

CONTINUITY AND STABILITY OF ENTROPY SOLUTIONS UNDER QUALITATIVE NONLINEARITY ASSUMPTIONS

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ABSTRACT. We consider bounded entropy solutions of $\operatorname{div}_x A(u) = 0$ on an open subset of $\mathbb{R}^N$, $N \geq 2$. The flux $A \in C^m(\mathbb{R}; \mathbb{R}^N)$, $m \geq \max\{N, 3\}$, is only assumed to satisfy the qualitative non-degeneracy condition that $\xi \cdot A$ is non-constant on every non-empty open interval of states, for every $\xi \neq 0$. We prove that entropy solutions are continuous outside the jump set of De Lellis, Otto, and Westdickenberg (*Arch. Ration. Mech. Anal.*, 2003). We also prove an L^1 -to- L^∞ stability estimate: on a ball, the uniform distance between an entropy solution and a continuous function is controlled by their L^1 distance on a larger ball, through a modulus depending only on the flux and on the modulus of continuity of the reference function. The nonlinearity assumption is sharp and strictly weaker than the power-law non-degeneracy in the continuity result of Silvestre (*Comm. Pure Appl. Math.*, 2019) and in the stability result of Golding (*C. R. Math. Acad. Sci. Paris*, 2024), at the price of more regularity of the flux. The key observation for the proof is that the qualitative condition yields quantitative non-degeneracy on a subinterval of every interval of values of the solution, where velocity-averaging estimates can be localized.

1. INTRODUCTION AND MAIN RESULTS

We study the regularity properties of *entropy solutions* of scalar conservation laws. Let $N \geq 2$ and $\Omega \subset \mathbb{R}^N$ be open, and let $I \subset \mathbb{R}$ be a compact interval of positive length. We consider

$$(1.1) \quad \operatorname{div}_x A(u) = 0 \quad \text{in } \Omega,$$

where $A \in C^m(\mathbb{R}; \mathbb{R}^N)$, with $m \geq \max\{N, 3\}$, and $u \in L^\infty(\Omega; I)$.¹ A function $u \in L^\infty(\Omega; I)$ is an *entropy solution* of (1.1) if, for every convex entropy $\eta \in C^2(\mathbb{R})$, the flux q_η defined up to a constant by $q'_\eta = \eta' a$, where $a := A'$ is the *velocity* associated with the flux, satisfies

$$(1.2) \quad \operatorname{div}_x q_\eta(u) \leq 0 \quad \text{in the sense of distributions in } \Omega.$$

We use the *kinetic formulation* of Lions, Perthame, and Tadmor [LPT94] to describe entropy dissipation (see also the monograph [Per02]): for an entropy solution u , there exists a non-negative Radon measure μ on $\Omega \times \mathbb{R}$, supported in $\Omega \times I$, such that

$$(1.3) \quad a(v) \cdot \nabla_x f = \partial_v \mu \quad \text{in the sense of distributions on } \Omega \times \mathbb{R},$$

where the additional variable v represents a state value and

$$(1.4) \quad f(x, v) := \mathbf{1}_{\{\inf I < v < u(x)\}}.$$

The measure μ encodes the failure of the entropy inequalities to be equalities: for every convex $\eta \in C^2(\mathbb{R})$,

$$(1.5) \quad \langle -\operatorname{div}_x q_\eta(u), \varphi \rangle = \int_{\Omega \times I} \varphi(x) \eta''(v) d\mu(x, v) \quad (\varphi \in C_c^\infty(\Omega)).$$

Following [DOW03, Definition 3, Eq. (12)], we define the set where the spatial marginal $\mu(\cdot \times I)$ has positive upper codimension-one density by

$$(1.6) \quad J := \left\{ x \in \Omega : \limsup_{r \downarrow 0} \frac{\mu(B_r(x) \times I)}{r^{N-1}} > 0 \right\}.$$

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¹ The usual time-dependent equation is included by regarding time as one of the coordinates and replacing a spatial flux F by $s \mapsto (s, F(s))$. Thus a time-dependent equation in n spatial dimensions has $N = n + 1$ and requires $m \geq \max\{n + 1, 3\}$ in the present argument. The case $N = 2$ corresponds to one spatial dimension.

In [DOW03, Theorem 1], De Lellis, Otto, and Westdickenberg proved that J is countably $(N-1)$ -rectifiable and that entropy solutions have vanishing mean oscillation outside J , under the genuine nonlinearity assumption

$$(1.7) \quad |\{v \in I : a(v) \cdot \xi = 0\}| = 0 \quad \text{for every } \xi \neq 0.$$

Later, for $A \in C^2(\mathbb{R}; \mathbb{R}^N)$, Silvestre proved continuity outside J under the quantitative non-degeneracy assumption

$$(1.8) \quad |\{v \in I : |a(v) \cdot \xi| < \delta\}| \leq C_I \delta^\alpha \quad (|\xi| = 1, \delta > 0),$$

where $0 < \alpha \leq 1$ and $C_I = C_I(a, I, \alpha) < \infty$ are fixed (see [Sil19, Assumption 1.1 and Theorems 1.2 and 6.1]).

Whether every point outside J is a Lebesgue point under the sole assumption (1.7) was recently discussed as an open problem for *quasi-entropy solutions* in [AMT26, Section 4.1] (see also [DOW03; COW08]).

In this note, for entropy solutions, we prove the stronger conclusion of continuity outside J for C^m fluxes, with $m \geq \max\{N, 3\}$, under the following weaker nonlinearity assumption (see Remark 1.2):

$$(1.9) \quad \begin{aligned} &\text{for every } \xi \in \mathbb{R}^N \setminus \{0\}, \quad v \mapsto \xi \cdot A(v) \\ &\text{is non-constant on every non-empty open interval contained in } I. \end{aligned}$$

Theorem 1.1. *Let $N \geq 2$ and $m \geq \max\{N, 3\}$ be integers, let $\Omega \subset \mathbb{R}^N$ be open, and let $I \subset \mathbb{R}$ be a compact interval of positive length. Let $A \in C^m(\mathbb{R}; \mathbb{R}^N)$ satisfy (1.9). Every entropy solution $u \in L^\infty(\Omega; I)$ of (1.1) has vanishing essential oscillation at every point outside J :*

$$(1.10) \quad \lim_{r \downarrow 0} \left(\operatorname{ess\,sup}_{B_r(x)} u - \operatorname{ess\,inf}_{B_r(x)} u \right) = 0 \quad \text{for every } x \in \Omega \setminus J.$$

In particular, u has a representative $\tilde{u} : \Omega \rightarrow I$ which is continuous at every point of $\Omega \setminus J$.

Remark 1.2 (Comparison of the nonlinearity conditions). Condition (1.8) implies (1.7). Indeed, fix $\xi \neq 0$ and apply (1.8) to $\xi/|\xi|$. For every $\delta > 0$, $\{v \in I : a(v) \cdot \xi = 0\} \subset \left\{v \in I : \left|a(v) \cdot \frac{\xi}{|\xi|}\right| < \delta\right\}$. Letting $\delta \downarrow 0$ in (1.8) shows that the exact zero set has measure zero. In turn, (1.7) implies (1.9). If $v \mapsto \xi \cdot A(v)$ were constant on a non-empty open interval $V \subset I$, then $\xi \cdot a(v) = \frac{d}{dv}(\xi \cdot A(v)) = 0$ for every $v \in V$, and hence the zero set of $a(\cdot) \cdot \xi$ would have positive measure.

Moreover, (1.9) is strictly weaker than (1.7), as the following example shows. Let $I = [0, 1]$ and let $E \subset [0, 1]$ be a closed nowhere-dense set of positive Lebesgue measure containing 0 and 1. Define $g = 0$ on E and outside $[0, 1]$, and on each complementary interval (b, c) set $g(v) = \exp\left(-\frac{1}{(v-b)(c-v)}\right)$. Then $g \in C^\infty(\mathbb{R})$: its derivatives vanish to every order at the end-points of each gap, and their suprema on gaps tend to zero faster than any power of the gap length. Moreover, g is non-constant on every non-empty open subinterval of I . For any $N \geq 2$, let $a(v) = (1, g(v), \dots, g(v)^{N-1})$. If $\xi \cdot A$ were constant on an open interval $V \subset I$, then the polynomial $\xi_1 + \xi_2 t + \dots + \xi_N t^{N-1}$ would vanish on $g(V)$, an interval of positive length, forcing $\xi = 0$. Thus (1.9) holds. On the other hand, for $\xi = e_2$ the zero set of $a(v) \cdot \xi$ contains E , so (1.7), and hence (1.8), fails. This example also has the evolutionary form $A(v) = (v, F(v))$.

