Regularity and Variationality of Solutions to Hamilton-Jacobi Equations. part II: variationality, existence, uniqueness

Andrea C. G. Mennucci ⋆

November 17, 2006

Abstract

We formulate an Hamilton–Jacobi partial differential equation

$$H(x, Du(x)) = 0$$

on a n dimensional manifold M, with assumptions of convexity of the sets $\{p : H(x,p) \le 0\} \subset T^*_x M$, for all x.

In this paper we reduce the above problem to a simpler problem: this shows that u may be built using an asymmetric distance (this is a generalization of the "distance function" in Finsler Geometry): this brings forth a 'completeness' condition, and a Hopf–Rinow theorem adapted to Hamilton–Jacobi problems. The 'completeness' condition implies that u is the unique viscosity solution to the above problem.

When H is moreover of class $C^{1,1}$, we show how the completeness condition is equivalent to a condition expressed using the characteristics flow.

1 Introduction

In this article we will study the Dirichlet Hamilton–Jacobi PDE

$$\begin{cases} H(x, Du(x)) = 0 & \text{in } M \setminus K \\ u(x) = u_0(x) & \text{when } x \in K. \end{cases} \tag{1.1}$$

where

- M is a connected boundaryless smooth differentiable manifold of dimension n;
- H will be a real function defined on the cotangent bundle T^*M, such that

$$Z_x \stackrel{\text{def}}{=} \{p \in T^*_x M \mid H(x,p) \le 0\} \tag{1.2}$$

is convex for all $x \in M$.

⋆Scuola Normale Superiore Piazza dei Cavalieri 7, 56126 Pisa, Italy. This work was partially done while visiting LIDS in MIT, supported by an Intel grant. **This is a 2nd version of this paper**, rewritten for clarity; the 1st version was available as [32]

- K will be a closed subset of M
- and u_0 will be a continuous real function defined on K.

In the first part [31] we studied the regularity properties of a generalized solution u. The main aim of this second part is to prove results on the existence and uniqueness of the solution u.

1.1 Notation

To start this introduction, we fix some notations.

We will use the notation $p \cdot v$ to mean that a covector $p \in T_x^* M$ is applied to a vector $v \in T_x M$.

If $g : \mathbb{R} \times M \to \mathbb{R}$ is a regular function, $g = g(t, x)$, we will write $\dot{g}$ for $\frac{\partial g}{\partial t}$.

Definition 1.3 (limit at infinity) *Given a topological space M and a $f : M \to \mathbb{R}$, we define*

$$\liminf_{x \to \infty} f(x) \stackrel{def}{=} \sup_{C \subset\subset M} \inf_{y \notin C} f(x)$$

where $C \subset\subset M$ are compact subsets (indeed this is called "the liminf for x exiting all compact sets").

We conclude with a remark on definitions

Definition 1.4 *Let $f : \Omega \to \mathbb{R}$ where $\Omega \subset \mathbb{R}^n$ is convex. We define that*

1. ***"f is strongly convex"*** *when $f \in C^2$ and the Hessian $D^2 f(x) = \frac{\partial^2 f}{\partial x x}(x)$ is positive definite $\forall x$; whereas*
2. ***"f is strictly convex"*** *when*

$$f(\lambda x + (1 - \lambda) y) < \lambda f(x) + (1 - \lambda) f(y)$$

for all $x, y \in \Omega$, $0 < \lambda < 1$

We must warn the reader that some authors use different definitions (and call the first definition "strictly convex"): this unfortunately happens also in some papers referenced from this paper.

A similar convention will be followed regarding convex sets.

1.2 The eikonal equation as a general model

1.2.1 the eikonal equation

For the sake of this section, let (M, g) be smooth, connected, boundaryless, Riemannian manifold of dimension n. The **length** of a Lipschitz curve $\gamma : [\alpha, \beta] \to M$ is defined by

$$\operatorname{len}^g \gamma \stackrel{\text{def}}{=} \int_\alpha^\beta |\dot{\gamma}(t)|_g \, dt \quad ;$$

then we can define **distance** $d^g(x, y)$ as the inf of $\operatorname{len} \gamma$ in the class of all Lipschitz curves connecting x to y; let moreover **minimal geodesic** be a curve that minimize $\operatorname{len}^g \gamma$ given the endpoints.

The following theorem summarizes existence and uniqueness and variationality results for the **eikonal equation**

$$\begin{cases} |\nabla u| - 1 = 0 & \text{in } M \setminus K\,, \\ u = u_0 & \text{on } K \end{cases} \tag{1.5}$$

Theorem 1.6 **(existence)** *If*

$$u_0(x) \le d^g(y,x) + u_0(y)$$

for all $x, y \in K$, then the value function

$$V(x) = \inf_{z \in K} \big(u_0(z) + d^g(z,x) \big) \tag{1.6.$\star$}$$

is a viscosity solution of (1.5).

