

REGULARITY FOR THE PRESCRIBED VERTICAL MEAN CURVATURE SYSTEM IN THE PLANE

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ABSTRACT. We prove everywhere regularity for Lipschitz solutions to the prescribed vertical mean curvature system in the plane, where the codimension is arbitrary and the mean curvature is in L^r , $r > 2$.

1. INTRODUCTION

In this paper we study the regularity of Lipschitz maps $u = (u^1, \dots, u^n) : B_1 \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$ that solve, in the sense of distributions, the system

$$\frac{1}{\sqrt{\det g}} \partial_i (\sqrt{\det(g)} g^{ij} \partial_j u^l) = f^l, \quad l = 1, \dots, n. \quad (1.1)$$

Here $g_{ij} = \delta_{ij} + \partial_i u \cdot \partial_j u$, $i, j = 1, \dots, m$, and $f = (f^1, \dots, f^n) \in L^r(B_1, \mathbb{R}^n)$, $r \geq 1$.

If $u \in C^2$, system (1.1) says that the vertical projection of the mean curvature vector to the graph of u (that is, the projection to the last n variables in $\mathbb{R}^m \times \mathbb{R}^n$) is f . For this reason, we call (1.1) the *prescribed vertical mean curvature system*. We also note that when $u \in C^2$, the horizontal projection $\tilde{f}$ of the mean curvature vector to the graph of u is determined by f and the fact that the mean curvature vector is normal to the graph of u , i.e. $\tilde{f} = -Du^T f$. Equivalently, when $u \in C^2$ solves (1.1), it also solves the system

$$\frac{1}{\sqrt{\det g}} \partial_i (\sqrt{\det(g)} g^{ij}) = -f \cdot \partial_j u, \quad j = 1, \dots, m. \quad (1.2)$$

Remark 1.1. More generally, whenever $v \in \text{Lip} \cap W^{2,1}(B_1, \mathbb{R}^n)$ and F is a smooth function on $\mathbb{R}^{n \times m}$, we have the implication

$$\partial_i (F_{p_i^l}(Dv)) = H^l, \quad l = 1, \dots, n \quad \implies \quad \partial_i (F(Dv) \delta_{ij} - F_{p_i^l}(Dv) v_j^l) = -H^l v_j^l, \quad j = 1, \dots, m. \quad (1.3)$$

Here $F_{p_i^l}$ denotes the derivative of F in the p_i^l direction. The system on the left reflects that v is a critical point of the energy

$$E(v) = \int_{B_1} F(Dv) + (H, v) dx \quad (1.4)$$

with respect to outer variations (i.e., $E(v + t\varphi) = E(v) + o(t)$ for all maps $\varphi \in C_c^\infty(B_1, \mathbb{R}^n)$), and the system on the right reflects that v is a critical point of E with respect to domain variations (i.e., $E(v(x + tV)) = E(v) + o(t)$ for all vector fields $V \in C_c^\infty(B_1, \mathbb{R}^m)$). If $D^2 F > 0$ and $H \in L_{\text{loc}}^2$, then Lipschitz solutions to the system on the left hand side of (1.3) are in $W_{\text{loc}}^{2,2}$ by standard regularity theory, and hence solve the system on the right hand side of (1.3). In this paper we are considering the area integrand $F(p) = \sqrt{\det(I + p^T p)}$. This integrand is polyconvex (a convex function of the sub-determinants of p), and it is convex if and only if $\min\{m, n\} = 1$.

If u is Lipschitz and both systems (1.1) and (1.2) are assumed to hold with $f \in L_{\text{loc}}^r$, then the graph of u has mean curvature bounded in L_{loc}^r in the sense of varifolds and the regularity of u is well-understood in all dimensions. Namely, provided $r > m$, we have $u \in W_{\text{loc}}^{2,r}$ away from a closed singular set of Hausdorff dimension at most $m - 4$ [7, Theorem 4.4]. This bound on the size of the singular set is optimal even for

$f = 0$, due to the Lawson-Osserman cone [16, Theorem 7.1]. Key tools in this setting that are not available when we only assume that (1.1) holds are the monotonicity formula and Allard's ε -regularity theorem [1].

When u is Lipschitz and we only assume that the system (1.1) holds, the situation is considerably more delicate, unless we are in the codimension one case $n = 1$. In that case solutions to (1.1) are minimizers of an energy of the form (1.4), and hence also solve (1.2), and thus belong to $W_{\text{loc}}^{2,r}(B_1, \mathbb{R}^n)$ when $r > m$, see [7, Theorem 4.4 and Remarks]. Celebrated examples of S. Müller-V. Šverák [19] and L. Székelyhidi [22] show that Lipschitz outer critical points of strongly quasiconvex, resp. polyconvex energies, can be extremely wild, e.g. nowhere C^1 (even when $m = n = 2$). Thus, positive regularity results for Lipschitz outer critical points of polyconvex energies must rely on additional structure.

In their seminal work on the minimal surface system, H. B. Lawson and R. Osserman conjectured that Lipschitz solutions to (1.1) also solve (1.2) when $f = 0$, see [16, Conjecture 2.1]. In [14] we proved the Lawson-Osserman conjecture in the planar case $m = 2$, for arbitrary codimension n . The purpose of this paper is to generalize this result to the case of nonzero right hand side. Our first main theorem is:

Theorem 1.2. *If u is a Lipschitz solution to (1.1) in $B_1 \subset \mathbb{R}^2$ and $f \in L^r(B_1, \mathbb{R}^n)$ with $r > 2$, then $u \in W_{\text{loc}}^{2,r}(B_1, \mathbb{R}^n)$, hence $u \in C_{\text{loc}}^{1,1-m/r}(B_1, \mathbb{R}^n)$ and solves (1.2).*

Standard examples from Sobolev space theory show that we cannot expect that $u \in C^1$ when $r \leq 2$, even in codimension $n = 1$. For example, $u(x_1, x_2) \doteq x_1 \sin(\log \log(1/|x|))$ is a Lipschitz function on a neighborhood of 0 in $\mathbb{R}^2$ that solves (1.1) with $f \in L^2$, but u is not C^1 .

It would be extremely interesting to investigate what happens when $m \geq 3$. The counterexamples of [19, 22] rely on showing that special sets of matrices known as T_N configurations exist in the differential inclusion associated to the energy under investigation. It is natural to seek such configurations in the differential inclusions associated to the area functional. Through the same techniques used to show Theorem 1.2, we prove the following rigidity statement for *approximate solutions* to our PDE:

Theorem 1.3. *Assume $(u_j)_j$ is a sequence of equibounded Lipschitz maps in $\text{Lip}(B_1, \mathbb{R}^n)$, $B_1 \subset \mathbb{R}^2$, whose weak-* limit is the Lipschitz map u , and assume*

$$\text{div}(D\mathcal{A}(Du_j)) = \text{div}(F_j)$$

with F_j equibounded in $L^\infty(B_1)$ and converging strongly to 0 in $L^1(B_1)$. Then, $(Du_j)_j$ converges strongly in L^1 and u is a solution to $\text{div}(D\mathcal{A}(Du)) = 0$.

The absence of T_N configurations in the inclusion set associated to critical points of the area functional when $m = 2$ is deduced in Remark 6.2 as a corollary of this theorem. The same rigidity question remains open in higher dimensions. It would also be interesting to study the regularity of solutions to (1.1) that are not assumed to be Lipschitz, but to lie in $W^{1,p}$ for some $p \geq 1$. We plan to investigate these directions in future work.

We now describe our strategy for proving Theorem 1.2. The starting point is the same as that in our proof for the case $f = 0$ in [16]. Namely, by the use of isothermal coordinates (solving a Beltrami equation) we obtain a quasiconformal map ψ such that the map $v = u \circ \psi$ solves the system

$$\Delta v = \det D\psi h \circ \psi, \quad \text{where } h \doteq \sqrt{\det g} f.$$

When $f = 0$ we conclude that v is harmonic; in particular, $v \in C^1$, which allowed us in [16] to decompose B_1 into various regions where regularity of u could be inferred for different reasons. However, when f is nonzero, the above system for v does not immediately give enough regularity to proceed in this manner. Instead, the key observation is that the system solved by ψ implies that (v, ψ) is a conformal map into $\mathbb{R}^{n+2}$. As a consequence, the Hopf differentials of v and ψ (see Section 3 for the definition) sum to zero. After establishing an ellipticity property of the Hopf differential on quasiconformal matrices, we can use the regularity properties of v from the system above to infer regularity properties of ψ that are better than

those we started with. We can then feed these back into the system above for v , bootstrapping until we get that both v and ψ are $C^{1,\alpha}$. The desired regularity for u then follows at the points $\psi(y)$ such that $D\psi(y)$ is nonzero. In the final step of the proof we show that the set where $D\psi = 0$ is in fact empty, using ideas related to unique continuation originating in work of P. Hartman and A. Wintner [12].

Remark 1.4. In the last step of our proof we show the differential inequality $|\Delta\psi| \leq |Dv||h \circ \psi||D\psi|$. In particular, when $f = 0$, ψ is harmonic, thus smooth with an isolated set of critical points. This gives a new proof, different from that in [16], of the regularity of u in the case that $f = 0$.

Remark 1.5. The Hopf differential also plays an important role in the theory of harmonic maps of Riemann surfaces, see [13].

The paper is organized as follows. In Section 2 we establish notation and discuss some preliminary results about the structure of the area functional and quasiconformal maps. In Section 3 we prove Theorem 1.2, using a few results whose proofs are delayed to later sections. In Sections 4 and 5 we prove those results, which include the ellipticity of the Hopf differential on quasiconformal matrices (Section 4) and the bootstrapping argument alluded to above (Section 5). In Section 6 we use these techniques to show rigidity of approximate solutions, namely Theorem 1.3. Finally, in Appendix A we provide for the reader's convenience a proof of the Hartman-Wintner result with regularity hypotheses relaxed so as to apply in our setting.

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AI STATEMENT

AI was not used in any part of the manuscript: the mathematical arguments were developed solely by the authors, who also wrote the entirety of the manuscript.

2. PRELIMINARIES AND NOTATION

2.1. Notation. We denote by X^T the transpose of $X \in \mathbb{R}^{n \times m}$. If $n = m$ and X is invertible, we denote by X^{-1} the inverse of X . We also write $\text{cof}(X)$ for the cofactor matrix of X , which satisfies

$$\text{cof}(X)X = X \text{cof}(X) = \det(X) \text{Id}_m. \quad (2.1)$$

Explicitly, in the case $n = 2$, if

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{then } \text{cof}(X) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

The standard Euclidean product of vectors $a, b \in \mathbb{R}^n$ is denoted by (a, b) , while the standard Euclidean product of matrices $A, B \in \mathbb{R}^{n \times m}$ is denoted by $\langle A, B \rangle$. The associated Euclidean norm is denoted both for vectors a and matrices A as $|a|$ and $|A|$.