If A is real-analytic, then (1.9) already implies (1.8), so that Silvestre's theorem applies. Indeed, for every $v_0 \in I$ the vectors $a^{(j)}(v_0)$, $j \geq 0$, span $\mathbb{R}^N$: otherwise some $\xi \neq 0$ would be orthogonal to all of them, and by analyticity $\xi \cdot a \equiv 0$ near v_0 , contradicting (1.9). Hence there is $k(v_0)$ such that $a(v_0), \dots, a^{(k(v_0))}(v_0)$ span $\mathbb{R}^N$; by compactness of I and continuity of the derivatives, a single k works on all of I , and [Sil19, Proposition 2.6] yields (1.8) with $\alpha = 1/k$.

For time-dependent equations in one spatial dimension, Bianchini and Marconi [BM17, Proposition 4.3 and Corollary 4.6] obtained continuity outside their jump set when the scalar spatial flux has no affine interval (which is the one-dimensional evolutionary counterpart of (1.9)).

Liverani's master's thesis [Liv19, Theorem 4.2.15] gives an alternative proof of continuity outside J in one spatial dimension under (1.7), by upgrading the vanishing mean oscillation property of [DOW03] through the Lagrangian representation of [BBM17]; for two spatial dimensions it sketches an argument that relies on an unproved structural assumption on the set of Lagrangian curves [Liv19, Assumption 4.3.4].

The present note gives a different argument, which works in every dimension $N \geq 2$ and uses the nonlinearity of the flux only through the qualitative condition (1.9). To our knowledge, [Theorem 1.1](#) is the first continuity result for multi-dimensional scalar conservation laws under a purely qualitative nonlinearity assumption. The assumption cannot be weakened: if (1.9) fails, there are entropy-conservative solutions with jumps, so that $J = \emptyset$ while continuity fails (see [Remark 1.9](#)). The price we pay is the regularity $A \in C^m$, with $m \geq \max\{N, 3\}$ (cf. [Remark 2.5](#)).

Our second result is a local stability estimate of L^1 -to- L^∞ type: on the unit ball, the uniform distance between an entropy solution u and a continuous reference function v is controlled by their L^1 distance on the ball of radius two, through a modulus that depends only on the flux, on the range I , and on the modulus of continuity of v . Under the power-law condition (1.8), two estimates of this type are known: Silvestre [[Sil19](#), Theorem 6.1 and Remark 6.3] proved a quantitative estimate when the reference function is a constant, and Golding [[Gol24](#), Theorem 2] a power estimate when the reference function is a Lipschitz entropy solution. Here we replace (1.8) by the qualitative condition (1.9) and allow reference functions with an arbitrary prescribed modulus of continuity,² which need not be solutions.

Theorem 1.3. *Let $N \geq 2$ and $m \geq \max\{N, 3\}$ be integers, let $I \subset \mathbb{R}$ be a compact interval of positive length, and let $A \in C^m(\mathbb{R}; \mathbb{R}^N)$ satisfy (1.9). For every modulus ρ there exists a modulus $\omega_{A,I,\rho}$ with the following property. If $u \in L^\infty(B_2(0); I)$ is an entropy solution of (1.1) and $v : B_2(0) \rightarrow I$ satisfies*

$$|v(x) - v(y)| \leq \rho(|x - y|) \quad (x, y \in B_2(0)),$$

then

$$(1.11) \quad \|u - v\|_{L^\infty(B_1(0))} \leq \omega_{A,I,\rho}(\|u - v\|_{L^1(B_2(0))}).$$

Besides the higher regularity of the flux, the price for replacing (1.8) by (1.9) is the loss of an explicit rate in [Theorem 1.3](#).

We now list some consequences of [Theorem 1.3](#). First, we note that convergence of bounded entropy solutions in L^1_{loc} towards a continuous limit improves to convergence in L^∞_{loc} .

Corollary 1.4. *Let $N \geq 2$ and $m \geq \max\{N, 3\}$ be integers, let $\Omega \subset \mathbb{R}^N$ be open, let $I \subset \mathbb{R}$ be a compact interval of positive length, and let $A \in C^m(\mathbb{R}; \mathbb{R}^N)$ satisfy (1.9). Let $u_j \in L^\infty(\Omega; I)$ be entropy solutions and let $v \in C(\Omega; I)$. If $u_j \rightarrow v$ in $L^1_{\text{loc}}(\Omega)$, then $u_j \rightarrow v$ in $L^\infty_{\text{loc}}(\Omega)$. More generally, if $v_j : \Omega \rightarrow I$ have a common modulus of continuity on every compact subset and $u_j - v_j \rightarrow 0$ in $L^1_{\text{loc}}(\Omega)$, then $u_j - v_j \rightarrow 0$ in $L^\infty_{\text{loc}}(\Omega)$.*

For Lipschitz reference functions, the dependence on the domain can be made explicit. The following corollary is the qualitative counterpart of [[Gol24](#), Theorem 2], which, under (1.8) and for a reference function that is a Lipschitz entropy solution, gives the power estimate $\|u - v\|_{L^\infty(\Omega')} \leq C(1 + dL)^\eta d^{-N\gamma} \|u - v\|_{L^1(\Omega)}^\gamma$, with exponents $\gamma \in (0, 1)$ and $\eta \in (0, \alpha/2)$, on space-time cylinders.

Corollary 1.5. *Let $N \geq 2$ and $m \geq \max\{N, 3\}$ be integers, let $\Omega \subset \mathbb{R}^N$ be open, let $I \subset \mathbb{R}$ be a compact interval of positive length, and let $A \in C^m(\mathbb{R}; \mathbb{R}^N)$ satisfy (1.9). There is a modulus $\omega_{A,I}$ such that the following holds. Let $u : \Omega \rightarrow I$ be an entropy solution, let $v : \Omega \rightarrow I$ be Lipschitz with constant L , assume $u - v \in L^1(\Omega)$, and let $\Omega' \Subset \Omega$. For every finite $d > 0$ with $d \leq \text{dist}(\Omega', \mathbb{R}^N \setminus \Omega)$,*

$$(1.12) \quad \|u - v\|_{L^\infty(\Omega')} \leq \omega_{A,I} \left(\frac{2^N(1 + dL)^N}{d^N} \|u - v\|_{L^1(\Omega)} \right).$$

We now turn to the large-time behavior of entropy solutions of the time-dependent equation

$$(1.13) \quad \partial_t u + \text{div}_x F(u) = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n.$$

Here we set $N = n + 1$ and $A(s) = (s, F(s))$: an entropy solution of (1.13) is an entropy solution of (1.1) on $\Omega = (0, \infty) \times \mathbb{R}^n$, and in this notation (1.9) is equivalent to

$$(1.14) \quad \xi \cdot F \text{ is not affine on any non-empty open interval in } I, \text{ for every } \xi \neq 0.$$

Global entropy solutions of (1.13) are understood in the sense of Kruřkov [[Kru70](#)], with their representatives in $C([0, \infty); L^1_{\text{loc}}(\mathbb{R}^n))$; see [[Pan05](#), Corollary 7.1].

² By a modulus of continuity we mean a continuous non-decreasing function $\rho : [0, \infty) \rightarrow [0, \infty)$ with $\rho(0) = 0$.

The first consequence is a qualitative counterpart of [Gol24, Corollary 4]: an entropy solution at finite initial L^1 distance from a reference solution whose spatial Lipschitz constant decays like $1/t$ approaches it uniformly as $t \rightarrow \infty$. The decay assumption holds, for instance, for self-similar solutions $v(t, x) = U(x/t)$ with U Lipschitz. Under (1.8), [Gol24, Corollary 4] gives the rate $C(1 + \Gamma)^\eta (t^{-n} \|u_0 - v_0\|_{L^1(\mathbb{R}^n)})^\gamma$ for some $\gamma, \eta > 0$; here the power γ is replaced by the modulus $\omega_{F,I}$, while the quantity $t^{-n} \|u_0 - v_0\|_{L^1(\mathbb{R}^n)}$ on which it acts is the same.