(uniqueness) *If (M, g) is complete, and u_0 is bounded from below, then V is the unique solution in the class $\mathcal{F}$ of continuous functions f that are bounded from below;*

(variationality) *and moreover the problem (1.6.$\star$) admits a minimizing geodesic curve for each x.*

The above theorem may be proved by the methods found in [27].

Remark 1.7 *The restriction to lower bounded functions is necessary: indeed $|x|$ and $-|x|$ are both viscosity solutions of Problem (1.5) with $M = \mathbb{R}^n$ and $K = \{0\}, u_0 = 0$. Moreover, the completeness of M plays an important rôle here: if M is the open unit ball of $\mathbb{R}^n$ the same example shows that the uniqueness does not hold.*

Completeness of the Riemannian Manifold may be tested by any of the equivalent relations in the renowned Hopf–Rinow

Theorem 1.8 (Hopf-Rinow) *the following conditions are equivalent:*

1. *the metric space (M, d^g) is complete*
2. *bounded closed sets are compact*
3. *Lets call* "EL-geodesic" *a curve $\gamma(t)$ satisfying the Euler–Lagrange O.D.E. for the action functional*

$$\int_\alpha^\beta |\dot\gamma(t)|_g^2 \, dt \quad ;$$

then any such curve be prolonged to $t \to \pm\infty$.

moreover any of the above implies that minimal geodesics *do exist.*

The statement (3) is sometimes called *"geodesic completeness"*.

1.2.2 a general "eikonal equation"

We want to extend the results of the previous section to the equation (1.1), and view it as a sort of "generalized eikonal equation". We now outline the development of this paper.

In the first part (section 2) we address the most general case, as defined in the beginning of the introduction:

- in §2.1 we briefly recall the definition of Viscosity solutions in a manifold;
- In §2.2, as done in other papers (e.g. [7]), we define

$$\sigma(x,v) \stackrel{\text{def}}{=} \sup \left\{ p \cdot v \mid p \in Z_x \right\} \tag{1.9}$$

where Z_x was defined in eqn. (1.2). Assuming

Hypotheses 1.10 (Z) *Suppose Z is closed. Let $A = \mathring{Z}$ and let A_x be the* fiber-wise slicing *of A:*

$$A_x \stackrel{def}{=} \{p \in T_x^* M \mid (x,p) \in A\}$$

We suppose that, for all x, Z_x is nonempty, convex and compact and $Z_x = \overline{A_x}$.

Then σ is continuous and locally bounded from above.

We then define (as in [13] and many other papers)

$$S(x,y) = \inf \int_0^1 \sigma(\xi(s), \dot{\xi}(s)) ds \tag{1.11}$$

This object should play the rôle that d^g was playing in theorem 1.6.

- Unfortunately, it may be the case that $S \equiv -\infty$ (as in Example 2.8); to avoid this, we will assume in 2.10 that

Hypothesis 1.12 (($\exists$u)) *There exists a smooth function $\underline{u}$ on M such that*

$$H(x, d\underline{u}(x)) < 0$$

(that we call strict subsolution [1] *of problem (1.1)).*

- The idea is then to define a Finsler metric, and a sort of "asymmetric distance function"

$$b(x,y) = S(x,y) - \underline{u}(x) + \underline{u}(y)$$

- then in §2.3 we will present theorems that precisely mimick the existence part of 1.6.

To proceed further in the generalization of 1.6, we need to define what "complete" means; we then proceed with these steps:

- we provide in §2.4 a brief compendium of the theory of *"asymmetric metric spaces"* (as found in Busemann's [5], [6]; or in [29]), and in particular remark as such spaces admit a Hopf–Rinow-like theorem 2.31; we will add some comments that are useful when studying (1.1).

[1] note that we do need to speak of "strict subsolution in the viscosity sense", since such a solution may be mollified, as shown in Lemma 6.3 in [13]

- We will then be able to state the **Hopf-Rinow–like theorem 2.34 for Hamilton-Jacobi equations**; where one of the equivalent ways of saying “complete” will be that there exists a *strict subsolution* $\underline{u}$ *such that*

$$\liminf_{y\to\infty} S(y,x) + \underline{u}(y) = \infty \tag{1.13}$$

- Supposing that the problem is “complete”, we will prove eventually a comparison theorem and then a uniqueness–variationality theorem 2.43.

The combination of the theorems 2.23 and 2.43 will be an evident generalization of 1.6 above.

