Solution of the Mumford–Shah conjecture

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Author's note

On September 9, 2026, I obtained a proof of the Mumford–Shah conjecture, formulated in 1989 [MS89], using ChatGPT Astra. A shorter version of the manuscript was produced on September 11.

I had been checking the argument and had planned to simplify it and improve its exposition before making it public. However, today, September 22, I saw OpenAI's announcement of September 21, reporting that an internal model had solved more than 100 long-standing open problems in mathematics. The announcement does not provide a complete list of these problems, and I do not know whether the Mumford–Shah conjecture is among them. This prompted me to make the current AI-generated draft publicly available now, to document the chronology of this work. I will continue checking the mathematical details, improving the exposition, and simplifying the proof. I believe a more elementary proof is possible and I am working on it.

How the manuscript was developed

I began working on regularity for the Mumford–Shah functional during my PhD, focusing on a variant with Dirichlet boundary conditions. My earlier results are presented in [Dea24].

In April 2026, I made further progress on this boundary problem. That work has not yet been made public.

In September 2026, I decided to use ChatGPT Pro to investigate both the conjecture arising from my PhD research and the main Mumford–Shah conjecture. I uploaded my PhD thesis, some books on the subject, and a LaTeX file containing my recent unpublished work.

On September 5, 2026, I gave Astra a detailed prompt in Work mode, specifying a research protocol and asking it to solve the Mumford–Shah conjecture. After six hours, the model had not produced a complete proof.

On September 6, I asked Astra to address the conjecture formulated in my PhD thesis, concerning the Mumford–Shah functional with Dirichlet boundary conditions. After one hour and twenty minutes, it produced a proof building on my recent progress.

On September 9, I returned to the conversation I had started on September 5 and provided Astra with the LaTeX manuscript of the new boundary regularity result. I asked it to use this result to prove the main conjecture. The model initially suggested several possible approaches. I then asked it to pursue those approaches or develop other original ideas, allowing it to work for up to six hours. After two hours and thirty minutes, it produced a complete proof of the Mumford–Shah conjecture.

The resulting manuscript was 37 pages long and included proofs of several well-known results. After further exchanges, I asked Astra to produce a shorter version. The manuscript shared here is the version obtained on September 11.

The strategy of the proof is to classify generalized global minimizers through a second-variation argument that combines inner and outer variations.

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Abstract

We present a compact-translation argument for the classification of planar Mumford–Shah generalized global minimizers. Two perpendicular translations of a compact isolated portion of the crack, independently relaxed in the displacement, saturate a full-plane Hodge identity. The resulting normal conditions imply a holomorphic rigidity statement which excludes that portion. Established classification theorems then leave only the constant, pure-jump, triple-junction, and crack-tip models. The planar interior regularity conjecture follows from the known equivalence between this classification and local regularity. No homogeneity assumption is imposed on the generalized global minimizer.

1 Introduction and main result

The planar Mumford–Shah conjecture concerns the local geometry of a minimizing discontinuity set: each point should lie on a regular arc, be an endpoint, or be a junction of three arcs meeting at equal angles. The regularity theory reduces this question to the classification of generalized global minimizers; see De Lellis–Focardi [DLF, Theorem 1.4.5 and the following paragraph]. This reduction includes regularity at crack tips, and does not require blow-ups to be homogeneous.

Our argument addresses the remaining global classification question. If the crack disconnects the plane, the David–Léger theorem already classifies it. If the complement is connected, we exclude a nonempty compact portion which is both open and closed relative to the crack. The new step combines relaxed second variations for two translations with the Hodge decomposition on the plane. A holomorphic decomposition then turns equality in those variations into a contradiction. A topological observation and the Bonnet–David classification complete the proof. We use the existing compactness and regularity theory by reference throughout.

