ZERO DIFFUSION-DISPERSION LIMIT FOR THE BENJAMIN–ONO–BURGERS EQUATION: THE REGIME $\delta = o(\varepsilon)$

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Abstract. We prove that solutions of the Benjamin–Ono–Burgers equation converge to the entropy solution of the inviscid Burgers equation as the viscosity ε and dispersion δ tend to zero subject to $\delta = o(\varepsilon)$. This improves the previously known condition $\delta = O(\varepsilon^{3/2})$. The main new ingredient is a local spacetime L^4 estimate obtained from the one-dimensional interaction identity associated with the Varadhan functional, applied to the conservation law and its quadratic entropy balance.

1. Introduction and main result

We consider the Cauchy problem

$$\begin{cases} \partial_t u_{\varepsilon,\delta} + u_{\varepsilon,\delta}\partial_x u_{\varepsilon,\delta} = \varepsilon\partial_x^2 u_{\varepsilon,\delta} - \delta\mathcal{H}\partial_x^2 u_{\varepsilon,\delta}, & t>0,\ x\in\mathbb{R}, \\ u_{\varepsilon,\delta}(0,x) = u_{0,\varepsilon,\delta}(x), & x\in\mathbb{R}, \end{cases} \tag{1.1}$$

where $\varepsilon > 0$, $\delta > 0$, and the Hilbert transform is defined by

$$\mathcal{H}v(x) = \frac{1}{\pi}\,\mathrm{p.v.}\int_{\mathbb{R}} \frac{v(y)}{y-x}\,\mathrm{d}y, \qquad \widehat{\mathcal{H}v}(\xi) = i\,\mathrm{sgn}(\xi)\widehat{v}(\xi). \tag{1.2}$$

For $H^2(\mathbb{R})$ data, (1.1) has a unique global solution in $C([0,\infty); H^2(\mathbb{R}))$; see [Guo+11; Mol13]. Moreover, for each fixed $\varepsilon, \delta > 0$, the solution is smooth for positive times.

We study its limit as diffusion ε and dispersion δ vanish simultaneously.

In [Coc+25], we proved the convergence of the family $\{u_{\varepsilon,\delta}\}$ to the unique entropy solution of

$$\begin{cases} \partial_t u + u\partial_x u = 0, & t>0,\ x\in\mathbb{R}, \\ u(0,x) = u_0(x), & x\in\mathbb{R}, \end{cases} \tag{1.3}$$

under the balance condition $\delta = O(\varepsilon^{3/2})$.

The compactness argument used in [Coc+25] is based on the L^p compensated-compactness framework introduced by Schonbek [Sch82] and refined by Lu [Lu89]. It amounts to the following criterion (cf. [Coc+25, Lemma 1.2]). On every bounded open set $\Omega \subset (0,\infty)\times\mathbb{R}$, let $\{v_j\}_{j\ge 1}$ be bounded in $L^4(\Omega)$. For $\eta \in C^2(\mathbb{R})$ with compactly supported derivative, define $q_\eta(z) := \int_0^z r\eta'(r)\,\mathrm{d}r$. If

$$\{\partial_t \eta(v_j) + \partial_x q_\eta(v_j)\}_j \quad \text{is relatively compact in } H^{-1}_{\mathrm{loc}}(\Omega) \tag{1.4}$$

for every such η, then a subsequence converges almost everywhere and strongly in $L^p_{\mathrm{loc}}(\Omega)$ for every $1 \le p < 4$.

The entropy-production estimates in [Coc+25] already give the compactness in (1.4) under $\delta = O(\varepsilon)$, while the dispersive entropy error vanishes under $\delta = o(\varepsilon)$. The stronger restriction $\delta = O(\varepsilon^{3/2})$ entered only through a global-in-space, pointwise-in-time L^4 estimate. That estimate was

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obtained from a modification of the higher-order conserved quantity I_4 of the inviscid Benjamin–Ono flow. Differentiating the modified functional produces, among other terms, a constant multiple of

$$\delta\varepsilon \int_{\mathbb{R}} (\partial_x u_{\varepsilon,\delta})^2 \mathcal{H}\partial_x u_{\varepsilon,\delta}\,\mathrm{d}x. \tag{1.5}$$

Estimating (1.5) by the one-dimensional Gagliardo–Nirenberg inequality causes precisely the loss from $O(\varepsilon)$ to $O(\varepsilon^{3/2})$.

