CONVEXITY FROM EXPANDING GRADIENT PERTURBATIONS OF THE IDENTITY

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Abstract. Let $\Omega \subset \mathbb{R}^d$ be open. We consider functions $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ whose gradient perturbations of the identity $x \mapsto x + t\nabla u(x)$ are expanding for almost every $t > 0$, in the sense that they push the Lebesgue measure on Ω forward to a measure dominated by Lebesgue measure. We prove that $\nabla u \in \mathrm{BV}_{\mathrm{loc}}(\Omega; \mathbb{R}^d)$, that the distributional Hessian D^2u is a positive-semidefinite matrix-valued measure satisfying $|D^2u| \leq \Delta u$, and consequently that u has a locally convex representative. This settles a conjecture of Bianchini and Talamini (arXiv:2603.18819, 2026), which originates from a question of Liu and Pego (*Pure Appl. Anal.*, 2025).

1. Introduction and main results

Consider a body $\Omega \subset \mathbb{R}^d$ that breaks, up to a null set, into countably many pieces, each of which moves with its own constant velocity. Such *rigidly breaking flows* were studied by Liu and Pego in [LP25]. If the velocity field is the gradient of a convex potential u, then the maps $x \mapsto x + t\nabla u(x)$ do not decrease pairwise distances [LP25, Lemma 1.2], so the pieces never collide. Liu and Pego asked whether, conversely, collision avoidance forces the potential to be convex under mild regularity assumptions [LP25, Remark 1.6]. In this note, we answer this question under the weaker requirement that the maps $x \mapsto x + t\nabla u(x)$ do not compress Lebesgue measure.

Let $\Omega \subset \mathbb{R}^d$ be open. Following [BT26, Definitions 2.1 and 2.2], we use the following terminology. A measurable map $T \colon \Omega \to \mathbb{R}^d$ is *measure preserving on a measurable set* $F \subset \Omega$ if, for every measurable set $D \subset \Omega$, the image $T(F \cap D)$ is measurable and

$$T_\#\big(\mathcal{L}^d \llcorner (F \cap D)\big) = \mathcal{L}^d \llcorner T(F \cap D). \tag{1.1}$$

After fixing a measurable representative of ∇u, we say that $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ satisfies the *measure-preserving condition (MPC)* if there exists a fixed full-measure set $F \subset \Omega$ such that

$$T_t := \mathrm{Id} + t\nabla u \quad \text{is measure preserving on } F \text{ for almost every } t > 0. \tag{1.2}$$

This condition is unchanged if F is replaced by a measurable subset of F that still has full measure in Ω.

A measurable map $T \colon \Omega \to \mathbb{R}^d$ is called *expanding* if

$$T_\#(\mathcal{L}^d \llcorner \Omega) \leq \mathcal{L}^d \qquad \text{as measures on } \mathbb{R}^d; \tag{1.3}$$

see [BT26, Definition 3.1]. If T is measure preserving on a full-measure set F, then

$$T_\#(\mathcal{L}^d \llcorner \Omega) = T_\#(\mathcal{L}^d \llcorner F) = \mathcal{L}^d \llcorner T(F) \leq \mathcal{L}^d,$$

so T is expanding (see also [BT26, Remark 3.2]). Moreover, if $D \subset \Omega$ is measurable and T is expanding, then

$$T_\#(\mathcal{L}^d \llcorner D) \leq T_\#(\mathcal{L}^d \llcorner \Omega) \leq \mathcal{L}^d. \tag{1.4}$$

Unlike pointwise injectivity and the MPC on a prescribed set, the expansion property (1.3) depends only on the almost-everywhere equivalence class of T.[1]

The motivating example for these notions is given by the *rigidly breaking potential flows* studied in [LP25]. Suppose that $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ is affine on each member of a countable family of pairwise

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[1] Condition (1.3) retains only a measure-theoretic trace of the pointwise expansion property of monotone maps: if $A \colon \Omega \to \mathbb{R}^d$ is monotone, then $|x - y + t(A(x) - A(y))| \geq |x - y|$ for all $t > 0$, and the (sub)gradient of a convex function is monotone [Min64]. For the fine properties of monotone maps and the structure of their distributional derivatives, see [AA99].

disjoint open sets $\{A_i\}_{i\in\mathbb{N}}$ whose union $A := \bigcup_{i\in\mathbb{N}} A_i$ has full measure in Ω. Let p_i be the gradient of the affine function on A_i, and choose the representative $\nabla u(x) = p_i$ for every $x \in A_i$. The sets A_i represent rigid pieces moving with velocities p_i. The motion is collision-free at time $t > 0$ when $T_t|_A$ is injective.

At every collision-free time, T_t is measure preserving on A. Indeed, for every measurable $D \subset \Omega$, injectivity makes the sets $(A_i \cap D) + tp_i$ pairwise disjoint, and hence

$$\begin{aligned}(T_t)_\#\big(\mathcal{L}^d \llcorner (A\cap D)\big) &= \sum_{i\in\mathbb{N}} \mathcal{L}^d \llcorner \big((A_i \cap D) + tp_i\big)\\ &= \mathcal{L}^d \llcorner T_t(A\cap D).\end{aligned} \tag{1.5}$$

Consequently, collision avoidance for almost every $t > 0$ implies the measure-preserving condition (1.2), with $F = A$.

If u is convex on a convex open set, then the map $x \mapsto x + t\nabla u(x)$, formed with the classical gradient, is injective on the differentiability set of u for every $t > 0$ (see [LP25, Lemma 1.2]). A converse, for rigidly breaking flows, is the following conjecture.

Conjecture 1.1. Let $\Omega \subset \mathbb{R}^d$ be open, and let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be affine on each member of a countable family of pairwise disjoint open sets $\{A_i\}_{i\in\mathbb{N}}$ whose union A has full measure in Ω, and choose the representative of ∇u that equals the gradient of the affine function on each A_i. If $(\mathrm{Id} + t\nabla u)|_A$ is injective for almost every $t > 0$, then u has a locally convex representative in Ω.

For Lipschitz u, this was formulated in [BT26, Conjecture 1.1], following the question raised by Liu and Pego in [LP25, Remark 1.6]. We have stated a stronger $W^{1,1}_{\mathrm{loc}}$ version. For a continuous potential on a bounded convex domain, locally affine on the maximal open full-measure set, and under injectivity at every sufficiently small positive time, Liu and Pego (see [LP25, Theorem 1.4]) proved the implication in each of three cases: $d = 1$; finitely many affine components; or $u \in C^1$.

Since collision avoidance implies the measure-preserving condition, Bianchini and Talamini in [BT26, Section 2] considered the following more general question.[2]

Conjecture 1.2. Let $\Omega \subset \mathbb{R}^d$ be open and let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ satisfy the measure-preserving condition (1.2). Then u has a locally convex representative in Ω.

In [BT26, Theorem 2.3], they proved the implication under the additional assumption

$$\nabla u \in \mathrm{BV}_{\mathrm{loc}}(\Omega;\mathbb{R}^d); \tag{1.6}$$

and then deduced the corresponding statement for rigidly breaking flows, i.e., Conjecture 1.1 for Lipschitz potentials with $\nabla u \in \mathrm{BV}_{\mathrm{loc}}$ (see [BT26, Theorem 1.2]).

