DYNAMICAL STABILITY FOR THE PERIODIC MODIFIED MULLINS-SEKERKA FLOW

DANIELE DE GENNARO AND ANNA KUBIN

ABSTRACT. We prove dynamical stability in arbitrary dimension for the modified Mullins–Sekerka flow, the gradient flow of the sharp-interface Ohta–Kawasaki energy on the flat torus. Specifically, we show that if an initial set has the same volume as a strictly stable critical set for the energy and is sufficiently close to it in $C^{3,\alpha}$, then the flow exists for all times and converges exponentially fast, in every C^k norm, to a translate of that critical set. The proof relies on a quantitative Alexandrov-type estimate for strictly stable critical sets of the energy.

1. INTRODUCTION

The Ohta–Kawasaki energy was introduced in [28] to describe microphase separation in diblock copolymer melts. Given a set $E \subset \mathbb{T}^N$, the sharp-interface Ohta–Kawasaki energy is defined as

$$\mathcal{J}(E) := P(E) + \gamma \int_{\mathbb{T}^N} \int_{\mathbb{T}^N} G(x,y)\big(u_E(x) - m\big)\big(u_E(y) - m\big)\,\mathrm{d}x\,\mathrm{d}y, \tag{1.1}$$

where $\gamma \geq 0$, $u_E := 2\chi_E - 1$, $m = \fint_{\mathbb{T}^N} u_E$ and G denotes the Green function of the Laplacian in $\mathbb{T}^N$. The two terms in (1.1) model competing effects: the perimeter penalizes the creation of interfaces, while the nonlocal term accounts for long-range repulsive interactions. Their competition gives rise to the formation of nontrivial periodic patterns. Critical points and local minimizers of the Ohta–Kawasaki functional have been extensively studied. In particular, explicit equilibria and their stability properties were analyzed in [29, 30], the relation between strict stability and local minimality was studied in [8, 2], and periodic locally minimizing critical points were constructed in [9]. We also refer to [3, 7, 14] for results on the structure and distribution of minimizers in the small-volume and droplet regimes. We emphasize, however, that there is still no characterization of critical points under general assumptions.

A natural evolution associated with (1.1) is the modified Mullins–Sekerka flow, also known as the nonlocal Mullins–Sekerka flow. More precisely, a smooth family of sets $(E_t)_{t\geq 0}$ is said to evolve by the modified Mullins–Sekerka flow if

$$\begin{cases} V_t = [\partial_{\nu_t} w_t] & \text{on } \partial E_t, \\ \Delta w_t = 0 & \text{in } \mathbb{T}^N \setminus \partial E_t, \\ w_t = H_{E_t} + 4\gamma v_{E_t} & \text{on } \partial E_t, \\ -\Delta v_{E_t} = u_{E_t} - m & \text{in } \mathbb{T}^N, \end{cases} \tag{1.2}$$

with v_E having zero average. Here V_t denotes the outer normal velocity of ∂E_t and $[\partial_{\nu_t} w_t]$ the jump of the normal derivative across the interface. When $\gamma = 0$, system (1.2) reduces to the classical Mullins–Sekerka flow [25]. This flow arises as the sharp-interface limit of the evolution associated with the diffuse Ohta–Kawasaki energy. The derivation of this flow as a sharp-interface limit of the Ohta–Kawasaki phase-field equation via a singular perturbation argument was first established in [27], and the rigorous convergence of the phase-field model to the sharp-interface evolution was proved in [23]. Short-time well-posedness for the classical Mullins–Sekerka problem was established in [6, 17], while the corresponding modified problem was treated in [13].

Like the classical Mullins–Sekerka flow, system (1.2) preserves the volume of the evolving sets and dissipates the energy $\mathcal{J}$. More precisely, every smooth solution of (1.2) satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}|E_t| = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}(E_t) = -\int_{\mathbb{T}^N} |\nabla w_t|^2 \,\mathrm{d}x. \tag{1.3}$$

Moreover, this system can be interpreted as the gradient flow of $\mathcal{J}$ with respect to a suitable metric on the space of interfaces. This gradient-flow interpretation has been developed in the smooth setting in [26, 23] and, more recently, at the level of weak solutions in [16].

In view of this gradient-flow structure, it is natural to expect that static stability of a critical point should imply dynamical stability. This question was addressed in [1] in dimensions two and three. More precisely, the authors proved that a solution of the modified Mullins–Sekerka flow starting sufficiently close to a strictly stable critical set exists for all time and converges exponentially fast, in a suitable Sobolev norm, to a translate of the reference set. Their argument is variational and is based on differentiating the dissipation (1.3), exploiting the positivity of the second variation and controlling the nonlinear remainder terms. We also refer to [12] for a detailed account of this result.

The arguments of [1], however, rely on interpolation estimates depending crucially on the dimension. A different approach was introduced in [11] for the discrete volume-preserving mean curvature flow on the flat torus, extending the ideas developed in [24] in the Euclidean setting. This approach was subsequently employed in [10] to prove, in arbitrary dimension, the dynamical stability of strictly stable sets for both the volume-preserving mean curvature flow and the surface diffusion flow. The key ingredient in these results is a quantitative Alexandrov-type estimate.

Quantitative versions of Alexandrov's theorem were first obtained in [22, 20, 24]. These results play an important role in the analysis of geometric evolutions, as they provide the correct estimates to show compactness for curvature-driven gradient flows, including the mean curvature flow, surface diffusion flow and Mullins–Sekerka flow. In the context of weak solutions to geometric flows, such estimates, coupled with a characterization of the critical points of the energy, can be used to prove the asymptotic convergence of weak solutions towards critical points under low regularity of the initial data. For more recent results regarding Alexandrov-type estimates, we refer to [19, 18, 21, 5].

The purpose of the present paper is to develop this strategy for the modified Mullins–Sekerka flow. In particular, we prove the stability of the flow in arbitrary dimension, thereby removing the dimensional restriction in [1]. Moreover, our result yields exponential convergence to a translate of the reference set in every C^k-norm, whereas the result of [1] establishes convergence only in a suitable Sobolev norm. Our main result is the following.

Theorem 1.1. *Let $E \subset \mathbb{T}^N$ be a strictly stable critical set for $\mathcal{J}$, and let $\alpha \in (0,1)$. There exist $\delta > 0$ and constants $c > 0$, $C_k > 0$, $k \in \mathbb{N}$, with the following property.*

If $E_0 \subset \mathbb{T}^N$ is a $C^{3,\alpha}$-set satisfying

$$|E_0| = |E|, \qquad \mathrm{dist}_{C^{3,\alpha}}(E_0, E) \le \delta,$$

then the modified Mullins–Sekerka flow starting from E_0 admits a unique smooth solution $(E_t)_{t\geq 0}$ for all positive times. Moreover, there exists $\sigma_\infty \in \mathbb{T}^N$ such that

$$\mathrm{dist}_{C^k}(E_t, E+\sigma_\infty) \leq C_k e^{-ct}, \qquad \textit{for all } k \in \mathbb{N}, t \geq 1.$$

We now outline the main steps of the proof. We first prove a local quantitative Alexandrov estimate for the Ohta–Kawasaki energy. More precisely, let E be a strictly stable critical set and suppose that F can be represented as a normal graph over E with height function f sufficiently small. We show the following inequality

$$\|f\|_{H^1(\partial E)} \leq C \left\| H_F + 4\gamma v_F - \overline{H_F + 4\gamma v_F} \right\|_{L^2(\partial F)}. \tag{1.4}$$

In particular, if $(E_t)_{t\in[0,T^*)}$ denotes the modified Mullins–Sekerka flow starting from an initial set E_0 sufficiently close to E, then this estimate, combined with the dissipation identity (1.3), yields exponential decay of the energy gap $\mathcal{J}(E_t) - \mathcal{J}(E)$ up to the extinction time T^*.