The observation of this note is that a local space-time L^4 estimate follows directly from an interaction identity. After localization by a positive, exponentially decaying weight, we pair the conservative form of (1.1) with its quadratic entropy balance. The determinant of their fluxes contains the coercive term $u_{\varepsilon,\delta}^4/12$. The remaining terms are controlled by the L^2 energy identity together with the weighted L^4 quantity when $\delta = O(\varepsilon)$.

Our main result is the following.

Proposition 1.1 (Local L^4 estimate)**.** *Fix $T > 0$, $\Lambda > 0$, $M \geq 0$, and a compact set $K \subset \mathbb{R}$. There is a constant $C = C(T, K, \Lambda, M)$ with the following property. If $u_{0,\varepsilon,\delta} \in H^2(\mathbb{R})$ and*

$$0 < \varepsilon \leq 1, \qquad 0 < \delta \leq \Lambda\varepsilon, \qquad \|u_{0,\varepsilon,\delta}\|_{L^2(\mathbb{R})} \leq M, \tag{1.6}$$

then the solution of (1.1) satisfies

$$\int_0^T \int_K |u_{\varepsilon,\delta}(t,x)|^4\,\mathrm{d}x\,\mathrm{d}t \leq C. \tag{1.7}$$

Using Proposition 1.1 in the arguments of [Coc+25] gives the following conclusion.

Theorem 1.2 (Diffusion-dispersion limit)**.** *Let $u_0 \in L^2(\mathbb{R})$, and let $\delta = \delta(\varepsilon) > 0$ satisfy*

$$\frac{\delta(\varepsilon)}{\varepsilon} \longrightarrow 0 \qquad \text{as } \varepsilon \downarrow 0. \tag{1.8}$$

Assume that, for every $0 < \varepsilon \leq 1$,

$$u_{0,\varepsilon,\delta(\varepsilon)} \in H^2(\mathbb{R}), \qquad \sup_{0<\varepsilon\leq 1} \left\|u_{0,\varepsilon,\delta(\varepsilon)}\right\|_{L^2(\mathbb{R})} < \infty, \tag{1.9}$$

and

$$u_{0,\varepsilon,\delta(\varepsilon)} \longrightarrow u_0 \quad \text{strongly in } L^2_{\mathrm{loc}}(\mathbb{R}) \quad \text{as } \varepsilon \downarrow 0. \tag{1.10}$$

If $u_{\varepsilon,\delta(\varepsilon)}$ is the solution of (1.1), then

$$u_{\varepsilon,\delta(\varepsilon)} \longrightarrow u \quad \text{strongly in } L^p_{\mathrm{loc}}([0,\infty) \times \mathbb{R}) \quad \text{for every } 1 \leq p < 4, \tag{1.11}$$

where u is the unique Kružkov entropy solution of (1.3) in $L^\infty_{\mathrm{loc}}([0,\infty); L^2(\mathbb{R}))$.

2. Proof of the L^4-estimate

To prove the spacetime L^4 estimate, we use the following version of the one-dimensional interaction identity associated with the Varadhan interaction functional [Tar08, Chapter 22].

Lemma 2.1 (Interaction identity)**.** *Let $A, B, D, E, P, Q \in C^1((0,T) \times \mathbb{R})$. Suppose that*

$$A, D \in C([0,T]; L^1(\mathbb{R})), \qquad B, E, P, Q \in L^1((0,T) \times \mathbb{R}), \tag{2.1}$$

that $AE, BD \in L^1((0,T) \times \mathbb{R})$, and that

$$\partial_t A + \partial_x B = P, \qquad \partial_t D + \partial_x E = Q \tag{2.2}$$

hold in $\mathcal{D}'((0,T) \times \mathbb{R})$. Define

$$\mathcal{I}(t) := \iint_{x<y} A(t,x) D(t,y)\,\mathrm{d}x\,\mathrm{d}y. \tag{2.3}$$

Then $\mathcal{I} \in W^{1,1}(0,T)$ and, for almost every $t \in (0,T)$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(t) = {} & \int_{\mathbb{R}} (AE - BD)(t,x)\,\mathrm{d}x \\ & + \iint_{x<y} \big(P(t,x)D(t,y) + A(t,x)Q(t,y)\big)\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{2.4}$$

