

# GLOBAL MINIMALITY OF LAMELLAE FOR ISOTROPIC OHTA-KAWASAKI AND YUKAWA ENERGIES

DARIO REGGIANI

**ABSTRACT.** We establish a sharp comparison principle for the sharp-interface Ohta–Kawasaki energy and its Yukawa-screened counterpart in dimensions  $2 \leq d \leq 5$ , for  $m = 0$ , and under periodic and zero-flux Neumann boundary conditions. Whenever the finite optimal stripe width is compatible with the domain, we prove global minimality of periodic lamellae and characterize all minimizers. The proof combines a positive radial comparison measure, a sharp Fourier bound, and rigidity for a directional translation inequality in  $BV$ . We also identify an obstruction to this measure construction in dimensions  $d \geq 6$ .

## 1. INTRODUCTION

The competition between interfacial energy and nonlocal repulsion is a fundamental mechanism underlying pattern formation and symmetry breaking. Interfacial energy favours the formation of large domains, whereas repulsive interactions favour their subdivision into smaller regions. The balance between these effects can select a characteristic length scale and give rise to ordered patterns, including droplets, cylinders, and lamellae. A central example is the Ohta–Kawasaki model, introduced in [40] to describe microphase separation in diblock copolymer melts.

A diblock copolymer consists of two chemically distinct polymer blocks joined by a covalent bond. The incompatibility of the two blocks favours segregation, while their connectivity prevents separation into macroscopically large pure phases. The Ohta–Kawasaki functional describes this mechanism through a local phase-separation energy coupled to a nonlocal term of Coulomb type (see the derivation from a mean-field description in [10]).

In the standard formulation, the diffuse-interface energy takes the form

$$\int_{\Omega} \left( \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \right) dx + \gamma \langle u - m, (-\Delta)^{-1}(u - m) \rangle_{L^2(\Omega)}.$$

Here  $u$  is an order parameter describing the local composition,  $W$  is a suitable double-well potential,  $\varepsilon$  measures the thickness of the transition layers, and

$$m = \frac{1}{|\Omega|} \int_{\Omega} u \, dx$$

is the prescribed average composition. In the sharp-interface description, the order parameter takes values in  $\{-1, 1\}$  and the local contribution becomes an interfacial energy. Absorbing the surface-tension constant into  $J > 0$ , the corresponding functional is

$$J|Du|(\Omega) + \gamma \langle u - m, (-\Delta)^{-1}(u - m) \rangle_{L^2(\Omega)}, \quad u \in BV(\Omega; \{-1, 1\}).$$

A closely related class of models is obtained by replacing the Coulomb interaction with a screened Coulomb, or Yukawa, interaction. At the operator level, this amounts to replacing the inverse Laplacian by

$$R_{\lambda} = (-\Delta + \lambda^2)^{-1}, \quad \lambda > 0.$$

Screened repulsive interactions are particularly relevant to charged colloidal suspensions and models of protein solutions. In these systems, short-range attraction can compete with electrostatic repulsion screened by the surrounding medium, leading to aggregation and spatially modulated structures (see [48, 7]).

The present paper addresses the sharp-interface case with  $m = 0$ . That is

$$\mathcal{E}_{\lambda, J, \gamma}(u) = J|Du|(\Omega) + \gamma \langle u, R_{\lambda} u \rangle_{L^2(\Omega)},$$

with periodic or homogeneous Neumann boundary conditions. For  $\lambda = 0$ , we work with zero-average configurations and interpret  $R_0$  as the inverse Laplacian on that subspace. For  $\lambda > 0$ , we retain the constant mode of the resolvent and impose no average constraint (still taking  $m = 0$  in the definition of our energy). Our interest is in the exact selection of lamellar configurations: periodically alternating parallel layers whose width is determined by the competition between the two terms.

**1.1. Overview of the literature.** Early rigorous results on periodic pattern selection were obtained in one dimension. Müller [38] proved periodicity of minimizers for a related singularly perturbed variational problem. For the diblock copolymer functional, Ren and Wei [41, 42] studied local minimizers and the emergence of increasingly fine periodic oscillations in singular limits.

In higher dimensions, identifying the geometry of minimizers remains a major challenge, owing to the nonlocal interactions and the wider range of competing configurations. A fundamental result of Alberti, Choksi, and Otto [4] shows that the energy of minimizers of the sharp-interface Ohta–Kawasaki functional is approximately uniformly distributed over sufficiently large subdomains (see also [46]). This establishes a rigorous form of spatial homogeneity at scales above the characteristic domain size.

Concerning critical points and local minimality, Ren and Wei [43] studied wriggled lamellar solutions and their stability in the diblock copolymer problem. Choksi and Sternberg [11] derived the first and second variations of the sharp-interface nonlocal isoperimetric functional, providing a framework for the analysis of specific morphologies. Acerbi, Fusco, and Morini [2] subsequently proved that strict positivity of the second variation implies quantitative local minimality with respect to  $L^1$  perturbations (see also [31] for the Neumann case and [21] for the anisotropic counterpart). Cristoferi [15] further developed the theory of periodic critical points and local minimizers, constructing locally minimizing configurations near strictly stable periodic constant-mean-curvature hypersurfaces. Additional criteria for the stability and local minimality of lamellar configurations were obtained by Acerbi, Chen, and Choi [1].

Global minimality has also been established in particular geometric or parameter regimes. Sternberg and Topaloglu [47] proved that, on the flat two-dimensional torus, a single lamellar region is globally minimizing for sufficiently weak nonlocal interaction and an appropriate range of volume fractions. Morini and Sternberg [37] proved that minimizers in sufficiently thin domains are striped, with a sequence of transitions in the optimal number of layers as the interaction strength varies. These results identify regimes in which the geometry of the domain or a perturbation of the local isoperimetric problem allows a complete description of minimizers.

A complementary literature concerns the small-mass and dilute regimes. The geometry of individual droplets and the minimality of balls at small mass are studied in [12, 32, 33, 30, 6, 20], while effective energies, droplet distributions, and confinement are analysed in [39, 8, 9, 27, 28, 34, 35, 3].

An analogous competition arises in spin systems with short-range ferromagnetic and long-range antiferromagnetic interactions, where lamellar patterns, consisting of stripes or slabs, are expected to minimize the energy in suitable regimes near the ferromagnetic transition. For these systems and related mean-field models, reflection positivity and localization have led to periodic-ordering results [23, 24], asymptotic optimality of stripes and slabs [25], and exact striped or slabbed ground states near the transition for sufficiently rapidly decaying interactions [26].

For anisotropic continuum energies, Goldman and Runa [29] identified periodic stripes as minimizers of a limiting functional obtained by  $\Gamma$ -convergence, while Daneri and Runa [17] proved exact periodic stripe formation. Related results extend to diffuse-interface models [16] and Yukawa-type interactions with  $\ell^1$  anisotropy [18]. Finally, Daneri and Runa [19] recently obtained the first rigorous proof of spontaneous lamellar pattern formation for fully isotropic continuum energies with competing local and nonlocal interactions, establishing the global minimality of periodic lamellae near the critical interaction strength for suitable power-law kernels.

Despite this progress, to the best of our knowledge, the exact selection of lamellae for the Ohta–Kawasaki functional and its screened variants remain largely open in dimension  $d \geq 2$  beyond the special geometric or perturbative regimes described above.

**1.2. Overview of the results and proof strategy.** In the present paper we address this gap. In particular, we establish a sharp lower bound for the screened and unscreened energies in dimensions  $2 \leq d \leq 5$ , under periodic and zero-flux Neumann boundary conditions. The comparison holds for every admissible binary  $BV$  configuration and yields the global minimality of periodic lamellae whenever

their optimal width is compatible with the domain. We also characterize all equality cases, obtaining a complete description of the global minimizers under the corresponding compatibility condition.

The relevant length scale is the width  $h_*$  that minimizes the energy density among periodic stripes. It is characterized by

$$J = \gamma j_\lambda(h_*),$$

where  $j_\lambda$  is defined in (3.5). In the unscreened case, this gives the explicit scale

$$h_* = \left( \frac{12J}{\gamma} \right)^{1/3}.$$

In the screened case, a finite optimal width exists precisely when  $0 < J/\gamma < \lambda^{-3}$ , and it is unique. Our main estimate is

$$\mathcal{E}_{\lambda, J, \gamma}^b(u) \geq \gamma \sigma_\lambda(h_*) L^d, \quad b \in \{\text{per}, N\},$$

where  $\gamma \sigma_\lambda(h_*)$  is the optimal stripe energy density, given by (3.6). Thus the value obtained by optimizing the one-dimensional stripe family provides a lower bound for the full multidimensional problem.

On the torus  $\mathbb{T}_L^d$ , the lower bound is attained if and only if there exists  $a \in \mathbb{Z}^d \setminus \{0\}$  such that  $|a| = L/(2h_*)$ . Whenever this condition holds, all global minimizers are of the form

$$u(x) = \text{sign} \cos \left( \frac{2\pi}{L} a \cdot x + \varphi \right), \quad |a| = \frac{L}{2h_*}, \quad \varphi \in \mathbb{R}.$$

The admissible directions therefore include oblique stripes, according to the geometry of the frequency lattice. For the Neumann problem on  $Q_L = (0, L)^d$ , equality holds precisely when  $n = L/h_*$  is a positive integer, and the complete class of global minimizers is

$$u(x) = \pm \text{sign} \cos \left( \frac{\pi n x_j}{L} \right), \quad j \in \{1, \dots, d\}.$$

The boundary phase intervals have width  $h_*/2$ , while consecutive interfaces are separated by  $h_*$ . In the screened case, no constraint on the average is imposed: attainment of the sharp bound itself forces the two phases to have equal volume. The precise statements are given in Theorems 3.1 and 3.2.

The comparison rests on the construction of a finite positive radial measure supported in  $\overline{B_{h_*}}$ . Its mass and first absolute directional moments reproduce, respectively, the optimal stripe energy density and the ratio  $J/\gamma$ . Its Fourier transform lies below the multiplier  $(\lambda^2 + |\xi|^2)^{-1}$  of the interaction operator, with equality at every odd harmonic of the stripe frequency  $\pi/h_*$ . The construction of this measure is reminiscent of the auxiliary-function method of Cohn and Elkies [13] and its sharp realizations in the sphere-packing problems in dimensions eight and twenty-four [49, 14]. The common principle is to impose suitable constraints in physical and Fourier space so that the resulting bound is saturated by the candidate optimal configuration. These properties connect a geometric estimate for translations of  $BV$  functions (appearing also in [22, 36]) with a spectral estimate for the nonlocal energy. The resulting energy excess admits an exact decomposition into two nonnegative terms, measuring the loss in the translation inequality and the loss in the Fourier comparison.

The equality analysis relies on a rigidity property of the directional translation inequality. In particular, we prove that a non-constant binary  $BV$  function whose translation defect vanishes for all sufficiently short displacements must have complete parallel interfaces. More precisely, vanishing of the defect in  $B_{h_*}$  forces the interfaces to be separated by at least  $h_*$ . The vanishing spectral defect then gives anti-periodicity at distance  $h_*$  and fixes every layer width to be exactly  $h_*$ . This yields the periodic classification. Even reflection transfers the comparison and its equality cases to the Neumann problem, where the reflection symmetries further determine the stripe direction and phase.

The construction of the comparison measure also explains the dimensional range of the argument. In the Appendix, we show that the prescribed Fourier multipliers cannot be the Fourier transform of a finite positive measure in any dimension  $d \geq 6$ . This identifies an obstruction to extending the present comparison construction to higher dimensions.

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**AI usage statement.** The main ideas underlying this work were suggested by OpenAI's ChatGPT 6 Astra following a discussions. The author subsequently worked through the arguments and rewrote the proofs in their present form, including a section explaining the intuition behind the approach. LLM tools were then used for language editing to improve the clarity of the exposition. The author takes full responsibility for the mathematical content of the paper, including all statements and proofs.

## 2. NOTATIONS AND PRELIMINARIES

*Geometric notation.* The symbols  $|x|$  and  $x \cdot y$  denote the Euclidean norm and scalar product, and  $e_1, \dots, e_d$  are the coordinate unit vectors. For  $r > 0$  and  $x \in \mathbb{R}^d$ , we set

$$B_r(x) := \{y \in \mathbb{R}^d : |y - x| < r\}, \quad B_r := B_r(0), \quad \mathbb{S}^{d-1} := \partial B_1.$$

We write  $B_r^d$  when the ambient dimension needs to be specified. For  $e \in \mathbb{S}^{d-1}$ , the orthogonal complement of  $e$  is  $e^\perp := \{y \in \mathbb{R}^d : y \cdot e = 0\}$ .

We denote by  $\mathcal{L}^d$  the Lebesgue measure on  $\mathbb{R}^d$  and by  $\mathcal{H}^k$  the  $k$ -dimensional Hausdorff measure, normalized to agree with  $k$ -dimensional Lebesgue measure on affine  $k$ -planes. For a measurable set  $E$ , we write  $|E| := \mathcal{L}^d(E)$ ,  $\mathbf{1}_E$  for its characteristic function, and  $E \triangle F := (E \setminus F) \cup (F \setminus E)$  for symmetric difference. Equalities between measurable sets or functions are understood up to Lebesgue-null sets unless otherwise stated.