The variational problem

Let $\Omega \subset \mathbb{R}^2$ be bounded and open, let $g \in L^\infty(\Omega;\mathbb{R})$, and let $\lambda \geq 0$. A pair (K,u) is *admissible* if $K \subset \Omega$ is relatively closed, $\mathcal{H}^1(K) < \infty$, and $u \in W^{1,2}(\Omega \setminus K;\mathbb{R})$. Its energy is

$$\mathcal{E}_\lambda(K,u) = \int_{\Omega\setminus K} \left(|\nabla u|^2 + \lambda|u-g|^2\right)\,\mathrm{d}x + \mathcal{H}^1(K). \tag{1.1}$$

We call (K,u) a *compact-perturbation minimizer* if

$$\mathcal{E}_\lambda(K,u) \leq \mathcal{E}_\lambda(L,v)$$

for every admissible (L,v) for which some compact $C \subset \Omega$ satisfies

$$L \setminus C = K \setminus C, \qquad v = u \quad \text{a.e. on } \Omega \setminus (C \cup K \cup L). \tag{1.2}$$

There is no restriction on the topology of competitors in this definition. We call K *reduced* if

$$\mathcal{H}^1(K \cap B_r(x)) > 0 \quad \text{whenever } x \in K \text{ and } \overline{B_r(x)} \subset \Omega. \tag{1.3}$$

For a global crack, the same definition is used in every ball.

Theorem 1.1 (Planar interior regularity)**.** *Let (K,u) be an admissible compact-perturbation minimizer of (1.1), and suppose that K is reduced. For every $x \in K$ there are $r > 0$ with $\overline{B_r(x)} \subset \Omega$, an integer $m \in \{1,2,3\}$, positive numbers $\ell_1,\dots,\ell_m$, and injective maps*

$$\gamma_i \in C^1([0,\ell_i];\mathbb{R}^2), \qquad i = 1,\dots,m,$$

with

$$|\gamma_i'(s)| = 1 \quad (0 \le s \le \ell_i), \qquad \gamma_i(0) = x, \tag{1.4}$$
$$\gamma_i(\ell_i) \in \partial B_r(x), \qquad \gamma_i([0,\ell_i)) \subset B_r(x),$$
$$K \cap \overline{B_r(x)} = \bigcup_{i=1}^{m} \gamma_i([0,\ell_i]), \tag{1.5}$$
$$\gamma_i([0,\ell_i]) \cap \gamma_j([0,\ell_j]) = \{x\} \qquad (i \neq j).$$

Endpoint derivatives are one-sided. The outward unit tangents $\tau_i = \gamma_i'(0)$ *satisfy*

$$\tau_1 + \tau_2 = 0 \quad (m = 2), \qquad \tau_i \cdot \tau_j = -\frac{1}{2} \quad (i \neq j,\ m = 3). \tag{1.6}$$

The set of points with $m = 1$ *or* $m = 3$ *is locally finite in* Ω*.*

We refer to (1.4)–(1.6) as an *exact local star*. The assertion concerns the entire crack in a closed ball, including the boundary intersections. In particular it is stronger than a parametrization of only a relatively open regular portion.

Theorem 1.1 follows from the global classification in Theorem 3.4 and the cited regularity theory, as explained in Section 2. Thus the proof ends with that classification; no additional endpoint or improvement-of-flatness argument is needed here.

2 Reduction to a global classification

All references to the established theory below use the published monograph [DLF]. We first match its standard formulation to the hypotheses of Theorem 1.1, which do not assume either rectifiability of K or boundedness of u.

2.1 Local normalization

Lemma 2.1. *Every point of* Ω *has a ball neighborhood* $V \Subset \Omega$ *on which there is a bounded displacement* U *such that* $(K \cap V, U)$ *is a reduced absolute minimizer in the sense of [DLF, Definition 1.2.1],*

$$\nabla U = \nabla u \quad \text{a.e. in } V \setminus K, \qquad K \cap V = \overline{J_U}^{V}, \qquad \mathcal{H}^1((K \cap V) \setminus J_U) = 0.$$

Here J_U *denotes the approximate jump set of* $U \in SBV^2(V)$*. If* $\lambda > 0$*, one may take* $U = u$ *on* $V \setminus K$*.*

Proof. We recall the local reduction, since it avoids adding hypotheses to the main theorem. The coarea inequality for a set of finite $\mathcal{H}^1$ measure, followed by Sobolev slicing, provides a circle compactly inside Ω which meets K in finitely many points and on each complementary arc carries a $W^{1,2}$ trace of u. These finitely many traces are bounded. Truncate u inside the circle at a level M bounding both the traces and $\|g\|_\infty$, and leave it unchanged outside. Matching traces make this a Sobolev competitor. Both bulk terms decrease, so minimality forces equality and hence preservation of the gradient. If $\lambda > 0$, strict decrease of $|u - g|^2$ on the truncated set gives equality of the functions as well. The replacement is still a minimizer, and we restrict it to a smaller concentric ball V.