In the general case, the Hopf-Rinow–HJ theorem 2.34 will mimick only the first two statements in 1.8, since the third statement requires a Euler–Lagrange O.D.E.; so in the second part (section 3) we moreover suppose

Hypothesis 1.14 (`(Hnd)`) $H \in C^{1,1}$ *in a neighbourhood of* $\{H = 0\}$*, and*

$$\forall (x,p) \in T^*M, \;\; H(x,p) = 0 \Longrightarrow \frac{\partial}{\partial p} H(x,p) \neq 0 \tag{1.14.$\star$}$$

With this additional hypothesis, we can define the Hamiltonian flow, and use it in theorem 3.6 to add a *“geodesic completeness”* statement to theorem 2.34. To prove 3.6,

- in §3.3 we transform the problem (1.1) to a simpler problem (3.7) where the hamiltonian $\hat{H}$ is positive and homogeneous (see 3.30 for details);

- in §3.5 we study the relationship between the Hamiltonian flows of the problems (1.1) and (3.7)

- in §3.6 we define $L : TM \to [0,\infty)$ as the Legendre-Fenchel dual to $\hat{H}$;

- since L is C^1 but not $C^{1,1}$ regular in general, in §3.7 we outline how some basic facts about the Legendre-Fenchel transform do hold true;

- so in §3.9 so we can view (M, L) as a sort of “Finsler Geometry” (altough in weaker regularity assumptions than what usually done); this brings forth a Hopf–Rinow theorem that immediatly proves 3.6.

Eventually in in the third part (section 4) we assume that $H(x,\cdot)$ is strongly convex, and we discuss some further properties. (We also remark that [31] contains regularity results for (1.1) that hold in similar hypotheses to §4; see also §5.4.2).

2 Convex Hamilton-Jacobi equation

We now reintroduce briefly some concepts from the first part [31] and from [29]. And we proceed to integrate this whole in a study of solutions to the Hamilton-Jacobi equation (1.1), in the hypothesis given in introduction.

2.1 Viscosity solutions on manifolds

We reference the reader to [27] or [31] for the standard definition of the *superdifferential* $\partial^+ u(x)$ and the *subdifferential* $\partial^- u(x)$, and the definition of *viscosity solutions* on a manifold.

We want to remark that the viscosity solution depends only on the sign of H:

Proposition 2.1 *Consider the two problems (1.1) and*

$$\begin{cases} \tilde{H}(x, Du(x)) = 0 & \textit{in } M \setminus K \\ u(x) = u_0(x) & \textit{when } x \in K. \end{cases} \tag{2.1.$\star$}$$

if

$$\textit{sign} H(x,p) = \textit{sign} \tilde{H}(x,p) \quad \forall x, p \in T^*M$$

then the definition of viscosity solution immmediatly implies that (1.1) and (2.1.⋆) have the same viscosity solutions

This fact is trivially proved; at the same time, it is not widely exploited in the literature (with some exceptions; to cite some examples, [30], [7], [24]).

With the above remark in mind, we naturally come to the idea of defining the viscosity solutions to the Hamilton-Jacobi problem using a set $Z \subset T^*M$:

Definition 2.2 (Viscosity solutions by inclusion) *Let $Z \subset T^*M$, define $Z_x = Z \cap T^*_x M$.*

We say that a continuous function u is a viscosity solution *of differential inclusion*

$$(x, Du(x)) \in Z \tag{2.2.$\star$}$$

in the open set $\Omega \subset M$ if for every $x \in \Omega$,

$$\begin{cases} \partial^+ u(x) \subset \overline{Z}_x \\ \partial^- u(x) \subset \overline{T^*_x M \setminus Z_x} \end{cases} \tag{2.2.$\star\star$}$$

If only the first condition is satisfied (resp. the second), u is called a viscosity subsolution *(resp. a* viscosity supersolution*).*

(Actually, we may say that (2.2.⋆⋆) is the Hamilton–Jacobi equation that we are really studying in this paper).

Remark 2.3 *Assume that $Z = \{H \le 0\}$ and Z is closed: then any solution of (2.2.⋆⋆) will be also a viscosity solution of (1.1), but not viceversa, as shown in this simple example: $u \equiv 0$ is a viscosity solution to $H(x, du(x)) = 0$ when*

$$H(x,p) \stackrel{def}{=} \max\{0, (|p| - 1)\}$$

but 0 is not a solution to (2.2.⋆⋆), since

$$0 \in \mathring{Z}_x = \{p \mid |p| < 1\}$$

(contradicting the second condition in (2.2.⋆⋆)).

Obviously, the two problems coincide if we further assume that

$$\mathring{Z} = \{H < 0\} \tag{2.3.$\star$}$$

(or equivalently $\partial Z = \{H = 0\}$).

Remark 2.4 *Intuitively, the remark 2.1 should hold also for solutions defined using the method of characteristics (such as* min *solution, that was introduced in the first part [31]); but the situation is slightly more complicated; indeed, if for example*

$$\tilde{H}(x,p) = H(x,p)^3$$

then the problem (2.1.⋆) is degenerate, $\tilde{H}(x,p) = 0 \Rightarrow \frac{\partial}{\partial p}\tilde{H} = \frac{\partial}{\partial x}\tilde{H} = 0$ *so that the characteristics (3.2) are constant in* t*: it is impossible to use them to define a solution.*

For this reason we will introduce a condition `(Hnd)`*: see equation (1.14.⋆), lemma 3.12 and 3.13.*

2.2 Finsler metrics

Following [29] and Siconolfi [38], we define

$$Z \overset{\text{def}}{=} \{(x,p) \in T^*M \mid H(x,p) \le 0\} \tag{2.5}$$

We then define, for any $x \in M$, the *figuratrix* set Z_x by slicing Z along the fibers of T^*M:

$$Z_x \overset{\text{def}}{=} Z \cap T^*_x M = \{p \in T^*_x M \mid (x,p) \in Z\} = \{p \in T^*_x M \mid H(x,p) \le 0\} \tag{2.6}$$

(exactly as we did in (1.2) in introduction). Z_x can be seen as a set-valued map $x \mapsto Z_x$ from M to the fibers of T^*M (see the discussion in §A.ii).

