

NON-LEBESGUE POINTS OUTSIDE THE JUMP SET OF FINITE-ENTROPY SOLUTIONS TO SCALAR CONSERVATION LAWS

NICOLA DE NITTI

ABSTRACT. We construct a bounded weak solution of a one-dimensional scalar conservation law, with a C^2 flux that has non-zero second derivative at some state, whose non-Lebesgue points outside the entropy jump set form a compact, purely 1-unrectifiable set E of Hausdorff dimension 1 that is not $\mathcal{H}^1$ - σ -finite. For each convex entropy, the entropy production is a finite signed measure, and the one-dimensional density of its total variation vanishes uniformly on E . Moreover, the mean oscillation of the solution vanishes uniformly on E , and E is the image of a product of two Cantor sets under a bi-Lipschitz map. In particular, we answer a question of Marconi (*Calc. Var. Partial Differential Equations*, 2022) about Burgers' equation, and disprove a conjecture of De Lellis and Otto (*J. Eur. Math. Soc.*, 2003) about the eikonal equation.

1. INTRODUCTION AND MAIN RESULTS

Scalar conservation laws

$$(1.1) \quad \partial_t u + \operatorname{div}_x f(u) = 0, \quad \text{in } \Omega_T,$$

where $\Omega_T := (0, T) \times \mathbb{R}^d$, $T > 0$, and $f: \mathbb{R} \rightarrow \mathbb{R}^d$ is a smooth *flux function*, exhibit a tension between non-uniqueness and regularization effects, both caused by nonlinearity. Even smooth initial data can develop discontinuities in finite time, so we must allow distributional solutions. In turn, distributional solutions need not be unique, and an admissibility condition is needed to select one. We refer to [Coc24; GR91; HR15] for an introduction to conservation laws.

A *bounded weak solution* of (1.1) is a bounded measurable function u that satisfies (1.1) in the sense of distributions on Ω_T . For a locally Lipschitz *entropy* function $\eta: \mathbb{R} \rightarrow \mathbb{R}$, we define the associated entropy flux $q_\eta: \mathbb{R} \rightarrow \mathbb{R}^d$ by $q_\eta(s) := \int_0^s \eta'(v) f'(v) dv$, and the *entropy production*

$$\mu_\eta := \partial_t \eta(u) + \operatorname{div}_x q_\eta(u).$$

We say that u has *finite entropy production* if μ_η is a finite signed Radon measure on Ω_T for every convex $\eta \in C^2(\mathbb{R})$. Solutions with locally finite signed entropy production arise in the variational analysis of entropy-cost functionals for scalar conservation laws [Bel+10, Proposition 2.3 and Theorem 2.5] and in the analysis of large deviations for stochastic scalar conservation laws [Mar10, Section 2.5 and Theorem 2.8].

A bounded weak solution u of (1.1) is an *entropy solution* if

$$(1.2) \quad \mu_\eta \leq 0 \quad \text{in the sense of distributions, for every convex } \eta.$$

Every non-negative distribution is a Radon measure, so by (1.2) every μ_η is a locally finite non-positive Radon measure. Entropy solutions arise as limits of vanishing-viscosity approximations, front-tracking approximations, monotone finite-difference schemes, etc. For the Cauchy problem with bounded initial datum, the entropy condition selects a unique solution by the theory of Kruřkov [Kru70].

While nonlinearity destroys uniqueness in the absence of an entropy condition, it also regularizes, in a way that linear transport does not. For $d = 1$ and a linear flux $f(s) = bs$, the solution $u(t, x) = u_0(x - bt)$ has exactly the regularity of its initial datum. If instead $f \in C^2(\mathbb{R})$ satisfies $f'' \geq c_f > 0$, bounded entropy solutions satisfy Oleřnik's one-sided estimate,

$$(1.3) \quad \partial_x u(t, \cdot) \leq \frac{1}{c_f t} \quad \text{in the sense of distributions on } \mathbb{R}, \text{ for almost every } t \in (0, T),$$

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and are therefore in $BV_{\text{loc}}((0, T) \times \mathbb{R})$. Without uniform convexity, for instance when f'' vanishes at some state, and in several space dimensions, Oleřnik's estimate (1.3) is no longer available. However, nonlinearity still has a regularizing effect (cf. [COW08]). Namely, for weak solutions with finite entropy production, in [DOW03, Theorem 1], De Lellis, Otto, and Westdickenberg proved a BV-like fine structure under the *genuine nonlinearity* assumption (1.4) on the flux $f \in C^{2,1}(\mathbb{R}; \mathbb{R}^d)$:

$$(1.4) \quad \mathcal{L}^1(\{v \in \mathbb{R} : \tau + \xi \cdot f'(v) = 0\}) = 0 \quad \text{for every } (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^d \text{ with } (\tau, \xi) \neq (0, 0).$$

The *kinetic formulation* (see, e.g., [LPT94, Theorem 1] or [DOW03, Proposition 1]) describes this structure. For a bounded weak solution u of (1.1), a *kinetic measure* is a locally finite signed Radon measure m on $\Omega_T \times \mathbb{R}$, compactly supported in the state variable, such that

$$(1.5) \quad \partial_t \chi(v, u) + f'(v) \cdot \nabla_x \chi(v, u) = \partial_v m \quad \text{in the sense of distributions on } \Omega_T \times \mathbb{R},$$

where

$$(1.6) \quad \chi(v, s) := \mathbb{1}_{\{0 < v < s\}} - \mathbb{1}_{\{s < v < 0\}}.$$

Such a measure, when it exists, is unique.

Let $\chi_0 \in C_c^\infty(\mathbb{R})$ be equal to 1 on an interval containing 0, the range of u , and the v -support of m . Testing (1.5) against $\eta'(v)\chi_0(v)\varphi(z)$, with $\varphi \in C_c^\infty(\Omega_T)$, and using smooth approximation in the state variable, we obtain, for every convex $\eta \in C^2(\mathbb{R})$,

$$\mu_\eta(U) = - \int_{U \times \mathbb{R}} \eta''(v) dm(z, v) \quad (U \Subset \Omega_T \text{ Borel}).$$

We define the marginal of $|m|$ on Ω_T and the *entropy jump set* by

$$(1.7) \quad \nu(U) := |m|(U \times \mathbb{R}) \quad (U \subset \Omega_T \text{ Borel}),$$

and

$$(1.8) \quad J := \left\{ z \in \Omega_T : \limsup_{r \downarrow 0} \frac{\nu(B_r(z))}{r^d} > 0 \right\},$$

where $B_r(z)$ is the open ball of radius r centered at z .

By [DOW03, Theorem 1], if $f \in C^{2,1}(\mathbb{R}; \mathbb{R}^d)$ satisfies (1.4) and u is a bounded weak solution with finite entropy production, then J is countably $\mathcal{H}^d$ -rectifiable, u has strong one-sided traces¹ at $\mathcal{H}^d$ -almost every point of J , and every point outside J is a VMO point. We recall that, for a ball $B \Subset \Omega_T$,

$$u_B := \frac{1}{|B|} \int_B u(z) dz, \quad \text{osc}(u; B) := \frac{1}{|B|} \int_B |u(z) - u_B| dz,$$

and say that $z \in \Omega_T$ is a *VMO point* if $\text{osc}(u; B_r(z)) \rightarrow 0$ as $r \downarrow 0$.

In the particular case of Burgers' equation, that is, $d = 1$ and $f(u) = u^2/2$, if $u \in C([0, T]; L^1(\mathbb{R}))$ has finite entropy production, then the marginal ν of the total variation of its kinetic measure is concentrated on a countable union of Lipschitz curves [Mar22b, Theorem 1]. More recently, for $d = 1$ and every flux satisfying (1.4), Ancona, Marconi, and Talamini proved a concentration result of the same kind, on a countably $\mathcal{H}^1$ -rectifiable set, for every bounded weak solution whose entropy productions are locally finite measures [AMT26, Definitions 2.1 and 4.1 and Theorem 4.5].

For entropy solutions (that is, when the entropy productions are non-positive), further information is available.

In one space dimension, by [BM17, Lemma 4.2, Proposition 4.3, and Corollary 4.6], the oscillation of an entropy solution outside a rectifiable jump set is confined to intervals on which the flux is affine, so that every point outside that set is a Lebesgue point when the flux has no affine interval. We recall that a point $z \in \Omega_T$ is a *Lebesgue point* if some $\bar{u} \in \mathbb{R}$ satisfies

$$\lim_{r \downarrow 0} \frac{1}{|B_r(z)|} \int_{B_r(z)} |u(y) - \bar{u}| dy = 0.$$

The number $\bar{u}$ is unique, and we call it the *Lebesgue value* of u at z . We denote by $N(u)$ the set of non-Lebesgue points. At a VMO point z , u has a Lebesgue value if and only if $\lim_{r \downarrow 0} \text{osc}(u; B_r(z))$ exists, and then the two coincide (see [DOW03, Remark 1]).

¹ That is, at $\mathcal{H}^d$ -almost every $z \in J$ there exist an approximate normal $n(z)$ to J and two values $u^\pm(z)$ such that the averages of $|u - u^\pm(z)|$ over the two half-balls $\{y \in B_r(z) : \pm(y - z) \cdot n(z) > 0\}$ tend to zero as $r \downarrow 0$.

Under the power-type nonlinearity assumption of [Sil19, Theorem 1.2], an entropy solution in every space dimension has a representative continuous outside the entropy jump set J of (1.8). For $f \in C^{\max\{d+1,3\}}$, this continuity also holds under the weaker nonlinearity assumption that no projection of $s \mapsto (s, f(s))$ onto a non-zero vector is constant on a non-empty open interval, as shown in [DeN26, Theorem 1.1].

The question addressed in this paper is how much of this additional regularity survives when the entropy productions are finite signed measures, no longer required to be non-positive. In particular, we ask whether such solutions have a Lebesgue value at $\mathcal{H}^d$ -almost every point outside the jump set.

In [LO18, Corollary 4], Lamy and Otto proved that every bounded weak solution u of Burgers' equation whose quadratic entropy production is a Radon measure satisfies $\dim_{\mathcal{H}} N(u) \leq 1$. Marconi proved in [Mar22a, Theorem 1.1] that, under power-type nonlinearity assumptions on f , every bounded weak solution u of (1.1) whose entropy production μ_η is a locally finite Radon measure for every convex entropy η satisfies $\dim_{\mathcal{H}} N(u) \leq d$.

Non-Lebesgue points outside J do occur.² In [Mar22a, Section 4, Example], Marconi produced a point of $N(u) \setminus J$ by gluing forward and backward entropy evolutions along a time slice: the datum has vanishing mean oscillation at one point but no Lebesgue value at that point. Thus the VMO regularity of [DOW03, Theorem 1] cannot in general be upgraded to the existence of a Lebesgue value, and the question becomes the size of $N(u) \setminus J$. Marconi asked in [Mar22a, Section 4] whether the set of non-Lebesgue points outside the jump set is nevertheless negligible for the codimension-one Hausdorff measure, and, in particular, for Burgers' equation, whether $\mathcal{H}^1(N(u) \setminus J) = 0$.

We answer this question in the negative. From now on, $d = 1$, so that $\Omega_T = (0, T) \times \mathbb{R}$, and $f \in C^2(\mathbb{R})$. For every such flux with non-zero second derivative at some state s_* , in particular for Burgers' equation, we construct a bounded weak solution of

$$(1.9) \quad \partial_t u + \partial_x f(u) = 0 \quad \text{in } \Omega_T$$

with finite entropy production for which $N(u) \setminus J$ is not $\mathcal{H}^1$ - σ -finite.³ In particular, $\mathcal{H}^1(N(u) \setminus J) = \infty$, because a set of finite $\mathcal{H}^1$ -measure is $\mathcal{H}^1$ - σ -finite.

Our main result, in its simplest form, is as follows.

Main Theorem. *Let $f \in C^2(\mathbb{R})$, let $s_* \in \mathbb{R}$ satisfy $f''(s_*) \neq 0$, and let $T > 0$ and $a > 0$. There exist a bounded weak solution u of (1.9) and a finite signed kinetic measure m of u with these properties. The function $u - s_*$ has compact support in $[0, T] \times \mathbb{R}$, and*

$$(1.10) \quad \|u - s_*\|_{L^\infty(\Omega_T)} \leq a, \quad u - s_* \in C([0, T]; L^1(\mathbb{R})), \quad u(0, \cdot) = s_*.$$

For every convex $\eta: \mathbb{R} \rightarrow \mathbb{R}$, the entropy production μ_η is a finite signed Radon measure on Ω_T . The set $E := N(u) \setminus J$ of non-Lebesgue points of u outside its entropy jump set is a compact subset of Ω_T that is purely 1-unrectifiable,⁴ has Hausdorff dimension 1, and is not $\mathcal{H}^1$ - σ -finite. Moreover,

$$(1.11) \quad \limsup_{r \downarrow 0} \sup_{z \in E} \left(\text{osc}(u; B_r(z)) + \frac{\nu(B_r(z))}{r} \right) = 0,$$

where ν is defined by (1.7), and $\lim_{r \downarrow 0} \sup_{z \in E} |\mu_\eta|(B_r(z))/r = 0$ for every convex η .

The construction has to meet three main requirements: at the points $z \in E$, the averages of u on small balls must fail to converge while both the mean oscillation and $\nu(B_r(z))/r$ tend uniformly to zero, so that $E \subset N(u) \setminus J$; the entropy productions must remain finite measures; and E must not be $\mathcal{H}^1$ - σ -finite. We replace an increasing shock by a finite shock pattern containing two nearby increasing shock segments, and we prescribe independently the direction in which their common center (the mean of the two states) moves, the ratio of the new half-jump (half the size of the jump) to the old one, the number of smaller parallelograms placed along them, in which the replacement is iterated, and the total duration of these parallelograms. The centers move back and forth across a fixed interval of states infinitely many times, and between two consecutive changes of direction

² A related example is given by Goldman and Merlet in [GM26, Proposition 3.20]: they construct a bounded gradient field $v = \nabla h$ on an open subset of $\mathbb{R}^2$ for which $\mu := \partial_1 v_2 = \partial_2 v_1 = \partial_1 \partial_2 h$ is a finite measure and $|\mu|(B_r(0))/r \rightarrow 0$ as $r \downarrow 0$, although 0 is a non-Lebesgue point of both components of v .

³ A set is $\mathcal{H}^1$ - σ -finite if it is a countable union of sets of finite $\mathcal{H}^1$ -measure.

⁴ A set is *purely 1-unrectifiable* if its intersection with every Lipschitz curve has zero $\mathcal{H}^1$ -measure.

they come close to a fixed state during a bounded number of steps of the construction only, so that the contributions of the successive crossings to the entropy productions are summable, uniformly in the state. At the same time, the growth of the sum of the cell durations with the generation makes E not $\mathcal{H}^1$ - σ -finite. We present the main ideas of the construction in [Section 2](#).

The [Main Theorem](#) is a qualitative version of our main result, [Theorem 1.1](#), which contains more precise information. The entropy productions can be made arbitrarily small, with $|\mu_\eta|(\Omega_T) \leq \delta \eta''(I)$ for a prescribed $\delta > 0$, where $I := [s_* - a, s_* + a]$ and η'' is the second derivative of η in the sense of distributions, and $|\mu_\eta|$ and ν vanish on E . The kinetic measure m is disintegrated in the state variable. The behavior of u on every time slice is described. Finally, for an explicit gauge g_W , E supports a probability measure λ such that $\lambda(B_r(z)) \asymp g_W(r)$ uniformly for $z \in E$ and sufficiently small r . Consequently, E has positive finite Hausdorff and packing measures for that gauge. To measure the size of E , we use Hausdorff and packing measures $\mathcal{H}^h$ and $\mathcal{P}^h$ with respect to a gauge h , that is, a continuous increasing function $h: [0, \varrho_0) \rightarrow [0, \infty)$ with $h(0) = 0$, for some $\varrho_0 > 0$.

Theorem 1.1. *Let $f \in C^2(\mathbb{R})$ and $s_* \in \mathbb{R}$ satisfy $f''(s_*) \neq 0$. Let $(W_n)_{n \geq 1}$ be a positive non-decreasing sequence with $W_n \rightarrow \infty$ as $n \rightarrow \infty$ for which there exists $C_W > 0$ such that*

$$(1.12) \quad 1 \leq \frac{W_{n+1}}{W_n} \leq 1 + \frac{C_W}{n} \quad \text{for all sufficiently large } n, \quad \sum_{n=1}^{\infty} W_n^{-1/2} = \infty.$$

Let $T > 0$. We define

$$(1.13) \quad \begin{aligned} \widetilde{W}(s) &:= W_n(s/n)^{\log(W_{n+1}/W_n)/\log((n+1)/n)} & (n \leq s \leq n+1), \\ L_r &:= \log(e^2 T/r), & g_W(r) := \frac{r}{\widetilde{W}(L_r/\log L_r)} \quad (0 < r < T), \quad g_W(0) := 0. \end{aligned}$$

For every $a > 0$ and $\delta > 0$, there exist a bounded weak solution u of [\(1.9\)](#), a finite signed kinetic measure m satisfying [\(1.5\)](#), a compact set $E \subseteq \Omega_T$ of non-Lebesgue points of u outside its entropy jump set J , and a Borel probability measure λ supported on E , with these properties, where $I := [s_* - a, s_* + a]$.

(i) For every convex $\eta: \mathbb{R} \rightarrow \mathbb{R}$, the entropy production μ_η is a finite signed Radon measure and we have

$$(1.14) \quad \|u - s_*\|_{L^\infty(\Omega_T)} \leq a, \quad |\mu_\eta|(\Omega_T) \leq \delta \eta''(I), \quad |\mu_\eta|(E) = 0,$$

where η'' is the distributional second derivative of η , a non-negative Radon measure. The support of $u - s_*$ is a compact subset of $[0, T] \times \mathbb{R}$, and

$$(1.15) \quad u - s_* \in C([0, T]; L^1(\mathbb{R})), \quad u(0, \cdot) = s_*.$$

There exist finite signed Radon measures M_v on Ω_T , $v \in \mathbb{R}$, such that

$$(1.16) \quad m(dz, dv) = M_v(dz) dv, \quad \mu_\eta = - \int_I M_v d\eta''(v),$$

where $v \mapsto M_v$ is continuous in total variation, vanishes outside I , and satisfies

$$(1.17) \quad \sup_{v \in \mathbb{R}} |M_v|(\Omega_T) \leq \delta, \quad |M_v|(E) = 0 \quad (v \in \mathbb{R}).$$

(ii) For the marginal ν defined in [\(1.7\)](#),

$$(1.18) \quad \nu(E) = 0, \quad \limsup_{r \downarrow 0} \sup_{z \in E} \frac{\nu(B_r(z))}{r} = 0.$$

In addition,

$$(1.19) \quad \limsup_{r \downarrow 0} \sup_{\substack{z \in E \\ v \in \mathbb{R}}} \frac{|M_v|(B_r(z))}{r} = 0.$$

Consequently, for every convex $\eta: \mathbb{R} \rightarrow \mathbb{R}$,

$$(1.20) \quad \limsup_{r \downarrow 0} \sup_{z \in E} \frac{|\mu_\eta|(B_r(z))}{r} = 0.$$

(iii) We have

$$(1.21) \quad E = N(u) \setminus J, \quad \limsup_{r \downarrow 0} \sup_{z \in E} \text{osc}(u; B_r(z)) = 0.$$

(iv) The set E is purely 1-unrectifiable, satisfies $\dim_{\mathcal{H}} E = 1$, and is not $\mathcal{H}^1$ - σ -finite. There exist $c_\lambda, C_\lambda > 0$ and $r_0 \in (0, T)$, depending only on f, s_*, T, a, δ , and $(W_n)_n$, such that

$$(1.22) \quad \begin{aligned} \lambda(B_r(z)) &\leq C_\lambda g_W(r) \quad (z \in \mathbb{R}^2, 0 < r < r_0), \\ \lambda(B_r(z)) &\geq c_\lambda g_W(r) \quad (z \in E, 0 < r < r_0). \end{aligned}$$

The gauge g_W is continuous and increasing for small r ,⁵ and

$$(1.23) \quad 0 < \mathcal{H}^{g_W}(E) < \infty, \quad 0 < \mathcal{P}^{g_W}(E) < \infty.$$

In particular,

$$(1.24) \quad \mathcal{H}^1(N(u) \setminus J) = \infty.$$

(v) The spatial sections and the time projection of E are denoted by

$$(1.25) \quad E_t := \{x \in \mathbb{R} : (t, x) \in E\}, \quad K := \{t \in (0, T) : E_t \neq \emptyset\}.$$

The set K is compact and has zero Lebesgue measure. For $t \in [0, T] \setminus K$, the function $u(t, \cdot)$ is piecewise constant with finitely many jumps. If $t \in K$, then the set E_t is perfect,⁶ has Hausdorff dimension zero, and satisfies $E_t \subset N(u(t, \cdot))$, the set of non-Lebesgue points of $u(t, \cdot)$ in $\mathbb{R}$.