We write $L^p(\Omega)$ if we consider L^p scalar-valued functions, and $L^p(\Omega, \mathbb{R}^k)$ if we consider $\mathbb{R}^k$ -valued functions. If we want to emphasize where the image of the map lies, say in $U \subset \mathbb{R}^k$, we will write $L^p(\Omega, U)$. We use the same convention for other functional spaces, such as Sobolev and Hölder spaces. When it is clear from the context, we will sometimes shorten the notation $\|h\|_{L^p(\Omega, \mathbb{R}^n)}$ to $\|h\|_p$.

2.2. The area functional. Here we recall some properties of the structure of the area functional. A more detailed explanation can be found in [14, Section 2.2]. As in that reference, we restrict our attention to the case in which the domain has dimension $m = 2$. As recalled in the introduction, the energy associated to the area integrand on graphs is

$$\mathbb{E}(u) \doteq \int_{B_1} \mathcal{A}(Du) dx, \quad u \in \text{Lip}(B_1, \mathbb{R}^n),$$

where the area integrand on graphs $\mathcal{A} : \mathbb{R}^{n \times 2} \rightarrow \mathbb{R}$ reads as

$$\mathcal{A}(Z) \doteq \sqrt{\det(M(Z)^T M(Z))}, \quad \text{where} \quad M(Z) \doteq \begin{pmatrix} \text{id}_2 \\ Z \end{pmatrix}, \quad \text{and } Z \in \mathbb{R}^{n \times 2}.$$

We say that a map $u \in \text{Lip}(B_1, \mathbb{R}^n)$ has mean curvature $f \in L^p_{\text{loc}}(B_1, \mathbb{R}^n)$, $p \geq 1$, if:

$$\begin{cases} \partial_i(\sqrt{\det g} g^{ij} \partial_j u^l) = \sqrt{\det g} f^l, & l = 1, \dots, n, \\ \partial_i(\sqrt{\det g} g^{ij}) = -\sqrt{\det g} g \partial_j u^i f^i, & j = 1, 2, \end{cases} \quad (2.2)$$

$$(2.3)$$

where $(g^{ij}) = (g_{ij})^{-1}$, $g_{ij} = \delta_{ij} + (Du^i, Du^j)$. System (2.2)-(2.3) is equivalent to asking that the graph of u , i.e. $\Gamma_u = \{(x, u(x)) : x \in B_1\}$, has mean curvature bounded in L^p in the sense of varifolds, see for instance [20, Section 4.2], [5, Section 6.3]. If we introduce

$$g(Z) \doteq \text{id}_2 + Z^T Z \quad \text{and} \quad B(Z) \doteq \sqrt{\det g(Z)} g^{-1}(Z), \quad (2.4)$$

we can rewrite system (2.2)-(2.3) in matrix notation:

$$\begin{cases} \text{div}(Du B(Du)) = \sqrt{\det g} f, \\ \text{div}(B(Du)) = -\sqrt{\det g} Du^T f, \end{cases} \quad (2.5)$$

$$(2.6)$$

where the divergence is taken row-wise. Finally, if we make use of the algebraic identity

$$D\mathcal{A}(Z) = Z B(Z), \quad \forall Z \in \mathbb{R}^{n \times 2}, \quad (2.7)$$

we can finally rewrite (2.5) as

$$\text{div}(D\mathcal{A}(Du)) = h \doteq \sqrt{\det g} f. \quad (2.8)$$

As $u \in \text{Lip}(B_1, \mathbb{R}^n)$, $\det g$ is a measurable function bounded from above and below, hence $h \in L^p(B_1, \mathbb{R}^n)$ if and only if $f \in L^p(B_1, \mathbb{R}^n)$. In the sequel we will find convenient to use h instead of f when performing calculations and estimates.

2.3. Quasiconformal mappings. We recall some useful properties of quasiconformal mappings.

Definition 2.1. Let $\Omega \subset \mathbb{R}^2$ be open. The map $\varphi \in W^{1,2}(\Omega, \mathbb{R}^2)$ is $(K-)$ quasiregular if there exists $K \geq 1$ such that

$$|D\varphi|^2(x) \leq K \det(D\varphi(x)), \quad \text{for a.e. } x \in \Omega. \quad (2.9)$$

We say that it is $(K-)$ quasiconformal if, in addition, φ is a homeomorphism¹ between Ω and the open set $\varphi(\Omega)$.

Theorem 2.2. Let $\varphi \in W^{1,2}(\Omega_1, \Omega_2)$ be quasiregular and $\psi \in W^{1,2}(\Omega_2, \Omega_3)$ be quasiconformal with $\Omega_3 = \psi(\Omega_2)$. Then:

- (1) $\psi^{-1} : \Omega_3 \rightarrow \Omega_2$ is a $W^{1,2}(\Omega_3, \Omega_2)$ quasiconformal mapping (with the same quasiregularity constant K of ψ , compare (2.9)), with gradient

$$D(\psi^{-1}(y)) = (D\psi)^{-1}(\psi^{-1}(y)), \quad \text{for a.e. } y \in \Omega_3;$$

- (2) the chain rule holds: for every $f \in W^{1,2}(\Omega_2)$, $f \circ \varphi \in W^{1,2}(\Omega_1)$ with

$$D(f \circ \varphi)(x) = Df(\varphi(x)) D\varphi(x), \quad \text{for a.e. } x \in \Omega_1;$$

¹Note that every quasiregular map is continuous, compare Theorem 2.2(4).

- (3) the composition $\psi \circ \varphi$ is quasiregular;
- (4) there exists $p = p(K) > 2$ such that $\varphi \in W_{\text{loc}}^{1,p}(\Omega_1, \Omega_2)$, if K is the quasiregularity constant of (2.9), with bounds depending solely on the $W^{1,2}$ norm of φ ;
- (5) $\det D\psi \neq 0$ a.e..

Assertion (1) can be found in [4, Theorem 9.1] and (2) is the content of [4, Lemma 9.6]. (3) is proved in [4, Theorem 9.4]. Property (4) is true in view of [4, Theorem 5.1], see also [3] for the sharp result. Finally, (5) can be found in [4, Theorem 7.2].

2.4. Complex derivatives. In some points of the proof it is convenient to use holomorphic and anti-holomorphic coordinates. To recall them, let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The conformal and anti-conformal parts are defined as follows, in matrix and complex notation:

$$A_{co} = \frac{1}{2} \begin{pmatrix} a+d & b-c \\ c-b & a+d \end{pmatrix}, \quad [A]_{\mathcal{H}} \doteq \frac{1}{2}[(a+d) + i(c-b)]$$

and

$$A_{aco} = \frac{1}{2} \begin{pmatrix} a-d & b+c \\ b+c & d-a \end{pmatrix}, \quad [A]_{\overline{\mathcal{H}}} \doteq \frac{1}{2}[(a-d) + i(c+b)].$$

In particular, we have the fundamental identity

$$Az = [A]_{\mathcal{H}}z + [A]_{\overline{\mathcal{H}}}\bar{z},$$

where the left-hand side Az is a matrix multiplication and the right-hand side is a sum of complex products of complex numbers. We have the following relations:

$$\det(A) = |[A]_{\mathcal{H}}|^2 - |[A]_{\overline{\mathcal{H}}}|^2, \quad |A|^2 = 2|[A]_{\mathcal{H}}|^2 + 2|[A]_{\overline{\mathcal{H}}}|^2. \quad (2.10)$$

Let $\Omega \subset \mathbb{R}^2$ be open. For a differentiable function $f : \Omega \rightarrow \mathbb{R}$, we can set

$$\partial_z f = f_z \doteq \frac{1}{2}(\partial_1 f - i\partial_2 f) \text{ and } \partial_{\bar{z}} f = f_{\bar{z}} \doteq \frac{1}{2}(\partial_1 f + i\partial_2 f) = \overline{f_z}. \quad (2.11)$$

For $f = (f_1, f_2) : \Omega \rightarrow \mathbb{R}^2$, we define

$$f_z \doteq \partial_z f_1 + i\partial_z f_2 \text{ and } f_{\bar{z}} \doteq \partial_{\bar{z}} f_1 + i\partial_{\bar{z}} f_2. \quad (2.12)$$

More explicitly, in the latter case we can write:

$$f_z = [Df]_{\mathcal{H}} = \frac{1}{2}[(\partial_1 f_1 + \partial_2 f_2) + i(\partial_1 f_2 - \partial_2 f_1)], \quad f_{\bar{z}} = [Df]_{\overline{\mathcal{H}}} = \frac{1}{2}[(\partial_1 f_1 - \partial_2 f_2) + i(\partial_1 f_2 + \partial_2 f_1)]. \quad (2.13)$$

This notation also yields another way to read (2.9). First of all, assume that $Z \in \mathbb{R}^{2 \times 2}$ is a quasiconformal matrix, i.e. that for some $K > 0$

$$|Z|^2 \leq K \det(Z).$$

Then, due to (2.10):

$$|Z|^2 \leq K \det(Z) \iff 2|[Z]_{\mathcal{H}}|^2 + 2|[Z]_{\overline{\mathcal{H}}}|^2 \stackrel{(2.10)}{\leq} K(|[Z]_{\mathcal{H}}|^2 - |[Z]_{\overline{\mathcal{H}}}|^2),$$

and the latter is true if and only if there exists $\kappa = \kappa(K) \in [0, 1)$ such that

$$|[Z]_{\overline{\mathcal{H}}}|^2 \leq \kappa |[Z]_{\mathcal{H}}|^2. \quad (2.14)$$

Using (2.14) we see that $f \in W_{\text{loc}}^{1,2}(\Omega, \mathbb{R}^2)$ is quasiregular if and only if there exists $\kappa \in [0, 1)$ such that

$$|f_{\bar{z}}|^2(z) \leq \kappa |f_z|^2(z) \text{ for a.e. } z \in \Omega. \quad (2.15)$$

We have the following result, which can be checked by hand and that we will use freely in the paper:

Lemma 2.3. *Let $f, g : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be sufficiently differentiable functions. Then,*

- (1) $f_{z\bar{z}} = f_{\bar{z}z} = \frac{1}{4}\Delta f$;
- (2) $(f)_z = \overline{f_{\bar{z}}}$ and $(f)_{\bar{z}} = \overline{f_z}$;
- (3) $(fg)_z = f_z g + f g_z$, $(fg)_{\bar{z}} = \overline{f_z} g + f \overline{g_z}$;

3. PROOF OF THEOREM 1.2

The proof is divided into steps. Following (2.8), we consider $u \in \text{Lip}(B_1, \mathbb{R}^n)$ solving

$$\text{div}(D\mathcal{A}(Du)) = h, \quad \text{for } h \in L^r(B_1, \mathbb{R}^n), r > 2. \quad (3.1)$$

Our goal is to prove that $u \in W_{\text{loc}}^{2,r}$, but it suffices to show that $u \in C^1(B_1, \mathbb{R}^n)$. Indeed, denoting $G_z(x) = G(x+z) - G(x)$ for any map G and any z , we can consider $g \in W^{1,r}$ with $h = \text{div}(g)$ and write:

$$\text{div}(g_z) = \text{div}(D\mathcal{A}(Du)(x+z) - D\mathcal{A}(Du)(x)) = \text{div} \left(\left[\int_0^1 D^2\mathcal{A}((1-t)Du(x) + tDu(x+z))dt \right] Du_z \right).$$

If $u \in C^1$, we can combine [8, Theorems 4.4-7.3] to deduce the $W_{\text{loc}}^{2,r}$ regularity of u , using that $\mathcal{A}$ fulfills a uniform Legendre-Hadamard condition on compact sets [23, Lemma 6.7].