Corollary 1.6. *Let $F \in C^m(\mathbb{R}; \mathbb{R}^n)$ with $m \geq \max\{n + 1, 3\}$ satisfy (1.14). Let u, v be global entropy solutions of (1.13) with values in I and initial data u_0, v_0 , respectively, such that $u_0 - v_0 \in L^1(\mathbb{R}^n)$. Suppose that v is locally Lipschitz for positive times and that $\sup_{t>0} t \|\nabla_x v(t, \cdot)\|_{L^\infty(\mathbb{R}^n)} \leq \Gamma$. Then there exists a modulus $\omega_{F,I}$ such that*

$$(1.15) \quad \|u(t, \cdot) - v(t, \cdot)\|_{L^\infty(\mathbb{R}^n)} \leq \omega_{F,I} \left(C_n (1 + C_F \Gamma)^n \frac{\|u_0 - v_0\|_{L^1(\mathbb{R}^n)}}{t^n} \right) \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

where $C_F := \sqrt{1 + \|F'\|_{L^\infty(I)}^2}$ and $C_n := 4 \cdot 8^n$.

In the proof of Corollary 1.6, the Lipschitz bound ensures that v is nearly constant on space-time balls of radius comparable to t . The next corollary assumes this property directly, and thus covers reference solutions that are not Lipschitz, to which [Gol24, Corollary 4] does not apply (see [Gol24, Remark 5] for a discussion of the decay assumption on $\|\nabla_x v(t, \cdot)\|_{L^\infty(\mathbb{R}^n)}$).

Corollary 1.7. *Let $F \in C^m(\mathbb{R}; \mathbb{R}^n)$ with $m \geq \max\{n + 1, 3\}$ satisfy (1.14), and let u, v be global entropy solutions of (1.13) with values in I and initial data u_0, v_0 such that $u_0 - v_0 \in L^1(\mathbb{R}^n)$. Suppose that v has a continuous representative for positive times and that, for every $\eta > 0$, there are $r_\eta \in (0, 1/4)$ and $t_\eta > 0$ such that*

$$(1.16) \quad \sup_{z \in B_{2r_\eta t}(t, x)} |v(z) - v(t, x)| \leq \eta \quad (t \geq t_\eta, x \in \mathbb{R}^n).$$

Then

$$\|u(t, \cdot) - v(t, \cdot)\|_{L^\infty(\mathbb{R}^n)} \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Finally, we apply Corollary 1.7 to planar self-similar solutions, which include the rarefaction waves connecting two constant states. Their profiles are continuous but need not be Lipschitz: for a scalar flux $f = e \cdot F$ that is strictly convex on $[u_-, u_+]$, the rarefaction profile agrees with $(f')^{-1}$ inside the fan and fails to be Lipschitz where f'' vanishes. Corollary 1.6, which requires v to be Lipschitz, therefore does not apply to them in general, while Corollary 1.7 does. Planar rarefaction waves with Lipschitz profiles are the main application of [Gol24, Corollary 4]; Corollary 1.8 removes the Lipschitz requirement on the profile, at the price of the rate.

Corollary 1.8. *Let $F \in C^m(\mathbb{R}; \mathbb{R}^n)$ with $m \geq \max\{n + 1, 3\}$ satisfy (1.14). Let $e \in \mathbb{R}^n$ be a unit vector, let $U : \mathbb{R} \rightarrow I$ be continuous, and suppose that*

$$v(t, x) := U((e \cdot x)/t)$$

is an entropy solution of (1.13). Then U is constant on $(-\infty, -R]$ and on $[R, \infty)$, where $R := \|F'\|_{L^\infty(I)}$; the function v is the global entropy solution of (1.13) with initial datum

$$v_0 := U(-\infty) \mathbf{1}_{\{e \cdot x < 0\}} + U(+\infty) \mathbf{1}_{\{e \cdot x > 0\}};$$

and every global entropy solution u of (1.13) with values in I and initial datum u_0 with $u_0 - v_0 \in L^1(\mathbb{R}^n)$ satisfies

$$\|u(t, \cdot) - v(t, \cdot)\|_{L^\infty(\mathbb{R}^n)} \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Remark 1.9 (Necessity of the nonlinearity condition). If (1.9) fails, then neither the continuity conclusion (1.10) of Theorem 1.1 nor the stability estimate (1.11) of Theorem 1.3 (already for constant references, $\rho \equiv 0$) can hold for all entropy solutions with values in I .

Suppose that $\xi \cdot A$ is constant on an open interval $V \subset I$, with $\xi \neq 0$, and choose $c < c + h$ in V . For every entropy flux q_η ,

$$\xi \cdot (q_\eta(c + h) - q_\eta(c)) = \int_c^{c+h} \eta'(v) \xi \cdot A'(v) dv = 0.$$

Consequently $u(x) = c + h\mathbb{1}_{\{x \cdot \xi > 0\}}$ satisfies every entropy equality. Its kinetic measure is zero and $J = \emptyset$, although it has a jump. Similarly, $u_k(x) = c + h\mathbb{1}_{\{0 < x \cdot \xi < 1/k\}}$ are entropy-conservative, converge to c in L^1_{loc} , and retain an L^∞ discrepancy h on every ball centered on $\{x \cdot \xi = 0\}$.

Fluxes that are affine on an interval of states are also the obstruction to strong precompactness and to the existence of strong traces of u itself; see [KV07; Pan07] for the theory of traces without non-degeneracy assumptions and [Mit26] for a recent discussion.

1.1. Strategy of the proofs. We prove the stability result, Theorem 1.3, first, in Section 3, and then deduce Theorem 1.1 from it in Section 4; the tools needed for both proofs are collected in Section 2.

The point of the argument is that quantitative non-degeneracy is never needed on the whole state interval. Assumption (1.9) and the existence of N continuous derivatives of A imply, by a Wronskian argument, that every non-empty open interval of states contains a compact subinterval K on which the power-law estimate (1.8) holds with I replaced by K (Lemma 2.3). On such an interval, Silvestre's averaging estimate and De Giorgi-type iteration can be run with the velocity variable localized to K : if two levels $\ell_0 < U$ lie in the interior of K and the mass of u above ℓ_0 is small on a ball, then u cannot exceed U on the half-ball (Lemma 2.4).

To prove Theorem 1.3, we freeze the continuous reference v on small balls and apply Lemma 2.4 on bands between its frozen value and that value plus ε (by reflection we obtain also the corresponding lower bound). A finite net of states and a covering of $B_1(0)$ make the resulting smallness condition on $\|u - v\|_{L^1}$ uniform in the reference function; these conditions define the modulus $\omega_{A,I,\rho}$.

Corollaries 1.4–1.8 are deduced from Theorem 1.3 by translation and dilation: if u is an entropy solution of (1.1) on Ω , then $u(z + r \cdot)$ is an entropy solution on $(\Omega - z)/r$. For the time-dependent equation, this is combined with Kruřkov's L^1 -contraction, which bounds the L^1 -distance of two solutions on space–time balls of radius comparable to t by their initial L^1 -distance.

To prove Theorem 1.1, we rescale the solution around a point $x_0 \notin J$. Owing to Panov's compactness theorem [Pan16, Theorem 1.1] (Lemma 2.1), which requires only (1.9), a subsequence of the rescaled solutions converges strongly in L^1 on every fixed ball. The definition of J implies that the rescaled entropy dissipation tends to zero. A rigidity result for entropy-conservative solutions on $\mathbb{R}^N$ (Lemma 2.2), which under (1.7) is due to De Lellis, Otto, and Westdickenberg [DOW03, Proposition 6(b)], yields that the limit is a constant c . This gives convergence to c in L^1 but not yet continuity, since values a fixed distance from c could persist on sets of vanishing measure. Theorem 1.3, applied with the constant reference c , upgrades this convergence to L^∞ convergence on the half-ball and hence gives vanishing essential oscillation at x_0 .

2. LOCAL ESTIMATES, COMPACTNESS, AND RIGIDITY

In this section, we collect the four ingredients needed for the proofs of Theorem 1.1 and Theorem 1.3.

First, we recall that the qualitative condition (1.9) is sufficient for Panov's compactness theorem [Pan16, Theorem 1.1], which we use in the following form.