The proof of (1.4) follows the strategy of [11]. We compare two expansions of the first variation of $\mathcal{J}$ around the reference set E and then exploit the coercivity of the second variation $\delta^2\mathcal{J}(E)$. A related approach has recently been developed in [4] for the fractional perimeter energy, where the authors additionally require to prove a uniform coercivity estimate for $\delta^2\mathcal{J}(F)$ holding for sets F in a neighborhood of the stable set E.

Finally, we prove that the extinction time is infinite and that the flow converges to a translate of E in every C^k-norm, for all $k \in \mathbb{N}$. For this purpose, we establish short-time regularity estimates that are uniform for initial data with bounded $C^{3,\alpha}$-norm. This is done combining the approaches of [13] and [6]. After a suitable change of unknown that transforms the moving-boundary problem into a problem on a fixed domain, the nonlocal contribution appears as a lower-order inhomogeneous term in the elliptic problem associated with the Mullins–Sekerka system. The results of [6] can thus be adapted to obtain a uniform lower bound for the existence time, together with parabolic Schauder estimates. The crucial point is that the estimates depend only on an upper bound for the $C^{3,\alpha}$-norm of the initial graph.

These estimates together with the exponential decay of the energy allows us to restart the flow on successive intervals of uniform length and thereby obtain global existence. The energy decay, combined with interpolation, then yields the exponential convergence in every C^k-norm, up to translations. Finally, we can show that also the translations converge exponentially fast, yielding the convergence of the whole flow to a single translate of E.

2. Preliminaries

In this section we collect some preliminary results used throughout the paper.

We consider the sharp-interface Ohta–Kawasaki energy

$$\mathcal{J}(E) := P(E) + F(E) := P(E) + \gamma \int_{\mathbb{T}^N} \int_{\mathbb{T}^N} G(x,y)(u_E(x) - m)(u_E(y) - m)\,\mathrm{d}x\,\mathrm{d}y, \tag{2.1}$$

where $u_E = \chi_E - \chi_{E^c} = 2\chi_E - 1$ and $m = \fint_{\mathbb{T}^N} u_E$. We recall that G is the Green's function of the Laplacian in the flat torus, i.e., for every $x \in \mathbb{T}^N$, it is the unique solution to

$$\begin{cases} -\Delta_y G(x,y) = \delta_x(y) - 1, & y \in \mathbb{T}^N \\ \int_{\mathbb{T}^N} G(x,y)\,\mathrm{d}y = 0, & y \in \mathbb{T}^N. \end{cases} \tag{2.2}$$

Note that $G(x,y) = G(y,x)$. Indeed, by integration by parts, $0 = \int G(x,\cdot)\Delta G(y,\cdot) - G(y,\cdot)\Delta G(x,\cdot) = G(x,y) - G(y,x)$. Moreover, the translation invariance of the Laplacian implies that $G(x,y) = G(x-y,0)$.

The potential G admits the decomposition

$$G(x,y) = h(x-y) + r(x-y),$$

where h satisfies $h(\rho) = C_N \rho^{2-N}$ for $N \geq 3$ and $h(\rho) = -\log(|\rho|)/2\pi$ for $N = 2$ in a neighborhood of zero, while r is smooth and one-periodic. This can be proved by looking for a solution to (2.2) of the form $\eta h + r$, where η is a periodic cut-off function with compact support. In particular, h satisfies the following Hölder estimate

$$|h(\rho_1) - h(\rho_2)| \leq C \max\left\{\frac{1}{|\rho_1|^{N-2+\beta}}, \frac{1}{|\rho_2|^{N-2+\beta}}\right\} |\rho_1 - \rho_2|^\beta \tag{2.3}$$

for every $\rho_1,\ \rho_2 \in \mathbb{T}^N \setminus \{0\}$, where C is a positive constant independent of ρ_1 and ρ_2.

We define

$$v_E(x) := \int_{\mathbb{T}^N} G(x,y)(u_E(y) - m)\,\mathrm{d}y, \tag{2.4}$$

which satisfies

$$-\Delta v_E = u_E - m \quad \text{in} \quad \mathbb{T}^N, \quad \int_{\mathbb{T}^N} v_E\,\mathrm{d}x = 0.$$

Observe that, since G has zero mean, we have $v_E(x) = 2\int_E G(x,y)\,\mathrm{d}y$ and

$$\mathcal{J}(E) = P(E) + 4\gamma \int_E \int_E G(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

We recall the following estimates.

Lemma 2.1. *Let $E \subset \mathbb{T}^N$ be a C^2 set. Then, it holds*

$$\sup_{x \in \partial E} \int_{\partial E} |G(x,y)|\,\mathrm{d}\mathcal{H}^{N-1}_y < +\infty, \quad \|v_E\|_{C^{1,\beta}(\partial E)} < +\infty \text{ for all } \beta \in (0,1). \tag{2.5}$$

The first bound can be shown using the integrability of G over ∂E, the second one follows from elliptic regularity estimates since $u_E - m \in L^\infty(\mathbb{T}^N)$.

2.1. **Stable sets.** We recall the expressions for the first and second variations of $\mathcal{J}$ and the notion of stable sets used in the rest of the paper.

Let $E \subset \mathbb{T}^N$ be a set of class C^2 and $X : \mathbb{T}^N \to \mathbb{R}^N$ be a vector field of class C^2. Consider the associated flow $\Phi : \mathbb{T}^N \times (-1,1) \to \mathbb{T}^N$ defined by $\frac{\partial \Phi}{\partial t} = X(\Phi),\ \Phi(\cdot,0) = Id$. We define the *first and second variation of $\mathcal{J}$ at E* with respect to Φ, respectively, by

$$\delta\mathcal{J}(E)[X] := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathcal{J}(E(t)), \quad \delta^2\mathcal{J}(E)[X] := \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0} \mathcal{J}(E(t)),$$

where $E(t) = \Phi(\cdot,t)(E)$. Given a set E of class C^2, we use the notation ν_E, H_E, B_E to denote the outer normal, scalar mean curvature and second fundamental form of E, respectively. Also, given a vector field $X : \mathbb{T}^N \to \mathbb{R}^N$ we denote by X_τ its tangential component along ∂E, defined as $X - (X \cdot \nu_E)\nu_E$. Similarly, div_τ denotes the tangential divergence on ∂E.

Given a function $\varphi \in C^2_c(\mathbb{T}^N)$ we will use the notation

$$\delta\mathcal{J}(E)[\varphi] := \delta\mathcal{J}(E)[\varphi \nu_E],$$

where ν_E denotes an extension of ν_E to a neighborhood of ∂E.

Theorem 2.2. [2, Theorem 3.1] *If E and X are as above, we have*

$$\delta\mathcal{J}(E)[X] = \int_{\partial^* E} (H_E + 4\gamma v_E)\nu_E \cdot X \,\mathrm{d}\mathcal{H}^{N-1}, \tag{2.6}$$

and

$$\begin{aligned}\delta^2\mathcal{J}(E)[X] = &\int_{\partial E} \left(|D_\tau(X\cdot\nu_E)|^2 - |B_E|^2(X\cdot\nu_E)^2\right)\,\mathrm{d}\mathcal{H}^{N-1}\\ &+ 8\gamma\int_{\partial E}\int_{\partial E} G(x,y)(X\cdot\nu_E)(x)(X\cdot\nu_E)(y)\,\mathrm{d}\mathcal{H}^{N-1}(x)\,\mathrm{d}\mathcal{H}^{N-1}(y)\\ &+ 4\gamma\int_{\partial E}\partial_{\nu_E} v_E (X\cdot\nu_E)^2\,\mathrm{d}\mathcal{H}^{N-1} - \int_{\partial E}(4\gamma v_E + H_E)\mathrm{div}_\tau(X_\tau(X\cdot\nu_E))\,\mathrm{d}\mathcal{H}^{N-1}\\ &+ \int_{\partial E}(4\gamma v_E + H_E)(\mathrm{div}X)(X\cdot\nu_E)\,\mathrm{d}\mathcal{H}^{N-1}.\end{aligned}$$

Definition 2.3. Let E be a set of class C^1. Given a measurable function $f : \partial E \to \mathbb{R}$ such that $\|f\|_{L^\infty(\partial E)}$ is sufficiently small, we set

$$\partial E_f := \{x + f(x)\nu_E(x) : x \in \partial E\}, \tag{2.7}$$

and we call E_f the *normal deformation* of E with height function f.