For clarity, the strong-to-SBV argument applies even before rectifiability is known. A bounded Sobolev function off a relatively closed set of finite $\mathcal{H}^1$ measure extends to BV; this follows by cutting it off near that set with cutoffs of uniformly bounded $W^{1,1}$ seminorm and support of vanishing area. Its singular derivative is supported on K, and its Cantor part vanishes there because $\mathcal{H}^1(K) < \infty$. Thus $U \in SBV^2(V)$ and $J_U \subset K$. The latter inclusion is pointwise: on disks avoiding K, the planar $W^{1,2}$ Poincaré inequality excludes approximate jump points.

The usual compact approximation argument now compares with the jump length instead of the possibly larger crack length; see [DLF, Appendix B.4] and [DFP, Theorem C]. Here are the relevant localization details. Given a bounded SBV^2 competitor w equal to U outside B_a, where $B_a \Subset B_b \Subset V$ are concentric balls, approximate $w|_{B_b}$ by uniformly bounded functions w_j, smooth off compact C^1 manifolds $M_j \Subset B_b$ (possibly with boundary), with $J_{w_j} \subset M_j$, $\mathcal{H}^1(M_j \setminus J_{w_j}) = 0$, strong L^2 convergence of values and gradients, and $\mathcal{H}^1(M_j) \to \mathcal{H}^1(J_w \cap B_b)$. Choose a cutoff η equal to one on B_a and zero near ∂B_b, and use

$$v_j = \eta w_j + (1-\eta)U, \qquad L_j = (K \setminus B_a) \cup M_j.$$

Outside B_b use the original pair. Radii may be chosen to have zero crack length on their boundary. The annulus is unchanged in the limit, since $w = U$ there. Taking $j \to \infty$ in minimality yields

$$\int_{B_b} (|\nabla U|^2 + \lambda|U-g|^2) + \mathcal{H}^1(K \cap B_a) \le \int_{B_b} (|\nabla w|^2 + \lambda|w-g|^2) + \mathcal{H}^1(J_w \cap B_b).$$

Keeping w fixed, let $a \uparrow b$. Taking $w = U$ first gives $\mathcal{H}^1(K \setminus J_U) = 0$ locally. Reducedness and relative closedness then imply $K = \overline{J_U}$ in V. The same comparison, with unbounded competitors first truncated, proves weak minimality. In particular K is rectifiable. Finally, a finite-energy competitor in the locally Sobolev class of [DLF, Definition 1.2.1] can be truncated to a bound for U and g; on the bounded ball it then belongs to $W^{1,2}$ and is an admissible competitor in the original formulation. This gives the asserted absolute minimality. $\square$

2.2 The established global theory

We use *generalized global minimizer* exactly as in [DLF, Definition 2.2.4]: it is a triple $(u, K, \{p_{kl}\})$ arising from rescaled absolute minimizers. The extended-real parameters p_{kl} record relative additive constants on the connected components of $\mathbb{R}^2 \setminus K$. They are part of the definition and are retained. In particular, no invariance under critical dilations is included in this terminology.

The compactness theorem [DLF, Theorem 2.2.3] supplies a generalized global limit along a subsequence of every sequence of interior rescalings. It gives local Hausdorff convergence of cracks and separate convergence of Dirichlet energies and length measures. Its limits are reduced, rectifiable, and have locally finite length; their displacement belongs to $W^{1,2}(B_R \setminus K)$ for every $R > 0$. For every generalized global minimizer, the upper density bound [DLF, Lemma 2.1.2] gives

$$\int_{B_R(z)\setminus K} |\nabla u|^2 + \mathcal{H}^1(K \cap B_R(z)) \le 2\pi R \qquad (z \in \mathbb{R}^2,\ R > 0). \tag{2.1}$$

Moreover, [DLF, Corollary 3.1.5] gives a relatively open regular subset of K of full $\mathcal{H}^1$ measure. Near each of its points the entire crack is a single embedded C^1 arc.