In this note, we prove that (1.6) follows already from the expansion property (1.3) of T_t at almost every time $t > 0$, and then adapt the argument of [BT26, Theorem 2.3] to obtain convexity. The novelty lies in the first implication.

Theorem 1.3 (Regularity and convexity of expanding gradient maps)**.** *Let $\Omega \subset \mathbb{R}^d$ be open and let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$. Assume that*

$$T_t := \mathrm{Id} + t\nabla u \quad \textit{is expanding for almost every } t > 0, \tag{1.7}$$

i.e., T_t satisfies (1.3) for almost every $t > 0$. Then Δu is a non-negative locally finite Radon measure on Ω, i.e., $\Delta u \in \mathcal{M}^+_{\mathrm{loc}}(\Omega)$, and $\nabla u \in \mathrm{BV}_{\mathrm{loc}}(\Omega;\mathbb{R}^d)$. Moreover, the distributional Hessian $D^2u = D(\nabla u)$ is a positive-semidefinite matrix-valued Radon measure. Consequently, u has a locally convex representative (i.e., a representative that is convex on every ball $B \Subset \Omega$) and

$$|D^2 u| \le \Delta u \qquad \textit{as Radon measures in } \Omega, \tag{1.8}$$

where $|D^2u|$ denotes the total variation of D^2u with respect to the Frobenius norm. If, in addition, there exist a time $t_0 > 0$ and a set $F \subset \Omega$ of full measure such that the map T_{t_0} is measure preserving on F in the sense of (1.1) (in particular, if u satisfies the measure-preserving condition (1.2)), then D^2u is singular with respect to $\mathcal{L}^d$.

Corollary 1.4. *Conjectures 1.1–1.2 are true.*

[2] In [BT26] the question is posed for $u \in W^{1,1}(\Omega)$.

Theorem 1.3 shows that the expansion property (1.7) forces both the regularity $\nabla u \in \mathrm{BV}_{\mathrm{loc}}(\Omega;\mathbb{R}^d)$ and the convexity of u. In forthcoming work, we will address the corresponding stability question, namely whether a quantitative failure of (1.7) controls the distance of u from the convex functions. The proof of Theorem 1.3 is given in Section 3.

Remark 1.5 (Sharpness). The regularity in Theorem 1.3 cannot be improved to $\nabla u \in W^{1,1}_{\mathrm{loc}}$, and (1.8) is sharp. For $u(x) = |x_1|$, the map T_t translates the two half-spaces $\{\pm x_1 > 0\}$ by $\pm t e_1$, hence is expanding for every $t > 0$, and $D^2 u = 2\, e_1 \otimes e_1\, \mathcal{H}^{d-1} \llcorner \{x_1 = 0\}$, so that $|D^2 u| = \Delta u$.

Remark 1.6 (On the set of times). By Lemma 2.1 below, hypothesis (1.7) is equivalent to the expansion property (1.3) for T_t at *every* $t > 0$, and it may be weakened to the expansion property for all t in a dense subset of $(0,\infty)$. For $d \geq 2$, density cannot be dropped: if $G \subset (0,\infty)$ misses an interval (a,b), then there exists a non-convex quadratic u on $\mathbb{R}^d$, depending on x_1 and x_2 only, such that T_t is expanding for every $t \in G$. Indeed, an invertible linear map L is expanding if and only if $|\det L| \geq 1$, because $L_{\#}(\mathcal{L}^d \llcorner \Omega) = |\det L|^{-1}\, \mathcal{L}^d \llcorner L(\Omega)$. Choose $\tau \in (a,b)$ and $0 < q < \min\{\tau, \tau - a, b - \tau\}$, and let $u(x) \coloneqq \frac{1}{2}(-x_1^2/\tau + x_2^2/q)$. Then

$$\det(\mathbb{I} + tD^2 u) = \Big(1 - \frac{t}{\tau}\Big)\Big(1 + \frac{t}{q}\Big) = 1 + \frac{t}{\tau q}(\tau - q - t),$$

and $|\det(\mathbb{I} + tD^2u)| < 1$ exactly when $\tau - q < t < \frac{1}{2}\big(\tau - q + \sqrt{(\tau-q)^2 + 8\tau q}\big)$. The right endpoint lies in $(\tau, \tau + q)$, since $(\tau+q)^2 < \tau^2 + 6\tau q + q^2 < (\tau + 3q)^2$; hence all the times at which T_t is not expanding lie in (a,b), and T_t is expanding for every $t \in G$. Taking $G = (0,a) \cup (b,\infty)$ shows that, for $d \geq 2$, neither $0 \in \overline{G}$ nor the unboundedness of G, nor both together, can replace density. In dimension $d = 1$, on the other hand, it suffices that $0 \in \overline{G}$, because the first-variation argument of [BT26, Proposition 3.3] uses the expansion property only along a sequence $t_j \downarrow 0$ and gives $u'' \geq 0$; and this cannot be dropped either, since for $a \coloneqq \inf G > 0$ the concave function $u(x) \coloneqq -x^2/a$ satisfies $|1 - 2t/a| \geq 1$ for every $t \in G$.

With Theorem 1.3, we can settle Conjectures 1.1–1.2.

Proof of Corollary 1.4. If u satisfies the measure-preserving condition (1.2), then, by the computation following (1.3), T_t is expanding at almost every $t > 0$, so Theorem 1.3 gives a locally convex representative of u. This proves Conjecture 1.2.

Under the assumptions of Conjecture 1.1, injectivity of $T_t|_A$ at almost every $t > 0$ gives the measure-preserving condition (1.2) with $F = A$ by (1.5), and Conjecture 1.2 applies. $\square$

Theorem 1.3 can be complemented by a quantitative estimate. For a ball $B \Subset \Omega$, let

$$\mathcal{E}(u;B) \coloneqq \inf_{a \in \mathbb{R}^d} \fint_B |\nabla u - a| \,\mathrm{d}x \tag{1.9}$$

be the L^1 *gradient excess* of u on B;[3] it measures the oscillation of ∇u on B, and it plays a central role in the proof of Theorem 1.3. For convex potentials, a class which by Theorem 1.3 contains every potential satisfying (1.7), the excess is controlled by the Laplacian mass at the same scale, with a sharp constant. For $k \geq 1$, we write $\omega_k \coloneqq \mathcal{L}^k(B_1(0))$ for the measure of the unit ball in $\mathbb{R}^k$, and $\omega_0 \coloneqq 1$.

Proposition 1.7 (Sharp excess bound for convex potentials)**.** *Let $\Omega \subset \mathbb{R}^d$ be open and let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be such that $\nabla u \in \mathrm{BV}_{\mathrm{loc}}(\Omega;\mathbb{R}^d)$ and $D^2 u \geq 0$.*[4] *Then, for every ball $B_r(x_0) \Subset \Omega$,*

$$\mathcal{E}(u;B_r(x_0)) \leq \fint_{B_r(x_0)} \Big|\nabla u - \fint_{B_r(x_0)} \nabla u \,\mathrm{d}y\Big| \,\mathrm{d}x \leq \frac{1}{2\omega_{d-1}} \frac{\Delta u(B_r(x_0))}{r^{d-1}}. \tag{1.10}$$

Equality holds in both inequalities for $u(x) = |x_1|$ on every ball centered on $\{x_1 = 0\}$.

