Remarks on an overdetermined boundary value problem

A. Farina & B. Kawohl

January 22, 2007

Abstract: *We modify and extend proofs of Serrin's symmetry result for overdetermined boundary value problems from the Laplace-operator to a general quasilinear operator and remove a strong ellipticity assumption in* [9] *and a growth assumption in* [5] *on the diffusion coefficient A, as well as a starshapedness assumption on Ω in* [4].

1 Introduction and Result

Consider the overtermined elliptic boundary value problem

$$-\mathrm{div}(A(|\nabla u|)\nabla u) = 1 \quad \text{in } \Omega \tag{1.1}$$

$$u = 0, \quad \text{and} \quad |\nabla u| = c \quad \text{on } \partial\Omega \tag{1.2}$$

on a connected bounded domain $\Omega \subset \mathbb{R}^N$, and suppose that the function $A : (0,\infty) \to [0,\infty)$ satisfies the regularity requirement

$$A \in C^2(0,+\infty) \tag{1.3}$$

and the (possibly degenerate) ellipticity condition

$$\lim_{t\to 0^+} tA(t) = 0 \ , \qquad \big(tA(t)\big)' > 0 \qquad \text{for } t > 0, \tag{1.4}$$

It was shown in [4] that these assumptions are sufficient to prove the existence of a radially symmetric solution to (1.1) and (1.2) if Ω is a ball in $\mathbb{R}^N$. Moreover, the assumptions imply that for general Ω with Lipschitz boundary there exist weak $C^1_0(\Omega)$ solutions u of (1.1), and that they are of class C^2 outside their set of critical points $\{x \in \Omega,\ |\nabla u(x)| = 0\}$.

It is well known that under suitable additional assumptions on A and Ω the ball is the *only* domain on which a solution exists. In the present note we review and generalize results of this nature and give a more geometric proof of the following Theorem:

Theorem 1.1 *If the overdetermined elliptic boundary value problem* (1.1) (1.2) *has a weak $C_0^1(\overline{\Omega})$-solution in a connected bounded domain $\Omega \subset \mathbb{R}^N$ with sufficiently smooth boundary $\partial\Omega$, and if A satisfies the above assumptions* (1.3) *and* (1.4)*, then Ω is a ball.*

A weak solution satisfies

$$\int_\Omega A(|\nabla u|)\nabla u \cdot \nabla \varphi \; dx = \int_\Omega \varphi \; dx \qquad \text{for every } \varphi \in C_0^1(\Omega) \tag{1.5}$$

and minimizes the functional

$$E(v) := \int_\Omega \{B(|\nabla v|) - v\} \; dx \qquad \text{with} \quad B(t) := \int_0^t sA(s) \; ds \tag{1.6}$$

on $W_0^{1,\infty}(\Omega)$ or a on suitable Sobolev-Orlicz space (with norm given in terms of the strictly convex function B). Because of (1.4) B is strictly convex and any minimizer of E is unique.

For classical solutions of strongly elliptic equations Theorem 1.1 is a celebrated result of Serrin [14]. To prove it, Serrin introduced the PDE community to Alexandrov's moving plane method, and the proof applied to even more general equations with classical solutions. For $A(|\nabla u|) \equiv 1$ Weinberger [17] provided a much simpler proof, and there have been several attempts to extend Weinberger's approach to more general equations. Philippin succeded in [9] for quasilinear (nondegenerate) equations, in which A was bounded above and below by positive constants. Garofalo and Lewis [5] proved it for a more general class of (possibly) degenerate equations with growth assumptions on A, including the p-Laplacian for $p \in (1,\infty)$, and for a somewhat weaker form of the Neumann boundary condition which did not require any explicit smoothness assumptions on $\partial\Omega$. In [4] Fragalà, Gazzola and Kawohl were able to provide a fairly simple and geometric proof that applies to degenerate equations such as

$$-\text{div}\left(\frac{|\nabla u|^{p-2}}{(1+|\nabla u|^2)^{q/2}}\nabla u\right) = 1 \qquad \text{in } \Omega, \tag{1.7}$$

with $p \in (1,\infty)$ and $q \in [0, p-1]$. This class of equations is not covered by any of the papers [14, 17, 9, 5]. However, the proof in [4] required an additional starshapedness assumption on Ω if the dimension N of Ω is larger than 2. In the present paper we can now also remove the starshapedness assumption. A totally different line of reasoning was pursued in [2], where Brock and Henrot used continuous Steiner symmetrization and domain derivative to study an overdetermined problem under the assumption that Ω is a convex domain and for a class of quasilinear operators of p-Laplacian-type.