For  $\ell > 0$ , the flat torus  $\mathbb{T}_\ell^d := \mathbb{R}^d / (\ell\mathbb{Z})^d$  is endowed with the measure induced by Lebesgue measure, so  $|\mathbb{T}_\ell^d| = \ell^d$ . A function on  $\mathbb{T}_\ell^d$  is identified with its  $\ell$ -periodic lift to  $\mathbb{R}^d$ , and we use the same symbol for both. Integration over the torus means integration over a fundamental cell, for instance  $[0, \ell)^d$ . Translations on the torus are understood through this periodic lift. If  $\Omega$  is a measurable domain or a torus with  $0 < |\Omega| < \infty$ , the average of  $u \in L^1(\Omega)$  is

$$\bar{u} := \frac{1}{|\Omega|} \int_\Omega u \, dx.$$

The domain of the average will always be clear from the context. The pairing  $\langle u, v \rangle$  denotes  $\int_\Omega uv \, dx$  for real-valued  $L^2$  functions. We use the usual Sobolev space  $H^1(\Omega)$  of  $L^2$  functions with first weak derivatives in  $L^2$ ; on a torus, functions and test functions are periodic.

*Measures, restrictions, and projections.* For a Radon measure  $\mu$  and a Borel set  $A$ , its restriction is  $(\mu \llcorner A)(B) := \mu(A \cap B)$ . The support  $\text{supp } \mu$  of a positive measure is the set of points whose every open neighborhood has positive  $\mu$ -measure. For a Borel map  $P : \mathbb{R}^d \rightarrow \mathbb{R}^k$ , the pushforward  $P_\# \mu$  is defined by  $P_\# \mu(A) := \mu(P^{-1}(A))$ . Equivalently,

$$\int_{\mathbb{R}^k} f(y) \, d(P_\# \mu)(y) = \int_{\mathbb{R}^d} f(Px) \, d\mu(x)$$

for every nonnegative Borel function  $f$  and every positive measure  $\mu$ . When  $P$  is a coordinate projection,  $P_\# \mu$  is also called the corresponding marginal of  $\mu$ .

A positive measure is *radial* if  $R_\# \mu = \mu$  for every orthogonal transformation  $R$ , and *even* if  $\mu(-A) = \mu(A)$  for every Borel set  $A$ . Radial measures are even.

*Functions of bounded variation and perimeter.* For an open set  $\Omega \subset \mathbb{R}^d$ , the space  $BV(\Omega)$  consists of the functions  $u \in L^1(\Omega)$  whose distributional gradient  $Du$  is a finite  $\mathbb{R}^d$ -valued Radon measure. Thus

$$\int_\Omega u \, \text{div } \phi \, dx = - \int_\Omega \phi \cdot dDu \quad \text{for every } \phi \in C_c^1(\Omega; \mathbb{R}^d).$$

We denote by  $|Du|$  its total variation measure and equip  $BV(\Omega)$  with the norm  $\|u\|_{L^1(\Omega)} + |Du|(\Omega)$ . The notation  $BV_{\text{loc}}$  means that these conditions hold locally. On a torus, the definition uses smooth periodic test fields. In particular, the variation on a torus includes any jump across the identified faces of a fundamental cell. On an open cube,  $|Du|(Q_L)$  is the relative variation inside the cube and contains no boundary contribution from its faces.

On the bounded domains considered here, a measurable set  $E$  has finite perimeter in  $\Omega$  if  $\mathbf{1}_E \in BV(\Omega)$ , and  $P(E; \Omega) := |D\mathbf{1}_E|(\Omega)$ . We write  $BV(\Omega; \{-1, 1\})$  for the  $BV$  functions taking only the values  $-1$  and  $1$  almost everywhere. For such a function, with  $E := \{u = 1\}$ , one has

$$u = 2\mathbf{1}_E - 1, \quad Du = 2D\mathbf{1}_E, \quad |Du|(\Omega) = 2P(E; \Omega).$$

If  $\partial^* E$  denotes the reduced boundary of  $E$ , the perimeter measure satisfies  $|D\mathbf{1}_E| = \mathcal{H}^{d-1} \llcorner (\partial^* E \cap \Omega)$ .

When a specific representative of a measurable set in  $\mathbb{R}^d$  is needed, its density-one representative is

$$E^{(1)} := \left\{ x \in \mathbb{R}^d : \lim_{r \downarrow 0} \frac{|E \cap B_r(x)|}{|B_r|} = 1 \right\}.$$

*Fourier conventions.* On  $T = \mathbb{T}_\ell^d$ , with volume  $V = \ell^d$ , the frequency lattice is  $\mathcal{K}_\ell := 2\pi\mathbb{Z}^d/\ell$ . For  $U \in L^2(T)$  we use the normalized coefficients

$$\widehat{U}(q) := \frac{1}{V} \int_T U(x) e^{-iq \cdot x} dx, \quad q \in \mathcal{K}_\ell.$$

With this convention,

$$U = \sum_{q \in \mathcal{K}_\ell} \widehat{U}(q) e^{iq \cdot x} \quad \text{in } L^2(T), \quad \int_T |U|^2 dx = V \sum_{q \in \mathcal{K}_\ell} |\widehat{U}(q)|^2.$$

In particular,  $\widehat{U}(0) = \overline{U}$ , so zero mean is equivalent to the absence of the constant Fourier mode. For real-valued  $U$ , one has  $\widehat{U}(-q) = \overline{\widehat{U}(q)}$ .

For a finite measure  $\mu$  on  $\mathbb{R}^d$ , we instead use the un-normalized Euclidean Fourier transform

$$\widehat{\mu}(\xi) := \int_{\mathbb{R}^d} e^{-i\xi \cdot z} d\mu(z), \quad \xi \in \mathbb{R}^d.$$

If  $d\mu = k(z) dz$ , we also write  $\widehat{k} = \widehat{\mu}$ . For an even positive measure,  $\widehat{\mu}$  is real-valued; for a radial measure it depends only on  $|\xi|$ . If  $P : \mathbb{R}^d \rightarrow \mathbb{R}^k$  is a coordinate projection, then

$$\widehat{P_\# \mu}(\eta) = \widehat{\mu}(P^\top \eta), \quad \eta \in \mathbb{R}^k,$$

where  $P^\top \eta$  inserts zero entries in the omitted coordinates. This identity follows directly from the definition of pushforward.

For a finite positive measure  $\mu$  and a periodic function  $U \in L^2(T)$ , convolution is defined on the periodic lift by

$$(\mu * U)(x) := \int_{\mathbb{R}^d} U(x - z) d\mu(z).$$

It is well-defined almost everywhere and satisfies

$$\|\mu * U\|_{L^2(T)} \leq \mu(\mathbb{R}^d) \|U\|_{L^2(T)}, \quad \widehat{\mu * U}(q) = \widehat{\mu}(q) \widehat{U}(q).$$

If  $\mu$  is even,  $U(x - z)$  may equivalently be replaced by  $U(x + z)$ . For real-valued  $U$  and even positive  $\mu$ , Parseval's identity therefore gives

$$\langle U, \mu * U \rangle = V \sum_{q \in \mathcal{K}_\ell} \widehat{\mu}(q) |\widehat{U}(q)|^2.$$

*One-dimensional functions and striped sets.* We call a binary function on  $\mathbb{R}^d$  *one-dimensional* or *striped* if there exist  $\nu \in \mathbb{S}^{d-1}$  and  $g \in BV_{\text{loc}}(\mathbb{R}; \{-1, 1\})$  such that

$$U(x) = g(x \cdot \nu) \quad \text{for almost every } x \in \mathbb{R}^d.$$

For a function on a torus this definition refers to its periodic lift. The positive phase is then  $\{x : x \cdot \nu \in I\}$ , where  $I := \{t : g(t) = 1\}$ . If  $g$  is nonconstant, its jumps describe complete parallel interfaces, and the regions between consecutive interfaces are the layers or stripes. The distance between consecutive interfaces is the layer width. Constant functions are included as degenerate one-dimensional functions.

By *periodic stripes of width  $h > 0$*  we mean the profiles

$$U_{h, \nu, \varphi}(x) := \text{sign} \cos\left(\frac{\pi}{h} x \cdot \nu + \varphi\right), \quad \nu \in \mathbb{S}^{d-1}, \quad \varphi \in \mathbb{R},$$

or their positive-phase sets. Their interfaces are

$$x \cdot \nu = h \left( k + \frac{1}{2} - \frac{\varphi}{\pi} \right), \quad k \in \mathbb{Z}.$$

Both phases have layer width  $h$ , whereas the full period in the normal direction is  $2h$ : translation by  $h\nu$  exchanges the phases, and translation by  $2h\nu$  preserves them. Such a profile defines a function on  $\mathbb{T}_\ell^d$  precisely when

$$\ell\nu_i \in 2h\mathbb{Z} \quad (i = 1, \dots, d), \quad \text{equivalently} \quad \frac{\ell\nu}{2h} \in \mathbb{Z}^d.$$

We will use the word "width" always referring to the distance between consecutive interfaces in the full profile.

### 3. MAIN RESULTS

**3.1. The functionals and the admissible classes.** Let  $Q_L := (0, L)^d$  and  $\mathbb{T}_L^d := \mathbb{R}^d / (L\mathbb{Z})^d$ , with  $L > 0$ .

For  $\lambda \geq 0$  and either  $\Omega = Q_L$  or  $\Omega = \mathbb{T}_L^d$ , we define the admissible classes

$$\mathcal{A}_\lambda(\Omega) = \begin{cases} BV(\Omega; \{-1, 1\}), & \lambda > 0, \\ \{u \in BV(\Omega; \{-1, 1\}) : \int_\Omega u = 0\}, & \lambda = 0. \end{cases} \quad (3.1)$$

For  $\lambda > 0$ , let  $R_\lambda^{\text{per}}$  and  $R_\lambda^N$  denote the inverses of the operator  $(-\Delta + \lambda^2)$  with periodic and zero-flux Neumann boundary conditions, respectively. We remark that in the Neumann case,  $w = R_\lambda^N u$  means

$$\int_{Q_L} (\nabla w \cdot \nabla \varphi + \lambda^2 w \varphi) \, dx = \int_{Q_L} u \varphi \, dx \quad \text{for every } \varphi \in H^1(Q_L). \quad (3.2)$$

The definition for  $R_\lambda^{\text{per}}$  uses periodic  $H^1$  functions and test functions instead.

When  $\lambda = 0$ , the inverse is defined only if the average of  $u$  on  $\Omega$  is zero, and the solution is normalized imposing  $\int_\Omega w = 0$ . On the other hand, if  $\lambda > 0$ , then we get  $\bar{w} = \bar{u}/\lambda^2$ , and no additional normalization is imposed.

For  $b \in \{\text{per}, N\}$  then the screened and unscreened Ohta-Kawasaki energies are

$$N_\lambda^b(u) = \langle u, R_\lambda^b u \rangle, \quad \mathcal{E}_{\lambda, J, \gamma}^b(u) = J|Du|(\Omega) + \gamma N_\lambda^b(u), \quad J, \gamma > 0. \quad (3.3)$$

In particular, it holds that

$$N_\lambda^b(u) = \int_\Omega (|\nabla w|^2 + \lambda^2 w^2) \, dx. \quad (3.4)$$

For  $h > 0$  we define the following quantities (they will be motivated by the computations in Section 4)

$$j_\lambda(h) := \begin{cases} \frac{\sinh(\lambda h) - \lambda h}{\lambda^3(1 + \cosh(\lambda h))}, & \lambda > 0, \\ \frac{h^3}{12}, & \lambda = 0, \end{cases} \quad (3.5)$$

$$\sigma_\lambda(h) := \begin{cases} \frac{\cosh(\lambda h) - 1}{\lambda^2(1 + \cosh(\lambda h))}, & \lambda > 0, \\ \frac{h^2}{4}, & \lambda = 0. \end{cases} \quad (3.6)$$

Assuming  $\gamma = 1$ , the quantity  $j_\lambda(h)$  represents the parameter  $J$  in (3.3) such that periodic stripes of width  $h$  are optimal among striped competitors and  $\sigma_\lambda(h)$  is the energy density associated to such optimal stripes.

**3.2. Periodic and Neumann minimizers.** We now present the main results of this paper, both in the screened and unscreened case as well as in the periodic and Neumann case.

**Theorem 3.1** (Periodic comparison and equality cases). *Let  $2 \leq d \leq 5$ ,  $\lambda \geq 0$ , and  $L, h, \gamma > 0$ . Set  $J = \gamma j_\lambda(h)$ . Then every  $u \in \mathcal{A}_\lambda(\mathbb{T}_L^d)$  satisfies*

$$\mathcal{E}_{\lambda, J, \gamma}^{\text{per}}(u) \geq \gamma \sigma_\lambda(h) L^d. \quad (3.7)$$

*Equality holds if and only if*

$$u(x) = \text{sign} \cos\left(\frac{2\pi}{L} a \cdot x + \varphi\right), \quad a \in \mathbb{Z}^d \setminus \{0\}, \quad |a| = \frac{L}{2h}, \quad \varphi \in \mathbb{R}. \quad (3.8)$$

Consequently, if such an integer vector exists, the minimum equals  $\gamma\sigma_\lambda(h)L^d$ , and (3.8) is the complete class of global minimizers.