We always assume that Z is closed (then any slice Z_x is closed).

We recall that we defined *slicewise support function* $\sigma : TM \to [-\infty,\infty]$ to be the support function of the set Z_x, in eqn. (1.9).

Lemma 2.7 *Let* $A = \mathring{Z}$ *and let* A_x *be the slicing of* A*:*

$$A_x \overset{\text{def}}{=} \{p \in T^*_x M \mid (x,p) \in A\}$$

Suppose that A_x *is non-empty and* $Z_x = \overline{A_x}$ *for all* x*. Then* σ *is the support of* A_x*, namely*

$$\sigma(x,v) = \sup\left\{p \cdot v \mid p \in A_x\right\} \tag{2.7.⋆}$$

and σ *is lower–semi–continuous.*

Suppose moreover that Z_x *is convex and compact for all* x*: then* σ *is continuous and locally bounded.*

Proof. Since $Z_x = \overline{A_x}$ then (2.7.⋆) holds.

We use the definition of Kuratowski convergence, and the results, exposited in appendix A. By A.13 we know that $x \mapsto A_x$ is l.s.c.: then by A.15 and (2.7.⋆), σ is l.s.c.

If moreover every Z_x is convex and compact, then, by A.12, $x \mapsto Z_x$ is u.s.c.; since $\liminf_{x\to\overline{x}} Z_x \supset \liminf_{x\to\overline{x}} A_x \supset A_{\overline{x}} \neq \emptyset$ then by A.16 we know that σ is u.s.c. and locally bounded. □

We conclude that when the hypotheses `(Z)` in 1.10 hold then σ is continuous and locally bounded from above.

We recall the definition given in (1.11)

$$S(x,y) = \inf \int_0^1 \sigma(\xi(s), \dot\xi(s)) ds \tag{1.11}$$

where the infimum is computed in the class of all locally Lipschitz ξ with given extrema $\xi(0) = x, \xi(1) = y$. This quantity $S(x,y)$ does not depend on $\underline{u}$, but it may fail to be positive (and hence to be an asymmetric distance, as is defined in section 2.4).

Unfortunately, it may be the case that $S \equiv -\infty$, as in this simple example:

Example 2.8 *Let $M = S^1 = \mathbb{R}/\mathbb{Z}$ and let $Z_x = [1,2]$ for all x; then $\sigma(x,v) = 2v$ for $v > 0$ and $\sigma(x,v) = v$ for $v < 0$. Given any $x, y \in M$, the winding curves $\gamma_n(t) = x + t(-n + y - x)$ prove that $S(x,y) = -\infty$. See fig.1.*

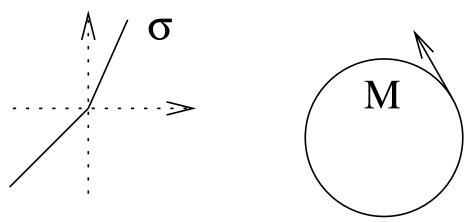

Figure 1: Example 2.8

We conclude this section with remarks on the conditions above

Example 2.9 *Let $M = \mathbb{R}$ and*

$$A_x \overset{def}{=} \begin{cases} (-1,1) & \text{if } x \le 0 \\ (-1+1/x, 1+1/x) & \text{if } x > 0 \end{cases}$$

and let $Z_x = \overline{A_x}$; then A is open, any slice Z_x is convex and compact, but Z is not closed, and σ is not locally bounded and continuous.

2.2.1 Finsler Metric from Z and $\underline{u}$

As aforementioned, we (almost always) assume in this paper that $(\exists\underline{u})$; this may be stated in two equivalent ways:

Lemma 2.10 ($(\exists\underline{u})$) *There exists a smooth function $\underline{u}$ on M such that*

$$d\underline{u}(x) \in A_x \ \ \forall x \ ; \tag{2.10.$\star$}$$

or (as said in remark 2.3) if $\partial Z = \{H = 0\}$, equivalently

$$H(x, d\underline{u}(x)) < 0$$

This $\underline{u}$ is called a *strict subsolution*. Note that $\underline{u} + c$ is again a *strict subsolution*, for any constant c. A discussion of this condition is in §5.2.