We recall that, due to the translation invariance of the functional $\mathcal{J}$, the second variation degenerates along flows of the form $\Phi(x,t) = x + t\eta$, where $\eta \in \mathbb{R}^N$. In view of this, it is convenient to introduce the subspace $T(\partial E)$ of $\tilde{H}^1(\partial E) := \left\{\varphi \in H^1(\partial E) : \int_{\partial E}\varphi\,\mathrm{d}\mathcal{H}^{N-1} = 0\right\}$ generated by the functions $\nu_i = \nu\cdot e_i$ for $i = 1,\dots,N$. Its L^2-orthogonal subspace will be denoted by $T^\perp(\partial E)$ and is given by

$$T^\perp(\partial E) = \Big\{\varphi \in \tilde{H}^1(\partial E) : \int_{\partial E}\varphi\nu_i\,\mathrm{d}\mathcal{H}^{N-1} = 0,\ i = 1,\dots,N\Big\}.$$

We are now in a position to define the class of sets to which our main result applies.

Definition 2.4. We say that a smooth set $E \subset \mathbb{T}^N$ is *a stable set* (for the energy $\mathcal{J}$) if it is a critical set for $\mathcal{J}$, i.e.,

$$\delta\mathcal{J}(E)[\varphi] = 0, \qquad \forall\varphi \in \tilde{H}^1(\partial E),$$

and has strictly positive second variation, i.e.,

$$\delta^2\mathcal{J}(E)[\varphi] > 0, \qquad \forall\varphi \in T^\perp(\partial E)\setminus\{0\}.$$

We observe that, if E is a stable set and we consider volume-preserving variations, then the second variation of $\mathcal{J}$ at E reduces to (see [2])

$$\begin{aligned}\delta^2\mathcal{J}(E)[X] = &\int_{\partial E}\left(|D_\tau(X\cdot\nu_E)|^2 - |B_E|^2(X\cdot\nu_E)^2\right)\,\mathrm{d}\mathcal{H}^{N-1}\\ &+ 8\gamma\int_{\partial E}\int_{\partial E} G(x,y)(X\cdot\nu_E)(x)(X\cdot\nu_E)(y)\,\mathrm{d}\mathcal{H}^{N-1}(x)\,\mathrm{d}\mathcal{H}^{N-1}(y)\\ &+ 4\gamma\int_{\partial E}\partial_\nu v_E(X\cdot\nu_E)^2\,\mathrm{d}\mathcal{H}^{N-1}.\end{aligned}$$

We recall the following coercivity estimate.

Lemma 2.5. [2, Lemma 3.6] *Assume that E is a stable set for $\mathcal{J}$. Then there exists $m_0 > 0$ such that*

$$\delta^2\mathcal{J}(E)[\varphi] \geq m_0\|\varphi\|^2_{H^1(\partial E)} \qquad \text{for all } \varphi \in T^\perp(\partial E).$$

As a corollary, we deduce a coercivity estimate for functions which are approximately orthogonal to $T(\partial E)$. The proof follows the same argument as that of [11, Lemma 2.9]. It is based on the continuity in $H^1(\partial E) \times H^1(\partial E)$ of the bilinear form associated with $\delta^2\mathcal{J}(E)$ and it is thus omitted.

Lemma 2.6. *Assume that E is a stable set for $\mathcal{J}$. Then, there exist $\delta, C > 0$ such that if $\varphi \in H^1(\partial E) \cap C^1(\partial E)$ satisfies $\|\varphi\|_{C^1(\partial E)} \leq \delta$ and*

$$\Big|\int_{\partial E} \varphi \, \mathrm{d}\mathcal{H}^{N-1}\Big| + \Big|\int_{\partial E} \varphi\nu_E \, \mathrm{d}\mathcal{H}^{N-1}\Big| \leq \delta\|\varphi\|_{L^2(\partial E)},$$

then it holds

$$\delta^2\mathcal{J}(E)[\varphi] \geq \frac{m_0}{2}\|\varphi\|^2_{H^1(\partial E)}, \tag{2.8}$$

where m_0 is defined in Lemma 2.5.

3. Quantitative Alexandrov Theorem

The aim of this section is to prove the following stability estimate.

Theorem 3.1. *Let $E \subset \mathbb{T}^N$ be a stable set. There exist $\delta^* \in (0, 1/2)$ and $C > 0$, depending only on E, with the following property: for any $f \in C^1(\partial E) \cap H^2(\partial E)$ such that $\|f\|_{C^1(\partial E)} \leq \delta^*$ and satisfying*

$$\left|\int_{\partial E} f \, \mathrm{d}\mathcal{H}^{N-1}\right| \leq \delta^*\|f\|_{L^2(\partial E)}, \qquad \left|\int_{\partial E} f\nu_E \, \mathrm{d}\mathcal{H}^{N-1}\right| \leq \delta^*\|f\|_{L^2(\partial E)}, \tag{3.1}$$

we have

$$\|f\|_{H^1(\partial E)} \leq C\|H_{E_f}(\cdot + f(\cdot)\nu_E(\cdot)) - \widetilde{H}_{E_f} + 4\gamma(v_{E_f}(\cdot + f(\cdot)\nu_E(\cdot)) - \widetilde{v}_{E_f})\|_{L^2(\partial E)}, \tag{3.2}$$

where we have set

$$\widetilde{H}_{E_f} := \fint_{\partial E} H_{E_f}(x + f(x)\nu_E(x)) \, \mathrm{d}\mathcal{H}^{N-1}(x), \quad \widetilde{v}_{E_f} := \fint_{\partial E} v_{E_f}(x + f(x)\nu_E(x)) \, \mathrm{d}\mathcal{H}^{N-1}(x).$$

Remark 3.2. We remark that the conditions (3.1) are related to geometric conditions satisfied by the set E_f. The first assumption is simply the first order condition ensuring that $|E| = |E_f|$. In some special geometries, the second condition is satisfied if E_f has the same barycenter as E. We refer to [11] for further details.

Proof. For ease of notation, we will often omit the integration measure when it is clear from context.

As a first step we develop the first variation of the energy $\mathcal{J} = P + \gamma F$. We note that the perimeter term can be treated as in the proof of [11, Theorem 1.3], hence we only focus on the nonlocal term. Moreover, without loss of generality, we can assume that the right-hand side of (3.2) is less than or equal to one, otherwise the estimate is trivial.