Consequently,

$$
\left| \int_0^T \int_{\mathbb{R}} (AE - BD)\,\mathrm{d}x\,\mathrm{d}t \right| \le |\mathcal{I}(0)| + |\mathcal{I}(T)| + \|D\|_{L^\infty(0,T;L^1)} \|P\|_{L^1((0,T)\times\mathbb{R})} + \|A\|_{L^\infty(0,T;L^1)} \|Q\|_{L^1((0,T)\times\mathbb{R})}. \tag{2.5}
$$

For fields that are compactly supported in space, this is the interaction identity of Golse and Perthame [GP13, Section 2]. In Lemma 2.1, the fields are only required to be integrable in space rather than compactly supported, and the time interval is finite, so that the endpoint values $\mathcal{I}(0)$ and $\mathcal{I}(T)$ appear in (2.5) instead of vanishing. The general case is reduced to the compactly supported one by truncation in space (the integrability assumptions (2.1) guarantee that the truncation errors vanish in the limit).

With Lemma 2.1 at hand, we can prove Proposition 1.1.

Proof. Step 1. *Localized balance laws.* Set

$$
\chi(x) = \exp\big(1 - \sqrt{1+x^2}\big).
$$

Then

$$
0 < \chi \le 1, \qquad \chi, \chi' \in L^2(\mathbb{R}), \qquad |\chi'| \le \chi,
$$

and

$$
\inf_{x \in K} \chi(x) > 0
$$

for every non-empty compact set $K \subset \mathbb{R}$.

Define the conservative and quadratic-entropy fluxes by

$$
\begin{aligned}
F_{\varepsilon,\delta} &= \frac{u_{\varepsilon,\delta}^2}{2} - \varepsilon \partial_x u_{\varepsilon,\delta} + \delta \mathcal{H} \partial_x u_{\varepsilon,\delta}, \\
G_{\varepsilon,\delta} &= \frac{u_{\varepsilon,\delta}^3}{3} - \varepsilon u_{\varepsilon,\delta} \partial_x u_{\varepsilon,\delta} + \delta u_{\varepsilon,\delta} \mathcal{H} \partial_x u_{\varepsilon,\delta}.
\end{aligned}
$$

The equation in conservative form and the identity obtained by multiplying it by $u_{\varepsilon,\delta}$ are

$$
\begin{aligned}
&\partial_t u_{\varepsilon,\delta} + \partial_x F_{\varepsilon,\delta} = 0, \\
&\partial_t \left(\frac{u_{\varepsilon,\delta}^2}{2} \right) + \partial_x G_{\varepsilon,\delta} = -\varepsilon (\partial_x u_{\varepsilon,\delta})^2 + \delta \partial_x u_{\varepsilon,\delta} \mathcal{H} \partial_x u_{\varepsilon,\delta}.
\end{aligned} \tag{2.6}
$$

We localize only the first balance law. Indeed, $u_{\varepsilon,\delta}^2/2$ is already integrable by the $L^2(\mathbb{R})$ energy estimate, whereas $u_{\varepsilon,\delta}$ need not belong to $L^1(\mathbb{R})$. Localizing the first law makes its density integrable and already places one factor of χ in the determinant.

Introduce

$$
\begin{aligned}
A &= \chi u_{\varepsilon,\delta}, \qquad & B &= \chi F_{\varepsilon,\delta}, \qquad & D &= \frac{u_{\varepsilon,\delta}^2}{2}, \\
E &= G_{\varepsilon,\delta}, \qquad & P &= \chi' F_{\varepsilon,\delta}, \qquad & Q &= -\varepsilon (\partial_x u_{\varepsilon,\delta})^2 + \delta \partial_x u_{\varepsilon,\delta} \mathcal{H} \partial_x u_{\varepsilon,\delta}.
\end{aligned}
$$

Here P is the error created by differentiating the spatial weight, while Q is the entropy-production term. Thus (2.6) takes exactly the form required in Lemma 2.1:

$$
\partial_t A + \partial_x B = P, \qquad \partial_t D + \partial_x E = Q.
$$

Step 2. *Uniform bounds required by the interaction estimate.* The quantitative estimate in Lemma 2.1 contains the time-endpoint values $\mathcal{I}(0)$ and $\mathcal{I}(T)$ of the interaction functional, together with the $L^\infty(0,T;L^1(\mathbb{R}))$ norms of A, D and the $L^1((0,T)\times\mathbb{R})$ norms of P, Q. We now bound these quantities, uniformly in (ε, δ).