**Theorem 3.2** (Neumann comparison and equality cases). *Let  $2 \leq d \leq 5$ ,  $\lambda \geq 0$ , and  $L, h, \gamma > 0$ . Set  $J = \gamma j_\lambda(h)$ . Then every  $u \in \mathcal{A}_\lambda(Q_L)$  satisfies*

$$\mathcal{E}_{\lambda, J, \gamma}^N(u) \geq \gamma\sigma_\lambda(h)L^d. \quad (3.9)$$

*Equality is possible if and only if  $n = L/h$  is a positive integer. In this case its complete class is*

$$u(x) = \pm \operatorname{sign} \cos\left(\frac{\pi n x_j}{L}\right), \quad j \in \{1, \dots, d\}. \quad (3.10)$$

*These functions are exactly the global minimizers.*

**Remark 3.3.** For  $\lambda > 0$ , neither theorem assumes zero mean. Attainment of the sharp comparison bound forces it (see Corollary 6.3). On the other hand, for  $\lambda = 0$ , zero mean is necessary for the inverse Laplacian to exist.

**Remark 3.4.** The condition  $L/(2h) \in \mathbb{N}$  is sufficient for coordinate-aligned periodic stripes, however, being the problem isotropic, on a multidimensional torus the lattice condition in (3.8) also allows oblique stripes.

**3.3. Ideas of the proof.** In this subsection we present the main ideas for the derivation of inequalities (3.7) and (3.9).

Consider first a  $\{-1, 1\}$ -valued function  $U$  on a torus  $T$  of volume  $V$ . Write  $DU = \theta|DU|$  and introduce

$$\mathcal{T}_U(z) = \int_T |U(x+z) - U(x)| dx, \quad \mathcal{A}_U(z) = \int_T |z \cdot \theta(x)| d|DU|(x). \quad (3.11)$$

The directional translation inequality (see Lemma 6.1) is

$$\mathcal{T}_U(z) \leq \mathcal{A}_U(z). \quad (3.12)$$

Heuristically, the perimeter estimate counts every interface crossing along a segment. The endpoint difference only records whether the two endpoints have different phases. Two crossings may return to the original phase and therefore create a strict loss in (3.12).

Suppose now to have an even positive finite measure  $\mu$  on  $\mathbb{R}^d$  such that

$$\mu(\mathbb{R}^d) = \sigma_\lambda(h), \quad \int_{\mathbb{R}^d} |z \cdot \nu| d\mu(z) = j_\lambda(h) \quad \text{for every } \nu \in \mathbb{S}^{d-1}. \quad (3.13)$$

The identity  $ab = 1 - |a - b|$  for  $a, b \in \{-1, 1\}$  readily gives

$$\langle U, \mu * U \rangle = \sigma_\lambda(h)V - \int_{\mathbb{R}^d} \mathcal{T}_U(z) d\mu(z) \geq \sigma_\lambda(h)V - j_\lambda(h)|DU|(T), \quad (3.14)$$

where the convolution is evaluated on the periodic extension of  $U$ . For a one dimensional striped set with layer width equal or larger than  $h$ , we have that  $\mathcal{T}_U(z) = \mathcal{A}_U(z)$  for every  $|z| \leq h$ . Thus, if the support of  $\mu$  coincides with  $\overline{B_h}$  and its Lebesgue density is positive on  $B_h$ , the inequality in (3.14) is an equality if and only if  $U$  is a striped set with layer width equal or larger than  $h$ .

Moreover, to have a control on the quantity on the left hand side of (3.14) using the energy, we need to impose  $N_\lambda^{\text{per}}(U) \geq \langle U, \mu * U \rangle$ . A sufficient requirement is

$$\widehat{\mu}(\xi) \leq \frac{1}{\lambda^2 + |\xi|^2}. \quad (3.15)$$

For  $\lambda = 0$  this is imposed only at  $\xi \neq 0$ , and the zero-mean constraint removes the constant mode. Combining it with (3.14) yields the desired lower bound.

For the periodic striped set with width  $h$  and normal  $\nu$ , the nonzero Fourier frequencies are odd multiples of  $(\pi/h)\nu$ . To have spectral sharpness therefore we need to impose

$$\widehat{\mu}((2k+1)\pi\nu/h) = \frac{1}{\lambda^2 + ((2k+1)\pi/h)^2}, \quad k \in \mathbb{Z}. \quad (3.16)$$

In this way the measure  $\mu$  is designed to yield equality for the candidate periodic striped set while giving a lower bound for every other competitor. In Section 5 we will show that for  $2 \leq d \leq 5$  a construction of such a measure is possible.

We now present a sketch on how to construct the measure  $\mu$  starting from the one dimensional case. For  $\lambda > 0$  the whole-line fundamental solution is given by

$$G_\lambda(t) = \frac{e^{-\lambda|t|}}{2\lambda}, \quad (-\partial_t^2 + \lambda^2)G_\lambda = \delta_0.$$

Using a symmetric finite difference

$$k_1(t) = aG_\lambda(t) - \frac{b}{2}(G_\lambda(t-h) + G_\lambda(t+h)). \quad (3.17)$$

For  $t \geq h$ , we get  $k_1(t) = aG_\lambda(t) - b \cosh(\lambda h)G_\lambda(t)$ . By evenness the same cancellation condition applies for  $t \leq -h$ . Thus compact support in  $[-h, h]$  requires

$$a = b \cosh(\lambda h).$$

At an odd stripe frequency  $\xi = (2k+1)\pi/h$ , we get

$$\widehat{k}(\xi) = \frac{a - b \cos(\xi h)}{\xi^2 + \lambda^2} = \frac{a+b}{\xi^2 + \lambda^2}.$$

Hence, to match  $\widehat{G}_\lambda(\xi)$  we need  $a+b=1$ . Writing  $C_{\lambda,h} := \cosh(\lambda h)$ , these equations give

$$a = \frac{C_{\lambda,h}}{1+C_{\lambda,h}}, \quad b = \frac{1}{1+C_{\lambda,h}}. \quad (3.18)$$

For  $0 \leq t < h$ , plugging the values of  $a$  and  $b$  gives

$$k_1(t) = \frac{C_{\lambda,h}e^{-\lambda t} - e^{-\lambda h} \cosh(\lambda t)}{2\lambda(1+C_{\lambda,h})} = \frac{\sinh(\lambda(h-t))}{2\lambda(1+C_{\lambda,h})}.$$

Therefore,

$$k_1(t) = \frac{\sinh(\lambda(h-|t|))}{2\lambda(1+C_{\lambda,h})} \mathbf{1}_{\{|t|<h\}}, \quad (3.19)$$

which is positive on  $(-h, h)$ , and

$$m_{\lambda,h}(\xi) := \widehat{k}_1(\xi) = \frac{C_{\lambda,h} - \cos(h|\xi|)}{(1+C_{\lambda,h})(\lambda^2 + |\xi|^2)}. \quad (3.20)$$

Our candidate measure is thus  $d\mu := k_1(t)dt$ , and the radial expression in (3.20) is the candidate Fourier multiplier in every dimension.

To generalize the construction in higher dimension, a key observation is that a striped set sees only the one-dimensional marginal of a kernel. This suggests constructing radial measures whose projection onto any line is (3.19).

There is a fast way to recover the radial density in  $\mathbb{R}^3$ , projection through two transverse dimensions gives

$$k_d(r) = 2\pi \int_r^h s k_{d+2}(s) ds, \quad k_{d+2}(r) = -\frac{k'_d(r)}{2\pi r} \quad \text{for } 0 < r < h. \quad (3.21)$$

In particular, (3.19) gives

$$k_3(r) = \frac{\cosh(\lambda(h-r))}{4\pi(1+C_{\lambda,h})r} \mathbf{1}_{\{0<r<h\}}. \quad (3.22)$$

In general however, the construction for different dimensions will be done using a more involved relation. The full construction is proved in Proposition 5.1. Finally, in the limit  $\lambda \downarrow 0$ , we can recover the relevant quantities in the unscreened case. This will be also done in Proposition 5.1.

#### 4. CONTINUOUS OPTIMIZATION OF ONE-DIMENSIONAL STRIPES

We now focus on the optimization in one dimension and the derivation of the relevant quantities in (3.5) and (3.6). Although the results in this section are already known, we present them for completeness.

**4.1. The periodic cell problem.** For each  $h > 0$ , let  $s_h(x) := \text{sign} \cos(\pi x/h)$  on the one-dimensional torus  $\mathbb{T}_{2h}^1$ . Define  $w_{\lambda,h} := R_{\lambda}^{\text{per}} s_h$  as its periodic potential, normalized to have zero mean when  $\lambda = 0$ . We define the *periodic stripe energy density* to be the energy of one full period divided by its length:

$$e_{\lambda,J,\gamma}(h) := \frac{1}{2h} \mathcal{E}_{\lambda,J,\gamma}^{\text{per}}(s_h) = \frac{J}{2h} |Ds_h|(\mathbb{T}_{2h}^1) + \frac{\gamma}{2h} \int_{-h}^h s_h(x) w_{\lambda,h}(x) dx. \quad (4.1)$$

For  $\lambda > 0$ ,  $w$  is even and anti-periodic, that is

$$w(-x) = w(x), \quad w(x+h) = -w(x).$$

On the positive layer  $[-h/2, h/2]$ , the equation and boundary values are

$$-w'' + \lambda^2 w = 1, \quad w'(0) = 0, \quad w(\pm h/2) = 0.$$

Solving the boundary-value problem gives the solution

$$p_{\lambda,h}(x) = \frac{1}{\lambda^2} \left( 1 - \frac{\cosh(\lambda x)}{\cosh(\lambda h/2)} \right), \quad |x| \leq h/2. \quad (4.2)$$

We extend it by setting  $w(x+kh) = (-1)^k p_{\lambda,h}(x)$ . Notice that the interface values are zero, and the derivatives match because

$$p'_{\lambda,h}(h/2) = -\frac{\tanh(\lambda h/2)}{\lambda} = -p'_{\lambda,h}(-h/2).$$

Thus, the extension is  $C^1$  and solves the equation distributionally. Moreover, anti-periodicity gives zero mean.

Using the fact that  $s_h w$  has period  $h$ , the nonlocal energy per unit length is given by

$$n_{\lambda}(h) := \frac{1}{h} \int_{-h/2}^{h/2} p_{\lambda,h}(x) dx = \frac{1}{\lambda^2} - \frac{1}{\lambda^2 h \cosh(\lambda h/2)} \frac{2 \sinh(\lambda h/2)}{\lambda} = \frac{1}{\lambda^2} - \frac{2}{\lambda^3 h} \tanh(\lambda h/2). \quad (4.3)$$

A similar computation for  $\lambda = 0$  gives

$$p_{0,h}(x) = \frac{h^2}{8} - \frac{x^2}{2}.$$

The same antiperiodic extension is  $C^1$ , has zero mean, and solves  $-w'' = s_h$ . Direct integration gives

$$n_0(h) = \frac{1}{h} \int_{-h/2}^{h/2} \left( \frac{h^2}{8} - \frac{x^2}{2} \right) dx = \frac{1}{h} \left( \frac{h^3}{8} - \frac{h^3}{24} \right) = \frac{h^2}{12}. \quad (4.4)$$

There are two jumps of size two in each cell of length  $2h$ . The total energy density is therefore

$$e_{\lambda,J,\gamma}(h) = \frac{2J}{h} + \gamma n_{\lambda}(h). \quad (4.5)$$

**4.2. Continuous minimization.** For  $\lambda = 0$ ,

$$e'_{0,J,\gamma}(h) = -\frac{2J}{h^2} + \frac{\gamma h}{6}, \quad e''_{0,J,\gamma}(h) = \frac{4J}{h^3} + \frac{\gamma}{6} > 0.$$

The energy diverges at both 0 and  $+\infty$ . Hence, its unique global minimizer is given by

$$h_* = \left( \frac{12J}{\gamma} \right)^{1/3}, \quad e_{0,J,\gamma}(h_*) = \frac{\gamma h_*^2}{4}. \quad (4.6)$$

For  $\lambda > 0$ , define for brevity  $a := \lambda h/2$ . Differentiating (4.3) yields

$$n'_{\lambda}(h) = \frac{2}{\lambda^3 h^2} \tanh a - \frac{1}{\lambda^2 h} \text{sech}^2 a.$$

Consequently,

$$e'_{\lambda,J,\gamma}(h) = \frac{2\gamma}{h^2} \left[ \frac{\tanh a - a \text{sech}^2 a}{\lambda^3} - \frac{J}{\gamma} \right] = \frac{2\gamma}{h^2} \left[ j_{\lambda}(h) - \frac{J}{\gamma} \right]. \quad (4.7)$$

Above we used the expression for  $j_{\lambda}(h)$  in (3.5), since  $\sinh(2a)/(1 + \cosh(2a)) = \tanh a$  and  $2a/(1 + \cosh(2a)) = a \text{sech}^2 a$ . Moreover,

$$j'_{\lambda}(h) = \frac{h}{2\lambda} \text{sech}^2 a \tanh a > 0, \quad \lim_{h \rightarrow 0^+} j_{\lambda}(h) = 0, \quad \lim_{h \rightarrow +\infty} j_{\lambda}(h) = \lambda^{-3}. \quad (4.8)$$

Thus a finite continuous optimal width exists exactly when  $0 < J/\gamma < \lambda^{-3}$ . It is unique and is characterized by

$$J = \gamma j_\lambda(h_*). \quad (4.9)$$

Indeed, the derivative in (4.7) is negative before  $h_*$  and positive after it, proving global minimality within the continuous periodic stripe family. At this width  $h_*$  we have

$$e_{\lambda,J,\gamma}(h_*) = \frac{\gamma}{\lambda^2} - \frac{2\gamma}{\lambda^3 h_*} \left( \frac{\lambda h_*}{2} \right) \operatorname{sech}^2(\lambda h_*/2) = \frac{\gamma}{\lambda^2} \tanh^2(\lambda h_*/2) = \gamma \sigma_\lambda(h_*). \quad (4.10)$$

If  $J/\gamma \geq \lambda^{-3}$ , the energy is strictly decreasing at every finite width and converges to  $\gamma/\lambda^2$  as  $h \rightarrow \infty$ ; and there is no minimizer in the stripe family.