The sequence $W_n = n$ satisfies (1.12) with $C_W = 1$, and the Main Theorem follows from Theorem 1.1 with $W_n = n$ and $\delta = 1$. Other particular choices of $(W_n)_n$, together with the gauges g_W to which they lead, are discussed in Section 9.

Using Theorem 1.1 with $f(v) := -\sqrt{1-v^2}$ near zero (if u solves (1.9) with this flux and ψ satisfies $\partial_x \psi = u$ and $\partial_t \psi = -f(u)$, then $|\nabla \psi|^2 = u^2 + f(u)^2 = 1$), we also deal with the eikonal equation, for an open set $U \subset \mathbb{R}^2$ with coordinates (t, x) ,

$$(1.26) \quad |\nabla \psi| = 1 \quad \text{in } U.$$

For a Lipschitz solution ψ of (1.26), the field $w := (-\partial_x \psi, \partial_t \psi)$ satisfies $|w| = 1$ almost everywhere in U and $\text{div } w = 0$ in the sense of distributions on U . A map $\Phi \in C_c^\infty(\mathbb{R}^2; \mathbb{R}^2)$ is an *eikonal entropy* if $\text{div } \Phi(w) = 0$ for every open set $\mathcal{O} \subset \mathbb{R}^2$ and every smooth field $w: \mathcal{O} \rightarrow \mathbb{R}^2$ satisfying $|w| = 1$ and $\text{div } w = 0$ [DO03, Definition 1]. We consider essentially bounded fields $w: U \rightarrow \mathbb{R}^2$ such that

$$(1.27) \quad \begin{aligned} |w| &= 1 \quad \text{a.e. in } U, \quad \text{div } w = 0, \\ \text{div } \Phi(w) &\in \mathcal{M}_{\text{loc}}(U) \quad \text{for every eikonal entropy } \Phi, \end{aligned}$$

in the sense of distributions, where $\mathcal{M}_{\text{loc}}(U)$ is the space of locally finite signed Radon measures on U . By [DO03, Theorem 1(a)], every such field w has a jump set J_w that is $\mathcal{H}^1$ - σ -finite and rectifiable. De Lellis and Otto conjectured that $\mathcal{H}^1$ -almost every point of $U \setminus J_w$ is a Lebesgue point of w (cf. [DO03, Conjecture 1(b')]). Let σ_w be a signed measure on $U \times (\mathbb{R}/2\pi\mathbb{Z})$ satisfying the kinetic formulation of the eikonal equation [LM24, Eq. (3)], and let $\nu_w(A) := |\sigma_w|(A \times (\mathbb{R}/2\pi\mathbb{Z}))$ be the marginal of $|\sigma_w|$ on U [LM24, Eq. (4)]. In [LM24, Theorem 1], Lamy and Marconi proved that $\mathcal{H}^1$ -almost every point z with $\nu_w(B_r(z)) = o(r^{1+\alpha})$ as $r \downarrow 0$, for some $\alpha > 0$, is a Lebesgue point of w ; in particular, the set $N(w)$ of non-Lebesgue points of w satisfies $\dim_{\mathcal{H}} N(w) \leq 1$. By [LM24, Remark 2], $\mathcal{H}^1$ -almost every point z with $\nu_w(B_r(z)) = O(r |\log r|^{-14})$ as $r \downarrow 0$ is also a Lebesgue point of w . In [LM26, Theorem 1.1], Lamy and Marconi proved the concentration of the entropy productions $\text{div } \Phi(w)$ on J_w for every eikonal entropy Φ whose restriction to $\mathbb{S}^1$ is odd up to an additive constant, under the additional assumption that U is bounded and $w \in B_{p,\infty}^{1/p}(U)$ for some $1 \leq p < 3$.

⁵ For a continuous increasing gauge h with $h(0) = 0$, the Hausdorff measure with gauge h is $\mathcal{H}^h(A) := \lim_{\varrho \downarrow 0} \inf \{ \sum_i h(\text{diam } U_i) : A \subset \bigcup_i U_i, \text{diam } U_i \leq \varrho \}$. The packing measure uses the same diameter convention: $\mathcal{P}_\varrho^h(A)$ is the supremum of $\sum_i h(\text{diam } B_i)$ over all countable families of pairwise disjoint closed balls B_i centered in A with $\text{diam } B_i \leq \varrho$, $\mathcal{P}_0^h(A) := \lim_{\varrho \downarrow 0} \mathcal{P}_\varrho^h(A)$, and $\mathcal{P}^h(A) := \inf \{ \sum_i \mathcal{P}_0^h(A_i) : A \subset \bigcup_i A_i \}$.

⁶ A set is *perfect* if it is closed and has no isolated points.

As a corollary of [Theorem 1.1](#), we show that $N(w) \setminus J_w$ is not $\mathcal{H}^1$ - σ -finite for some field w of this class, which disproves the conjecture of De Lellis and Otto.

Corollary 1.2. *Let $T > 0$, let $(W_n)_n$ be positive and non-decreasing, with $W_n \rightarrow \infty$ as $n \rightarrow \infty$, and satisfy [\(1.12\)](#), and let $\Gamma \subset \mathbb{S}^1$ be an arc containing $(0, 1)$ in its interior. There exist a 1-Lipschitz solution $\psi: \Omega_T \rightarrow \mathbb{R}$ of [\(1.26\)](#) in $U = \Omega_T$, a finite non-negative Radon measure ν on Ω_T , a compact set $E \Subset \Omega_T$, and a Borel probability measure λ supported on E , with these properties. The field $w := (-\partial_x \psi, \partial_t \psi)$ takes values in Γ and satisfies [\(1.27\)](#) in Ω_T .*

(i) *For every eikonal entropy Φ ,*

$$(1.28) \quad \begin{aligned} |\operatorname{div} \Phi(w)| &\leq C_{\text{eik}} \|\Phi\|_{C^2(\mathbb{R}^2)} \nu, \\ \nu(E) &= 0, \quad \limsup_{r \downarrow 0} \sup_{z \in E} \frac{\nu(B_r(z))}{r} = 0, \end{aligned}$$

where $C_{\text{eik}} := 4/3 + (4/3)^{3/2}$ and

$$\|\Phi\|_{C^2(\mathbb{R}^2)} := \sup_{y \in \mathbb{R}^2} (|\Phi(y)| + |D\Phi(y)| + |D^2\Phi(y)|),$$

with the operator norms of $D\Phi(y): \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and $D^2\Phi(y): \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$.

(ii) *The points of E are non-Lebesgue points of w at which the mean oscillation of w vanishes uniformly:*

$$(1.29) \quad E \subset N(w), \quad \limsup_{r \downarrow 0} \sup_{z \in E} \operatorname{osc}(w; B_r(z)) = 0.$$

(iii) *The set E and the measure λ satisfy [\(1.22\)](#) and [\(1.23\)](#), and E is purely 1-unrectifiable, satisfies $\dim_{\mathcal{H}} E = 1$, and is not $\mathcal{H}^1$ - σ -finite.*

(iv) *For every $\mathcal{H}^1$ - σ -finite set $Z \subset \Omega_T$, in particular for the jump set J_w of w in [\[DO03, Theorem 1\(a\)\]](#), the set $N(w) \setminus Z$ is not $\mathcal{H}^1$ - σ -finite, and*

$$(1.30) \quad \mathcal{H}^1(N(w) \setminus Z) = \infty.$$

2. OVERVIEW OF THE PROOF

We construct the solution u of [Theorem 1.1](#) as the limit of piecewise constant weak solutions $u^{(n)}$ of [\(1.9\)](#) whose discontinuities lie on finitely many straight segments, for $n \geq N_0$ with an integer N_0 fixed in [Section 4](#); we call n the *generation*. For each n , we also construct finitely many disjoint compact parallelograms, the *cells* of generation n . Each cell contains one segment of an increasing shock of $u^{(n)}$, from the state $S_n - \varepsilon_n$ to the state $S_n + \varepsilon_n$, along a line whose slope is the Rankine–Hugoniot speed of these states. We call S_n the center and ε_n the half-jump; both are the same in every cell of generation n . A cell of generation n has duration ℓ_n (the length of its time interval) and spatial half-width $\kappa_* \varepsilon_n \ell_n$. Here $\kappa_* = f''(s_*) > 0$ is fixed in [Section 3](#), where we reduce to the case $f''(s_*) > 0$.

To pass from generation n to generation $n+1$, we change $u^{(n)}$ only inside the cells of generation n . In each such cell Q , we replace $u^{(n)}$ by a piecewise constant weak solution containing two parallel increasing shock segments with center $S_{n+1} = S_n + \beta_n \varepsilon_n / 32$, where $\beta_n \in \{-1, 1\}$, and half-jump $\varepsilon_{n+1} = q_n \varepsilon_n$, where $q_n \leq 1$ is close to 1, together with finitely many auxiliary shocks inside Q ; this change is the replacement in Q . Along the two new shocks we choose $2M_n$ disjoint cells of duration ℓ_{n+1} ; they are the cells of generation $n+1$ inside Q , called the *children* of Q , and Q is their *parent*. We label the children of Q by the pairs (j, i) , where $j \in \{0, 1\}$ indicates the new shock and $i \in \{1, \dots, M_n\}$ the position along it. Two new shocks, rather than one, are needed for the size of E : with one new shock per cell, E would lie on a Lipschitz curve over a Lebesgue-null set of times, hence $\mathcal{H}^1(E) = 0$. The cells of all later generations inside Q are its *descendants*, and the set E is the intersection over n of the unions of the cells of generation n .

Idea of the construction. Let $z \in E$ and $\ell_n \leq r < \ell_{n-1}$. Outside a subset of $B_r(z)$ of area at most $C\varepsilon_n r^2$, the solution u differs from S_n by at most $C\varepsilon_n$ (see [\(8.4\)](#) and [\(8.5\)](#)), so that

$$(2.1) \quad \frac{1}{|B_r(z)|} \int_{B_r(z)} |u - S_n| \, dy \leq C\varepsilon_n.$$

We choose the half-jumps and the signs β_n so that

$$(2.2) \quad \varepsilon_n \longrightarrow 0, \quad (S_n)_n \text{ is bounded and does not converge.}$$

By (2.1) and (2.2), $\text{osc}(u; B_r(z)) \leq 2C\varepsilon_n \rightarrow 0$. Taking $r = \ell_n$ in (2.1) gives $|u_{B_{\ell_n}}(z) - S_n| \leq C\varepsilon_n \rightarrow 0$, so these averages do not converge. Hence $z \in N(u)$ (see Fig. 3).

The centers move by $|S_{n+1} - S_n| = \varepsilon_n/32$ and must travel between $s_* - b$ and $s_* + b$ infinitely often, for a fixed small $b > 0$. Hence we need $\sum_n \varepsilon_n = \infty$. Since the entropy productions of $u^{(n)}$ are measures supported on its shocks, we estimate them cell by cell. The replacement in a cell of generation n contributes to the measure M_v in (1.16) a measure with total variation at most $C\ell_n \varepsilon_n^2$. This contribution vanishes unless $|S_n - v| \leq 8\varepsilon_n$ by (7.6). The D_n cells of generation n have total duration $D_n \ell_n = TW_n/(16W_{N_0})$ by (4.12), so generation n contributes a measure of total variation at most $CTW_n \varepsilon_n^2 \mathbb{1}_{\{|S_n - v| \leq 8\varepsilon_n\}}/W_{N_0}$. With $a_n = W_n^{-1/2}$, $A_n = e + \sum_{j \leq n} a_j$, and $\varepsilon_n = a_n/A_n$, we have $W_n \varepsilon_n^2 = A_n^{-2}$, and the hypothesis $\sum_n W_n^{-1/2} = \infty$ gives both $\sum_n \varepsilon_n = \infty$ and $A_n \rightarrow \infty$. Between two consecutive changes of sign of β_n , the center comes within $8\varepsilon_n$ of a fixed state only for a bounded number of generations, and A_n grows by at least the factor e^{32b} . Hence the contributions at each state are summable (Lemma 4.2).

Finally, the growth of the total duration $D_n \ell_n \propto W_n$ makes E large: the probability measure λ on E of Theorem 1.1(iv) satisfies $\lambda(B_r(z)) \leq Cr/(D_n \ell_n)$ for $\ell_n \leq r < \ell_{n-1}$ (Proposition 6.2), so $\lambda(B_r(z))/r \rightarrow 0$ uniformly on E and E is not $\mathcal{H}^1$ - σ -finite by the mass distribution principle (6.18).

Outline of the paper. The paper is organized as follows (see Fig. 1 for a summary of the dependencies among sections).

In Section 3, we construct the replacement in one cell. We write the solution as $u = \partial_x h$, where the potential h satisfies $\partial_t h + f(\partial_x h) = 0$ almost everywhere in Ω_T . Finitely many minima and maxima of affine solutions of this equation are again solutions (see (3.4)). After normalizing the flux near s_* , we insert the piecewise affine potential $V_{\beta,q}^F$ of (3.6), whose spatial derivative has two parallel increasing shocks with states $\beta/32 \pm q$, where $\beta \in \{-1, 1\}$ is the sign of the change of center and $q \in [15/16, 1]$ is the ratio of the half-jumps. By Proposition 3.1, for a parent shock with center s , half-jump ε , and duration L , the $2M$ children have center $s + \beta\varepsilon/32$, half-jump $q\varepsilon$, and total duration RL , and the four parameters β , q , M , and R can be prescribed independently (see Fig. 2). The potential is unchanged on the boundary of the cell, so its spatial derivative is again a weak solution.

In Section 4, we choose ε_n , M_n , ℓ_n , and the signs β_n , and we prove Lemma 4.2, which bounds the sum of $W_j \varepsilon_j^2$ over the generations $j \geq n$ with S_j near a fixed state by $CW_n \varepsilon_n^2$.

In Section 5, we iterate the replacement of Section 3 in every cell of every generation, starting from a solution with one increasing shock. Every sequence $\omega = ((j_n, i_n))_{n \geq N_0} \in \Sigma$ of labels of one child per generation determines nested cells $Q_n(\omega)$ with one common point $\Pi(\omega)$. We set $E := \Pi(\Sigma)$ and $\lambda := \Pi_\# \rho$, where ρ is the product of the uniform probabilities on the choices. For $z = \Pi(\omega)$, the ball $B_{\ell_{n+1}}(z)$ lies in $Q_n(\omega)$ and meets no shock of $u^{(n)}$ by (5.10) (see Fig. 4). The functions $u^{(n)} - s_*$ converge in $C([0, T]; L^1(\mathbb{R}))$ to $u - s_*$.

In Section 6, we study the size of E . The time coordinate of the point $\Pi(\omega)$ common to the cells $Q_n(\omega)$ depends only on the indices i_n , and its spatial coordinate, after the subtraction of a function φ of the time coordinate, only on the indices j_n . Hence $E = \Psi(K \times \mathcal{C})$, where $K \subset (0, T)$ is the set of time coordinates of the points of E , a Cantor set, $\mathcal{C} \subset \mathbb{R}$ is a Cantor set, and $\Psi(t, y) = (t, \varphi(t) + y)$ is bi-Lipschitz (Lemma 6.1). Since the indices i_n and j_n are independent random variables on (Σ, ρ) , λ is the image under Ψ of a product measure. Counting the intervals of each factor that meet a ball gives $\lambda(B_r(z)) \asymp g_W(r)$ (Propositions 6.2 and 6.3), the Hausdorff and packing measures in Theorem 1.1(iv), and the number of cells meeting a ball (Corollary 6.4). Since $\mathcal{L}^1(K) = \mathcal{L}^1(\mathcal{C}) = 0$, E is purely 1-unrectifiable.

In Section 7, for a cell Q of generation n , we bound the change of the measures $\zeta_k^{(n)} = (\partial_t + f'(k)\partial_x)\mathbb{1}_{\{u^{(n)} \geq k\}}$ in Q by $C\ell_n \varepsilon_n$, and the change of their integrals $M_v^{(n)} = \int_{-\infty}^v \zeta_k^{(n)} dk$ by $C\ell_n \varepsilon_n^2$ (see (7.4) and (7.6)). Summing over the generations with $|S_n - v| \leq 8\varepsilon_n$ by Lemma 4.2, we prove Theorem 1.1(i).

In Section 8, for $z \in E$ and $\ell_n \leq r < \ell_{n-1}$, we prove (2.1) and bound $|M_v|(B_r(z))$ and $\nu(B_r(z))$ by $C\varepsilon_n^2 r$, using the number of cells meeting $B_r(z)$. These bounds prove Theorems 1.1(ii)

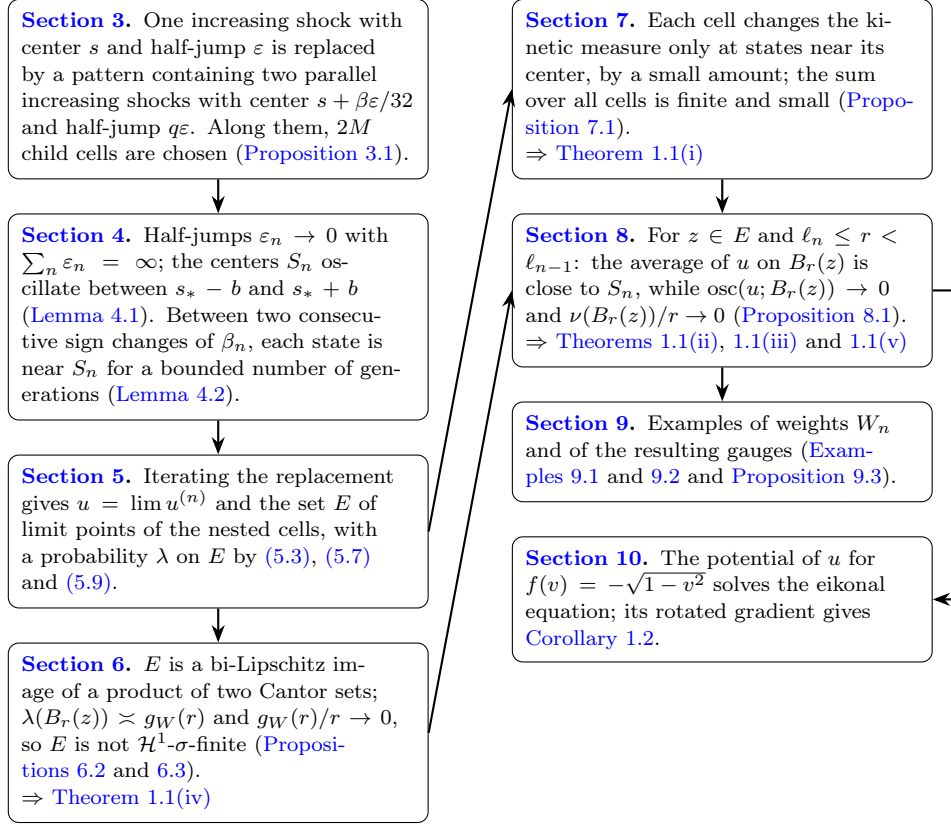


FIGURE 1. Structure of the proof of Theorem 1.1 and Corollary 1.2.

and 1.1(iii). A corresponding one-dimensional estimate on intervals centered at the points of E_t , for $t \in K$, proves Theorem 1.1(v).

In Section 9, we give explicit weights satisfying (1.12) and compute the corresponding gauges g_W .

Finally, in Section 10, we apply the construction to the flux $f(v) = -\sqrt{1-v^2}$ of (10.1) and prove Corollary 1.2.

Constants. In Sections 3, 4, 5, 6, 7, 8 and 9, C denotes a positive constant that depends only on f , s_* , the number b fixed in Section 3, $(W_n)_n$, and C_W ; its value may change from line to line. In particular, C does not depend on N_0 , n , j , r , z , k , v , α , T , or δ . We write C_Θ if C may also depend on Θ , and $X \asymp Y$ if $C^{-1}Y \leq X \leq CY$. The lower bounds on N_0 needed for the geometric and scale estimates depend only on f , s_* , b , and $(W_n)_n$, and only the final choice enforcing (7.13) may also depend on T and δ .