Multiplying the equation by $u_{\varepsilon,\delta}$ and integrating in x, the nonlinear term is a spatial derivative and the viscous term gives the usual dissipation. Since $u_{\varepsilon,\delta}$ is real-valued, the skew-adjointness of $\mathcal{H}$ gives

$$
\int_{\mathbb{R}} \partial_x u_{\varepsilon,\delta}\, \mathcal{H} \partial_x u_{\varepsilon,\delta}\,\mathrm{d}x = 0,
$$

so the dispersive term does not contribute to the energy identity. Consequently,

$$\|u_{\varepsilon,\delta}(t)\|^2_{L^2(\mathbb{R})} + 2\varepsilon \int_0^t \|\partial_x u_{\varepsilon,\delta}(s)\|^2_{L^2(\mathbb{R})}\, \mathrm{d}s = \|u_{0,\varepsilon,\delta}\|^2_{L^2(\mathbb{R})}. \tag{2.7}$$

Using also the $L^2(\mathbb{R})$-isometry of $\mathcal{H}$, we obtain

$$\begin{aligned} &\sup_{0\le t\le T} \|u_{\varepsilon,\delta}(t)\|_{L^2(\mathbb{R})} \le M, \\ &\int_0^T \|\partial_x u_{\varepsilon,\delta}(t)\|^2_{L^2(\mathbb{R})}\, \mathrm{d}t = \int_0^T \|\mathcal{H}\partial_x u_{\varepsilon,\delta}(t)\|^2_{L^2(\mathbb{R})}\, \mathrm{d}t \le \frac{M^2}{2\varepsilon}. \end{aligned} \tag{2.8}$$

These estimates first control the two densities A and D. Indeed,

$$\|A(t)\|_{L^1(\mathbb{R})} \le \|\chi\|_{L^2(\mathbb{R})} M, \qquad \|D(t)\|_{L^1(\mathbb{R})} \le \frac{M^2}{2}.$$

Recall that the interaction functional in Lemma 2.1 is

$$\mathcal{I}(t) = \iint_{x<y} A(t,x)D(t,y)\, \mathrm{d}x\, \mathrm{d}y.$$

It follows that

$$|\mathcal{I}(t)| \le \|A(t)\|_{L^1(\mathbb{R})} \|D(t)\|_{L^1(\mathbb{R})} \le \frac{\|\chi\|_{L^2(\mathbb{R})}}{2} M^3.$$

Thus the two time-endpoint values $\mathcal{I}(0)$ and $\mathcal{I}(T)$ are uniformly bounded.

We next estimate the two right-hand sides P and Q. For P, the quadratic part of $F_{\varepsilon,\delta}$ is controlled using $|\chi'| \le 1$ and the energy bound. The derivative terms are controlled using $\chi' \in L^2(\mathbb{R})$, followed by Cauchy–Schwarz in time. This gives

$$\begin{aligned} \|P\|_{L^1((0,T)\times\mathbb{R})} \le& \frac{TM^2}{2} + \varepsilon \|\chi'\|_{L^2(\mathbb{R})} T^{1/2} \left(\int_0^T \|\partial_x u_{\varepsilon,\delta}(t)\|^2_{L^2(\mathbb{R})}\, \mathrm{d}t \right)^{1/2} \\ &+ \delta \|\chi'\|_{L^2(\mathbb{R})} T^{1/2} \left(\int_0^T \|\mathcal{H}\partial_x u_{\varepsilon,\delta}(t)\|^2_{L^2(\mathbb{R})}\, \mathrm{d}t \right)^{1/2} \\ \le& \frac{TM^2}{2} + \frac{M \|\chi'\|_{L^2(\mathbb{R})} T^{1/2}}{\sqrt{2}} \left(\sqrt{\varepsilon} + \frac{\delta}{\sqrt{\varepsilon}} \right). \end{aligned}$$