The starting point of the proof is precisely the same as the one of [14, Proposition 3.3], i.e. as in [14, Lemma 4.1] we start by factorizing u as:

$$u = v \circ \varphi, \quad (3.2)$$

where $v \in W^{1,2}(\varphi(B_1), \mathbb{R}^n)$ is shown to solve an auxiliary elliptic system (3.5) and $\varphi \in W^{1,2}(B_1, \mathbb{R}^2)$ is a quasiconformal map. The details are contained in Step 1. In Step 2 we will study the regularity of v and of $\psi = \varphi^{-1}$. This exploits auxiliary results that will be proved in the forthcoming sections, Lemma 4.1, Theorem 5.1 and Corollary 5.2. From the results of Step 2, we will infer through (3.2) that u is $C^{1,\alpha}$ outside of a singular set $\psi(S)$. In Step 3 we will show how to remove this singular set with the help of Theorem A.1 and hence how to conclude the proof of the ongoing Theorem 1.2.

Step 1: factorization of u . We show that u can be factorized as (3.2). Let

$$g \doteq \text{id}_2 + Du^T Du = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \quad \text{and} \quad \mu(z) \doteq \frac{E - G + 2iF}{E + G + 2\sqrt{EG - F^2}}.$$

$u \in \text{Lip}(B_1, \mathbb{R}^n)$ implies $\text{id}_2 \leq g \leq \Lambda \text{id}_2$ at a.e. point of B_1 in the sense of quadratic forms. This yields $|\mu|(z) \leq k = k(\Lambda) < 1$ for a.e. $z \in B_1$. Therefore, after extending μ to 0 outside B_1 , we can exploit [2, Theorem 5.3.4] to find a quasiconformal solution $\varphi \in W_{\text{loc}}^{1,2}(\mathbb{R}^2, \mathbb{R}^2)$ to $\varphi_{\bar{z}} = \mu\varphi_z$ a.e. in $\mathbb{R}^2$. Passing to real notation, this is equivalent to:

$$g = \rho D\varphi^T D\varphi, \quad \text{with } \rho \doteq \frac{\sqrt{\det(g)}}{\det(D\varphi)}, \quad \text{a.e. in } B_1. \quad (3.3)$$

For details, see [2, Theorem 10.1.1]. Set $U \doteq \varphi(B_1)$, which is open since φ is injective. As said, $v \in W^{1,2}(U, \mathbb{R}^n)$ defined by $u \circ \varphi^{-1}$ solves an auxiliary elliptic system: we write, for any $\eta \in C_c^\infty(U, \mathbb{R}^n)$,

$$\begin{aligned} \int_U \langle Dv, D\eta \rangle dy &= \int_U \langle Du \circ \varphi^{-1} (D\varphi)^{-1} \circ \varphi^{-1}, D\eta \rangle dy \\ &= \int_{B_1} \det(D\varphi) \langle Du(D\varphi)^{-1}, D\eta \circ \varphi \rangle dx = \int_{B_1} \det(D\varphi) \langle Du(D\varphi)^{-1}, D(\eta \circ \varphi)(D\varphi)^{-1} \rangle dx \\ &= \int_{B_1} \det(D\varphi) \langle Du(D\varphi)^{-1} (D\varphi)^{-T}, D(\eta \circ \varphi) \rangle dx \stackrel{(3.3)}{=} \int_{B_1} \sqrt{\det(g)} \langle Du g^{-1}, D(\eta \circ \varphi) \rangle dx \\ &\stackrel{(2.4)-(2.7)}{=} \int_{B_1} \langle D\mathcal{A}(Du), D(\eta \circ \varphi) \rangle dx \stackrel{(3.1)}{=} \int_{B_1} \langle h, \eta \circ \varphi \rangle dx. \end{aligned} \quad (3.4)$$

In the last line we used the quasiconformality of φ and Theorem 2.2(2) to infer $\eta \circ \varphi \in W_0^{1,2}(B_1, \mathbb{R}^n)$, which is an admissible test map for the weak formulation of (3.1). Hence:

$$\int_U \langle Dv, D\eta \rangle dy = - \int_{B_1} (h, \eta \circ \varphi) dx. \quad (3.5)$$

Using the area formula, we see that the latter is just the weak formulation of

$$\Delta v = h \circ \psi \det(D\psi),$$

a form that we will exploit in (3.14).

Step 2: regularity of v and ψ . Bound the right-hand side of (3.5) through Hölder inequality:

$$\left| \int_{B_1} (h, \eta \circ \varphi) dx \right| \leq \|h\|_r \|\eta \circ \varphi\|_{r'}.$$

We let $\psi \doteq \varphi^{-1}$, which is quasiconformal due to Theorem 2.2(1). After the change of variables $x = \psi(y)$,

$$\|\eta \circ \varphi\|_{r'}^{r'} = \int_{B_1} |\eta(\varphi(x))|^{r'} dx = \int_U |\eta(y)|^{r'} \det(D\psi)(y) dy. \quad (3.6)$$

Let $p > 2$ be such that $D\psi \in L_{\text{loc}}^p(U, \mathbb{R}^2)$, see Theorem 2.2(4). We can then bound the previous integral as:

$$\int_U |\eta(y)|^{r'} \det(D\psi)(y) dy \leq \left(\int_U |\eta(y)|^{r'\kappa} dy \right)^{\frac{1}{\kappa}} \left(\int_{\text{spt}(\eta)} \det(D\psi)^{\frac{p}{2}} dy \right)^{\frac{2}{p}}. \quad (3.7)$$

Here, $\kappa = \kappa(p)$ is the Hölder conjugate of $p/2$, namely

$$\kappa(p) \doteq \frac{p}{p-2}.$$

Combining (3.5)-(3.6)-(3.7), we find that

$$\left| \int_U \langle Dv, D\eta \rangle dy \right| \leq C \|\eta\|_{L^{r'\kappa}(U, \mathbb{R}^n)},$$

for some C that depends on p , ψ and on $\text{dist}(\text{spt}(\eta), \partial U)$. Therefore, if β is the Hölder conjugate of $r'\kappa(p)$:

$$D\psi \in L_{\text{loc}}^p(U, \mathbb{R}^2) \implies \Delta v \in L_{\text{loc}}^\beta(U, \mathbb{R}^n), \quad \text{for } \beta = (r'\kappa(p))' = \frac{r'p}{(r'-1)p+2}. \quad (3.8)$$

In addition to this information, we can find another relation between v and ψ . Considering (3.3), we write

$$\text{id}_2 + Du^T Du = g = \rho D\varphi^T D\varphi.$$

Using (3.2) and the chain rule we have, a.e. in B_1 :

$$\text{id}_2 + D\varphi^T Dv^T \circ \varphi Dv \circ \varphi D\varphi = \rho D\varphi^T D\varphi.$$

Inverting $D\varphi$ and using that $(D\varphi)^{-1}(x) = D\psi(\varphi(x))$ by Theorem 2.2(1), we can write

$$D\psi^T(\varphi(x)) D\psi(\varphi(x)) + Dv^T(\varphi(x)) Dv(\varphi(x)) = \rho \text{id}_2.$$

Finally we deduce that, for $a \doteq \rho \circ \psi(y)$:

$$D\psi^T(y) D\psi(y) + Dv^T(y) Dv(y) = a(y) \text{id}_2, \quad \text{for a.e. } y \in U. \quad (3.9)$$

This tells us that the traceless parts of the symmetric matrices $D\psi^T D\psi$ and $Dv^T Dv$ coincide a.e., up to a sign. Let us introduce² the Hopf differential operator $H : \mathbb{R}^{k \times 2} \rightarrow \mathbb{R}^{2 \times 2}$, which yields, for all $Z \in \mathbb{R}^{k \times 2}$, the traceless part of $Z^T Z$:

$$H(Z) \doteq Z^T Z - \frac{|Z|^2}{2} \text{id}_2. \quad (3.10)$$

²The Hopf differential arises when considering domain variations for the Dirichlet energy, in the sense that a map u which is critical with respect to domain variations of the Dirichlet energy fulfills $\text{div } H(Du) = 0$.

Here k is any natural number and for us $k = 2$ or $k = n$ but, with a small abuse of notation, we will not denote H differently for different k . Relation (3.9) then yields

$$H(D\psi)(y) = -H(Dv)(y), \quad \text{for a.e. } y \in U. \quad (3.11)$$

One last observation is that $u \in \text{Lip}(B_1, \mathbb{R}^n)$ and the chain rule Theorem 2.2(2) implies

$$|Dv(y)| = |Du(\psi(y))D\psi(y)| \leq C|D\psi(y)| \quad \text{a.e. in } U. \quad (3.12)$$

Consider then the system obtained combining (3.8), (3.11) and (3.12). In Theorem 5.1 we will show that this system implies that the map v actually belongs to $W_{\text{loc}}^{2,2+\varepsilon}(U, \mathbb{R}^n)$, for some $\varepsilon > 0$. Next, in Corollary 5.2, we will infer that also $\psi \in W_{\text{loc}}^{2,2+\varepsilon}(U, \mathbb{R}^2)$ for a possibly smaller $\varepsilon > 0$. This in particular yields that the gradients of v and ψ are α -Hölder continuous. Now the classical inverse function theorem implies that $\varphi = \psi^{-1}$ belongs as well to $C^{1,\alpha}(E, \mathbb{R}^2)$ on the open set

$$E = B_1 \setminus \psi(S), \quad \text{where } S \doteq \{y \in U : \det(D\psi(y)) = 0\} = \{y \in U : |D\psi|(y) = 0\},$$

where equality between definitions of S holds since ψ is quasiregular. Since $v \in C^{1,\alpha}(U, \mathbb{R}^n)$, (3.2) readily implies that $u \in C^{1,\alpha}(E, \mathbb{R}^n)$. To conclude that $u \in C^{1,\alpha}(B_1, \mathbb{R}^n)$ we need to show that $S = \emptyset$.