3. REPLACEMENTS OF AN INCREASING SHOCK

A *shock segment* of a piecewise constant function, briefly a *shock*, is a segment $\{(t, x_0 + \sigma t) : t_1 \leq t \leq t_2\}$ across which the function jumps from a left state u_- to a right state $u_+ \neq u_-$; its *duration* is $t_2 - t_1$. It is *increasing* if $u_- < u_+$. Its *center* is $(u_- + u_+)/2$, and its *half-jump* is $|u_+ - u_-|/2$. The main step replaces an increasing shock segment inside a parallelogram around it, the cell of Section 2, leaving the solution unchanged outside. The new solution contains two parallel increasing shock segments on which we choose the children for the next generation. The common center and half-jump of these shocks, and the sum of the child durations, can be prescribed separately. Normalizing the flux near s_* makes the same bounds apply to every replacement.

We set $\kappa_* := f''(s_*) > 0$.⁷ Next, we choose a state interval on which the normalized flux is uniformly close to $v^2/2$, with tolerance $\delta_0 := 1/10240$.⁸ We choose $0 < b < 1$ with $4b < a$ so that

$$(3.1) \quad I_f := [s_* - 4b, s_* + 4b], \quad 36 \sup_{\xi \in I_f} \left| \frac{f''(\xi)}{\kappa_*} - 1 \right| \leq \delta_0,$$

and we take the largest such b of the form 2^{-k} with $k \in \mathbb{N}$. For $0 < \varepsilon \leq 1$ and $[s - 8\varepsilon, s + 8\varepsilon] \subset I_f$, we define the speed of the shock with states $s \pm \varepsilon$ and the normalized flux by

$$\sigma_{s,\varepsilon} := \frac{f(s + \varepsilon) - f(s - \varepsilon)}{2\varepsilon}, \quad F_{s,\varepsilon}(v) := \frac{f(s + \varepsilon v) - f(s) - \varepsilon \sigma_{s,\varepsilon} v}{\kappa_* \varepsilon^2}.$$

By Taylor's formula with integral remainder,

$$(3.2) \quad \begin{aligned} F_{s,\varepsilon}(0) &= 0, & F_{s,\varepsilon}(1) &= F_{s,\varepsilon}(-1), \\ \left| F_{s,\varepsilon}(v) - \frac{v^2}{2} \right| &\leq \frac{v^2 + |v|}{2} \sup_{\xi \in I_f} \left| \frac{f''(\xi)}{\kappa_*} - 1 \right| \leq \delta_0 \quad (|v| \leq 8). \end{aligned}$$

The factor 36 in (3.1) is the maximum of $(v^2 + |v|)/2$ on $[-8, 8]$. Neither bound in (3.2) depends on s or ε .

We construct locally Lipschitz potentials satisfying

$$(3.3) \quad \partial_t h + f(\partial_x h) = 0 \quad \text{almost everywhere in } \Omega_T.$$

Then $u = \partial_x h$ satisfies (1.9) in the sense of distributions, since

$$\partial_t u + \partial_x f(u) = \partial_x (\partial_t h + f(\partial_x h)) = 0.$$

For a continuous flux F and $v, c \in \mathbb{R}$, we call $(t, x) \mapsto c + vx - F(v)t$ an *affine potential with state v* ; its gradient is $(-F(v), v)$. Functions obtained from finitely many affine potentials by finitely many minima and maxima satisfy

$$(3.4) \quad \partial_t H + F(\partial_x H) = 0 \quad \text{almost everywhere in } \mathbb{R}^2,$$

because a finite minimum or maximum of Lipschitz functions coincides with one of them at every point, and two Lipschitz functions that coincide on a set have the same gradient almost everywhere on that set (see also [AK66, Lemma 1.1]). We use this fact to change a potential inside a parallelogram without changing it outside, as in the nested quadrilateral constructions of [ADM99, Section 3, pp. 338–340].⁹

We impose the normalization and parameter bounds

$$(3.5) \quad \begin{aligned} F(0) &= 0, & F(1) &= F(-1), & \max_{|v| \leq 8} \left| F(v) - \frac{v^2}{2} \right| &\leq \delta_0, \\ \beta &\in \{-1, 1\}, & \frac{15}{16} &\leq q \leq 1, & M \in \mathbb{N}, & M \geq 512, & 1 \leq R \leq \frac{11}{10}. \end{aligned}$$

The sign β is the direction in which the center moves, q is the ratio of successive half-jumps, M is the number of time intervals on each new shock, and R is the ratio of the total child duration to the parent duration. Since the ratios q_n and R_n used in Section 4 tend to 1 by (4.5), we only need bounds on q and R near 1, and we fix explicit ones. With $R \leq 11/10 < 6/5$, the M time intervals of length $R/(2M)$ are disjoint subsets of $[1/5, 4/5]$, with gaps at least $1/11$ times their length by (3.14). The lower bound on q is used in $1/q \leq 16/15$ and $q^2 \geq (15/16)^2 = 225/256$ in (3.12) and (3.13). Finally, $M \geq 512$ gives the child duration $R/(2M) \leq (11/10)/1024 = 11/10240$, hence also the child half-width at most $11/10240$; this keeps each child inside a strip of (3.11) around one of the two new shocks.

For the states $\beta/32 \pm q$, we define the mean flux and the shock speed

$$A_{\beta,q}^F := \frac{F(\beta/32 + q) + F(\beta/32 - q)}{2}, \quad \theta_{\beta,q}^F := \frac{F(\beta/32 + q) - F(\beta/32 - q)}{2q},$$

⁷ It suffices to treat $f''(s_*) > 0$. If $f''(s_*) < 0$, we construct a solution $\tilde{u}$ for the flux $-f$ and define $u(t, x) := \tilde{u}(t, -x)$; then $\partial_t u + \partial_x f(u) = (\partial_t \tilde{u} - \partial_x f(\tilde{u}))(t, -x) = 0$ in the sense of distributions, and the reflection preserves balls, the time coordinate, and all the measures and norms in the statements of Theorem 1.1 and Corollary 1.2.

⁸ The value of δ_0 is not optimized: it is used in (3.10), (3.12) and (3.13), which hold for every $\delta_0 \leq 1/500$, and its denominator is that of the bound $11/10240$ on the half-width of the children in (3.7).

⁹ Hamilton–Jacobi differential inclusions are used to construct bounded weak solutions of scalar conservation laws that are nowhere continuous in the interior of their support in [VK18, Theorem 1.2].

and the potentials

$$\begin{aligned}
(3.6) \quad & H_0^F(t, x) := |x| - F(1)t, \\
& V_{\beta, q}^F(t, x) := \frac{1}{16} + \frac{\beta x}{32} - A_{\beta, q}^F t + q \min_{j \in \{0, 1\}} \left| x - \theta_{\beta, q}^F t - \frac{\beta j}{64} \right|, \\
& P_\beta^F(t, x) := 27 + 8\beta x - F(8\beta)t, \\
& H_1^{F, \beta, q} := \max\{H_0^F, \min\{0, P_\beta^F, V_{\beta, q}^F\}\}.
\end{aligned}$$

The segments $\{x = \theta_{\beta, q}^F t + \beta j/64, 1/5 \leq t \leq 4/5\}$, $j \in \{0, 1\}$, are the two new shocks, and their spatial separation at each common time is $1/64$. Their speed $\theta_{\beta, q}^F$ is within $16\delta_0/15$ of $\beta/32$, so for $t \geq 1/5$ they lie at distance more than $1/256$ from the shock $\{x = 0\}$ of $\partial_x H_0^F$ by (3.12). The intercept $1/16$ gives $V_{\beta, q}^F > H_0^F$ and $V_{\beta, q}^F < 0$ near the new shocks by (3.13). Since $P_\beta^F - H_0^F \leq 27 - (F(8\beta) - 8F(1))t$ where $H_0^F < 0$, with $F(8\beta) - 8F(1) \geq 28 - 9\delta_0$, the modification ends before the time $27/(28 - 9\delta_0) < 1$ by (3.10). The cutoffs 0 and P_β^F localize the modification, and the outer maximum with H_0^F restores H_0^F outside it. The margins $[0, 1/5)$ and $(4/5, 1]$ separate the time interval $[1/5, 4/5]$ from the times at which the modification begins and ends.¹⁰

Proposition 3.1. *Under (3.5), the potential $H_1^{F, \beta, q}$ satisfies (3.4) and equals H_0^F on the boundary of $[0, 1] \times [-1, 1]$. For $j \in \{0, 1\}$ and $1 \leq i \leq M$, we define*

$$\begin{aligned}
(3.7) \quad & \ell := \frac{R}{2M}, \quad t_i := \frac{1}{5} + \left(i - \frac{1}{2}\right) \frac{3}{5M}, \quad I_i := [t_i - \ell/2, t_i + \ell/2], \\
& Q_{ji} := \left\{ (t, x) : t \in I_i, \quad \left| x - \theta_{\beta, q}^F t - \frac{\beta j}{64} \right| \leq q\ell \right\}.
\end{aligned}$$

The $2M$ cells Q_{ji} are pairwise disjoint. On a neighborhood of each cell, $H_1^{F, \beta, q}$ is the maximum of the two affine potentials with states $\beta/32 - q$ and $\beta/32 + q$. With $g := 1/11$, $c_{\text{sep}} := 1/128$, and $c_0 := 1/256$, whose values come from the margins in (3.12) and (3.14), we have $\min I_{i+1} - \max I_i \geq g\ell$ for $1 \leq i < M$, and $\text{dist}(Q_{0i} \cap \{t = \tau\}, Q_{1i} \cap \{t = \tau\}) \geq c_{\text{sep}}$ for $\tau \in I_i$. Every child satisfies

$$(3.8) \quad Q_{ji} \subset \{(t, x) : c_0 \leq t \leq 1 - c_0, \quad c_0 \leq |x| \leq 1 - c_0\}.$$

Almost everywhere in $\mathbb{R}^2$, $\partial_x H_1^{F, \beta, q} \in [-8, 8]$ and $|\partial_x H_1^{F, \beta, q} - \partial_x H_0^F| \leq 9$. The discontinuities of $\partial_x H_1^{F, \beta, q}$ and $\partial_x H_0^F$ in the closure of $\{H_1^{F, \beta, q} \neq H_0^F\}$ lie on at most

$$(3.9) \quad C_* := 1 + \binom{8}{2}^2$$

shock segments, which are contained in $[0, 1] \times [-1, 1]$, so that each has duration at most 1. These constants are independent of F , β , q , M , and R .

Proof. Step 1: Equation and support. Every affine piece in (3.6) has gradient $(-F(v), v)$ for one of the states

$$v \in \{-1, 1, 0, 8\beta, \beta/32 - q, \beta/32 + q\}.$$

For the four pieces in $V_{\beta, q}^F$, we use

$$A_{\beta, q}^F \pm q\theta_{\beta, q}^F = F(\beta/32 \pm q).$$

Thus $H_1^{F, \beta, q}$ satisfies (3.4). Where $H_1^{F, \beta, q} > H_0^F$, we have $H_0^F < 0$ and $H_0^F < P_\beta^F$, and hence

$$\begin{aligned}
(3.10) \quad & |x| < F(1)t, \quad t > 0, \\
& 0 < P_\beta^F - H_0^F \leq 27 - (F(8\beta) - 8F(1))t, \\
& t < \frac{27}{28 - 9\delta_0} < \frac{39}{40}, \quad |x| < \frac{1}{2}.
\end{aligned}$$

Since $H_1^{F, \beta, q} \geq H_0^F$, we obtain $H_1^{F, \beta, q} = H_0^F$ outside $(0, 39/40) \times (-1/2, 1/2)$, in particular on the boundary of $[0, 1] \times [-1, 1]$.

¹⁰ None of the values $1/64$, $1/32$, $1/16$, 8 , and 27 is essential; they are one explicit choice for which the strict inequalities in (3.10), (3.12) and (3.13) hold.

Step 2: *The two new shocks.* The child half-width satisfies $q\ell \leq \ell = R/(2M) \leq (11/10)/(2 \cdot 512) = 11/10240$. On the closed strips

$$(3.11) \quad \frac{1}{5} \leq t \leq \frac{4}{5}, \quad \min_{j \in \{0,1\}} \left| x - \theta_{\beta,q}^F t - \frac{\beta j}{64} \right| \leq \frac{11}{10240},$$

we obtain

$$(3.12) \quad \begin{aligned} \left| \theta_{\beta,q}^F - \frac{\beta}{32} \right| &\leq \frac{\delta_0}{q} \leq \frac{16\delta_0}{15}, \\ |x| &\leq \frac{4}{5} \left(\frac{1}{32} + \frac{16\delta_0}{15} \right) + \frac{1}{64} + \frac{11}{10240} < \frac{3}{64}, \\ |x| &\geq \frac{1}{5} \left(\frac{1}{32} - \frac{16\delta_0}{15} \right) - \frac{11}{10240} > \frac{1}{256}. \end{aligned}$$

By (3.5) and $q \leq 1$,

$$\left| F(1) - \frac{1}{2} \right| \leq \delta_0, \quad \left| A_{\beta,q}^F - \frac{1}{2048} - \frac{q^2}{2} \right| \leq \delta_0, \quad F(1) - A_{\beta,q}^F \geq -\frac{1}{2048} - 2\delta_0.$$

Hence, using $-|x| \leq \beta x \leq |x|$, $|x| < 3/64$, $1/5 \leq t \leq 4/5$, $q^2 \geq 225/256$, $q \min_j |x - \theta_{\beta,q}^F t - \beta j/64| \leq 11/10240$, and $F(8\beta) \leq 32 + \delta_0$,

$$(3.13) \quad \begin{aligned} V_{\beta,q}^F - H_0^F &\geq \frac{1}{16} - \frac{33}{32} \cdot \frac{3}{64} - \frac{1}{2560} - \frac{8}{5}\delta_0 > \frac{1}{100}, \\ V_{\beta,q}^F &\leq \frac{1}{16} + \frac{3}{2048} - \frac{1}{2} \cdot \frac{225}{256} \cdot \frac{1}{5} + \frac{11}{10240} + \frac{4}{5}\delta_0 < -\frac{1}{50}, \\ P_{\beta}^F &\geq 27 - \frac{3}{8} - \frac{4}{5}(32 + \delta_0) > 1. \end{aligned}$$

Therefore $H_1^{F,\beta,q} = V_{\beta,q}^F$ on a neighborhood of the strips. On the strip around the line with index j , we have $|x - \theta_{\beta,q}^F t - \beta(1-j)/64| \geq 1/64 - 11/10240 > 11/10240$, so on this strip only the term with index j attains the minimum in $V_{\beta,q}^F$. Each line

$$x = \theta_{\beta,q}^F t + \frac{\beta j}{64}, \quad j \in \{0,1\}, \quad \frac{1}{5} \leq t \leq \frac{4}{5},$$

is a shock segment of $\partial_x H_1^{F,\beta,q}$ with left state $\beta/32 - q$ and right state $\beta/32 + q$.

Step 3: *Cells and number of shocks.* For the intervals and widths in (3.7),

$$(3.14) \quad \begin{aligned} t_1 - \ell/2 &= \frac{1}{5} + \frac{1}{M} \left(\frac{3}{10} - \frac{R}{4} \right) > \frac{1}{5}, \quad t_M + \ell/2 < \frac{4}{5}, \\ t_{i+1} - t_i - \ell &= \left(\frac{6}{5R} - 1 \right) \ell \geq \frac{\ell}{11}, \quad \frac{1}{64} - 2q\ell > \frac{1}{128}. \end{aligned}$$

By (3.14), $(t_{i+1} - t_i - \ell)/\ell \geq g$ and $1/64 - 2q\ell > c_{\text{sep}}$. Moreover, by (3.12), every Q_{ji} lies in $\{1/5 \leq t \leq 4/5, 1/256 \leq |x| \leq 3/64\}$, which gives (3.8).

After identical affine functions and coincident equality lines have been identified, the eight affine functions in (3.6) (two in H_0^F , the constant 0, P_{β}^F , and four in $V_{\beta,q}^F$) have at most $\binom{8}{2}$ equality lines; two affine potentials with distinct states are equal along a line whose slope is the Rankine–Hugoniot speed of these states. Each line is divided into at most $\binom{8}{2}$ pieces by the others. Adding the segment of the shock $\{x = 0\}$ of $\partial_x H_0^F$ in the closure of $\{H_1^{F,\beta,q} \neq H_0^F\}$ gives (3.9). Since $\partial_x H_0^F = \pm 1$ and $|\partial_x H_1^{F,\beta,q}| \leq 8$, we have $|\partial_x H_1^{F,\beta,q} - \partial_x H_0^F| \leq 9$. $\square$

For an increasing shock segment of (1.9) with states $s \pm \varepsilon$ on the line $x = \sigma_{s,\varepsilon} t + b_0$, initial time t_0 , and duration L , we define the cell

$$(3.15) \quad Q := \{(t, x) : t_0 \leq t \leq t_0 + L, \quad |x - \sigma_{s,\varepsilon} t - b_0| \leq \kappa_* \varepsilon L\},$$

whose projection $[t_0, t_0 + L]$ onto the t -axis is the time interval of Q .

Let h^{old} satisfy (3.3) and equal the maximum of the two affine potentials with states $s \pm \varepsilon$ on a neighborhood of Q . Inside Q , we replace h^{old} by $h + c$, where the constant $c \in \mathbb{R}$ is chosen so that $h + c = h^{\text{old}}$ on ∂Q , and

$$(3.16) \quad h(t, x) = sx - f(s)t + \kappa_* \varepsilon^2 L H_1^{F,s,\varepsilon,\beta,q} \left(\frac{t - t_0}{L}, \frac{x - \sigma_{s,\varepsilon} t - b_0}{\kappa_* \varepsilon L} \right).$$

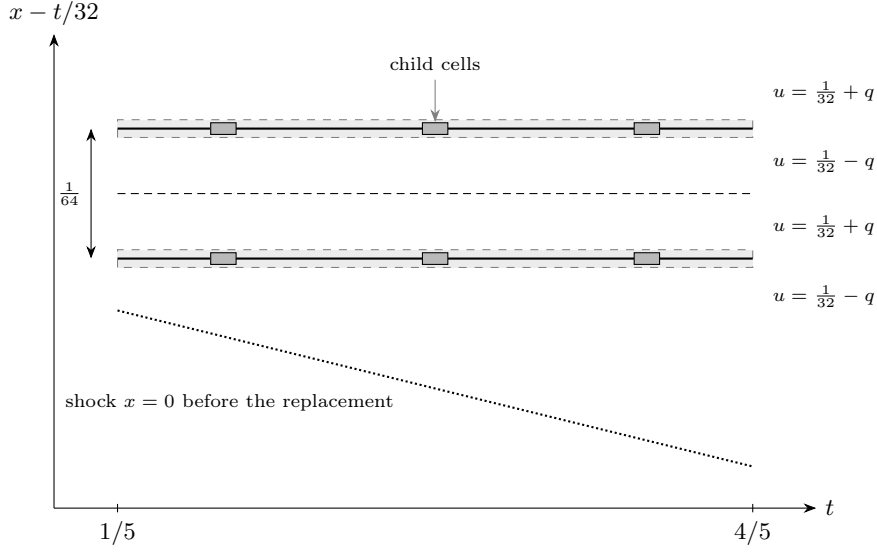


FIGURE 2. The replacement in the normalized cell for $F(v) = v^2/2$ and $\beta = 1$, drawn in the coordinates $(t, x - t/32)$, in which lines of speed $1/32$ are horizontal. Before the replacement, $\partial_x H_0^F$ has one increasing shock along $x = 0$ (dotted), from the state -1 to the state 1 . After the replacement, for $1/5 \leq t \leq 4/5$, the function $u = \partial_x H_1^{F,\beta,q}$ has two parallel increasing shocks separated by $1/64$ in the spatial coordinate (thick), from $1/32 - q$ to $1/32 + q$, and a decreasing shock along the dashed line halfway between them; the states are those of $\partial_x V_{\beta,q}^F$ in (3.6). The children of (3.7) (dark rectangles) lie in the shaded strips of (3.11). Only three of the M children on each shock are shown; their widths and durations are not to scale.