[3] The infimum in (1.9) is attained, since the objective is continuous in a and coercive. For fixed radius, the excess is measurable as a function of the center, because the infimum may be taken over $a \in \mathbb{Q}^d$ and each resulting average is a measurable function of the center.

[4] Equivalently, u has a locally convex representative: a distribution whose second derivative is a positive-semidefinite matrix-valued measure coincides, on every convex open subset, with a convex function [Dud77]; the converse is classical.

The proof of Proposition 1.7 is given in Section 4: since $D^2u \geq 0$, the directional derivatives $\nabla u \cdot \theta$ are BV functions whose total variation is controlled by Δu, and a sharp L^1 Poincaré inequality on balls [Cia89, Theorem 1], averaged over $\theta \in \mathbb{S}^{d-1}$, gives the estimate.

1.1. **Outline of the proof.** The case $d = 1$ is elementary: by [BT26, Proposition 3.3] (see also Lemma 2.2), $u'' = \Delta u$ is a non-negative measure, so u' has a non-decreasing $\mathrm{BV}_{\mathrm{loc}}$ representative and u has a locally convex representative. For $d \geq 2$, the proof of Theorem 1.3 proceeds as follows. The starting point is that Δu is a non-negative Radon measure (Lemma 2.2, which is [BT26, Proposition 3.3]). This alone does not give $\nabla u \in \mathrm{BV}_{\mathrm{loc}}$ (see the examples recalled after Lemma 2.2), but the missing information comes from the expansion property. To make this precise, we measure the oscillation of ∇u on a ball B by the L^1 gradient excess $\mathcal{E}(u; B)$ introduced in (1.9). Roughly speaking, $\nabla u \in \mathrm{BV}_{\mathrm{loc}}$ if and only if $\mathcal{E}(u; B_r(x))$ is of order r on average, uniformly as $r \downarrow 0$ (cf. the nonlocal characterizations of the total variation in [Dáv02; Pon04]). The first stage is an excess-decay estimate, from which such a bound follows by iteration: for every fixed $\theta \in (0, 1)$ and $\eta > 0$,

$$\mathcal{E}(u; B_{\theta r}(x_0)) \leq \eta\, \mathcal{E}(u; B_r(x_0)) + C(d, \theta, \eta)\, \frac{\Delta u(B_r(x_0))}{r^{d-1}} \tag{1.11}$$

(Proposition 2.5). In turn, the proof of (1.11) is based on two ingredients: (1) a harmonic-approximation estimate (Lemma 2.3), saying that a function with small Laplacian mass is close, at the level of gradients, to a harmonic function: on every ball there is a harmonic h with

$$\int_{B_r(x_0)} |\nabla u - \nabla h| \,\mathrm{d}x \leq r\, \Delta u(B_r(x_0)); \tag{1.12}$$

and (2) a rigidity statement (Lemma 2.4), saying that a C^2 superharmonic function h for which $\mathrm{Id} + s\nabla h$ is expanding for arbitrarily small $s > 0$ must be affine. With these two facts, (1.11) follows by contradiction: if it failed, rescaling by the excess on the larger ball would produce a sequence of potentials with unit excess and vanishing Laplacian mass; by (1.12) and compactness, the sequence converges to a harmonic function h with positive excess, and the expansion property, which holds at every positive time by Lemma 2.1 and is invariant under the rescaling, passes to h at every $s > 0$. By Lemma 2.4, h is affine, which is incompatible with its positive excess.

To pass from excess decay to a BV bound, we integrate (1.11) over all centers x in a compact set K and iterate it along the radii $r_n = \theta^n R$. In this way we obtain a bound of order r for $\int_K \mathcal{E}(u; B_r(x)) \,\mathrm{d}x$ at every scale $0 < r < R$. Since $\int D\rho_\varepsilon = 0$ for a standard mollifier ρ_ε, this bounds $D(\nabla u * \rho_\varepsilon)$ in $L^1(K)$ uniformly in ε, and $\nabla u \in \mathrm{BV}_{\mathrm{loc}}$ follows by letting $\varepsilon \downarrow 0$.

Once we know that $\nabla u \in \mathrm{BV}_{\mathrm{loc}}$, and thus that D^2u is a measure, we can follow the proof of [BT26, Theorem 2.3] (where this regularity was instead assumed) to prove $D^2u \geq 0$ and thus the conclusion; the difference compared to [BT26] is that we only have the expansion property at our disposal, instead of measure preservation, but the argument is the same. The absolutely continuous part of the measure D^2u is handled through the area formula, which converts the expansion property into $|\det(\mathbb{I} + tD^2u)| \geq 1$ almost everywhere, for every $t > 0$; measure preservation would give equality here, but the inequality suffices: if $D^2u(x)$ had a negative eigenvalue λ, then $\det(\mathbb{I} + tD^2u(x))$ would vanish at $t = -1/\lambda > 0$. The singular part of D^2u is handled through Alberti's rank-one theorem [Alb93; DR16; MV19]: by the symmetry of D^2u, the polar of its singular part is $\pm n \otimes n$ for a unit vector n, and since the trace of D^2u is $\Delta u \geq 0$ the sign is $+$, so that the singular part of D^2u is non-negative as well. In conclusion, $D^2u \geq 0$, which implies the existence of a locally convex representative of u.

2. Preparatory results

We first observe that the set of times at which the expansion property (1.3) holds is closed, so that hypothesis (1.7) is in fact a statement about every positive time: a set of full measure in $(0, \infty)$ is dense, and a closed dense subset of $(0, \infty)$ is all of $(0, \infty)$.

Lemma 2.1 (Closedness in time). *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ and suppose that $\mathrm{Id} + t\nabla u$ is expanding for every t in a dense set $G \subset (0, \infty)$. Then $\mathrm{Id} + t\nabla u$ is expanding for every $t > 0$.*

Proof. Fix $t > 0$ and $t_j \in G$ with $t_j \to t$. For every $0 \le \zeta \in C_c(\mathbb{R}^d)$ and every x, we have $\zeta(x + t_j\nabla u(x)) \to \zeta(x + t\nabla u(x))$, so Fatou's lemma and the expansion property (1.3) at time t_j give

$$\int_\Omega \zeta\big(x + t\nabla u(x)\big)\,\mathrm{d}x \le \liminf_{j\to\infty} \int_\Omega \zeta\big(x + t_j\nabla u(x)\big)\,\mathrm{d}x \le \int_{\mathbb{R}^d} \zeta\,\mathrm{d}y,$$

which is the expansion property (1.3) at time t. $\square$

We next recall (see [BT26, Proposition 3.3]) that the expansion property yields that u is subharmonic.

Lemma 2.2 (Subharmonicity)**.** *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$, and assume that* $\mathrm{Id} + t\nabla u$ *is expanding for almost every $t > 0$. Then*

$$\Delta u \ge 0 \qquad \text{in } \mathscr{D}'(\Omega). \tag{2.1}$$

In particular, Δu is a non-negative locally finite Radon measure.

