SOLUTION OF THE MUMFORD–SHAH CONJECTURE

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Abstract. We prove the Mumford–Shah conjecture, formulated in 1989, by completing the classification of blow-up limits of minimizers in the plane. The remaining case concerns limits whose discontinuity set is noncompact and has connected complement. In this setting, we rule out bounded connected components using second-variation inequalities obtained from combined inner and outer variations associated with two perpendicular translations of a compact subset separated from the rest of the discontinuity set. Established results then imply that the discontinuity set is connected, and the known classification for connected discontinuity sets completes the proof.

1. Introduction and main result

The Mumford–Shah functional was introduced as a variational model for image segmentation [MS89]. Its unknowns are a function u, representing a piecewise smooth approximation of a given image g, and a set K, representing its edges. The model balances the Dirichlet energy of u away from K, the length of K, and the discrepancy between u and the image. This competition leads to a free discontinuity problem: both the function and the set across which it may jump are determined by minimization.

Let $\Omega \subset \mathbb{R}^2$ be open, let $g \in L^\infty(\Omega;\mathbb{R})$, and let $\lambda \geq 0$. For a relatively closed, countably $\mathcal{H}^1$-rectifiable set $K \subset \Omega$ and a function $u \in W^{1,2}_{\mathrm{loc}}(\Omega \setminus K;\mathbb{R})$, we consider

$$\mathcal{E}_\lambda(u,K;U) = \int_{U\setminus K} \left(|\nabla u|^2 + \lambda |u-g|^2\right)\,\mathrm{d}x + \mathcal{H}^1(K\cap U) \tag{1.1}$$

for bounded open sets $U \subset \Omega$. The term $\lambda \int_{U\setminus K} |u-g|^2\,\mathrm{d}x$ is called the *fidelity term*: it penalizes the discrepancy between u and the prescribed datum g, with weight λ. The Dirichlet term penalizes large spatial gradients, whereas the length term makes discontinuities costly; allowing jumps can, however, reduce the cost of approximating sharp transitions in g.

The central regularity question concerns the geometry of the minimizing set K. The conjecture of Mumford and Shah predicts that, in the interior of the domain, K is a locally finite union of C^1 arcs, which may have free endpoints and may meet only in triple junctions with angles of 120°. Thus a point of K should have precisely one of three local configurations: a regular arc, a loose end, or a triple junction. The difficulty is to derive this description from minimality without assuming in advance that K is connected, has finitely many components, or already consists of curves. In this paper we prove the planar interior conjecture by completing the classification of generalized global minimizers.

Background and earlier results. The existence theory rests on formulations that allow discontinuity sets with no prescribed smoothness. De Giorgi and Ambrosio [DGA88] introduced the space of special functions of bounded variation, and Ambrosio [Amb90] developed an existence theory for free discontinuity energies in this setting. The density estimates and essential closedness theorem of De Giorgi, Carriero and Leaci [DGCL89] link the weak

formulation to minimization over pairs (u, K) with K closed. Dal Maso, Morel and Solimini [DMMS92] developed a complementary geometric approach to existence and approximation. These results provide a framework in which the regularity conjecture can be studied without building its conclusion into the admissible class; see also [AFP].

A major part of the subsequent theory consists of ε-regularity results: sufficiently good approximation by an expected model at one scale implies regularity at smaller scales. David [Dav96] established such results near straight lines and triple junctions. The work of Ambrosio–Pallara [AP97] and Ambrosio–Fusco–Pallara [AFP97] developed partial regularity in arbitrary dimension. In particular, the discontinuity set of a planar minimizer is locally a regular arc at $\mathcal{H}^1$-almost every point. This almost-everywhere description does not by itself control the exceptional points or exclude the accumulation of small connected components.

The loose-end case required a further development. Even when the set was known to consist of a single arc ending at an interior point, the existence of a tangent at that endpoint was a separate issue. De Lellis, Focardi and Ghinassi [DLFG21] proved an ε-regularity theorem near the crack-tip model, establishing regularity up to the loose end and vanishing curvature there; see also the earlier work of Andersson and Mikayelyan [AM19]. The treatment of the fidelity term, together with a unified account of the planar theory, is given in the monograph of De Lellis and Focardi [DLF]. Earlier systematic accounts include [Dav05]. Another line of progress concerns higher integrability: De Lellis and Focardi [DLF13] proved that ∇u belongs locally to L^p for some $p > 2$ in the plane, and De Philippis and Figalli [DPF14] established the corresponding result in arbitrary dimension.

Further contributions to the regularity theory include higher regularity of the regular part of the discontinuity set [AFP99], flatness and finiteness results [Leg99], quantitative density lower bounds [LF13], and monotonicity formulae for more general free discontinuity problems [BL14]. For planar minimizers of the fidelity-free functional, Fischer [Fis22] obtained a monotonicity formula for the David–Léger entropy truncated at the sharp level $3/2$, without a connectedness assumption on the discontinuity set, and derived an optimal lower bound on the energy density at points where u is not smooth. In dimension three, regularity near minimal cones [Lem11], rigidity of global minimizers [Lem15], and stationarity of crack-front configurations [LM18] have also been investigated. More recently, Zhang [Zha25] gave an alternative proof of a local square-root growth estimate at tip points, extending an estimate of David and Léger. Labourie and Lemenant [LL26] proved that a planar minimizer u has at most three distinct limit values at each point of K, when approached from $\Omega \setminus K$, answering a question of De Giorgi.