Lemma 2.1. *Let $N \geq 2$, let $\Omega \subset \mathbb{R}^N$ be open, and let $I \subset \mathbb{R}$ be a compact interval of positive length. Let $A \in C^1(\mathbb{R}; \mathbb{R}^N)$ satisfy (1.9). Every sequence of entropy solutions $u_j \in L^\infty(\Omega; I)$ of (1.1) has a subsequence converging strongly in $L^1_{\text{loc}}(\Omega)$ to an entropy solution with values in I .*

For bounded solutions on the whole space with zero entropy dissipation, the rigidity result of [DOW03, Proposition 6(b)] holds.³

Lemma 2.2. *Let $A \in C^1(\mathbb{R}; \mathbb{R}^N)$, set $a := A'$, let $I \subset \mathbb{R}$ be a compact interval of positive length, and assume (1.9). If $u_\infty \in L^\infty(\mathbb{R}^N; I)$ satisfies*

$$(2.1) \quad a(v) \cdot \nabla_x \mathbb{1}_{\{v < u_\infty(x)\}} = 0 \quad \text{in the sense of distributions on } \mathbb{R}^N \times \text{int } I,$$

then u_∞ is almost everywhere constant.

³ The cited proposition assumes (1.7), but an inspection of its proof shows that the argument works under (1.9). Indeed, the proof uses nonlinearity only to select N linearly independent velocities in each non-empty open interval of states. This selection follows from (1.9), since $\text{span}\{a(v) : v \in V\} = \mathbb{R}^N$ for every such interval $V \subset I$.

Although (1.9) does not imply the global quantitative estimate (1.8), the existence of at least N continuous derivatives of A yields such an estimate on a smaller interval inside every open interval of states.

Lemma 2.3. *Let $N \geq 2$ and $m \geq N$ be integers, let $I \subset \mathbb{R}$ be a compact interval of positive length, and let $A \in C^m(\mathbb{R}; \mathbb{R}^N)$, with $a := A'$. Assume (1.9). Then every non-empty open interval $V \subset I$ contains a compact interval K of positive length for which there is a constant $0 < C_K = C_K(a, K) < \infty$ such that*

$$(2.2) \quad |\{v \in K : |a(v) \cdot \xi| < \delta\}| \leq C_K \delta^{1/(N-1)} \quad (|\xi| = 1, \delta > 0).$$

Proof. If $\det[a, a', \dots, a^{(N-1)}]$ vanished identically on V , [Hag26, Theorem 2.3], applied to the components of a , would give a nonempty open subinterval $W \subset V$ and a constant vector $\xi \neq 0$ with $\xi \cdot a = 0$ on W . Then $\xi \cdot A$ would be constant there, contrary to (1.9). The theorem applies because $a \in C^{N-1}$.

The determinant is therefore non-zero at some point of V and, by continuity, on a compact interval $K \subset V$ of positive length. In particular,

$$\min_{v \in K, |\xi|=1} \max_{0 \leq j \leq N-1} |\xi \cdot a^{(j)}(v)| > 0.$$

Now [Sil19, Proposition 2.6] yields (2.2) with exponent $1/(N-1)$. $\square$

The next lemma localizes Silvestre's averaging and De Giorgi iteration argument [Sil19, Proposition 5.1 and Theorem 6.1] to a compact state interval K whose interior contains the two truncation levels.

Lemma 2.4. *Let $N \geq 2$, let $I \subset \mathbb{R}$ be a compact interval of positive length, and let $A \in C^3(\mathbb{R}; \mathbb{R}^N)$, with $a := A'$. Let $K \subset I$ be a compact interval of positive length, and assume*

$$(2.3) \quad |\{v \in K : |a(v) \cdot \xi| < \delta\}| \leq C_K \delta^\alpha \quad (|\xi| = 1, \delta > 0),$$

where $0 < \alpha \leq 1$ and $C_K < \infty$. Fix $\ell_0 < U$ with $[\ell_0, U] \subset \text{int } K$. There exists $\varepsilon_0 = \varepsilon_0(N, I, K, \alpha, C_K, \|a\|_{C^2(I)}, \ell_0, U) > 0$ such that every entropy solution $u \in L^\infty(B_2(0); I)$ satisfies

$$(2.4) \quad \int_{B_2(0)} (u - \ell_0)_+ dx \leq \varepsilon_0 \implies u \leq U \text{ almost everywhere in } B_1(0).$$

Proof. Step 1: Truncations and their kinetic equations. Fix $h > 0$ with $[\ell_0, U + h] \subset \text{int } K$ and choose $\chi \in C_c^\infty(\text{int } K)$ with $0 \leq \chi \leq 1$ and $\chi = 1$ on $[\ell_0, U + h]$. For $\ell \in [\ell_0, U]$, define

$$(2.5) \quad f_\ell(x, v) := \mathbb{1}_{\{\ell < v < u(x)\}}, \quad w_\ell(x) := \int_K \chi(v) f_\ell(x, v) dv.$$

Because $\chi = 1$ on $[\ell, \ell + h]$ and $h \leq |K| \leq |I|$, we have

$$(2.6) \quad \frac{h}{|I|} (u - \ell)_+ \leq \min\{(u - \ell)_+, h\} \leq w_\ell \leq (u - \ell)_+.$$

By [Sil19, Lemma 2.2], $\max\{u, \ell\}$ is an *entropy subsolution* of (1.1) in the convex set $B_2(0)$, that is, $\text{div}_x q_\eta(\max\{u, \ell\}) \leq 0$ in the sense of distributions in $B_2(0)$ for every convex non-decreasing $\eta \in C^2(\mathbb{R})$. Since the entropy fluxes of the shifted flux $s \mapsto A(\ell + s) - A(\ell)$ are, up to constants, the functions $s \mapsto q_\eta(\ell + s)$, the function $(u - \ell)_+ = \max\{u, \ell\} - \ell$ is a non-negative entropy subsolution for the shifted flux. Applying [Sil19, Proposition 4.1] to $(u - \ell)_+$, whose kinetic function is $\mathbb{1}_{\{0 < s < (u - \ell)_+(x)\}}$, and returning to the variable $v = \ell + s$ gives

$$(2.7) \quad a(v) \cdot \nabla_x f_\ell = \partial_v \mu_{0,\ell} - \mu_{1,\ell} \quad \text{in the sense of distributions on } B_2(0) \times \mathbb{R},$$

for two non-negative Radon measures $\mu_{0,\ell}$ and $\mu_{1,\ell}$ supported in $B_2(0) \times [\ell, \sup I]$ (these measures are constructed in the proof of [Sil19, Proposition 4.1]).

Let $1 \leq r < R \leq 2$ and choose $\varphi = \varphi_{r,R} \in C_c^\infty(B_{(r+R)/2}(0))$ equal to one on $B_r(0)$, with $0 \leq \varphi \leq 1$ and $|\nabla \varphi| \leq c_N/(R-r)$, where $c_N = c_N(N)$. Then

$$(2.8) \quad \int \varphi d\mu_{0,\ell} + \int \varphi d\mu_{1,\ell} + \int_{\mathbb{R}^N \times \mathbb{R}} |a(v) \cdot \nabla \varphi| f_\ell dx dv \leq \frac{C_{\text{kin}}}{R-r} \int_{B_R(0)} (u - \ell)_+ dx, \\ \|\varphi f_\ell\|_{L^2(\mathbb{R}^N \times \mathbb{R})}^2 \leq \int_{B_R(0)} (u - \ell)_+ dx,$$

with $C_{\text{kin}} := (2 + |I|) c_N \|a\|_{L^\infty(I)}$, which does not depend on ℓ . Indeed, testing (2.7) against $\varphi(x)\psi(v)$, with $\psi \in C_c^\infty(\mathbb{R})$ equal to one on I , so that the term $\partial_v \mu_{0,\ell}$ does not contribute, gives

$$\int \varphi d\mu_{1,\ell} = \int_{\mathbb{R}^N \times \mathbb{R}} f_\ell a(v) \cdot \nabla \varphi dx dv,$$

so that this term and the third one are bounded by $\|a\|_{L^\infty(I)} \|\nabla \varphi\|_{L^\infty} \int_{B_R(0)} (u - \ell)_+ dx$; the proof of [Sil19, Lemma 4.3], applied with the test function φ to the measure $\mu_{0,\ell}$ constructed in [Sil19, Proposition 4.1], bounds $\int \varphi d\mu_{0,\ell}$ by $\|(u - \ell)_+\|_{L^\infty} \leq |I|$ times the same quantity; and the last bound follows from $f_\ell^2 = f_\ell$ and $\varphi^2 \leq \mathbf{1}_{B_R(0)}$.