Since $0 < \varepsilon \le 1$ and $\delta \le \Lambda\varepsilon$, this gives

$$\|P\|_{L^1((0,T)\times\mathbb{R})} \le C(T, \Lambda, M).$$

For Q, the $L^2(\mathbb{R})$-isometry of $\mathcal{H}$ and (2.8) give

$$\begin{aligned} \|Q\|_{L^1((0,T)\times\mathbb{R})} \le& \varepsilon \int_0^T \|\partial_x u_{\varepsilon,\delta}(t)\|^2_{L^2(\mathbb{R})}\, \mathrm{d}t \\ &+ \delta \int_0^T \|\partial_x u_{\varepsilon,\delta}(t)\|_{L^2(\mathbb{R})} \|\mathcal{H}\partial_x u_{\varepsilon,\delta}(t)\|_{L^2(\mathbb{R})}\, \mathrm{d}t \\ =& (\varepsilon + \delta) \int_0^T \|\partial_x u_{\varepsilon,\delta}(t)\|^2_{L^2(\mathbb{R})}\, \mathrm{d}t \\ \le& \frac{1+\Lambda}{2} M^2. \end{aligned}$$

Step 3. *Verification of the remaining interaction hypotheses.* The balance laws required by Lemma 2.1 were established in Step 1, and Step 2 proved that

$$P, Q \in L^1((0,T)\times\mathbb{R}).$$

It remains to verify

$$A, D \in C([0,T]; L^1(\mathbb{R})), \qquad B, E \in L^1((0,T)\times\mathbb{R}),$$

that

$$AE, BD \in L^1((0,T)\times\mathbb{R}),$$

and that the six fields have the required C^1 regularity on $(0,T)\times\mathbb{R}$. We verify these conditions in this order.

Continuity of the densities. The continuity of $u_{\varepsilon,\delta}$ into $L^2(\mathbb{R})$ gives

$$A, D \in C([0,T]; L^1(\mathbb{R})).$$

More precisely,

$$\|A(t)-A(s)\|_{L^1(\mathbb{R})} \le \|\chi\|_{L^2(\mathbb{R})} \|u_{\varepsilon,\delta}(t)-u_{\varepsilon,\delta}(s)\|_{L^2(\mathbb{R})},$$

while

$$\|D(t)-D(s)\|_{L^1(\mathbb{R})} \le \frac{1}{2}\left(\|u_{\varepsilon,\delta}(t)\|_{L^2(\mathbb{R})} + \|u_{\varepsilon,\delta}(s)\|_{L^2(\mathbb{R})}\right) \|u_{\varepsilon,\delta}(t)-u_{\varepsilon,\delta}(s)\|_{L^2(\mathbb{R})}.$$

Fixed-parameter integrability. The remaining conditions require only finiteness for each fixed pair (ε,δ), not estimates uniform in those parameters. We may therefore use the assumed continuity of $u_{\varepsilon,\delta}$ into $H^2(\mathbb{R})$ and the embedding $H^2(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$. In particular,

$$\int_0^T\int_{\mathbb{R}} \chi|u_{\varepsilon,\delta}|^4 \,\mathrm{d}x\,\mathrm{d}t \le T \sup_{0\le t\le T}\|u_{\varepsilon,\delta}(t)\|^2_{L^\infty(\mathbb{R})} \sup_{0\le t\le T}\|u_{\varepsilon,\delta}(t)\|^2_{L^2(\mathbb{R})} < \infty.$$

Integrability of the fluxes. Using the definitions of B, E, the preceding fixed-parameter $L^\infty(\mathbb{R})$ bound, and (2.8), we find

$$\begin{aligned}
\int_0^T\int_{\mathbb{R}} |B|\,\mathrm{d}x\,\mathrm{d}t &\le \frac{TM^2}{2} + (\varepsilon+\delta)\|\chi\|_{L^2(\mathbb{R})} T^{1/2}\left(\frac{M^2}{2\varepsilon}\right)^{1/2} < \infty,\\
\int_0^T\int_{\mathbb{R}} |E|\,\mathrm{d}x\,\mathrm{d}t &\le \frac{TM^2}{3}\sup_{0\le t\le T}\|u_{\varepsilon,\delta}(t)\|_{L^\infty(\mathbb{R})} + (\varepsilon+\delta)MT^{1/2}\left(\frac{M^2}{2\varepsilon}\right)^{1/2} < \infty.
\end{aligned}$$