In what follows, we will first outline the proof, because then the individual steps will not be obscured by technicalities. While steps 1 and 3 are more or less technical refinements of known methods, we should point out that in step 2 we uncover hidden geometric information in the constancy of P with geometric rather than analytic arguments.

Step 1: Set $\Phi(t) := 2\int_0^t (A(s) + sA'(s))s\,ds$. If u solves (1.1) (1.2), then the function

$$P(x) := \Phi(|\nabla u(x)|) + \frac{2}{N}u(x)$$

attains its maximum over $\overline{\Omega}$ on the boundary $\partial\Omega$. Therefore either $P(x) < \Phi(c)$ on a set of positive measure, or $P(x) \equiv \Phi(c)$.

Step 1 was performed in [4] in full detail, first for strongly elliptic and then after a regularization for degenerate elliptic operators under the assumptions (1.3) and (1.4). To prove Step 1 one has to derive a differential inequality for P in the spirit of [8] or [15]. Under stronger assumptions Step 1 was also done in [9] and [5].

Step 2: In the second case that $P(x) \equiv \Phi(c)$ the function u must be radial and radially decreasing. One way to see this goes via isoparametric surfaces. One notes that the identity $\Phi(|\nabla u(x)|) + \frac{2}{N}u(x) = \Phi(c)$ gives rise to a first order semilinear equation $|\nabla u| = g(u)$ in Ω. This and the second order differential equation imply that all level sets of u are isoparametric surfaces, i.e. all their principal curvatures are either elements of a set $\{0,\ \kappa(u)\}$ with only two elements. Because of the Dirichlet boundary conditions the principal curvatures of a given level surface are all identically nonzero and thus the level sets are concentric spheres, see [6, Theorem 5] or [16]. For readers who do not like isoparametric surfaces we give an *alternative geometric proof of Step 2* in Section 2, while the traditional purely analytical way to derive radial symmetry from the constancy of P is briefly explained in Section 4. It is interesting to note that both geometric versions of proof of Step 2 require (1.3) and (1.4) to hold.

Step 3: To rule out the first case from Step 1 (i.e. $P(x) < \Phi(c)$ on a set of positive measure), we show via certain integral identities (named after Rellich [13], Pohožaev [10] and Pucci-Serrin [12]), that

$$\int_\Omega P(x)\ dx = \Phi(c)|\Omega|. \tag{1.8}$$

One essential part of this step can be found in [9] or [5], however, there it was derived under stronger assumptions on A than here. Since [9] and [5] use different notation, we shall give the relatively short but full details below in Section 3 and explain why one can pass from nodegenerate equations and classical solutions to degenerate equations and weak solutions. Again assumptions (1.3) and (1.4) are crucial.

2 Alternative Proof of Step 2

If $P \equiv \Phi(c)$ then $\Phi(|\nabla u|(x)) = \Phi(c) - \frac{2}{N}u(x)$. In this situation the strict monotonicity of the map $t \to tA(t)$ can be used to make $|\nabla u|$ explicit as a function g of u

$$|\nabla u(x)| = \Phi^{-1}\left(\Phi(c) - \frac{2}{N}u(x)\right) =: g(u(x)), \tag{2.1}$$

and the regularity of A is needed to render $g(u)$ of class C^1 on the interval $(0, \max u)$. Since Φ is strictly monotone, ∇u vanishes only in points where u attains its maximum on Ω. Therefore $\nu = -\frac{\nabla u}{|\nabla u|}$ is well defined on the open set $U := \{x \in \Omega \mid u(x) \in (0, \max u)\}$. Observe that $u_\nu = -|\nabla u| = -g(u)$ and that (1.1) can be rewritten as

$$-A(|\nabla u|)\Delta u - |\nabla u|A'(|\nabla u|)u_{\nu\nu} = 1 \quad \text{in } U, \tag{2.2}$$