The connection between local regularity and global geometry comes from blow-up analysis. The rescaling

$$K_{x,r} = \frac{K - x}{r}, \qquad u_{x,r}(y) = r^{-1/2} u(x + ry)$$

puts the Dirichlet energy and the length on the same scale:

$$\mathcal{E}_0(u_{x,r}, K_{x,r}; B_R) = r^{-1} \mathcal{E}_0(u, K; B_{rR}(x)).$$

The fidelity term disappears in the limit as $r \downarrow 0$. Compactness, with suitable normalizations of the additive constants on different complementary components, leads to generalized global minimizers of the fidelity-free energy. The expected models are the constant configuration, the pure jump across a straight line, the triple junction, and the crack tip. The latter consists of a half-line and a function with square-root growth; its precise normalization is recalled in (2.2). The compactness and ε-regularity theory makes the interior conjecture equivalent to the assertion that these models exhaust the generalized global minimizers [DLF, Theorem 1.4.5].

Several fundamental classification results establish this assertion under additional geometric hypotheses. Bonnet [Bon96] introduced a monotonicity argument and classified global minimizers whose discontinuity set is connected. Bonnet and David [BD01] proved that the crack tip is globally minimizing and obtained a stronger rigidity theorem, which applies when all but at most one connected component of K are contained in a common compact set. David and Léger [DL02] classified the case in which K disconnects the plane, leaving only the straight line and the triple junction, and showed that K has at most one unbounded connected component. The forms of these results used below are recalled in Section 2. Together they isolate the remaining difficulty in configurations with disconnected K and connected complement, for which bounded components cannot all be confined to one compact set.

There is also a related regularity problem at the boundary of the ambient domain. For prescribed Dirichlet data, the author [Dea24] studied boundary blow-ups and proved an ε-regularity theorem near the tangent line to the boundary. The result proved here concerns interior points; the boundary problem involves a different class of limiting minimizers.

Main result. We now specify the local minimality convention in the statement. We call (u, K) *admissible* if its energy is locally finite, that is, $\mathcal{E}_\lambda(u, K; U) < \infty$ for every open $U \Subset \Omega$. We call an admissible pair (u, K) an *absolute minimizer* if, for every bounded open $U \subset \Omega$,

$$\mathcal{E}_\lambda(u, K; U) \le \mathcal{E}_\lambda(v, L; U)$$

for every admissible (v, L) for which some compact $C \subset U$ satisfies

$$L \setminus C = K \setminus C, \qquad v = u \quad \text{a.e. on } \Omega \setminus (C \cup K \cup L). \tag{1.2}$$

This is absolute minimality in [DLF, Section 1.1 and Definition 1.2.1]; competitor sets L are rectifiable, but there is no restriction on their topology. We call K *reduced* if

$$\mathcal{H}^1(K \cap B_r(x)) > 0 \quad \text{whenever } x \in K \text{ and } \overline{B_r(x)} \subset \Omega. \tag{1.3}$$

When $\Omega = \mathbb{R}^2$, the same definition is used in every ball. This condition requires K to coincide locally with the support of its length measure, excluding extraneous points that would obscure a geometric statement.

Theorem 1.1 (Planar interior regularity)**.** *Let (u, K) be an absolute minimizer of (1.1), and suppose that K is reduced. For every $x \in K$ there are $r > 0$ with $\overline{B_r(x)} \subset \Omega$, an integer $m \in \{1, 2, 3\}$, positive numbers $\ell_1, \dots, \ell_m$, and injective maps*

$$\gamma_i \in C^1([0, \ell_i]; \mathbb{R}^2), \qquad i = 1, \dots, m,$$

with

$$|\gamma_i'(s)| = 1 \quad (0 \le s \le \ell_i), \qquad \gamma_i(0) = x, \tag{1.4}$$
$$\gamma_i(\ell_i) \in \partial B_r(x), \qquad \gamma_i([0, \ell_i)) \subset B_r(x),$$

$$K \cap \overline{B_r(x)} = \bigcup_{i=1}^{m} \gamma_i([0, \ell_i]), \tag{1.5}$$
$$\gamma_i([0, \ell_i]) \cap \gamma_j([0, \ell_j]) = \{x\} \qquad (i \ne j).$$

Endpoint derivatives are one-sided. The outward unit tangents $\tau_i = \gamma_i'(0)$ satisfy

$$\tau_1 + \tau_2 = 0 \quad (m = 2), \qquad \tau_i \cdot \tau_j = -\frac{1}{2} \quad (i \ne j,\ m = 3). \tag{1.6}$$

The set of points with $m = 1$ or $m = 3$ is locally finite in Ω.

Outline of the proof. The proof reduces to the classification of reduced generalized global minimizers. By the established results recalled above, it remains to consider noncompact K with connected complement $\mathbb{R}^2 \setminus K$. Our strategy is to use second variations to exclude bounded connected components of K. Indeed, the David–Léger theorem gives at most one unbounded component; once every component is unbounded, K is connected and the known classification applies.

