

STRONG NONLOCAL-TO-LOCAL CONVERGENCE FOR GENERAL TRAFFIC MODELS

NICOLA DE NITTI AND HARSH PRASAD

ABSTRACT. We consider a nonlocal conservation law for traffic flow in which drivers adjust their speed to a weighted average of the density ahead. We prove that, as the observation scale tends to zero, the nonlocal averages converge strongly in L^p_{loc} , with $1 \leq p < \infty$, to the entropy solution of the local Lighthill–Whitham–Richards model. The initial density is any bounded function, the velocity is only required to be Lipschitz and non-increasing, and the kernel is anisotropic, non-negative, normalized, monotone, and of bounded variation. The main point of the proof is that the drivers’ speed does not oscillate on the observation scale. This is shown by a blow-up argument and a Liouville property of the Burgers equation, and it allows us to pass to the limit in the entropy inequalities.

1. INTRODUCTION AND MAIN RESULTS

Traffic models based on conservation of vehicles relate the flux to the density and to the speed of the drivers. In the local models of Lighthill–Whitham and Richards [LW55; Ric56], the speed depends on the density at the same position. We consider a nonlocal variant in which drivers respond to an average of the traffic ahead:

$$(1.1) \quad \begin{cases} \partial_t u_\varepsilon + \partial_x (u_\varepsilon V(w_\varepsilon)) = 0, & t > 0, x \in \mathbb{R}, \\ u_\varepsilon(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

where, for an observation scale $\varepsilon > 0$, w_ε is the nonlocal average of the density

$$(1.2) \quad \begin{aligned} w_\varepsilon(t, x) &:= (\gamma_\varepsilon * u_\varepsilon)(t, x) = \int_{-\infty}^0 \gamma(y) u_\varepsilon(t, x - \varepsilon y) dy, \\ \gamma_\varepsilon(x) &:= \varepsilon^{-1} \gamma(x/\varepsilon), \end{aligned}$$

the convolution being in space. The initial datum satisfies

$$(1.3) \quad u_0 \in L^\infty(\mathbb{R}), \quad 0 \leq u_0 \leq 1.$$

These bounds represent normalized traffic states: zero corresponds to an empty road and one to bumper-to-bumper traffic. The velocity is Lipschitz and non-increasing,

$$(1.4) \quad V \in W^{1,\infty}([0, 1]), \quad V'(\xi) \leq 0 \quad \text{for almost every } \xi \in [0, 1].$$

We extend V constantly beyond the endpoints when a function on $\mathbb{R}$ is needed. No sign condition on V is imposed. The kernel satisfies

$$(1.5) \quad \begin{aligned} \gamma &\in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \quad \gamma \geq 0, \quad \int_{\mathbb{R}} \gamma(z) dz = 1, \\ \text{ess supp } \gamma &\subset (-\infty, 0], \quad \gamma \text{ is non-decreasing on } (-\infty, 0). \end{aligned}$$

The support condition means that drivers look downstream; monotonicity gives greater weight to traffic closer to their position.

Under (1.3), (1.4) and (1.5), (1.1) has a unique bounded distributional solution, with

$$(1.6) \quad 0 \leq u_\varepsilon, w_\varepsilon \leq 1, \quad u_\varepsilon \in C([0, \infty); L^1_{\text{loc}}(\mathbb{R})).$$

This follows from [Col+23, Proposition 2.1(i)–(ii)]. By [Col+23, Corollary 2.2(i)–(ii)], w_ε has a Lipschitz representative with initial value $\gamma_\varepsilon * u_0$.

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The question is whether the density perceived over a shrinking observation region approaches the *entropy solution* of the local model

$$(1.7) \quad \begin{cases} \partial_t u + \partial_x f(u) = 0, & t > 0, x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

where

$$(1.8) \quad f(\xi) := \xi V(\xi), \quad \xi \in [0, 1].$$

The entropy solution of (1.7) is the unique weak solution with values in $[0, 1]$ satisfying, for every $k \in [0, 1]$ and every non-negative $\varphi \in C_c^1([0, \infty) \times \mathbb{R})$,

$$(1.9) \quad \int_0^\infty \int_{\mathbb{R}} [u - k] \partial_t \varphi + \operatorname{sgn}(u - k) (f(u) - f(k)) \partial_x \varphi \, dx \, dt + \int_{\mathbb{R}} |u_0(x) - k| \varphi(0, x) \, dx \geq 0.$$

The formal limit of the averaging operator does not by itself justify convergence of the solutions of (1.1) to the entropy solution of (1.7). Indeed, while numerical experiments by Amorim, Colombo, and Teixeira [ACT15, Section 3.3, Figures 5–6], with symmetric interaction kernels, suggested such convergence as the kernels concentrated at the origin, Colombo, Crippa, and Spinolo [CCS19, Section 5] subsequently constructed counterexamples to convergence in general.