This change, together with the choice of the $2M$ children in (3.20), is the replacement in Q with parameters (β, q, M, R) . At a point where $\partial_x H_1^{F_{s,\varepsilon},\beta,q} = v$ in the normalized coordinates of (3.16), we have

$$(3.17) \quad \begin{aligned} \partial_x h &= s + \varepsilon v, \\ \partial_t h &= -f(s) - \varepsilon \sigma_{s,\varepsilon} v - \kappa_* \varepsilon^2 F_{s,\varepsilon}(v) = -f(s + \varepsilon v). \end{aligned}$$

The function equal to $h + c$ in Q and to h^{old} outside Q is locally Lipschitz, because $h + c = h^{\text{old}}$ on ∂Q , and by (3.17) it satisfies (3.3). Such a constant c exists: by (3.17), the potential obtained from (3.16) with $H_0^{F_{s,\varepsilon}}$ in place of $H_1^{F_{s,\varepsilon},\beta,q}$ has the same two affine gradients and the same shock line as h^{old} , so it differs from h^{old} by a constant on Q , and $H_1^{F_{s,\varepsilon},\beta,q} = H_0^{F_{s,\varepsilon}}$ on the boundary of $[0, 1] \times [-1, 1]$ by Proposition 3.1.

In the original coordinates, the two new shocks lie on

$$(3.18) \quad x_j(t) = \sigma_{s,\varepsilon} t + b_0 + \kappa_* \varepsilon \left(\theta_{\beta,q}^{F_{s,\varepsilon}}(t - t_0) + \frac{\beta j L}{64} \right), \quad j \in \{0, 1\}.$$

Their speed and states satisfy

$$(3.19) \quad \begin{aligned} \sigma_{s,\varepsilon} + \kappa_* \varepsilon \theta_{\beta,q}^{F_{s,\varepsilon}} &= \sigma_{s+\beta\varepsilon/32, q\varepsilon}, \\ u_- &= s + \beta\varepsilon/32 - q\varepsilon, \quad u_+ = s + \beta\varepsilon/32 + q\varepsilon. \end{aligned}$$

We reuse the symbols of (3.7) for the cells in the original coordinates. For child duration $\ell := RL/(2M)$, these cells are

$$(3.20) \quad \begin{aligned} t_i &:= t_0 + L \left(\frac{1}{5} + \left(i - \frac{1}{2} \right) \frac{3}{5M} \right), & I_i &:= [t_i - \ell/2, t_i + \ell/2], \\ Q_{ji} &:= \{(t, x) : t \in I_i, \quad |x - x_j(t)| \leq \kappa_* q \varepsilon \ell\}. \end{aligned}$$

The common shock center, the common half-jump, and the sum of the child durations are

$$(3.21) \quad s + \frac{\beta\varepsilon}{32}, \quad q\varepsilon, \quad 2M\ell = RL.$$

Thus Q_{ji} has the form (3.15), with (s, ε, t_0, L) replaced by $(s + \beta\varepsilon/32, q\varepsilon, t_i - \ell/2, \ell)$ and with central line $x = x_j(t)$. The central line of the parent cell Q is $x = \sigma_{s,\varepsilon}t + b_0$. By Proposition 3.1, the time gaps are at least $g\ell$, the spatial gaps at equal times are at least $c_{\text{sep}}\kappa_*\varepsilon L$, and every child lies in

$$\{(t, x) : t_0 + c_0L \leq t \leq t_0 + (1 - c_0)L, \quad c_0\kappa_*\varepsilon L \leq |x - \sigma_{s,\varepsilon}t - b_0| \leq (1 - c_0)\kappa_*\varepsilon L\}.$$

In Q , the new function $\partial_x h$ takes values in $[s - 8\varepsilon, s + 8\varepsilon]$; it differs from $\partial_x h^{\text{old}}$ only in Q , by at most 9ε . The discontinuities of the new and of the old function in Q lie on at most C_* shock segments, each of duration at most L .

Remark 3.2. Every point z in a child or any of its descendants satisfies

$$(3.22) \quad \text{dist}(z, \partial Q \cup \{x = \sigma_{s,\varepsilon}t + b_0\}) \geq \min \left\{ c_0L, \frac{c_0\kappa_*\varepsilon L}{\sqrt{1 + \sigma_{s,\varepsilon}^2}} \right\} \geq c_{\text{dist},f}\varepsilon L,$$

where $c_{\text{dist},f} := c_0 \min \left\{ 1, \frac{\kappa_*}{\sqrt{1 + \|f'\|_{L^\infty(I_f)}^2}} \right\} > 0$.

4. SCALES AND BOUNDED NON-CONVERGENT CENTERS

We choose half-jumps ε_n with $\sum_n \varepsilon_n = \infty$ and signs β_n such that $\liminf_n S_n = s_* - b$ and $\limsup_n S_n = s_* + b$, and we bound the sum of $W_j \varepsilon_j^2$ over the generations j with S_j near a fixed state (Lemma 4.2).

4.1. The scales. Replacing C_W by a larger constant, we may assume that the ratio bound in (1.12) holds for every $n \geq 1$:

$$(4.1) \quad W_n \nearrow \infty, \quad 1 \leq \frac{W_{n+1}}{W_n} \leq 1 + \frac{C_W}{n}, \quad \sum_{n=1}^{\infty} W_n^{-1/2} = \infty.$$

We choose $K_f = K_f(f, s_*, b) > 0$ so that

$$(4.2) \quad K_f \geq 512, \quad \frac{11}{20K_f} \leq \min \left\{ \frac{c_{\text{dist},f}}{2}, \frac{\kappa_*}{2048} \right\}.$$

We define

$$(4.3) \quad a_n := W_n^{-1/2}, \quad A_0 := e, \quad A_n := e + \sum_{j=1}^n a_j, \\ \varepsilon_n := \frac{a_n}{A_n}, \quad M_n := \left\lceil \frac{K_f}{\varepsilon_n} \right\rceil,$$

and

$$(4.4) \quad q_n := \frac{\varepsilon_{n+1}}{\varepsilon_n}, \quad R_n := \frac{W_{n+1}}{W_n}.$$

By (4.3), $W_n \varepsilon_n^2 = A_n^{-2}$. Since $(a_n)_n$ is non-increasing and $(A_n)_n$ is increasing, $(\varepsilon_n)_n$ is non-increasing, and, for all sufficiently large n ,

$$(4.5) \quad 0 < \varepsilon_n \leq \frac{1}{n}, \quad A_{n-1} \geq \frac{1}{2}A_n, \\ \varepsilon_n \leq \log A_n - \log A_{n-1} \leq 2\varepsilon_n, \\ q_n = \left(\frac{W_n}{W_{n+1}} \right)^{1/2} \frac{A_n}{A_{n+1}} \longrightarrow 1, \quad R_n \longrightarrow 1, \quad \sum_{n=1}^{\infty} \varepsilon_n = \infty.$$

The logarithmic bounds in (4.5) hold because $A_{n-1} \geq A_n/2$ and

$$\log A_n - \log A_{n-1} = \int_{A_{n-1}}^{A_n} \frac{ds}{s} \in \left[\frac{a_n}{A_n}, \frac{a_n}{A_{n-1}} \right].$$

By (4.1),

$$W_n \leq Cn^{C_W}, \quad A_n \leq e + na_1 \leq Cn.$$

Consequently,

$$(4.6) \quad C^{-1}n^{-(1+C_W)} \leq \varepsilon_n \leq \frac{1}{n}.$$

We choose N_0 so large that q_n , R_n , and M_n satisfy (3.5) for every $n \geq N_0$, that (6.8) holds for every $n > N_0$ (its threshold does not depend on N_0), that (7.13) holds (its constant does not depend on N_0), and

$$(4.7) \quad \varepsilon_{N_0} \leq b/8.$$

The bound (4.7) gives $b + 9\varepsilon_{N_0} < 3b$ in Lemma 4.1. Increasing N_0 does not change any constant in this section. We define

$$(4.8) \quad D_{N_0} := 1, \quad D_n := \prod_{j=N_0}^{n-1} (2M_j),$$

and

$$(4.9) \quad \ell_{N_0} := T/16, \quad \ell_{n+1} := \frac{R_n}{2M_n} \ell_n.$$

The number ℓ_{N_0} is the duration of the cell of generation N_0 , the root cell of Section 5; its value $T/16$ is not essential, and any duration smaller than $T/8$ would serve. The bound $K_f \geq 512$ in (4.2) gives $M_n \geq 512$, as required in (3.5), and the factor $11/20$ in (4.2) appears because

$$\frac{\ell_{n+1}}{\ell_n} = \frac{R_n}{2M_n} \leq \frac{11}{10} \frac{\varepsilon_n}{2K_f}.$$

By (3.5), (4.3) and (4.9), we obtain

$$(4.10) \quad \frac{\ell_{n+1}}{\ell_n} \asymp \varepsilon_n, \quad \ell_{n+1} \leq \frac{1}{2} \ell_n,$$

and

$$(4.11) \quad \frac{D_{n+1}\ell_{n+1}}{D_n\ell_n} = R_n,$$

hence

$$(4.12) \quad D_n\ell_n = \frac{TW_n}{16W_{N_0}}.$$

By (4.2), the duration of a child satisfies

$$(4.13) \quad \begin{aligned} \ell_{n+1} &\leq \frac{c_{\text{dist},f}}{2} \varepsilon_n \ell_n, \\ 16\ell_{n+1} &\leq \frac{\kappa_* \varepsilon_n \ell_n}{128}. \end{aligned}$$

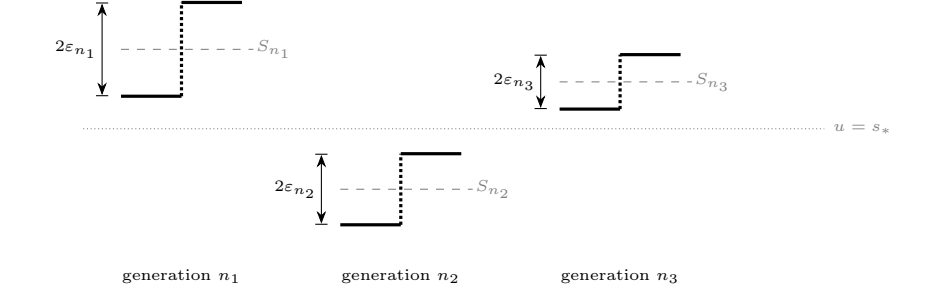
We use the first bound to obtain $B_{\ell_{n+1}}(z) \subset \text{int } Q_n(\omega)$ in (5.10). The second bound compares 16 child durations with the gap $c_{\text{sep}}\kappa_*\varepsilon_n\ell_n = \kappa_*\varepsilon_n\ell_n/128$ between the children on the two new shocks of a cell of generation n ; it separates the points of $\mathcal{C}$ with different indices j_n in the proof of Proposition 6.2, where any factor larger than 2 in place of 16 would do, and we take 16 as a convenient power of 2. Accordingly, $2048 = 16 \cdot 128$ in (4.2).

4.2. The centers. Starting at s_* , we move the center upward until $S_n \geq s_* + b$, then downward until $S_n \leq s_* - b$, and repeat. More precisely, we set $\beta_{N_0} := 1$ and, for $n > N_0$, $\beta_n := -\beta_{n-1}$ if $\beta_{n-1}(S_n - s_*) \geq b$ and $\beta_n := \beta_{n-1}$ otherwise, where

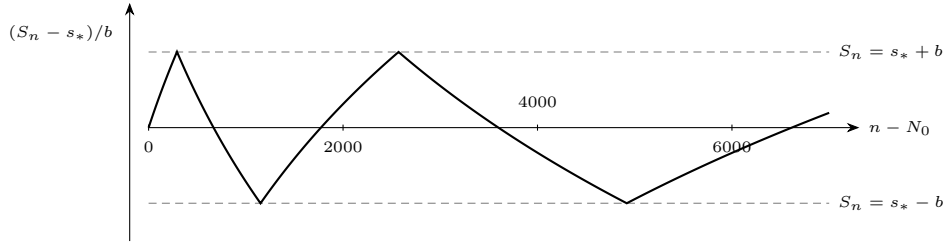
$$(4.14) \quad S_{N_0} := s_*, \quad S_{n+1} := S_n + \frac{\beta_n \varepsilon_n}{32}, \quad \beta_n \in \{-1, 1\}.$$

Since $\sum_n \varepsilon_n = \infty$ by (4.5), the sign β_n changes infinitely often. The center crosses $s_* + b$ or $s_* - b$ by at most $\varepsilon_n/32 \leq \varepsilon_{N_0}/32$, so

$$(4.15) \quad |S_n - s_*| \leq b + \frac{\varepsilon_{N_0}}{32}, \quad \limsup_{n \rightarrow \infty} S_n = s_* + b, \quad \liminf_{n \rightarrow \infty} S_n = s_* - b.$$



(A) Profiles with states $S_{n_i} \pm \varepsilon_{n_i}$ at generations $n_1 < n_2 < n_3$. The half-jumps shrink while the centers move.



(B) The centers S_n of (4.14) for $W_n = n$, $N_0 = 1004$, $b = 1/256$, and $s_* = 0$ (so that b has the form 2^{-k} required in (3.1) and $\varepsilon_{N_0} \leq b/8$ as in (4.7)), computed from the recursion and plotted every 25 indices together with every index at which the sign β_n changes. By (4.15), the centers lie in $[s_* - b - \varepsilon_{N_0}/32, s_* + b + \varepsilon_{N_0}/32]$.

FIGURE 3. The centers and the half-jumps of the nested cells.

Lemma 4.1. *The set of limit points of $(S_n)_n$ is $[s_* - b, s_* + b]$, and $[S_n - 8\varepsilon_n, S_n + 8\varepsilon_n] \subset (s_* - 3b, s_* + 3b) \subset I_f$ for $n \geq N_0$.*

Proof. Let $y \in [s_* - b, s_* + b]$. Between two consecutive changes of sign, $|S_n - y| \leq \varepsilon_n/32$ for some n ; since $\varepsilon_n \rightarrow 0$, y is a limit point of $(S_n)_n$. On the other hand, if $S_n \notin [s_* - b, s_* + b]$ and $m < n$ is the largest index with $S_m \in [s_* - b, s_* + b]$, then, by (4.14) and the choice of the signs, the distance to $[s_* - b, s_* + b]$ does not increase from index $m+1$ to index n , so $\text{dist}(S_n, [s_* - b, s_* + b]) \leq \varepsilon_m/32$; since $m \rightarrow \infty$ as $n \rightarrow \infty$, this distance tends to zero, and $(S_n)_n$ has no limit points outside $[s_* - b, s_* + b]$. Finally, by (4.7) and (4.15),

$$|S_n - s_*| + 8\varepsilon_n \leq b + 9\varepsilon_{N_0} < 3b.$$

□

4.3. The centers near a fixed state. In Section 7, the contribution of the cells of generation n to M_v vanishes unless $|S_n - v| \leq 8\varepsilon_n$, and its total variation is at most $CTW_n \varepsilon_n^2 / W_{N_0}$ (see (4.12) and (7.6)). The next lemma sums these bounds.

Lemma 4.2. *For every $\Theta > 0$, there exists $C_\Theta > 0$, with the dependencies in Section 2, such that*

$$(4.16) \quad \sup_{\alpha \in \mathbb{R}} \sum_{j=n}^{\infty} W_j \varepsilon_j^2 \mathbf{1}_{\{|S_j - \alpha| \leq \Theta \varepsilon_j\}} \leq C_\Theta W_n \varepsilon_n^2 \quad (n \geq N_0).$$

Proof. Step 1: Indices near α between two sign changes. We fix an interval of indices on which the centers are monotone, let k be the first index satisfying $|S_k - \alpha| \leq \Theta \varepsilon_k$, and choose an integer $N_\Theta > 128\Theta$. Since $q_n \rightarrow 1$, for all sufficiently large k ,

$$\varepsilon_{k+N_\Theta} \geq \frac{1}{2} \varepsilon_k, \quad |S_{k+N_\Theta} - S_k| = \frac{1}{32} \sum_{j=k}^{k+N_\Theta-1} \varepsilon_j \geq \frac{N_\Theta}{64} \varepsilon_k > 2\Theta \varepsilon_k,$$

provided $k + N_\Theta$ belongs to the same interval of monotonicity. Any index $j \geq k + N_\Theta$ in that interval with $|S_j - \alpha| \leq \Theta\varepsilon_j$ would satisfy, by the monotonicity of the centers,

$$2\Theta\varepsilon_k < |S_{k+N_\Theta} - S_k| \leq |S_j - S_k| \leq \Theta(\varepsilon_j + \varepsilon_k) \leq 2\Theta\varepsilon_k,$$

a contradiction. Including the finitely many indices k for which $\varepsilon_{k+N_\Theta} \geq \varepsilon_k/2$ need not hold, we obtain at most C_Θ such indices per interval, independently of N_0 and α .

Step 2: Sum over the sign changes. Let $n_1 < n_2$ be consecutive sign-change indices, that is, consecutive indices n with $\beta_n \neq \beta_{n-1}$. The choice of the signs gives $\beta_{n-1}(S_n - s_*) \geq b$ for $n \in \{n_1, n_2\}$, with opposite signs β_{n_1-1} and β_{n_2-1} . By (4.5), (4.7) and (4.14),

$$\begin{aligned} \sum_{j=n_1}^{n_2-1} \varepsilon_j &= 32|S_{n_2} - S_{n_1}| \geq 64b, \\ \log \frac{A_{n_2}}{A_{n_1}} &\geq \sum_{j=n_1+1}^{n_2} \varepsilon_j \geq 64b - \varepsilon_{n_1} \geq 32b, \\ W_{n_2} \varepsilon_{n_2}^2 &\leq e^{-64b} W_{n_1} \varepsilon_{n_1}^2. \end{aligned}$$

In the first line, $64b = 32 \cdot 2b$ because $|S_{n_2} - S_{n_1}| \geq 2b$, the length of $[s_* - b, s_* + b]$; in the second line, $64b - \varepsilon_{n_1} \geq 32b$ because $\varepsilon_{n_1} \leq b/8$. The initial incomplete interval contributes at most $C_\Theta W_n \varepsilon_n^2$, since $W_j \varepsilon_j^2 = A_j^{-2}$ is non-increasing in j . Summing the remaining intervals as a geometric series, we obtain

$$\sum_{j=n}^{\infty} W_j \varepsilon_j^2 \mathbf{1}_{\{|S_j - \alpha| \leq \Theta\varepsilon_j\}} \leq C_\Theta W_n \varepsilon_n^2 \left(1 + \sum_{i=0}^{\infty} e^{-64bi} \right) \leq C_\Theta W_n \varepsilon_n^2.$$

□

5. THE SOLUTION AND THE SET E

Let $\varepsilon := \varepsilon_{N_0}$. We define

$$h^{(N_0)}(t, x) := \min \left\{ s_* x - f(s_*)t, \max_{\pm} \{ (s_* \pm \varepsilon)x - f(s_* \pm \varepsilon)t \} \right\}, \quad u^{(N_0)} := \partial_x h^{(N_0)}.$$

For $t > 0$, we have $u^{(N_0)} = s_* - \varepsilon$ for $a_- t < x < \sigma_{s_*, \varepsilon} t$, $u^{(N_0)} = s_* + \varepsilon$ for $\sigma_{s_*, \varepsilon} t < x < a_+ t$, and $u^{(N_0)} = s_*$ otherwise, where

$$a_- := \frac{f(s_*) - f(s_* - \varepsilon)}{\varepsilon}, \quad a_+ := \frac{f(s_* + \varepsilon) - f(s_*)}{\varepsilon}, \quad \sigma_{s_*, \varepsilon} = \frac{a_- + a_+}{2}.$$

The *root cell* is the cell (3.15) with $s = s_*$, $b_0 = 0$, $t_0 = T/2$, and $L = \ell_{N_0} = T/16$; it is the only cell of generation N_0 . By (3.1), $f'' \geq \kappa_*/2$ on I_f , so, for $t \geq T/2$, the spatial distance at fixed t from its central line $x = \sigma_{s_*, \varepsilon} t$ to each of the lines $x = a_{\pm} t$ is

$$\frac{a_+ - a_-}{2} t \geq \frac{\kappa_* \varepsilon t}{4} \geq \frac{\kappa_* \varepsilon T}{8} > \frac{\kappa_* \varepsilon T}{16}.$$

Thus the root cell lies in $\{a_- t < x < a_+ t\}$, where $h^{(N_0)}$ is the maximum of the two affine potentials with states $s_* \pm \varepsilon$, and in $[T/2, 9T/16] \times \mathbb{R}$. The start time $T/2$ keeps $u = u^{(N_0)}$ for $t < T/2$, which we use in (5.5), and gives the lower bound $\kappa_* \varepsilon T/8$.