while

$$\Delta u = u_{\nu\nu} + (N-1)Hu_\nu. \tag{2.3}$$

Here H is the mean curvature of the level set of u. From (2.2) and (2.3) we can extract

$$H = H(u_\nu, u_{\nu\nu}) = H(g(u), u_{\nu\nu}),$$

and to see that H depends only on u, we have to express $u_{\nu\nu}$ in terms of u. On one hand

$$\frac{\partial}{\partial\nu}(|\nabla u|^2) = 2u_\nu u_{\nu\nu}$$

and on the other hand because of (2.1)

$$\frac{\partial}{\partial\nu}(|\nabla u|^2) = 2g(u)g'(u)u_\nu,$$

so that $u_{\nu\nu} = g(u)g'(u)$ is in fact related to u. Here the differentiability of g enters into the proof. So upon performing obvious algebraic operations (2.1) and (2.2) lead to

$$H = \frac{1 + g(u)g'(u)[A(g(u)) + g(u)A'(g(u))]}{(N-1)g(u)A(g(u))} =: h(u) \quad \text{in } U, \tag{2.4}$$

But this identity just says that every level set of u at height between zero and $\max u$ is a set of constant mean curvature. By Alexandrov's classical result [1] each connected component of it must then be a sphere. In particular Ω must now be simply connected, because otherwise a particular level set, say $\{x \in \Omega;\ u(x) = \delta\}$, would contain two nested spheres of equal radius, a contradiction. Therefore each level set consists of exactly one sphere, and because of (2.1) these spheres are concentric.

3 Proof of Step 3

For the proof of Step 3 we observe (using $B(t) := \int_0^t sA(s)\ ds$ and integration by parts) that

$$\begin{aligned}\Phi(t) &= 2\int_0^t sA(s)\ ds + 2\int_0^t s^2 A'(s)\ ds \\ &= 2B(t) + 2[-\int_0^t 2sA(s)\ ds + t^2 A(t)] \\ &= -2B(t) + 2t^2 A(t),\end{aligned}$$

so that

$$P(x) = 2A(|\nabla u|)|\nabla u|^2 - 2B(|\nabla u|) + \frac{2}{N}u. \tag{3.1}$$

Now we test the differential equation first with u and then with the scalar product $(x, \nabla u)$ and integrate by parts. The first integration gives

$$\int_\Omega A(|\nabla u|)|\nabla u|^2\ dx = \int_\Omega u\ dx. \tag{3.2}$$

Then we test (1.1) with $(x, \nabla u)$. This gives

$$-\int_\Omega \mathrm{div}(A(|\nabla u|)\nabla u)\ (x, \nabla u)\ dx = \int_\Omega (x, \nabla u)\ dx. \tag{3.3}$$

The right hand side of (3.3) transforms as follows

$$\int_\Omega (x, \nabla u)\ dx = -\int_\Omega (\mathrm{div} x)\ u\ dx = -N\int_\Omega u\ dx,$$

while (formally) for the left hand side of (3.3) we have (with $B(t) := \int_0^s tA(s)\ ds$)

$$\begin{aligned}&\int_\Omega A(|\nabla u|)\nabla u \nabla(x, \nabla u)\ dx - \int_{\partial\Omega} A(a)u_\nu(x, \nabla u)\ ds \\ =\ &\int_\Omega A(|\nabla u|)\left[|\nabla u|^2 + (x, \nabla(\frac{|\nabla u|^2}{2}))\right]\ dx - \int_{\partial\Omega} A(c)c^2(x, \nu)\ ds \\ =\ &\int_\Omega A(|\nabla u|)|\nabla u|^2 + (x, \nabla B(|\nabla u|))\ dx - A(c)c^2 N|\Omega| \\ =\ &\int_\Omega [A(|\nabla u|)|\nabla u|^2 - NB(|\nabla u|)]\ dx + \int_{\partial\Omega} B(c)(x, \nu)\ ds - A(c)c^2 N|\Omega| \\ =\ &\int_\Omega N[\frac{1}{N}A(|\nabla u|)|\nabla u|^2 - B(|\nabla u|)]\ dx - N[A(c)c^2 - B(c)]\ |\Omega|,\end{aligned}$$