Consider the map $(x, s) \mapsto x + s\nu_E(x)$ defined in a tubular neighborhood of ∂E and denote by $J(x, s)$ its Jacobian. Recall the Taylor expansion $J(x, s) = 1 + H_E(x)s + O(s^2)$. We also consider $\Phi_f(x) := x + f(x)\nu_E(x)$ for $x \in \partial E$ and let $\varphi \in H^1(\partial E)$. By the first variation formula (2.6) and the area formula, we have

$$\delta F(E_f)[\varphi] = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} F(E_{f+\varepsilon\varphi}) = 4\gamma \int_{\partial E} v_{E_f}(\Phi_f(x))\varphi(x) J(x, f(x))\,\mathrm{d}\mathcal{H}^{N-1}_x.$$

We now expand the difference $v_{E_f} \circ \Phi_f - v_E$. Recalling that $v_E(x) = 2\int_E G(x,y)\,\mathrm{d}y$ and using the normal parametrization coordinates, we infer

$$v_{E_f}(\Phi_f(x)) = v_E(\Phi_f(x)) + 2\int_{\partial E}\int_0^{f(y)} G(\Phi_f(x), y + s\nu_E(y))J(y,s)\,\mathrm{d}s\,\mathrm{d}\mathcal{H}^{N-1}_y. \tag{3.3}$$

Since $v_E \in C^{1,\beta}(\mathbb{T}^N)$, we deduce

$$v_E \circ \Phi_f - v_E = \partial_{\nu_E} v_E\, f + O(|f|^{1+\beta}).$$

We next consider the second term in (3.3). Using the decomposition $G = h + r$ and the Hölder estimate (2.3), for $|s| \le |f(y)|$ we have

$$\big|h\big(\Phi_f(x) - y - s\nu_E(y)\big) - h(x-y)\big| \le C\frac{|\Phi_f(x) - x - s\nu_E(y)|^\beta}{|x-y|^{N-2+\beta}} \le C\frac{\|f\|^\beta_{C^0(\partial E)}}{|x-y|^{N-2+\beta}},$$

where the denominators in (2.3) are estimated using that the projection is Lipschitz in a fixed tubular neighborhood of ∂E

$$|x - y| \le C\,|\Phi_f(x) - y - s\nu_E(y)|\,.$$

Moreover, r satisfies the above estimate with $\|f\|_{C^0(\partial E)}$ in place of $\|f\|^\beta_{C^0(\partial E)}$ as it is smooth. Since $N - 2 + \beta < N - 1$, the kernel $|x-y|^{-(N-2+\beta)}$ is integrable on ∂E, uniformly with respect to x. Therefore, using Cauchy-Schwartz, we deduce

$$\begin{aligned}
&\Big|\int_{\partial E}\int_{\partial E}\int_0^{f(y)} \big(G(\Phi_f(x), y + s\nu_E(y)) - G(x,y)\big)J(y,s)\varphi(x)\,\mathrm{d}s\,\mathrm{d}\mathcal{H}^{N-1}_y\,\mathrm{d}\mathcal{H}^{N-1}_x\Big| \\
&\le C\|f\|^\beta_{C^0(\partial E)}\int_{\partial E}\int_{\partial E}\frac{(|f(y)| + O(|f(y)|^2))|\varphi(x)|}{|x-y|^{N-2+\beta}}\,\mathrm{d}\mathcal{H}^{N-1}_y\,\mathrm{d}\mathcal{H}^{N-1}_x \\
&\le C\|f\|^\beta_{C^0(\partial E)}\left(\int_{\partial E}\int_{\partial E}\frac{|f(y)|^2}{|x-y|^{N-2+\beta}}\right)^{\frac12}\left(\int_{\partial E}\int_{\partial E}\frac{|\varphi(x)|^2}{|x-y|^{N-2+\beta}}\right)^{\frac12} \\
&\le C\|f\|^\beta_{C^0(\partial E)}\|f\|_{L^2(\partial E)}\|\varphi\|_{L^2(\partial E)}.
\end{aligned}$$

Finally, using the Taylor expansion $J(\cdot, f) = 1 + H_E f + O(f^2)$, we get

$$\begin{aligned}
&4\gamma\int_{\partial E}\big(v_{E_f}\circ\Phi_f - v_E\big)\varphi(1 + H_E f)\,\mathrm{d}\mathcal{H}^{N-1} \\
&= 4\gamma\int_{\partial E}\partial_{\nu_E} v_E\, f\varphi\,\mathrm{d}\mathcal{H}^{N-1} + 8\gamma\int_{\partial E}\int_{\partial E} G(x,y)f(y)\varphi(x)\,\mathrm{d}\mathcal{H}^{N-1}_y\,\mathrm{d}\mathcal{H}^{N-1}_x \\
&\quad + O\big(\|f\|^\beta_{C^0(\partial E)}\|f\|_{L^2(\partial E)}\|\varphi\|_{L^2(\partial E)}\big).
\end{aligned} \tag{3.4}$$

Let us introduce the notation $\mathcal{R}_f(\varphi)$ for all error terms satisfying

$$|\mathcal{R}_f(\varphi)| \le C(\delta^*)^\beta\|f\|_{L^2(\partial E)}\|\varphi\|_{L^2(\partial E)}.$$

Coupling (3.4) together with [11, equation (3.13)], we obtain

$$
\begin{aligned}
&\int_{\partial E}(H_{E_f}\circ\Phi_f - H_E + 4\gamma(v_{E_f}\circ\Phi_f - v_E))\varphi(1+H_Ef+R)\\
&= \int_{\partial E}(\nabla f\cdot\nabla\varphi - |B_E|^2 f\varphi) + 8\gamma\int_{\partial E}\int_{\partial E}G(x,y)f(y)\varphi(x) + 4\gamma\int_{\partial E}\partial_\nu v_E f\varphi\\
&\quad + \int_{\partial E} h\cdot\nabla\varphi + O(|\nabla f|^2)\varphi + \mathcal{R}_f(\varphi),
\end{aligned}
\tag{3.5}
$$

where, using the notation of [11], $R = O(f^2)+O(|\nabla f|^2)$ and $|h| \le C(|f|+|\nabla f|^2)|\nabla f|$. Choosing $\varphi = 1$ in (3.5) yields

$$
\int_{\partial E}(H_{E_f}\circ\Phi_f - H_E + 4\gamma(v_{E_f}\circ\Phi_f - v_E))(1+H_Ef+R) = \mathcal{R}_f(1) + \int_{\partial E}O(|f|)+O(|\nabla f|^2).
$$

Since E is a critical set for $\mathcal{J}$, $H_E + 4\gamma v_E$ is constant on ∂E, hence

$$
\begin{aligned}
&|(\widetilde{H}_{E_f} - H_E) + 4\gamma(\widetilde{v}_{E_f} - v_E)|\\
&\le \left|\fint((H_{E_f}\circ\Phi_f - \widetilde{H}_{E_f}) + 4\gamma(v_{E_f}\circ\Phi_f - \widetilde{v}_{E_f}))(H_Ef+R)\right|\\
&\quad + \left|\fint((\widetilde{H}_{E_f} - H_E) + 4\gamma(\widetilde{v}_{E_f} - v_E))(H_Ef+R)\right| + \mathcal{R}_f(1) + \int_{\partial E}O(|f|)+O(|\nabla f|^2)\\
&\le C\delta^*\|(H_{E_f}\circ\Phi_f - \widetilde{H}_{E_f}) + 4\gamma(v_{E_f}\circ\Phi_f - \widetilde{v}_{E_f})\|_{L^2(\partial E)}\\
&\quad + C\delta^*|(\widetilde{H}_{E_f} - H_E) + 4\gamma(\widetilde{v}_{E_f} - v_E)| + C\|f\|_{H^1(\partial E)},
\end{aligned}
\tag{3.6}
$$

where we used Hölder's inequality, and $C > 0$ denotes a constant only depending on E. In particular, the previous inequality entails

$$
|(\widetilde{H}_{E_f} - H_E) + 4\gamma(\widetilde{v}_{E_f} - v_E)| \le C\delta^*\|(H_{E_f}\circ\Phi_f - \widetilde{H}_{E_f}) + 4\gamma(v_{E_f}\circ\Phi_f - \widetilde{v}_{E_f})\|_{L^2(\partial E)} + C\|f\|_{H^1(\partial E)}.
$$