Colombo, Crippa, Marconi, and Spinolo [Col+21] proved convergence for initial data of bounded variation, bounded away from zero, with a lower one-sided bound on the initial slope, under further restrictions on the velocity and kernel. They also showed that allowing vacuum can destroy uniform total variation bounds for the densities. Earlier, for monotone initial data and under additional assumptions on the velocity and on the one-sided kernel, Keimer and Pflug [KP19] proved strong $L_{\text{loc}}^1((0, T) \times \mathbb{R})$ convergence of the nonlocal density to the entropy solution using preservation of spatial monotonicity. Oleřnik-type estimates, which give one-sided bounds on spatial difference quotients at positive times, provide another compactness mechanism [Coc+24].

Bressan and Shen [BS20; BS21] treated exponential kernels of the form $\gamma_\varepsilon(y) := \varepsilon^{-1} e^{y/\varepsilon} \mathbf{1}_{(-\infty, 0]}(y)$, and uniformly positive data of bounded variation. The identity $\varepsilon \partial_x w_\varepsilon = w_\varepsilon - u_\varepsilon$ allows them to rewrite the averaging relation as an auxiliary differential equation and obtain a hyperbolic relaxation system. For the same class of kernels, in [Coc+23, Theorems 3.2 and 4.2], Coclite, Coron, De Nitti, Keimer, and Pflug proved convergence of both u_ε and w_ε for initial data of bounded variation, assuming that f is strictly concave on the interval spanned by the initial density. Their proof uses a uniform bound on the total variation of w_ε . For kernels convex on $(-\infty, 0)$, Colombo, Crippa, Marconi, and Spinolo [Col+23, Theorems 1.1 and 1.3] proved total variation estimates and convergence of the averages for initial data of bounded variation, with a convergence rate when the kernel has a finite first moment. Later, De Nitti and Huang [DH26, Theorem 1.1] extended strong convergence to the densities themselves by treating the relation between the density and the derivative of its average as a Volterra integral equation. In [DH, Theorem 2.1], they also considered an entropy-consistent numerical scheme and proved that its discrete nonlocal averages converge to the local entropy solution as the mesh size and observation scale tend to zero along arbitrary paths.

Keimer and Pflug [KP26, Theorem 4.5 and Remark 4.6] proved weak-* convergence of the densities and their nonlocal averages to the entropy solution under (1.3), (1.4) and (1.5). They studied spatial primitives and the Hamilton–Jacobi equations obtained by integrating the conservation law in space and identified the weak limit without establishing strong convergence of the averages.

More recently, Coclite, De Nitti, and Huang [CDH25] proved convergence for every kernel in (1.5) when V is affine, and for general V when the averaging measure $\gamma(z) \, dz$ is absolutely continuous with respect to the positive part of the distributional derivative $D\gamma$, with a genuine nonlinearity condition on f in the latter case.

We use the entropy balance derived in [CDH25] to prove the following strong convergence result.

Theorem 1.1. *Assume (1.3), (1.4) and (1.5). Let u_ε solve (1.1), let w_ε be given by (1.2), and let u be the entropy solution of (1.7). Then, as $\varepsilon \downarrow 0$,*

$$(1.10) \quad w_\varepsilon \longrightarrow u \quad \text{strongly in } L_{\text{loc}}^p([0, \infty) \times \mathbb{R}) \quad (1 \leq p < \infty),$$

$$(1.11) \quad u_\varepsilon \xrightarrow{*} u \quad \text{in } L^\infty((0, T) \times \mathbb{R}) \quad (\text{for all } T > 0).$$

The new conclusion is strong convergence for every velocity and kernel satisfying (1.4) and (1.5), without a genuine-nonlinearity condition on f or an absolute-continuity relation between the kernel and its derivative.

Outline of the proof. Our proof combines a rescaling argument with Young measures. We pass to the limit in the full family of entropy inequalities and use a comparison theorem for the resulting Young measures (somewhat in the spirit of [EGH00]). This replaces the compensated compactness argument of [CDH25], which uses relations between weak limits of products of entropy and flux quantities to exclude oscillations.