Step 3: $S = \emptyset$ and conclusion of the proof. To show that S is empty, we need to use the claimed regularity of v and ψ to rewrite (3.5) and (3.11). We notice in fact that (3.5) is equivalent to

$$\int_U \langle Dv, D\eta \rangle dy = - \int_U (h \circ \psi, \eta) \det(D\psi) dy \quad (3.13)$$

after the usual change of variables $y = \varphi(x)$. Since $v \in W_{\text{loc}}^{2,2+\varepsilon}(U, \mathbb{R}^n)$, we can actually write (3.13) in its strong form a.e., denoting $F \doteq h \circ \psi$:

$$\Delta v = F \det(D\psi). \quad (3.14)$$

A simple change of variables and the fact that $D\psi \in L_{\text{loc}}^\infty$ since $\psi \in W_{\text{loc}}^{2,2+\varepsilon}(U, \mathbb{R}^2)$ implies that

$$F \det(D\psi)^{\frac{1}{2}} \in L_{\text{loc}}^r(U, \mathbb{R}^n). \quad (3.15)$$

Since $v \in W_{\text{loc}}^{2,2+\varepsilon}(U, \mathbb{R}^n)$ and $\psi \in W_{\text{loc}}^{2,2+\varepsilon}(U, \mathbb{R}^2)$, we take the divergence of (3.11) pointwise a.e. to get:

$$D\psi^T \Delta \psi = -Dv^T \Delta v \stackrel{(3.14)}{=} -Dv^T F \det(D\psi).$$

As $D\psi$ is invertible almost everywhere by Theorem 2.2(5), we can rewrite the last equation as

$$\Delta \psi = -\text{cof}^T(D\psi) Dv^T F, \quad \text{a.e. in } U. \quad (3.16)$$

We can now bound:

$$|\Delta \psi| \leq C|F||D\psi||Dv| \stackrel{(3.12)}{\leq} C|D\psi|^2|F|, \quad \text{a.e. in } U. \quad (3.17)$$

Setting $g \doteq |D\psi||F|$, we see from (3.15) and the fact that ψ is quasiregular that $g \in L_{\text{loc}}^r(U)$. Thus, we rewrite the previous equation as:

$$|\Delta \psi|(y) \leq Cg(y)|D\psi|(y), \quad \text{for a.e. } y \text{ in } U. \quad (3.18)$$

Adapting an argument of [12, Theorem 1-2] from the case $g \in L_{\text{loc}}^\infty(U)$ to the case $g \in L_{\text{loc}}^r(U)$ and $r > 2$, we will show in Appendix A that any non-constant $W_{\text{loc}}^{2,2+\varepsilon}(U, \mathbb{R}^2)$ map ψ fulfilling (3.18) admits a precise asymptotic expansion at points of S . Assuming by contradiction that $S \neq \emptyset$ in our case, we can use this expansion to find contradict the quasiconformality of ψ . Let us show this.

Assume without loss of generality $0 \in S$ and $\psi(0) = 0$, and up to scaling that ψ is defined in B_1 . By Theorem A.1, and using notation (A.1), we find that

$$\partial_z \psi_1 = a_1 z^k + o(|z|^k) \quad \text{and} \quad \partial_z \psi_2 = a_2 z^k + o(|z|^k), \quad (3.19)$$

for some $k \geq 1$, $a_1, a_2 \in \mathbb{C}$ with $|a_1| + |a_2| \neq 0$. The map ψ is quasiregular, hence its gradient fulfills (2.15) for $\kappa \in (0, 1)$. Making use of (2.11), we can write at every point of B_1 :

$$|\partial_{\bar{z}}\psi|^2 = |\partial_{\bar{z}}\psi_1 + i\partial_{\bar{z}}\psi_2|^2 \stackrel{(2.11)}{=} |\partial_z\psi_1 - i\partial_z\psi_2|^2 \leq \kappa |\partial_z\psi|^2 \stackrel{(2.11)}{=} \kappa |\partial_z\psi_1 + i\partial_z\psi_2|^2.$$

Substituting (3.19) in this inequality we readily find that

$$|a_1 - ia_2|^2 \leq \kappa |a_1 + ia_2|^2. \quad (3.20)$$

Write, for a $R \in C^1(B_1, \mathbb{R}^2)$:

$$\psi(z) = (a_1 + ia_2) \frac{z^{k+1}}{k+1} + (\bar{a}_1 + i\bar{a}_2) \frac{\bar{z}^{k+1}}{k+1} + R(z). \quad (3.21)$$

We claim that

$$R(z) = o(|z|^{k+1}). \quad (3.22)$$

To see this, we can observe that

$$(a_1 + ia_2)z^k + o(|z|^k) \stackrel{(3.19)}{=} \partial_z\psi_1(z) + i\partial_z\psi_2(z) \stackrel{(2.12)}{=} \partial_z\psi(z) \stackrel{(3.21)}{=} (a_1 + ia_2)z^k + \partial_z R(z).$$

Thus we deduce $\partial_z R(z) = o(|z|^k)$. An analogous computation yields $\partial_{\bar{z}} R(z) = o(|z|^k)$. Due to (2.10), we then have $|DR|(z) = o(|z|^k)$. Since $\psi(0) = 0$, (3.21) yields $R(0) = 0$, and hence (3.22) holds. Now define

$$\tilde{\psi}(z) \doteq \frac{k+1}{a_1 + ia_2} \psi(z) \quad \text{and} \quad \lambda(z) \doteq \frac{\bar{a}_1 + i\bar{a}_2}{a_1 + ia_2} + \frac{(k+1)R(z)}{(a_1 + ia_2)\bar{z}^{k+1}}.$$

Notice that $a_1 + ia_2 \neq 0$ by (3.20) and the fact that $|a_1| + |a_2| \neq 0$. Moreover, $\tilde{\psi}$ is still quasiconformal, and, assuming $|z| \leq \rho$, we find $\delta > 0$ such that

$$|\lambda|(z) \leq 1 - \delta \quad (3.23)$$

by (3.20) and (3.22). Thus (3.21) can be rewritten as $\tilde{\psi}(z) = z^{k+1} + \lambda(z)\bar{z}^{k+1}$. Consider now the homotopy

$$H(t, z) \doteq z^{k+1} + t\lambda(z)\bar{z}^{k+1}, \quad \text{so that } H(0, z) = z^{k+1} \text{ and } H(1, z) = \tilde{\psi}(z).$$

Pick any regular value $y \in \tilde{\psi}(B_1)$ for $\tilde{\psi}$ with $0 < |y| \leq \frac{\delta}{2}\rho^{k+1}$. Such y exists since $\tilde{\psi}$ is a homeomorphism with $\tilde{\psi}(0) = 0$. Our choice of $|y|$ implies that $y \notin H(t, \partial B_\rho)$ for any $t \in [0, 1]$. Thus, the homotopy invariance of the Browder degree (see [17, Section 5] for reference) yields

$$\deg(\tilde{\psi}, y, B_\rho) = \deg(z^{k+1}, y, B_\rho) = k+1 > 1, \quad (3.24)$$

but, since $\tilde{\psi}$ is quasiconformal, $\deg(\tilde{\psi}, y, B_\rho)$ is either 1 or 0, depending on whether $\tilde{\psi}^{-1}(y) \in B_\rho$ or not. In either case, we find a contradiction with (3.24) and we conclude the proof.

Remark 3.1. An alternate approach to Step 3 that avoids using the precise asymptotic expansion of ψ is to use that S consists of isolated points (also a consequence of Theorem A.1). In particular, u is $C^{1,\alpha}$ away from a set of isolated points. It follows that u solves (2.2)-(2.3) since isolated points are removable for this system, and one can then conclude using [7, Theorem 4.4]. However, it is interesting that one can show that S is empty using the full strength of the expansion, so we have chosen to do so in the course of the proof.

4. AN ALGEBRAIC INEQUALITY

In the proof of Theorems 5.1-5.2, we need an inequality of algebraic nature which quantifies the ellipticity of the Hopf differential (3.10) when restricted to quasiconformal matrices. This regularizing property of the Hopf differential is also at the core of [13]. Although the proof is different and much shorter in this case, this section is analogous to [14, Section 6], and is inspired by the work of V. Šverák [21, Theorem 3] (see also [23, Theorem 6.5] for further applications of these ideas). The main difference between [14, Section 6] and the situation at hand is that in [14] we were interested in ellipticity properties of the inner variations for the area functional, while here we study similar properties of the Hopf differential, which arises as the matrix field associated to inner variations of the Dirichlet energy. We refer the reader to [11, Lemma 4.6]

and [15, Section 9.1] for analogous computations.

Before stating the inequality, we rewrite the Hopf differential using complex notation:

$$[H(X)]_{\mathcal{H}} = 0 \text{ and } [H(X)]_{\overline{\mathcal{H}}} = 2\overline{[X]_{\mathcal{H}}}[X]_{\overline{\mathcal{H}}}, \quad \text{for any } X \in \mathbb{R}^{2 \times 2}. \quad (4.1)$$

The main result of this section is following:

Lemma 4.1. *There exists $C > 0$ depending on $K > 0$ such that:*

$$|X|^2 \leq K \det X, |Y|^2 \leq K \det Y \implies |X - Y|^2 \leq C \det(X - Y) + C \frac{|H(X) - H(Y)|^2}{\max\{|X|, |Y|\}^2}. \quad (4.2)$$

Proof. The constant C appearing in the next computations will always depend solely on K and can change line-by-line. We start by noticing that, by symmetry, we can assume that $|Y| \leq |X|$ and, in addition, that

$$0 < |Y| \leq |X|. \quad (4.3)$$

Indeed if not $Y = 0$ and (4.2) is true simply by quasiconformality of X , up to taking $C \geq K$.