Thus

$$B, E \in L^1((0,T)\times\mathbb{R}).$$

Integrability of the cross-products. From the definitions of A, B, D, E,

$$|AE| + |BD| \le \chi\left(\frac{7}{12}|u_{\varepsilon,\delta}|^4 + \frac{3\varepsilon}{2}|u_{\varepsilon,\delta}|^2|\partial_x u_{\varepsilon,\delta}| + \frac{3\delta}{2}|u_{\varepsilon,\delta}|^2|\mathcal{H}\partial_x u_{\varepsilon,\delta}|\right).$$

The fourth-power term is integrable by the fixed-parameter estimate above. The remaining terms are integrable because

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}} \chi|u_{\varepsilon,\delta}|^2\left(|\partial_x u_{\varepsilon,\delta}| + |\mathcal{H}\partial_x u_{\varepsilon,\delta}|\right)\mathrm{d}x\,\mathrm{d}t\\
&\quad\le \sup_{0\le t\le T}\|u_{\varepsilon,\delta}(t)\|_{L^\infty(\mathbb{R})} MT^{1/2}\\
&\qquad\times\left[\left(\int_0^T \|\partial_x u_{\varepsilon,\delta}(t)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}t\right)^{1/2} + \left(\int_0^T \|\mathcal{H}\partial_x u_{\varepsilon,\delta}(t)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}t\right)^{1/2}\right] < \infty.
\end{aligned}$$

Therefore

$$AE, BD \in L^1((0,T)\times\mathbb{R}).$$

Interior regularity. Positive-time smoothing gives

$$A, B, D, E, P, Q \in C^1((0,T)\times\mathbb{R}).$$

Together with the balance laws from Step 1, the uniform $L^1((0,T)\times\mathbb{R})$ bounds for P, Q from Step 2, and the continuity and integrability properties verified above, this establishes every hypothesis of Lemma 2.1.

We may now apply (2.5). The first two terms on its right-hand side are the time-endpoint values $\mathcal{I}(0)$ and $\mathcal{I}(T)$; the remaining terms involve the localization error P and the entropy-production term Q. Thus

$$
\begin{aligned}
\left|\int_0^T\int_{\mathbb{R}}(AE-BD)\,\mathrm{d}x\,\mathrm{d}t\right| \leq& |\mathcal{I}(0)|+|\mathcal{I}(T)| \\
&+\sup_{0\leq t\leq T}\|D(t)\|_{L^1(\mathbb{R})}\,\|P\|_{L^1((0,T)\times\mathbb{R})} \\
&+\sup_{0\leq t\leq T}\|A(t)\|_{L^1(\mathbb{R})}\,\|Q\|_{L^1((0,T)\times\mathbb{R})} \\
\leq& C(T,\Lambda,M).
\end{aligned} \tag{2.9}
$$

Step 4. *The interaction determinant.* We now compute the determinant whose integral is controlled by (2.9):

$$
\begin{aligned}
AE-BD &= \chi\left(u_{\varepsilon,\delta}G_{\varepsilon,\delta}-\frac{u_{\varepsilon,\delta}^2}{2}F_{\varepsilon,\delta}\right) \\
&= \chi\left(\frac{u_{\varepsilon,\delta}^4}{12}-\frac{\varepsilon}{2}u_{\varepsilon,\delta}^2\partial_x u_{\varepsilon,\delta}+\frac{\delta}{2}u_{\varepsilon,\delta}^2\mathcal{H}\partial_x u_{\varepsilon,\delta}\right).
\end{aligned} \tag{2.10}
$$

The fixed-parameter integrability established in Step 3 shows that every term in (2.10) is integrable.

Step 5. *Control of the error terms and conclusion.* For the viscous remainder, we use

$$u_{\varepsilon,\delta}^2\partial_x u_{\varepsilon,\delta}=\frac{1}{3}\partial_x\left(u_{\varepsilon,\delta}^3\right)$$

and integrate by parts so that the derivative falls on the weight. For each $t\in[0,T]$, the inclusion $u_{\varepsilon,\delta}(t)\in H^2(\mathbb{R})$, together with the properties of χ, implies

$$\chi u_{\varepsilon,\delta}^3\in W^{1,1}(\mathbb{R}).$$

Hence its limits at both ends of $\mathbb{R}$ vanish, and the integration by parts produces no boundary term. Using $|\chi'|\leq\chi$ and the Cauchy–Schwarz inequality, we obtain

