon = \alpha(t) dt + \sqrt{\varepsilon} \sigma_{\text{per}} \left(\frac{X_t^\varepsilon}{\varepsilon} \right) dB_t, \quad (8.2)$$

for a scalar-valued periodic function σ_{per} , which we assume to be bounded, bounded away from zero, and sufficiently regular. The case of a suitable, matrix-valued σ_{per} could likewise be considered, but we leave aside this further extension. Equation (8.2) is an immediate generalization to (8.1), but also a non-trivial extension since, in sharp contrast to (8.1), the equation is implicit in X_t^ε and its solution cannot be simply expressed in terms of α and B_t as in the case (8.1), where $X_t^\varepsilon = \int_0^t \alpha(s) ds + \sqrt{\varepsilon} B_t$. Equation (8.2) arises in the stochastic-control interpretation of the Hamilton-Jacobi equation

$$\lambda U_\varepsilon(x) - \frac{1}{2} \varepsilon \sigma_{\text{per}}^2 \left(\frac{x}{\varepsilon} \right) \Delta U_\varepsilon(x) + H_{\text{per}} \left(\frac{x}{\varepsilon}, \nabla U_\varepsilon(x) \right) = \psi(x). \quad (8.3)$$

Our proof for (8.1) may be adjusted to address the more general case (8.2), and to establish that the homogenized limit of (8.3) takes the form (1.2), but now with the homogenized Hamiltonian H_{hom} that is the Legendre transform of

$$f_{\text{hom}}(\xi) = \liminf_{T \rightarrow +\infty} \frac{1}{T} \inf \left\{ \int_0^T \mathbb{E} \left(\frac{1}{2} |\beta(t)|^2 + V_{\text{per}}(Y_t) \right) dt \right\}, \quad (8.4)$$

instead of (1.5). The infimum in (8.4) is taken, as for (1.5) on the Brownian motion B_t and the filtration $\mathcal{F}_t$, with all the necessary properties of adaptation described for (1.5), and (this is the new feature) over all the pairs $(\beta(t), Y_t)$ that satisfy

$$dY_t = \beta(t) dt + \sigma_{\text{per}}(Y_t) dB_t \quad \text{and} \quad \mathbb{E} \left(\frac{1}{T} \int_0^T \beta(s) ds \right) = \xi, \quad (8.5)$$

where, of course, β plays the role of $\dot{w}$ in our earlier setting. The step of the proof that has essentially to be adjusted is the construction of the (almost) corrector. The concatenation of the trajectories at the stopping times is not performed in terms of the pair $(w(t), B_t)$ but of the pair $(\beta(t), Y_t)$. Notice indeed that our earlier construction, can itself be entirely expressed as a concatenation of $(\dot{w}(t), w(t) + B_t)$ and thus can be extended to our new setting. Correspondingly, Proposition 9.2 should be extended, this time using the properties of the partial differential equation $-\frac{1}{2} \sigma_{\text{per}}^2(x) \Delta u = 1$, fully similar, in this respect, to those of $-\frac{1}{2} \Delta u = 1$, given our assumptions of non degeneracy of σ_{per} .

Another class of extensions we might consider for our work consists of general Hamiltonians $H_{\text{per}}(x, p)$ that are not quadratic in p , or that do not even separately depend on the two variables p and x , as was the case when $H_{\text{per}}(x, p) = \frac{1}{2} |p|^2 - V_{\text{per}}(x)$. The latter Hamiltonian $H_{\text{per}}(x, p)$ being convex in p , the oscillating Hamilton-Jacobi equation (1.1) we consider derives from a stochastic control theory problem, and we use this property intensively in the present work. The running cost involved in this control problem and the drift within the random trajectories respectively write as $L_{\text{per}}(x, \alpha) = \frac{1}{2} |\alpha|^2 + V_{\text{per}}(x)$ and $b_{\text{per}}(x, \alpha) = \alpha$. The general link between the Hamiltonian $H_{\text{per}}(x, p)$, the Lagrangian $L_{\text{per}}(x, \alpha)$, and the drift $b_{\text{per}}(x, \alpha)$ is given by the Legendre transformation

$$H_{\text{per}}(x, p) = \sup_{\alpha} \{ \langle b_{\text{per}}(x, \alpha), p \rangle - L_{\text{per}}(x, \alpha) \}.$$

Since our arguments proceed from Γ -convergence type arguments and are thus performed on the control-theoretic formulation of the problem, it is intuitive that other suitable running costs and drifts may be considered. Provided they enjoy the right analytic properties, our arguments will apply. Definite conclusions are yet to be obtained but we believe that, for instance, our arguments and results carry over to the case when $b_{\text{per}}(x, \alpha)$ is kept to be α but $L_{\text{per}}(x, \alpha)$ is now

- periodic and continuous in x , with a modulus of continuity uniform in α ,
- there exist $c_1, c_2 > 0$ and $r > 1$ such that $c_1|\alpha|^r \leq L(x, \alpha) \leq c_2(1 + |\alpha|^r)$, and
- $L(x, \cdot)$ is convex for all x .