**4.3. Neumann cells and compatibility with the domain.** On  $(0, L)$ , let  $n = L/h \in \mathbb{N}$  and set

$$u_n(x) := \operatorname{sign} \cos(\pi n x / L).$$

The interfaces are at  $(k + \frac{1}{2})h$ ,  $k = 0, \dots, n-1$ . Notice also that the endpoint phase intervals have length  $h/2$ , whereas all interior intervals have length  $h$ . There are exactly  $n$  interior jumps, giving  $|Du_n|((0, L)) = 2n$ .

Restrict the potential constructed above to  $(0, L)$ . By evenness and anti-periodicity of  $w$ , we have

$$w'(0) = 0, \quad w'(L) = w'(nh) = (-1)^n w'(0) = 0.$$

Thus the restriction of  $w$  to  $(0, L)$  solves

$$-w'' + \lambda^2 w = u_n \quad \text{in } (0, L), \quad w'(0) = w'(L) = 0.$$

Moreover, evenness and anti-periodicity imply  $w(h-x) = -w(x)$ , and hence

$$\int_0^h w(x) dx = 0.$$

Consequently,

$$\int_0^L w(x) dx = \sum_{k=0}^{n-1} \int_{kh}^{(k+1)h} w(x) dx = \sum_{k=0}^{n-1} (-1)^k \int_0^h w(x) dx = 0.$$

In particular, when  $\lambda = 0$ , this restriction satisfies the required mean-zero normalization.

Finally, both  $s_h$  and  $w$  are anti-periodic with period  $h$ , so their product is  $h$ -periodic. Since  $u_n = s_h|_{(0,L)}$  and  $L = nh$ , we obtain

$$\frac{1}{L} \int_0^L u_n(x) w(x) dx = \frac{1}{h} \int_0^h s_h(x) w(x) dx = \frac{1}{h} \int_{-h/2}^{h/2} p_{\lambda,h}(x) dx = n_\lambda(h).$$

Hence,

$$\frac{\mathcal{E}_{\lambda,J,\gamma}^N(u_n)}{L} = \frac{2J}{h} + \gamma n_\lambda(h). \quad (4.11)$$

Thus, for every  $h$  with  $L/h \in \mathbb{N}$ , the Neumann stripe  $u_{L/h}$  has the same energy per unit length as the periodic stripe of width  $h$ . In particular, if  $L/h_* \in \mathbb{N}$ , then  $u_{L/h_*}$  attains the optimal periodic stripe energy density  $e_{\lambda,J,\gamma}(h_*)$ .

## 5. CONSTRUCTION OF THE POSITIVE COMPARISON MEASURE

In this section we proceed with the construction of a positive measure which will be crucial in the derivation of the sharp comparison. We will start with its expression in dimension five and derive the lower dimensional ones by projection. Then, we will show they satisfy the desired properties.

**Proposition 5.1** (Radial comparison measures). *Let  $h > 0$ ,  $\lambda \geq 0$ , and  $1 \leq d \leq 5$ . There is a finite positive radial measure  $\mu_{\lambda,h}^{(d)}$  on  $\mathbb{R}^d$ , supported in  $\overline{B}_h$ , whose absolutely continuous part has strictly positive density  $\mathcal{L}^d$ -almost everywhere in  $B_h$ . Its Fourier transform is*

$$\widehat{\mu_{\lambda,h}^{(d)}}(\xi) = \frac{\cosh(\lambda h) - \cos(h|\xi|)}{(1 + \cosh(\lambda h))(\lambda^2 + |\xi|^2)}. \quad (5.1)$$

For  $\lambda = 0$  the expression above reduces to  $(1 - \cos(h|\xi|))/(2|\xi|^2)$ , and  $\widehat{\mu_{0,h}}(0) = h^2/4$ . Its mass and first absolute directional moments are

$$\mu_{\lambda,h}^{(d)}(\mathbb{R}^d) = \sigma_\lambda(h), \quad \int_{\mathbb{R}^d} |z \cdot \nu| d\mu_{\lambda,h}^{(d)}(z) = j_\lambda(h) \quad (\nu \in \mathbb{S}^{d-1}). \quad (5.2)$$

*Proof.* We first define the measure in dimension five and verify its projections and Fourier transform. Define

$$C_{\lambda,h} := \cosh(\lambda h), \quad A_{\lambda,h} := \frac{1}{8\pi^2(1 + C_{\lambda,h})}.$$

Define for  $r = |z|$

$$d\mu_{\lambda,h}^{(5)}(z) := A_{\lambda,h} \left[ \frac{\lambda \sinh(\lambda(h-r))}{r^2} + \frac{\cosh(\lambda(h-r))}{r^3} \right] \mathbf{1}_{\{0 < r < h\}} dz + \frac{A_{\lambda,h}}{h^2} d\mathcal{H}^4 \llcorner \partial B_h. \quad (5.3)$$

Every coefficient is nonnegative, and the bulk density is positive for  $0 < r < h$ . Near the origin the absolutely continuous part is bounded by  $Cr^{-3}$ , which is integrable in  $\mathbb{R}^5$ . The absolutely continuous part is bounded near  $r = h$ , and the surface measure is finite. Thus (5.3) defines a positive finite measure. We will define  $\mu_{\lambda,h}^{(d)} := (\pi_d)_\# \mu_{\lambda,h}^{(5)}$  for  $d \leq 4$ , where  $\pi_d : \mathbb{R}^5 \rightarrow \mathbb{R}^d$  is the coordinate projection.

*Step 1: projection from dimension five to dimension three.* Write  $z = (x, y) \in \mathbb{R}^3 \times \mathbb{R}^2$  and  $r = |x| < h$ . Integration of the absolutely continuous part over  $y$  gives

$$2\pi A_{\lambda,h} \int_r^h \left[ \frac{\lambda \sinh(\lambda(h-s))}{s} + \frac{\cosh(\lambda(h-s))}{s^2} \right] ds = 2\pi A_{\lambda,h} \left[ \frac{\cosh(\lambda(h-r))}{r} - \frac{1}{h} \right]. \quad (5.4)$$

The identity follows from

$$\frac{d}{ds} \frac{\cosh(\lambda(h-s))}{s} = -\frac{\lambda \sinh(\lambda(h-s))}{s} - \frac{\cosh(\lambda(h-s))}{s^2}.$$

To compute the projection of the spherical part, we parameterize  $\partial B_h \subset \mathbb{R}^5 = \mathbb{R}^3 \times \mathbb{R}^2$  by

$$\Phi(x, \vartheta) = (x, t(x) \cos \vartheta, t(x) \sin \vartheta), \quad t(x) := \sqrt{h^2 - |x|^2},$$

where  $x \in B_h \subset \mathbb{R}^3$  and  $0 < \vartheta < 2\pi$ . This parametrization covers the sphere up to an  $\mathcal{H}^4$ -null set. For  $i = 1, 2, 3$ , we have  $\partial_i t(x) = -\frac{x_i}{t(x)}$ , and therefore the tangent vectors are given by

$$\partial_i \Phi = \left( e_i, -\frac{x_i}{t} \cos \vartheta, -\frac{x_i}{t} \sin \vartheta \right), \quad \partial_\vartheta \Phi = (0, -t \sin \vartheta, t \cos \vartheta).$$

Their scalar products satisfy

$$\partial_i \Phi \cdot \partial_j \Phi = \delta_{ij} + \frac{x_i x_j}{t^2}, \quad |\partial_\vartheta \Phi|^2 = t^2,$$

whereas  $\partial_i \Phi \cdot \partial_\vartheta \Phi = x_i \cos \vartheta \sin \vartheta - x_i \sin \vartheta \cos \vartheta = 0$ . Thus the Gram matrix is block diagonal:

$$G = (D\Phi)^\top D\Phi = \begin{pmatrix} I_3 + \frac{xx^\top}{t^2} & 0 \\ 0 & t^2 \end{pmatrix}.$$

The matrix  $I_3 + xx^\top/t^2$  acts as the identity on  $x^\perp$  and has eigenvalue  $1 + |x|^2/t^2$  in the direction of  $x$ . Consequently,

$$\det G = t^2 \det \left( I_3 + \frac{xx^\top}{t^2} \right) = t^2 \left( 1 + \frac{|x|^2}{t^2} \right) = t^2 + |x|^2 = h^2.$$

The surface Jacobian is therefore  $J_4 \Phi = \sqrt{\det G} = h$ , so the surface element is  $h dx d\vartheta$ .

Let  $\pi : \mathbb{R}^3 \times \mathbb{R}^2 \rightarrow \mathbb{R}^3$  denote the coordinate projection. For every nonnegative Borel function  $\varphi$ , the surface change-of-variables formula gives

$$\int_{\partial B_h} \varphi(\pi(z)) d\mathcal{H}^4(z) = \int_{B_h} \int_0^{2\pi} \varphi(x) h d\vartheta dx = 2\pi h \int_{B_h} \varphi(x) dx.$$

Hence,  $\pi_\#(\mathcal{H}^4 \llcorner \partial B_h) = 2\pi h \mathbf{1}_{B_h} dx$ . After multiplication by the spherical coefficient  $A_{\lambda,h}/h^2$ , its contribution to the projected density is therefore

$$\frac{A_{\lambda,h}}{h^2} 2\pi h = \frac{2\pi A_{\lambda,h}}{h}.$$

Adding this to the projected bulk density in (5.4), we obtain, for  $0 < r < h$ ,

$$2\pi A_{\lambda,h} \left[ \frac{\cosh(\lambda(h-r))}{r} - \frac{1}{h} \right] + \frac{2\pi A_{\lambda,h}}{h} = \frac{2\pi A_{\lambda,h}}{r} \cosh(\lambda(h-r)).$$

The three-dimensional projection has therefore density

$$k_{\lambda,h}^{(3)}(r) = \frac{\cosh(\lambda(h-r))}{4\pi(1+C_{\lambda,h})r} \mathbf{1}_{\{0 < r < h\}}. \quad (5.5)$$

*Step 2: projection from dimension three onto a line.* For  $|t| < h$ , integrate over the two transverse coordinates:

$$k_{\lambda,h}^{(1)}(t) = 2\pi \int_{|t|}^h s k_{\lambda,h}^{(3)}(s) ds = \frac{1}{2(1+C_{\lambda,h})} \int_{|t|}^h \cosh(\lambda(h-s)) ds = \frac{\sinh(\lambda(h-|t|))}{2\lambda(1+C_{\lambda,h})} \quad (\lambda > 0). \quad (5.6)$$

It vanishes outside  $[-h, h]$ . At  $\lambda = 0$  the same integral gives

$$k_{0,h}^{(1)}(t) = \frac{(h-|t|)_+}{4}. \quad (5.7)$$

*Step 3: the Fourier transform.* For  $\rho = |\xi| > 0$ , spherical integration in dimension three yields

$$\widehat{\mu_{\lambda,h}^{(3)}}(\xi) = \frac{4\pi}{\rho} \int_0^h r k_{\lambda,h}^{(3)}(r) \sin(\rho r) dr = \frac{1}{(1+C_{\lambda,h})\rho} \int_0^h \cosh(\lambda(h-r)) \sin(\rho r) dr. \quad (5.8)$$

To compute the quantity above, let  $f(r) = \cosh(\lambda(h-r))$  and  $I := \int_0^h f(r) \sin(\rho r) dr$ . Since  $f'' = \lambda^2 f$ ,  $f(0) = C_{\lambda,h}$ ,  $f(h) = 1$ , and  $f'(h) = 0$ , two integrations by parts give

$$\begin{aligned} \lambda^2 I &= \int_0^h f''(r) \sin(\rho r) dr = [f'(r) \sin(\rho r)]_0^h - \rho \int_0^h f'(r) \cos(\rho r) dr \\ &= -\rho [f(r) \cos(\rho r)]_0^h - \rho^2 I = \rho(C_{\lambda,h} - \cos(\rho h)) - \rho^2 I. \end{aligned}$$

Therefore

$$I = \frac{\rho(C_{\lambda,h} - \cos(\rho h))}{\lambda^2 + \rho^2},$$

which proves (5.1) in dimension three. This computation also covers  $\lambda = 0$  for  $\rho > 0$ . Continuity of the Fourier transform at the origin supplies the stated values there.