Given a choice of a strict subsolution $\underline{u}$, we define then $F : TM \to [0, \infty]$ so that $F(x, \cdot)$ is the support function of the set Z_x corrected by $d\underline{u}(x)$,

$$F(x,v) \stackrel{\text{def}}{=} \sigma(x,v) - v \cdot d\underline{u}(x) = \tag{2.11}$$
$$= \sup\left\{p \cdot v \mid (p + d\underline{u}(x)) \in Z_x\right\} \tag{2.12}$$

By the definition it is clear that $F \geq 0$, and $F(x,v) = 0$ iff $v = 0$: indeed, we know that $d\underline{u}(x) \in \mathring{Z}_x$ (and so $\mathring{Z}_x \neq \emptyset$).

The hypotheses that were used in all lemmas in this section are exactly those that were anticipated in introduction in 1.10; we summarize all the above results in a theorem:

Theorem 2.13 *Assume* (Z) *in 1.10. Choose a strict subsolution $\underline{u}$, and define F. Then F is a **Finsler metric**, satisfying*

- $F \geq 0$*, and* $F(x,v) = 0$ *iff* $v = 0$*,*
- F *is continuous and locally bounded.*

2.2.2 Asymmetric distance from Z and $\underline{u}$

We define the *length* $\operatorname{len}^L \gamma$ of a locally Lipschitz curve $\xi : [0,1] \to M$ as

$$\operatorname{len}^L \gamma = \int_0^1 F(\xi(s), \dot{\xi}(s))\, ds \tag{2.14}$$

We present here this definition, and postpone to §2.4 other arguments.

Definition 2.15 $b : M \times M \to \mathbb{R}^+$ *is an* asymmetric distance *if* b *satisfies*

- $b \geq 0$ *and* $b(x,y) = 0$ *iff* $x = y$
- $b(x,y) \leq b(x,z) + b(z,y) \quad \forall x,y,z \in M.$

As in §2.x in [29], we then define the asymmetric distance

$$b(x,y) = \inf \operatorname{len}^L \gamma \tag{2.16}$$

where the infimum is computed in the class of all locally Lipschitz ξ with given extrema $\xi(0) = x, \xi(1) = y$.

Under the hypotheses (Z) from 1.10 above, b is an asymmetric distance. Further properties are listed in the proposition 2.33.

b is itself also called a *Finsler metric*,[2] and is naturally associated to Hamilton-Jacobi equations; see [38] and references therein.

By (2.11),(2.14),(2.16),(1.11) and direct calculation

$$S(x,y) = b(x,y) + \underline{u}(x) - \underline{u}(y) \tag{2.17}$$

This means that, when the strict subsolution exists, then S will also be locally bounded from below.

Other consequences will be explored in §2.5.1.

[2] whereas in this paper we usually prefer to call F "the metric" and b "the distance"

2.3 Viscosity solutions of the HJ equation

Hypotheses 2.18 *Let* $Z \stackrel{def}{=} \{H \le 0\}$*; we assume in this section that* $\partial Z = \{H = 0\}$ *(otherwise in the following we would obtain solutions of problem (2.2.⋆) but possibly not of the original problem (1.1)). We assume* `(Z)` *from 1.10. We assume that a* strict subsolution $\underline{u}$ *exists: so* $S(x, y)$ *is continuous and locally bounded.*

We know that

Proposition 2.19 *Assume 2.18. Fix* $a \in M$*. The function*

$$u(x) \stackrel{def}{=} S(a, x) \tag{2.19.$\star$}$$

is a viscosity solution of $H(x, Du(x)) = 0$ *if* $x \ne a$*, and a viscosity subsolution of* $H(x, Du(x)) = 0$ *for all* x*.*

the proof being standard [3].

We define the *value function*

$$V(x) \stackrel{\mathrm{def}}{=} \inf \Big(u_0(\xi(0)) + \int_0^1 \sigma(\xi(s), \dot\xi(s)) ds \Big) \tag{2.20}$$

where the infinimum is in the class of Lipschitz paths ξ with $\xi(1) = x$ and $\xi(0) \in K$. we can rewrite V also as

$$\begin{aligned} V(x) &= \underline{u}(x) + \inf_{z \in K} \big(u_0(z) - \underline{u}(z) + b(z, x) \big) && (2.21) \\ &= \inf_{z \in K} (u_0(z) + S(z, x)) && (2.22) \end{aligned}$$

This last formula is the inf-convolution of the solution (2.19.⋆): then this builds a solution to (1.1):

Theorem 2.23 *Assume 2.18.*

Then V *is a viscosity solution to* $H(x, du(x)) = 0$ *on* $M \setminus K$*.*

If moreover

$$u_0(x) \le S(y, x) + u_0(y) \tag{2.23.$\star$}$$

for all $x, y \in K$*, then* $V = u_0$ *on* K*: so* V *is the viscosity solution to (1.1).*

Proof. Again, the proof is standard, and may be carried on in many different fashions; for example: V solves a "minimum time problem"

$$V(x) = \inf_{\gamma, \gamma(T) = x} (u_0(\gamma(0)) + T)$$

where the infimum is computed in the class of locally Lipschitz curves $\gamma : \mathbb{R}^+ \to M$ s.t. $\gamma(T) = x$ for a $T > 0$ and s.t. $F(\gamma(s), \dot\gamma(s)) \le 1$ for almost all s; then the proof follows from prop. 2.3 in ch. IV in [4]. □