The children of a cell are labeled by the pairs (j, i) of (3.20), where $j \in \{0, 1\}$ labels the shock $x = x_j(t)$ and $i \in \{1, \dots, M_n\}$ labels the time interval I_i . We let

$$\mathcal{A}_n := \{0, 1\} \times \{1, \dots, M_n\}, \quad \Sigma_n := \prod_{j=N_0}^{n-1} \mathcal{A}_j, \quad \Sigma := \prod_{n \geq N_0} \mathcal{A}_n.$$

By Kolmogorov's extension theorem (see [Dur19, Theorem 2.1.21]), the uniform probabilities on the finite sets $\mathcal{A}_n$ define a product probability ρ on Σ . For $n > N_0$ and $w = (w_{N_0}, \dots, w_{n-1}) \in \Sigma_n$, let $[w] := \{\omega \in \Sigma : \omega_j = w_j \text{ for } N_0 \leq j < n\}$. By (4.8),

$$(5.1) \quad \rho([w]) = \prod_{j=N_0}^{n-1} (2M_j)^{-1} = D_n^{-1}.$$

Let $\mathcal{Q}_{N_0}$ consist of the root cell. For $n \geq N_0$, we define $h^{(n+1)}$ from $h^{(n)}$ by the replacement (3.16) in every $Q \in \mathcal{Q}_n$, with $(s, \varepsilon, L) = (S_n, \varepsilon_n, \ell_n)$ and $(\beta, q, M, R) = (\beta_n, q_n, M_n, R_n)$, and we let $\mathcal{Q}_{n+1}$ consist of the children Q_{ji} , $(j, i) \in \mathcal{A}_n$, of all $Q \in \mathcal{Q}_n$. By (3.21), (4.4), (4.9) and (4.14), every cell of generation $n+1$ has center S_{n+1} , half-jump ε_{n+1} , and duration ℓ_{n+1} . Each $w = (w_{N_0}, \dots, w_{n-1}) \in \Sigma_n$ determines one cell $Q_w \in \mathcal{Q}_n$: starting from the root cell, w_j labels a child of the cell of generation j for $N_0 \leq j < n$. Every $h^{(n)}$ is continuous and piecewise affine and satisfies (3.3), so $u^{(n)} := \partial_x h^{(n)}$ is a weak solution. For $Q \in \mathcal{Q}_n$, let $\Delta u_Q := (u^{(n+1)} - u^{(n)})\mathbf{1}_Q$. By Proposition 3.1 and Lemma 4.1, almost everywhere in Ω_T ,

$$(5.2) \quad |u^{(n)} - s_*| \leq 3b < a, \quad |\Delta u_Q| \leq 9\varepsilon_n \mathbf{1}_Q, \quad |Q| = 2\kappa_* \varepsilon_n \ell_n^2.$$

Let U_n be the union of the cells in $\mathcal{Q}_n$, and set

$$(5.3) \quad E := \bigcap_{n \geq N_0} U_n.$$

The sets U_n are compact and nested. By (4.10), (4.11) and (5.2) and $q_n \leq 1$, $R_n \leq 11/10$,

$$|U_n| = 2\kappa_* D_n \varepsilon_n \ell_n^2, \quad \frac{|U_{n+1}|}{|U_n|} = R_n q_n \frac{\ell_{n+1}}{\ell_n} \leq \frac{11}{20}.$$

Hence $|E| \leq |U_n| \rightarrow 0$. If $z \in \Omega_T \setminus E$, then $z \notin U_{n_z}$ for some n_z ; since U_{n_z} is compact and $u^{(n)} = u^{(n_z)}$ outside U_{n_z} for $n \geq n_z$, the functions $u^{(n)}$, $n \geq n_z$, coincide on a neighborhood of z . We define $u := \lim_n u^{(n)}$ on $\Omega_T \setminus E$ and $u := s_*$ on E . Almost everywhere in Ω_T ,

$$(5.4) \quad |u - s_*| \leq 3b < a, \quad |u - u^{(n)}| \leq 6b \mathbf{1}_{U_n}, \quad \|u - u^{(n)}\|_{L^1(\Omega_T)} \leq 6b|U_n| \rightarrow 0.$$

By (5.4) and the Lipschitz continuity of f on I_f , $\|f(u^{(n)}) - f(u)\|_{L^1(\Omega_T)} \rightarrow 0$, so u is a weak solution of (1.9). Since $u = u^{(N_0)}$ outside the root cell, which lies in $[T/2, 9T/16] \times \mathbb{R}$ and in $\{a_-t < x < a_+t\}$,

$$(5.5) \quad \begin{aligned} \text{supp}(u(t, \cdot) - s_*) &\subseteq [a_-t, a_+t] && \text{for almost every } t \in (0, T), \\ \|u(t, \cdot) - s_*\|_{L^1(\mathbb{R})} &= \varepsilon(a_+ - a_-)t \leq \|f''\|_{L^\infty(I_f)} \varepsilon^2 t && \text{for } 0 < t < T/2. \end{aligned}$$

In particular, $\lim_{t \downarrow 0} \|u(t, \cdot) - s_*\|_{L^1(\mathbb{R})} = 0$.

For every $t \in [0, T]$, we take $u^{(n)}(t, \cdot)$ to be the derivative of $x \mapsto h^{(n)}(t, x)$, which exists for almost every $x \in \mathbb{R}$. For these representatives, the bounds in (5.2) hold for almost every $x \in \mathbb{R}$ at every time t , including the times at which shocks meet, because $x \mapsto h^{(n)}(t, x)$ is continuous and piecewise affine with slopes among the values of $u^{(n)}$ on the regions adjacent to the line $\{t\} \times \mathbb{R}$. At each time t , the cells of $\mathcal{Q}_n$ that meet $\{t\} \times \mathbb{R}$ have the same time interval, and their number is at most 2^{n-N_0} , one for each $(j_{N_0}, \dots, j_{n-1}) \in \{0, 1\}^{n-N_0}$. Hence, for $n' \geq n$, since $|u^{(n')} - u^{(n)}| \leq 6b \mathbf{1}_{U_n}$ and $2^{n-N_0} \ell_n \leq \ell_{N_0}$ by (4.10),

$$(5.6) \quad \begin{aligned} \mathcal{L}^1(\{x : (t, x) \in U_n\}) &\leq 2^{n-N_0+1} \kappa_* \varepsilon_n \ell_n, \\ \sup_{t \in [0, T]} \|u^{(n')}(t, \cdot) - u^{(n)}(t, \cdot)\|_{L^1(\mathbb{R})} &\leq 12b\kappa_* 2^{n-N_0} \varepsilon_n \ell_n \leq 12b\kappa_* \ell_{N_0} \varepsilon_n \rightarrow 0. \end{aligned}$$

Each $u^{(n)}$ is constant on each of finitely many polygonal regions, whose boundaries consist of shock segments with speeds in $[-\|f'\|_{L^\infty(I_f)}, \|f'\|_{L^\infty(I_f)}]$ and of parts of $\{t=0\}$ and $\{t=T\}$, and $|u^{(n)} - s_*| \leq 3b$; hence $t \mapsto u^{(n)}(t, \cdot) - s_*$ is Lipschitz from $[0, T]$ to $L^1(\mathbb{R})$. By (5.6), $(u^{(n)} - s_*)_n$ is a Cauchy sequence in $C([0, T]; L^1(\mathbb{R}))$; by (5.4), its limit is a representative of $u - s_*$, which we use from now on:

$$(5.7) \quad u^{(n)} - s_* \rightarrow u - s_* \quad \text{in } C([0, T]; L^1(\mathbb{R})).$$

For $\omega \in \Sigma$, let $Q_{N_0}(\omega)$ be the root cell and, for $n > N_0$, $Q_n(\omega) := Q_w$ for $w = (\omega_{N_0}, \dots, \omega_{n-1}) \in \Sigma_n$. Its central line has slope $\sigma_{S_n, \varepsilon_n}$, with $|\sigma_{S_n, \varepsilon_n}| \leq \|f'\|_{L^\infty(I_f)}$, and its half-width is $\kappa_* \varepsilon_n \ell_n$, so

$$(5.8) \quad \text{diam } Q_n(\omega) \leq C \ell_n.$$

Since the sets $Q_{n+1}(\omega) \subset Q_n(\omega)$ are compact and $\text{diam } Q_n(\omega) \rightarrow 0$, we have $\bigcap_n Q_n(\omega) = \{\Pi(\omega)\}$ for some $\Pi(\omega) \in \Omega_T$. If ω and ω' agree on the indices $N_0, \dots, n-1$, then $|\Pi(\omega) - \Pi(\omega')| \leq C \ell_n$, so

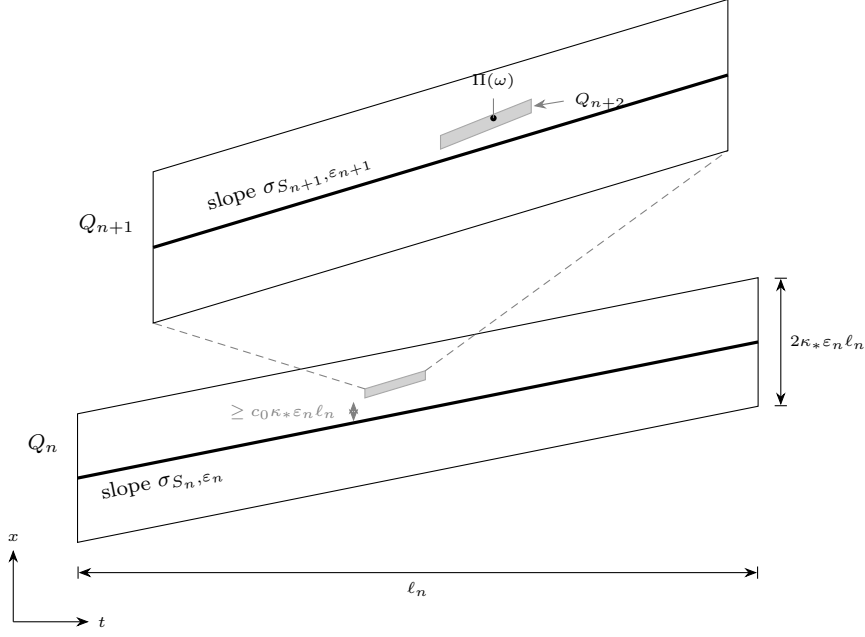


FIGURE 4. Two consecutive cells Q_n and Q_{n+1} . The shocks of successive generations (heavy lines) have different speeds, and the limit point $\Pi(\omega)$ lies off the central line of every cell.

Π is continuous; it is injective because the children of a cell are pairwise disjoint (Proposition 3.1). The cells of $\mathcal{Q}_n$ are pairwise disjoint, so each $z \in E$ lies in exactly one cell of $\mathcal{Q}_n$ for every n ; hence

$$(5.9) \quad E = \Pi(\Sigma), \quad \lambda := \Pi_{\#}\rho.$$

Thus $E \Subset \Omega_T$ is compact, λ is a Borel probability measure, and $\lambda(Q_w) = \rho([w]) = D_n^{-1}$ for every $n > N_0$ and $w \in \Sigma_n$, by (5.1), because $\Pi^{-1}(Q_w) = [w]$; also $\lambda(Q_{N_0}(\omega)) = 1 = D_{N_0}^{-1}$.

By the distances in (3.22) and the first bound in (4.13), every $z = \Pi(\omega)$ satisfies

$$(5.10) \quad \begin{aligned} B_{\ell_{n+1}}(z) &\subset \text{int } Q_n(\omega), \\ |u^{(n)} - S_n| &= \varepsilon_n \quad \text{almost everywhere on } B_{\ell_{n+1}}(z), \\ B_{\ell_{n+1}}(z) &\text{ meets no shock of } u^{(n)}. \end{aligned}$$

For every $n \geq N_0$, no shock of $u^{(n)}$ meets E .

6. THE SIZE OF E

The set E is the image of a product of two Cantor sets, one in the time variable and one in the space variable: a point's time coordinate depends only on the labels i_n , and its spatial coordinate, after subtracting a Lipschitz function of time, depends only on the labels j_n . Since ρ is a product measure, the sequences $(i_n)_n$ and $(j_n)_n$ of $\omega = ((j_n, i_n))_n \in \Sigma$ are independent random variables on the probability space (Σ, ρ) , so that λ is the image of a product measure as well.

6.1. A product of two Cantor sets. We call the distinct time intervals of the cells in $\mathcal{Q}_n$ the generation- n time intervals. By (3.20), these intervals depend only on $i_{N_0}, \dots, i_{n-1}$. They are pairwise disjoint, and each interval of generation $n+1$ lies in one of generation n (Proposition 3.1). Every cell meets E , since it contains $\Pi(\omega)$ for every sequence ω whose first indices label that cell. Since the sets U_n are compact and nested, the time projection K of (1.25) is

$$(6.1) \quad K = \bigcap_{n \geq N_0} \bigcup_{\text{generation-}n \text{ time intervals } \mathcal{I}} \mathcal{I}.$$

For $t \in K$, let $\tau_n(t)$ be the left endpoint of its unique generation- n time interval. Applying (3.18) and (3.19) in every generation, starting from the central line $x = \sigma_{S_{N_0}, \varepsilon_{N_0}} t$ of the root cell, we

separate the displacement due to the indices i_n from the displacement due to the indices j_n :

$$(6.2) \quad \begin{aligned} \varphi(t) &:= \sigma_{S_{N_0}, \varepsilon_{N_0}} t + \sum_{n \geq N_0} (\sigma_{S_{n+1}, \varepsilon_{n+1}} - \sigma_{S_n, \varepsilon_n})(t - \tau_n(t)), \quad t \in K, \\ \mathcal{C} &:= \left\{ \sum_{n \geq N_0} \frac{\kappa_* \beta_n \varepsilon_n \ell_n}{64} j_n : j_n \in \{0, 1\} \right\}, \\ E &= \{(t, \varphi(t) + y) : t \in K, y \in \mathcal{C}\}. \end{aligned}$$

The series defining φ converges uniformly on K , and the series defining $\mathcal{C}$ converges uniformly in $(j_n)_n$, because $(\varepsilon_n)_n$ is non-increasing and $\ell_{n+1} \leq \ell_n/2$ by (4.10), and

$$(6.3) \quad \begin{aligned} |\sigma_{S_{n+1}, \varepsilon_{n+1}} - \sigma_{S_n, \varepsilon_n}| &\leq \|f''\|_{L^\infty(I_f)} \left(|S_{n+1} - S_n| + \frac{|\varepsilon_{n+1} - \varepsilon_n|}{2} \right) \leq C\varepsilon_n, \\ 0 \leq t - \tau_n(t) &\leq \ell_n, \quad \sum_{j \geq n} \varepsilon_j \ell_j \leq 2\varepsilon_n \ell_n. \end{aligned}$$

The map $\Psi(t, y) = (t, \varphi(t) + y)$ sends $K \times \mathcal{C}$ onto E . The next lemma extends φ to $\mathbb{R}$ and shows that Ψ is bi-Lipschitz.

Lemma 6.1. *The function φ is Lipschitz on K , with constant independent of N_0 and T . We extend φ affinely on each bounded connected component of $\mathbb{R} \setminus K$, by $\varphi(\min K)$ on $(-\infty, \min K]$, and by $\varphi(\max K)$ on $[\max K, \infty)$. Then the map*

$$\Psi(t, y) := (t, \varphi(t) + y)$$

is a bi-Lipschitz map of $\mathbb{R}^2$ onto itself.

Proof. Let $t, t' \in K$, and let $k+1$ be the first generation at which their time intervals differ. By the separation of the time intervals in Proposition 3.1,

$$(6.4) \quad |t - t'| \geq g\ell_{k+1}, \quad \tau_j(t) = \tau_j(t') \quad (N_0 \leq j \leq k).$$

Owing to (6.4), the differences between the terms of (6.2) at t and at t' with $n \leq k$ sum to $(\sigma_{S_{k+1}, \varepsilon_{k+1}} - \sigma_{S_{N_0}, \varepsilon_{N_0}})(t - t')$, and the terms with $n > k$ are bounded by (6.3) at t and at t' . With $|\sigma_{S_{k+1}, \varepsilon_{k+1}}| \leq \|f'\|_{L^\infty(I_f)}$ and $\ell_{k+1} \leq |t - t'|/g$, we obtain

$$\begin{aligned} |\varphi(t) - \varphi(t')| &\leq |\sigma_{S_{k+1}, \varepsilon_{k+1}}| |t - t'| + C\varepsilon_{k+1} \ell_{k+1} \\ &\leq (\|f'\|_{L^\infty(I_f)} + C\varepsilon_{N_0}/g) |t - t'|. \end{aligned}$$

Affine interpolation preserves the Lipschitz constant. The inverse map is $(t, x) \mapsto (t, x - \varphi(t))$. $\square$

Let λ_K be the image of ρ under the map sending ω to the first coordinate of $\Pi(\omega)$, and let $\lambda_{\mathcal{C}}$ be the image of ρ under $((j_n, i_n))_n \mapsto \sum_{n \geq N_0} \kappa_* \beta_n \varepsilon_n \ell_n j_n / 64$. The map $(j_n)_n \mapsto \sum_{n \geq N_0} \kappa_* \beta_n \varepsilon_n \ell_n j_n / 64$ is injective (cf. Step 2 of the proof of Proposition 6.2). Since $(i_n)_n$ and $(j_n)_n$ are independent random variables on (Σ, ρ) and, by (6.2), $\Pi(\omega) = \Psi(t, y)$ with t depending only on $(i_n)_n$ and y only on $(j_n)_n$, λ is the image of $\lambda_K \otimes \lambda_{\mathcal{C}}$ under Ψ , $\lambda_K(\mathcal{I}) = 2^{n-N_0}/D_n$ for every generation- n time interval $\mathcal{I}$, and, for fixed $j_{N_0}, \dots, j_{n-1} \in \{0, 1\}$, the points of $\mathcal{C}$ with these indices have $\lambda_{\mathcal{C}}$ -measure $2^{-(n-N_0)}$.

The set K is covered by the $D_n/2^{n-N_0}$ generation- n time intervals, each of length ℓ_n . Moreover, by (6.3), the points of $\mathcal{C}$ with fixed indices $j_{N_0}, \dots, j_{n-1}$ form a set of diameter at most $\sum_{k \geq n} \kappa_* \varepsilon_k \ell_k / 64 \leq \kappa_* \varepsilon_n \ell_n / 32$, so $\mathcal{C}$ is covered by 2^{n-N_0} intervals of length $\kappa_* \varepsilon_n \ell_n / 32$. By $W_n \leq Cn^{C_w}$ and (4.6), (4.10) and (4.12), for every $s > 0$,

$$(6.5) \quad \begin{aligned} \mathcal{L}^1(K) &\leq \frac{D_n \ell_n}{2^{n-N_0}} = \frac{TW_n}{16W_{N_0} 2^{n-N_0}} \longrightarrow 0, \\ 2^{n-N_0} (\kappa_* \varepsilon_n \ell_n / 32)^s &\longrightarrow 0, \quad \dim_{\mathcal{H}} \mathcal{C} = 0, \quad \mathcal{L}^1(\mathcal{C}) = 0. \end{aligned}$$

Both K and $\mathcal{C}$ are Cantor sets: they are compact, they are Lebesgue-null by (6.5) and hence totally disconnected, and they are perfect, because every generation- n time interval contains $M_n \geq 2$ generation- $(n+1)$ time intervals, each of which meets K , and every point of $\mathcal{C}$ is the limit of the points of $\mathcal{C}$ that differ from it in one index j_n . A product of two compact Lebesgue-null subsets of $\mathbb{R}$ is purely 1-unrectifiable: if $\gamma = (\gamma_1, \gamma_2): [0, 1] \rightarrow \mathbb{R}^2$ is Lipschitz and $S_\gamma := \gamma^{-1}(K \times \mathcal{C})$, then,

by the area formula [AFP00, Theorem 2.71] applied to γ_1 and γ_2 , and because $\gamma_1(S_\gamma) \subset K$ and $\gamma_2(S_\gamma) \subset \mathcal{C}$ are Lebesgue-null,

$$\mathcal{H}^1(\gamma(S_\gamma)) \leq \int_{S_\gamma} |\gamma'| ds \leq \sum_{i=1}^2 \int_{S_\gamma} |\gamma'_i| ds = \sum_{i=1}^2 \int_{\gamma_i(S_\gamma)} \#(S_\gamma \cap \gamma_i^{-1}(y)) dy = 0.$$

Since both Ψ and Ψ^{-1} are Lipschitz, the inverse image of a Lipschitz curve meets $K \times \mathcal{C}$ in an $\mathcal{H}^1$ -null set. Applying Ψ gives

$$(6.6) \quad \mathcal{H}^1(E \cap \Lambda) = 0 \quad \text{for every Lipschitz curve } \Lambda \subset \mathbb{R}^2.$$

For each compact interval $\mathcal{I} \subset (0, T) \setminus K$, the unions in (6.1) are compact and decrease in n , so $(\mathcal{I} \times \mathbb{R}) \cap U_n = \emptyset$ for some n ; hence

$$(6.7) \quad u = u^{(n)} \quad \text{on } \mathcal{I} \times \mathbb{R} \quad \text{for some } n = n(\mathcal{I}).$$

6.2. The measure of balls, the number of cells, and the size of E . We now compare $\lambda(B_r(z))$ with r at the scale ℓ_n , using the product structure of (6.2).