Testing now (3.5) with $\varphi = f$, and using Hölder's and Young's inequalities together with the above estimate, we get for every $\eta > 0$

$$
\begin{aligned}
&\int_{\partial E}(|\nabla f|^2 - |B_E|^2 f^2) + 8\gamma\int_{\partial E}\int_{\partial E}G(x,y)f(x)f(y) + 4\gamma\int_{\partial E}\partial_\nu v_E f^2\\
&= \int_{\partial E}((H_{E_f}\circ\Phi_f - H_E) + 4\gamma(v_{E_f}\circ\Phi_f - v_E))f(1+H_Ef+R) + C(\delta^*)^\beta\|f\|^2_{H^1(\partial E)}\\
&\le C\|(H_{E_f}\circ\Phi_f - \widetilde{H}_{E_f}) + 4\gamma(v_{E_f}\circ\Phi_f - \widetilde{v}_{E_f})\|_{L^2(\partial E)}\|f\|_{L^2(\partial E)} + C(\delta^*)^\beta\|f\|^2_{H^1(\partial E)}\\
&\quad + |(\widetilde{H}_{E_f} - H_E) + 4\gamma(\widetilde{v}_{E_f} - v_E)|\left|\int_{\partial E}f(1+H_Ef+R)\right|\\
&\le \frac{1}{\eta}C^2\|(H_{E_f}\circ\Phi_f - \widetilde{H}_{E_f}) + 4\gamma(v_{E_f}\circ\Phi_f - \widetilde{v}_{E_f})\|^2_{L^2(\partial E)} + \eta\|f\|^2_{H^1(\partial E)} + C(\delta^*)^\beta\|f\|^2_{H^1(\partial E)},
\end{aligned}
$$

where in the last inequality we also used the first assumption in (3.1). The conclusion follows by taking δ^*, η sufficiently small and using Lemma 2.6.

∎

4. Dynamical Stability

In this section we employ the stability estimate of Theorem 3.1 to prove the dynamical stability of stable sets under the modified Mullins-Sekerka flow.

We start by recalling that a smooth family of sets $(E(t))_t \subset \mathbb{T}^N$ with $E(0) = E_0 \subset \mathbb{T}^N$, defined on some (maximal) time interval $(0, T^*)$, is a solution of the *modified Mullins-Sekerka flow* starting from E_0 if it satisfies

$$\begin{cases} V(x,t) = [\partial_{\nu_{E(t)}} w(x,t)] & x \in \partial E(t),\, t \in (0,T^*), \\ \Delta w(x,t) = 0 & x \in \mathbb{T}^N \setminus \partial E(t),\, t \in (0,T^*), \\ w(x,t) = H_{E(t)}(x) + 4\gamma v(x,t) & x \in \partial E(t),\, t \in (0,T^*), \\ -\Delta v(x,t) = u_{E(t)}(x) - \fint_{\mathbb{T}^N} u_{E(t)}(y)\,\mathrm{d}y\,, & x \in \mathbb{T}^N,\, t \in (0,T^*), \end{cases} \tag{4.1}$$

with initial condition $E(0) = E_0$, where both w and v are subject to periodic boundary conditions and v has zero average. We recall that V stands for the outer normal velocity of the moving boundary ∂E, $u_{E(t)} := 2\chi_{E(t)} - 1$ and $[\partial_{\nu_{E(t)}} w(\cdot,t)]$ denotes the jump of the normal derivative of $w(\cdot,t)$ at $\partial E(t)$, i.e., $[\partial_{\nu_{E(t)}} w(\cdot,t)] := \partial_{\nu_{E(t)}} w(\cdot,t)^+ - \partial_{\nu_{E(t)}} w(\cdot,t)^-$, with $w(\cdot,t)^+$ and $w(\cdot,t)^-$ denoting the restrictions of $w(\cdot,t)$ to $\mathbb{T}^N \setminus \overline{E(t)}$ and $E(t)$, respectively.

4.1. **The nonlocal term is a perturbation.** In this section, we combine the results of [13] and [6] to obtain existence and Schauder estimates for solutions to (4.1) starting from a sufficiently smooth set.

Let $E \subset \mathbb{T}^N$ be a fixed smooth reference set, and set $\Gamma := \partial E$, $\Omega^- := E$ and $\Omega^+ := \mathbb{T}^N \setminus \overline{E}$. Let ν denote the outer unit normal to E. For a height function $h : \Gamma \times [0,T] \to \mathbb{R}$, sufficiently small in C^1, we set

$$\Phi_h(p,t) := p + h(p,t)\nu(p), \quad \Gamma_h(t) := \{p + h(p,t)\nu(p) : p \in \Gamma\}.$$

We denote by E_h the set enclosed by Γ_h, and write $\Omega_h^- := E_h$ and $\Omega_h^+ := \mathbb{T}^N \setminus \overline{E}_h$. We consider the modified Mullins–Sekerka system

$$\begin{cases} V_h = [\partial_{\nu_h} w_h] & \text{on } \Gamma_h, \\ \Delta w_h = 0 & \text{in } \mathbb{T}^N \setminus \Gamma_h, \\ w_h = H_{\Gamma_h} + 4\gamma v_h & \text{on } \Gamma_h, \\ -\Delta v_h = u_{E_h} - m_h & \text{in } \mathbb{T}^N, \\ \fint_{\mathbb{T}^N} v_h = 0, \end{cases} \tag{4.2}$$

where $u_{E_h} := 2\chi_{E_h} - 1$ and $m_h := \fint_{\mathbb{T}^N} u_{E_h}$.

Define $z_h := w_h - 4\gamma v_h$. Since $\Delta w_h = 0$ and $\Delta v_h = -u_{E_h} + m_h$, we have $\Delta z_h = 4\gamma(u_{E_h} - m_h)$ in $\mathbb{T}^N \setminus \Gamma_h$ and $z_h = H_{\Gamma_h}$ on Γ_h. Moreover, since $u_{E_h} - m_h \in L^\infty(\mathbb{T}^N)$, elliptic regularity yields $v_h \in W^{2,p}(\mathbb{T}^N)$ for every $1 \le p < \infty$ and thus $v_h \in C^{1,\beta}(\mathbb{T}^N)$ for every $\beta \in (0,1)$. In particular, we have $[\partial_{\nu_h} v_h] = 0$ on Γ_h and $[\partial_{\nu_h} z_h] = [\partial_{\nu_h} w_h]$. Thus (4.2) is equivalent to

$$\begin{cases} V_h = [\partial_{\nu_h} z_h] & \text{on } \Gamma_h, \\ \Delta z_h = 4\gamma(u_{E_h} - m_h) & \text{in } \mathbb{T}^N \setminus \Gamma_h, \\ z_h = H_{\Gamma_h} & \text{on } \Gamma_h. \end{cases} \tag{4.3}$$

Following [6, Section 2], we apply the Hanzawa transformation [15] to rewrite (4.3) on the fixed domains $\Omega^\pm$. Let d be the signed distance from Γ, and let π be the nearest-point projection onto Γ. Choose $\rho > 0$ such that the level sets of d are as regular as Γ in the tubular neighborhood $\{|d| < 2\rho\}$, and choose $\zeta \in C_c^\infty((-2,2))$ a standard cutoff function with $\zeta \equiv 1$ on $(-1,1)$ and $0 \le \zeta \le 1$. We define the Hanzawa transform $X_h : \mathbb{T}^N \times [0,T] \to \mathbb{T}^N$ by

$$
\begin{aligned}
X_h(x,t) &:= x + \zeta\Big(\frac{d(x)}{\rho}\Big) h(\pi(x),t)\nu(\pi(x)) \qquad && \text{if } |d(x)| < 2\rho,\\
X_h(x,t) &:= x && \text{if } |d(x)| \ge 2\rho.
\end{aligned}
$$

If $\sup_{t\in[0,T]} \|h(\cdot,t)\|_{C^1(\Gamma)}$ is sufficiently small, $X_h(\cdot,t)$ is a diffeomorphism on $\mathbb{T}^N$, uniformly in t, and it satisfies $X_h(\Gamma,t) = \Gamma_h(t)$ and $X_h(\Omega^\pm,t) = \Omega_h^\pm(t)$.