[3] in the $\mathbb{R}^n$ case, see for example Theorem 2.1 in [38]; or see Prop.4.2 in [13]

2.4 Asymmetric metric spaces

We now deviate from the main argument of the paper, that is Hamilton–Jacobi theory, to provide a brief compendium of the theory of asymmetric metric spaces. For the sake of this section, M may be a generic set. Let b be an **asymmetric distance** as was defined in 2.15. We call the pair (M, b) an **asymmetric metric space**. We agree that b defines a topology τ on M, generated by the families of **forward** and **backward** open balls

$$B^+(x, \varepsilon) \stackrel{\text{def}}{=} \{y \mid b(x, y) < \varepsilon\}, \quad B^-(x, \varepsilon) \stackrel{\text{def}}{=} \{y \mid b(y, x) < \varepsilon\}$$

that is, the topology is generated by the symmetric distance

$$d(x, y) \stackrel{\text{def}}{=} b(x, y) \vee b(y, x) \tag{2.24}$$

Remark 2.25 *b is also known as a* "quasi metric" *or* "ostensible metric"*; see for example Kelly [20], Reilly, Subrahmanyam and Vamanamurthy [17] [4] Fletcher and Lindgren [15, (pp 176-181)], Künzi [22].*

The differences between what we present here and what is discussed in those reference is mainly in the choice of the topology and of the definition of "Cauchy sequences" and "completeness" (a detailed explanation is in sec. §2.vi in [29]).

Remark 2.26 *If we would add to the definition 2.15 this additional statement:*

- $\forall (x_n) \subset M, x \in M,\ b(x_n, x) \to 0 \iff b(x, x_n) \to 0$

then the space (M, b) would be a "general metric spaces"*, as defined by Busemann [5], [6].*

This third hypothesis is equivalent to saying that the topology τ generated by the symmetric distance (2.24) may be generated by forward balls only (or backwards balls only).

If the topology τ is locally compact, then the above hypothesis is satisfied (by proposition 2.15 from [29]).

We define that

Definition 2.27 *A sequence $(x_n) \subset M$ is called "*forward Cauchy*" if*

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } \forall n, m,\ \ m \ge n \ge N,\ \ b(x_n, x_m) < \varepsilon \tag{2.27.$\star$}$$

*We say that (M, b) is "*forward complete*" if any* forward Cauchy *sequence (x_n) converges to a point x.* [5]

The above definitions agree with those used in Finsler Geometry (as defined in ch. VI in [3]).

We induce from b the length $\operatorname{len}^b \gamma$ of a continuous curve $\gamma : [\alpha, \beta] \to M$, by using the total variation

$$\operatorname{len}^b \gamma \stackrel{\text{def}}{=} \sup_T \sum_{i=1}^n b\big(\gamma(t_{i-1}), \gamma(t_i)\big) \tag{2.28}$$

where the sup is carried out over all finite subsets $T = \{t_0, \cdots, t_n\}$ of $[\alpha, \beta]$ and $t_0 \le \cdots \le t_n$.

[4] [17] provides also a wide discussion of the references on *quasi metrics*

[5] idem est, $x_n \to x$ according to the topology τ: cf. 2.3 and §2.vi.1 in [29] on the notion of convergence

We define b^g

$$b^g(x,y) = \inf \operatorname{len}^b \gamma \tag{2.29}$$

where the inf is taken in the class of all continuous curves γ connecting x to y. If the $\inf$ is a minimum, the curve providing the minimum is called a *geodesic*.

If the space (M,b) is Lipschitz-arcwise connected, then it is easily proved that b^g is an asymmetric distance.

Definition 2.30 *We say that the (asymmetric) metric space (M,b) is **a path-metric space**, or that b is **intrinsic**, if $b = b^g$.*

We now state a Hopf–Rinow-like theorem:

Theorem 2.31 *Suppose that (M,b) is path-metric, and forward-locally compact (that is, for any x there exists $\varepsilon > 0$ s.t. $\{y \mid b(x,y) \le \varepsilon\}$ is compact); then the following are equivalent*

- *(M,b) is forward complete*
- *forward-bounded and closed sets are compact*

and both imply that (M,b) admits geodesics

A similar statement holds for "backward" conditions (also cf. 2.35). This theorem is proved in 2.38 in [29]; with the stronger hypothesis that (M,b) be locally compact, it is proved in I.8 in [6].

Remark 2.32 *The* forward-local compactness *is called* weak local compactness *in [6]; on this hypothesis, see the notes at the end of section I of [6]; or page 7 in Zaustinsky's [39]; or Phadke's [34] – where though a* weak global compactness *is addressed.*

2.4.1 Finsler metric as asymmetric space

Lets now jump back to the definition of b as Finsler distance, that was given in §2.2.2.