so that after multiplication with 2/N equation (3.3) can also be written as

$$\int_\Omega \left[\frac{2}{N}A(|\nabla u|)|\nabla u|^2 - 2B(|\nabla u|) + 2u\right] dx = \left[2A(c)c^2 - 2B(c)\right]\ |\Omega|, \tag{3.4}$$

that is, using (3.1) and (3.2), as

$$\int_\Omega P(x)\ dx = \Phi(c)|\Omega|. \tag{3.5}$$

This completes the proof of Step 3 in the (regular) case that A satisfies the stronger assumptions $A : [0,\infty) \to (0,\infty)$, $A \in C^2[0,\infty)$ and $(tA(t))' > 0$ for $t \geq 0$.

In the degenerate case, where $A \in C^2(0,\infty)$ and $(tA(t))' > 0$ only for $t > 0$, we simply observe that only (3.3) requires a certain regularity of u. However, as shown in Theorem 2 of [3] equation (3.4) still holds if u is only of class $C^1(\overline{\Omega})$. As a function of ∇u, the function $B(|\nabla u|)$ is strictly convex due to (1.4). Subsequently also (3.5) holds in the degenerate case.

4 Concluding remarks

Step 1, the maximum principle part, applies also to equations involving the right hand side $w(|\nabla u|^2)f(u)$ instead of 1 in (1.1). In fact in [15, Theorem 7.3] Sperb considers the elliptic equation

$$-\text{div}\left(a(|\nabla u|^2)\nabla u\right) = w(|\nabla u|^2)f(u) \quad \text{in } \Omega \tag{4.6}$$

with $w > 0$ and introduces the functions $\Phi(t) := \int_0^t [a(s) + 2sa'(s)]/w(s)\, ds$ and $F(t) = \int_0^t f(s)\ ds$ and

$$P(x) := \Phi\left(|\nabla u(x)|^2\right) + \frac{2}{N}F\big(u(x)\big)\ .$$

If u solves (1.1) (1.2) and under suitable assumptions on $a \geq 0$, $w > 0$ and $f > 0$ he shows that P attains its maximum on the boundary. For $w \equiv 1$ the assumption on f is $f' \leq 0$.

Clearly, if $P \equiv const$ in Ω, also Step 2 of the proof goes through. Therefore there is a chance for improvement of our result to more general equations than (1.1).

If one proceeds with Step 3 for the special case $w \equiv 1$, one arrives at

$$\int_\Omega [P(x) + R(x)]\ dx = \Phi(c^2)\ |\Omega|, \tag{4.7}$$

where the remainder term $R(x)$ is given by

$$\int_\Omega R(x)\ dx = \left(2 - \frac{2}{N}\right) \int_\Omega [F(u) - uf(u)]\ dx. \tag{4.8}$$

But now the first case from Step 1 can apparently only be ruled out if $F(u) \le uf(u)$. If $f'(u) \le 0$ as required in Step 1, however, F is concave and $F(u) \ge uf(u)$. So for constant w this observation limits the method of proof to constant functions f.

Nevertheless it is not excluded that the method could be extended to nonconstant w.

Another way to reach the conclusion of Step 3 was pursued in [17, 9, 5]. These authors analyzed the consequences of equality in the differential inequality for P and came to the conclusion that mixed second derivatives of u had to vanish and pure second derivatives had to coincide. From this one can derive that u depends only on $|x|$ (modulo translations of the origin).

Still another way to reach the conclusion of Step 3 was followed in [4], where as a consequence of Step 1 the relation $P_\nu \ge 0$ on $\partial\Omega$ was exploited. It leads to a uniform bound on the mean curvature H of $\partial\Omega$, namely $H \le [NcA(c)]^{-1}$. The strategy was then to show that the bound is sharp everywhere on $\partial\Omega$.

In our paper we have assumed smoothness of $\partial\Omega$ out of convenience, because our main goal was the removal of assumptions on the differential operator and on the domain geometry. For nonsmooth, say Lipschitz boundaries, the Neumann condition can only hold a.e. on $\partial\Omega$, and then results of [11] seem to be useful.