We start by reducing the proof of (4.2) to the proof of a similar inequality. To state it, we need to use the complex coordinates introduced in Section 2.4. For simplicity, let us denote

$$a \doteq [X]_{\mathcal{H}}, \quad a' \doteq [X]_{\overline{\mathcal{H}}}, \quad b \doteq [Y]_{\mathcal{H}}, \quad b' \doteq [Y]_{\overline{\mathcal{H}}}.$$

We claim that there exists $C = C(K) > 0$ and $\lambda = \lambda(K) \in [0, 1)$ such that

$$|a' - b'|^2 \leq \lambda |a - b|^2 + C \frac{|\bar{a}a' - \bar{b}b'|^2}{|a|^2}. \quad (4.4)$$

Let us first show how the claim implies the Lemma. Notice that

$$\max\{|X|^2, |Y|^2\} = |X|^2 \stackrel{(2.10)}{=} 2|a|^2 + 2|a'|^2 \stackrel{(2.14)}{\leq} C|a|^2. \quad (4.5)$$

We fix $T = T(K) > 0$ large enough so that

$$(1 - T) + \lambda(1 + T) < 0. \quad (4.6)$$

We can do so since $\lambda = \lambda(K)$ appearing in (4.4) belongs to $[0, 1)$. We can then compute:

$$\begin{aligned} 2^{-1}|X - Y|^2 &\stackrel{(2.10)}{=} |a - b|^2 + |a' - b'|^2 \\ &\stackrel{(2.10)}{=} (1 - T)|a - b|^2 + (1 + T)|a' - b'|^2 + T \det(X - Y) \\ &\stackrel{(4.4)}{\leq} (1 - T)|a - b|^2 + \lambda(1 + T)|a - b|^2 + T \det(X - Y) + (1 + T)C \frac{|\bar{a}a' - \bar{b}b'|^2}{|a|^2} \\ &\stackrel{(4.6)}{\leq} T \det(X - Y) + (1 + T)C \frac{|\bar{a}a' - \bar{b}b'|^2}{|a|^2} \stackrel{(4.1)-(4.5)}{\leq} T \det(X - Y) + C \frac{|H(X) - H(Y)|^2}{\max\{|X|, |Y|\}^2} \end{aligned}$$

This shows the validity of (4.2) assuming (4.4). Now, as a final step, we show (4.4). Using the fact that $Y \neq 0$, (4.3), we can estimate for an $\varepsilon = \varepsilon(K) > 0$ to be fixed later,

$$\begin{aligned} |a' - b'|^2 &= \left| \frac{a'\bar{a}}{\bar{a}} - \frac{b'\bar{b}}{\bar{b}} \right|^2 = \left| \frac{a'\bar{a} - b'\bar{b}}{\bar{a}} + \left(\frac{1}{\bar{a}} - \frac{1}{\bar{b}} \right) b'\bar{b} \right|^2 \\ &\leq \left(1 + \frac{1}{\varepsilon^2} \right) \left| \frac{a'\bar{a} - b'\bar{b}}{\bar{a}} \right|^2 + (1 + \varepsilon^2) |a - b|^2 \frac{|b'|^2}{|a|^2}. \end{aligned} \quad (4.7)$$

By (4.3) and (2.10), we have that $|b'|^2 + |b|^2 \leq |a'|^2 + |a|^2$. Using (2.14) in this inequality, we find

$$(1 + \kappa^{-1})|b'|^2 \leq (1 + \kappa)|a|^2 \implies |b'|^2 \leq \kappa|a|^2. \quad (4.8)$$

Let us remark once again that $\kappa = \kappa(K) \in (0, 1)$. We can then find $\varepsilon = \varepsilon(K) > 0$ such that $\lambda \doteq (1 + \varepsilon^2)\kappa < 1$. For any such $\varepsilon > 0$, we see that (4.7) implies (4.4) through (4.8) and we conclude the proof. $\square$

5. AUXILIARY RESULTS

Theorem 5.1. *Let $r > 2$ and let $v \in W^{1,2}(B_1, \mathbb{R}^n)$, $\psi \in W^{1,2}(B_1, \mathbb{R}^2)$, $f \in L^r(B_1, \mathbb{R}^n)$. Assume ψ is quasiconformal and that the maps are linked through (3.8), (3.11) and (3.12) for all $p > 2$, namely:*

$$\begin{cases} D\psi \in L_{\text{loc}}^p(B_1, \mathbb{R}^2) \Rightarrow \Delta v \in L_{\text{loc}}^\beta(B_1, \mathbb{R}^n), & \text{for } \beta = \beta(p) = \frac{r'p}{(r'-1)p+2}, \end{cases} \quad (5.1)$$

$$\begin{cases} H(D\psi) = -H(Dv) \text{ a.e. in } B_1, \end{cases} \quad (5.2)$$

$$\begin{cases} |Dv(y)| \leq C|D\psi(y)| \text{ a.e. in } B_1. \end{cases} \quad (5.3)$$

Then, there exists $\varepsilon > 0$ such that $v \in W_{\text{loc}}^{2,2+\varepsilon}(B_1, \mathbb{R}^n)$.

Proof. The proof is based on an iterative argument. Namely, starting with a given regularity $D\psi \in L_{\text{loc}}^{p_i}(B_1, \mathbb{R}^2)$ for some $p_i > 2$, we can employ (5.1) to deduce higher regularity of v . From (5.2), Lemma 4.1 and a few additional observations we can then infer $D\psi \in L_{\text{loc}}^{p_{i+1}}(B_1, \mathbb{R}^2)$ with $p_{i+1} > p_i$, compare (5.13). The goal is to show that, for sufficiently large i , $\beta(p_i) > 2$ and conclude the assertion through (5.1) and standard Calderón-Zygmund theory. The iterative argument is hidden in the definition of σ of (5.14), but it is the underlying idea of the proof. We start by describing one step of the iteration.

Assume $D\psi \in L_{\text{loc}}^p(B_1, \mathbb{R}^2)$ for some $p > 2$. The existence of a starting p for which this is true is guaranteed by Theorem 2.2(4). We notice preliminarily that

$$p > \frac{4}{2-r'} \implies \beta(p) > 2. \quad (5.4)$$

Thus we can, without loss of generality, assume that $2 < p < \frac{4}{2-r'}$.

Consider $\eta \in C_c^\infty(B_1)$ with $\eta \equiv 1$ on $B_{\rho'}$ and $\text{spt}(\eta) \subset B_\rho \subset B_1$. We employ (4.2) with $X = D\psi(x+h)$ and $Y = D\psi(x)$, where $|h| < (1-\rho)/2$. For any map f , we set $f_h(x) \doteq f(x+h) - f(x)$. Since the determinant is a Null Lagrangian (see [18, Theorem 2.3(ii)]):

$$0 = \int_{B_1} \det(D(\eta(x)\psi_h(x)))dx = \int_{B_1} \eta^2 \det(D\psi_h)dx + 2 \int_{B_1} \eta \langle \psi_h \otimes D\eta, \text{cof}^T(D\psi_h) \rangle dx. \quad (5.5)$$

Using (4.2) and routine estimates, see for instance [23, Proposition 6.6], this leads to

$$\int_{B_1} \eta^2 |D\psi_h|^2 dx \leq C \int_{B_1} |\eta| |\psi_h| |D\eta| |D\psi_h| dx + C \int_{B_1} \eta^2 \frac{|H(D\psi(x+h)) - H(D\psi(x))|^2}{\max\{|D\psi(x+h)|, |D\psi(x)|\}^2} dx. \quad (5.6)$$

In what follows, C is a constant which depends on p and on ρ and is independent of h , and will change line-by-line. We observe that every component of $H(X)$ is a quadratic form, compare (3.10). Thus:

$$\begin{aligned} |H(D\psi(x+h)) - H(D\psi(x))|^2 &\stackrel{(5.2)}{=} |H(Dv(x+h)) - H(Dv(x))|^2 \\ &\leq C \max\{|Dv|(x+h), |Dv|(x)\}^2 |Dv_h|^2(x) \\ &\stackrel{(5.3)}{\leq} C \max\{|D\psi|(x+h), |D\psi|(x)\}^2 |Dv_h|^2(x), \end{aligned}$$

where the first inequality is true precisely due to the fact that H is a quadratic form. This and Young's inequality allow us to estimate (5.6) as

$$\int_{B_1} \eta^2 |D\psi_h|^2 dx \leq C \int_{B_1} |D\eta|^2 |\psi_h|^2 dx + C \int_{B_1} \eta^2 |Dv_h|^2(x) dx. \quad (5.7)$$

As $\psi \in W^{1,2}(B_1, \mathbb{R}^2)$, we can bound the first addendum simply by:

$$\int_{B_1} \eta^2 |D\psi_h|^2 dx \leq C|h|^2 + C \int_{B_1} \eta^2 |Dv_h|^2(x) dx. \quad (5.8)$$

Estimating the second addendum requires more work. By (5.1), $v \in W_{\text{loc}}^{2,\beta}(B_1, \mathbb{R}^n)$. Set $f^h(x) \doteq \eta(x)|Dv_h|(x)$ and choose $\theta \in (0, 1)$ as

$$\theta \doteq \frac{4}{\beta} \left[1 - \frac{1}{\beta} \right]. \quad (5.9)$$

We have $2 = \theta\beta + (1 - \theta)\beta^*$ and, using Hölder's inequality, we get:

$$\int_{B_1} |f^h|^2(x) dx \leq \left(\int_{B_1} |f^h|^\beta(x) dx \right)^\theta \left(\int_{B_1} |f^h|^{\beta^*}(x) dx \right)^{1-\theta} \leq C \left(\int_{B_1} |f^h|^\beta(x) dx \right)^\theta, \quad (5.10)$$

where here

$$C = \sup_{h \in B_{\frac{1-\rho}{2}}} \left(\int_{B_1} |f^h|^{\beta^*}(x) dx \right)^{1-\theta} < +\infty.$$

As $Dv \in W_{\text{loc}}^{1,\beta}(B_1, \mathbb{R}^n)$, from (5.10) we can then estimate (5.8) as:

$$\int_{B_{\rho'}} |D\psi_h|^2 dx \leq \int_{B_1} \eta^2 |D\psi_h|^2 dx \leq C|h|^{\theta\beta}. \quad (5.11)$$

This means that $D\psi$ belongs to the local version of Besov space $B_{\infty}^{\frac{\theta\beta}{2}, 2}$, and we could already invoke embedding theorems concerning Besov spaces. Let us, instead, work with more classical fractional Sobolev spaces, which we denote by $H^t = W^{t,2}$. (5.11) implies that $D\psi \in H^t(B_{\rho'}, \mathbb{R}^2)$ for all $t < t_0 \doteq \frac{\beta\theta}{2}$. Indeed,

$$\begin{aligned} \int_{B_{\rho'} \times B_{\rho'}} \frac{|D\psi(x) - D\psi(y)|^2}{|x - y|^{2(1+t)}} dx dy &= \int_{B_{\rho'}} \int_{B_{\rho'}} \frac{|D\psi(x) - D\psi(y)|^2}{|x - y|^{2(1+t)}} dy dx \\ &= \int_{B_{\rho'}} \int_{B_{\frac{1-\rho}{2}}(x)} \frac{|D\psi(x) - D\psi(y)|^2}{|x - y|^{2(1+t)}} dy dx \\ &\quad + \int_{B_{\rho'}} \int_{B_{\rho'} \setminus B_{\frac{1-\rho}{2}}(x)} \frac{|D\psi(x) - D\psi(y)|^2}{|x - y|^{2(1+t)}} dy dx \\ &\leq \int_{B_{\rho'}} \int_{B_{\frac{1-\rho}{2}}} \frac{|D\psi(x) - D\psi(x+h)|^2}{|h|^{2(1+t)}} dh dx + C \|D\psi\|_2^2. \end{aligned}$$