Considering more general drifts $b_{\text{per}}(x, \alpha)$ that may depend upon x (say, Lipschitz continuous in x , uniformly in α) is likely to be more delicate. Likewise is the case when these extensions are combined with a diffusion σ_{per} that is not taken constant any longer, as in the beginning of this section. On the contrary, the stability result of Section 6 should extend unchanged to the setting of this section, since its proof only uses periodicity of V_{per} and non-negativity of V_0 .

9 Appendix: properties of the Brownian motion

As mentioned in the Introduction, this appendix presents two classical results. The first result, namely the following ‘concatenation property’ is a standard result in the theory of stochastic processes. We provide a proof here for the convenience of the reader.

Proposition 9.1. *Consider B_t^1, B_t^2 two Brownian motions, valued in $\mathbb{R}^d$, adapted to two independent filtrations $\mathcal{F}_t^1$ and $\mathcal{F}_t^2$, and let τ be a stopping time for the filtration $\mathcal{F}_t^1$. Then the process*

$$\tilde{B}_t = \begin{cases} B_t^1 & \text{when } t \leq \tau, \\ B_{t-\tau}^2 + B_\tau^1 & \text{when } t \geq \tau, \end{cases} \quad (9.1)$$

is a Brownian motion, adapted to the filtration $\tilde{\mathcal{F}}_t := \sigma(\mathcal{F}_t^1, \mathcal{F}_t^2)$.

Proof. We are going to use a *coupling method*. For this purpose, we first introduce the concatenation mapping $F: (C(\mathbb{R}_+, \mathbb{R}^d) \times \mathbb{R}_+) \times C_0(\mathbb{R}_+, \mathbb{R}^d) \rightarrow C(\mathbb{R}_+, \mathbb{R}^d)$, where C_0 denotes continuous functions with value 0 in 0, defined by

$$F((a, T), b)(t) = \begin{cases} a(t) & \text{when } t \leq T, \\ b(t - T) + a(T) & \text{when } t \geq T. \end{cases} \quad (9.2)$$

This function is deterministic, measurable and actually continuous. Note that $\tilde{B}_t = F((B_t^1, \tau), B_t^2)$; that is, for P -almost every $\omega \in \Omega$ we have $\tilde{B}_t(\omega) = F((B^1(\omega), \tau(\omega)), B^2(\omega))(t)$, where $B^i(\omega)(t) = B_t^i(\omega)$. From this, we can deduce that $\tilde{B}_t$ is adapted to $\tilde{\mathcal{F}}_t$. Note moreover that τ is also a stopping time for $\tilde{\mathcal{F}}_t$.

Let $\{U_t\}_{t \geq 0}$ be the stochastic process with values in $(\mathbb{R}^d \times \mathbb{R}_+) \times \mathbb{R}^d$ defined by the pair

$$U_t = ((\tilde{B}_{t \wedge \tau}, \tau), \tilde{B}_{t+\tau} - \tilde{B}_\tau), \quad (9.3)$$

and let $\{V_t\}_{t \geq 0}$ be the stochastic process with values in $(\mathbb{R}^d \times \mathbb{R}_+) \times \mathbb{R}^d$ defined by the pair

$$V_t = ((B_{t \wedge \tau}^1, \tau), B_{t+\tau}^1 - B_\tau^1). \quad (9.4)$$

By well-known properties of the Brownian motions, the trajectories of U_t and V_t are almost surely elements of the functional space $C(\mathbb{R}_+, \mathbb{R}^d \times \mathbb{R}^d)$. Therefore, the stochastic processes U_t and V_t define two random functions with values in $(C(\mathbb{R}_+, \mathbb{R}^d) \times \mathbb{R}_+) \times C(\mathbb{R}_+, \mathbb{R}^d)$ denoted by U and V .