Proof. Let $U \Subset \Omega$ be an open set. By (1.4), the restriction of $\mathrm{Id} + t\nabla u$ to U is expanding for almost every $t > 0$. Since $u \in W^{1,1}(U)$, [BT26, Proposition 3.3] gives $\Delta u \ge 0$ in $\mathscr{D}'(U)$. As $U \Subset \Omega$ is arbitrary, (2.1) follows. Finally, every positive distribution is a locally finite non-negative Radon measure (see [DK10, Theorem 3.18]). $\square$

Subharmonicity alone, however, does not imply $\nabla u \in \mathrm{BV}_{\mathrm{loc}}$: the fundamental solution of the Laplacian has a Hessian with a principal-value part, which is not a measure (see the discussion after [BT26, Proposition 2.4]), and even the additional information that $\partial_{11}u \ge 0$ and $\partial_{22}u \ge 0$ (separate convexity) does not help: in [CFM05, Theorem 3] a Lipschitz, separately convex function on $\mathbb{R}^2$ whose gradient is not in $\mathrm{BV}_{\mathrm{loc}}$ is constructed (see also [FT23, Theorem 3.1] for a Lipschitz subharmonic function on $B_1(0) \subset \mathbb{R}^2$ with the same property). The additional regularity comes from the expansion property (1.3) at all times, through the two lemmas that follow. The first says that if Δu has small mass on a ball, then ∇u is close in L^1 to the gradient of a harmonic function on that ball; this is a linear estimate, which holds for any signed measure Δu and does not use the expansion property, and it is the linear counterpart of the comparison estimate in [DM11, Lemma 3.3, Eq. (3.24)].

Lemma 2.3 (L^1 harmonic approximation)**.** *Let $d \ge 2$, let $B_r(x_0) \subset \mathbb{R}^d$, and let $u \in W^{1,1}(B_r(x_0))$. Suppose that Δu is a finite signed Radon measure in $B_r(x_0)$. Then there exists a harmonic function h in $B_r(x_0)$ such that*

$$\int_{B_r(x_0)} |\nabla u - \nabla h|\,\mathrm{d}x \le r\,|\Delta u|(B_r(x_0)). \tag{2.2}$$

Proof. Extend Δu by zero to $\mathbb{R}^d$, and let Φ_d be the fundamental solution normalized by $\Delta\Phi_d = \delta_0$. Thus

$$|\nabla\Phi_d(z)| = \frac{1}{|\mathbb{S}^{d-1}|}|z|^{1-d} \qquad (z \ne 0).$$

The Newtonian potential $w = \Phi_d * \Delta u$ belongs to $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$ and satisfies $\Delta w = \Delta u$ in $B_r(x_0)$. For every $y \in B_r(x_0)$, since $z \mapsto |z|^{1-d}$ is radially decreasing and $B_r(x_0) - y$ has the same measure as $B_r(0)$,

$$\int_{B_r(x_0)} |x-y|^{1-d}\,\mathrm{d}x \le \int_{B_r(0)} |z|^{1-d}\,\mathrm{d}z = |\mathbb{S}^{d-1}|\,r.$$

Tonelli's theorem, applied to the total variation of Δu, then gives

$$\begin{aligned}\int_{B_r(x_0)} |\nabla w(x)|\,\mathrm{d}x &\le \int_{B_r(x_0)}\int_{B_r(x_0)} |\nabla\Phi_d(x-y)|\,\mathrm{d}|\Delta u|(y)\,\mathrm{d}x \\ &\le |\Delta u|(B_r(x_0)) \sup_{y\in B_r(x_0)} \int_{B_r(x_0)} \frac{|x-y|^{1-d}}{|\mathbb{S}^{d-1}|}\,\mathrm{d}x \le r\,|\Delta u|(B_r(x_0)).\end{aligned}$$

Set $h = u - w$ in $B_r(x_0)$. Then $\Delta h = 0$ in $\mathscr{D}'(B_r(x_0))$, so Weyl's lemma gives a smooth harmonic representative of h and the preceding estimate is (2.2).[5] □

Next, we show that a harmonic function, or more generally a smooth superharmonic one, whose gradient perturbations of the identity are expanding must be affine; in contrast to Lemma 2.3, it is here that the expansion property enters.

Lemma 2.4 (Expanding superharmonic potentials are affine)**.** *Let $U \subset \mathbb{R}^d$ be open and let $h \in C^2(U)$ satisfy $\Delta h \le 0$. Suppose that $0 \in \overline{G}$ for a set $G \subset (0,\infty)$ and that, for every $s \in G$ and every ball $B \Subset U$,*

$$(\mathrm{Id} + s\nabla h)_{\#}(\mathcal{L}^d \llcorner B) \le \mathcal{L}^d. \tag{2.4}$$

Then h is affine on every connected component of U.

Proof. Fix $x_0 \in U$. Choose $s \in G$ sufficiently small that $\mathbb{I} + sD^2h(x_0)$ is positive definite. By continuity and the inverse function theorem, there exists an open neighborhood V of x_0 on which

$$F_s(y) := y + s\nabla h(y)$$

is a C^1-diffeomorphism onto its image. Shrinking V if necessary,

$$\det\big(\mathbb{I} + sD^2h(y)\big) > 0 \qquad \text{in } V.$$

Choose $r_0 > 0$ with $\overline{B_{r_0}(x_0)} \subset V$. If $0 < r < r_0$, then $F_s(B_r(x_0))$ is open and hence Borel.

Applying (2.4) with $B = B_r(x_0)$ and evaluating the resulting measure inequality on $F_s(B_r(x_0))$, we obtain

$$\mathcal{L}^d(B_r(x_0)) \le \mathcal{L}^d(F_s(B_r(x_0))).$$

The change-of-variables formula therefore gives

$$1 \le \frac{\mathcal{L}^d(F_s(B_r(x_0)))}{\mathcal{L}^d(B_r(x_0))} = \fint_{B_r(x_0)} \det\big(\mathbb{I} + sD^2h(y)\big)\,\mathrm{d}y.$$

Letting $r \downarrow 0$ yields

$$\det\big(\mathbb{I} + sD^2h(x_0)\big) \ge 1. \tag{2.5}$$

On the other hand, superharmonicity gives

$$\operatorname{tr}\big(\mathbb{I} + sD^2h(x_0)\big) = d + s\Delta h(x_0) \le d.$$

Since $\mathbb{I} + sD^2h(x_0)$ is positive definite, the arithmetic–geometric mean inequality applied to its eigenvalues gives

$$1 \le \det\big(\mathbb{I} + sD^2h(x_0)\big) \le \left(\frac{\operatorname{tr}(\mathbb{I} + sD^2h(x_0))}{d}\right)^d \le 1.$$

Equality in the arithmetic–geometric mean inequality implies $\mathbb{I} + sD^2h(x_0) = \mathbb{I}$. Since $s > 0$, it follows that $D^2h(x_0) = 0$. As x_0 was arbitrary, $D^2h = 0$ in U, and hence h is affine on every connected component of U. □

We can now combine Lemmas 2.3 and 2.4 into a quantitative decay statement for the gradient excess $\mathcal{E}(u;B)$ defined in (1.9): for a fixed $\theta \in (0,1)$, the excess on the smaller ball $B_{\theta r}(x_0)$ is at most an arbitrarily small fraction η of the excess on $B_r(x_0)$, plus a multiple, depending on η, of $\Delta u(B_r(x_0))/r^{d-1}$. The argument is inspired by the excess-decay estimates in [DM09] and [DM11, Lemma 3.6].