Proposition 6.2. *For all $n > N_0$ larger than a threshold that depends only on f , s_* , b , and $(W_n)_n$, and for $\ell_n \leq r < \ell_{n-1}$,*

$$(6.8) \quad C^{-1} \frac{r}{D_n \ell_n} \leq \lambda(B_r(z)) \leq C \frac{r}{D_n \ell_n} \quad (z \in E), \quad \text{where} \quad \frac{r}{D_n \ell_n} = \frac{16W_{N_0}}{T} \frac{r}{W_n}.$$

The upper bound also holds for every $z \in \mathbb{R}^2$.

Proof. Step 1: Time intervals. Distinct time intervals of generation n are separated by at least $g\ell_n$: at the generation where they first differ, their separation is at least $g\ell_j$ for some $j \leq n$. An interval of radius r therefore meets at most $C(1 + r/\ell_n)$ such intervals.

For the lower bound, let $t \in K$. The generation- $(n-1)$ time interval containing t contains M_{n-1} generation- n time intervals, whose midpoints $t_1 < \dots < t_{M_{n-1}}$ satisfy $t_{i+1} - t_i = 3\ell_{n-1}/(5M_{n-1}) = 6\ell_n/(5R_{n-1}) \in [12\ell_n/11, 6\ell_n/5]$ by (3.20) and (4.9). Moreover, t is within $\ell_n/2$ of one midpoint, and at least $(M_{n-1} - 1)/2$ of the other midpoints lie on one side of that midpoint. Hence at least

$$C^{-1} \min\{M_{n-1}, 1 + r/\ell_n\} \geq C^{-1} r/\ell_n$$

midpoints lie within distance Cr of t , for a fixed C . The entire corresponding time intervals lie in $(t - Cr, t + Cr)$ because $\ell_n \leq r$. We count these complete intervals, each of λ_K -measure $2^{n-N_0}/D_n$, and obtain

$$(6.9) \quad \begin{aligned} \lambda_K((t - r, t + r)) &\leq C \frac{2^{n-N_0} r}{D_n \ell_n}, \\ \lambda_K((t - Cr, t + Cr)) &\geq C^{-1} \frac{2^{n-N_0} r}{D_n \ell_n}. \end{aligned}$$

The measure λ_K has no atoms, because $\lambda_K(\{t\}) \leq 2^{n-N_0}/D_n \rightarrow 0$ for every $t \in \mathbb{R}$.

Step 2: The indices j_n . By (3.5) and (4.9), $\ell_{k+1}/\ell_k = R_k/(2M_k) \leq (11/10)/1024 = 11/10240$, and $(\varepsilon_k)_k$ is non-increasing. Hence

$$\sum_{k>j} \varepsilon_k \ell_k \leq \varepsilon_j \ell_j \sum_{i=1}^{\infty} (11/10240)^i < \frac{1}{2} \varepsilon_j \ell_j.$$

If two points of $\mathcal{C}$ first differ at index j , their distance is therefore at least

$$\frac{\kappa_* \varepsilon_j \ell_j}{64} - \sum_{k>j} \frac{\kappa_* \varepsilon_k \ell_k}{64} \geq \frac{\kappa_* \varepsilon_j \ell_j}{128}.$$

If $n = N_0 + 1$, the upper bound in (6.10) holds because $\lambda_{\mathcal{C}} \leq 1$. If $n \geq N_0 + 2$ and two points of $\mathcal{C}$ first differ at an index $j \leq n - 2$, then, by the second bound in (4.13), their distance is at least $\kappa_* \varepsilon_{n-2} \ell_{n-2}/128 \geq 16\ell_{n-1} > 2r$. Hence all points of $\mathcal{C} \cap (y - r, y + r)$ have the indices $j_{N_0}, \dots, j_{n-2}$ of y . Two points of $\mathcal{C}$ with the same indices $j_{N_0}, \dots, j_{n-1}$ have distance at most $\kappa_* \varepsilon_n \ell_n/32 < \ell_n \leq r$ for all sufficiently large n . Thus

$$(6.10) \quad 2^{-(n-N_0)} \leq \lambda_{\mathcal{C}}((y - r, y + r)) \leq 2^{-(n-N_0-1)} \quad (y \in \mathcal{C}).$$

Step 3: Balls. Multiplying (6.9) and (6.10), we bound the product measure of squares, with a fixed dilation for the lower bound. By Lemma 6.1, the image of each square is contained in a ball of comparable radius, and the inverse image of each ball is contained in a square of comparable radius. Since $\ell_{n+1}/\ell_n \rightarrow 0$ and $W_{n+1}/W_n \rightarrow 1$, multiplication of a sufficiently small radius r by a fixed constant changes the index n with $\ell_n \leq r < \ell_{n-1}$ by at most one and changes r/W_n by a bounded factor; if the dilated radius exceeds ℓ_{N_0} , the required bounds hold because $\lambda_K \leq 1$, $\lambda_C \leq 1$, and $D_{N_0+1}\ell_{N_0+1} = R_{N_0}\ell_{N_0} \asymp \ell_{N_0}$. Hence the fixed dilations can be removed, which proves (6.8). If $z' \in B_r(z) \cap E$, then $B_r(z) \subset B_{2r}(z')$; if $B_r(z) \cap E = \emptyset$, then $\lambda(B_r(z)) = 0$. Hence the upper bound holds for every $z \in \mathbb{R}^2$. $\square$

The next proposition converts the bounds of Proposition 6.2 into bounds in terms of the gauge g_W of (1.13) and into the Hausdorff and packing measures used in Theorem 1.1(iv).

Proposition 6.3. *The gauge g_W in (1.13) is continuous and increasing for small r , and*

$$(6.11) \quad \lim_{r \downarrow 0} \frac{g_W(cr)}{g_W(r)} = c \quad (c > 0).$$

There exists $C_g = C_g(f, s_*, b, T, N_0, (W_n)_n) \geq 1$ such that, for all sufficiently small $r > 0$,

$$(6.12) \quad \begin{aligned} \lambda(B_r(z)) &\leq C_g g_W(r) & (z \in \mathbb{R}^2), \\ \lambda(B_r(z)) &\geq C_g^{-1} g_W(r) & (z \in E). \end{aligned}$$

For every Borel set $A \subset E$,

$$(6.13) \quad C_g^{-1} \lambda(A) \leq \mathcal{H}^{g_W}(A), \quad \mathcal{P}^{g_W}(A) \leq C_g \lambda(A).$$

Proof. In this proof, C_g and the constants implicit in $\asymp$ may change from line to line, with the dependencies stated above. By (1.13) and (4.1),

$$(6.14) \quad 0 \leq \frac{x \widetilde{W}'(x)}{\widetilde{W}(x)} \leq C_g \quad \text{for almost every } x \geq 1.$$

Using (4.3), (4.6) and (4.9), we obtain, as $n \rightarrow \infty$ with N_0 fixed,

$$(6.15) \quad \begin{aligned} \log \frac{\ell_{N_0}}{\ell_n} &= \sum_{j=N_0}^{n-1} \log(2M_j) - \log \frac{W_n}{W_{N_0}} \asymp n \log n, \\ \ell_n \leq r < \ell_{n-1} &\implies n \asymp \frac{L_r}{\log L_r}, \quad \widetilde{W}\left(\frac{L_r}{\log L_r}\right) \asymp W_n. \end{aligned}$$

The sum in (6.15) is comparable to $n \log n$ because $\log(2M_j) \asymp \log j$ by (4.3) and (4.6), whereas $\log W_n = O(\log n)$ because $W_n \leq Cn^{C_W}$. The last comparison in (6.15) follows by integrating (6.14) between the comparable arguments n and $L_r/\log L_r$. For $\ell_n \leq r < \ell_{n-1}$, we have $g_W(r) \asymp r/W_n$ by (6.15), so we obtain (6.12) by (6.8). Differentiating, we find

$$(6.16) \quad 1 \leq \frac{d \log g_W(r)}{d \log r} \leq 1 + \frac{C_g}{L_r} \quad \text{for almost every } r \in (0, r_1), \text{ for some } r_1 \in (0, T).$$

Integrating (6.16) between r and cr , we obtain (6.11).

For a Borel set $A \subset E$ and an open set $U \supset A$, we apply [Sim16, Theorem 2.1, Eqs. (2.1)–(2.2)] to $\lambda \llcorner U$ and to g_W , extended to a continuous increasing gauge on $(0, \infty)$ with $g_W(2r) \leq 3g_W(r)$ for every $r > 0$, for instance by $g_W(r_1)(r/r_1)^{1+C_g/L_{r_1}}$ for $r \geq r_1$, after decreasing r_1 so that $2^{C_g/L_{r_1}} \leq 3/2$, which is possible by (6.16); the two-sided bounds on $\lambda(B_r(z))$ required there hold for $z \in A$ and small r by (6.12). If $z \in A$ and $0 < r < \text{dist}(z, \mathbb{R}^2 \setminus U)$, then $(\lambda \llcorner U)(B_r(z)) = \lambda(B_r(z))$. Hence

$$C_g^{-1} \lambda(A) \leq \mathcal{H}^{g_W}(A), \quad \mathcal{P}^{g_W}(A) \leq C_g \lambda(U).$$

Taking the infimum over open $U \supset A$, we obtain (6.13). $\square$

The upper bound in (6.8) also bounds the number of cells that meet a ball, which we use in Section 8.

Corollary 6.4. *For every $n > N_0$ and every ball $B_r(z)$ with $0 < r < \ell_{n-1}$, and for $n = N_0$ and every ball $B_r(z)$, the number $N_n(B_r(z))$ of generation- n cells meeting the ball satisfies*

$$(6.17) \quad N_n(B_r(z)) \leq C \left(1 + \frac{r}{\ell_n}\right).$$

Proof. If a generation- n cell meets $B_r(z)$, its intersection with E lies in $B_{r+C\ell_n}(z)$ by (5.8). These intersections are disjoint and have λ -measure D_n^{-1} . From (6.8), applied at $\max\{r, \ell_n\}$ and its fixed multiples (as in Step 3 of the proof of Proposition 6.2, a fixed multiple of the radius changes the generation by at most one and the bound by a constant factor),

$$\frac{N_n(B_r(z))}{D_n} \leq \lambda(B_{r+C\ell_n}(z)) \leq \frac{C(r + \ell_n)}{D_n \ell_n}.$$

By the choice of N_0 in Section 4, the bounds (6.8) hold for every $n > N_0$, so the bound (6.17) holds for every $n > N_0$. For $n = N_0$, $N_{N_0}(B_r(z)) \leq 1$. $\square$

We can now prove the statements of Theorem 1.1 on the size of E .

Proof of Theorem 1.1(iv). By (6.6), E is purely 1-unrectifiable. The number b depends only on f , s_* , and a ; if K_f is the smallest number and N_0 the smallest integer admissible in Section 4, then N_0 depends only on f , s_* , a , T , δ , and $(W_n)_n$. By Proposition 6.3, the bounds (1.22) and (1.23) hold with $C_\lambda := C_g$, $c_\lambda := C_g^{-1}$, and r_0 the radius below which the bounds (6.12) hold, and these depend only on f , s_* , T , a , δ , and $(W_n)_n$. For $s > 0$, the generation- n cells cover E , and

$$\sum_{Q \in \mathcal{Q}_n} (\text{diam } Q)^{1+s} \leq C_s D_n \ell_n^{1+s} = \frac{C_s T}{16 W_{N_0}} W_n \ell_n^s \rightarrow 0,$$

where C_s also depends on s ; here $W_n \ell_n^s \rightarrow 0$ because $W_n \leq C n^{C_W}$ and $\log(\ell_{N_0}/\ell_n) \asymp n \log n$ by (6.15). Hence $\mathcal{H}^{1+s}(E) = 0$ for every $s > 0$, and $\dim_{\mathcal{H}} E \leq 1$. For $0 < s < 1$,

$$\frac{g_W(r)}{r^s} \leq \frac{r^{1-s}}{W_1} \rightarrow 0, \quad \mathcal{H}^s(E) = \infty,$$

by (6.13), so $\dim_{\mathcal{H}} E = 1$.

Let h be a gauge with $h(2r) \leq C_h h(r)$ for all sufficiently small $r > 0$, for some $C_h \geq 1$. We claim that

$$(6.18) \quad \limsup_{r \downarrow 0} \sup_{z \in \mathbb{R}^2} \frac{\lambda(B_r(z))}{h(r)} = 0 \implies \lambda(A) = 0 \text{ for every Borel set } A \subset \mathbb{R}^2 \text{ with } \mathcal{H}^h(A) < \infty.$$

To prove the claim, we fix $\varrho > 0$ and cover A by sets U_i with $\text{diam } U_i \leq \varrho$ and $\sum_i h(\text{diam } U_i) \leq \mathcal{H}^h(A) + 1$. A set U_i of positive diameter lies in the ball $B_{2 \text{diam } U_i}(z_i)$ for any $z_i \in U_i$, and a set U_i of diameter zero is a point, which is λ -null because $\lambda(B_r(z)) \leq h(r) \sup_{z'} \lambda(B_r(z'))/h(r) \rightarrow 0$ as $r \downarrow 0$. Hence

$$\begin{aligned} \lambda(A) &\leq \sum_i \lambda(B_{2 \text{diam } U_i}(z_i)) \leq \sup_{0 < r \leq 2\varrho} \sup_{z \in \mathbb{R}^2} \frac{\lambda(B_r(z))}{h(r)} \sum_i h(2 \text{diam } U_i) \\ &\leq C_h \sup_{0 < r \leq 2\varrho} \sup_{z \in \mathbb{R}^2} \frac{\lambda(B_r(z))}{h(r)} (\mathcal{H}^h(A) + 1), \end{aligned}$$

and the right-hand side tends to zero as $\varrho \downarrow 0$. Since $\mathcal{H}^h$ is Borel regular, every set of finite $\mathcal{H}^h$ -measure is contained in a Borel set of the same measure; since $\lambda(E) = 1$, the set E is therefore not $\mathcal{H}^h$ - σ -finite whenever the hypothesis of (6.18) holds. For $h(r) = r$, the hypothesis holds by (6.12), because

$$\frac{g_W(r)}{r} = \frac{1}{\widetilde{W}(L_r/\log L_r)} \rightarrow 0.$$

Hence E is not $\mathcal{H}^1$ - σ -finite. A set of finite $\mathcal{H}^1$ -measure is $\mathcal{H}^1$ - σ -finite, so $\mathcal{H}^1(E) = \infty$; since $E = N(u) \setminus J$ by Proposition 8.1, we obtain (1.24). $\square$

7. ENTROPY PRODUCTIONS AND THE KINETIC MEASURE

In this section, we construct the kinetic measure of the solutions from [Section 5](#) in the form $m(dz, dv) = M_v(dz) dv$, where each M_v is a finite signed measure on Ω_T . We prove that the measures $M_v^{(n)}$ defined in [\(7.1\)](#) below converge to M_v in total variation, uniformly in v , and that $v \mapsto M_v$ is continuous in total variation.

7.1. The entropy production in one cell. For the piecewise constant solution $u^{(n)}$, we define the measures $\zeta_k^{(n)}$ and their integrals $M_v^{(n)}$ in the state variable by

$$(7.1) \quad \zeta_k^{(n)} := (\partial_t + f'(k)\partial_x)\mathbb{1}_{\{u^{(n)} \geq k\}}, \quad M_v^{(n)} := \int_{-\infty}^v \zeta_k^{(n)} dk.$$

These are finite signed measures on Ω_T . On a shock segment of $u^{(n)}$ with left state s_1 , right state s_2 , and speed σ , the density of $\zeta_k^{(n)}$ with respect to the time variable along the segment is

$$(7.2) \quad (f'(k) - \sigma)(\mathbb{1}_{\{s_2 \geq k\}} - \mathbb{1}_{\{s_1 \geq k\}}), \quad \sigma = \frac{f(s_2) - f(s_1)}{s_2 - s_1}.$$

For an increasing shock segment ($s_1 < s_2$), the integral of [\(7.2\)](#) over $k \in (-\infty, v)$ is $-G_{s_1, s_2}^f(v)$, where G_{s_1, s_2}^f is the difference between the chord of f over $[s_1, s_2]$ and f :

$$G_{s_1, s_2}^f(v) := \left[f(s_1) + \frac{v - s_1}{s_2 - s_1} (f(s_2) - f(s_1)) - f(v) \right] \mathbb{1}_{\{s_1 < v < s_2\}}.$$

Thus, on an increasing shock segment $\{(t, x(t)) : t_1 \leq t \leq t_2\}$, the restriction of $m^{(n)} := M_v^{(n)} dv$ to the segment times $\mathbb{R}$ is

$$(7.3) \quad ((t, v) \mapsto (t, x(t), v))_{\#} (-\mathbb{1}_{[t_1, t_2]}(t) G_{s_1, s_2}^f(v) dt dv).$$

For a decreasing shock segment ($s_1 > s_2$), the density in [\(7.3\)](#) is $+\mathbb{1}_{[t_1, t_2]}(t) G_{s_2, s_1}^f(v)$. In particular, $|M_v^{(n)}|(\{z\}) = 0$ for every $z \in \Omega_T$. As $f'' > 0$ on I_f , we have $G_{s_1, s_2}^f > 0$ on (s_1, s_2) whenever $s_1 < s_2$ in I_f .

Because $u^{(n+1)} - u^{(n)} = \sum_{Q \in \mathcal{Q}_n} \Delta u_Q$ and Δu_Q vanishes outside Q , the measure $\zeta_k^{(n+1)} - \zeta_k^{(n)}$ is concentrated on U_n . For $Q \in \mathcal{Q}_n$, we define

$$\Delta \zeta_{k, Q} := (\zeta_k^{(n+1)} - \zeta_k^{(n)}) \llcorner Q, \quad \text{so that} \quad \zeta_k^{(n+1)} - \zeta_k^{(n)} = \sum_{Q \in \mathcal{Q}_n} \Delta \zeta_{k, Q}.$$

All states in Q lie in $[S_n - 8\varepsilon_n, S_n + 8\varepsilon_n]$. If σ is the speed of a shock segment in Q and k lies between its two states, then $|f'(k) - \sigma| \leq 16\|f''\|_{L^\infty(I_f)}\varepsilon_n$. Hence, by [\(3.9\)](#) and [\(7.2\)](#),

$$(7.4) \quad \|\Delta \zeta_{k, Q}\|_{\text{TV}} \leq C\ell_n \varepsilon_n \mathbb{1}_{\{|S_n - k| \leq 8\varepsilon_n\}}.$$

Since, for $s, s' \in \mathbb{R}$,

$$\int_{\mathbb{R}} (\mathbb{1}_{\{s \geq k\}} - \mathbb{1}_{\{s' \geq k\}}) dk = s - s', \quad \int_{\mathbb{R}} f'(k)(\mathbb{1}_{\{s \geq k\}} - \mathbb{1}_{\{s' \geq k\}}) dk = f(s) - f(s'),$$

we have

$$\int_{\mathbb{R}} (\zeta_k^{(n+1)} - \zeta_k^{(n)}) dk = \partial_t(u^{(n+1)} - u^{(n)}) + \partial_x(f(u^{(n+1)}) - f(u^{(n)})) = 0.$$

Restricting to Q , we obtain

$$(7.5) \quad \int_{\mathbb{R}} \Delta \zeta_{k, Q} dk = 0.$$

We define the primitive of $\Delta \zeta_{k, Q}$ in k by

$$\Delta M_{v, Q} := \int_{-\infty}^v \Delta \zeta_{k, Q} dk = - \int_v^\infty \Delta \zeta_{k, Q} dk.$$

The map $v \mapsto \Delta M_{v, Q}$ is continuous in total variation, vanishes for $|S_n - v| > 8\varepsilon_n$ (for $v > S_n + 8\varepsilon_n$ by [\(7.5\)](#)), and satisfies

$$(7.6) \quad \|\Delta M_{v, Q}\|_{\text{TV}} \leq C\ell_n \varepsilon_n^2 \mathbb{1}_{\{|S_n - v| \leq 8\varepsilon_n\}}.$$

The constants C in (7.4) and (7.6) have the dependencies fixed in Section 2 (in particular, they do not depend on N_0).