A bounded connected component need not be isolated from the other components, so it cannot necessarily be translated independently. However, Lemma 3.3 shows that it is contained in a nonempty compact subset $S \subset K$ which is both open and closed relative to K. Since K is noncompact, S is a proper subset of K. Moreover, compactness of S and closedness of $K \setminus S$ imply that the two sets have positive distance. We can therefore translate S while keeping the rest of K fixed, preserving its length. Proposition 3.1 rules out such a set by combining the second-variation inequalities for two perpendicular translations, while allowing u to adjust to each deformation.

Our starting point, Lemma 3.2, is a second-variation estimate for a generalized global minimizer with connected complement $D = \mathbb{R}^2 \setminus K$, valid for every deformation field $X \in C_c^\infty(\mathbb{R}^2;\mathbb{R}^2)$. No smoothness of K in the region affected by the deformation is required. We consider the combined inner and outer variations

$$\Phi_t = \mathrm{Id} + tX, \qquad K_t = \Phi_t(K), \qquad u_t \circ \Phi_t = u + t\varphi,$$

where $\varphi \in \mathscr{T}_K$ and

$$\mathscr{T}_K = \{\varphi \in W^{1,2}(D) : \varphi = 0 \text{ a.e. outside some compact subset of } \mathbb{R}^2\}.$$

The coefficient of t^2 in the energy expansion is

$$Q_X + \|\nabla\varphi\|_2^2 + 2\langle m_X, \nabla\varphi\rangle.$$

Here Q_X is the coefficient for the pure inner variation, obtained by setting $\varphi = 0$, while m_X couples the deformation to the correction of u through the inner product $\langle m_X, \nabla\varphi\rangle$ above. Explicitly,

$$\begin{aligned} m_X &= ((\operatorname{div} X)I - DX - (DX)^T)\nabla u, \\ Q_X &= \int_D \left(|(DX)^T\nabla u|^2 - \det(DX)|\nabla u|^2\right)\,\mathrm{d}x \\ &\quad + \frac{1}{2}\int_K \left(|DX\,\tau|^2 - (\tau\cdot DX\,\tau)^2\right)\,\mathrm{d}\mathcal{H}^1, \end{aligned}$$

where τ is a unit approximate tangent to K. Norms and inner products are taken in $L^2(\mathbb{R}^2;\mathbb{R}^2)$.

For each fixed X, inner variations alone give $Q_X \geq 0$. The freedom to choose φ gives a stronger condition: completing the square and taking the infimum over $\varphi \in \mathscr{T}_K$ yields

$$Q_X \geq \|\Pi_K m_X\|_2^2,$$

where Π_K is the orthogonal projection onto the L^2 closure of the gradients of functions in $\mathscr{T}_K$. Thus the optimization quantifies how much the quadratic cost of a prescribed deformation can be reduced by adjusting u.

Second variations of the Mumford–Shah functional were also studied by Cagnetti, Mora and Morini [CMM08]. Their analysis concerns deformations supported in the regular part of the discontinuity set, with the function chosen to minimize the Dirichlet energy on the deformed domain with fixed boundary values.

We then apply this general estimate to $X_1 = \chi e_1$ and $X_2 = \chi e_2$, where χ is a smooth compactly supported cutoff equal to one near S and zero near $K \setminus S$. These fields generate the two perpendicular translations of S. Their pure inner quadratic costs Q_{X_1} and Q_{X_2} are already nonnegative, so the essential information comes from the projection term obtained by optimizing over φ. The associated fields m_{X_1} and m_{X_2} are related by a 90° rotation and satisfy

$$Q_{X_1} + Q_{X_2} = \|m_{X_1}\|_2^2 = \|m_{X_2}\|_2^2.$$

Let Π denote the orthogonal projection onto the L^2 closure of the gradients of functions in $C_c^\infty(\mathbb{R}^2)$. The Hodge decomposition on $\mathbb{R}^2$ gives

$$\|\Pi m_{X_1}\|_2^2 + \|\Pi m_{X_2}\|_2^2 = Q_{X_1} + Q_{X_2}.$$

Since the restrictions to D of functions in $C_c^\infty(\mathbb{R}^2)$ belong to $\mathscr{T}_K$, the space defining Π is contained in the space defining Π_K. Thus replacing Π by Π_K can only increase each squared norm. On the other hand, the second-variation inequalities bound $\|\Pi_K m_{X_i}\|_2^2$ by Q_{X_i} for $i = 1, 2$. Summing these inequalities forces equality throughout. Let v be a real scalar potential such that

$$\nabla v = \Pi m_{X_1}.$$

It satisfies $\Delta v = \operatorname{div} m_{X_1}$, so it is harmonic near K, where $m_{X_1} = 0$. The equality conditions imply that both the normal and tangential derivatives of v vanish along every regular arc of K. Thus $\nabla v = 0$ on these arcs.

To connect this information back to u, we use the equation for the potential to split one derivative of u into two harmonic functions. One extends through S and the other through $K \setminus S$. The vanishing conditions and unique continuation make both functions constant. It follows that ∇u is constant, and the linear energy-growth bound forces it to vanish. The classification of minimizers with vanishing Dirichlet energy then contradicts the presence of S.

This excludes bounded connected components in the remaining case and completes the classification in Theorem 3.4. The planar regularity statement, Theorem 1.1, then follows by the existing compactness and regularity theory, including regularity at loose ends.

Section 2 records the precise results from that theory needed for this implication. Section 3 proves compact translation rigidity and its topological consequence, and completes the global classification.