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[5] We can also quote [LSW63, Eq. (5.2) and Theorem 5.1, pp. 58–59], which implies that, for every $1 < q < d/(d-1)$, there exists a function $w \in W_0^{1,q}(B_r(x_0))$ satisfying $\Delta w = \Delta u$ and

$$\|\nabla w\|_{L^q(B_r(x_0))} \le C_{d,q} r^{1-d+d/q} |\Delta u|(B_r(x_0)). \tag{2.3}$$

Choose $q = 2d/(2d-1)$. Hölder's inequality and (2.3) yield

$$\begin{aligned}\int_{B_r(x_0)} |\nabla w|\,\mathrm{d}x &\le \mathcal{L}^d(B_r(0))^{1-1/q}\|\nabla w\|_{L^q(B_r(x_0))}\\ &\le C_d r^{d(1-1/q)} r^{1-d+d/q}|\Delta u|(B_r(x_0))\\ &= C_d r\,|\Delta u|(B_r(x_0)).\end{aligned}$$

This route gives the stronger integrability $W_0^{1,q}$ but a non-sharp constant.

Proposition 2.5 (Excess improvement). *Let $d \geq 2$, let $\Omega \subset \mathbb{R}^d$ be open, and let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$. Suppose that $\mathrm{Id} + t\nabla u$ is expanding for almost every $t > 0$, so that Δu is a non-negative Radon measure by Lemma 2.2. For every $\theta \in (0,1)$ and every $\eta > 0$, there exists $C = C(d, \theta, \eta)$ such that, for every ball $B_r(x_0) \Subset \Omega$,*

$$\mathcal{E}(u; B_{\theta r}(x_0)) \leq \eta \mathcal{E}(u; B_r(x_0)) + C \frac{\Delta u(B_r(x_0))}{r^{d-1}}. \tag{2.6}$$

Proof. Step 1. *Decay under small normalized mass.* We claim that there exists $\varepsilon_0 = \varepsilon_0(d, \theta, \eta) > 0$ such that the following implication holds for every open set $\Omega \subset \mathbb{R}^d$, every $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ satisfying (1.7), and every ball $B_r(x_0) \Subset \Omega$:

$$\frac{\Delta u(B_r(x_0))}{r^{d-1}} \leq \varepsilon_0 \mathcal{E}(u; B_r(x_0)) \quad \Longrightarrow \quad \mathcal{E}(u; B_{\theta r}(x_0)) \leq \eta \mathcal{E}(u; B_r(x_0)). \tag{2.7}$$

If $\mathcal{E}(u; B_r(x_0)) = 0$, then ∇u is constant almost everywhere on $B_r(x_0)$, so the conclusion is immediate.

Suppose that the claim is false. For every $n \in \mathbb{N}$, there then exist an open set $\Omega_n \subset \mathbb{R}^d$, a function $u_n \in W^{1,1}_{\mathrm{loc}}(\Omega_n)$ satisfying (1.7), and a ball $B_{r_n}(x_n) \Subset \Omega_n$ such that

$$\frac{\Delta u_n(B_{r_n}(x_n))}{r_n^{d-1} e_n} \leq \frac{1}{n}, \qquad \mathcal{E}(u_n; B_{\theta r_n}(x_n)) > \eta e_n, \tag{2.8}$$

where

$$e_n := \mathcal{E}(u_n; B_{r_n}(x_n)) > 0.$$

Step 2. *Blow-up and harmonic limit.* Choose $a_n \in \mathbb{R}^d$ attaining the excess on $B_{r_n}(x_n)$, and define on $(\Omega_n - x_n)/r_n$

$$v_n(y) := \frac{u_n(x_n + r_n y) - r_n a_n \cdot y}{r_n e_n}. \tag{2.9}$$

Then

$$\nabla v_n(y) = \frac{\nabla u_n(x_n + r_n y) - a_n}{e_n}.$$

The choice of a_n, scaling, and (2.8) give

$$\begin{aligned} &\mathcal{E}(v_n; B_1(0)) = \fint_{B_1(0)} |\nabla v_n| \,\mathrm{d}y = 1, \\ &\Delta v_n(B_1(0)) = \frac{\Delta u_n(B_{r_n}(x_n))}{r_n^{d-1} e_n} \leq \frac{1}{n}, \qquad \mathcal{E}(v_n; B_\theta(0)) > \eta. \end{aligned} \tag{2.10}$$

Let h_n be the harmonic function in $B_1(0)$ given by Lemma 2.3. Then

$$\|\nabla v_n - \nabla h_n\|_{L^1(B_1(0))} \leq \Delta v_n(B_1(0)) \longrightarrow 0. \tag{2.11}$$

After subtracting the average of h_n over $B_1(0)$, which does not change its gradient, (2.10) and (2.11) and Poincaré's inequality give

$$\sup_n \|h_n\|_{W^{1,1}(B_1(0))} < \infty.$$

The mean-value property therefore bounds (h_n) locally uniformly in $B_1(0)$. By Harnack's convergence theorem (a locally bounded family of harmonic functions is normal), after passing to a subsequence,

$$h_n \longrightarrow h \qquad \text{in } C^\infty_{\mathrm{loc}}(B_1(0))$$

for some harmonic function h (see, e.g., [ABR01, Theorems 1.23 and 2.6]). Combining this with (2.11), we obtain

$$\nabla v_n \longrightarrow \nabla h \quad \text{strongly in } L^1(B_\sigma(0)) \quad \text{for every } 0 < \sigma < 1. \tag{2.12}$$

The excess is continuous under strong L^1 convergence of the gradients: for $v, w \in W^{1,1}(B)$,

$$|\mathcal{E}(v; B) - \mathcal{E}(w; B)| \leq \fint_B |\nabla v - \nabla w| \,\mathrm{d}x. \tag{2.13}$$

Hence (2.10), (2.12) and (2.13) give

$$\mathcal{E}(h; B_\theta(0)) \geq \eta. \tag{2.14}$$

In particular, h is not affine.

Step 3. *Expansion property and rigidity of the limit.* By Lemma 2.1, $\mathrm{Id} + t\nabla u_n$ is expanding for every $t > 0$ and every n. From (2.9),

$$x_n + r_n\big(y + s\nabla v_n(y)\big) = \left(\mathrm{Id} + \frac{r_n s}{e_n}\nabla u_n\right)(x_n + r_n y) - \frac{r_n s}{e_n} a_n. \tag{2.15}$$

Translations and simultaneous dilations of source and target preserve the expansion property (1.3). It follows that $\mathrm{Id} + s\nabla v_n$ is expanding for every $s > 0$ and every n.