The entropy balance from [CDH25], recalled in Section 2 below, consists of a non-negative dissipation and a remainder in divergence form. The dissipation controls velocity differences against the kernel derivative, whereas the remainder uses the kernel itself. To bridge this gap, we show that velocity oscillations over space–time displacements of order ε vanish locally in measure (see Proposition 3.2). We argue by contradiction: if they did not, we could find centers where the oscillation persists but the dissipation is small, and rescaling around them would produce a Lipschitz solution W of $\partial_\tau W + V(W)\partial_x W = 0$ on the whole plane. Its velocity $V(W)$ solves Burgers' equation and is therefore constant by the rigidity result of [DOW03, Proposition 6(b)] (recalled in Lemma 3.1), which contradicts the persistence of oscillations at the center.

This estimate makes the entropy remainder vanish. The Young measure generated by w_ε then satisfies the local entropy inequalities with initial measure $\delta_{u_0(x)}$. The comparison theorem in Lemma 4.1 (see [BGŠ17, Theorem 3.1]), following [DiP85], identifies it with $\delta_{u(t,x)}$ and yields strong convergence.

The rescaling argument and the use of rigidity for solutions with zero entropy production were inspired by De Lellis, Otto, and Westdickenberg [DOW03, Section 4.3 and Proposition 6(b)].

2. THE ENTROPY BALANCE

We work with the non-decreasing right-continuous representative of γ on $(-\infty, 0)$. Its positive Stieltjes derivative is the finite measure

$$(2.1) \quad \begin{aligned} \mu_\gamma &:= D\gamma|_{(-\infty, 0)}, & \mu_\gamma((-\infty, 0)) &= \gamma(0^-), \\ D\gamma &= \mu_\gamma - \gamma(0^-)\delta_0, & 0 < \gamma(0^-) &< \infty. \end{aligned}$$

Here $D\gamma$ denotes the distributional derivative of the kernel extended by zero on $(0, \infty)$, and δ_0 is the Dirac delta at zero.

For $\eta \in C^2([0, 1])$, we define the *entropy flux* $q_\eta(r) := \int_0^r \eta'(z)(V(z) + zV'(z)) dz$, characterized by $q'_\eta = \eta' f'$, and set

$$(2.2) \quad H_\eta(a | b) := \int_a^b \eta''(z)(V(z) - V(b)) dz,$$

which is non-negative when η is convex, since V is non-increasing.

The entropy balance of [CDH25, Proposition 3.1] reads

$$(2.3) \quad \partial_t \eta(w_\varepsilon) + \partial_x q_\eta(w_\varepsilon) = -\partial_x R_{\varepsilon, \eta} - D_{\varepsilon, \eta}$$

in the sense of distributions on $(0, \infty) \times \mathbb{R}$, where

$$(2.4) \quad \begin{aligned} R_{\varepsilon, \eta}(t, x) &:= \int_{-\infty}^0 \gamma(y) u_\varepsilon(t, x - \varepsilon y) \int_{w_\varepsilon(t, x)}^{w_\varepsilon(t, x - \varepsilon y)} \eta'(r) V'(r) dr dy, \\ D_{\varepsilon, \eta}(t, x) &:= \frac{1}{\varepsilon} \int_{(-\infty, 0)} u_\varepsilon(t, x - \varepsilon y) H_\eta(w_\varepsilon(t, x) | w_\varepsilon(t, x - \varepsilon y)) d\mu_\gamma(y). \end{aligned}$$

We note that $D_{\varepsilon, \eta} \geq 0$ when η is convex, and that $R_{\varepsilon, \eta}$ is bounded by $\|\eta'\|_{L^\infty([0, 1])}(V(0) - V(1))$. Since w_ε is Lipschitz on $[0, \infty) \times \mathbb{R}$ with $w_\varepsilon(0, \cdot) = \gamma_\varepsilon * u_0$, (2.3) can also be tested against non-negative $\varphi \in C_c^1([0, \infty) \times \mathbb{R})$ that do not vanish at $t = 0$ (truncate in time and use monotone convergence for the dissipation); the initial trace then appears as $\int_{\mathbb{R}} \eta(\gamma_\varepsilon * u_0) \varphi(0, x) dx$. We use (2.3) in this form below.