Step 2: Localized averaging. We prove that there are $\theta = \theta(\alpha) \in (0, 1)$, $p > 1$, and $C_{\text{av}} = C_{\text{av}}(N, I, K, \alpha, C_K, \|a\|_{C^2(I)}, \chi, h)$ such that

$$(2.9) \quad \frac{1}{p} = \frac{1 + \theta}{2} - \frac{\theta}{2N},$$

$$\|(u - \ell)_+\|_{L^p(B_r(0))} \leq \frac{C_{\text{av}}}{(R - r)^\theta} \left(\int_{B_R(0)} (u - \ell)_+ dx \right)^{(1+\theta)/2} \quad (1 \leq r < R \leq 2, \ell \in [\ell_0, U]).$$

The constant is independent of ℓ, r, R, u .

Fix r, R, φ as in Step 1 and write $K = [k_-, k_+]$. Multiplying (2.7) by φ ,

$$a(v) \cdot \nabla_x (\varphi f_\ell) = \partial_v (\varphi \mu_{0,\ell}) + \nu_\ell, \quad \nu_\ell := -\varphi \mu_{1,\ell} + (a(v) \cdot \nabla \varphi) f_\ell dx dv,$$

in the sense of distributions on $\mathbb{R}^N \times \mathbb{R}$, where ν_ℓ is a finite signed measure. For a finite signed measure ν on $\mathbb{R}^N \times K$, let

$$\langle P\nu, \psi \rangle := \int_{\mathbb{R}^N \times K} \left(\int_t^{k_+} \psi(x, v) dv \right) d\nu(x, t);$$

then $P\nu$ is a finite signed measure, $\|P\nu\|_{\mathcal{M}} \leq |K| \|\nu\|_{\mathcal{M}}$, and $\partial_v P\nu = \nu$ in the sense of distributions on $\mathbb{R}^N \times \text{int } K$. Hence, restricting ν_ℓ to $\mathbb{R}^N \times K$,

$$(2.10) \quad a(v) \cdot \nabla_x (\varphi f_\ell) = \partial_v G_\ell \quad \text{in the sense of distributions on } \mathbb{R}^N \times \text{int } K,$$

where $G_\ell := (\varphi \mu_{0,\ell})|_{\mathbb{R}^N \times K} + P\nu_\ell$ and (2.8) gives

$$\|G_\ell\|_{\mathcal{M}} \leq \frac{(1 + |K|)C_{\text{kin}}}{R - r} \int_{B_R(0)} (u - \ell)_+ dx, \quad \|\varphi f_\ell\|_{L^2(\mathbb{R}^N \times K)} \leq \left(\int_{B_R(0)} (u - \ell)_+ dx \right)^{1/2}.$$

As in [Sil19, Proposition 5.1], we apply [TT07, Averaging Lemma 2.1] to (2.10) on $\mathbb{R}^N \times K$, with symbol $L(i\xi, v) = i a(v) \cdot \xi$, homogeneous of order one, data $\varphi f_\ell \in L^2$, source $\partial_v G_\ell$ with G_ℓ a bounded measure, and averaging function $\chi \in C_c^\infty(\text{int } K)$.⁴ It gives $\theta = \theta(\alpha) \in (0, 1)$ such that, for $s := \theta/2$ and $1/q_\theta := (1 + \theta)/2$, the average $\int_K \varphi f_\ell \chi dv = \varphi w_\ell$ satisfies

$$\|\varphi w_\ell\|_{W^{s,q_\theta}(\mathbb{R}^N)} \leq C \left(\|\varphi f_\ell\|_{L^2(\mathbb{R}^N \times K)}^{1-\theta} \|G_\ell\|_{\mathcal{M}}^\theta + \|\varphi w_\ell\|_{L^{q_\theta}(\mathbb{R}^N)} \right),$$

with $C = C(N, K, \alpha, C_K, \|a\|_{C^2(K)}, \chi)$; the second term on the right accounts for the low spatial frequencies and for the passage from the local norm in [TT07, Eq. (2.5)] to the global one, which is possible because φw_ℓ vanishes outside $B_2(0)$. By the Sobolev embedding $W^{s,q_\theta}(\mathbb{R}^N) \subset L^p(\mathbb{R}^N)$ with $1/p = 1/q_\theta - s/N = (1 + \theta)/2 - \theta/(2N)$, by the bounds above, by $\|\varphi w_\ell\|_{L^{q_\theta}} \leq |I|^{1-1/q_\theta} \left(\int_{B_R(0)} (u - \ell)_+ dx \right)^{1/q_\theta}$ (recall $0 \leq w_\ell \leq |I|$ and $\varphi w_\ell \leq \mathbf{1}_{B_R(0)}(u - \ell)_+$), and by $R - r \leq 1$,

$$\|w_\ell\|_{L^p(B_r(0))} \leq \|\varphi w_\ell\|_{L^p(\mathbb{R}^N)} \leq \frac{C'}{(R - r)^\theta} \left(\int_{B_R(0)} (u - \ell)_+ dx \right)^{(1+\theta)/2},$$

where $C' = C'(N, I, K, \alpha, C_K, \|a\|_{C^2(I)}, \chi)$. Together with the lower bound in (2.6), this is (2.9) with $C_{\text{av}} := (|I|/h) C'$.

⁴ The hypotheses of [TT07, Averaging Lemma 2.1] hold in our context: the coefficients of the symbol are of class C^2 in v on K ; the modified symbol $a(v) \cdot \xi/|\xi|$ has the truncation property uniformly in v by [TT07, Section 2.4]; and the non-degeneracy condition [TT07, Eq. (2.4)] on K is (2.3) with $\leq \delta$ in place of $< \delta$, which holds with $2^\alpha C_K$ in place of C_K . In the proof of [TT07, Averaging Lemma 2.1], the kinetic equation [TT07, Eq. (2.3)] is used only after multiplication by the averaging function and integration by parts in v , so the proof works although (2.10) holds on $\mathbb{R}^N \times \text{int } K$ only.

Step 3: De Giorgi iteration. For $k \geq 0$, define

$$(2.11) \quad \ell_k := U - (U - \ell_0)2^{-k}, \quad r_k := 1 + 2^{-k}.$$

The levels increase from ℓ_0 to U , and the radii decrease from 2 to 1; since every ℓ_k lies in $[\ell_0, U]$, (2.9) applies at each step with the same constant. Since $\ell_{k+1} - \ell_k = (U - \ell_0)2^{-k-1}$ and $r_k - r_{k+1} = 2^{-k-1}$, Chebyshev's inequality gives

$$|B_{r_{k+1}}(0) \cap \{u > \ell_{k+1}\}| \leq \frac{2^{k+1}}{U - \ell_0} \int_{B_{r_k}(0)} (u - \ell_k)_+ dx,$$

and Hölder's inequality together with (2.9), applied at level ℓ_k with $r = r_{k+1}$ and $R = r_k$, gives

$$(2.12) \quad \begin{aligned} & \int_{B_{r_{k+1}}(0)} (u - \ell_{k+1})_+ dx \\ & \leq \|(u - \ell_k)_+\|_{L^p(B_{r_{k+1}}(0))} |B_{r_{k+1}}(0) \cap \{u > \ell_{k+1}\}|^{1-1/p} \\ & \leq C_{\text{av}} 2^{(k+1)\theta} \left(\int_{B_{r_k}(0)} (u - \ell_k)_+ dx \right)^{(1+\theta)/2} \left(\frac{2^{k+1}}{U - \ell_0} \int_{B_{r_k}(0)} (u - \ell_k)_+ dx \right)^{1-1/p} \\ & = C_{\text{av}} (U - \ell_0)^{-(1-1/p)} 2^{(k+1)(\theta+1-1/p)} \left(\int_{B_{r_k}(0)} (u - \ell_k)_+ dx \right)^{1+\theta/(2N)}, \end{aligned}$$

where the last equality uses

$$(2.13) \quad \frac{1+\theta}{2} + 1 - \frac{1}{p} = 1 + \frac{\theta}{2N} > 1.$$