The last integral is finite due to (5.11) and our choice of t_0 . Thus, $D\psi \in H^t(B_{\rho'}, \mathbb{R}^2)$, see [6, Section 2]. From the fractional Sobolev embeddings [6, Theorem 6.5] we find that then $D\psi \in L_{\text{loc}}^\tau(B_{\rho'}, \mathbb{R}^2)$ for $\tau = \frac{4}{2-2t} = \frac{2}{1-t}$. In other words, if $\tau_0 \doteq \frac{2}{1-t_0}$ then, due to the arbitrariness of $\rho > 0$,

$$D\psi \in L_{\text{loc}}^{\tau_0 - \delta}(B_1, \mathbb{R}^2), \quad \forall \delta > 0.$$

We can write more explicitly

$$\frac{1}{\tau_0} = \frac{1}{2} - \frac{t_0}{2} = \frac{1}{2} - \frac{\beta\theta}{4} \stackrel{(5.9)}{=} \frac{1}{2} - \left[1 - \frac{1}{\beta} \right] = \frac{1}{\beta} - \frac{1}{2} = \frac{1}{\beta^*}, \text{ i.e. } \tau_0 = \beta^* \stackrel{(5.1)}{=} \frac{2r'p}{4 + (r' - 2)p}. \quad (5.12)$$

By hand, we can check that:

$$\tau_0(p) \stackrel{(5.4)}{>} p, \quad \forall p \in \left(2, \frac{4}{2-r'} \right) \quad \text{and} \quad \tau_0(p) > \frac{4}{2-r'}, \quad \forall p \in \left(\frac{8}{4 - (r')^2}, \frac{4}{2-r'} \right). \quad (5.13)$$

Let now

$$\sigma \doteq \sup \{ s > 1 : D\psi \in L_{\text{loc}}^s(B_1, \mathbb{R}^2) \}. \quad (5.14)$$

As remarked above, $\sigma > 2$ due to Theorem 2.2(4). We claim

$$\sigma > \frac{8}{4 - (r')^2}. \quad (5.15)$$

If this is not the case, then

$$\sigma \leq \frac{8}{4 - (r')^2} = \frac{2}{2 + r'} \frac{4}{2 - r'} < \frac{4}{2 - r'}.$$

Thus one could repeat the reasoning above with $p < \sigma$ but very close to σ , and use the first inequality of (5.13) to get a contradiction with the fact that σ is a supremum. Since (5.15) is verified, we can repeat once again the reasoning above with any $p \in \left(\frac{8}{4 - (r')^2}, \frac{4}{2 - r'}\right)$ for which $D\psi \in L^p_{\text{loc}}(B_1, \mathbb{R}^2)$ to obtain that $\sigma > \frac{4}{2 - r'}$ by the second inequality of (5.13) and conclude using (5.4). $\square$

Corollary 5.2. *Under the same assumptions of Theorem 5.1, it holds $\psi \in W^{2,2+\varepsilon}_{\text{loc}}(B_1, \mathbb{R}^2)$ for some $\varepsilon > 0$.*

Proof. By Theorem 5.1 we have $v \in W^{2,2+\varepsilon}_{\text{loc}}(B_1, \mathbb{R}^n)$ for some $\varepsilon > 0$. We can then use (5.8) to estimate

$$\int_{B_1} \eta^2 |D\psi_h|^2 dx \stackrel{(5.8)}{\leq} C|h|^2 + C \int_{B_1} \eta^2 |Dv_h|^2(x) dx \leq C|h|^2,$$

from which we obtain $\psi \in W^{2,2}_{\text{loc}}(B_1, \mathbb{R}^2)$. To get the required higher integrability of $D^2\psi$, we employ Gehring's Lemma [9, Proposition 5.1] in a standard way, see [23, Proposition 6.6] or [14, Theorem 6.2]. $\square$

6. RIGIDITY FOR APPROXIMATE SOLUTIONS

In this section we show Theorem 1.3. Let us first introduce approximate solutions. Consider a rotation J of 90 degrees, namely

$$J \doteq \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix},$$

and define the inclusion set

$$K = \left\{ Y \in \mathbb{R}^{(2n) \times 2} : Y = \begin{pmatrix} X \\ D\mathcal{A}(X)J \end{pmatrix} \right\}. \quad (6.1)$$

Notice that $w = (u, v) : B_1 \rightarrow \mathbb{R}^n \times \mathbb{R}^n$ solves $Dw \in K$ a.e. if and only if $\text{div}(D\mathcal{A}(Du)) = 0$. This is the standard way of turning *exact solutions* to a PDE into *exact solutions* to a differential inclusion (and vice versa). As said, we will consider *approximate solutions*, i.e. we are interested in showing that given any sequence of equi-Lipschitz maps $(w_j)_j$, weakly-* converging to $w \in \text{Lip}(B_1, \mathbb{R}^{2n})$

$$\text{dist}(Dw_j, K) \rightarrow 0 \text{ in } L^1(B_1) \implies Dw_j \rightarrow Dw \text{ in } L^1(B_1, \mathbb{R}^{2n}). \quad (6.2)$$

Equivalent rephrasings of this property are more convenient, compare to the statement of Theorem 1.3. The equivalence of these two statements is given by the next:

Lemma 6.1. *Let K be defined in (6.1). Then, the following are equivalent:*

- (1) (6.2) holds;
- (2) for every sequence $(u_j)_j$ weakly-* converging to a Lipschitz map u with

$$\text{div}(D\mathcal{A}(Du_j)) = \text{div}(F_j), \text{ for } F_j \text{ equibounded in } L^\infty(B_1), F_j \rightarrow 0 \text{ in } L^1(B_1), \quad (6.3)$$

$(Du_j)_j$ converges strongly in L^1 towards Du ;

Proof. We start assuming (2). Consider $(w_j)_j$ as in (6.2), and define $w_j(x) = (u_j(x), v_j(x))$ with $u_j, v_j : B_1 \rightarrow \mathbb{R}^n$ equi-Lipschitz. Since $\text{dist}(Dw_j, K) \rightarrow 0$ strongly in $L^1(B_1)$, we can set $F_j \doteq D\mathcal{A}(Du_j) + Dv_j J$, which is equibounded in $L^\infty(B_1, \mathbb{R}^{n \times 2})$, and infer $F_j \rightarrow 0$ strongly in $L^1(B_1, \mathbb{R}^{n \times 2})$. Thus,

$$\text{div}(D\mathcal{A}(Du_j)) = \text{div}(D\mathcal{A}(Du_j) + Dv_j J) = \text{div}(F_j),$$

and hence by (6.3) $(Du_j)_j$ converges strongly in L^1 . As $Dv_j = -F_j J + D\mathcal{A}(Du_j)J$, a sum of two strongly convergent sequences, also $(Dv_j)_j$ strongly converges in L^1 , and hence $(Dw_j)_j$ does, as wanted. Conversely, assume that (1), and thus (6.2), is true. Consider $(u_j)_j$ as in (2), and define v_j through (6.3), namely

$$Dv_j(x) = D\mathcal{A}(Du_j)(x)J - F_j(x)J, \quad \text{for a.e. } x \text{ and for all } j.$$

Setting $w_j = (u_j, v_j)$, this map is easily seen to fulfill the assumptions of property (6.2), and hence converges strongly due to our assumption (1). It follows that (2) holds. $\square$

Remark 6.2. By combining Theorem 1.3 and Lemma 6.1, we obtain that every *gradient Young measure* compactly supported in the inclusion set K defined in (6.1) must be trivial. In particular, the set K does not contain any T_N configuration, the *laminates* used in [19, 22] to construct irregular critical points. For the sake of brevity we do not provide the details, but we refer the reader to [18, Section 3] or to [10, Section 4.1] for the theory of Young measures, and to [19, 22] for notions on laminates and T_N configurations.

6.1. Proof of Theorem 1.3. Consider, as in the statement, a sequence $(u_j)_j$ weakly-* converging to $u \in \text{Lip}(B_1, \mathbb{R}^n)$ and satisfying

$$\text{div}(D\mathcal{A}(Du_j)) = \text{div}(F_j). \quad (6.4)$$

It suffices to show that Du_j converges strongly in $L^1(B_r)$, for any fixed $r < 1$. Thus, consider r fixed. We start as in Step 1 of Theorem 1.2, i.e. by factorizing u_j as

$$u_j = v_j \circ \varphi_j. \quad (6.5)$$

Indeed we let $g_j \doteq \text{id}_2 + Du_j^T Du_j$ in B_1 and $g_j \equiv \text{id}_2$ in B_1^c . Since g_j are equibounded, uniformly positive-definite symmetric tensors, we can find a K -quasiconformal solution $\varphi_j \in W_{\text{loc}}^{1,2}(\mathbb{R}^2, \mathbb{R}^2)$ (for some K independent of j) fulfilling

$$g_j = \rho_j D\varphi_j^T D\varphi_j, \quad \text{a.e. in } \mathbb{R}^2, \quad \text{with } \rho_j \doteq \frac{\sqrt{\det(g_j)}}{\det(D\varphi_j)}. \quad (6.6)$$

Compare to (3.3) and the details of that section. It is convenient to choose φ_j to be the *normalized* solution to the Beltrami equation associated to g_j , see [2, Theorem 5.3.4], namely, using complex notation,

$$\varphi_j(0) = 0, \varphi_j(1) = 1, \text{ and } \varphi_j(\infty) = \infty \quad \forall j. \quad (6.7)$$

Through this choice, $(\varphi_j)_j$ admits at least one weak $W_{\text{loc}}^{1,2}(\mathbb{R}^2, \mathbb{R}^2)$ accumulation point φ , which is still K -quasiconformal and normalized [2, Theorem 3.9.4]. We assume without loss of generality that φ is the weak $W_{\text{loc}}^{1,2}(\mathbb{R}^2, \mathbb{R}^2)$ limit of $(\varphi_j)_j$. The same exact reasoning can be employed on $\psi_j \doteq \varphi_j^{-1}$, which are still K -quasiconformal by Theorem 2.2(1) and fulfill (6.7). Assume, again without loss of generality, that ψ is the only weak $W_{\text{loc}}^{1,2}(\mathbb{R}^2, \mathbb{R}^2)$ quasiconformal limit of this sequence. The local uniform convergence provided by Theorem 2.2(4) shows that $\psi = \varphi^{-1}$. Taking now any fixed $s < t \in (r, 1)$, we consider the sets $U_j = \varphi_j(B_1)$, $U_t = \varphi(B_t)$, and $U_s = \varphi(B_s)$, which are open since φ_j and φ are homeomorphisms. A simple contradiction argument shows that:

$$\overline{\varphi_j(B_r)} \subset U_s \subset \overline{U_s} \subset U_t \subset \overline{U_t} \subset U_j, \quad \text{for all } j \in \mathbb{N} \text{ sufficiently large.} \quad (6.8)$$

For simplicity, and without loss of generality, we will assume that (6.8) holds for all j .