If $D = \mathbb{R}^2 \setminus K$ is connected, the exterior separation condition on competitors in [DLF, Definition 2.2.2] is vacuous. Consequently [DLF, Theorem 2.2.3(ii)] makes (K, u) an ordinary absolute minimizer of the fidelity-free energy on the plane. In particular, all compactly supported displacement variations and smooth material variations used in Section 3 are admissible. No assertion about homogeneity is used in these facts.

The geometric models are the empty crack, a line, and three half-lines meeting at 120°, with vanishing bulk gradient, and the canonical crack tip. Up to translation, rotation, sign, and an additive constant, the latter is

$$K = \{(r,0) : r \geq 0\}, \qquad u(r,\theta) = \sqrt{\frac{2}{\pi}}\sqrt{r}\cos\frac{\theta}{2}, \quad r > 0, \quad 0 < \theta < 2\pi. \tag{2.2}$$

The component parameters in the disconnected models are interpreted as in [DLF, Section 2.4]. We shall use the following classification results:

(i) If $\mathbb{R}^2 \setminus K$ is disconnected, the minimizer is a pure jump or a triod [DLF, Theorem 4.4.1].

(ii) The crack has at most one unbounded connected component [DLF, Corollary 4.4.3].

(iii) If all but at most one connected component of K lie in one common compact set, the minimizer is elementary or a canonical crack tip [DLF, Theorem 1.4.6, first paragraph].

These are the established David–Léger and Bonnet–David classification results, in the form needed for generalized limits.

Deduction of Theorem 1.1 from Theorem 3.4. Lemma 2.1 reduces the assertion locally to reduced absolute minimizers in the standard theory. Theorem 3.4 classifies every generalized global minimizer, so [DLF, Theorem 1.4.5] gives the Mumford–Shah regularity conclusion. The local regularity statements, including the prescribed angles and C^1 regularity up to the endpoints, are contained in [DLF, Theorem 1.3.3]. They allow a bounded measurable fidelity datum. The source normalization $\lambda \leq 1$ is harmless: with $a = (1+\lambda)^{-1/2}$, the change

$$x = x_0 + ay, \qquad \widehat{u}(y) = a^{-1/2}u(x_0 + ay), \qquad \widehat{g}(y) = a^{-1/2}g(x_0 + ay)$$

replaces the coefficient by $\widehat{\lambda} = \lambda a^2 \leq 1$.

To obtain the exact closed-ball formulation, parametrize the finitely many local C^1 arms regularly by $q_i(t)$, with $q_i(0) = x$. Then $\frac{d}{dt}|q_i(t) - x| \to |q_i'(0)| > 0$ as $t \downarrow 0$. A sufficiently small circle therefore meets each initial arm exactly once and misses all remaining pieces. Restriction and arclength reparametrization give (1.4)–(1.6). In such a neighborhood every point other than the center lies in the interior of one arc. Endpoints and triple junctions consequently have no interior accumulation point, which gives local finiteness. □

3 Compact translations and global classification

The central argument is a rigidity statement for compact pieces of a global crack. It uses two independent translations, each accompanied by an optimal first-order correction of the displacement. Their second-order costs exhaust a full-plane Hodge identity. Equality then gives two independent normal conditions on every regular portion of the crack.

3.1 Admissible variations and their relaxation

For a closed set $K \subset \mathbb{R}^2$, put $D = \mathbb{R}^2 \setminus K$ and define the real test space

$$\mathscr{T}_K = \{\varphi \in W^{1,2}(D) : \varphi = 0 \text{ a.e. outside some compact subset of } \mathbb{R}^2\}.$$

The compact support in this definition is a support in the plane; it may meet K. We shall use the closed subspace

$$\mathcal{V}_K = \overline{\{\nabla\varphi : \varphi \in \mathscr{T}_K\}}^{L^2(\mathbb{R}^2;\mathbb{R}^2)}$$

and its orthogonal projection Π_K. Sets of zero planar measure are ignored when identifying vector fields on D with vector fields on the plane.