Let $\varepsilon_0 > 0$ be defined by

$$C_{\text{av}} (U - \ell_0)^{-(1-1/p)} 2^{\theta+1-1/p} \varepsilon_0^{\theta/(2N)} = 2^{-\frac{2N}{\theta}(\theta+1-1/p)}.$$

If $\int_{B_2(0)} (u - \ell_0)_+ dx \leq \varepsilon_0$, then

$$(2.14) \quad \int_{B_{r_k}(0)} (u - \ell_k)_+ dx \leq \varepsilon_0 2^{-k \frac{2N}{\theta}(\theta+1-1/p)} \quad (k \geq 0).$$

Indeed, for $k = 0$ this is the assumption, and if (2.14) holds for some k , then (2.12) and the choice of ε_0 give

$$\begin{aligned} & \int_{B_{r_{k+1}}(0)} (u - \ell_{k+1})_+ dx \\ & \leq C_{\text{av}} (U - \ell_0)^{-(1-1/p)} 2^{(k+1)(\theta+1-1/p)} \varepsilon_0^{1+\theta/(2N)} 2^{-k(\theta+1-1/p)(\frac{2N}{\theta}+1)} \\ & = \left[C_{\text{av}} (U - \ell_0)^{-(1-1/p)} 2^{\theta+1-1/p} \varepsilon_0^{\theta/(2N)} \right] \varepsilon_0 2^{-k \frac{2N}{\theta}(\theta+1-1/p)} = \varepsilon_0 2^{-(k+1) \frac{2N}{\theta}(\theta+1-1/p)}. \end{aligned}$$

Since $B_1(0) \subset B_{r_k}(0)$ and $\ell_k < U$, (2.14) gives $\int_{B_1(0)} (u - U)_+ dx \leq \int_{B_{r_k}(0)} (u - \ell_k)_+ dx \rightarrow 0$, that is, $u \leq U$ almost everywhere in $B_1(0)$. Finally, ε_0 depends only on C_{av} , θ , N , and $U - \ell_0$, and χ and h were chosen in terms of K , ℓ_0 , U ; this concludes the proof. $\square$

Remark 2.5 (Regularity of the flux). We assumed $A \in C^m(\mathbb{R}; \mathbb{R}^N)$, with $m \geq \max\{N, 3\}$. The dimension-dependent requirement $m \geq N$ is needed in Lemma 2.3; the additional requirement $m \geq 3$ is used in the averaging step of Lemma 2.4.

3. PROOF OF THE STABILITY RESULT

The proof of Theorem 1.3 follows the outline given in Section 1.1 and rests on Lemmas 2.3 and 2.4. For fixed $\varepsilon > 0$, we select finitely many intervals of states on which Lemma 2.4 applies (Step 1), prove the estimate on a single ball where v is nearly constant (Step 2), and cover $B_1(0)$ by such balls (Step 3); the modulus $\omega_{A,I,\rho}$ is then assembled from the resulting smallness conditions (Step 4).

Proof of Theorem 1.3. Fix $0 < \varepsilon < |I|$; for $\varepsilon \geq |I|$ there is nothing to prove, since $|u - v| \leq |I|$.

Step 1: Discretization of the state interval. Let $\{c_i\}_{i=1}^{n_\varepsilon} \subset I$ be a finite $\varepsilon/32$ -net of I : every point of I lies within distance $\varepsilon/32$ of some c_i . Fix i and suppose that $\sup I - c_i \geq \varepsilon/2$. Then $V_i^+ := (c_i + \varepsilon/4, c_i + \varepsilon/2)$ is a non-empty open interval contained in I . By Lemma 2.3, there is a compact interval $K_i^+ \subset V_i^+$ of positive length on which (2.3) holds with $\alpha = 1/(N-1)$ and some constant $C_{K_i^+}$. Since $K_i^+ \subset V_i^+ \subset \text{int } I$, we may choose $\ell_i < U_i$ with $[\ell_i, U_i] \subset \text{int } K_i^+$ and apply Lemma 2.4; let $\kappa_i^+ > 0$ be the constant ε_0 given by that lemma. Note that

$$(3.1) \quad c_i + \varepsilon/4 < \ell_i < U_i < c_i + \varepsilon/2.$$

If $\sup I - c_i < \varepsilon/2$, no construction is needed above c_i . Symmetrically, if $c_i - \inf I \geq \varepsilon/2$, we apply the same construction to the reflected flux $\tilde{A}(v) := -A(-v)$ on the interval $-I$ and at the state $-c_i$. This flux is again of class C^m and satisfies (1.9) on $-I$. We obtain levels $\ell_i^- < U_i^-$ and a constant $\kappa_i^- > 0$. Set $\kappa_\varepsilon := \min\{1, \min_i \kappa_i^+, \min_i \kappa_i^-\} \in (0, 1]$, where each minimum runs over the finitely many indices i for which the corresponding constant was constructed. All these choices depend only on A, I, N , and ε .

Step 2: The estimate on a single ball. We claim the following. Let $z \in \mathbb{R}^N$ and $r > 0$, let u be an entropy solution on $B_{2r}(z)$ with values in I , and let $v : B_{2r}(z) \rightarrow I$ satisfy $|v - c_i| \leq \varepsilon/16$ on $B_{2r}(z)$ for some i . Then

$$(3.2) \quad r^{-N} \int_{B_{2r}(z)} |u - v| \, dy \leq \kappa_\varepsilon \implies |u - v| \leq \varepsilon \quad \text{almost everywhere in } B_r(z).$$

We first prove the upper bound $u - v \leq \varepsilon$. If $\sup I - c_i < \varepsilon/2$, then $u - v \leq \sup I - c_i + \varepsilon/16 < \varepsilon$ on $B_{2r}(z)$. If $\sup I - c_i \geq \varepsilon/2$, then $v \leq c_i + \varepsilon/16 < \ell_i$ on $B_{2r}(z)$ by (3.1), and therefore

$$(3.3) \quad (u - \ell_i)_+ \leq (u - v)_+ \leq |u - v| \quad \text{on } B_{2r}(z).$$

The rescaled function $u_r(y) := u(z + ry)$ is an entropy solution on $B_2(0)$ with values in I , and by (3.3)

$$\int_{B_2(0)} (u_r - \ell_i)_+ \, dy = r^{-N} \int_{B_{2r}(z)} (u - \ell_i)_+ \, dy \leq r^{-N} \int_{B_{2r}(z)} |u - v| \, dy \leq \kappa_\varepsilon \leq \kappa_i^+.$$

Lemma 2.4 gives $u_r \leq U_i$ almost everywhere in $B_1(0)$, that is, $u \leq U_i < c_i + \varepsilon/2$ almost everywhere in $B_r(z)$, and so $u - v < \varepsilon/2 + \varepsilon/16 < \varepsilon$ there. If $c_i - \inf I < \varepsilon/2$, then $v - u \leq c_i - \inf I + \varepsilon/16 < \varepsilon$. Otherwise, the lower bound $v - u \leq \varepsilon$ follows by applying the upper bound to $-u, -v$, the flux $\tilde{A}$, the interval $-I$, and the state $-c_i$: the reflection $u \mapsto -u$ maps entropy solutions with flux A and values in I to entropy solutions with flux $\tilde{A}$ and values in $-I$, the remaining hypotheses of the claim are invariant under it, and the same computation gives $\int_{B_2(0)} (-u_r - \ell_i^-)_+ \, dy \leq \kappa_\varepsilon \leq \kappa_i^-$. This proves (3.2).