Acknowledgement: This proof was essentially discovered during a visit of A. Farina in Cologne in Dec. 2005 and during the workshop "Symmetries of solutions of nonlinear elliptic PDEs" in Vienna in January 2006. We thank Gérard Philippin for bringing [9] to our attention and the referee for helpful suggestions about the exposition. This research was financially supported in part by the ESF-Programme "Global and geometrical aspects of nonlinear partial differential equations (GLOBAL)".

References

[1] Alexandrov, A.D., A characteristic property of the spheres, Ann. Mat. Pura Appl. **58** (1962) 303–354.

[2] Brock,F., Henrot, A., A symmetry result for an overdetermined elliptic problem using continuous rearrangement and domain derivative, Rend. Circ. Mat. Palermo **51** (2002) 375–390.

[3] Degiovanni, M., Musesti, A. & Squassina, M., On the regularity of solutions in the Pucci-Serrin identity, Calculus of Variations **18** (2003) 317–334.

[4] Fragalà, I., Gazzola, F. & Kawohl, B., Overdetermined boundary value problems with possibly degenerate ellipticity: a geometric approach. Math. Zeitschr., **???# /*"!@!"*/# ????**

[5] Garofalo, N. &. Lewis, J.L., A symmetry result related to some overdetermined boundary value problems, Amer. J. Math. **111** (1989) 9–33.

[6] Kawohl, B., Symmetrization - or how to prove symmetry of solutions to partial differential equations, in: *Partial Differential Equations, Theory and Numerical Solution,* Eds.: W.Jäger, J.Nečas, O.John, K.Najzar, J.Stara, Chapman & Hall CRC Research Notes in Math. **406** London (1999) 214–229.

[7] Lieberman, G., Boundary regularity for solutions of degenerate elliptic equations. Nonlinear Anal. **12** (1988) 1203–1219.

[8] Payne, L.A. & Philippin, G.A., Some maximum principles for nonlinear elliptic equations in divergence form with applications to capillary surfaces and to surfaces of constant mean curvature, Nonlinear Anal. **3** (1979) 193–211.

[9] Philippin, G.A., Applications of the maximum principle to a variety of problems involving elliptic differential equations, in: *Maximum principles and eigenvalue problems in partial differential equations (Knoxville, TN, 1987),* Ed. P.W. Schaefer, Longman Sci. Tech., Pitman Res. Notes Math. Ser., **175** Harlow (1988) 34–48.

[10] S.J. Pohožaev, Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$, Soviet Math. Doklady **6** (1965) 1408–1411.

[11] J. Prajapat, Serrin's result for domains with a corner or cusp, Duke Math. J. **91** (1998) 29–31.

[12] Pucci, P. & Serrin, J., A general variational identity, Indiana Univ. Math. J., **35** (1986) 681–703.

[13] Rellich, F., Darstellung der Eigenwerte von $\Delta u + \lambda u = 0$ durch ein Randintegral. Math. Zeitschr., **. 46** (1940). 635–636.

[14] Serrin, J., A symmetry problem in potential theory, Arch. Ration. Mech. Anal. **43** (1971) 304–318.

[15] Sperb, R., *Maximum Principles and Applications*, Academic Press (1981)

[16] Thorbergsson, G., A survey on isoparametric hypersurfaces and their generalizations. in: *Handbook of differential geometry, Vol. I* , Eds.: F.J. E. Dillen, L. C. A.Verstraelen. North-Holland, Amsterdam (2000) 963–995.

[17] Weinberger, H., Remark on the preceding paper of Serrin, Arch. Ration. Mech. Anal. **43** (1971) 319–320.

Mathematics Subject Classification (2000). 35J65, 35B35

Keywords. symmetry, overdetermined boundary problem, isoparametric surfaces, Rellich identity, Pohozaev identity, Pucci-Serrin identity

Author's addresses:

Alberto Farina
LAMFA, CNRS UMR 6140, Université de Picardie Jules Verne,
Faculté de Mathématiques et d'Informatique
33, rue Saint-Leu
80039 Amiens, France
alberto.farina@u-picardie.fr

Bernd Kawohl
Mathematisches Institut
Universität zu Köln
D-50923 Köln, Germany
kawohl@mi.uni-koeln.de