For the quadratic entropy $\eta_2(\xi) := \xi^2/2$, we have, for every compact rectangle $Q \subset [0, \infty) \times \mathbb{R}$,

$$(2.5) \quad \int_Q D_{\varepsilon, \eta_2}(t, x) dx dt \leq C_Q.$$

The bound follows by testing (2.3) with a non-negative $\varphi \in C_c^1([0, \infty) \times \mathbb{R})$ equal to one on Q : all other terms, including the initial trace when Q meets $\{t = 0\}$, are bounded uniformly in ε , so C_Q depends only on Q , V , and γ .

Moreover, by [CDH25, Eq. (4.4)],

$$(2.6) \quad |V(a) - V(b)|^2 \leq 2\|V'\|_{L^\infty([0,1])} H_{\eta_2}(a | b) \quad (a, b \in [0, 1]).$$

Differentiating the spatial convolutions using (2.1) gives (cf. [Col+23, proof of Corollary 2.2])

$$(2.7) \quad \begin{aligned} \varepsilon \partial_x w_\varepsilon(t, x) &= \int_{(-\infty, 0)} u_\varepsilon(t, x - \varepsilon y) d\mu_\gamma(y) - \gamma(0^-) u_\varepsilon(t, x), \\ \varepsilon \partial_t w_\varepsilon(t, x) &= - \int_{(-\infty, 0)} u_\varepsilon(t, x - \varepsilon y) V(w_\varepsilon(t, x - \varepsilon y)) d\mu_\gamma(y) + \gamma(0^-) u_\varepsilon(t, x) V(w_\varepsilon(t, x)). \end{aligned}$$

These identities hold in the sense of distributions and almost everywhere. In particular, w_ε has a Lipschitz representative, and

$$(2.8) \quad \begin{aligned} \varepsilon \|\partial_x w_\varepsilon\|_{L^\infty((0, \infty) \times \mathbb{R})} &\leq \gamma(0^-), \\ \varepsilon \|\partial_t w_\varepsilon\|_{L^\infty((0, \infty) \times \mathbb{R})} &\leq \gamma(0^-) (\max\{V(0), 0\} - \min\{V(1), 0\}). \end{aligned}$$

The second bound follows because $u_\varepsilon V(w_\varepsilon)$ lies between $\min\{V(1), 0\}$ and $\max\{V(0), 0\}$. Finally, from (2.7) we get

$$(2.9) \quad \begin{aligned} \varepsilon (\partial_t w_\varepsilon + V(w_\varepsilon) \partial_x w_\varepsilon)(t, x) \\ = \int_{(-\infty, 0)} u_\varepsilon(t, x - \varepsilon y) [V(w_\varepsilon(t, x)) - V(w_\varepsilon(t, x - \varepsilon y))] d\mu_\gamma(y). \end{aligned}$$

3. VANISHING VELOCITY OSCILLATION ON THE OBSERVATION SCALE

The argument of this section rests on the following rigidity result, which is a special case of [DOW03, Definition 5 and Proposition 6(b)].

Lemma 3.1 (Rigidity for entire Lipschitz Burgers solutions). *If $Z \in W^{1, \infty}(\mathbb{R}^2)$ satisfies*

$$\partial_\tau Z + \partial_x \left(\frac{Z^2}{2} \right) = 0 \quad \text{in the sense of distributions on } \mathbb{R}^2,$$

then Z is constant.

The remainder $R_{\varepsilon, \eta}$ in (2.4) is an average, against the kernel, of velocity differences $V(w_\varepsilon(t, x - \varepsilon y)) - V(w_\varepsilon(t, x))$ over shifts of order ε . The dissipation bound (2.5) controls such differences only against the kernel derivative μ_γ , not against γ itself. The next proposition closes this gap: it shows that the oscillation of $V(w_\varepsilon)$ over space–time displacements of order ε tends to zero locally in measure, which will give $R_{\varepsilon, \eta} \rightarrow 0$ in L^1_{loc} (see Step 1 of the proof of Theorem 1.1).

Proposition 3.2. *Assume (1.3), (1.4) and (1.5). For $M > 0$ and $t > \varepsilon M$, define*

$$(3.1) \quad \omega_{\varepsilon, M}(t, x) := \sup_{|\tau| \leq M, |y| \leq M} |V(w_\varepsilon(t + \varepsilon \tau, x + \varepsilon y)) - V(w_\varepsilon(t, x))|.$$

For every compact $K \subset (0, \infty) \times \mathbb{R}$ and every $1 \leq p < \infty$,

$$(3.2) \quad \lim_{\varepsilon \downarrow 0} \int_K \omega_{\varepsilon, M}(t, x)^p dx dt = 0,$$

where $\omega_{\varepsilon, M}$ is defined on K as soon as εM is smaller than the lower time of K .