Our claim $U_t \stackrel{\mathcal{L}}{\sim} V_t$ follows from the following three properties:

(i) U_t also writes as

$$U_t = ((B_{t \wedge \tau}^1, \tau), B_t^2), \quad (9.5)$$

respectively by definition of the first argument and because the second argument writes as $\tilde{B}_{t+\tau} - \tilde{B}_\tau = (B_t^2 + B_\tau^1) - B_\tau^1 = B_t^2$;

(ii) by the strong Markov property, the second argument of V_t is a Brownian motion W_t , independent from $(B_{t \wedge \tau}^1, \tau)$, so that

$$V_t = ((B_{t \wedge \tau}^1, \tau), W_t); \quad (9.6)$$

(iii) the two arguments within each pair are independent from one another, so the joint law of each pair is the product of the laws of the arguments. Moreover, for every finite family $0 \leq t_1 \leq \dots \leq t_M$ by a similar argument we can see that the joint law in $((\mathbb{R}^d \times \mathbb{R}_+) \times \mathbb{R}^d)^M$ of $(U_{t_1}, \dots, U_{t_M})$ is the same as the joint law of $(V_{t_1}, \dots, V_{t_M})$.

We then deduce that the random maps U and V have the same law in $(C(\mathbb{R}_+, \mathbb{R}^d) \times \mathbb{R}_+) \times C(\mathbb{R}_+, \mathbb{R}^d)$.

We finally remark that, since the concatenation mapping F is deterministic and continuous, the maps $F(U)$ and $F(V)$ have the same law in $C(\mathbb{R}_+, \mathbb{R}^d)$. Since

$$\tilde{B} = F(U) \quad \text{and} \quad B^1 = F(V),$$

where $\tilde{B}$ and B^1 denote the continuous functions $t \mapsto \tilde{B}_t$ and $t \mapsto B_t^1$, respectively, we conclude that $\tilde{B}$ and B^1 have the same law in $C(\mathbb{R}_+, \mathbb{R}^d)$. This implies that $\tilde{B}$ is a Brownian motion. $\square$

The second property of the Brownian motion allows one to show that solutions to minimum problems can be extended until they ‘almost’ reach a point in $\mathbb{Z}^d$, up to extending their domain in a controlled way, and up to a set of small probability. This shows that such solutions have a sort of almost-periodic behaviour, which can be used for the construction of correctors, as in the deterministic almost-periodic case.

Proposition 9.2. *Given a Brownian motion B_t on $\mathbb{R}^d$, a radius r , and a time length T , then there exists $c(r, T) > 0$, with $c(r, T) \rightarrow 0$ if $T \rightarrow +\infty$, such that*

$$P(\{\omega \in \Omega : \text{dist}(B_t(\omega), \mathbb{Z}^d) \leq r \text{ for all } t \in [t_0, t_0 + T]\}) \leq c(r, T)$$

for each time t_0 .

Proof. The issue is related to the capacity of the ball on the periodic torus. We let W_t , with $W_0 = 0$, denote the projection of the Brownian motion B_t onto the torus $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$.

We first fix a closed ball $E_r \subset \mathbb{T}^d$ of radius $r > 0$ and, for $x \in E_r^c := \mathbb{T}^d \setminus E_r$, we let

$$\tau_x := \inf\{t \geq 0 : x + W_t \in E_r\} \quad (9.7)$$

denote the first time at which the Brownian enters E_r , which is a stopping time for the filtration generated by the Brownian. A simple application of the Feynman-Kac representation formula for a steady-state elliptic equation, with homogeneous Dirichlet boundary conditions on ∂E_r and periodic conditions on the torus, implies that the function $u(x) = \mathbb{E}[\tau_x]$ solves on $\mathbb{T}^d$ the equation

$$\begin{cases} -\frac{1}{2}\Delta u = 1 & x \in E_r^c, \\ u = 0 & x \in E_r. \end{cases}$$

We then have

- (a) the solution u is bounded and we may define $U_r := \sup_{x \in \mathbb{T}^d} u(x) < \infty$;
- (b) this bound only depends on the ambient dimension d and the radius r of the ball E .

In the following we use the shorthand $\{\tau_x > T_0\}$ for $\{\omega \in \Omega : \tau_x(\omega) > T_0\}$, and similar notation, for subsets of Ω .

For any time T_0 , we have $u(x) = \mathbb{E}[\tau_x] \geq T_0 P(\{\tau_x > T_0\})$, so that, upon taking the supremum over $x \in \mathbb{T}^d$, and noting that $P(\{\tau_x > T_0\}) = P(\{x + W_t \in E_r^c \text{ for all } t \in [0, T_0]\})$, we obtain

$$\sup_{x \in \mathbb{T}^d} P(\{x + W_t \in E_r^c \text{ for all } t \in [0, T_0]\}) \leq \frac{U_r}{T_0}. \quad (9.8)$$