7.2. Convex entropies and the kinetic measure. Summing the bounds of Section 7.1 over all cells, we obtain the kinetic measure of u and its entropy productions.

Proposition 7.1. *There exist finite signed measures M_v on Ω_T , $v \in \mathbb{R}$, such that $\sup_{v \in \mathbb{R}} \|M_v^{(n)} - M_v\|_{\text{TV}} \rightarrow 0$, the map $v \mapsto M_v$ is continuous in total variation, and $M_v = 0$ for $v \notin (s_* - a, s_* + a)$. Moreover,*

$$(7.7) \quad \sup_{v \in \mathbb{R}} \|M_v\|_{\text{TV}} \leq CT\varepsilon_{N_0}^2, \quad |M_v|(E) = 0.$$

The measures $m^{(n)} := M_v^{(n)} dv$ converge in total variation to $m := M_v dv$, and (u, m) satisfies (1.5). For every continuous η whose distributional second derivative $D^2\eta$ is locally a finite signed measure, the entropy productions $\mu_\eta^{(n)} := \partial_t \eta(u^{(n)}) + \partial_x q_\eta(u^{(n)})$ converge in total variation to μ_η , and

$$(7.8) \quad \begin{aligned} \mu_\eta &= - \int_I M_v dD^2\eta(v), \\ |\mu_\eta|(A) &\leq \sup_{v \in I} |M_v|(A) |D^2\eta|(I) \quad \text{for every Borel set } A \subseteq \Omega_T. \end{aligned}$$

In particular, for every convex $\eta: \mathbb{R} \rightarrow \mathbb{R}$,

$$(7.9) \quad \|\mu_\eta\|_{\text{TV}} \leq CT\varepsilon_{N_0}^2 \eta''(I), \quad |\mu_\eta|(E) = 0.$$

Proof. Step 1: Uniform convergence with respect to v . By (4.12) and (7.6) and Lemma 4.2,

$$(7.10) \quad \begin{aligned} \sup_{v \in \mathbb{R}} \sum_{j=n}^{\infty} \sum_{Q \in \mathcal{Q}_j} \|\Delta M_{v,Q}\|_{\text{TV}} &\leq \frac{CT}{W_{N_0}} \sup_{v \in \mathbb{R}} \sum_{j=n}^{\infty} W_j \varepsilon_j^2 \mathbf{1}_{\{|s_j - v| \leq 8\varepsilon_j\}} \\ &\leq \frac{CTW_n \varepsilon_n^2}{W_{N_0}} \rightarrow 0. \end{aligned}$$

Here $W_n \varepsilon_n^2 = A_n^{-2} \rightarrow 0$ by (4.3) and (4.5). The three shocks of $u^{(N_0)}$, on the lines $x = a_-t$, $x = \sigma_{s_*, \varepsilon}t$, and $x = a_+t$, have total duration at most $3T$, and their states lie within ε_{N_0} of s_* . Hence

$$(7.11) \quad \sup_{v \in \mathbb{R}} \|M_v^{(N_0)}\|_{\text{TV}} \leq CT\varepsilon_{N_0}^2.$$

We define $M_v := M_v^{(N_0)} + \sum_{j \geq N_0} \sum_{Q \in \mathcal{Q}_j} \Delta M_{v,Q}$; by (7.10), the series converges in total variation uniformly in v , and $\sup_v \|M_v^{(n)} - M_v\|_{\text{TV}} \rightarrow 0$. By (7.10) with $n = N_0$ and (7.11), we obtain the first bound in (7.7). As each $v \mapsto M_v^{(n)}$ is continuous in total variation and vanishes outside I , its uniform limit has these properties. Each $M_v^{(n)}$ is concentrated on the shocks of $u^{(n)}$, which do not meet E by (5.10); hence $|M_v|(E) \leq \|M_v - M_v^{(n)}\|_{\text{TV}} \rightarrow 0$.

Step 2: The kinetic equation and entropy productions. We have

$$(7.12) \quad \|m^{(n)} - m\|_{\text{TV}} \leq 2a \sup_{v \in I} \|M_v^{(n)} - M_v\|_{\text{TV}} \rightarrow 0.$$

By (1.6), $\chi(v, s) = \mathbf{1}_{\{s > v\}} - \mathbf{1}_{\{0 > v\}}$ for $v \notin \{0, s\}$, so $\partial_t \chi(v, u^{(n)}) + f'(v) \partial_x \chi(v, u^{(n)}) = \zeta_v^{(n)} dv$ in the sense of distributions on $\Omega_T \times \mathbb{R}$, and $\partial_v m^{(n)} = \zeta_v^{(n)} dv$ by (7.1); thus $(u^{(n)}, m^{(n)})$ satisfies (1.5). Since

$$\int_{\mathbb{R}} |\chi(v, s) - \chi(v, s')| dv = |s - s'|,$$

we obtain

$$\|\chi(\cdot, u^{(n)}) - \chi(\cdot, u)\|_{L^1(\Omega_T \times \mathbb{R})} = \|u^{(n)} - u\|_{L^1(\Omega_T)} \rightarrow 0$$

by (5.4). Together with (7.12), this proves (1.5) for (u, m) . For a continuous η with locally finite $D^2\eta$, which is locally the difference of two convex functions and hence locally Lipschitz, we have, for $s \in \mathbb{R}$,

$$\eta(s) - \eta(0) = \int_{\mathbb{R}} \eta'(k)(\mathbf{1}_{\{s \geq k\}} - \mathbf{1}_{\{0 \geq k\}}) dk, \quad q_\eta(s) = \int_{\mathbb{R}} \eta'(k)f'(k)(\mathbf{1}_{\{s \geq k\}} - \mathbf{1}_{\{0 \geq k\}}) dk,$$

so that $\mu_\eta^{(n)} = \int_I \eta'(k)\zeta_k^{(n)} dk$. Integrating by parts in k , we obtain

$$\mu_\eta^{(n)} = - \int_I M_v^{(n)} dD^2\eta(v).$$

The boundary terms vanish because $M_{s_*-a}^{(n)} = M_{s_*+a}^{(n)} = 0$. We then have

$$\left| \mu_\eta^{(n)} + \int_I M_v dD^2\eta(v) \right| (\Omega_T) \leq \sup_{v \in I} \|M_v^{(n)} - M_v\|_{TV} |D^2\eta|(I) \rightarrow 0.$$

Since η and q_η are Lipschitz on I , by (5.4),

$$\|\eta(u^{(n)}) - \eta(u)\|_{L^1(\Omega_T)} + \|q_\eta(u^{(n)}) - q_\eta(u)\|_{L^1(\Omega_T)} \rightarrow 0.$$

Consequently, $\mu_\eta^{(n)} \rightarrow \partial_t \eta(u) + \partial_x q_\eta(u) = \mu_\eta$ in the sense of distributions. Therefore $\mu_\eta = - \int_I M_v dD^2\eta(v)$ and $\mu_\eta^{(n)} \rightarrow \mu_\eta$ in total variation, which gives (7.8); with (7.7), we also obtain (7.9). $\square$

We can now prove the first statement of Theorem 1.1.

Proof of Theorem 1.1(i). We choose N_0 so large that

$$(7.13) \quad CT\varepsilon_{N_0}^2 \leq \delta,$$

where C is the constant in (7.7), which is independent of N_0 . By (7.9) and (7.13), $|\mu_\eta|(\Omega_T) \leq \delta \eta''(I)$ and $|\mu_\eta|(E) = 0$ for every convex η . Moreover, by Proposition 7.1 and (7.13), $m(dz, dv) = M_v(dz) dv$, $\mu_\eta = - \int_I M_v d\eta''(v)$, $\sup_v |M_v|(\Omega_T) \leq \delta$, and $|M_v|(E) = 0$; these are (1.16) and (1.17). In addition, $\|u - s_*\|_{L^\infty(\Omega_T)} \leq 3b < a$ by (5.4). For almost every $t \in (0, T)$, $u(t, \cdot) - s_*$ vanishes outside $[a_-t, a_+t]$, so the support of $u - s_*$ lies in the compact set $\{(t, x) : 0 \leq t \leq T, a_-t \leq x \leq a_+t\}$, and $\|u(t, \cdot) - s_*\|_{L^1(\mathbb{R})} \rightarrow 0$ as $t \downarrow 0$ by (5.5). Finally, $u - s_* \in C([0, T]; L^1(\mathbb{R}))$ by (5.7). These are (1.14) and (1.15). $\square$

8. LOCAL BEHAVIOR ON E

The next proposition collects the local estimates at the points of E .

Proposition 8.1. *For $n \geq N_0 + 2$, $\ell_n \leq r < \ell_{n-1}$, and $z \in E$,*

$$(8.1) \quad \frac{1}{|B_r(z)|} \int_{B_r(z)} |u(y) - S_n| dy \leq C\varepsilon_n,$$

$$(8.2) \quad \sup_{v \in \mathbb{R}} \frac{|M_v(B_r(z))|}{r} + \frac{\nu(B_r(z))}{r} \leq C\varepsilon_n^2.$$

Moreover, $E = N(u) \setminus J$, $\lim_{r \downarrow 0} \sup_{z \in E} \text{osc}(u; B_r(z)) = 0$, and, for every $z \in E$, the set of limit points of $u_{B_r(z)}$ as $r \downarrow 0$ is $[s_* - b, s_* + b]$. For each $c \in [s_* - b, s_* + b]$, there exists a strictly increasing sequence $(n_j)_j$ such that

$$(8.3) \quad \lim_{j \rightarrow \infty} \sup_{z \in E} |u_{B_{\ell_{n_j}}}(z) - c| = 0.$$

Proof. Step 1: Averages. By (5.10), $u^{(n-2)}$ is constant on $B_r(z)$, with $|u^{(n-2)} - S_{n-2}| = \varepsilon_{n-2}$ on $B_r(z)$. Moreover, by (5.2), $|u^{(n)} - u^{(n-2)}| \leq 9(\varepsilon_{n-2} + \varepsilon_{n-1})$, and by (4.14), $|S_n - S_{n-2}| \leq (\varepsilon_{n-2} + \varepsilon_{n-1})/32$. Hence

$$(8.4) \quad |u^{(n)} - S_n| \leq \varepsilon_{n-2} + \left(9 + \frac{1}{32}\right)(\varepsilon_{n-2} + \varepsilon_{n-1}) \leq C\varepsilon_n \quad \text{almost everywhere on } B_r(z).$$

Here $\varepsilon_{n-1} \leq \frac{16}{15}\varepsilon_n$ and $\varepsilon_{n-2} \leq (\frac{16}{15})^2\varepsilon_n$, because $q_j \geq 15/16$ by (3.5). By (5.2) and (6.17) and $r \geq \ell_n$,

$$(8.5) \quad |U_n \cap B_r(z)| \leq C \frac{r}{\ell_n} \varepsilon_n \ell_n^2 \leq C \varepsilon_n r^2.$$

Since $|u - u^{(n)}| \leq 6b\mathbb{1}_{U_n}$ by (5.4), $|B_r(z)| = \pi r^2$, and $\text{osc}(u; B) \leq 2 \int_B |u - S_n| dy$ for every ball B , we obtain

$$\frac{1}{|B_r(z)|} \int_{B_r(z)} |u - S_n| dy \leq C \varepsilon_n + \frac{6b|U_n \cap B_r(z)|}{\pi r^2} \leq C \varepsilon_n, \quad \text{osc}(u; B_r(z)) \leq 2C \varepsilon_n.$$

Step 2: Measure densities. The measures $M_v^{(n-2)}$ vanish on $B_r(z)$ by (5.10). By (6.17), $N_{n-2}(B_r(z)) + N_{n-1}(B_r(z)) \leq C$. For $j \in \{n-2, n-1\}$ and $Q \in \mathcal{Q}_j$, the measure $\Delta M_{v,Q}$ is supported on at most C_* shock segments by (3.9), with the absolute value of the density of each old or new shock contribution bounded by $\sup_v G_{s_1, s_2}^f(v) \leq \|f''\|_{L^\infty(I_f)}(s_2 - s_1)^2/8 \leq C\varepsilon_j^2$ with respect to the time variable by (7.3), because all states $s_1 < s_2$ in Q lie in $[S_j - 8\varepsilon_j, S_j + 8\varepsilon_j]$; on a segment where an old and a new shock overlap, the absolute value of the density of the difference is at most the sum of the absolute values of the two densities. The part of each segment in $B_r(z)$ has duration at most $2r$.

For $j \geq n$, every $Q \in \mathcal{Q}_j$ with $Q \cap B_r(z) \neq \emptyset$ lies in a cell of generation n that meets $B_r(z)$. Each generation- n cell contains D_j/D_n generation- j cells. By (4.12), (4.16), (6.17) and (7.6),

$$\sup_v \frac{|M_v|(B_r(z))}{r} \leq C(\varepsilon_{n-2}^2 + \varepsilon_{n-1}^2) + \frac{C}{W_n} \sup_v \sum_{j=n}^{\infty} W_j \varepsilon_j^2 \mathbb{1}_{\{|v - S_j| \leq 8\varepsilon_j\}} \leq C\varepsilon_n^2.$$

Since $M_v = 0$ for $v \notin I_f$ by Lemma 4.1 and (7.5), $\nu(B_r(z)) \leq \int_{I_f} |M_v|(B_r(z)) dv \leq |I_f| \sup_v |M_v|(B_r(z)) = 8b \sup_v |M_v|(B_r(z))$, which gives the bound for ν in (8.2). In particular, $E \cap J = \emptyset$.

Step 3: Non-Lebesgue points. From (8.1),

$$\sup_{z \in E} |u_{B_{\ell_n}(z)} - S_n| \leq C \varepsilon_n \rightarrow 0.$$

By Lemma 4.1, for each $c \in [s_* - b, s_* + b]$ there exist $n_j \uparrow \infty$ with $S_{n_j} \rightarrow c$, which gives (8.3). Since $(S_n)_n$ does not converge, neither does $(u_{B_{\ell_n}(z)})_n$, so $E \subset N(u)$. By Lemma 4.1, $\text{dist}(S_n, [s_* - b, s_* + b]) \rightarrow 0$, so, by (8.1), $\lim_{r \downarrow 0} \sup_{z \in E} \text{dist}(u_{B_r(z)}, [s_* - b, s_* + b]) = 0$.

Conversely, let $z \in N(u) \setminus E$. A neighborhood of z is disjoint from U_n for some n . On this neighborhood, $u = u^{(n)}$ and $m = m^{(n)}$, so $z \in N(u^{(n)})$. Since $u^{(n)}$ is constant on each of finitely many polygonal regions whose boundaries in Ω_T consist of shock segments, z lies on a shock segment of $u^{(n)}$, possibly at one of its endpoints. By (7.3), on every shock segment of $u^{(n)}$ in this neighborhood, with states $s_1 \neq s_2$, $s_- := \min\{s_1, s_2\}$, and $s_+ := \max\{s_1, s_2\}$, the measure ν has the constant density $\int_{s_-}^{s_+} G_{s_-, s_+}^f(v) dv > 0$ with respect to the time variable, because $f'' > 0$ on I_f . For a shock segment of speed σ containing z , its part in $B_r(z)$ has duration at least $r/\sqrt{1 + \sigma^2}$ for all sufficiently small $r > 0$. Hence $z \in J$, because

$$\limsup_{r \downarrow 0} \frac{\nu(B_r(z))}{r} > 0.$$

We conclude that $N(u) \setminus J = E$. □

With Propositions 7.1 and 8.1, we can now prove the remaining statements of Theorem 1.1.

Proof of Theorems 1.1(ii) and 1.1(iii). By Proposition 7.1, $\nu(E) \leq \int_I |M_v|(E) dv = 0$. For $z \in E$ and $\ell_n \leq r < \ell_{n-1}$, by Proposition 8.1, $\text{osc}(u; B_r(z)) \leq C\varepsilon_n$ and $\sup_v |M_v|(B_r(z))/r + \nu(B_r(z))/r \leq C\varepsilon_n^2$; since $\varepsilon_n \rightarrow 0$ and $E = N(u) \setminus J$ by Proposition 8.1, we obtain (1.18), (1.19) and (1.21). Finally, for convex η , (1.16) gives $|\mu_\eta| \leq \int_I |M_v| d\eta''(v)$, so $|\mu_\eta|(B_r(z)) \leq \eta''(I) \sup_v |M_v|(B_r(z))$, and (1.19) gives (1.20). □

Proof of Theorem 1.1(v). The set K is compact, as the projection of the compact set E , and $\mathcal{L}^1(K) = 0$ by (6.5). By (6.7), $u(t, \cdot)$ is piecewise constant with finitely many jumps for $t \in$

$(0, T) \setminus K$. Moreover, $u(0, \cdot) = s_*$ by (1.15), and $u(T, \cdot) = u^{(N_0)}(T, \cdot)$ because the root cell lies in $[T/2, 9T/16] \times \mathbb{R}$. For $t \in K$,

$$E_t = \varphi(t) + \mathcal{C}.$$

Since $\mathcal{C}$ is perfect by Section 6.1 and $\dim_{\mathcal{H}} \mathcal{C} = 0$ by (6.5), the set E_t is perfect and $\dim_{\mathcal{H}} E_t = 0$.

We fix $t \in K$ and $x \in E_t$, and let $\Pi(\omega) = (t, x)$. By (5.10), the segment $\{t\} \times (x - \ell_{n+1}, x + \ell_{n+1})$ lies in $Q_n(\omega)$ and meets no shock of $u^{(n)}$, and $|u^{(n)} - S_n| = \varepsilon_n$ on it. Since $|\Delta u_Q| \leq 9\varepsilon_n$ by (5.2),

$$|u^{(n+1)}(t, \cdot) - S_n| \leq 10\varepsilon_n \quad \text{almost everywhere on } (x - \ell_{n+1}, x + \ell_{n+1}).$$

The cells of generation $n+1$ that meet $\{t\} \times (x - \ell_{n+1}, x + \ell_{n+1})$ are among the two children Q_{0i} and Q_{1i} of $Q_n(\omega)$ with $t \in I_i$, and $\mathcal{L}^1(\{y : (t, y) \in Q_{0i} \cup Q_{1i}\}) = 4\kappa_* \varepsilon_{n+1} \ell_{n+1}$ by (3.20). For every $N > n+1$, $|u^{(N)}(t, \cdot) - u^{(n+1)}(t, \cdot)| \leq 6b\mathbf{1}_{U_{n+1}}(t, \cdot)$ almost everywhere on $\mathbb{R}$. Letting $N \rightarrow \infty$ by (5.7), we obtain

$$(8.6) \quad \frac{1}{2\ell_{n+1}} \int_{x-\ell_{n+1}}^{x+\ell_{n+1}} |u(t, y) - S_n| dy \leq 10\varepsilon_n + 12b\kappa_* \varepsilon_{n+1} \leq C\varepsilon_n \rightarrow 0.$$

If x were a Lebesgue point of $u(t, \cdot)$, the centers S_n would converge by (8.6), contrary to Lemma 4.1. $\square$

9. EXPLICIT WEIGHTS AND GAUGES

We first give two explicit choices of $(W_n)_n$ satisfying (1.12) and compute the corresponding gauges g_W .