For a generalized global minimizer with connected complement, absolute minimality gives the weak equation

$$\int_D \nabla u \cdot \nabla \varphi = 0 \qquad (\varphi \in \mathscr{T}_K) \tag{3.1}$$

and the inner variation identity

$$\int_D (|\nabla u|^2 \operatorname{div} X - 2\nabla u \cdot DX \nabla u) + \int_K \tau \cdot DX\tau \, \mathrm{d}\mathcal{H}^1 = 0. \tag{3.2}$$

Here $X \in C_c^\infty(\mathbb{R}^2;\mathbb{R}^2)$ and τ is a unit approximate tangent to K. These are the usual variational identities; see [DLF, Section 2.5]. Minimality also gives second-order stability under the material variations

$$\Phi_t = \operatorname{Id} + tX, \qquad K_t = \Phi_t(K), \qquad u_t \circ \Phi_t = u + t\varphi. \tag{3.3}$$

Differences of the infinite global energies are evaluated on a bounded region containing the support of the variation.

Proposition 3.1 (Compact translation rigidity)**.** *Let $(u, K, \{p_{kl}\})$ be a reduced generalized global minimizer with connected complement. There is no nonempty compact proper subset of K which is both open and closed relative to K.*

Lemma 3.2 (Relaxed second variation)**.** *For $X \in C_c^\infty(\mathbb{R}^2;\mathbb{R}^2)$, write $A = DX$, $p = \nabla u$, and*

$$m_X = ((\operatorname{tr} A)I - A - A^T)p, \tag{3.4}$$

$$Q_X = \int_D (|A^T p|^2 - \det A\,|p|^2) + \frac{1}{2}\int_K (|A\tau|^2 - (\tau \cdot A\tau)^2) \, \mathrm{d}\mathcal{H}^1. \tag{3.5}$$

For a generalized global minimizer with connected complement,

$$Q_X \geq \|\Pi_K m_X\|_{L^2}^2. \tag{3.6}$$

Proof. Pulling back the bulk integral in (3.3) gives the coefficient matrix

$$\begin{aligned} M_t &= \det(I + tA)(I + tA)^{-1}(I + tA)^{-T} \\ &= I + t((\operatorname{tr} A)I - A - A^T) + t^2(AA^T - \det A\, I) + O(t^3). \end{aligned}$$

The second-order identity follows from the two-dimensional Cayley–Hamilton formula $A^2 - (\operatorname{tr} A)A + \det A\, I = 0$. Expanding $(p + t\nabla\varphi) \cdot M_t(p + t\nabla\varphi)$ and the tangential Jacobian $|(I + tA)\tau|$, the coefficient of t^2 in the energy difference is

$$Q_X + \|\nabla\varphi\|_2^2 + 2\langle m_X, \nabla\varphi\rangle.$$

The coefficient of t vanishes by (3.1) and (3.2). The remainder estimates are justified on a fixed bounded set: DX is bounded, u has finite local energy and φ has finite Sobolev norm. In particular no differentiation of u across K is involved.

Second-order stability makes this quadratic expression nonnegative for every actual test φ. Its infimum over the closure of their gradients is $Q_X - \|\Pi_K m_X\|_2^2$. This proves (3.6). The argument does not require the projected vector field itself to possess a globally square-integrable potential. $\square$

3.2 Two translations and equality in the Hodge decomposition

Proof of Proposition 3.1. Suppose that $S \subsetneq K$ is nonempty, compact and relatively open and closed. Its nonempty complement $K \setminus S$ is closed and has positive distance from S. We may therefore choose a real $\chi \in C_c^\infty(\mathbb{R}^2)$ such that

$$\chi = 1 \quad \text{near } S, \qquad \chi = 0 \quad \text{near } K \setminus S.$$

Thus $\operatorname{supp} \nabla\chi$ is a compact subset of D. Reducedness and relative openness imply

$$0 < \mathcal{H}^1(S) < \infty. \tag{3.7}$$

Set $X_1 = \chi e_1$ and $X_2 = \chi e_2$. These vector fields translate S rigidly and fix the remainder near the crack. Their derivative matrices vanish on a neighborhood of K, so their surface second variations vanish exactly. Let $\mathsf{J}(a, b) = (-b, a)$. Formula (3.4) gives

$$\begin{aligned} m := m_{X_1} &= (-\chi_x u_x - \chi_y u_y, -\chi_y u_x + \chi_x u_y), \\ m_{X_2} &= \mathsf{J}m, \\ Q_{X_1} &= \int_D u_x^2 |\nabla\chi|^2, \qquad Q_{X_2} = \int_D u_y^2 |\nabla\chi|^2. \end{aligned} \tag{3.8}$$