2. Reduction to a global classification

All references to the established theory below use the published monograph [DLF].

We use *generalized global minimizer* exactly as in [DLF, Definition 2.2.4]: it is a triple $(u, K, \{p_{kl}\})$ arising from rescaled absolute minimizers. The extended-real parameters p_{kl} record relative additive constants on the connected components of $\mathbb{R}^2 \setminus K$. They are part of the definition; when $\mathbb{R}^2 \setminus K$ is connected, the only parameter is $p_{11} = 0$.

The compactness theorem below requires a uniform L^∞ bound on the minimizers before rescaling [DLF, Assumption 2.2.1(iv)]. For blow-ups of a fixed absolute minimizer (u, K) at an interior point, we obtain this bound by working on a sufficiently small ball. The maximum principle [DLF, Lemma 2.1.1] gives local boundedness of u when $\lambda > 0$. When $\lambda = 0$, a local truncation gives a bounded function $\widetilde{u}$ such that $(\widetilde{u}, K)$ is still an absolute minimizer and $\nabla \widetilde{u} = \nabla u$. We apply the compactness theorem to this bounded minimizer.

The compactness theorem [DLF, Theorem 2.2.3] supplies a generalized global limit $(u, K, \{p_{kl}\})$ along a subsequence of every sequence of interior rescalings. It gives local

Hausdorff convergence to K and separate convergence of Dirichlet energies and length measures. The limiting set K is reduced, countably $\mathcal{H}^1$-rectifiable, and locally of finite length, while $u \in W^{1,2}(B_R \setminus K)$ for every $R > 0$. For every generalized global minimizer $(u, K, \{p_{kl}\})$, the upper density bound [DLF, Lemma 2.1.2] gives

$$\int_{B_R(z)\setminus K} |\nabla u|^2 + \mathcal{H}^1(K \cap B_R(z)) \le 2\pi R \qquad (z \in \mathbb{R}^2,\ R > 0). \tag{2.1}$$

Moreover, [DLF, Corollary 3.1.5] gives a relatively open subset of regular points of K of full $\mathcal{H}^1$ measure. At each such point x, there is $r > 0$ such that $K \cap B_r(x)$ is a single embedded C^1 arc.

If $D = \mathbb{R}^2 \setminus K$ is connected, the exterior separation condition on competitors in [DLF, Definition 2.2.2] is vacuous. Consequently [DLF, Theorem 2.2.3(ii)] makes (u, K) an ordinary absolute minimizer of the fidelity-free energy on the plane. In particular, the compactly supported combined inner and outer variations used in Section 3 are admissible.

A generalized global minimizer $(u, K, \{p_{kl}\})$ is called *elementary* if $\nabla u = 0$ almost everywhere on $\mathbb{R}^2 \setminus K$. By [DLF, Theorem 2.4.1], in this case K is either empty, a straight line (the pure-jump model), or the union of three half-lines with a common endpoint and pairwise angles of 120° (the triple-junction model). In each case, u is constant on every connected component of $\mathbb{R}^2 \setminus K$. The other canonical model is the crack tip. Up to translation, rotation, sign, and an additive constant, its set K and function u are given by

$$\begin{aligned} K &= \{(r, 0) : r \ge 0\}, \\ u(r, \theta) &= \sqrt{\frac{2}{\pi}}\sqrt{r}\cos\frac{\theta}{2}, \quad r > 0, \quad 0 < \theta < 2\pi. \end{aligned} \tag{2.2}$$

The parameters p_{kl} in the two disconnected models are interpreted as in [DLF, Section 2.4]. For a generalized global minimizer $(u, K, \{p_{kl}\})$, we shall use the following classification results:

(i) If $\mathbb{R}^2 \setminus K$ is disconnected, then $\nabla u = 0$ almost everywhere on $\mathbb{R}^2 \setminus K$, and K is either a straight line or the triple-junction configuration described above [DLF, Theorem 4.4.1].
(ii) K has at most one unbounded connected component [DLF, Corollary 4.4.3].
(iii) If all but at most one connected component of K lie in one common compact set, then the minimizer is elementary or (u, K) is the canonical crack tip (2.2), up to the transformations specified above [DLF, Theorem 1.4.6, first paragraph].

These are the established David–Léger and Bonnet–David classification results, in the form needed for generalized limits.

By [DLF, Theorem 1.4.5], it suffices to prove the classification stated in Theorem 3.4. After the usual local normalization and rescaling, the regularity theorem [DLF, Theorem 1.3.3] gives Theorem 1.1. Along each sufficiently short C^1 branch issuing from x, the distance from x increases strictly. We may therefore choose a smaller ball whose boundary meets each branch exactly once. Parametrizing the resulting closed arcs by arclength gives (1.4)–(1.5). The local description at every point of K also gives a neighborhood containing no endpoints or triple junctions other than, possibly, that point itself. Since K is relatively closed in Ω, such points cannot accumulate in Ω and are therefore locally finite.

3. Compact translations and global classification

We first prove the compact translation rigidity result and then use it to complete the classification of generalized global minimizers.