We define $v_j \doteq u_j \circ \psi_j$. As in (3.12), we notice that:

$$|Dv_j(y)| \leq C|D\psi_j(y)| \quad \text{a.e. in } U. \quad (6.9)$$

Throughout, C is a constant that varies from line to line and which is independent of j . Since $(v_j)_j$ is also equibounded in $L^\infty(U_j, \mathbb{R}^2)$, (6.9) implies that $(v_j)_j$ is equibounded in $W^{1,2}(U_j, \mathbb{R}^2)$ and we can assume that it has a weak $W^{1,2}$ limit v on U_t . We will show that

$$v_j \text{ converges strongly in } W^{1,2}(U_s, \mathbb{R}^n) \text{ to } v. \quad (6.10)$$

We compute the equation solved by v_j , similarly to what we did in (3.4). For any $\eta \in C_c^\infty(U_j, \mathbb{R}^2)$:

$$\int_{U_j} \langle Dv_j, D\eta \rangle dy = \int_{B_1} \sqrt{\det(g_j)} \langle Du_j g_j^{-1}, D(\eta \circ \varphi_j) \rangle dx. \quad (6.11)$$

Now, combining (2.4) and (2.7), we obtain from (6.4) and (6.11) that:

$$\int_{U_j} \langle Dv_j, D\eta \rangle dy = \int_{B_1} \langle F_j, D(\eta \circ \varphi_j) \rangle dx. \quad (6.12)$$

After a change of variable we finally get:

$$\int_{U_j} \langle Dv_j, D\eta \rangle dy = \int_{B_1} \langle F_j, D\eta(\varphi_j) D\varphi_j \rangle dx = \int_{U_j} \langle F_j \circ \psi_j, D\eta \operatorname{cof}(D\psi_j) \rangle dy. \quad (6.13)$$

Set $G_j \doteq F_j \circ \psi_j \operatorname{cof}(D\psi_j)^T$. In the sense of distributions in U_j , v_j solves

$$\Delta v_j = \operatorname{div}(G_j). \quad (6.14)$$

Using the fact that ψ_j is K -quasiconformal, we can bound:

$$\int_{U_j} |G_j|^2 dx \leq C \int_{U_j} |F_j \circ \psi_j|^2 |D\psi_j|^2 dx \leq C \int_{U_j} |F_j \circ \psi_j|^2 \det(D\psi_j) dx = C \int_{B_1} |F_j|^2 dx.$$

Hence, $\lim_{j \rightarrow \infty} \|G_j\|_{L^2(U_j)} = 0$. From this, (6.8) and (6.14) we obtain that v is harmonic, and thus $v'_j \doteq v_j - v$ again solves (6.14) in U_t . Testing this PDE with $\eta^2 v'_j$, for any $\eta \in C_c^\infty(U_t)$, we get to:

$$\int_{U_t} |Dv'_j|^2 \eta^2 dx \leq C \left(\int_{U_t} |v'_j|^2 dx + \int_{U_j} |G_j|^2 dx \right), \quad (6.15)$$

which shows (6.10).

We now show that, up to a subsequence,

$$\varphi_j \text{ converges strongly and pointwise a.e. in } W^{1,2}(B_s, \mathbb{R}^2) \text{ to } \varphi. \quad (6.16)$$

Performing the same computations that brought us to (3.11), we infer

$$H(D\psi_j)(y) = -H(Dv_j)(y), \quad \text{for a.e. } y \in U_j. \quad (6.17)$$

Using (4.2)-(6.9)-(6.17) and the quadratic structure of $H(\cdot)$, we have, for a.e. $y \in U_t$ and for all $j, m \in \mathbb{N}$,

$$|D\psi_j(y) - D\psi_m(y)|^2 \leq C \det(D\psi_j(y) - D\psi_m(y)) + C |H(Dv_j(y)) - H(Dv_m(y))|$$

With computations analogous to the ones of (5.5), we obtain, for all $\eta \in C_c^\infty(U_t)$:

$$\int_{U_t} |D\psi_j(y) - D\psi_m(y)|^2 \eta^2 dy \leq C(\eta) \left(\int_{U_t} |\psi_j(y) - \psi_m(y)|^2 dy + \int_{U_t} |H(Dv_j(y)) - H(Dv_m(y))| \eta^2 dy \right).$$

Using (6.10), we infer that $(D\psi_j)_j$ is a Cauchy sequence in $L^2(U_s, \mathbb{R}^2)$. Hence, $(\psi_j)_j$ converges to ψ in $W^{1,2}(U_s, \mathbb{R}^2)$ and thus, up to a subsequence, $(D\psi_j)_j$ converges pointwise a.e. towards $D\psi$ on U_s . Now set

$$f_j = \frac{|D\psi_j|^2}{\det(D\psi_j)},$$

which is well-defined by Theorem 2.2(5) and an equibounded sequence in $L^\infty(U_s)$ since $(\psi_j)_j$ is quasiconformal with uniform quasiconformality constant. We then have:

$$\lim_j \int_{B_s} |D\varphi_j|^2 dx = \lim_j \int_{U_s} |(D\psi_j)^{-1}|^2 \det(D\psi_j) dy = \lim_j \int_{U_s} f_j dy = \int_{U_s} \frac{|D\psi|^2}{\det(D\psi)} dy = \int_{B_s} |D\varphi|^2 dx,$$

from the dominated convergence theorem applied to the sequence $(f_j)_j$. This shows (6.16).

We are finally ready to conclude. Let $\alpha \in (0, 1)$ be sufficiently small. Recalling (6.10) we have, for all j ,

$$\int_{B_r} |Dv_j(\varphi_j(x))|^\alpha dx = \int_{B_r} |Dv_j(y)|^\alpha \det(D\psi_j(y)) dy \leq C,$$

due to Hölder inequality and Theorem 2.2(4), provided α is sufficiently small. Furthermore, we estimate

$$\begin{aligned} \int_{B_r} |Dv_j(\varphi_j) - Dv(\varphi)|^\alpha dx &\leq \int_{B_r} |Dv_j(\varphi_j) - Dv(\varphi_j)|^\alpha dx + \int_{B_r} |Dv(\varphi_j) - Dv(\varphi)|^\alpha dx \\ &= \int_{\varphi_j(B_r)} |Dv_j - Dv|^\alpha \det(D\psi_j) dy + \int_{B_r} |Dv(\varphi_j) - Dv(\varphi)|^\alpha dx. \end{aligned}$$

The first integral converges to zero due to (6.8)-(6.10) and Hölder inequality, provided α is sufficiently small, and the second converges to zero since v is harmonic and $(\varphi_j)_j$ converges uniformly to φ on B_r . In particular, up to non-relabelled subsequences, $Dv_j(\varphi_j) \rightarrow Dv(\varphi)$ for a.e. $x \in B_r$. Through (6.5), we can write $Du_j(x) = Dv_j(\varphi_j(x))D\varphi_j(x)$, which converges a.e. in B_r to $Dv(\varphi(x))D\varphi(x) \in L^2(B_r, \mathbb{R}^{n \times 2})$. Thus the dominated convergence theorem implies that Du_j converges strongly in $L^2(B_r)$ to $Dv(\varphi(x))D\varphi(x)$. In particular, we must have that $Du(x) = Dv(\varphi(x))D\varphi(x)$ for a.e. $x \in B_r$. This reasoning shows that for every subsequence of (u_j) , there exists a further subsequence converging strongly in $W^{1,2}(B_r, \mathbb{R}^n)$ to u , from which we deduce the required strong convergence and hence the proof.

APPENDIX A. HARTMAN AND WINTNER'S ARGUMENT REVISITED

In the following, we will use the classical notations $o(|z - x|^t)$, $O(|z - x|^t)$ to denote C^0 functions defined on a neighborhood of the point x such that

$$\lim_{z \rightarrow x} \frac{o(|z - x|^t)}{|z - x|^t} = 0 \text{ and } |O(|z - x|^t)| \leq C|z - x|^t, \quad \text{for some } C > 0. \quad (\text{A.1})$$

Furthermore, for a map $w : B_1 \rightarrow \mathbb{R}^n$, we will denote by w_j its j -th component. Finally, for a continuous $h : \Gamma \rightarrow \mathbb{C}$, where Γ is the image of a regular parametrization $\Gamma : [0, 1] \rightarrow \mathbb{C}$, we set

$$\int_\Gamma h d\gamma = \int_0^1 h(\Gamma(t))\Gamma'(t)dt \quad \text{and} \quad \int_\Gamma |h|d|\gamma| = \int_\Gamma |h(\Gamma(t))|\Gamma'(t)|dt.$$

Theorem A.1. *Let $n \geq 1$, $r > 2$, $g \in L^r(B_1)$, $B_1 \subset \mathbb{R}^2$, and $w \in W_{\text{loc}}^{2,2+\gamma} \cap C^{1,\gamma}(B_1, \mathbb{R}^n)$ be a solution to*

$$|\Delta w|(z) \leq g(z)|Dw|(z) \quad \text{for a.e. } z \in B_1. \quad (\text{A.2})$$

Then either $Dw \equiv 0$ or the zeroes of Dw are isolated. In this case, if $x \in B_1$ is such that $Dw(x) = 0$, there exists $k = k(x) \in \mathbb{N}$ and $a_j \in \mathbb{C}$ such that, for all $j = 1, \dots, n$,

$$\partial_z w_j(z) = a_j(z - x)^k + o(|z - x|^k),$$

and $a_{j_0} \neq 0$ for some j_0 .

Theorem A.1 follows from adapting the arguments of [12]. In particular, the proofs of Propositions A.2-A.3 are essentially the same as the ones of [12, Theorem 1-Theorem 2] respectively.