Fix $s > 0$, a measurable set D with $D \Subset B_1(0)$, and a function $0 \le \zeta \in C_c^\infty(\mathbb{R}^d)$. By (1.4),

$$\int_D \zeta\big(y + s\nabla v_n(y)\big)\,\mathrm{d}y \le \int_{\mathbb{R}^d} \zeta(z)\,\mathrm{d}z.$$

Moreover, (2.12) yields

$$\begin{aligned}\left|\int_D \zeta\big(y + s\nabla v_n(y)\big)\,\mathrm{d}y - \int_D \zeta\big(y + s\nabla h(y)\big)\,\mathrm{d}y\right| \\ \le s\|\nabla\zeta\|_{L^\infty}\|\nabla v_n - \nabla h\|_{L^1(D)} \longrightarrow 0.\end{aligned}$$

Passing to the limit gives

$$\int_D \zeta\big(y + s\nabla h(y)\big)\,\mathrm{d}y \le \int_{\mathbb{R}^d} \zeta(z)\,\mathrm{d}z.$$

By the characterization of inequalities between Radon measures using non-negative C_c^∞ test functions, and since D was arbitrary,

$$(\mathrm{Id} + s\nabla h)_\#(\mathcal{L}^d \llcorner D) \le \mathcal{L}^d.$$

Thus h satisfies the hypotheses of Lemma 2.4 with $G = (0, \infty)$, and hence is affine. This contradicts (2.14) and proves (2.7).

Step 4. *Removal of the smallness assumption.* If the premise of (2.7) holds, its conclusion implies (2.6). Otherwise,

$$\mathcal{E}(u; B_r(x_0)) < \frac{1}{\varepsilon_0}\frac{\Delta u(B_r(x_0))}{r^{d-1}}.$$

For every $a \in \mathbb{R}^d$,

$$\mathcal{E}(u; B_{\theta r}(x_0)) \le \fint_{B_{\theta r}(x_0)} |\nabla u - a|\,\mathrm{d}x \le \theta^{-d}\fint_{B_r(x_0)} |\nabla u - a|\,\mathrm{d}x.$$

Taking the infimum over a and using the preceding inequality gives

$$\mathcal{E}(u; B_{\theta r}(x_0)) < \frac{\theta^{-d}}{\varepsilon_0}\frac{\Delta u(B_r(x_0))}{r^{d-1}}.$$

Thus (2.6) holds with $C = \theta^{-d}/\varepsilon_0$. □

3. Proof of the main theorem

Proof of Theorem 1.3. By Lemma 2.2, $\Delta u \in \mathcal{M}^+_{\mathrm{loc}}(\Omega)$. Let $K \Subset \Omega$ be compact and, for $R > 0$, define $K_R := \{x \in \mathbb{R}^d : \mathrm{dist}(x, K) < R\}$. Choose $R > 0$ such that $\overline{K_R} \Subset \Omega$.

Step 1. *The one-dimensional case.* Suppose first that $d = 1$. Since $u'' = \Delta u$ is a non-negative locally finite measure, $u' \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ with $D(u') = u'' = \Delta u \ge 0$, so that $D^2 u \ge 0$ and (1.8) holds with equality; moreover u' has a non-decreasing representative and u has a locally convex representative. Assume henceforth that $d \ge 2$.

Step 2. *The integrated excess inequality.* Let $\theta = 1/8$ and $\eta = \theta/2 = 1/16$. For every $0 < r \le R$, Proposition 2.5 and Tonelli's theorem give

$$\begin{aligned}\int_K \mathcal{E}(u; B_{\theta r}(x))\,\mathrm{d}x &\le \eta\int_K \mathcal{E}(u; B_r(x))\,\mathrm{d}x + \frac{C}{r^{d-1}}\int_K \Delta u(B_r(x))\,\mathrm{d}x \\ &= \eta\int_K \mathcal{E}(u; B_r(x))\,\mathrm{d}x + \frac{C}{r^{d-1}}\int_{K_r} \mathcal{L}^d(K \cap B_r(y))\,\mathrm{d}\Delta u(y) \\ &\le \eta\int_K \mathcal{E}(u; B_r(x))\,\mathrm{d}x + C\,\mathcal{L}^d(B_1(0))\,r\,\Delta u(K_R). \qquad (3.1)\end{aligned}$$

<u>Step 3.</u> *Iteration across all scales.* Write the last term of (3.1) as $C_0\, r\, \Delta u(K_R)$, where $C_0 := C\, \mathcal{L}^d(B_1(0))$ depends only on d, since θ and η are now fixed, and let $r_n = \theta^n R$. Dividing (3.1) by $r_{n+1} = \theta r_n$ and using $\eta/\theta = 1/2$ gives

$$\frac{1}{r_{n+1}} \int_K \mathcal{E}(u; B_{r_{n+1}}(x))\, \mathrm{d}x \le \frac{1}{2r_n} \int_K \mathcal{E}(u; B_{r_n}(x))\, \mathrm{d}x + 8C_0 \Delta u(K_R). \tag{3.2}$$

The same constant C_0 applies for every n. Induction therefore gives, for all $n \ge 0$,

$$\begin{aligned} \frac{1}{r_n} \int_K \mathcal{E}(u; B_{r_n}(x))\, \mathrm{d}x &\le \frac{2^{-n}}{R} \int_K \mathcal{E}(u; B_R(x))\, \mathrm{d}x + 8C_0 \Delta u(K_R) \sum_{j=0}^{n-1} 2^{-j} \\ &= \frac{2^{-n}}{R} \int_K \mathcal{E}(u; B_R(x))\, \mathrm{d}x + 16C_0(1 - 2^{-n}) \Delta u(K_R), \end{aligned}$$

where the sum is empty when $n = 0$. In particular,

$$\sup_{n \ge 0} \frac{1}{r_n} \int_K \mathcal{E}(u; B_{r_n}(x))\, \mathrm{d}x \le \frac{1}{R} \int_K \mathcal{E}(u; B_R(x))\, \mathrm{d}x + 16C_0 \Delta u(K_R). \tag{3.3}$$

For any $0 < r \le R$, choose $n \ge 0$ such that $r_{n+1} \le r \le r_n$. Comparing the averages over the two concentric balls gives

$$\mathcal{E}(u; B_r(x)) \le \left(\frac{r_n}{r}\right)^d \mathcal{E}(u; B_{r_n}(x)).$$

After integration and division by r, this yields

$$\begin{aligned} \frac{1}{r} \int_K \mathcal{E}(u; B_r(x))\, \mathrm{d}x &\le \left(\frac{r_n}{r}\right)^{d+1} \frac{1}{r_n} \int_K \mathcal{E}(u; B_{r_n}(x))\, \mathrm{d}x \\ &\le \theta^{-(d+1)} \left(\frac{1}{R} \int_K \mathcal{E}(u; B_R(x))\, \mathrm{d}x + 16C_0 \Delta u(K_R) \right), \end{aligned}$$

where we used $r \ge \theta r_n$ and (3.3). Thus, with $C_1 := 8^{d+1} \max\{1, 16C_0\}$,

$$\sup_{0 < r \le R} \frac{1}{r} \int_K \mathcal{E}(u; B_r(x))\, \mathrm{d}x \le C_1 \left(\frac{1}{R} \int_K \mathcal{E}(u; B_R(x))\, \mathrm{d}x + \Delta u(K_R) \right). \tag{3.4}$$