Let

$$
Y_h := X_h^{-1}, \quad Z_h(y,t) := z_h(X_h(y,t),t).
$$

Since X_h maps the fixed domains $\Omega^\pm$ onto the corresponding moving sets $\Omega_h^\pm$, it holds

$$
u_{E_h}(X_h(y,t)) = u_E(y). \tag{4.4}
$$

Define the transformed elliptic operator (see [6] for details)

$$
\mathcal{L}_h Z_h := (\Delta_x (Z_h \circ Y_h)) \circ X_h = a_h^{ij} D_{ij} Z_h + b_h^i D_i Z_h,
$$

where $a_h^{ij} = \big(\nabla_x Y_h^i \cdot \nabla_x Y_h^j\big) \circ X_h$ and $b_h^i = (\Delta_x Y_h^i) \circ X_h$. Thus, for $h(\cdot,t) \in C^{3,\alpha}(\Gamma)$ in a sufficiently small C^1-neighborhood of the origin, $\mathcal{L}_h$ is uniformly elliptic and its coefficients depend smoothly on h. Under the previous change of unknowns, system (4.3) transforms into

$$
\begin{cases}
\mathcal{L}_h Z_h = 4\gamma(u_E - m_h) & \text{in } \Omega^\pm,\\
Z_h = H_{\Gamma_h} \circ \Phi_h & \text{on } \Gamma,\\
V_h \circ \Phi_h = [\partial_{\nu_h} Z_h] \circ \Phi_h & \text{on } \Gamma.
\end{cases} \tag{4.5}
$$

Note that the right-hand side of the first equation in (4.5) is constant in each of the two domains

$$
4\gamma(u_E - m_h) = \begin{cases} 4\gamma(1 - m_h) & \text{in } \Omega^-,\\ 4\gamma(-1 - m_h) & \text{in } \Omega^+. \end{cases}
$$

Moreover, the normal velocity V_h can be related to the time-derivative of the height function h as follows

$$
V_h \circ \Phi_h = \partial_t h(\nu \cdot (\nu_h \circ \Phi_h)).
$$

For $\|h\|_{C^1}$ sufficiently small, the term $\nu \cdot (\nu_h \circ \Phi_h)$ in non-zero. Hence, the third equation in (4.5) can be rewritten as

$$
\partial_t h = (\nu \cdot (\nu_h \circ \Phi_h))^{-1} [\partial_{\nu_h}(Z \circ Y_h)] \circ \Phi_h.
$$

After localization via a cut-off function η and flattening near a point of Γ, the operator $\mathcal{L}_h$ can be written as a perturbation of the Laplacian. For a localized unknown $U = \eta Z_h$, one can write

$$
\Delta U = \operatorname{div} F_{1,h} + F_{2,h}
$$

in the fixed half-spaces, for some suitable functions $F_{1,h}$ and $F_{2,h}$. For more details on the localization and freezing of the coefficients we refer to the proof of [6, Lemma 4.2]. In particular, $F_{2,h}$ consists of coefficients and cutoff errors together, as well as the additional Ohta–Kawasaki term

$$
F_{2,h}^{\mathrm{OK}} := 4\gamma\eta(u_E - m_h).
$$

To check that this additional term can be treated with the fixed point argument of [6], it is enough to verify that $F_{2,h}^{OK}$ satisfies the required estimates in the parabolic Hölder spaces.

Specifically, we refer to Lemma 3.4 and Section 5 in [6]. Since $m_h = 2|E_h| - 1$, the map $h \mapsto m_h$ is smooth in a sufficiently small C^0-neighborhood of the origin. Consequently, on every bounded subset of $C^{\alpha/3}([0,T]; C^0(\Gamma))$, we have

$$\|m_h\|_{C^{\alpha/3}([0,T])} \leq C$$

and, for every $h, \tilde{h}$ in a bounded subset of $C^{\alpha/3}([0,T]; C^0(\Gamma))$,

$$\|m_h - m_{\widetilde{h}}\|_{C^{\alpha/3}([0,T])} \leq C\|h - \widetilde{h}\|_{C^{\alpha/3}([0,T];C^0(\Gamma))}.$$

Since u_E is constant in each $\Omega^{\pm}$ and the cutoff η is smooth, it follows that

$$\|F^{\mathrm{OK}}_{2,h}\|_{C^{\alpha,\alpha/3}(\Omega^{\pm}\times[0,T])} \leq C\gamma$$

and

$$\|F^{\mathrm{OK}}_{2,h} - F^{\mathrm{OK}}_{2,\widetilde{h}}\|_{C^{\alpha,\alpha/3}(\Omega^{\pm}\times[0,T])} \leq C\gamma\|h - \widetilde{h}\|_{C^{\alpha/3}([0,T];C^0(\Gamma))}.$$

Hence, the additional term satisfies the boundedness and local Lipschitz properties required in the fixed-point scheme of [6, Section 5]. Moreover, being lower-order, it does not affect the principal third-order operator, and the fixed-point argument of [6] applies accordingly, with constants that may depend on γ but depend on the initial height function only through a prescribed $C^{3,\alpha}$-bound. This is the observation made in [13], adapted here to the framework of [6].

Once short-time existence is shown in $C^{3,\alpha}$, additional regularity follows by the bootstrap argument of [6, Section 5]. Note that the term m_h is now a fixed constant $m_h = m_{h_0}$, so $F^{\mathrm{OK}}_{2,h}$ does not appear in the higher-regularity computations.

Proposition 4.1. *Let $E \subset \mathbb{T}^N$ be a smooth set and let $\gamma, \alpha \geq 0$. Then, there exists $M_0 = M_0(E) > 0$ such that for every $M \in (0, M_0)$ and $\ell \in \mathbb{N}$ there are constants*

$$T = T(E,\alpha,\gamma,M) > 0, \quad C = C(E,\alpha,\gamma,M) > 0, \quad C_\ell = C_\ell(E,\alpha,\gamma,M) > 0$$

with the following property. For every $h_0 \in C^{3,\alpha}(\Gamma)$ with $\|h_0\|_{C^{3,\alpha}(\Gamma)} \leq M$, the modified Mullins–Sekerka flow with initial datum Γ_{h_0} admits a unique classical solution

$$\Gamma(t) = \partial E_{h(\cdot,t)}, \qquad t \in [0,T],$$

such that

$$h \in C([0,T]; C^{3,\alpha}(\Gamma)) \cap C^1([0,T]; C^{\alpha}(\Gamma)), \tag{4.6}$$

$$\sup_{0\leq t\leq T} \|h(\cdot,t)\|_{C^{3,\alpha}(\Gamma)} + \sup_{0\leq t\leq T} \|\partial_t h(\cdot,t)\|_{C^{\alpha}(\Gamma)} \leq C. \tag{4.7}$$

Moreover, for every integer $\ell \geq 1$ it holds

$$\sup_{0<t\leq T} t^{\ell/3}\|h(\cdot,t)\|_{C^{3+\ell,\alpha}(\Gamma)} \leq C_\ell. \tag{4.8}$$

We emphasize that in the result above all the constants are uniform with respect to h_0 as long as $\|h_0\|_{C^{3,\alpha}(\Gamma)} \leq M$.

4.2. **Proof of the Main Result.** We start by recalling a technical lemma whose proof follows from [2, Lemma 3.8] combined with [2, Theorem 1.1].