$$
\begin{aligned}
\frac{\varepsilon}{2}\left|\int_0^T\int_{\mathbb{R}}\chi u_{\varepsilon,\delta}^2\partial_x u_{\varepsilon,\delta}\,\mathrm{d}x\,\mathrm{d}t\right| =& \frac{\varepsilon}{6}\left|\int_0^T\int_{\mathbb{R}}\chi' u_{\varepsilon,\delta}^3\,\mathrm{d}x\,\mathrm{d}t\right| \\
\leq& \frac{\varepsilon}{6}\left(\int_0^T\int_{\mathbb{R}}\chi|u_{\varepsilon,\delta}|^4\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}\left(\int_0^T\int_{\mathbb{R}}\chi|u_{\varepsilon,\delta}|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2} \\
\leq& \frac{\varepsilon M T^{1/2}}{6}\left(\int_0^T\int_{\mathbb{R}}\chi|u_{\varepsilon,\delta}|^4\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}.
\end{aligned} \tag{2.11}
$$

For the dispersive remainder, we use spacetime Cauchy–Schwarz. Since $0<\chi\leq 1$, the weighted $L^2((0,T)\times\mathbb{R})$ norm of $\mathcal{H}\partial_x u_{\varepsilon,\delta}$ is controlled by the energy estimate. Hence

$$
\begin{aligned}
\frac{\delta}{2}\left|\int_0^T\int_{\mathbb{R}}\chi u_{\varepsilon,\delta}^2\mathcal{H}\partial_x u_{\varepsilon,\delta}\,\mathrm{d}x\,\mathrm{d}t\right| \leq& \frac{\delta}{2}\left(\int_0^T\int_{\mathbb{R}}\chi|u_{\varepsilon,\delta}|^4\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}\left(\int_0^T\int_{\mathbb{R}}\chi|\mathcal{H}\partial_x u_{\varepsilon,\delta}|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2} \\
\leq& \frac{\delta M}{2\sqrt{2\varepsilon}}\left(\int_0^T\int_{\mathbb{R}}\chi|u_{\varepsilon,\delta}|^4\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}.
\end{aligned} \tag{2.12}
$$

Substituting the determinant formula (2.10) into the interaction estimate (2.9), and then using (2.11) and (2.12), gives

$$\frac{1}{12}\int_0^T\int_{\mathbb{R}}\chi|u_{\varepsilon,\delta}|^4\,\mathrm{d}x\,\mathrm{d}t\leq C(T,\Lambda,M)+\left(\frac{\varepsilon M T^{1/2}}{6}+\frac{\delta M}{2\sqrt{2\varepsilon}}\right)\left(\int_0^T\int_{\mathbb{R}}\chi|u_{\varepsilon,\delta}|^4\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}.$$

The parameter restriction is used once more here. Since $0<\varepsilon\leq 1$ and $\delta\leq\Lambda\varepsilon$,

$$\frac{\varepsilon M T^{1/2}}{6}+\frac{\delta M}{2\sqrt{2\varepsilon}}\leq\frac{M T^{1/2}}{6}+\frac{\Lambda M}{2\sqrt{2}}.$$

Up to modifying the constant $C(T,\Lambda,M)>0$, we deduce that

$$\int_0^T\int_{\mathbb{R}}\chi|u_{\varepsilon,\delta}|^4\,\mathrm{d}x\,\mathrm{d}t\le C(T,\Lambda,M)\left(1+\left(\int_0^T\int_{\mathbb{R}}\chi|u_{\varepsilon,\delta}|^4\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}\right).$$

Since $X\le C(1+X^{1/2})$ implies $X\le 2C+C^2$ for $X\ge 0$, we conclude, up once again to modifying the constant, that

$$\int_0^T\int_{\mathbb{R}}\chi|u_{\varepsilon,\delta}|^4\,\mathrm{d}x\,\mathrm{d}t\le C(T,\Lambda,M). \tag{2.13}$$

In conclusion, using now that for every non-empty compact set $K\subset\mathbb{R}$,

$$\inf_{x\in K}\chi(x)>0,$$

we conclude that

$$\int_0^T\int_K|u_{\varepsilon,\delta}|^4\,\mathrm{d}x\,\mathrm{d}t\le\frac{1}{\inf_{x\in K}\chi(x)}\int_0^T\int_{\mathbb{R}}\chi|u_{\varepsilon,\delta}|^4\,\mathrm{d}x\,\mathrm{d}t\le C(T,K,\Lambda,M),$$

which proves (1.7). □

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