Using the Markov property of the Brownian motion, we now iterate formula (9.8). To this end, we indeed observe that the probability of not entering the ball E_r in, say, an interval of length $2T_0$ is smaller than that of not entering it in the first interval $[0, T_0]$, and then of still not entering E_r in $[T_0, 2T_0]$ provided that we take the sup of the latter over all the positions of the process at time T_0 . Mathematically, this intuition amounts to the following observations. We first remark that

$$\begin{aligned} & P(\{x + W_t \in E_r^c \text{ for all } t \in [0, 2T_0]\}) \\ &= P(\{x + W_t \in E_r^c \text{ for all } t \in [0, T_0]\} \cap \{x + W_t \in E_r^c \text{ for all } t \in [T_0, 2T_0]\}) \\ &= P(\{x + W_t \in E_r^c \text{ for all } t \in [0, T_0]\} \\ &\quad \cap \{x + W_{T_0} + (W_{s+T_0} - W_{T_0}) \in E_r^c \text{ for all } s \in [0, T_0]\}). \end{aligned} \quad (9.9)$$

We focus on the latter event

$$A_2 := \{x + W_{T_0} + (W_{s+T_0} - W_{T_0}) \in E_r^c \text{ for all } s \in [0, T_0]\}.$$

Using the Markov property, the increment $W_{s+T_0} - W_{T_0}$ is a Brownian motion, which is denoted by $\widetilde{W}_s$, that is independent from the original Brownian motion W_t on $[0, T_0]$ and thus of the event

$$A_1 := \{x + W_t \in E_r^c \text{ for all } t \in [0, T_0]\}.$$

In addition, writing the event

$$A_2 = \{x + W_{T_0} + \widetilde{W}_s \in E_r^c \text{ for all } s \in [0, T_0]\},$$

again by the Markov property, we note that A_2 depends on the event A_1 through the position $y = x + W_{T_0}$.

We therefore have

$$P(A_1 \cap A_2) = P(A_1)P(A_2|A_1) = P(A_1)P(A_2|x + W_{T_0}). \quad (9.10)$$

Now we have

$$P(A_2|x + W_{T_0}) \leq \sup_{y \in \mathbb{T}^d} P(\{y + \widetilde{W}_s \in E_r^c \text{ for all } s \in [0, T_0]\}). \quad (9.11)$$

Collecting (9.9), (9.10) and (9.11), we obtain

$$\begin{aligned} P(\{x + W_t \in E_r^c \text{ for all } t \in [0, 2T_0]\}) &\leq P(x + W_t \in E_r^c \text{ for all } t \in [0, T_0]) \times \\ &\quad \times \sup_{y \in \mathbb{T}^d} P(\{y + \widetilde{W}_s \in E_r^c \text{ for all } s \in [0, T_0]\}). \end{aligned} \quad (9.12)$$

Using (9.8) for each of the two factors in the right-hand side of (9.12), we have

$$\sup_{x \in \mathbb{T}^d} P(\{x + W_t \in E_r^c \text{ for all } t \in [0, 2T_0]\}) \leq \left(\frac{U_r}{T_0}\right)^2.$$

An iteration of this argument gives

$$\sup_{x \in \mathbb{T}^d} P(\{x + W_t \in E_r^c \text{ for all } t \in [0, kT_0]\}) \leq \left(\frac{U_r}{T_0}\right)^k. \quad (9.13)$$

We finally consider an arbitrary time T , which we may always assume larger than $2U_r$ since we are interested in the behavior of $P(\{x + W_t \in E_r^c \text{ for all } t \in [0, T]\})$ for T large. Choosing $T_0 = 2U_r$ and iterating k times where k is the integer part $k = \lfloor \frac{T}{T_0} \rfloor$, by (9.13) we obtain

$$\sup_{x \in \mathbb{T}^d} P(\{x + W_t \in E_r^c \text{ for all } t \in [0, kT_0]\}) \leq \left(\frac{1}{2}\right)^{\lfloor \frac{T}{T_0} \rfloor} \leq \left(\frac{1}{2}\right)^{\frac{T}{T_0} - 1},$$

thus, since $kT_0 = \lfloor \frac{T}{T_0} \rfloor T_0 \leq T$, that, *a fortiori*,

$$\sup_{x \in \mathbb{T}^d} P(\{x + W_t \in E_r^c \text{ for all } t \in [0, T]\}) \leq \left(\frac{1}{2}\right)^{\frac{T}{T_0}-1}.$$

It follows that

$$\sup_{x \in \mathbb{T}^d} P(\{x + W_t \in E_r^c \text{ for all } t \in [0, T]\}) \leq 2 \exp(-\lambda T) \quad (9.14)$$

for, say, the positive constant $\lambda = \frac{1}{2U_r} \log 2$. An optimal constant (the actual exponential rate) may be determined but (9.14) is sufficient for our purposes in the main body of the text: the probability $P(\{x + W_t \in E_r^c \text{ for all } t \in [0, T]\})$ is, uniformly in $x \in \mathbb{T}^d$, getting exponentially small as time grows. This concludes the proof of our claim. $\square$

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