Lemma 4.2. *Let $E \subset \mathbb{T}^N$ be a stable set and $p > N-1$. For every $\varepsilon > 0$, there exist constants $C > c > 0$, $\rho > 0$ such that the following holds. If $F \subset \mathbb{T}^N$ is a $W^{2,p}$ set satisfying*

$$|F| = |E|, \qquad \inf_{\sigma\in\mathbb{T}^N} \mathrm{dist}_{W^{2,p}}(E, F+\sigma) \le \rho,$$

then there exist $\sigma \in \mathbb{T}^N$ and $f \in W^{2,p}(\partial E)$ such that $F + \sigma = E_f$, $\|f\|_{W^{2,p}(\partial E)} \le C\rho$ and

$$\Big| \int_{\partial E} f \nu_E \,\mathrm{d}\mathcal{H}^{N-1} \Big| \le \varepsilon \|f\|_{L^2(\partial E)}, \qquad \Big| \int_{\partial E} f \,\mathrm{d}\mathcal{H}^{N-1} \Big| \le C\|f\|^2_{L^2(\partial E)}. \tag{4.9}$$

Moreover, it holds

$$c\|f\|^2_{L^1(\partial E)} \le \mathcal{J}(F) - \mathcal{J}(E) \le C\|f\|^2_{H^1(\partial E)}. \tag{4.10}$$

Remark 4.3. We recall that the function f is defined via a normal projection from the set F and by the implicit function theorem. In particular, following the proof of [2, Lemma 3.8], we deduce that, if F is of class C^k for $k \in \mathbb{N}$, then so is f, with bounds depending on the distance $\mathrm{dist}_{C^k}(F, E)$.

We are now able to show our main result concerning the dynamical stability of the flow.

Proof of Theorem 1.1. Let $M \in (0, M_0(E))$ with $M_0(E)$ being the constant given by Proposition 4.1. Throughout the proof, the constants $C > c > 0$ may depend on E, α and γ, but not on E_0, and may change from line to line.

By assumption there exists $h_0 \in C^{3,\alpha}(\partial E)$ such that $E_0 = E_{h_0}$ and $\|h_0\|_{C^{3,\alpha}(\partial E)} \le \delta$. For δ small, depending on M, Proposition 4.1 ensures that the modified Mullins–Sekerka flow E_t starting from E_0 exists for a positive time $T > 0$ and that there exists $h \in C([0,T]; C^{3,\alpha}(\Gamma)) \cap C^1([0,T]; C^{\alpha}(\Gamma))$ satisfying (4.6)-(4.8) such that $E_t = E_{h(\cdot,t)}$. Let us also remark that considering smaller δ does not decrease T.

Step 1: We start by proving that, as long as the flow exists, it satisfies

$$\mathcal{J}(E_t) - \mathcal{J}(E) \le Ce^{-ct}.$$

We recall the following identities holding along the smooth flow

$$\frac{\mathrm{d}}{\mathrm{d}t}|E_t| = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}(E_t) = -\int_{\mathbb{T}^N} |\nabla w_t|^2 \,\mathrm{d}x,$$

where w_t is the harmonic function satisfying $w_t = H_{E_t} + 4\gamma v_{E_t}$ on ∂E_t. By Poincare's inequality on $\mathbb{T}^N$ and trace estimates for w_t, we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}(E_t) \le -C\|H_{E_t} + 4\gamma v_{E_t} - \overline{H_{E_t} + 4\gamma v_{E_t}}\|^2_{L^2(\partial E_t)}. \tag{4.11}$$

Note that the constant C, which depends on E_t via the trace estimates for w_t, is uniform in a fixed C^2-neighborhood of E.

Let $\delta^* > 0$ be the constant given by Theorem 3.1, and let $\rho = \rho(\delta^*) > 0$ be the constant given by Lemma 4.2 for $\varepsilon \le \delta^*$. By the bound (4.7), we have

$$\|h(\cdot,t)\|_{C^0(\partial E)} \le \|h_0\|_{C^0(\partial E)} + t \sup_{0\le s\le T} \|\partial_t h(\cdot,s)\|_{C^0(\partial E)} \le \delta + Ct.$$

Hence, up to taking T and δ smaller, by interpolation, we obtain $\|h(\cdot,t)\|_{W^{2,p}(\partial E)} \le \min\{\delta^*, \rho\}$ for every $t \in (0,T)$. Lemma 4.2 then implies that there exist translations $\sigma_t \in \mathbb{T}^N$ and functions $f(\cdot,t) \in C^1(\partial E)$ such that $E_t + \sigma_t = E_{f(\cdot,t)}$. By (4.10), (4.11) and Theorem 3.1 we deduce

$$\begin{aligned}\mathcal{J}(E_t) - \mathcal{J}(E) &\leq C\|f(\cdot,t)\|^2_{H^1(\partial E)} \leq C\|H_{E_t} + 4\gamma v_{E_t} - \overline{H_{E_t} + 4\gamma v_{E_t}}\|^2_{L^2(\partial E_t)} \\ &\leq -C\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{J}(E_t) - \mathcal{J}(E)).\end{aligned}$$

Then Gronwall's inequality implies

$$\mathcal{J}(E_t) - \mathcal{J}(E) \leq (\mathcal{J}(E_0) - \mathcal{J}(E))e^{-Ct}.$$

Moreover, using the bounds of (4.10), we deduce

$$\|f(\cdot,t)\|^2_{L^1(\partial E)} \leq C\delta^2 e^{-Ct}. \tag{4.12}$$

Step 2: We adapt the restarting argument in [10, Theorem 0.1, Step 2]. By the uniform short-time existence result of Proposition 4.1, the functions $h(\cdot,t)$ are uniformly bounded in C^k for every $t \in [T/2, T)$ and $k \in \mathbb{N}$. Combining this bound with the exponential decay (4.12) and interpolation, we also deduce that $\|f(\cdot,t)\|_{C^k(\partial E)} \leq C_k\delta^{1/2}$ for every $k \in \mathbb{N}$ and $t \in (T/2, T)$, for some $C_k > 0$.

Up to taking δ smaller, we can thus restart the flow at time $T/2$ from $E_{f(\cdot,T/2)}$ and extend it up to the time $3T/2$. By translation invariance and uniqueness of strong solutions, we have extended the original flow E_t up to the time $3T/2$. We can then iterate this procedure to deduce the global existence of the flow.

Applying the same interpolation argument with arbitrary k, we obtain the exponential convergence in C^k to E of the translated sets $E_t + \sigma_t$.

Step 3: We consider the translations σ_t defined by Lemma 4.2. By compactness we can find $\tau \in \mathbb{T}^N$ and a sequence $t_n \to +\infty$ such that $\sigma_{t_n} \to \tau$ as $n \to +\infty$. In particular, it holds $E_{t_n} \to E - \tau$ in C^k, for every $k \in \mathbb{N}$. For F, $G \subset \mathbb{T}^N$, consider the dissipation

$$\mathcal{D}(F,G) := \int_{F\triangle G} \mathrm{dist}_{\partial G}(x)\,\mathrm{d}x = \int_F \mathrm{sd}_G\,\mathrm{d}x - \int_G \mathrm{sd}_G\,\mathrm{d}x,$$

where $\mathrm{sd}_G = \mathrm{dist}_G - \mathrm{dist}_{G^c}$ denotes the signed distance function to a set G. Following the computations in Step 3 of the proof of [1, Theorem 3.4], we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{D}(E_t, E-\tau) = -\int_{\mathbb{T}^N} \nabla\omega \cdot \nabla w_t\,\mathrm{d}x, \tag{4.13}$$

where ω denotes the harmonic extension of $\mathrm{sd}_{E-\tau}$ to $\mathbb{T}^N \setminus \partial E_t$. Note that, by elliptic estimates, $\|\nabla\omega\|_{L^2(\mathbb{T}^N)} \leq C\|\mathrm{sd}_{E-\tau}\|_{C^1(\partial E_t)} \leq C$. The constant C is uniform along the flow by the uniform C^2-bounds on the evolving sets E_t previously obtained. By the exponential decay (4.12) and elliptic estimates we deduce $\|\nabla w_t\|_{L^2} \leq Ce^{-Ct}$, in particular

$$\Big|\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{D}(E_t, E-\tau)\Big| \leq Ce^{-Ct}.$$

Hence, $\mathcal{D}(E_t, E-\tau)$ admits a limit as $t \to +\infty$. Since $E_{t_n} \to E-\tau$ we deduce that $\mathcal{D}(E_t, E-\tau) \to 0$ as $t \to +\infty$ and that the whole flow $E_t + \tau$ converges towards E. This concludes the proof as the exponential convergence follows from Step 2. $\square$

Acknowledgements. The authors thank Vesa Julin and Massimiliano Morini for helpful discussions and suggestions. D. De Gennaro was partially funded by the European Union: the European Research Council (ERC), through StG "ANGEVA", project number: 101076411. Views and opinions expressed are however those of the authors only and do not necessarily reflect those of the European Union or the European Research Council. Neither the European Union nor the granting authority can be held responsible for them. D. De Gennaro was supported by the Italian Ministry of University and Research (MUR) through the FIS 2 project SiGmA: "Singularities in Geometric Analysis: Minimal Surfaces and Mean Curvature Flows", project code FIS-2023-02962 (CUP G53C25000120001). A. Kubin research has been supported by the Austrian Science Fund (FWF) through grants 10.55776/F65, 10.55776/P35359, 10.55776/Y1292. Part of this contribution was completed while D. De Gennaro was visiting A. Kubin at the Technische Universität Wien.