Step 3: Covering argument. Choose $0 < r_\varepsilon \leq 1/4$ such that $\rho(2r_\varepsilon) \leq \varepsilon/32$. Fix $z \in B_1(0)$ and select c_i with $|v(z) - c_i| \leq \varepsilon/32$. Then $B_{2r_\varepsilon}(z) \subset B_{3/2}(0) \subset B_2(0)$ and, for $y \in B_{2r_\varepsilon}(z)$, $|v(y) - c_i| \leq \rho(2r_\varepsilon) + \varepsilon/32 \leq \varepsilon/16$. If $\|u - v\|_{L^1(B_2(0))} \leq r_\varepsilon^N \kappa_\varepsilon$, then $r_\varepsilon^{-N} \|u - v\|_{L^1(B_{2r_\varepsilon}(z))} \leq \kappa_\varepsilon$, and Step 2 yields $|u - v| \leq \varepsilon$ almost everywhere in $B_{r_\varepsilon}(z)$. Since $B_1(0)$ is covered by countably many of the balls $B_{r_\varepsilon}(z)$, $z \in B_1(0)$, we conclude that

$$(3.4) \quad \|u - v\|_{L^1(B_2(0))} \leq r_\varepsilon^N \kappa_\varepsilon \implies \|u - v\|_{L^\infty(B_1(0))} \leq \varepsilon.$$

The constant $r_\varepsilon^N \kappa_\varepsilon$ depends only on $A, I, N, \rho, \varepsilon$.

Step 4: Construction of the modulus. For $s \geq 0$, let $\Theta(s)$ be the supremum of $\|u - v\|_{L^\infty(B_1(0))}$ over all pairs (u, v) as in the statement with $\|u - v\|_{L^1(B_2(0))} \leq s$. Then $0 \leq \Theta \leq |I|$, Θ is non-decreasing, $\Theta(0) = 0$, and, by (3.4), $\Theta(s) \leq \varepsilon$ for $0 < s \leq r_\varepsilon^N \kappa_\varepsilon$ and every $\varepsilon \in (0, |I|)$; hence $\Theta(s) \rightarrow 0$ as $s \downarrow 0$. Define

$$\omega_{A,I,\rho}(s) := \int_1^{2s} \Theta(s\tau) \, d\tau = \frac{1}{s} \int_s^{2s} \Theta(t) \, dt \quad (s > 0), \quad \omega_{A,I,\rho}(0) := 0.$$

Since Θ is bounded and non-decreasing, $\omega_{A,I,\rho}$ is non-decreasing, continuous on $(0, \infty)$, and satisfies $\Theta(s) \leq \omega_{A,I,\rho}(s) \leq \Theta(2s)$; in particular, it is continuous at 0, so it is a modulus bounded by $|I|$. The inequality $\Theta \leq \omega_{A,I,\rho}$ is (1.11). $\square$

We can now prove Corollaries 1.4–1.8.

Proof of Corollary 1.4. It suffices to prove the second statement, since a continuous function is uniformly continuous on compact sets. Let $\Omega' \Subset \Omega$ be compact and cover it by finitely many balls $B_{r_i}(z_i)$, $i = 1, \dots, k$, with $\overline{B_{2r_i}(z_i)} \subset \Omega$. Let ρ be a common modulus of continuity of the v_j on the compact set $\bigcup_i \overline{B_{2r_i}(z_i)}$. For each i , the functions $u_j(z_i + r_i \cdot)$ are entropy solutions on $B_2(0)$ with values in I , and $v_j(z_i + r_i \cdot)$ has modulus $\rho_i(s) := \rho(r_i s)$ on $B_2(0)$. Theorem 1.3 gives

$$\|u_j - v_j\|_{L^\infty(B_{r_i}(z_i))} \leq \omega_{A,I,\rho_i}(r_i^{-N} \|u_j - v_j\|_{L^1(B_{2r_i}(z_i))}) \longrightarrow 0 \quad (j \rightarrow \infty).$$

Taking the maximum over the finitely many i gives $\|u_j - v_j\|_{L^\infty(\Omega')} \rightarrow 0$. $\square$

Proof of Corollary 1.5. Let $\omega_{A,I} := \omega_{A,I,\rho}$ be the modulus of Theorem 1.3 for $\rho(s) = s$. Fix $z \in \Omega'$ and set $r := d/[2(1+dL)]$. Then $2r \leq d$, so $B_{2r}(z) \subset \Omega$, and $y \mapsto v(z + ry)$ is Lipschitz on $B_2(0)$ with constant $rL \leq 1/2$. Theorem 1.3, applied to $u(z + r \cdot)$ and $v(z + r \cdot)$, gives

$$\|u - v\|_{L^\infty(B_r(z))} \leq \omega_{A,I}(r^{-N} \|u - v\|_{L^1(B_{2r}(z))}) \leq \omega_{A,I}\left(\frac{2^N(1+dL)^N}{d^N} \|u - v\|_{L^1(\Omega)}\right).$$

Since Ω' is covered by countably many of the balls $B_r(z)$, $z \in \Omega'$, this proves (1.12). $\square$

Proof of Corollary 1.6. Let u and v be entropy solutions of (1.1) on $(0, \infty) \times \mathbb{R}^n$ with $N = n + 1$ and $A(s) = (s, F(s))$; let $\omega_{F,I}$ be the modulus of Theorem 1.3 for this flux and $\rho(s) = s$. Since v is locally Lipschitz for positive times, (1.13) holds almost everywhere in the form $\partial_t v = -F'(v) \cdot \nabla_x v$, which together with the spatial Lipschitz bound gives

$$|\nabla_{t,x} v(\tau, x)| \leq C_F \Gamma / \tau \quad \text{for almost every } (\tau, x).$$

Fix (t, x) and let $r := \frac{t}{4(1+2C_F \Gamma)}$. Then $B_{2r}(t, x) \subset (t/2, 3t/2) \times \mathbb{R}^n$, so that $|\nabla_{t,x} v| \leq 2C_F \Gamma / t$ on $B_{2r}(t, x)$, and the function $y \mapsto v((t, x) + ry)$ is Lipschitz on $B_2(0)$ with constant $2rC_F \Gamma / t \leq 1/4$; in particular, it admits the modulus $\rho(s) = s$. We use the local contraction inequality of [Kru70, Theorem 1] and the local L^1 time continuity of entropy solutions; see [Pan05, Corollary 7.1]. Letting the spatial radius tend to infinity in the contraction inequality gives

$$(3.5) \quad \|u(t, \cdot) - v(t, \cdot)\|_{L^1(\mathbb{R}^n)} \leq \|u_0 - v_0\|_{L^1(\mathbb{R}^n)} \quad (t > 0).$$

By (3.5),

$$\begin{aligned} r^{-N} \int_{B_{2r}(t,x)} |u - v| \, d\tau \, dy &\leq 4r^{-n} \|u_0 - v_0\|_{L^1(\mathbb{R}^n)} \\ &= 4 \cdot 4^n (1 + 2C_F \Gamma)^n \frac{\|u_0 - v_0\|_{L^1(\mathbb{R}^n)}}{t^n} \\ &\leq C_n (1 + C_F \Gamma)^n \frac{\|u_0 - v_0\|_{L^1(\mathbb{R}^n)}}{t^n}. \end{aligned}$$

Applying Theorem 1.3, with $\rho(s) = s$, to the rescalings of u and v by the factor r around (t, x) gives $|u - v| \leq \omega_{F,I}(4r^{-n} \|u_0 - v_0\|_{L^1(\mathbb{R}^n)})$ almost everywhere in $B_r(t, x)$. Since $x \in \mathbb{R}^n$ is arbitrary, a countable subcover gives the same bound, with t fixed on the right-hand side, almost everywhere on $(t - r, t + r) \times \mathbb{R}^n$. Local L^1 time continuity then passes this bound from almost every nearby time slice to the slice at t , proving (1.15) for every $t > 0$. $\square$