To establish (5.1) in dimension five, write  $\mu_5 := \mu_{\lambda,h}^{(5)}$  and let  $\pi : \mathbb{R}^3 \times \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be the coordinate projection. Since  $\mu_3 := \pi_{\#} \mu_5$  has the density given in (5.5), for every  $\eta \in \mathbb{R}^3$ , by definition of pushforward

$$\widehat{\mu_3}(\eta) = \int_{\mathbb{R}^3} e^{-i\eta \cdot x} d(\pi_{\#} \mu_5)(x) = \int_{\mathbb{R}^5} e^{-i\eta \cdot \pi(z)} d\mu_5(z) = \int_{\mathbb{R}^5} e^{-i(\eta,0) \cdot z} d\mu_5(z) = \widehat{\mu_5}(\eta, 0).$$

Thus the Fourier transform of the projected measure is the restriction of  $\widehat{\mu_5}$  to  $\mathbb{R}^3 \times \{0\}$ . Moreover,  $\mu_5$  is invariant under orthogonal transformations. Its Fourier transform has the same invariance, that is, for every  $R \in O(5)$ ,

$$\widehat{\mu_5}(R\xi) = \int_{\mathbb{R}^5} e^{-i(R\xi) \cdot z} d\mu_5(z) = \int_{\mathbb{R}^5} e^{-i\xi \cdot R^T z} d\mu_5(z) = \widehat{\mu_5}(\xi).$$

Given any  $\xi \in \mathbb{R}^5$ , choose  $R \in O(5)$  such that  $R\xi = (\eta, 0)$  for some  $\eta \in \mathbb{R}^3$  with  $|\eta| = |\xi|$ . Applying the three-dimensional formula, we obtain

$$\widehat{\mu_5}(\xi) = \widehat{\mu_5}(\eta, 0) = \widehat{\mu_3}(\eta) = \frac{\cosh(\lambda h) - \cos(h|\xi|)}{(1 + \cosh(\lambda h))(\lambda^2 + |\xi|^2)}.$$

At  $\lambda = 0$  and  $\xi = 0$ , the expression is understood by continuity. This proves (5.1) in dimension five.

*Step 4: dimensions two and four.* Recall that  $\mu_{\lambda,h}^{(d)} = (\pi_d)_{\#} \mu_{\lambda,h}^{(5)}$  for  $d \in \{2, 4\}$ . These measures are positive, radial, and supported in  $\overline{B}_h$ . Moreover, reasoning as above, the Fourier projection identity gives

$$\widehat{\mu_{\lambda,h}^{(d)}}(\xi) = \widehat{\mu_{\lambda,h}^{(5)}}(\xi, 0), \quad \xi \in \mathbb{R}^d,$$

which proves (5.1) in both dimensions.

We now compute their densities. For a radial density  $k_{d+1}$  supported in  $B_h \subset \mathbb{R}^{d+1}$ , integration in the last coordinate gives

$$k_d(r) = 2 \int_0^{\sqrt{h^2-r^2}} k_{d+1}(\sqrt{r^2+t^2}) dt = 2 \int_r^h \frac{s k_{d+1}(s)}{\sqrt{s^2-r^2}} ds.$$

Applying this formula to the three-dimensional density (5.5), whose projection onto  $\mathbb{R}^2$  agrees with  $\mu_{\lambda,h}^{(2)}$ , yields

$$k_{\lambda,h}^{(2)}(r) = \frac{1}{2\pi(1+C_{\lambda,h})} \int_r^h \frac{\cosh(\lambda(h-s))}{\sqrt{s^2-r^2}} ds, \quad 0 < r < h. \quad (5.9)$$

In particular, when  $\lambda = 0$ ,

$$k_{0,h}^{(2)}(r) = \frac{1}{4\pi} \operatorname{arcosh}(h/r) = \frac{1}{4\pi} \log \frac{h + \sqrt{h^2-r^2}}{r}. \quad (5.10)$$

In dimension four, we project separately the absolutely continuous part and spherical parts of (5.3). Denoting its radial absolutely continuous part by  $b_{\lambda,h}$ , its projection has density

$$2 \int_0^{\sqrt{h^2-r^2}} b_{\lambda,h}(\sqrt{r^2+t^2}) dt.$$

For the spherical part, write  $\partial B_h \subset \mathbb{R}^5$  as the two graphs

$$t = \pm \sqrt{h^2 - |x|^2}, \quad x \in B_h \subset \mathbb{R}^4.$$

Each graph has surface Jacobian

$$\sqrt{1 + \left| \nabla \sqrt{h^2 - |x|^2} \right|^2} = \frac{h}{\sqrt{h^2 - |x|^2}}.$$

Thus the projection of  $(A_{\lambda,h}/h^2)\mathcal{H}^4 \llcorner \partial B_h$  has density  $2A_{\lambda,h}/(h\sqrt{h^2-r^2})$ . Consequently,

$$k_{\lambda,h}^{(4)}(r) = 2 \int_0^{\sqrt{h^2-r^2}} b_{\lambda,h}(\sqrt{r^2+t^2}) dt + \frac{2A_{\lambda,h}}{h\sqrt{h^2-r^2}}, \quad 0 < r < h. \quad (5.11)$$

Both projected measures are therefore absolutely continuous, with densities extended by zero outside  $B_h$ . The formulas show that these densities are strictly positive for  $0 < r < h$ , hence  $\mathcal{L}^d$ -almost everywhere in  $B_h$ . Their integrability also follows directly from finiteness of the original measure, since projection preserves total mass.

*Step 5: mass and first absolute directional moments.* Projection preserves total mass, and the line marginal in (5.6) is the same in every direction. For  $\lambda > 0$  we thus have

$$\mu_{\lambda,h}^{(d)}(\mathbb{R}^d) = \frac{1}{\lambda(1+C_{\lambda,h})} \int_0^h \sinh(\lambda(h-t)) dt = \frac{C_{\lambda,h}-1}{\lambda^2(1+C_{\lambda,h})} = \sigma_\lambda(h). \quad (5.12)$$

For any unit vector  $\nu$  we have

$$\int |z \cdot \nu| d\mu_{\lambda,h}^{(d)}(z) = \frac{1}{\lambda(1+C_{\lambda,h})} \int_0^h t \sinh(\lambda(h-t)) dt. \quad (5.13)$$

Integration by parts gives

$$\int_0^h t \sinh(\lambda(h-t)) dt = \left[ -\frac{t}{\lambda} \cosh(\lambda(h-t)) \right]_0^h + \frac{1}{\lambda} \int_0^h \cosh(\lambda(h-t)) dt = -\frac{h}{\lambda} + \frac{\sinh(\lambda h)}{\lambda^2}.$$

Plugging this identity into (5.13), yields  $j_\lambda(h)$  as defined in (3.5). For  $\lambda = 0$ , (5.7) gives directly

$$\int k_{0,h}^{(1)}(t) dt = \frac{1}{2} \int_0^h (h-t) dt = \frac{h^2}{4}, \quad \int |t| k_{0,h}^{(1)}(t) dt = \frac{1}{2} \int_0^h t(h-t) dt = \frac{h^3}{12}. \quad (5.14)$$

This concludes the proof.  $\square$

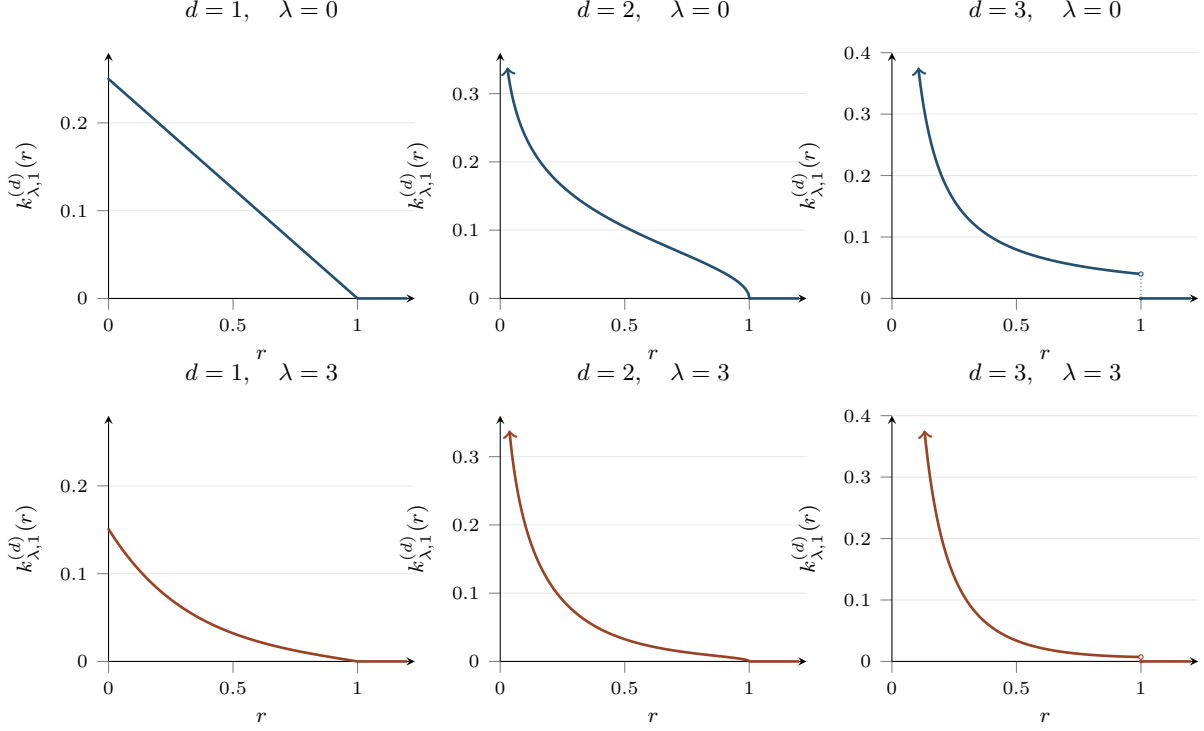


FIGURE 1. Radial profiles of the Lebesgue densities  $k_{\lambda,1}^{(d)}(r)$  of the comparison measures in Proposition 5.1, with  $h = 1$  and  $r = |z|$ . The first row corresponds to  $\lambda = 0$  and the second to  $\lambda = 3$ ; the columns correspond to  $d = 1, 2, 3$ . The vertical scale is the same within each column. Arrows indicate divergence as  $r \downarrow 0$  for  $d = 2, 3$ . All densities vanish for  $r > 1$ ; for  $d = 3$ , the open circle marks the positive left limit at  $r = 1$ .

## 6. THE SHARP ESTIMATE, RIGIDITY, AND PROOFS OF THE THEOREMS

In this section we prove the additional intermediate results needed to prove Theorems 3.1 and 3.2.

**6.1. The directional translation inequality.** We now state and prove a general result for  $BV$  function on the torus. A similar statement also appears in [36, Lemma 3.1] and [22, Proposition 11].

**Lemma 6.1** (Translation control for arbitrary  $BV$  functions). *Let  $T$  be the periodic torus and let  $U \in BV(T)$ . Write  $DU = \theta|DU|$  and define  $\mathcal{T}_U, \mathcal{A}_U$  by (3.11). Then, for every  $z \in \mathbb{R}^d$ .*

$$\mathcal{T}_U(z) \leq \mathcal{A}_U(z). \quad (6.1)$$

*In particular, the function  $\mathcal{D}_U(z) := \mathcal{A}_U(z) - \mathcal{T}_U(z)$  is nonnegative and continuous on  $\mathbb{R}^d$ .*

*Proof.* For smooth periodic  $U$ , the fundamental theorem of calculus gives

$$|U(x+z) - U(x)| \leq \int_0^1 |z \cdot \nabla U(x+tz)| dt.$$

Integrating over the torus and using translation invariance proves (6.1). For a general  $BV$  functions, let  $U_\varepsilon = \rho_\varepsilon * U$  be a periodic mollification converging to  $U$  in  $L^1(T)$ . The scalar measure  $z \cdot DU$  satisfies

$$\int_T |z \cdot \nabla U_\varepsilon| dx = \int_T |\rho_\varepsilon * (z \cdot DU)| dx \leq |z \cdot DU|(T) = \mathcal{A}_U(z).$$

Moreover,  $|\mathcal{T}_{U_\varepsilon}(z) - \mathcal{T}_U(z)| \leq 2\|U_\varepsilon - U\|_{L^1(T)}$ . Hence, passing to the limit proves the inequality.