Proof. Step 1: Selection of centers with small dissipation. The function in (3.1) is continuous and bounded by $V(0) - V(1)$. It therefore suffices to prove that it tends to zero in measure on every compact rectangle $Q \subset (0, \infty) \times \mathbb{R}$. If this fails, there are $a, b > 0$, a sequence $\varepsilon_n \downarrow 0$, and sets

$$E_n := \{(t, x) \in Q : \omega_{\varepsilon_n, M}(t, x) \geq a\}, \quad |E_n| \geq b.$$

We choose a compact rectangle $Q' \subset (0, \infty) \times \mathbb{R}$ whose interior contains Q . For large n , every square centered in Q with coordinate radius $\varepsilon_n^{3/4}$ lies in Q' . For $c \in Q$, set

$$F_n(c) := \frac{1}{\varepsilon_n^2} \int_{c + [-\varepsilon_n^{3/4}, \varepsilon_n^{3/4}]^2} \varepsilon_n D_{\varepsilon_n, \eta_2}(t, x) dx dt.$$

Each point of Q' lies in the squares of a set of centers of area at most $4\varepsilon_n^{3/2}$, so Fubini's theorem and (2.5) give

$$\int_{E_n} F_n(c) \, dc \leq \frac{4\varepsilon_n^{3/2}}{\varepsilon_n} C_{Q'} = 4C_{Q'} \varepsilon_n^{1/2}.$$

Since $|E_n| \geq b$, there is $(t_n, x_n) \in E_n$ with

$$(3.3) \quad F_n(t_n, x_n) \leq \frac{4C_{Q'}}{b} \varepsilon_n^{1/2}.$$

Step 2: The rescaled profile. Define

$$(3.4) \quad \begin{aligned} U_n(\tau, x) &:= u_{\varepsilon_n}(t_n + \varepsilon_n \tau, x_n + \varepsilon_n x), \\ W_n(\tau, x) &:= w_{\varepsilon_n}(t_n + \varepsilon_n \tau, x_n + \varepsilon_n x), \end{aligned}$$

for $\tau > -t_n/\varepsilon_n$. Since t_n is bounded below by the lower time of Q , which is positive, every bounded interval of τ is contained in this range for all large n . By (2.8), the functions W_n have uniform Lipschitz constants and take values in $[0, 1]$. The Arzelà–Ascoli theorem and a diagonal argument give $W_n \rightarrow W$ locally uniformly on $\mathbb{R}^2$ along a subsequence (not relabeled), with $W \in W^{1,\infty}(\mathbb{R}^2)$.

By (2.9),

$$(3.5) \quad \begin{aligned} &\partial_\tau W_n + V(W_n) \partial_x W_n \\ &= \int_{(-\infty, 0)} U_n(\tau, x - y) [V(W_n(\tau, x)) - V(W_n(\tau, x - y))] \, d\mu_\gamma(y). \end{aligned}$$

Define the rescaled dissipation density

$$\begin{aligned} \mathcal{D}_n(\tau, x) &:= \int_{(-\infty, 0)} U_n(\tau, x - y) H_{\eta_2}(W_n(\tau, x) \mid W_n(\tau, x - y)) \, d\mu_\gamma(y) \\ &= \varepsilon_n D_{\varepsilon_n, \eta_2}(t_n + \varepsilon_n \tau, x_n + \varepsilon_n x). \end{aligned}$$

The Cauchy–Schwarz inequality, $\mu_\gamma((-\infty, 0)) = \gamma(0^-)$, $0 \leq U_n \leq 1$, and (2.6) imply

$$(3.6) \quad |\partial_\tau W_n + V(W_n) \partial_x W_n|^2 \leq 2\gamma(0^-) \|V'\|_{L^\infty([0,1])} \mathcal{D}_n(\tau, x).$$