Proof of Corollary 1.7. As in the proof of Corollary 1.6, u and v are entropy solutions of (1.1) with $N = n + 1$ and $A(s) = (s, F(s))$, and (3.5) holds. Fix $0 < \varepsilon < |I|$ and let $\{c_i\}$ and κ_ε be the state net and the constant κ_ε of Step 1 of the proof of Theorem 1.3 for this flux. Set $\eta := \varepsilon/32$ and, for $t \geq t_\eta$, $r := r_\eta t$. Fix $x \in \mathbb{R}^n$. Since $r_\eta < 1/4$, we have $B_{2r}(t, x) \subset (t/2, 3t/2) \times \mathbb{R}^n$. Choose c_i with $|v(t, x) - c_i| \leq \varepsilon/32$; by (1.16), $|v - c_i| \leq \eta + \varepsilon/32 = \varepsilon/16$ on $B_{2r}(t, x)$. Moreover, (3.5) gives

$$r^{-N} \int_{B_{2r}(t,x)} |u - v| \, d\tau \, dy \leq 4(r_\eta t)^{-n} \|u_0 - v_0\|_{L^1(\mathbb{R}^n)},$$

which is at most κ_ε as soon as $t \geq T_\varepsilon := \max\left\{t_\eta, r_\eta^{-1} (4\|u_0 - v_0\|_{L^1(\mathbb{R}^n)}/\kappa_\varepsilon)^{1/n}\right\}$. For such t , Step 2 of the proof of Theorem 1.3, namely (3.2), yields $|u - v| \leq \varepsilon$ almost everywhere in $B_r(t, x)$. A countable subcover as x varies gives $|u - v| \leq \varepsilon$ almost everywhere in $(t - r, t + r) \times \mathbb{R}^n$, and local L^1 time continuity gives $\|u(t, \cdot) - v(t, \cdot)\|_{L^\infty(\mathbb{R}^n)} \leq \varepsilon$ for every $t \geq T_\varepsilon$. $\square$

Proof of Corollary 1.8. Since U is continuous, v is a continuous entropy solution of (1.1) on $\Omega := (0, \infty) \times \mathbb{R}^n$ with $A(s) = (s, F(s))$. By [Sil19, Lemma 10.1], which requires no nonlinearity assumption, a continuous entropy solution is constant along each segment of the line through a point $x_0 \in \Omega$ with direction $A'(v(x_0))$ that lies in Ω . Applied at $x_0 = (1, \xi e)$, $\xi \in \mathbb{R}$, this gives $v(t, \xi e + (t-1)F'(U(\xi))) = U(\xi)$ for every $t > 0$, that is, with $b_\xi := e \cdot F'(U(\xi))$,

$$(3.6) \quad U\left(b_\xi + \frac{\xi - b_\xi}{t}\right) = U(\xi) \quad (t > 0), \quad |b_\xi| \leq R.$$

If $\xi, \zeta > R$, then $t := (\xi - b_\xi)/(\zeta - b_\xi) > 0$ and (3.6) gives $U(\zeta) = U(\xi)$; the same holds when $\xi, \zeta < -R$, since numerator and denominator are then both negative. By continuity, U is constant on $(-\infty, -R]$ and on $[R, \infty)$.

Consequently, for $e \cdot x \neq 0$ we have $v(t, x) = v_0(x)$ as soon as $0 < t < |e \cdot x|/R$, so $v(t, \cdot) \rightarrow v_0$ in $L^1_{\text{loc}}(\mathbb{R}^n)$ as $t \downarrow 0$ by dominated convergence, the functions taking values in the compact interval I . By Kruřkov's uniqueness theorem, v is the global entropy solution with initial datum v_0 .

Next, for $\tau \in [1/2, 3/2]$, the map $(\tau, y) \mapsto \max\{-R, \min\{R, (e \cdot y)/\tau\}\}$ is Lipschitz with a constant depending only on R . Composing it with the uniformly continuous restriction of U to $[-R, R]$ proves uniform continuity of $(\tau, y) \mapsto U((e \cdot y)/\tau)$ on $[1/2, 3/2] \times \mathbb{R}^n$. Rescaling gives (1.16); apply Corollary 1.7. $\square$

4. PROOF OF THE CONTINUITY RESULT

We now combine Lemmas 2.1 and 2.2 with Theorem 1.3 to prove Theorem 1.1 by the blow-up argument outlined in Section 1.1. Up to the identification of the blow-up limit as a constant (Step 1 below), only the qualitative condition (1.9) is used, and the conclusion at that stage is the vanishing mean oscillation of u at x_0 , as in [DOW03, Theorem 1]; using Theorem 1.3, we then obtain the vanishing of the essential oscillation (Steps 2–3).

Proof of Theorem 1.1. Step 1: A constant limit at a point of zero density. Fix $x_0 \in \Omega \setminus J$. We construct a sequence $r_{j_k} \downarrow 0$ and a constant $c \in I$ such that

$$(4.1) \quad u(x_0 + r_{j_k} \cdot) \longrightarrow c \quad \text{strongly in } L^1_{\text{loc}}(\mathbb{R}^N).$$

Choose any sequence $r_j \downarrow 0$ with $B_{2r_j}(x_0) \subset \Omega$. We consider the rescaling

$$(4.2) \quad u_{r_j}(x) := u(x_0 + r_j x), \quad \mu_{r_j}(E \times H) := r_j^{1-N} \mu((x_0 + r_j E) \times H)$$

for Borel sets $E \subset (\Omega - x_0)/r_j$ and $H \subset I$. The rescaled solution u_{r_j} and measure μ_{r_j} satisfy (1.3) on the rescaled domain. For every fixed $R > 0$,

$$(4.3) \quad \mu_{r_j}(B_R(0) \times I) = r_j^{1-N} \mu(B_{Rr_j}(x_0) \times I) = R^{N-1} \frac{\mu(B_{Rr_j}(x_0) \times I)}{(Rr_j)^{N-1}} \longrightarrow 0.$$

Each fixed ball lies in the rescaled domain for all sufficiently large j . A diagonal extraction using Lemma 2.1 on bounded balls yields a subsequence r_{j_k} and $u_\infty \in L^\infty(\mathbb{R}^N; I)$ such that $u_{r_{j_k}} \rightarrow u_\infty$ strongly in L^1 on every bounded ball. Moreover, for every $R > 0$,

$$(4.4) \quad \int_{B_R(0)} \int_I \left| \mathbb{1}_{\{\inf I < v < u_{r_{j_k}}(x)\}} - \mathbb{1}_{\{\inf I < v < u_\infty(x)\}} \right| dv dx = \int_{B_R(0)} |u_{r_{j_k}} - u_\infty| dx \longrightarrow 0.$$

Passing to the limit in the kinetic equation using (4.3) and (4.4), we find that u_∞ satisfies (2.1). By Lemma 2.2, $u_\infty = c$ almost everywhere for some $c \in I$, which proves (4.1).

Step 2: Stability at the constant blow-up limit. Each $u_{r_{j_k}}$ is an entropy solution on $B_2(0)$ with values in I , and $\|u_{r_{j_k}} - c\|_{L^1(B_2(0))} \rightarrow 0$ by (4.1). Applying Theorem 1.3 with the constant $v \equiv c$ and the modulus $\rho \equiv 0$, we obtain

$$(4.5) \quad \|u(x_0 + r_{j_k} \cdot) - c\|_{L^\infty(B_1(0))} \longrightarrow 0.$$

Step 3: Continuity. By (4.5),

$$(4.6) \quad \text{ess sup}_{B_{r_{j_k}}(x_0)} u - \text{ess inf}_{B_{r_{j_k}}(x_0)} u \leq 2\|u_{r_{j_k}} - c\|_{L^\infty(B_1(0))} \longrightarrow 0.$$

Essential oscillation is non-decreasing as a function of the radius. Thus its limit as $r \downarrow 0$ is determined by any sequence of radii decreasing to zero, and (4.6) proves (1.10) at x_0 . This argument applies to each $x_0 \notin J$.

It remains to construct the representative. Let

$$\bar{u}(x) := \limsup_{r \downarrow 0} \operatorname{ess\,sup}_{B_r(x)} u, \quad \underline{u}(x) := \liminf_{r \downarrow 0} \operatorname{ess\,inf}_{B_r(x)} u \quad (x \in \Omega),$$

which exist by monotonicity; as in [Sil19, Section 2.2], $\bar{u}$ is upper semicontinuous, $\underline{u}$ is lower semicontinuous, and $\underline{u} \leq \bar{u}$ on Ω , while $\underline{u} = \bar{u}$ on $\Omega \setminus J$ by (1.10). Define $\tilde{u}(x) := u(x)$ if x is a Lebesgue point of u , with $u(x)$ its Lebesgue value, and $\tilde{u}(x) := \bar{u}(x)$ otherwise. Then $\tilde{u} = u$ almost everywhere, and $\underline{u} \leq \tilde{u} \leq \bar{u}$ on Ω , since a Lebesgue value is a limit of averages of u over small balls. The semicontinuity of the two envelopes and their equality on $\Omega \setminus J$ therefore imply that $\tilde{u}$ is continuous at every $x_0 \in \Omega \setminus J$. $\square$

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