Consequently

$$Q_{X_1} + Q_{X_2} = \|m\|_2^2. \tag{3.9}$$

The field m is smooth and compactly supported in D, since (3.1) makes u harmonic there.

Let $\mathcal{V}$ be the L^2 closure of gradients of real functions in $C_c^\infty(\mathbb{R}^2)$, and let Π be its orthogonal projection. Then $\mathcal{V} \subset \mathcal{V}_K$. The full-plane Fourier projection is the orthogonal projection onto each nonzero frequency direction; hence

$$\|\Pi m\|_2^2 + \|\Pi(\mathsf{J}m)\|_2^2 = \|m\|_2^2. \tag{3.10}$$

Combining (3.6), (3.9) and (3.10), we obtain

$$\begin{aligned} \|m\|_2^2 &\geq \|\Pi_K m\|_2^2 + \|\Pi_K(\mathsf{J}m)\|_2^2 \\ &\geq \|\Pi m\|_2^2 + \|\Pi(\mathsf{J}m)\|_2^2 = \|m\|_2^2. \end{aligned}$$

Both nonnegative projection differences vanish. Thus

$$R_1 = m - \Pi m \perp \mathcal{V}_K, \qquad R_2 = \mathsf{J}m - \Pi(\mathsf{J}m) \perp \mathcal{V}_K. \tag{3.11}$$

Normal conditions on a regular arc

Identify a vector (a_1, a_2) with $a_1 + ia_2$, and define the holomorphic complex gradient

$$F = u_x - iu_y \quad \text{on } D.$$

Then $m = -2F\partial_{\bar z}\chi$. Let $\mathcal{B} = \partial_z \partial_{\bar z}^{-1}$ be the Beurling transform on the plane. The Fourier formula for Π is

$$\Pi a = \frac{1}{2}(a + \overline{\mathcal{B}a}). \tag{3.12}$$

Equivalently, the right-hand side is curl-free and its complement is divergence-free, which characterizes the orthogonal decomposition in L^2. Because $\mathcal{B}$ is complex-linear, wherever $m = 0$ one has

$$R_1 = -\frac{1}{2}\overline{\mathcal{B}m}, \qquad R_2 = \frac{i}{2}\overline{\mathcal{B}m} = -\mathsf{J}R_1. \tag{3.13}$$

Moreover $\mathcal{B}m$ is holomorphic away from $\operatorname{supp} m$: distributionally $\partial_{\bar z}\mathcal{B}m = \partial_z m$. In particular both residual fields are smooth across K locally.

Choose a neighborhood in which the entire crack is a C^1 graph. A smooth ambient cutoff restricted to one side of that graph and extended by zero to the other side belongs to $\mathscr{T}_K$. Its support can be chosen away from the edge of the neighborhood, and its trace can be varied on a smaller subarc. Since each R_j is divergence-free, (3.11) and integration by parts give

$$R_1 \cdot \nu = R_2 \cdot \nu = 0$$

on the subarc. These equalities hold first as trace distributions and then pointwise by continuity. In view of (3.13), R_1 is orthogonal to two independent directions, so

$$\mathcal{B}m = 0 \quad \text{on every such regular subarc.} \tag{3.14}$$

Only local one-sided tests have been used; connectedness of the complement does not prevent those tests from having different traces on the two banks.