By the change of variables (3.4),

$$F_n(t_n, x_n) = \int_{[-\varepsilon_n^{-1/4}, \varepsilon_n^{-1/4}]^2} \mathcal{D}_n(\tau, x) \, dx \, d\tau,$$

and every compact set is eventually contained in this square. By (3.3) and (3.6), the right side of (3.5) tends to zero in $L^2_{\text{loc}}(\mathbb{R}^2)$. Writing the left side as $\partial_\tau W_n + \partial_x \int_0^{W_n} V(r) \, dr$ and using local uniform convergence, we obtain in the limit

$$(3.7) \quad \partial_\tau W + V(W) \partial_x W = 0 \quad \text{almost everywhere on } \mathbb{R}^2.$$

Step 3: The contradiction. Set $Z := V(W)$. The Lipschitz chain rule applied to (3.7) gives

$$\partial_\tau Z + \partial_x(Z^2/2) = 0 \quad \text{on } \mathbb{R}^2.$$

Since Z is bounded and Lipschitz on the whole plane, Lemma 3.1 implies that it is constant. Local uniform convergence then gives

$$\omega_{\varepsilon_n, M}(t_n, x_n) = \sup_{|\tau| \leq M, |x| \leq M} |V(W_n(\tau, x)) - V(W_n(0, 0))| \longrightarrow 0,$$

which contradicts $(t_n, x_n) \in E_n$. □

4. ENTROPY COMPARISON AND STRONG CONVERGENCE

We now assemble the proof of Theorem 1.1. Proposition 3.2 makes the remainder $R_{\varepsilon, \eta}$ vanish in L^1_{loc} , so that the entropy balance (2.3) passes to the limit along the Young measure generated by w_ε . The following reduction lemma, a consequence of [BGŚ17, Theorem 3.1, Eq. (3.2)],¹ then

¹ We apply [BGŚ17, Theorem 3.1] to ν and δ_v , with identity parametrization, zero source, and a Lipschitz extension of f . For $\rho(x) = e^{-\sqrt{1+x^2}}$ and $L = \|f'\|_{L^\infty([0,1])}$, set

$$A(t) = \int_{\mathbb{R}} \rho(x) \int_{[0,1]} |\lambda - v(t, x)| \, d\nu_{t,x}(\lambda) \, dx.$$

forces this Young measure to be the Dirac mass at the entropy solution, which amounts to strong convergence.

Lemma 4.1 (Entropy measure-valued reduction). *Let $T > 0$, $f \in W^{1,\infty}([0, 1])$, and let $v_0 : \mathbb{R} \rightarrow [0, 1]$ be measurable. Let v be the entropy solution of (1.7) with flux f and initial datum v_0 . Suppose that a measurable family of probability measures $\nu_{t,x}$ on $[0, 1]$, for $(t, x) \in (0, T) \times \mathbb{R}$, satisfies*

$$(4.1) \quad \int_0^T \int_{\mathbb{R}} \int_{[0,1]} [\eta(\lambda) \partial_t \varphi + q_\eta(\lambda) \partial_x \varphi] d\nu_{t,x}(\lambda) dx dt + \int_{\mathbb{R}} \eta(v_0(x)) \varphi(0, x) dx \geq 0$$

for every convex $\eta \in C^2([0, 1])$, with $q'_\eta = \eta' f'$, and every non-negative $\varphi \in C_c^1([0, T) \times \mathbb{R})$. Then $\nu_{t,x} = \delta_{v(t,x)}$ for almost every $(t, x) \in (0, T) \times \mathbb{R}$.

Proof of Theorem 1.1. Step 1: Vanishing of the entropy remainder. Since $V' \leq 0$, for every $\eta \in C^2([0, 1])$,

$$\left| \int_a^b \eta'(r) V'(r) dr \right| \leq \|\eta'\|_{L^\infty([0,1])} |V(a) - V(b)|.$$

For $t > \varepsilon M$, splitting the kernel integral at $-M$ therefore gives

$$(4.2) \quad |R_{\varepsilon,\eta}(t, x)| \leq \|\eta'\|_{L^\infty([0,1])} \left(\omega_{\varepsilon,M}(t, x) + (V(0) - V(1)) \int_{-\infty}^{-M} \gamma(y) dy \right).$$

Integrating over a compact $K \subset (0, \infty) \times \mathbb{R}$, letting first $\varepsilon \downarrow 0$ by Proposition 3.2 and then $M \rightarrow \infty$, gives $R_{\varepsilon,\eta} \rightarrow 0$ in $L^1(K)$. Since $|R_{\varepsilon,\eta}| \leq \|\eta'\|_{L^\infty([0,1])} (V(0) - V(1))$ everywhere, its integral over $[0, \delta] \times I$ is bounded by a constant times $\delta|I|$ for every bounded interval I . Letting $\delta \downarrow 0$ proves convergence in $L^1_{\text{loc}}([0, \infty) \times \mathbb{R})$.