We recall 3.6 and 3.7 from [29]:

Proposition 2.33 *Assume* `(Z)` *from 1.10. Assume the strict subsolution $\underline{u}$ exists. Define F and b as in §2.2.2.*

1. *The topology τ induced by d (cf. (2.24)) coincides with the topology of the manifold M; so the* asymmetric metric space *(M,b) that was defined in §2.2.2 was indeed locally compact. So (M,b) is also a* general metric space*, as defined by Busemann (see 2.26).*
2. *For any Lipschitz γ, $\operatorname{len}^L \gamma$ (defined in eqn. (2.14)) coincides with $\operatorname{len}^b \gamma$ (defined in eqn. (2.28)).*
3. *consequently, (M,b) is path-metric, that is, $b = b^g$.*
4. *For any $x \in M$ there exists a neighbourhood U where, $\forall y \in U$, there exists a minimal geodesic γ connecting x to y with*

$$F(\xi(s), \dot\xi(s)) = 1 \quad \forall s. \tag{2.33.$\star$}$$

2.4.2 Hopf-Rinow for Hamilton-Jacobi

We now use the theory above to state this version of the *Hopf-Rinow* theorem that is adapted to Hamilton-Jacobi problems

Theorem 2.34 (Hopf-Rinow for Hamilton-Jacobi) *Assume all hypotheses* `(Z)` *from 1.10. Choose a strict subsolution* $\underline{u}$*, and define* F *and* b *with it, as above. Then the conditions 1-4 here following are equivalent.*

1. (M, b) *is backward-complete,*
2. *backward bounded closed sets are compact,*
3. $$\liminf_{y\to\infty} b(y, x) = \infty, \tag{2.34.$\star$}$$
4. $$\liminf_{y\to\infty} S(y, x) + \underline{u}(y) = \infty \tag{1.13}$$
   (that was already presented in (1.13)).

We remark that in eqn. (2.34.$\star$), we did not write "$\forall x$" or "$\exists x$", since both statements would be quivalent; and similarly for (1.13).

If the above conditions hold, then, for any fixed $x, y \in M$*, there is a Lipschitz curve* ξ *connecting them that minimizes the length* $\text{len}^L\,\xi$ *defined in (2.14), and that such that* $F(\xi, \dot\xi)$ *is constant (cf. eqn.(2.33.$\star$)); this curve* ξ *is also a minimum for* $S(x, y)$ *in (1.11).*

Theorem 3.44 will add two more conditions to the above.

Proof. The equivalence $1 \iff 2$ of the first two statements, and the existence of geodesics, follows from the more general Hopf-Rinow theorem in 2.31. . The other equivalences are easy (and does not need any special hypotheses on (M, b)) they may be proven using the triangular inequality and the definition of the topology. Moreover (1.13) is just a rewriting of (2.34.$\star$). □

Remark 2.35 (Conjugate problems) *If we define a problem conjugate to (1.1) by using the Hamiltonian* $\overline{H}(x, p) \stackrel{def}{=} H(x, -p)$*, then we may restate all above theorems by using* forward *conditions, and using the conjugate distance* $\overline{b}(x, y) = b(y, x)$ *and the conjugate* $\overline{S}(x, y) = S(y, x)$*.*

2.5 Tilting of asymmetric distances

The realm of asymmetric distances admits a nice operation which is not allowed when dealing with symmetric metrics. Let b be an asymmetric distance on M, and let $\varphi : M \to \mathbb{R}$ be a function; we define that

$$\tilde{b}(x, y) = b(x, y) - \varphi(x) + \varphi(y) \tag{2.36}$$

is a *tilted version* of b by means of φ.

It is readily seen that $\tilde{b}$ satisfies the triangular inequality; if moreover $\tilde{b}(x, y) \geq 0$ (with equality only for $x = y$), then $\tilde{b}$ is an asymmetric distance; we have then proved

Proposition 2.37 *Suppose φ is strictly-1-Lipschitz with respect to b, that is,*

$$b(x,y) \geq \varphi(x) - \varphi(y) \quad \forall x,y, \text{ with equality only for } x = y \tag{2.37.$\star$}$$

then

$$\tilde{b}(x,y) \stackrel{def}{=} b(x,y) - \varphi(x) + \varphi(y)$$

is an asymmetric distance.

Note that by (2.37.$\star$) follows that

$$d(x,y) \geq |\varphi(x) - \varphi(y)|, \text{ with equality only for } x = y \tag{2.38}$$

that is, φ must be strictly-1-Lipschitz w.r.t. to d : then φ must be continuous.

The tilting relation is invertible: b is a *tilted version* of $\tilde{b}$ by means of $-\varphi$.

The tilting relation is also transitive: if $\tilde{b}$ is a *tilted version* of b by means of φ and $\tilde{\tilde{b}}$ is a *tilted version* of $\tilde{b}$ by means of $\tilde{\varphi}$, then $\tilde{\tilde{b}}$ is a *tilted version* of b by means of $\tilde{\varphi} + \varphi$.