Translation is continuous in  $L^1(T)$ , so  $\mathcal{T}_U$  is continuous. Finally,

$$|\mathcal{A}_U(z) - \mathcal{A}_U(z')| \leq |z - z'| |DU|(T).$$

This proves the continuity of  $\mathcal{D}_U$ . □

**6.2. The global comparison.** We are ready to show the sharp global comparison inequality which will entail Theorems 3.1 and 3.2.

**Proposition 6.2** (Sharp comparison on the torus). *Let  $1 \leq d \leq 5$ ,  $\lambda \geq 0$ ,  $h > 0$ , and let  $T = \mathbb{T}_\ell^d$  have volume  $V = \ell^d$ . For  $U \in \mathcal{A}_\lambda(T)$ ,*

$$N_\lambda^{\text{per}}(U) + j_\lambda(h)|DU|(T) \geq \sigma_\lambda(h)V. \quad (6.2)$$

More precisely, considering the measure  $\mu_{\lambda,h}^{(d)}$  given by Proposition 5.1, we get

$$\begin{aligned} N_\lambda^{\text{per}}(U) + j_\lambda(h)|DU|(T) - \sigma_\lambda(h)V \\ = \int_{\mathbb{R}^d} \mathcal{D}_U(z) d\mu_{\lambda,h}^{(d)}(z) + V \sum_{q \in (2\pi/\ell)\mathbb{Z}^d} \frac{1 + \cos(h|q|)}{(1 + \cosh(\lambda h))(\lambda^2 + |q|^2)} |\widehat{U}(q)|^2. \end{aligned} \quad (6.3)$$

For  $\lambda = 0$ , the sum omits  $q = 0$ . Both terms on the right are nonnegative.

*Proof.* Let  $\mu = \mu_{\lambda,h}^{(d)}$  be the measure from Proposition 5.1. Since  $\mu$  is finite, convolution with the periodic extension defines a bounded operator on  $L^2(T)$ , with norm bounded by  $\mu(\mathbb{R}^d)$ . Indeed, for trigonometric polynomials the convolution is given by  $\widehat{\mu}(q)$ . Density of such polynomials and boundedness extend this formula to  $L^2(T)$ .

For  $U$  with values in  $\{-1, 1\}$ , Fubini give

$$Q_\mu(U) := \langle U, \mu * U \rangle = \int_{\mathbb{R}^d} \int_T U(x)U(x+z) dx d\mu(z) = \sigma_\lambda(h)V - \int_{\mathbb{R}^d} \mathcal{D}_U(z) d\mu(z). \quad (6.4)$$

Using (5.2) gives

$$\int_{\mathbb{R}^d} \mathcal{D}_U(z) d\mu(z) = \int_T \left[ \int_{\mathbb{R}^d} |z \cdot \theta(x)| d\mu(z) \right] d|DU|(x) = j_\lambda(h)|DU|(T).$$

Since  $|\theta| = 1$  for  $|DU|$ -almost every  $x$ , we have

$$Q_\mu(U) + j_\lambda(h)|DU|(T) - \sigma_\lambda(h)V = \int_{\mathbb{R}^d} \mathcal{D}_U(z) d\mu(z) \geq 0. \quad (6.5)$$

Recall that for  $\lambda > 0$ ,

$$N_\lambda^{\text{per}}(U) = V \sum_{q \in (2\pi/\ell)\mathbb{Z}^d} \frac{|\widehat{U}(q)|^2}{\lambda^2 + |q|^2}.$$

For  $\lambda = 0$ , the same formula is valid with  $q \neq 0$  because the source has zero mean. By (5.1) we also infer that

$$\frac{1}{\lambda^2 + |q|^2} - \widehat{\mu}(q) = \frac{1 + \cos(h|q|)}{(1 + \cosh(\lambda h))(\lambda^2 + |q|^2)} \geq 0 \quad (6.6)$$

Subtracting the Fourier expression for  $Q_\mu$  gives the spectral term in (6.3). Combining with (6.5) proves the result.  $\square$

The next result entails that if  $\lambda > 0$  then equality forces  $\overline{U} = 0$ .

**Corollary 6.3** (The screened constant mode). *If  $\lambda > 0$ , then*

$$N_\lambda^{\text{per}}(U) + j_\lambda(h)|DU|(T) - \sigma_\lambda(h)V \geq \frac{V}{\lambda^2} \text{sech}^2\left(\frac{\lambda h}{2}\right) |\overline{U}|^2. \quad (6.7)$$

*In particular, equality in (6.2) forces  $\overline{U} = 0$ .*

*Proof.* Retaining the  $q = 0$  term in (6.3), its coefficient is

$$\frac{2}{(1 + \cosh(\lambda h))\lambda^2} = \lambda^{-2} \text{sech}^2\left(\frac{\lambda h}{2}\right) > 0.$$

$\square$

**6.3. Rigidity of zero translation defect.** In this subsection, we prove that if  $U \in BV(\mathbb{T}_\ell^d; \{-1, 1\})$  and its translation defect  $\mathcal{D}_U$  vanishes in a neighborhood of the origin, then  $U$  is one-dimensional.

**Lemma 6.4** (Zero translation defect implies parallel layers). *Let  $U \in BV(\mathbb{T}_\ell^d; \{-1, 1\})$  be non-constant up to  $\mathcal{L}^d$ -null sets, and let  $h > 0$ . Suppose*

$$\mathcal{D}_U(z) = 0 \quad \text{for } \mathcal{L}^d\text{-almost every } |z| < h. \quad (6.8)$$

*For the periodic lift to  $\mathbb{R}^d$ , let  $E_+$  and  $E_-$  be the density-one representatives of  $\{U = 1\}$  and  $\{U = -1\}$ , respectively. Then  $E_+$  and  $E_-$  are open, and their common boundary is a locally finite family of complete parallel affine hyperplanes. Consecutive planes have distance at least  $h$ . In particular,*

$$U(x) = g(\nu \cdot x) \quad \text{almost everywhere} \quad (6.9)$$

*for some unit vector  $\nu$  and  $g \in BV_{\text{loc}}(\mathbb{R}; \{-1, 1\})$  whose values alternate across the intervals.*

*Proof.* Set  $T := \mathbb{T}_\ell^d$ , and also denote by  $U$  its periodic lift to  $\mathbb{R}^d$ . By continuity of  $\mathcal{D}_U$  given by Lemma 6.1, we get that (6.8) holds for every  $|z| < h$ .

*Step 1: Short segments contain at most one jump.* Fix  $e \in \mathbb{S}^{d-1}$  and  $r \in (0, h)$ . By (6.8),

$$\mathcal{T}_U(re) = \mathcal{A}_U(re) = r \int_T |e \cdot \theta| d|DU| = r |D_e U|(T).$$

By the slicing theorem for  $BV$  functions (see [5, section 3.11]), for  $\mathcal{H}^{d-1}$ -almost every  $y \in e^\perp$ , the function

$$f_y(s) := U(y + se)$$

belongs to  $BV_{\text{loc}}(\mathbb{R}; \{-1, 1\})$ . It is therefore piecewise constant, with locally finitely many jumps, each of size two. Define

$$\Lambda_{e,r}(y + se) := \frac{1}{2} |Df_y|((s, s + r)).$$

By definition  $\Lambda_{e,r}(x)$  counts the jumps along the open segment from  $x$  to  $x + re$ , and

$$|U(x + re) - U(x)| \leq 2\Lambda_{e,r}(x) \quad \text{for almost every } x.$$

Let  $F := [0, \ell)^d$  be a half-open fundamental cell. Slicing and Tonelli's theorem give

$$\begin{aligned} 2 \int_F \Lambda_{e,r}(x) dx &= \int_{e^\perp} \int_{\mathbb{R}} \mathbf{1}_F(y + se) \int_{(s, s+r)} d|Df_y|(t) ds d\mathcal{H}^{d-1}(y) \\ &= \int_0^r \int_{\mathbb{R}^d} \mathbf{1}_F(z - \tau e) d|D_e U|(z) d\tau = \int_0^r |D_e U|(F + \tau e) d\tau \\ &= r |D_e U|(T), \end{aligned}$$

where the last equality follows from periodicity, since every translate of  $F$  is a fundamental cell for the periodic measure  $|D_e U|$ . Therefore,

$$\int_F (2\Lambda_{e,r}(x) - |U(x + re) - U(x)|) dx = \mathcal{A}_U(re) - \mathcal{T}_U(re) = \mathcal{D}_U(re) = 0.$$

Since the integrand is nonnegative, we infer that

$$2\Lambda_{e,r}(x) = |U(x + re) - U(x)| \leq 2 \quad \text{for } \mathcal{L}^d\text{-almost every } x.$$

Hence  $\Lambda_{e,r} \leq 1$   $\mathcal{L}^d$ -almost everywhere. By periodicity and Fubini's theorem, for  $\mathcal{H}^{d-1}$ -almost every  $y \in e^\perp$  and  $\mathcal{H}^1$ -almost every  $s \in \mathbb{R}$ , the restriction  $t \mapsto U(y + te)$  has at most one jump in  $(s, s + r)$ .

*Step 2: Local half-space structure.* Let  $B \subset \mathbb{R}^d$  be any open ball with  $\text{diam } B < h$ , and choose  $r \in (\text{diam } B, h)$ . For every direction  $e$ , Step 1 implies that  $\mathcal{H}^{d-1}$ -almost every line parallel to  $e$  contains at most one jump inside  $B$ . Thus both phases have interval sections in  $B$ , modulo one-dimensional null sets. Applying Lemma 6.5 to  $B \cap \{U = 1\}$ , we conclude that  $U$  is either constant almost everywhere in  $B$ , or

$$U = 2\mathbf{1}_H - 1 \quad \text{almost everywhere in } B$$

for a half-space  $H$  whose boundary meets  $B$ . In the nonconstant case, the density-one representatives satisfy

$$E_+ \cap B = B \cap \text{int } H, \quad E_- \cap B = B \cap \text{int}(H^c).$$

In the constant case, one representative fills  $B$  and the other is empty. In particular, setting  $\Gamma := \text{supp}|DU|$ , we infer that for every  $p \in \Gamma$  and every  $0 < \rho < h/2$ , there is an affine hyperplane  $P_p$  through  $p$  such that  $\Gamma \cap B_\rho(p) = P_p \cap B_\rho(p)$ . Since balls of diameter less than  $h$  cover  $\mathbb{R}^d$ , the local description shows that  $E_+$  and  $E_-$  are open and that  $\partial E_+ = \partial E_- = \Gamma$ .

*Step 3: Complete parallel hyperplanes and their spacing.* The local supporting hyperplane is unique and locally constant along  $\Gamma$ . Indeed, if  $q \in \Gamma \cap B_\rho(p)$ , then  $q \in P_p$ , so  $P_p \cap B_\rho(p) \cap B_\rho(q)$  is a relatively open neighborhood of  $q$  in  $P_p$ . It is contained in  $P_q$ , hence  $P_p = P_q$ . Consequently, each connected component  $C$  of  $\Gamma$  lies in a single affine hyperplane  $P$ . The local description makes  $C$  relatively open in  $P$ . It is also relatively closed in  $P$ , because connected components of the closed set  $\Gamma$  are closed. Since  $P$  is connected,  $C = P$ . Thus  $\Gamma$  is a union of complete affine hyperplanes. Distinct such hyperplanes are disjoint, since they belong to different connected components of  $\Gamma$ , and hence parallel.

Their mutual distances are at least  $h$ . Indeed, if two planes had distance  $a < h$ , a ball centered halfway between them, with radius in  $(a/2, h/2)$ , would meet both. This contradicts Step 2, since that ball has diameter less than  $h$  and can contain at most one planar interface. The family of planes is therefore locally finite.

*Step 4: One-dimensionality.* Since  $U$  is nonconstant,  $\Gamma$  is nonempty. Choose a common unit normal  $\nu$  and write

$$\Gamma = \bigcup_{t \in \mathcal{S}} \{x \in \mathbb{R}^d : x \cdot \nu = t\},$$

where  $\mathcal{S} \subset \mathbb{R}$  is closed and its distinct points have distance at least  $h$ . For every connected component  $I$  of  $\mathbb{R} \setminus \mathcal{S}$ , the corresponding slab

$$\{x \in \mathbb{R}^d : x \cdot \nu \in I\}$$

is connected and has zero variation measure. Hence  $U$  is constant almost everywhere on that slab. Define the function  $g$  on  $I$  to be this constant value. Then, by definition

$$U(x) = g(x \cdot \nu) \quad \mathcal{L}^d\text{-almost everywhere.}$$

The local half-space description shows that the values switch across every interface, so the phases alternate. Since  $\mathcal{S}$  is locally finite,  $g$  has only finitely many jumps on each bounded interval, and therefore  $g \in BV_{\text{loc}}(\mathbb{R}; \{-1, 1\})$ .  $\square$

The following result was used in the proof of Lemma 6.4. Although the result is probably standard, we could not find anywhere a rigorous proof, hence we sketch its proof.