AI usage statement

All the proofs contained in this manuscript were written by the authors, who take full responsibility for the correctness of the statements. This article does not contain any mathematical content generated by AI. AI tools were used solely for language editing, in particular to improve the fluency and clarity of the exposition, without contributing to the mathematical content.

References

[1] E. Acerbi, N. Fusco, V. Julin, and M. Morini. Nonlinear stability results for the modified Mullins–Sekerka and the surface diffusion flow. *J. Differential Geom.*, 113(1):1–53, 2019.

[2] E. Acerbi, N. Fusco, and M. Morini. Minimality via second variation for a nonlocal isoperimetric problem. *Comm. Math. Phys.*, 322(2):515–557, 2013.

[3] S. Alama, L. Bronsard, R. Choksi, and I. Topaloglu. Droplet phase in a nonlocal isoperimetric problem under confinement. *Commun. Pure Appl. Anal.*, 19(1):175–202, 2020.

[4] G. Alberti, G. Cozzi, A. Massaccesi, and J. Mirmina. Stability of the ball in isoperimetric inequalities between two fractional perimeters. *arXiv preprint arXiv:2605.07543*, 2026.

[5] V. Arya, D. De Gennaro, and A. Kubin. The asymptotic of the Mullins–Sekerka and the area-preserving curvature flow in the planar flat torus. *J. Differential Equations*, 451:Paper No. 113755, 25, 2026.

[6] X. Chen, J. Hong, and F. Yi. Existence uniqueness and regularity of classical solutions of the Mullins–Sekerka problem. *Communications in Partial Differential Equations*, 21(11-12):1705–1727, 1996.

[7] R. Choksi and M. A. Peletier. Small volume fraction limit of the diblock copolymer problem: I. Sharp-interface functional. *SIAM J. Math. Anal.*, 42(3):1334–1370, 2010.

[8] R. Choksi and P. Sternberg. On the first and second variations of a nonlocal isoperimetric problem. *Journal fur die Reine und Angewandte Mathematik*, 611:75, 2007.

[9] R. Cristoferi. On periodic critical points and local minimizers of the Ohta–Kawasaki functional. *Nonlinear Anal.*, 168:81–109, 2018.

[10] D. De Gennaro, A. Diana, A. Kubin, and A. Kubin. Stability of the surface diffusion flow and volume-preserving mean curvature flow in the flat torus. *Mathematische Annalen*, 390(3):4429–4461, 2024.

[11] D. De Gennaro and A. Kubin. Long time behaviour of the discrete volume preserving mean curvature flow in the flat torus. *Calc. Var. Partial Differential Equations*, 62(3):Paper No. 103, 39, 2023.

[12] S. Della Corte, A. Diana, and C. Mantegazza. Global existence and stability for the modified Mullins–Sekerka and surface diffusion flow. *Mathematics in Engineering*, 4(6):1–104, 2022.

[13] J. Escher and Y. Nishiura. Smooth unique solutions for a modified Mullins-Sekerka model arising in diblock copolymer melts. *Hokkaido Math. J.*, 31(1):137–149, 2002.

[14] D. Goldman, C. B. Muratov, and S. Serfaty. The Γ-limit of the two-dimensional Ohta–Kawasaki energy. I. Droplet density. *Arch. Ration. Mech. Anal.*, 210(2):581–613, 2013.

[15] E. Hanzawa. Classical solutions of the Stefan problem. *Tohoku Mathematical Journal, Second Series*, 33(3):297–335, 1981.

[16] S. Hensel and K. Stinson. Weak solutions of Mullins–Sekerka flow as a Hilbert space gradient flow. *Arch. Ration. Mech. Anal.*, 248(1):8, 2024.
[17] E. Joachim and S. Gieri. A center manifold analysis for the Mullins–Sekerka model. *Journal of Differential Equations*, 143(2):267–292, 1998.
[18] V. Julin, M. Morini, F. Oronzio, and E. Spadaro. A sharp quantitative Alexandrov inequality and applications to volume preserving geometric flows in 3d. *Archive for Rational Mechanics and Analysis*, 249(6):78, 2025.
[19] V. Julin, M. Morini, M. Ponsiglione, and E. Spadaro. The asymptotics of the area-preserving mean curvature and the Mullins–Sekerka flow in two dimensions. *Math. Ann.*, 2022.
[20] V. Julin and J. Niinikoski. Quantitative Alexandrov theorem and asymptotic behavior of the volume preserving mean curvature flow. *Analysis & PDE*, 16(3):679–710, 2023.
[21] E. Kim and D. Kwon. Area-preserving anisotropic mean curvature flow in two dimensions. *Calculus of Variations and Partial Differential Equations*, 64(1):27, 2025.
[22] B. Krummel and F. Maggi. Isoperimetry with upper mean curvature bounds and sharp stability estimates. *Calc. Var. Partial Differential Equations*, 56(2):Paper No. 53, 43, 2017.
[23] N. Q. Le. On the convergence of the Ohta–Kawasaki equation to motion by nonlocal Mullins–Sekerka law. *SIAM J. Math. Anal.*, 42(4):1602–1638, 2010.
[24] M. Morini, M. Ponsiglione, and E. Spadaro. Long time behavior of discrete volume preserving mean curvature flows. *J. Reine Angew. Math.*, 784:27–51, 2022.
[25] W. W. Mullins and R. F. Sekerka. Morphological stability of a particle growing by diffusion or heat flow. In *Fundamental contributions to the continuum theory of evolving phase interfaces in solids*, pages 75–81. Springer, Berlin, 1999.
[26] B. Niethammer and F. Otto. Ostwald ripening: The screening length revisited. *Calc. Var. Partial Differential Equations*, 13(1):33–68, 2001.
[27] Y. Nishiura and I. Ohnishi. Some mathematical aspects of the micro-phase separation in diblock copolymers. *Physica D: Nonlinear Phenomena*, 84(1-2):31–39, 1995.
[28] T. Ohta and K. Kawasaki. Equilibrium morphology of block copolymer melts. *Macromolecules*, 19(10):2621–2632, 1986.
[29] X. Ren and J. Wei. Existence and stability of spherically layered solutions of the diblock copolymer equation. *SIAM J. Appl. Math.*, 66(3):1080–1099, 2006.
[30] X. Ren and J. Wei. Many droplet pattern in the cylindrical phase of diblock copolymer morphology. *Rev. Math. Phys.*, 19(8):879–921, 2007.

(Daniele De Gennaro) Dipartimento di Matematica, Università degli Studi di Milano, Via Saldini 50, I-20133 Milano, Italy
Email address, D. De Gennaro: daniele.degennaro@unimi.it

(Anna Kubin) Institute of Analysis and Scientific Computing, Technische Universität Wien, Wiedner Haupstrasse 8-10, 1040 Vienna, Austria
Email address, Anna Kubin: anna.kubin@asc.tuwien.ac.at