3.1. **Admissible variations and optimization over corrections.** For a closed set $K \subset \mathbb{R}^2$ of zero two-dimensional Lebesgue measure, put $D = \mathbb{R}^2 \setminus K$ and define the real test space

$$\mathscr{T}_K = \{\varphi \in W^{1,2}(D) : \varphi = 0 \text{ a.e. outside some compact subset of } \mathbb{R}^2\}.$$

We shall use the closed subspace

$$\mathcal{V}_K = \overline{\{\nabla\varphi : \varphi \in \mathscr{T}_K\}}^{L^2(\mathbb{R}^2;\mathbb{R}^2)}$$

and its orthogonal projection Π_K. Sets of zero two-dimensional Lebesgue measure are ignored when identifying vector fields on D with vector fields on the plane.

Let $(u, K, \{p_{kl}\})$ be a generalized global minimizer with $\mathbb{R}^2 \setminus K$ connected. Absolute minimality gives the weak equation

$$\int_D \nabla u \cdot \nabla\varphi = 0 \qquad (\varphi \in \mathscr{T}_K) \tag{3.1}$$

and the inner variation identity

$$\int_D (|\nabla u|^2 \operatorname{div} X - 2\nabla u \cdot DX \nabla u) + \int_K \tau \cdot DX\tau \, \mathrm{d}\mathcal{H}^1 = 0. \tag{3.2}$$

Here $X \in C_c^\infty(\mathbb{R}^2;\mathbb{R}^2)$ and τ is a unit approximate tangent to K. These are the usual variational identities; see [DLF, Section 2.5]. Minimality also gives second-order stability under the combined inner and outer variations

$$\Phi_t = \mathrm{Id} + tX, \qquad K_t = \Phi_t(K), \qquad u_t \circ \Phi_t = u + t\varphi. \tag{3.3}$$

These reduce to inner variations when $\varphi = 0$ and to outer variations when $X = 0$. Differences of the infinite global energies are evaluated on a bounded region containing the support of the variation.

Proposition 3.1 (Compact translation rigidity)**.** *Let $(u, K, \{p_{kl}\})$ be a reduced generalized global minimizer with $\mathbb{R}^2 \setminus K$ connected. There is no nonempty compact proper subset of K which is both open and closed relative to K.*

Lemma 3.2 (Second variation optimized over corrections to u)**.** *Let $(u, K, \{p_{kl}\})$ be a generalized global minimizer with $\mathbb{R}^2 \setminus K$ connected. For $X \in C_c^\infty(\mathbb{R}^2;\mathbb{R}^2)$, define*

$$m_X = ((\operatorname{div} X)I - DX - (DX)^T)\nabla u, \tag{3.4}$$

$$\begin{aligned} Q_X = &\int_D (|(DX)^T\nabla u|^2 - \det(DX)|\nabla u|^2) \, \mathrm{d}x \\ &+ \frac{1}{2}\int_K (|DX\,\tau|^2 - (\tau \cdot DX\,\tau)^2) \, \mathrm{d}\mathcal{H}^1. \end{aligned} \tag{3.5}$$

Then

$$Q_X \geq \|\Pi_K m_X\|_{L^2}^2. \tag{3.6}$$

Proof. Fix $\varphi \in \mathscr{T}_K$, and choose a ball B containing the supports of X and φ compactly. For sufficiently small $|t|$, the map Φ_t in (3.3) is a diffeomorphism equal to the identity outside a compact subset of B; in particular, $\Phi_t(B) = B$. For the matrix calculation, write $A = DX$. The identity $u_t \circ \Phi_t = u + t\varphi$ and the Sobolev chain rule give

$$\nabla u_t(\Phi_t(x)) = (I + tA(x))^{-T}(\nabla u(x) + t\nabla\varphi(x)) \quad \text{for a.e. } x \in B \setminus K.$$

Changing variables $y = \Phi_t(x)$ in the Dirichlet energy, we obtain

$$\begin{aligned}\int_{B\setminus K_t} |\nabla u_t(y)|^2 \,\mathrm{d}y \\ &= \int_{B\setminus K} \left|(I+tA)^{-T}(\nabla u + t\nabla\varphi)\right|^2 \det(I+tA)\,\mathrm{d}x \\ &= \int_{B\setminus K} (\nabla u + t\nabla\varphi)\cdot M_t(\nabla u + t\nabla\varphi)\,\mathrm{d}x,\end{aligned}$$

where

$$\begin{aligned}M_t &= \det(I+tA)(I+tA)^{-1}(I+tA)^{-T} \\ &= I + t\big((\operatorname{tr}A)I - A - A^T\big) + t^2\big(AA^T - \det A\, I\big) + O(t^3).\end{aligned}$$

The second-order coefficient follows from the two-dimensional Cayley–Hamilton identity $A^2 - (\operatorname{tr}A)A + \det A\, I = 0$ and its transpose.