Finally, choosing $a = 0$ in the definition of the excess and applying Tonelli's theorem gives

$$\begin{aligned} \int_K \mathcal{E}(u; B_R(x))\, \mathrm{d}x &\le \int_K \fint_{B_R(x)} |\nabla u(y)|\, \mathrm{d}y\, \mathrm{d}x \\ &= \frac{1}{\mathcal{L}^d(B_R(0))} \int_{K_R} |\nabla u(y)|\, \mathcal{L}^d(K \cap B_R(y))\, \mathrm{d}y \\ &\le \int_{K_R} |\nabla u(y)|\, \mathrm{d}y. \end{aligned} \tag{3.5}$$

<u>Step 4.</u> *A preliminary mollified Hessian estimate.* Let

$$\rho \in C_c^\infty(B_1(0)), \qquad \rho \ge 0, \qquad \int_{\mathbb{R}^d} \rho\, \mathrm{d}x = 1, \qquad \rho_\varepsilon(x) := \varepsilon^{-d} \rho(x/\varepsilon).$$

For $x \in K$ and $0 < \varepsilon < R$, the cancellation identity $\int_{\mathbb{R}^d} D\rho_\varepsilon = 0$ gives, for every $a \in \mathbb{R}^d$,

$$D(\nabla u * \rho_\varepsilon)(x) = \int_{B_\varepsilon(x)} \big(\nabla u(y) - a\big) \otimes D\rho_\varepsilon(x - y)\, \mathrm{d}y.$$

Taking the infimum over a yields

$$|D(\nabla u * \rho_\varepsilon)(x)| \le \frac{C_{d,\rho}}{\varepsilon} \mathcal{E}(u; B_\varepsilon(x)).$$

After integration over K, this estimate and (3.4) and (3.5) give

$$\sup_{0 < \varepsilon < R} \int_K |D(\nabla u * \rho_\varepsilon)|\, \mathrm{d}x \le C_1 C_{d,\rho} \left(\frac{1}{R} \int_{K_R} |\nabla u|\, \mathrm{d}x + \Delta u(K_R) \right). \tag{3.6}$$

<u>Step 5.</u> *BV bound.* Let $U \Subset \Omega$ be an open set, set $K_U = \overline{U}$, and choose $R_U > 0$ such that $\overline{(K_U)_{R_U}} \Subset \Omega$. Fix a sequence $\varepsilon_j \downarrow 0$ with $\varepsilon_j < R_U$, so that $\nabla u * \rho_{\varepsilon_j} \to \nabla u$ in $L^1(U)$,

while (3.6), applied with $K = K_U$ and $R = R_U$, bounds $\int_U |D(\nabla u * \rho_{\varepsilon_j})|\,\mathrm{d}x$ uniformly in j. Let $\Phi \in C^1_c(U;\mathbb{R}^{d\times d})$ satisfy $|\Phi| \le 1$. Row-wise integration by parts gives

$$\left|\int_U \nabla u \cdot \operatorname{div}\Phi\,\mathrm{d}x\right| = \lim_{j\to\infty}\left|-\int_U D(\nabla u * \rho_{\varepsilon_j}) : \Phi\,\mathrm{d}x\right| \le \sup_j \int_U |D(\nabla u * \rho_{\varepsilon_j})|\,\mathrm{d}x < \infty.$$

Taking the supremum over all such Φ gives $|D(\nabla u)|(U) < \infty$ and, thus, $\nabla u \in \mathrm{BV}(U;\mathbb{R}^d)$. This shows that $\nabla u \in \mathrm{BV}_{\mathrm{loc}}(\Omega;\mathbb{R}^d)$.

Step 6. *Positivity of the Hessian and the sharp estimate.* We argue as in [BT26, Proof of Theorem 2.3], which requires $\Delta u \in \mathcal{M}^+_{\mathrm{loc}}(\Omega)$ and $\nabla u \in \mathrm{BV}_{\mathrm{loc}}(\Omega;\mathbb{R}^d)$. By Lebesgue's decomposition,

$$D^2u = H\mathcal{L}^d + (D^2u)^s,$$

where $H\,\mathcal{L}^d$ and $(D^2u)^s$ are respectively the absolutely continuous and singular parts of D^2u with respect to $\mathcal{L}^d$. The singular part is treated exactly as there ([BT26, Proof of Theorem 2.3, part (1)]): by Alberti's rank-one theorem [Alb93] and the symmetry of D^2u, the polar of $(D^2u)^s$ has the form $\lambda\, n\otimes n$ with $|n| = 1$ and $\lambda \in \{-1,1\}$, the polar having unit Frobenius norm; moreover, $\lambda = 1$ because $\lambda|(D^2u)^s|$ is the singular part of $\Delta u \ge 0$ (see also [FT23, Proposition 2.1 and Remark 2.2]).

For the absolutely continuous part, their proof uses the stronger measure-preserving condition to obtain $|\det(\mathbb{I}+tH)| = 1$ almost everywhere (see [BT26, Eqs. (2.5)–(2.8)]). Under our expansion property (1.7), the same argument gives only $|\det(\mathbb{I}+tH)| \ge 1$; however, inspecting the proof,[6] this still yields $H \ge 0$ almost everywhere in Ω.

Thus D^2u is positive semidefinite, and hence u has a locally convex representative by [Dud77].[7] Finally, for every positive-semidefinite matrix A, $|A| \le \operatorname{tr} A$, which, applied to the polar decomposition $D^2u = M\,|D^2u|$, whose density M is positive semidefinite $|D^2u|$-almost everywhere, yields $|D^2u| = |M|\,|D^2u| \le \operatorname{tr}(M)\,|D^2u| = \operatorname{tr}(D^2u) = \Delta u$, which is (1.8).

Step 7. *Singularity under measure preservation.* Assume now that T_{t_0} is measure preserving on a full-measure set F for some $t_0 > 0$. Then [BT26, Eqs. (2.6)–(2.7)] give $|\det(\mathbb{I}+t_0H)| = 1$ almost everywhere. Since $H \ge 0$, all factors in $\prod_i(1+t_0\lambda_i) = 1$ are at least 1, so the eigenvalues λ_i of H vanish, i.e., $H = 0$ almost everywhere and D^2u is singular with respect to $\mathcal{L}^d$. □

4. A sharp excess bound for convex potentials

In this section we prove Proposition 1.7. We use only the information that $D^2u \ge 0$, which makes D^2u absolutely continuous with respect to Δu, with a positive-semidefinite density of unit trace. The result follows from a sharp L^1 Poincaré-type inequality on balls (see [Cia89, Theorem 1]), applied to the directional derivatives $\nabla u\cdot\theta$ and averaged over $\theta \in \mathbb{S}^{d-1}$.