Holomorphic separation of the compact piece

For a smooth compactly supported complex function a, define

$$\mathcal{C}a(z) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{a(\zeta)}{z-\zeta}\,\mathrm{d}A(\zeta), \qquad \partial_{\bar z}\mathcal{C}a = a.$$

Put

$$h = \mathcal{C}(F\partial_{\bar z}\chi) = -\tfrac{1}{2}\mathcal{C}m.$$

The function h is smooth on the plane, holomorphic off $\operatorname{supp} m$, and satisfies $h(z) = O(|z|^{-1})$ uniformly at infinity. On a neighborhood of K,

$$h' = -\tfrac{1}{2}\mathcal{B}m. \tag{3.15}$$

The two functions

$$F_0 = (1-\chi)F + h, \qquad F_1 = \chi F - h \tag{3.16}$$

are holomorphic on D and sum to F. The first extends holomorphically through S, and the second through $K \setminus S$. Their domains of extension are

$$\mathbb{C} \setminus (K \setminus S), \qquad \mathbb{C} \setminus S,$$

respectively. Both domains are connected: they contain connected D, and every open component meets D because K has zero area.

By (3.7) and full-measure regularity, S contains a regular subarc. Near it $F_0 = h$, and (3.14)–(3.15) imply $F_0' = 0$ on that arc. The holomorphic identity theorem gives $F_0 = A$ for a complex constant A throughout its domain. Outside a large disk, $\chi = 0$, so

$$F = A - h = A + O(|z|^{-1}) \quad \text{on } D.$$

If $A \neq 0$, integration over the complement of the area-zero set K gives quadratic bulk-energy growth. This contradicts (2.1). Thus $A = 0$.

The nonempty set $K \setminus S$ also contains a positive-length regular subarc, by reducedness and full-measure regularity. Near that subarc $F_1 = -h$, so the same argument makes F_1 constant. Outside a large disk $F_1 = -h = O(|z|^{-1})$, and the constant is zero. Therefore $F = 0$ on D.

Contradiction with the length variation

The bulk gradient vanishes. For any fixed $a \in \mathbb{R}^2$, use

$$X(x) = \chi(x)(x - a)$$

in (3.2). On S, $DX = I$; near $K \setminus S$, $DX = 0$. Consequently the inner variation reduces to

$$0 = \int_S 1 \, \mathrm{d}\mathcal{H}^1 = \mathcal{H}^1(S) > 0,$$

a contradiction. This proves the proposition. □

3.3 Topological consequence and classification

Lemma 3.3. *Let $K \subset \mathbb{R}^2$ be closed and reduced, and assume that it is locally of finite length. If K has no nonempty compact relatively open-and-closed subset, then every connected component of K is unbounded.*

Proof. Suppose that C is a compact component, and choose R with $C \Subset B_R$. In the compact space $X = K \cap \overline{B}_R$, the component containing C is exactly C. A component of a compact Hausdorff space equals the intersection of its clopen neighborhoods. For each point of $X \cap \partial B_R$, choose such a neighborhood containing C and excluding that point. Compactness of $X \cap \partial B_R$ gives a finite intersection W of these neighborhoods which misses the sphere. If that sphere intersection is empty, take $W = X$.

The set W is compact and clopen in X. Its positive distance from ∂B_R makes it relatively open in K, and compactness makes it relatively closed in K. It is nonempty because it contains C, a contradiction. This argument includes singleton components and does not require a component to be isolated. Finally every component of a closed subset of the plane is closed, so every bounded component would be compact. □

Theorem 3.4 (Classification of generalized global minimizers)**.** *Every reduced planar generalized global minimizer is an elementary minimizer or a canonical crack tip. No homogeneity assumption is required.*

Proof. Let $(u, K, \{p_{kl}\})$ be a generalized global minimizer in the sense of [DLF, Definition 2.2.4]. If $\mathbb{R}^2 \setminus K$ is disconnected, [DLF, Theorem 4.4.1] gives a pure jump or a triod. If K is compact, including $K = \varnothing$, [DLF, Theorem 1.4.6, first paragraph] already applies, since all components are contained in a common compact set. Its only model with compact crack is the constant minimizer.

It remains to consider noncompact K with connected complement. Proposition 3.1 excludes every nonempty compact relatively open-and-closed subset of K, since any such subset would be proper. Lemma 3.3 therefore makes every component of K unbounded. By [DLF, Corollary 4.4.3] there is at most one such component, so K is connected. The first paragraph of [DLF, Theorem 1.4.6] applies again: the requirement that all but at most one component lie in a common compact set is automatic for a connected crack. This gives the asserted classification. □

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