For the length term, the area formula for countably rectifiable sets (see [AFP, Theorem 2.91]) gives

$$\mathcal{H}^1(K_t \cap B) = \int_{K\cap B} |(I+tA)\tau|\,\mathrm{d}\mathcal{H}^1,$$

where τ is a unit approximate tangent to K. The tangential Jacobian expands as

$$|(I+tA)\tau| = 1 + t\,\tau\cdot A\tau + \frac{t^2}{2}\big(|A\tau|^2 - (\tau\cdot A\tau)^2\big) + O(t^3).$$

Since DX is bounded and $|\tau| = 1$, the $O(t^3)$ remainder estimates in the expansions of M_t and the tangential Jacobian are uniform in the spatial variable. Since u and φ have finite Dirichlet energy on $B\setminus K$ and $\mathcal{H}^1(K\cap B) < \infty$, the integrated remainder is $o(t^2)$ for each fixed φ. Since the competitors are admissible for both signs of sufficiently small t, minimality implies that the differentiable function $t \mapsto \mathcal{E}_0(u_t, K_t; B)$ has a local minimum at $t = 0$. Thus its linear coefficient vanishes, and the preceding expansions give

$$\begin{aligned}\mathcal{E}_0(u_t, K_t; B) - \mathcal{E}_0(u, K; B) \\ &= t^2\big(Q_X + \|\nabla\varphi\|_2^2 + 2\langle m_X, \nabla\varphi\rangle\big) + o(t^2),\end{aligned}$$

where norms and inner products are in $L^2(\mathbb{R}^2;\mathbb{R}^2)$. Dividing the minimality inequality by t^2 and letting $t \to 0$ yields

$$Q_X + \|\nabla\varphi\|_2^2 + 2\langle m_X, \nabla\varphi\rangle \geq 0 \qquad (\varphi \in \mathscr{T}_K).$$

Since $\nabla\varphi \in \mathcal{V}_K$, orthogonal projection gives $\langle m_X, \nabla\varphi\rangle = \langle \Pi_K m_X, \nabla\varphi\rangle$. Completing the square,

$$\|\nabla\varphi\|_2^2 + 2\langle m_X, \nabla\varphi\rangle = \|\nabla\varphi + \Pi_K m_X\|_2^2 - \|\Pi_K m_X\|_2^2.$$

By the definition of $\mathcal{V}_K$, there are $\varphi_j \in \mathscr{T}_K$ such that $\nabla\varphi_j \to -\Pi_K m_X$ in $L^2(\mathbb{R}^2;\mathbb{R}^2)$. Applying the preceding inequality to φ_j and passing to the limit proves (3.6). The limit $t \to 0$ is taken first for each fixed test, so no remainder estimate uniform in j is needed. In particular, the projected field need not itself be the gradient of an element of $\mathscr{T}_K$. □

3.2. **Two translations and the Hodge decomposition.**

Proof of Proposition 3.1. Suppose that $S \subsetneq K$ is nonempty, compact and relatively open and closed. Its nonempty complement $K \setminus S$ is closed and has positive distance from S. We may therefore choose a real $\chi \in C_c^\infty(\mathbb{R}^2)$ such that

$$\chi = 1 \quad \text{near } S, \qquad \chi = 0 \quad \text{near } K \setminus S.$$

Thus $\operatorname{supp} \nabla\chi$ is a compact subset of D. Reducedness and relative openness imply

$$0 < \mathcal{H}^1(S) < \infty. \tag{3.7}$$

Set $X_1 = \chi e_1$ and $X_2 = \chi e_2$. The corresponding deformations translate S rigidly and are the identity in a neighborhood of $K \setminus S$. Let $\mathsf{J}(a,b) = (-b,a)$. Substituting into (3.4)–(3.5) gives

$$\begin{aligned} m := m_{X_1} &= (-\chi_x u_x - \chi_y u_y, -\chi_y u_x + \chi_x u_y), \\ m_{X_2} &= \mathsf{J}m, \\ Q_{X_1} &= \int_D u_x^2 |\nabla\chi|^2, \qquad Q_{X_2} = \int_D u_y^2 |\nabla\chi|^2. \end{aligned} \tag{3.8}$$

Consequently

$$Q_{X_1} + Q_{X_2} = \|m\|_2^2. \tag{3.9}$$

The field m is smooth and compactly supported in D, since (3.1) makes u harmonic there.

Let $\mathcal{V}$ be the L^2 closure of gradients of real functions in $C_c^\infty(\mathbb{R}^2)$, and let Π be its orthogonal projection. Then $\mathcal{V} \subset \mathcal{V}_K$. The planar L^2 Helmholtz–Hodge decomposition (see [Gal11, Section III.1]) gives

$$L^2(\mathbb{R}^2;\mathbb{R}^2) = \mathcal{V} \oplus^\perp \mathsf{J}\mathcal{V}.$$

Write

$$m = g + \mathsf{J}h, \qquad g, h \in \mathcal{V}.$$

Since $\mathcal{V}$ and $\mathsf{J}\mathcal{V}$ are orthogonal in L^2 and $\mathsf{J}^2 = -I$, we have

$$\Pi m = g, \qquad \Pi(\mathsf{J}m) = \Pi(\mathsf{J}g - h) = -h.$$

Therefore, using that J preserves the L^2 norm,

$$\begin{aligned} \|\Pi m\|_2^2 + \|\Pi(\mathsf{J}m)\|_2^2 &= \|g\|_2^2 + \|h\|_2^2 \\ &= \|g\|_2^2 + \|\mathsf{J}h\|_2^2 \\ &= \|m\|_2^2. \end{aligned} \tag{3.10}$$

Since $\mathcal{V} \subset \mathcal{V}_K$, for every $f \in L^2(\mathbb{R}^2;\mathbb{R}^2)$ we have $\Pi_K f - \Pi f \perp \mathcal{V}$, and hence

$$\|\Pi_K f\|_2^2 - \|\Pi f\|_2^2 = \|\Pi_K f - \Pi f\|_2^2 \geq 0.$$

Combining (3.6), (3.9) and (3.10), we obtain

$$\begin{aligned} \|m\|_2^2 &\geq \|\Pi_K m\|_2^2 + \|\Pi_K(\mathsf{J}m)\|_2^2 \\ &\geq \|\Pi m\|_2^2 + \|\Pi(\mathsf{J}m)\|_2^2 = \|m\|_2^2. \end{aligned}$$