Proof of Proposition 1.7. Step 1. *The case $d = 1$.* Since $u'' = \Delta u \ge 0$, u' has a non-decreasing representative on $B_r(x_0) = (x_0 - r, x_0 + r)$; let m and M be its infimum and supremum there, so that $M - m = \Delta u(B_r(x_0))$, and note that $\mu \coloneqq \fint_{B_r(x_0)} u'\,\mathrm{d}y \in [m, M]$. If $m = M$ there is nothing

[6] Fix a measurable representative of ∇u; the expansion property (1.3) does not depend on this choice. Let f^Λ and G^Λ be the Lipschitz maps and sets provided by the Lusin–Lipschitz property of BV maps as in [BT26, Proof of Theorem 2.3, part (2)], so that $f^\Lambda = \nabla u$ pointwise on G^Λ, $Df^\Lambda = H$ almost everywhere on G^Λ and $\mathcal{L}^d(\Omega\setminus G^\Lambda) \to 0$ as $\Lambda \to \infty$. For a measurable set $A \subset \Omega$ and a time t at which (1.3) holds, the set $T_t(A\cap G^\Lambda) = (\mathrm{Id} + tf^\Lambda)(A\cap G^\Lambda)$ is the image of a measurable set under a Lipschitz map, hence measurable, and the area formula for the Lipschitz map $\mathrm{Id} + tf^\Lambda$ (their Eq. (2.6)) and (1.4) evaluated on $T_t(A\cap G^\Lambda)$ give

$$\int_{A\cap G^\Lambda} |\det(\mathbb{I}+tH)|\,\mathrm{d}x = \int_{\mathbb{R}^d} \#\{x \in A\cap G^\Lambda : T_t(x) = y\}\,\mathrm{d}y \ge \mathcal{L}^d(T_t(A\cap G^\Lambda)) \ge \mathcal{L}^d(A\cap G^\Lambda).$$

In [BT26], measure preservation makes the multiplicity in the middle integral equal to one, whence equality; here only the inequality is available, but since A is arbitrary it yields $|\det(\mathbb{I}+tH)| \ge 1$ almost everywhere on G^Λ, hence on Ω as $\Lambda \to \infty$. Since (1.3) holds at every $t > 0$ by Lemma 2.1, for almost every $x \in \Omega$ the inequality $|\det(\mathbb{I}+tH(x))| \ge 1$ holds for all t in a countable dense subset of $(0,\infty)$, hence for all $t \ge 0$ by continuity (cf. [BT26, Eqs. (2.7)–(2.8)]). Since H is symmetric almost everywhere, a negative eigenvalue λ of $H(x)$ would make $\det(\mathbb{I}+tH(x))$ vanish at $t = -1/\lambda > 0$.

[7] For an alternative proof of this claim, we note that the mollifications of u have non-negative Hessian, hence are convex, and convex functions bounded in L^1 are locally uniformly Lipschitz, so by the Arzelà–Ascoli theorem a subsequence converges locally uniformly on each ball $B \Subset \Omega$; the limit is convex and, since $u * \rho_\varepsilon \to u$ in $L^1_{\mathrm{loc}}(\Omega)$, coincides with u almost everywhere, hence is a convex representative of u on B.

to prove. Otherwise, since $s \mapsto |s-\mu|$ is convex, it lies below its chord on $[m, M]$, and averaging that chord inequality over $B_r(x_0)$ gives

$$\fint_{B_r(x_0)} |u' - \mu| \,\mathrm{d}x \le \frac{2(\mu - m)(M-\mu)}{M-m} \le \frac{M-m}{2} = \frac{1}{2}\Delta u(B_r(x_0)),$$

which is (1.10) since $\omega_0 = 1$. Assume $d \ge 2$ from now on.

Step 2. *A scalar inequality.* For $v \in \mathrm{BV}(B_r(x_0))$, by [Cia89, Theorem 1] (which is stated for BV functions and in which equality holds for the characteristic function of a half-ball)

$$\int_{B_r(x_0)} \Big| v - \fint_{B_r(x_0)} v \,\mathrm{d}y \Big| \,\mathrm{d}x \le \frac{\omega_d}{2\omega_{d-1}}\, r\, |Dv|(B_r(x_0)). \tag{4.1}$$

Step 3. *Averaging over directions.* Since $D^2 u \ge 0$, $\Delta u(A) = 0$ implies $D^2 u(A) = 0$ for every Borel set A, so $D^2 u \ll \Delta u$, and its density P is positive semidefinite with $\operatorname{tr} P = 1$ for Δu-almost every point. For every $\theta \in \mathbb{S}^{d-1}$, we have $\nabla u \cdot \theta \in \mathrm{BV}(B_r(x_0))$ with $D(\nabla u \cdot \theta) = (D^2 u)\theta = P\theta\, \Delta u$, hence $|D(\nabla u \cdot \theta)|(B_r(x_0)) = \int_{B_r(x_0)} |P\theta| \,\mathrm{d}\Delta u$. Moreover, since $\int_{\mathbb{S}^{d-1}} |\theta \cdot e| \,\mathrm{d}\mathcal{H}^{d-1}(\theta) = 2\omega_{d-1}$ for every unit vector e,

$$\begin{aligned} |z| &= \frac{1}{2\omega_{d-1}} \int_{\mathbb{S}^{d-1}} |z \cdot \theta| \,\mathrm{d}\mathcal{H}^{d-1}(\theta) \qquad (z \in \mathbb{R}^d), \\ \int_{\mathbb{S}^{d-1}} |P\theta| \,\mathrm{d}\mathcal{H}^{d-1}(\theta) &\le \sum_{i=1}^{d} \lambda_i \int_{\mathbb{S}^{d-1}} |\theta \cdot e_i| \,\mathrm{d}\mathcal{H}^{d-1}(\theta) = 2\omega_{d-1}, \end{aligned} \tag{4.2}$$

where $\lambda_i \ge 0$ and e_i are the eigenvalues and eigenvectors of P. Applying (4.1) to $v = \nabla u \cdot \theta$, integrating over $\theta \in \mathbb{S}^{d-1}$ (Tonelli), and using (4.2),

$$\begin{aligned} 2\omega_{d-1} \int_{B_r(x_0)} \Big| \nabla u - \fint_{B_r(x_0)} \nabla u \,\mathrm{d}y \Big| \,\mathrm{d}x &\le \frac{\omega_d}{2\omega_{d-1}}\, r \int_{B_r(x_0)} \int_{\mathbb{S}^{d-1}} |P\theta| \,\mathrm{d}\mathcal{H}^{d-1}(\theta) \,\mathrm{d}\Delta u \\ &\le \omega_d\, r\, \Delta u(B_r(x_0)). \end{aligned}$$

Choosing $a = \fint_{B_r(x_0)} \nabla u \,\mathrm{d}y$ and dividing by $\mathcal{L}^d(B_r(x_0)) = \omega_d r^d$ proves (1.10).

Step 4. *Equality.* For $u(x) = |x_1|$ and $x_0 \in \{x_1 = 0\}$, we have $\nabla u = \operatorname{sign}(x_1) e_1$, $\fint_{B_r(x_0)} \nabla u \,\mathrm{d}y = 0$, $\int_{B_r(x_0)} |\nabla u| \,\mathrm{d}x = \omega_d r^d$, and by Remark 1.5 $\Delta u(B_r(x_0)) = 2\omega_{d-1} r^{d-1}$; moreover, $a = 0$ attains the infimum in (1.9), because $a \mapsto \fint_{B_r(x_0)} |\nabla u - a| \,\mathrm{d}x$ is convex and, by the symmetry $x_1 \mapsto -x_1$, even. Hence all three terms in (1.10) equal one. ☐

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