**Lemma 6.5** (Euclidean monotone sets). *Let  $B \subset \mathbb{R}^d$  be an open ball and let  $A \subset B$  be measurable. Suppose that, for almost every direction  $e \in \mathbb{S}^{d-1}$  and for  $\mathcal{H}^{d-1}$ -almost every  $y \in e^\perp$ , both*

$$A \cap (y + \mathbb{R}e) \quad \text{and} \quad (B \setminus A) \cap (y + \mathbb{R}e)$$

*agree, up to  $\mathcal{H}^1$ -null sets, with intervals, possibly empty. Then there exists a half-space  $H \subset \mathbb{R}^d$  such that*

$$|A \triangle (B \cap H)| = 0,$$

*allowing also the cases  $|A| = 0$  and  $|B \setminus A| = 0$ .*

*Proof.* If  $|A| = 0$  or  $|B \setminus A| = 0$ , the conclusion is immediate.

We first prove that  $A$  agrees almost everywhere with a convex set. The hypothesis concerning the sections of  $A$ , together with Fubini's theorem and the Blaschke–Petkantschin formula, implies

$$(1-t)x + ty \in A \quad \text{for almost every } (x, y, t) \in A \times A \times (0, 1). \quad (6.10)$$

Here and below, almost every refers to the appropriate product Lebesgue measure. To justify this implication, parametrize pairs of distinct points by their common line and their two coordinates on that line. On almost every such line  $\ell$ , the section  $A \cap \ell$  agrees almost everywhere with an interval  $I_\ell$ . For almost every pair of endpoints in  $A \cap \ell$ , both endpoints belong to  $I_\ell$  and are distinct. Their interpolating segment lies in  $I_\ell$ , and its points belong to  $A$  for almost every interpolation parameter. The Blaschke–Petkantschin formula then transfers this assertion to almost every pair in  $A \times A$  (see [45, Section 7.2]).

We now iterate (6.10). For  $k \geq 1$ , consider

$$\Delta_k^\circ = \left\{ (\lambda_1, \dots, \lambda_k) : \lambda_i > 0, \sum_{i=1}^k \lambda_i < 1 \right\}, \quad \lambda_0 = 1 - \sum_{i=1}^k \lambda_i.$$

Then

$$\sum_{i=0}^k \lambda_i x_i \in A \quad \text{for almost every } (x_0, \dots, x_k, \lambda) \in A^{k+1} \times \Delta_k^\circ. \quad (6.11)$$

We prove this by induction on  $k$ . The case  $k = 1$  is (6.10). For the induction step, write

$$\sum_{i=0}^k \lambda_i x_i = (1-t)z + tx_k, \quad z = \sum_{i=0}^{k-1} \Lambda_i x_i,$$

where  $t = \lambda_k$  and  $\Lambda_i = \lambda_i/(1-t)$ . By the induction hypothesis,  $z \in A$  for almost every choice of the preceding vertices and coefficients.

For each fixed  $\Lambda \in \Delta_{k-1}^\circ$ , the map

$$(x_0, \dots, x_k, t) \mapsto \left( \sum_{i=0}^{k-1} \Lambda_i x_i, x_k, t \right)$$

is a surjective affine map. Its inverse image of a Lebesgue-null set is therefore null. Consequently, the exceptional set in (6.10), applied to  $(z, x_k, t)$ , remains null after this substitution. Fubini's theorem and the smooth change of coefficient variables  $(\Lambda, t) \mapsto \lambda$  prove (6.11). Taking  $k = d$  and applying Fubini, we obtain

$$|\text{co}\{x_0, \dots, x_d\} \setminus A| = 0 \quad \text{for almost every } (x_0, \dots, x_d) \in A^{d+1}. \quad (6.12)$$

Indeed, for affinely independent vertices the barycentric map from  $\Delta_d^\circ$  onto the interior of their simplex is an invertible affine map. For affinely dependent vertices, the simplex has zero  $d$ -dimensional measure.

Let  $E := A^{(1)} \cap B$ , and  $K := \text{co } E$ . By the Lebesgue density theorem,  $|A \setminus K| = 0$ . We show that every simplex with vertices in  $E$  is contained in  $A$  up to a null set. Fix affinely independent  $p_0, \dots, p_d \in E$  and set  $S := \text{co}\{p_0, \dots, p_d\}$ . Every sufficiently small product neighborhood of  $(p_0, \dots, p_d)$  meets  $A^{d+1}$  in a set of positive measure. We can therefore choose tuples  $(x_0^{(n)}, \dots, x_d^{(n)})$  satisfying (6.12), with  $x_i^{(n)} \rightarrow p_i$  for every  $i$ . Set  $S_n = \text{co}\{x_0^{(n)}, \dots, x_d^{(n)}\}$ . Every point of  $\text{int } S$  belongs to  $S_n$  for all sufficiently large  $n$ . Since  $|\partial S| = 0$ , Fatou's lemma gives

$$|S \setminus A| \leq \liminf_{n \rightarrow \infty} |S_n \setminus A| = 0.$$

Choose a countable dense subset  $D \subset E$ . By Carathéodory's theorem,  $\text{co } D$  is a countable union of simplices with vertices in  $D$ . Hence  $|(\text{co } D) \setminus A| = 0$ . Moreover, we have  $\overline{\text{co } D} = \overline{K}$ . Convex sets with the same closure have the same interior, and the boundary of a convex set is Lebesgue-null. Thus  $|K \setminus \text{co } D| = 0$ , and consequently  $|K \setminus A| = 0$ . We thus have proved  $|A \triangle K| = 0$ . Notice also that  $K \subset B$ , since  $B$  is convex.

Applying the same argument to  $B \setminus A$  yields a convex set  $K' \subset B$  such that  $|(B \setminus A) \triangle K'| = 0$ . Both  $K$  and  $K'$  have nonempty interior, because both have positive measure. Their interiors are disjoint, since  $|K \cap K'| = 0$ . If one of them has empty interior, it is null and we choose  $H$  disjoint from  $B$  or containing  $B$ . Otherwise, the separation theorem for convex sets [44, Theorem 11.3] gives a half-space  $H$  with  $\text{int } K \subset H$  and  $\text{int } K' \subset \mathbb{R}^d \setminus H$ . Since  $|B \setminus (K \cup K')| = 0$  and convex sets have null boundary, we conclude  $|A \triangle (B \cap H)| = 0$ .  $\square$

#### 6.4. Proof of the periodic theorem.

We are finally ready to prove Theorem 3.1.

*Proof of Theorem 3.1.* Multiplying (6.2) by  $\gamma$  and using  $J = \gamma j_\lambda(h)$  proves (3.7).

Suppose now that equality holds in (3.7). This implies that both nonnegative terms in (6.3) vanish. The geometric integrand  $\mathcal{D}_U$  is continuous and nonnegative. Since the measure  $\mu_{\lambda, h}^{(d)}$  has a positive Lebesgue density almost everywhere in  $B_h$  by Proposition 5.1, the vanishing of its integral forces

$$\mathcal{D}_U(z) = 0 \quad \text{for every } |z| < h.$$

Indeed, a strictly positive value would give a positive lower bound on a small open ball and hence a strictly positive integral. The function  $U$  has zero mean. Indeed, this is imposed for  $\lambda = 0$ , and follows from Corollary 6.3 for  $\lambda > 0$ . It is therefore non-constant. Lemma 6.4 yields  $U(x) = g(\nu \cdot x)$  with parallel interfaces spaced at least  $h$  apart for some  $\nu \in \mathbb{S}^{d-1}$ .

Tangential translation invariance implies that every nonzero Fourier frequency with  $\widehat{U}(q) \neq 0$  is parallel to  $\nu$ . To verify this, for each  $v \perp \nu$  one has  $U(\cdot + v) = U$ , hence  $e^{iq \cdot v} \widehat{U}(q) = \widehat{U}(q)$ . Varying  $v$  gives  $q \perp \nu^\perp$ . Vanishing of the spectral term in (6.3) further implies that if  $\widehat{U}(q) \neq 0$  for  $q \neq 0$ , then  $h|q| \in (2\mathbb{N} - 1)\pi$ .

Since such  $q$ 's are parallel to  $\nu$ , we get that  $e^{iq \cdot (h\nu)} = -1$ . The zero Fourier coefficient vanishes, so uniqueness of Fourier series gives

$$U(x + h\nu) = -U(x) \quad \text{almost everywhere,} \quad g(t + h) = -g(t) \quad \text{almost everywhere.} \quad (6.13)$$

Thus, (6.13) exchanges the two phases and leaves their interface family invariant. That is, if  $S$  is the set of jump points of  $g$ , then  $S + h = S$ . Choose  $t_0 \in S$ . The set  $t_0 + h\mathbb{Z}$  is contained in  $S$ . No additional jump point can lie between two consecutive points of this grid, because distinct points have distance at least  $h$ . Hence  $S = t_0 + h\mathbb{Z}$ . The phases alternate, so  $g$  is a square wave with layer width exactly  $h$ . Its period group is exactly  $2h\mathbb{Z}$ . Using the fact that  $U$  is  $L$ -periodic in each coordinate, we infer that for every  $i = 1, \dots, d$ ,

$$U(x + Le_i) = g((x + Le_i) \cdot \nu) = g(x \cdot \nu + L\nu_i).$$

Therefore,  $g(t + L\nu_i) = g(t)$  for  $\mathcal{L}^1$ -almost every  $t$ . So  $L\nu_i$  must be a period of  $g$ , which in turn means  $L\nu_i = 2ha_i$  for some  $a_i \in \mathbb{Z}$ . Since  $|\nu| = 1$ ,

$$|a| = \frac{L}{2h}|\nu| = \frac{L}{2h} > 0,$$

so  $a \neq 0$  and the phase of the square wave gives exactly (3.8).

Conversely, let  $U$  have the form (3.8). Notice that by symmetry it has zero mean and variation  $|DU|(T) = 2V/h$ . Its Fourier support consists of odd multiples of  $2\pi a/L$ . In particular, for every  $q \in (2\pi/L)\mathbb{Z}^d$  with  $q \neq 0$  such that  $\widehat{U}(q) \neq 0$ , since  $|a| = L/2h$ , we have  $h|q| \in (2\mathbb{N} - 1)\pi$ . Thus the spectral term in (6.3) vanishes. If  $|z| \leq h$ , then  $|z \cdot \nu| \leq h$ , and direct translation of the square wave gives

$$\mathcal{T}_U(z) = \frac{2V}{h}|z \cdot \nu| = \mathcal{A}_U(z).$$

This implies that the geometric term in (6.3) also vanishes since the measure is supported in  $\overline{B}_h$ . This proves equality and completes the classification.  $\square$

**6.5. Even reflection and the Neumann functional.** Before proving Theorem 3.2 we need a preliminary result.

**Lemma 6.6** (Exact reflection identities). *Let  $u \in \mathcal{A}_\lambda(Q_L)$ . Define its even reflection on  $(-L, L)^d$  by*

$$U(x) := u(|x_1|, \dots, |x_d|),$$

*and extend it with period  $2L$  in every coordinate. Then*

$$|DU|(\mathbb{T}_{2L}^d) = 2^d |Du|(Q_L), \quad \int_{\mathbb{T}_{2L}^d} U = 2^d \int_{Q_L} u, \quad (6.14)$$

*and*

$$N_\lambda^{\text{per}}(U) = 2^d N_\lambda^N(u). \quad (6.15)$$

*Proof.* We first observe that on each open reflected copy of  $Q_L$ , reflection is an isometry and preserves total variation. Across a coordinate face  $x_i = 0$ , the two one-sided traces agree by evenness. Moreover, the same holds across the periodic identification  $x_i = -L \sim L$ . Therefore, there is no additional jump term on these faces (see e.g. [5, Theorem 3.84]). This entails  $U \in BV(\mathbb{T}_{2L}^d; \{-1, 1\})$  and (6.14).