Example 9.1. For $W_n = n$, the weights satisfy (1.12) with $C_W = 1$, since $W_{n+1}/W_n = 1 + 1/n$ and $\sum_n n^{-1/2} = \infty$. With $A_n = e + \sum_{j \leq n} j^{-1/2}$, the scales in (4.3) satisfy

$$A_n \sim 2\sqrt{n}, \quad \varepsilon_n \sim \frac{1}{2n}, \quad M_n \asymp n.$$

The exponent in (1.13) equals 1, so $\widetilde{W}(s) = s$ and $g_W(r) = r \log L_r / L_r$. By Proposition 6.3, there exists $C_{\text{lin}} = C_{\text{lin}}(f, s_*, T, a, N_0) \geq 1$ such that, for all sufficiently small $r > 0$,

$$C_{\text{lin}}^{-1} r \frac{\log L_r}{L_r} \leq \lambda(B_r(z)) \leq C_{\text{lin}} r \frac{\log L_r}{L_r} \quad (z \in E).$$

Example 9.2. For $W_n = (n \log(en))^2$, $n \geq 1$, the weights satisfy (1.12), since $\log(e(n+1))/\log(en) \leq 1 + 1/n$ gives $W_{n+1}/W_n \leq (1 + 1/n)^4 \leq 1 + 15/n$, and $\sum_n W_n^{-1/2} = \sum_n (n \log(en))^{-1} = \infty$; the factor e in $\log(en)$ only ensures $W_1 = 1 > 0$. We have

$$A_n \sim \log \log n, \quad \varepsilon_n \sim \frac{1}{n \log n \log \log n}, \quad \log(\ell_{N_0}/\ell_n) \sim n \log n.$$

The asymptotics of A_n and ε_n follow by integral comparison in (4.3). For the durations, we compute from (4.3) and (4.9)

$$\log(\ell_{N_0}/\ell_n) = \sum_{j=N_0}^{n-1} \log(2M_j) - \log(W_n/W_{N_0}) \sim \sum_{j=N_0}^{n-1} \log j \sim n \log n.$$

For $\ell_n \leq r < \ell_{n-1}$, we therefore have $L_r \sim n \log n$, hence $W_n \asymp L_r^2$ and $\varepsilon_n^2 \asymp (L_r \log \log L_r)^{-2}$. By (6.8) and (8.2), there exists $C_{\log} = C_{\log}(f, s_*, T, a, N_0) \geq 1$ such that, for all sufficiently small $r > 0$,

$$C_{\log}^{-1} \frac{r}{L_r^2} \leq \lambda(B_r(z)) \leq C_{\log} \frac{r}{L_r^2} \quad (z \in E), \quad \sup_{z \in E} \frac{\nu(B_r(z))}{r} \leq \frac{C_{\log}}{L_r^2 (\log \log L_r)^2}.$$

Finally, we give weights for which $0 < \mathcal{H}^{h_2}(E) < \infty$ for a gauge h_2 with

$$h_2(r) = o\left(r L_r^{-2} \prod_{j=3}^k (\log_j(e^2 T/r))^{-2}\right) \quad \text{as } r \downarrow 0, \quad \text{for every integer } k \geq 3,$$

where $\log_j$ is the j -fold logarithm.¹¹ To this end, we define

$$(9.1) \quad \tau(x) := \begin{cases} \log x, & 1 \leq x \leq e, \\ 1 + \tau(\log x), & x \geq e, \end{cases} \quad P(x) := \frac{1}{\tau'(x)}.$$

At $x = e$, both formulas in (9.1) give $\tau(e) = 1$ and $\tau'(e) = 1/e$. Since $\tau'(x) = \tau'(\log x)/x$ for $x \geq e$, induction over the intervals $[e, e^e], [e^e, e^{e^e}], \dots$ gives $\tau \in C^1([1, \infty))$ with $\tau' > 0$; moreover, $\tau(x) \rightarrow \infty$ as $x \rightarrow \infty$. Differentiating (9.1), we obtain

$$(9.2) \quad P(x) = x \quad (1 \leq x \leq e), \quad P(x) = xP(\log x) \quad (x \geq e).$$

For almost every $x \geq e$,

$$\frac{xP'(x)}{P(x)} = 1 + \frac{P'(\log x)}{P(\log x)}.$$

On $[e, e^e]$, $P(x) = x \log x$, so $xP'(x)/P(x) = 1 + 1/\log x \leq 2$. On each later interval $[e^e, e^{e^e}], \dots$, the recursion and the bound on the previous interval give $xP'(x)/P(x) \leq 1 + 2/\log x \leq 1 + 2/e < 2$. Together with $P(x) = x$ on $[1, e]$, induction gives that P is locally absolutely continuous and

$$(9.3) \quad P(x) \geq x, \quad \frac{1}{x} \leq \frac{P'(x)}{P(x)} \leq \frac{2}{x} \quad \text{for almost every } x \geq 1.$$

Proposition 9.3. *Let*

$$(9.4) \quad W_n := [P(n)(1 + \tau(n))]^2.$$

For $\gamma \geq 0$ and sufficiently small $r > 0$, we define

$$(9.5) \quad h_\gamma(r) := \frac{r}{P\left(\frac{L_r}{\log L_r}\right)^2 \left(1 + \tau\left(\frac{L_r}{\log L_r}\right)\right)^\gamma}.$$

We set $h_\gamma(0) := 0$. The sequence $(W_n)_n$ of (9.4) satisfies (1.12). Let E be the set given by Theorem 1.1 for these weights. Then

$$\begin{cases} E \text{ is not } \mathcal{H}^{h_\gamma}\text{-}\sigma\text{-finite,} & 0 \leq \gamma < 2, \\ 0 < \mathcal{H}^{h_2}(E) < \infty, & \gamma = 2, \\ \mathcal{H}^{h_\gamma}(E) = 0, & \gamma > 2. \end{cases}$$

For $0 \leq \gamma < 2$,

$$(9.6) \quad \limsup_{r \downarrow 0} \sup_{z \in \mathbb{R}^2} \frac{\lambda(B_r(z))}{h_\gamma(r)} = 0.$$

Moreover, for every integer $k \geq 3$ and every $\gamma \geq 0$,

$$(9.7) \quad \lim_{r \downarrow 0} \frac{h_\gamma(r)}{r} L_r^2 \prod_{j=3}^k (\log_j(e^2 T/r))^2 = 0.$$

Proof. Step 1: The weights and the gauge h_2 . By (9.3) and (9.4), $P(x) \geq x$, and $1 + \tau \geq 1$, we obtain, for almost every $x \geq 1$ and every $X \geq 1$,

$$(9.8) \quad 0 \leq \frac{d}{dx} \log(P(x)^2(1 + \tau(x))^2) = 2 \frac{P'(x)}{P(x)} + \frac{2}{P(x)(1 + \tau(x))} \leq \frac{6}{x},$$

$$\int_1^X \frac{1}{P(x)(1 + \tau(x))} dx = \log(1 + \tau(X)) \rightarrow \infty.$$

The function $W(x) := P(x)^2(1 + \tau(x))^2$ is increasing, with $W(x) \geq x^2$. Hence $W_n = W(n) \rightarrow \infty$, $1 \leq W_{n+1}/W_n \leq (1 + 1/n)^6 \leq 1 + 63/n$ by (9.8), and $\sum_{n \leq N} W_n^{-1/2} \geq \int_1^{N+1} \frac{dx}{P(x)(1 + \tau(x))} = \log(1 + \tau(N+1)) \rightarrow \infty$. Hence $(W_n)_n$ satisfies (4.1) with $C_W = 63$. On $[n, n+1]$, both $\bar{W}$ and W take values in $[W_n, W_{n+1}] \subset [W_n, 64W_n]$, so $\bar{W} \asymp W$. By Proposition 6.3,

$$(9.9) \quad h_2(r) \asymp g_W(r), \quad 0 < \mathcal{H}^{h_2}(E) < \infty.$$

¹¹ We write $\log_1 x := \log x$ and $\log_{j+1} x := \log(\log_j x)$ whenever these expressions are defined.

Step 2: Comparison with the gauge h_2 . For fixed $\gamma \geq 0$ and almost every $x \geq 1$,

$$0 \leq \frac{d}{dx} \log(P(x)^2(1 + \tau(x))^\gamma) \leq \frac{4 + \gamma}{x}.$$

Hence, as in (6.16), h_γ is continuous and increasing near zero, and $h_\gamma(2r) \leq C_\gamma h_\gamma(r)$ for all sufficiently small $r > 0$. There exists $C_{\text{rec}} = C_{\text{rec}}(f, s_*, \gamma, T, a, N_0) \geq 1$ such that, for all sufficiently small $r > 0$,

$$(9.10) \quad \frac{\lambda(B_r(z))}{h_\gamma(r)} \leq C_{\text{rec}} \left(1 + \tau\left(\frac{L_r}{\log L_r}\right)\right)^{\gamma-2} \quad (z \in \mathbb{R}^2).$$

If $\gamma < 2$, then $(1 + \tau(L_r/\log L_r))^{\gamma-2} \rightarrow 0$, so we obtain (9.6) from (9.10), and, by (6.18), E is not $\mathcal{H}^{h_\gamma}$ - σ -finite. For $\gamma > 2$, we use (9.9) and

$$\frac{h_\gamma(r)}{h_2(r)} = \left(1 + \tau\left(\frac{L_r}{\log L_r}\right)\right)^{2-\gamma} \rightarrow 0.$$

Given $\vartheta > 0$, we choose $\varrho > 0$ with $h_\gamma \leq \vartheta h_2$ on $(0, \varrho)$. Covers of E by sets of diameter at most ϱ give $\mathcal{H}^{h_\gamma}(E) \leq \vartheta \mathcal{H}^{h_2}(E)$, so $\mathcal{H}^{h_\gamma}(E) = 0$.

Step 3: Comparison with finite products of logarithms. We fix an integer $k \geq 3$. Applying (9.2) k times, to $x, \log_1 x, \dots, \log_{k-1} x$, which is possible once $\log_{k-1} x \geq e$, we obtain, for all sufficiently large x ,

$$(9.11) \quad P(x) = x \prod_{j=1}^{k-1} \log_j x P(\log_k x), \quad \frac{(x \prod_{j=1}^{k-1} \log_j x)^2}{P(x)^2} = \frac{1}{P(\log_k x)^2} \rightarrow 0.$$

For $s \rightarrow \infty$,

$$(9.12) \quad \frac{s}{\log s} \prod_{j=1}^{k-1} \log_j \left(\frac{s}{\log s}\right) \sim s \prod_{j=2}^{k-1} \log_j s.$$

Taking $s = L_r$ and $x = L_r/\log L_r$ in (9.11) and (9.12), and using $\log_j L_r = \log_{j+1}(e^2 T/r)$ and $(1 + \tau)^\gamma \geq 1$ in (9.5), we obtain (9.7). $\square$

10. THE EIKONAL EQUATION

We apply Theorem 1.1 to the flux $f(v) = -\sqrt{1-v^2}$, with $|v| \leq a$. Let u be the solution of (1.9) given by Theorem 1.1 for this flux, and let ψ satisfy $\partial_x \psi = u$ and $\partial_t \psi = -f(u)$. The field $w = (-\partial_x \psi, \partial_t \psi)$ of Corollary 1.2 is the gradient of ψ rotated by a right angle. The proof of Corollary 1.2 is based on two observations: $N(w) = N(u)$, and, for every eikonal entropy Φ , the distribution $\text{div } \Phi(w)$ is a finite signed measure satisfying $|\text{div } \Phi(w)| \leq C_{\text{eik}} \|\Phi\|_{C^2(\mathbb{R}^2)} \nu$, where ν is defined by (1.7).

Proof of Corollary 1.2. Step 1: The flux and the potential. Let $\mathcal{R}(v) := (-v, \sqrt{1-v^2})$ for $|v| < 1$. For $0 < a < 1$, the image $\mathcal{R}([-a, a])$ is the arc of $\mathbb{S}^1$ with endpoints $(\mp a, \sqrt{1-a^2})$ and midpoint $\mathcal{R}(0) = (0, 1)$. Since Γ contains $(0, 1)$ in its interior, we can fix $0 < a < 1/2$ so small that $\mathcal{R}([-a, a]) \subset \Gamma$; the bound $a < 1/2$ keeps the derivatives of $\mathcal{R}$ on $[-a, a]$ bounded independently of a . We take a smooth extension to $\mathbb{R}$ of

$$(10.1) \quad f(v) = -\sqrt{1-v^2} \quad (|v| \leq a).$$

With this flux, $s_* = 0$, this a , $\delta := 1$, and the prescribed T and weights, we apply Theorem 1.1, and we let $u^{(n)}$ and $h^{(n)}$ be the solutions and potentials of Section 5. By (5.2), $|u^{(n)}| \leq 3b < a$, so, by (3.3) and (10.1), almost everywhere in Ω_T ,

$$\nabla h^{(n)} = (\partial_t h^{(n)}, \partial_x h^{(n)}) = \left(\sqrt{1-(u^{(n)})^2}, u^{(n)}\right), \quad |\nabla h^{(n)}| = 1.$$

We obtain ψ as the limit of these potentials. By (3.16), the potential $h^{(n+1)}$ differs from $h^{(n)}$ only inside the generation- n cells, and these cells lie in the root cell. Hence all $h^{(n)}$ coincide outside the root cell. We fix $x_0 \in \mathbb{R}$ smaller than the second coordinate of every point of the root cell. Then

$$h^{(n)}(t, x) = h^{(N_0)}(t, x_0) + \int_{x_0}^x u^{(n)}(t, y) dy \quad ((t, x) \in \Omega_T),$$

and, by (5.7), the potentials $h^{(n)}$ converge uniformly on Ω_T to

$$\psi(t, x) := h^{(N_0)}(t, x_0) + \int_{x_0}^x u(t, y) dy.$$

Since Ω_T is convex, every $h^{(n)}$ is 1-Lipschitz, and so is ψ . The derivatives $\partial_t h^{(n)}$ converge to $\sqrt{1-u^2}$ in $L^1_{\text{loc}}(\Omega_T)$. Hence, almost everywhere in Ω_T ,

$$(10.2) \quad \nabla \psi = (\sqrt{1-u^2}, u), \quad w = (-\partial_x \psi, \partial_t \psi) = \mathcal{R}(u), \quad |w| = 1.$$

In particular, w takes values in $\mathcal{R}([-a, a]) \subset \Gamma$, and $\text{div } w = -\partial_t \partial_x \psi + \partial_x \partial_t \psi = 0$ in the sense of distributions.

Step 2: The eikonal entropies. For an eikonal entropy Φ , we define

$$\eta_\Phi(v) := \Phi_1(\mathcal{R}(v)), \quad q_\Phi(v) := \Phi_2(\mathcal{R}(v)) \quad (v \in [-a, a]),$$

and extend both functions smoothly to $\mathbb{R}$. We claim that

$$(10.3) \quad q'_\Phi(v) = f'(v)\eta'_\Phi(v) \quad (v \in [-a, a]).$$

To prove it, we fix $v_0 \in (-a, a)$ and consider the smooth solution of the scalar law with flux $-\sqrt{1-v^2}$ given, for $t > 0$ and $x \in \mathbb{R}$, by

$$\bar{u}(t, x) := \frac{x/t}{\sqrt{1+(x/t)^2}}, \quad \text{so that} \quad \frac{\bar{u}}{\sqrt{1-\bar{u}^2}} = \frac{x}{t}, \quad \partial_t \bar{u} + \frac{\bar{u}}{\sqrt{1-\bar{u}^2}} \partial_x \bar{u} = 0.$$

The field $\bar{w} := \mathcal{R}(\bar{u})$ is smooth on the open half-plane $\{t > 0\}$, with $|\bar{w}| = 1$ and $\text{div } \bar{w} = -(\partial_t \bar{u} + \bar{u}(1-\bar{u}^2)^{-1/2} \partial_x \bar{u}) = 0$. By the definition of an eikonal entropy, $\text{div } \Phi(\bar{w}) = 0$. Near the point $\bar{z} := (1, v_0(1-v_0^2)^{-1/2})$, where $\bar{u} = v_0$ and $\partial_x \bar{u} = (1-v_0^2)^{3/2} > 0$, the values of $\bar{u}$ lie in $(-a, a)$, and

$$0 = \text{div } \Phi(\bar{w}) = \partial_t \eta_\Phi(\bar{u}) + \partial_x q_\Phi(\bar{u}) = (q'_\Phi(\bar{u}) - f'(\bar{u})\eta'_\Phi(\bar{u}))\partial_x \bar{u}.$$

Since $\partial_x \bar{u}(\bar{z}) \neq 0$, we obtain $q'_\Phi(v_0) = f'(v_0)\eta'_\Phi(v_0)$. As $v_0 \in (-a, a)$ is arbitrary and both sides are continuous on $[-a, a]$, (10.3) holds. By (10.2), $\Phi(w) = (\eta_\Phi(u), q_\Phi(u))$. Since $q_\Phi - q_{\eta_\Phi}$ is constant on $[-a, a]$ by (10.3), $\text{div } \Phi(w) = \partial_t \eta_\Phi(u) + \partial_x q_{\eta_\Phi}(u) = \mu_{\eta_\Phi}$. We apply (7.8) to $\eta_\Phi \in C^2(\mathbb{R})$, with $m(dz, dv) = M_v(dz) dv$ concentrated on $\Omega_T \times [-a, a]$, and by the definition of ν in (1.7),

$$(10.4) \quad \text{div } \Phi(w) = - \int_{\mathbb{R}} \eta''_\Phi(v) dm(\cdot, v), \quad |\text{div } \Phi(w)| \leq \sup_{[-a, a]} |\eta''_\Phi| \nu \leq C_{\text{eik}} \|\Phi\|_{C^2(\mathbb{R}^2)} \nu.$$

Here, for $|v| \leq a < 1/2$,

$$\begin{aligned} \eta''_\Phi &= D^2 \Phi_1(\mathcal{R})[\mathcal{R}', \mathcal{R}'] + \nabla \Phi_1(\mathcal{R}) \cdot \mathcal{R}'', \\ |\mathcal{R}'(v)|^2 &= \frac{1}{1-v^2} \leq \frac{4}{3}, \quad |\mathcal{R}''(v)| = \frac{1}{(1-v^2)^{3/2}} \leq \left(\frac{4}{3}\right)^{3/2}, \end{aligned}$$

so $\sup_{[-a, a]} |\eta''_\Phi| \leq (4/3 + (4/3)^{3/2}) \|\Phi\|_{C^2(\mathbb{R}^2)} = C_{\text{eik}} \|\Phi\|_{C^2(\mathbb{R}^2)}$. By (1.18) and (10.4), we obtain (1.28).

Step 3: Non-Lebesgue points, mean oscillation, and σ -finiteness. The map $\mathcal{R}$ is bi-Lipschitz on $[-a, a]$, with inverse $(y_1, y_2) \mapsto -y_1$: since $|\mathcal{R}'(v)| = (1-v^2)^{-1/2}$,

$$|v - v'| \leq |\mathcal{R}(v) - \mathcal{R}(v')| \leq \frac{|v - v'|}{\sqrt{1-a^2}} \quad (v, v' \in [-a, a]).$$

Therefore, for $v_0 \in [-a, a]$ and a ball $B \Subset \Omega_T$,

$$(10.5) \quad \int_B |u - v_0| dz \leq \int_B |w - \mathcal{R}(v_0)| dz \leq \frac{1}{\sqrt{1-a^2}} \int_B |u - v_0| dz.$$

Since $w \in \mathcal{R}([-a, a])$ almost everywhere in Ω_T , we have $\int_{B_r(z)} |w - y| dz' \geq \text{dist}(y, \mathcal{R}([-a, a]))$ for every $y \in \mathbb{R}^2$. If y is a Lebesgue value of w at z , the left-hand side tends to zero, so $y = \mathcal{R}(v_0)$ with $v_0 = -y_1 \in [-a, a]$. By (10.5), v_0 is a Lebesgue value of u at z . Conversely, u takes values in $[-a, a]$, so a Lebesgue value v_0 of u lies in $[-a, a]$, and $\mathcal{R}(v_0)$ is a Lebesgue value of w . For the oscillation, we use $\text{osc}(w; B) \leq 2 \int_B |w - \mathcal{R}(v_0)| dz$ and (10.5) with $v_0 = u_B \in [-a, a]$. Therefore

$$(10.6) \quad N(w) = N(u), \quad \text{osc}(w; B) \leq \frac{2}{\sqrt{1-a^2}} \text{osc}(u; B).$$

By (10.6) and Theorem 1.1(iii), $E \subset N(u) = N(w)$ and

$$\sup_{z \in E} \text{osc}(w; B_r(z)) \leq \frac{2}{\sqrt{1-a^2}} \sup_{z \in E} \text{osc}(u; B_r(z)) \rightarrow 0 \quad (r \downarrow 0);$$

this is (1.29). The set E and the measure λ are those of Theorem 1.1, so they have the properties in Theorem 1.1(iv). If Z is $\mathcal{H}^1$ - σ -finite, then so is $E \cap Z$. Since a union of two $\mathcal{H}^1$ - σ -finite sets is $\mathcal{H}^1$ - σ -finite and E is not, neither $E \setminus Z$ nor its superset $N(w) \setminus Z$ is $\mathcal{H}^1$ - σ -finite. As a set of finite $\mathcal{H}^1$ -measure is $\mathcal{H}^1$ - σ -finite, we obtain (1.30). Since the jump set J_w of w given by [DO03, Theorem 1(a)] is $\mathcal{H}^1$ - σ -finite, the choice $Z = J_w$ is admissible in (1.30), and it gives $\mathcal{H}^1(N(w) \setminus J_w) = \infty$. $\square$

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