The first and last quantities coincide, so equality holds throughout. Using the preceding projection identity, we obtain

$$\|\Pi_K m - \Pi m\|_2^2 + \|\Pi_K(\mathsf{J}m) - \Pi(\mathsf{J}m)\|_2^2 = 0.$$

Both terms are nonnegative, so

$$\Pi_K m = \Pi m, \qquad \Pi_K(\mathsf{J}m) = \Pi(\mathsf{J}m).$$

Since $f - \Pi_K f \perp \mathcal{V}_K$ for every f, we conclude that

$$R_1 = m - \Pi m \perp \mathcal{V}_K, \qquad R_2 = \mathsf{J}m - \Pi(\mathsf{J}m) \perp \mathcal{V}_K. \tag{3.11}$$

A real scalar potential and the normal conditions. Since $\Pi m \in \mathcal{V}$, it is an L^2 limit of gradients of smooth, compactly supported functions, and hence $\operatorname{curl}(\Pi m) = 0$ in $\mathcal{D}'(\mathbb{R}^2)$. As the plane is simply connected, the Poincaré lemma supplies a real function $v \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$ such that

$$\nabla v = \Pi m, \qquad \Delta v = \operatorname{div} m \quad \text{in } \mathcal{D}'(\mathbb{R}^2). \tag{3.12}$$

The second identity follows because $m - \Pi m$ is orthogonal to all compactly supported smooth gradients. Since $m \in C^\infty_c(D;\mathbb{R}^2)$, elliptic regularity gives $v \in C^\infty(\mathbb{R}^2)$, and v is harmonic on a neighborhood of K.

Using the decomposition above, (3.11) and (3.12) give

$$R_1 = m - \nabla v = \mathsf{J}h, \qquad R_2 = \mathsf{J}g = \mathsf{J}\nabla v.$$

In particular, both residuals are smooth and divergence-free on the plane, and, near K, where $m = 0$,

$$R_1 = -\nabla v, \qquad R_2 = \mathsf{J}\nabla v. \tag{3.13}$$

Fix a regular point of K and choose, in suitable coordinates, a sufficiently small rectangle U around it such that $\Gamma := K \cap U$ is a single C^1 graph. The set $U \setminus K$ is the union of two Lipschitz domains U^+ and U^- lying on either side of Γ. Let ν denote the outward unit normal to U^+. For any real $\eta \in C^\infty_c(U)$, define on D

$$\varphi(x) = \begin{cases} \eta(x), & x \in U^+, \\ 0, & x \in D \setminus U^+. \end{cases}$$

Since η vanishes near ∂U, this definition creates no jump inside D: the only possible jump lies on $\Gamma \subset K$. Thus $\varphi \in W^{1,2}(D)$, with $\nabla\varphi = \mathbf{1}_{U^+}\nabla\eta$ almost everywhere on D. Its support in the plane is contained in $\operatorname{supp}\eta \Subset U$, so $\varphi \in \mathscr{T}_K$.

Using (3.11) and integration by parts on U^+, we obtain, for $j = 1, 2$,

$$\begin{aligned} 0 &= \int_D R_j \cdot \nabla\varphi \, \mathrm{d}x = \int_{U^+} R_j \cdot \nabla\eta \, \mathrm{d}x \\ &= \int_{\partial U^+} \eta \, R_j \cdot \nu \, \mathrm{d}\mathcal{H}^1 - \int_{U^+} \eta \operatorname{div} R_j \, \mathrm{d}x \\ &= \int_\Gamma \eta \, R_j \cdot \nu \, \mathrm{d}\mathcal{H}^1. \end{aligned}$$

Here the boundary term on $\partial U^+ \setminus \Gamma$ vanishes because η is supported compactly in U, and the volume term vanishes because $\operatorname{div} R_j = 0$. Since the identity holds for every $\eta \in C^\infty_c(U)$ and $R_j \cdot \nu$ is continuous on Γ, we conclude that

$$R_1 \cdot \nu = R_2 \cdot \nu = 0 \quad \text{on } \Gamma.$$

By (3.13),

$$\nabla v \cdot \nu = (\mathsf{J}\nabla v) \cdot \nu = 0.$$

The vectors ν and $\mathsf{J}\nu$ span the plane, so

$$\nabla v = 0 \quad \text{on every regular subarc of } K. \tag{3.14}$$

Harmonic separation of the compact piece. We use the following standard harmonic uniqueness fact: if w is harmonic on a connected open set $U \subset \mathbb{R}^2$ and $\nabla w = 0$ along a nontrivial embedded C^1 arc contained in U, then w is constant. Indeed, $w = c$ and $\partial_\nu w = 0$ on a connected subarc. Extending $w - c$ from one side by zero to the other gives a weakly harmonic function in a small neighborhood, by integration by parts. Interior elliptic regularity and the identity principle imply that this extension vanishes. Thus $w = c$ on an open subset of U, and hence throughout U.