Step 2: The limiting entropy inequalities. For every bounded interval I , the continuity of translations in $L^1_{\text{loc}}(\mathbb{R})$ and Lebesgue's dominated convergence theorem give

$$(4.3) \quad \|\gamma_\varepsilon * u_0 - u_0\|_{L^1(I)} \leq \int_{-\infty}^0 \gamma(y) \|u_0(\cdot - \varepsilon y) - u_0\|_{L^1(I)} dy \rightarrow 0.$$

Given any sequence $\varepsilon_n \downarrow 0$, the compact-range form of the Young-measure theorem in [Hun97, Theorem 1] gives a subsequence and a measurable family of probability measures $\nu_{t,x}$ on $[0, 1]$ such that, for every continuous $G : [0, 1] \rightarrow \mathbb{R}$,

$$(4.4) \quad G(w_{\varepsilon_n}) \xrightarrow{*} \int_{[0,1]} G(\lambda) d\nu_{t,x}(\lambda) \quad \text{in } L^\infty_{\text{loc}}([0, \infty) \times \mathbb{R}).$$

Fix $T > 0$ and test (2.3) against a non-negative $\varphi \in C_c^1([0, T) \times \mathbb{R})$, with the initial trace $\eta(\gamma_\varepsilon * u_0)$ as explained in Section 2. For convex $\eta \in C^2([0, 1])$, discard the non-negative dissipation and pass to the limit along the subsequence using Step 1 and (4.3) and (4.4). This gives (4.1) with $v_0 = u_0$.

Step 3: Comparison with the entropy solution. The flux $f(\xi) = \xi V(\xi)$ is Lipschitz on $[0, 1]$. Hence Lemma 4.1 yields $\nu_{t,x} = \delta_{u(t,x)}$ almost everywhere on $(0, T) \times \mathbb{R}$. Since T is arbitrary, this holds on $(0, \infty) \times \mathbb{R}$.

Step 4: Strong convergence of the averages. By [Hun97, Proposition 1], the extracted subsequence converges to u locally in measure. The bound $0 \leq w_\varepsilon \leq 1$ yields strong convergence in L^p_{loc} for every $1 \leq p < \infty$. By uniqueness of the entropy solution, every sequence $\varepsilon_n \downarrow 0$ has a subsequence along which $w_{\varepsilon_n} \rightarrow u$; by Urysohn's subsequence principle, the whole family converges as in (1.10).

The weak-* convergence in (1.11) follows from (1.10). For $\varphi \in L^1((0, T) \times \mathbb{R})$, Fubini's theorem gives

$$\int_0^T \int_{\mathbb{R}} (u_\varepsilon - w_\varepsilon) \varphi dx dt = \int_0^T \int_{\mathbb{R}} u_\varepsilon (\varphi - \tilde{\gamma}_\varepsilon * \varphi) dx dt, \quad \tilde{\gamma}_\varepsilon(x) := \gamma_\varepsilon(-x),$$

and the right side tends to zero because $0 \leq u_\varepsilon \leq 1$ and $\tilde{\gamma}_\varepsilon * \varphi \rightarrow \varphi$ in $L^1((0, T) \times \mathbb{R})$. Since the bounded family w_ε converges weakly-* to u by (1.10), so does u_ε . $\square$

Testing with compact approximations of ρ , using $|\rho'| \leq \rho$ and $|f(\lambda) - f(v)| \leq L|\lambda - v|$, gives $\partial_t(e^{-Lt}A) \leq 0$ in distributions on $(-\infty, T)$, with $A = 0$ at negative times because the initial data coincide. Since $A \geq 0$, necessarily $A = 0$.

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(N. De Nitti) POLITECNICO DI BARI, DIPARTIMENTO DI MECCANICA, MATEMATICA E MANAGEMENT, VIA E. ORABONA 4, 70125 BARI, ITALY.
Email address: nicola.denitti@poliba.it

(H. Prasad) UNIVERSITÄT BIELEFELD, FAKULTÄT FÜR MATHEMATIK, 33615 BIELEFELD, GERMANY.
Email address: hprasad@math.uni-bielefeld.de