Proposition A.2. *Let w fulfill the assumptions of Theorem A.1. Then, for all $k \geq 1$ the following holds:*

$$Dw(z) = o(|z - x|^{k-1}) \implies \lim_{z \rightarrow x} \frac{\partial_z w_j(z)}{(z - x)^k} \text{ exists for all } j. \quad (\text{A.3})$$

Proof. Without loss of generality, we assume that $x = 0$. Consider $0 < \rho < \frac{1}{2}$ to be chosen later. We take any $y \in B_\rho \setminus \{0\}$ and consider, for each j , $f_j(z) \doteq z^{-k}(z - y)^{-1}\partial_z w_j(z)$. Let $0 < \sigma < |y|$ be so small that $B_\sigma(y) \cap B_\sigma(0) = \emptyset$ and set $\Omega_\sigma \doteq B_\rho \setminus (B_\sigma(y) \cup B_\sigma(0))$. By the divergence theorem and Lemma 2.3, we see that

$$\int_{\partial\Omega_\sigma} f_j(z)d\gamma(z) = 2i \int_{\Omega_\sigma} \partial_{\bar{z}} f_j(z)dz = 2i \int_{\Omega_\sigma} z^{-k}(z - y)^{-1}\partial_{z\bar{z}} w_j(z)dz = \frac{i}{2} \int_{\Omega_\sigma} z^{-k}(z - y)^{-1}\Delta w_j(z)dz.$$

Using the structure of Ω_σ we get:

$$\int_{\partial B_\rho} f_j(z) d\gamma(z) - \int_{\partial B_\sigma(y)} f_j(z) d\gamma(z) - \int_{\partial B_\sigma} f_j(z) d\gamma(z) = \frac{i}{2} \int_{B_\rho \setminus (B_\sigma(y) \cup B_\sigma)} z^{-k} (z - y)^{-1} \Delta w_j(z) dz. \quad (\text{A.4})$$

We have:

$$\lim_{\sigma \rightarrow 0} \int_{\partial B_\sigma(y)} f_j(z) d\gamma(z) = 2\pi i y^{-k} \partial_z w_j(y).$$

Furthermore by assumption we know that $\partial_z w_j(z) = o(|z|^{k-1})$ and hence

$$\lim_{\sigma \rightarrow 0} \int_{\partial B_\sigma} f_j(z) d\gamma(z) = 0.$$

We thus let $\sigma \rightarrow 0$ in (A.4) to get:

$$\int_{\partial B_\rho} z^{-k} (z - y)^{-1} \partial_z w_j(z) d\gamma(z) - 2\pi i y^{-k} \partial_z w_j(y) = \frac{i}{2} \int_{B_\rho} z^{-k} (z - y)^{-1} \Delta w_j(z) dz. \quad (\text{A.5})$$

The last passage to the limit in the right-hand side is justified by the fact that, for all $y \in \mathbb{R}^2 \setminus \{0\}$,

$$z^{-k} (z - y)^{-1} \Delta w_j(z) \in L^1_{\text{loc}}(B_1).$$

To see this, notice that the only non-trivial pole is 0, but, since $Dw(z) = o(|z|^{k-1})$, we have for a.e. $z \in B_\rho$,

$$|z^{-k} (z - y)^{-1} \Delta w_j(z)| \stackrel{(\text{A.2})}{\leq} |z|^{-k} |z - y|^{-1} g(z) |Dw(z)| \leq C' |z|^{-1} |z - y|^{-1} g(z). \quad (\text{A.6})$$

The constant C' depends on k . Instead, $C > 0$ will denote a universal constant that can change line by line but that does not depend on k . Resuming our reasoning, the last function in (A.6) belongs to L^1_{loc} since $r > 2$. We need to estimate the right-hand side of (A.5). To do so, we first estimate (A.5) using (A.2):

$$|y^{-k} \partial_z w_j(y)| \leq \int_{\partial B_\rho} |z|^{-k} |z - y|^{-1} |\partial_z w_j(z)| |d\gamma|(z) + C \int_{B_\rho} |z|^{-k} |z - y|^{-1} g(z) |Dw(z)| dz.$$

Summing over j , we get to

$$|y|^{-k} |Dw(y)| \leq C \int_{\partial B_\rho} |z|^{-k} |z - y|^{-1} |Dw(z)| |d\gamma|(z) + C \int_{B_\rho} |z|^{-k} |z - y|^{-1} g(z) |Dw(z)| dz. \quad (\text{A.7})$$

For any $z_0 \in B_\rho \setminus \{0\}$, multiply by $g(y)|y - z_0|^{-1}$ this expression and integrate over B_ρ to get

$$\begin{aligned} \int_{B_\rho} |y|^{-k} |Dw(y)| g(y) |y - z_0|^{-1} dy &\leq C \int_{\partial B_\rho} |z|^{-k} \left[\int_{B_\rho} g(y) |y - z_0|^{-1} |z - y|^{-1} dy \right] |Dw(z)| |d\gamma|(z) \\ &\quad + C \int_{B_\rho} |z|^{-k} \left[\int_{B_\rho} g(y) |y - z_0|^{-1} |z - y|^{-1} dy \right] g(z) |Dw(z)| dz. \end{aligned} \quad (\text{A.8})$$

Notice that $|(z - y)(y - z_0)|^{-1} = |z - z_0|^{-1} |(z - y)^{-1} + (y - z_0)^{-1}|$, therefore:

$$\begin{aligned} \int_{B_\rho} g(y) |(z - y)(y - z_0)|^{-1} dy &= \int_{B_\rho} g(y) |z - z_0|^{-1} |(z - y)^{-1} + (y - z_0)^{-1}| dy \\ &\leq |z - z_0|^{-1} \int_{B_\rho} g(y) [|z - y|^{-1} + |y - z_0|^{-1}] dy. \end{aligned} \quad (\text{A.9})$$

We observe that, due to Hölder's inequality and the fact that $r > 2$, for all $b \in \overline{B_\rho}$

$$\int_{B_\rho} g(y) |y - b|^{-1} dy \leq \|g\|_{L^r} \left(\int_{B_\rho} |y - b|^{-r'} dy \right)^{\frac{1}{r'}} \leq C \left(\int_{B_{2\rho}(b)} |y - b|^{-r'} dy \right)^{\frac{1}{r'}} \leq C \rho^{\frac{2}{r'} - 1}.$$

Thus, we can continue (A.9) as:

$$\int_{B_\rho} g(y)|(z-y)(y-z_0)|^{-1}dy \leq C|z-z_0|^{-1}\rho^{\frac{2}{r}-1}. \quad (\text{A.10})$$

We can finally estimate (A.8):

$$\begin{aligned} \int_{B_\rho} g(y)|y-z_0|^{-1}|y|^{-k}|Dw(y)|dy &\leq C\rho^{\frac{2}{r}-1} \int_{\partial B_\rho} |z|^{-k}|z-z_0|^{-1}|Dw(z)|d|\gamma|(z) \\ &\quad + C\rho^{\frac{2}{r}-1} \int_{B_\rho} |z|^{-k}|z-z_0|^{-1}g(z)|Dw(z)|dz. \end{aligned} \quad (\text{A.11})$$

Observe that the second addendum is finite, due to the same bounds as in (A.6). Now fix ρ so small that

$$1 - C\rho^{\frac{2}{r}-1} > 0 \quad (\text{A.12})$$

and absorb the second addendum of the right-hand side of (A.11) in the left-hand side to get

$$\int_{B_\rho} g(z)|z-y|^{-1}|z|^{-k}|Dw(z)|dz \leq C\rho^{\frac{2}{r}-1} \int_{\partial B_\rho} |z|^{-k}|z-y|^{-1}|Dw(z)|d|\gamma|(z). \quad (\text{A.13})$$

Notice that we changed the name of the variable of integration from y to z in the first integral on the left-hand side, and we renamed z_0 as y . Going back to (A.7), this shows that $Dw = O(|y|^k)$. Hence, if $F_y(z) \doteq z^{-k}(z-y)^{-1}\Delta w_j(z)$, we can bound for some constants $c_1, c_2 > 0$:

$$|F_y(z)| \stackrel{(\text{A.2})}{\leq} c_1|z|^{-k}|z-y|^{-1}g(z)|Dw(z)| \leq c_2|z-y|^{-1}g(z),$$

Since $g \in L^r(B_1)$ and $r > 2$, $F_y(\cdot)$ has equibounded $L^p(B_\rho)$ norms for some $p > 1$. In addition, $\lim_{y \rightarrow 0} F_y(z) = F_0(z)$ for a.e. $z \in B_\rho$. Thus, we can pass to the limit in (A.5) to show the assertion, namely:

$$-2\pi i \lim_{y \rightarrow 0} y^{-k} \partial_z w_j(y) = \frac{i}{2} \int_{B_\rho} z^{-k-1} \Delta w_j(z) dz - \int_{\partial B_\rho} z^{-k-1} \partial_z w_j(z) d\gamma(z), \quad \forall j = 1, \dots, n.$$

□

Proposition A.3. *Let w fulfill the assumptions of Theorem A.1. If $Dw(z) = o(|z-x|^k)$ for some $x \in B_1$ and for all $k \in \mathbb{N}$, then $Dw \equiv 0$ in $B_r(x)$ for some $r > 0$.*

Proof. We again assume without loss of generality that $x = 0$, and moreover that $g \geq 1$ a.e in B_1 . Due to our hypotheses, we see that (A.13) now holds for $y = 0$, any (fixed) $\rho > 0$ which fulfills (A.12), and all k :

$$\int_{B_\rho} g(z)|z|^{-k-1}|Dw(z)|dz \leq C\rho^{\frac{2}{r}-1} \int_{\partial B_\rho} |z|^{-k-1}|Dw(z)|d|\gamma|(z) = C\rho^{\frac{2}{r}-2-k} \int_{\partial B_\rho} |Dw(z)|d|\gamma|(z). \quad (\text{A.14})$$

From the proof of Proposition A.2 we know that the constant C of (A.14) is independent of k . Using that $g \geq 1$ a.e. and dividing both sides of (A.14) by ρ^{-k-1} , we get:

$$\int_{B_\rho} \frac{\rho^{k+1}}{|z|^{k+1}} |Dw(z)|dz \leq C\rho^{\frac{2}{r}-1} \int_{\partial B_\rho} |Dw(z)|d|\gamma|(z). \quad (\text{A.15})$$

Since $|z| \leq \rho$ and the right-hand side does not depend on k , letting $k \rightarrow \infty$ shows $Dw \equiv 0$ on B_ρ by Fatou's Lemma, as wanted. □

Proof of Theorem A.1. Consider the closed set

$$Z = \{x \in B_1 : \forall r > 0 \text{ with } B_r(x) \subset B_1, B_r(x) \text{ contains at least two zeroes of } Dw\}.$$

Through Propositions A.2-A.3 we see that Z is open as well. Hence, either $Dw \equiv 0$ or the zeroes of Dw are isolated. In the latter case, the claimed asymptotic expansion of $\partial_z w_j$ is then a consequence of Proposition A.2, and the fact that $a_{j_0} \neq 0$ for some j_0 is a consequence of Proposition A.3. □

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