Using $u_{yy} = -u_{xx}$ and $u_{xy} = u_{yx}$ in (3.8), we compute on D

$$\operatorname{div} m = -(\Delta\chi)u_x - 2\nabla\chi \cdot \nabla u_x = -\Delta(\chi u_x). \tag{3.15}$$

Here u_x is harmonic on D. By (3.12) and (3.15), the real scalar functions

$$f_0 = (1-\chi)u_x - v, \qquad f_1 = \chi u_x + v \tag{3.16}$$

are harmonic on D and satisfy $f_0 + f_1 = u_x$. Near S, the first equals $-v$, so it extends harmonically through S. Near $K \setminus S$, the second equals v, so it extends harmonically through $K \setminus S$. Their respective domains of extension are

$$U_0 = \mathbb{R}^2 \setminus (K \setminus S), \qquad U_1 = \mathbb{R}^2 \setminus S.$$

Both domains are connected: each contains connected D, and every open component meets D because K has zero two-dimensional Lebesgue measure.

By (3.7) and full-measure regularity, S contains a regular subarc. Near that subarc $f_0 = -v$, and (3.14) gives $\nabla f_0 = 0$ there. The harmonic uniqueness fact therefore makes f_0 constant throughout U_0. The nonempty relatively open set $K \setminus S$ also contains a positive-length regular subarc, by reducedness and full-measure regularity. Near it $f_1 = v$, so the same argument makes f_1 constant throughout U_1.

It follows that $u_x = f_0 + f_1$ is constant on D. Harmonicity gives

$$\nabla u_y = (u_{xy}, u_{yy}) = (0, -u_{xx}) = 0.$$

Connectedness of D then makes u_y constant as well. Thus ∇u is a constant vector on D. Since K has zero two-dimensional Lebesgue measure, a nonzero constant gradient would give quadratic bulk-energy growth, contrary to (2.1). Consequently $\nabla u = 0$ on D. By the classification of elementary generalized global minimizers [DLF, Theorem 2.4.1], connectedness of D implies $K = \varnothing$, contradicting $\varnothing \neq S \subset K$. This proves the proposition. □

3.3. **Topological consequence and classification.**

Lemma 3.3. *Let $K \subset \mathbb{R}^2$ be closed. If K has no nonempty compact subset that is both relatively open and relatively closed in K, then every connected component of K is unbounded.*

Proof. Suppose that C is a bounded connected component of K. Since C is closed in K and K is closed in $\mathbb{R}^2$, the set C is closed and bounded in $\mathbb{R}^2$, hence compact. Choose R with $C \subset B_R$ and set

$$X := K \cap \overline{B}_R, \qquad E := X \cap \partial B_R.$$

The space X is compact, and $C \cap E = \varnothing$. Moreover, C is a connected component of X: it is connected and contained in X, and any connected subset of X strictly containing C would also be a connected subset of K, contradicting the maximality of C.

We use the standard fact [Eng89, Theorem 6.1.23] that a component of a compact Hausdorff space is the intersection of all subsets containing it that are both open and closed in that space. Thus, for each $y \in E$, there is a set $V_y \subset X$, open and closed in X, such that $C \subset V_y$

and $y \notin V_y$. The sets $X \setminus V_y$ form an open cover of E relative to X. Since E is compact, we can choose finitely many $y_1, \dots, y_N \in E$ such that

$$E \subset \bigcup_{i=1}^{N} (X \setminus V_{y_i}), \qquad W := \bigcap_{i=1}^{N} V_{y_i}.$$

Then W is open and closed in X and contains C. The covering property gives $W \cap E = \varnothing$, so $W \subset B_R$. If $E = \varnothing$, take $W = X$, which has the same properties.

Since W is closed in the compact space X, it is compact and hence closed in $\mathbb{R}^2$, so it is also relatively closed in K. Since W is open in X, write $W = O \cap X$ for some open set $O \subset \mathbb{R}^2$. As $W \subset B_R$, we have

$$W = K \cap (O \cap B_R),$$

so W is relatively open in K. Finally, W is nonempty because it contains C. Thus W is a nonempty compact subset of K that is both relatively open and relatively closed in K, contrary to the hypothesis. □

Theorem 3.4 (Classification of generalized global minimizers)**.**
Let $(u, K, \{p_{kl}\})$ be a reduced generalized global minimizer in $\mathbb{R}^2$. Then either $\nabla u = 0$ a.e. on $\mathbb{R}^2 \setminus K$, so that the minimizer is elementary, or (u, K) is the canonical crack tip (2.2), up to translation, rotation, sign, and an additive constant.

Proof. Let $(u, K, \{p_{kl}\})$ be a generalized global minimizer in the sense of [DLF, Definition 2.2.4]. If $\mathbb{R}^2 \setminus K$ is disconnected, [DLF, Theorem 4.4.1] gives a pure-jump or triple-junction model. If K is compact, including $K = \varnothing$, [DLF, Theorem 1.4.6, first paragraph] already applies, since all components are contained in a common compact set. Its only model with compact K is the constant minimizer.

It remains to consider noncompact K with connected complement. Proposition 3.1 excludes every nonempty compact relatively open-and-closed subset of K, since any such subset would be proper. Lemma 3.3 therefore makes every component of K unbounded. By [DLF, Corollary 4.4.3] there is at most one such component, so K is connected. The classification of generalized global minimizers with connected K [DLF, Theorem 1.4.6] gives the conclusion. □

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