To prove (6.15) let  $v = R_\lambda^N u$  on  $Q_L$  (with mean-zero normalization if  $\lambda = 0$ ), and let  $W$  be its even reflection. Since again the traces match along the faces, we get  $W \in H^1(\mathbb{T}_{2L}^d)$ . For a smooth periodic test function  $\psi$  defined on  $(-L, L)^d$ , set

$$\varphi(y) := \sum_{\varepsilon \in \{-1, 1\}^d} \psi(\varepsilon_1 y_1, \dots, \varepsilon_d y_d), \quad y \in Q_L.$$

Clearly  $\varphi \in H^1(Q_L)$  and satisfies

$$\partial_i \varphi(y) = \sum_{\varepsilon \in \{-1, 1\}^d} \varepsilon_i (\partial_i \psi)(\varepsilon y).$$

We now use  $\varphi$  in (3.2). Since  $\partial_i W(\varepsilon y) = \varepsilon_i \partial_i v(y)$ ,

$$\int_{\mathbb{T}_{2L}^d} (\nabla W \cdot \nabla \psi + \lambda^2 W \psi) dx = \int_{\mathbb{T}_{2L}^d} U \psi dx.$$

By density, the identity holds for every  $\psi \in H^1(\mathbb{T}_{2L}^d)$ . For  $\lambda > 0$  this characterizes  $R_\lambda^{\text{per}} U$ . For  $\lambda = 0$ ,  $U$  and the reflected potential both have zero mean, and the zero-mean solution is unique. Hence the reflected potential equals  $R_\lambda^{\text{per}} U$ . Therefore, using a change of variables,

$$N_\lambda^{\text{per}}(U) = \int_{\mathbb{T}_{2L}^d} U W dx = 2^d \int_{Q_L} uv dx = 2^d N_\lambda^N(u).$$

This concludes the proof.  $\square$

**6.6. Proof of the Neumann theorem.** We now prove Theorem 3.2, it will follow as a consequence of Theorem 3.1.

*Proof of Theorem 3.2.* Let  $u \in \mathcal{A}_\lambda(Q_L)$ , and like in Lemma 6.6, define its even extension by  $U(x) = u(|x_1|, \dots, |x_d|)$  for  $x \in (-L, L)^d$  followed by  $2L$ -periodic extension in each coordinate. By Lemma 6.6,

$$|DU|(\mathbb{T}_{2L}^d) = 2^d |Du|(Q_L), \quad N_\lambda^{\text{per}}(U) = 2^d N_\lambda^N(u).$$

This gives that  $\mathcal{E}_{\lambda, J, \gamma}^{\text{per}}(U) = 2^d \mathcal{E}_{\lambda, J, \gamma}^N(u)$ . Moreover,  $\bar{U} = \bar{u}$  by (6.14). In particular, when  $\lambda = 0$ , the mean-zero constraint is preserved, so  $U$  is an admissible periodic competitor. Since  $J = \gamma j_\lambda(h)$ , Proposition 6.2 on  $\mathbb{T}_{2L}^d$  gives  $\mathcal{E}_{\lambda, J, \gamma}^{\text{per}}(U) \geq \gamma \sigma_\lambda(h)(2L)^d$ . Dividing by  $2^d$  we obtain

$$\mathcal{E}_{\lambda, J, \gamma}^N(u) \geq \gamma \sigma_\lambda(h) L^d,$$

which is (3.9). For  $\lambda > 0$ , applying Corollary 6.3 to  $U$  gives

$$\mathcal{E}_{\lambda, J, \gamma}^N(u) - \gamma \sigma_\lambda(h) L^d \geq \frac{\gamma L^d}{\lambda^2} \text{sech}^2\left(\frac{\lambda h}{2}\right) |\bar{u}|^2.$$

Thus equality forces  $\bar{u} = 0$  also in the screened case.

Suppose now that equality holds in (3.9), that is

$$\mathcal{E}_{\lambda, J, \gamma}^{\text{per}}(U) = \gamma \sigma_\lambda(h)(2L)^d.$$

By Theorem 3.1 applied with domain size  $2L$ , we infer that there exist  $a \in \mathbb{Z}^d \setminus \{0\}$  and  $\varphi \in \mathbb{R}$  such that

$$U(x) = \text{sign} \cos(q \cdot x + \varphi), \quad q = \frac{\pi}{L} a, \quad |a| = \frac{L}{h}.$$

In particular,  $q \in \pi \mathbb{Z}^d / L$  and  $|q| = \pi / h$ .

For each  $i \in \{1, \dots, d\}$ , let us now define  $R_i(x_1, \dots, x_d) := (x_1, \dots, -x_i, \dots, x_d)$ . By construction of the even extension, we have that  $U(R_i x) = U(x)$   $\mathcal{L}^d$ -almost everywhere.

The interfaces of the periodic lift of  $U$ ,  $\Gamma := \text{supp } |DU|$  in  $\mathbb{R}^d$ , are given by

$$\Gamma = \bigcup_{k \in \mathbb{Z}} \left\{ x \in \mathbb{R}^d : q \cdot x + \varphi = \frac{\pi}{2} + k\pi \right\}.$$

In particular,  $\Gamma$  is a family of complete parallel hyperplanes with normal  $q$ . The identity  $U \circ R_i = U$  implies  $R_i \Gamma = \Gamma$ . On the other hand, reflecting these hyperplanes gives a family with normal  $R_i q$ . Since the interface family has a unique normal line,  $R_i q$  must be parallel to  $q$ . This implies  $R_i q = \pm q$  for every  $i = 1, \dots, d$ .

Suppose now that  $q_i \neq 0$  and  $q_k \neq 0$  for distinct indices  $i$  and  $k$ . Reflection in the  $i$ -th coordinate changes  $q_i$  to  $-q_i$ , so  $R_i q \neq q$ . It leaves  $q_k$  unchanged, so  $R_i q \neq -q$ . This is a contradiction. Therefore  $q$  has exactly one nonzero coordinate.

Since  $q \in (\pi/L) \mathbb{Z}^d$ , there exist  $j \in \{1, \dots, d\}$  and  $n \in \mathbb{N}$  with  $n \geq 1$  such that  $q = \pm \frac{\pi n}{L} e_j$ . Thus,  $nh = L$  by the condition on  $|q|$ . In particular,  $L/h \in \mathbb{N}$ .

Replacing  $\varphi$  by  $-\varphi$  if necessary, we may choose the positive orientation and write

$$U(x) = \text{sign} \cos\left(\frac{\pi n x_j}{L} + \varphi\right).$$

Set  $\kappa := \pi n/L$ . We now show that  $\varphi \in \pi\mathbb{Z}$ . The fundamental Fourier terms of the square wave are

$$\text{sign} \cos(\kappa x_j + \varphi) = \frac{2}{\pi} e^{i\varphi} e^{i\kappa x_j} + \frac{2}{\pi} e^{-i\varphi} e^{-i\kappa x_j} + (\text{harmonics of order } \geq 3),$$

with the expansion understood in  $L^2$ . Thus, with the normalized Fourier coefficients on the torus,

$$\widehat{U}(\kappa e_j) = \frac{2}{\pi} e^{i\varphi}, \quad \widehat{U}(-\kappa e_j) = \frac{2}{\pi} e^{-i\varphi}.$$

Reflection in the  $j$ th coordinate interchanges these two frequencies. Since  $U \circ R_j = U$ , their coefficients must agree, giving  $e^{i\varphi} = e^{-i\varphi}$ . It follows that  $\sin \varphi = 0$ , or equivalently,  $\varphi \in \pi\mathbb{Z}$ . Therefore,

$$U(x) = \pm \text{sign} \cos\left(\frac{\pi n x_j}{L}\right).$$

Restricting  $U$  to  $Q_L$  gives exactly (3.10).

Conversely, suppose now  $n = L/h \in \mathbb{N}$ , and let

$$u(x) = \varepsilon \text{sign} \cos\left(\frac{\pi n x_j}{L}\right), \quad \varepsilon \in \{-1, 1\}.$$

This function belongs to  $BV(Q_L; \{-1, 1\})$  and it has zero mean. Thus,  $u$  is admissible for every  $\lambda \geq 0$ , including  $\lambda = 0$ .

Its even periodic extension on  $\mathbb{T}_{2L}^d$  is given by

$$U(x) = \varepsilon \text{sign} \cos\left(\frac{\pi n x_j}{L}\right).$$

Indeed, this expression is even in every coordinate and  $2L$ -periodic. Its fundamental frequency is  $q = (\pi n/L)e_j$ , with  $|q| = \pi/h$ . It therefore belongs to the periodic equality class in Theorem 3.1. This implies that

$$\mathcal{E}_{\lambda, J, \gamma}^{\text{per}}(U) = \gamma \sigma_\lambda(h) (2L)^d.$$

Applying the reflection identities of Lemma 6.6 once more yields

$$\mathcal{E}_{\lambda, J, \gamma}^N(u) = 2^{-d} \mathcal{E}_{\lambda, J, \gamma}^{\text{per}}(U) = \gamma \sigma_\lambda(h) L^d.$$

These profiles attain the lower bound and are therefore global minimizers.  $\square$

## APPENDIX A. FAILURE OF THE CONSTRUCTION IN DIMENSION SIX AND ABOVE

In this appendix we show that the construction of the measure  $\mu$  in Proposition 5.1 with Fourier multipliers given by

$$m_{\lambda, h}(\xi) = \frac{\cosh(\lambda h) - \cos(h|\xi|)}{(1 + \cosh(\lambda h))(\lambda^2 + |\xi|^2)}, \quad (\text{A.1})$$

with the value at  $\xi = 0$  understood by continuity, is not possible for  $d \geq 6$ .

**Proposition A.1.** *For every  $h > 0$ ,  $\lambda \geq 0$ , and  $d \geq 6$ , there is no finite positive Borel measure on  $\mathbb{R}^d$  whose Fourier transform equals (A.1) at every real frequency.*

Before proving the Proposition, we need a preliminary lemma.

**Lemma A.2.** *Let  $\eta$  be a finite positive rotation-invariant measure on  $\mathbb{R}^6$ . Away from the origin, its projection onto  $\mathbb{R}^4$  has a radial density  $g: (0, \infty) \rightarrow [0, +\infty)$  satisfying*

$$g(r) \leq \frac{2\eta(\mathbb{R}^6)}{\pi^2 r^4} \quad \text{for } r > 0. \quad (\text{A.2})$$

*In particular,  $g$  is bounded on every annulus bounded away from the origin. It also has a nonincreasing radial representative.*

*Proof.* By rotation invariance,  $\eta$  decomposes into normalized surface measures on spheres, weighted by the distribution of its mass with respect to the radius, together with a possible atom at the origin. Let  $\nu$  denote this distribution on  $(0, \infty)$ .

The radius- $s$  sphere in  $\mathbb{R}^6$  is parameterized, away from a surface-null subset, by

$$(y, \vartheta) \mapsto \left(y, \sqrt{s^2 - |y|^2} \cos \vartheta, \sqrt{s^2 - |y|^2} \sin \vartheta\right), \quad y \in B_s^4, \quad 0 \leq \vartheta < 2\pi.$$

As in the projection calculation in Section 5, its surface element is  $s dy d\vartheta$ . Since  $\mathcal{H}^5(\partial B_s^6) = \pi^3 s^5$ , the projection of normalized surface measure has density

$$\frac{2\pi s}{\pi^3 s^5} \mathbf{1}_{\{|y| < s\}} = \frac{2}{\pi^2 s^4} \mathbf{1}_{\{|y| < s\}}.$$

Hence, for  $r > 0$

$$g(r) = \int_{(r, \infty)} \frac{2}{\pi^2 s^4} d\nu(s) \leq \frac{2\nu((0, \infty))}{\pi^2 r^4} \leq \frac{2\eta(\mathbb{R}^6)}{\pi^2 r^4} \quad (\text{A.3})$$

The quantity on the left hand side above is non-increasing in  $r$  by definition. This proves (A.2).  $\square$

*Proof of Proposition A.1.* Suppose first that a finite positive measure  $\eta$  on  $\mathbb{R}^6$  has Fourier transform  $m_{\lambda, h}$ . Since  $m_{\lambda, h}$  is radial, uniqueness of Fourier transforms of finite measures implies that  $\eta$  is rotation invariant.

We now project  $\eta$  onto  $\mathbb{R}^4$ . Its Fourier transform is the restriction of  $m_{\lambda, h}$  to  $\mathbb{R}^4$ . By Proposition 5.1, this is also the Fourier transform of  $\mu_{\lambda, h}^{(4)}$ , so the two projected measures must coincide. The four-dimensional density in (5.11) is

$$k_{\lambda, h}^{(4)}(r) = 2 \int_0^{\sqrt{h^2 - r^2}} b_{\lambda, h}(\sqrt{r^2 + t^2}) dt + \frac{2A_{\lambda, h}}{h\sqrt{h^2 - r^2}}, \quad 0 < r < h. \quad (\text{A.4})$$

Its integrand term is nonnegative. Therefore, recalling  $A_{\lambda, h} = 1/(8\pi^2(1 + C_{\lambda, h}))$ , with  $C_{\lambda, h} = \cosh(\lambda h)$ , we infer

$$k_{\lambda, h}^{(4)}(r) \geq \frac{2A_{\lambda, h}}{h\sqrt{h^2 - r^2}} \longrightarrow +\infty \quad \text{as } r \rightarrow h.$$

This contradicts Lemma A.2, which bounds the projected density uniformly on  $h/2 < r < h$ .

This proves nonexistence in dimension six. If a positive finite measure in dimension  $d > 6$  had transform  $m_{\lambda, h}$ , its projection onto a six-dimensional subspace would provide the excluded positive measure there. The conclusion thus follows for every  $d \geq 6$ .  $\square$

**Remark A.3.** Proposition A.1 excludes the existence of a measure as in Proposition 5.1 with Fourier multipliers given by (A.1) in dimensions  $d \geq 6$ . This means that it does not by itself exclude a different comparison mechanism.

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(Dario Reggiani) APPLIED MATHEMATICS MÜNSTER, UNIVERSITY OF MÜNSTER, EINSTEINSTRASSE 62, 48149 MÜNSTER, GERMANY

*Email address:* `dario.reggiani@uni-muenster.de`