As aforementioned, the *tilting operation* is not useful in the realm of symmetric distances: if both b and $\tilde{b}$ are symmetric distances, then φ must be constant, that is, $b = \tilde{b}$. It is instead possible to tilt a symmetric distance to obtain an asymmetric distance: for example, if b derives from a Riemannian geometry, then its tilted $\tilde{b}$ would be a *Randers metric*.[6]

2.5.1 Tilting of S to b

The equation

$$S(x,y) = b(x,y) + \underline{u}(x) - \underline{u}(y) \tag{(2.17)}$$

that we already saw in (2.17) says that b is obtained from S by a tilting operation.

This means that the family of all possible asymmetric metrics that we may associate to (1.1) using strict subsolutions, are equivalent up to tilting.

Unfortunately, tilted metrics are not equivalent w.r.t. completeness

Example 2.39 *Consider $M = \mathbb{R}$ and $b(x,y) = |x-y|$: then (M,b) is complete. Let*

$$\varphi(x) \stackrel{def}{=} \begin{cases} -x^2/(1+x) & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

φ satisfies (2.37.$\star$) since $-1 < \varphi' \leq 0$. Define $\tilde{b}$ as before: $(M,\tilde{b})$ is not forward complete since the sequence $x_n = n$ is forward Cauchy.

(The example 5.8 shows the same phenomenon.)

So, we cannot state a notion of "completeness of S, up to tilting".

2.6 Comparison Theorem

This section is devoted to the proof of this result.

[6] Randers metrics are a particular case of a Finsler metrics: they have the form $F(x,v) = \sqrt{\alpha_{i,j}(x)v^i v^j} + \beta_i(x)v^i$ where α is a Riemannian metric and β is a 1-form; see §1.3C in [3]; if β is exact then the distance corresponding to F is a tilted version of the Riemannian distance induced by α.

Theorem 2.40 *Assume* (Z) *from 1.10; suppose that a choice of strict subsolution* $\underline{u}$ *exists so that the asymmetric metric space* (M, b) *is backward complete.*

Suppose that u *is a viscosity subsolution and* v *is a supersolution of* $H(x, du(x)) = 0$ *for* $x \in M \setminus K$ *whereas* $u \le v$ *on* K*, and suppose that* $v \ge \underline{u} + c$ *for a constant* $c \in \mathbb{R}$*: then* $u \le v$*.*

To prove the above theorem, we will use the following result from prop. 4.3 in Camilli and Siconolfi [7] [7]

Proposition 2.41 (prop. 4.3 in [7]) *Let* $\Omega \subset M$ *be open. For any* $f : \Omega \to \mathbb{R}$ *continuous, we define the Clarke generalized differential as*

$$\partial f(x) \stackrel{def}{=} co\{p \in T_x^* M \mid \exists (x_n) \subset \Omega, \exists df(x_n) \stackrel{def}{=} p_n, (x_n, p_n) \to (x, p) \text{ in } T^* M\}$$

where $co(A)$ *is the convex envelope of a set* $A \subset T_x^* M$*. Consider a problem* $H(x, du(x)) = 0$ *such that 1.10 holds on* $Z \stackrel{def}{=} \{H \le 0\}$.

- v *is a supersolution of* $H(x, dv(x)) = 0$ *in* Ω *iff for any* $x \in \Omega$ *and any Lipshitz continuous* ϕ *which is subtangent to* v *at* x *there exists* $p \in \partial\phi(x)$ *such that* $H(x, p) \ge 0$
- *Any subsolution of* $H(x, du(x)) = 0$ *is Lipshitz continuous, and for all* $p \in \partial u(x)$ *we have* $H(x, p) \le 0$

Using this result we can simplify the proof in [27], and yet prove the general theorem 2.40 here proposed.

By means of the transformation in (3.8), we assume without loss of generality that $\underline{u} \equiv 0$, and we replace the problem at hand with the problem (3.7). Let

$$h(x, p) \stackrel{\text{def}}{=} \sqrt{\hat{H}(x, p)}$$

in the following.

As in the work of Kružhkov [21], we consider the transformed functions $\tilde{u} = -e^{-u}$ and $\tilde{v} = -e^{-v}$, which are respectively a viscosity subsolution and a supersolution of

$$\begin{cases} h(x, dv) + v = 0 & \text{in } M \setminus K, \\ v = -e^{-u_0} & \text{on } K \end{cases} \tag{2.42}$$

(see proposition 6 in [27]) moreover, $0 > \tilde{v} \ge -e^{-\inf v} \ge -e^{-c}$ and $\tilde{u} < 0$.

We establish a comparison result for this last problem (2.42): this clearly implies the above theorem. We fix $C \stackrel{\text{def}}{=} e^{-c}$. We argue by contradiction, and suppose that $\tilde{u}$ and $\tilde{v}$ are resp. a subsolution and a supersolution of (2.42), $0 > \tilde{v} \ge -C$, $\tilde{u} < 0$, and that at a point $\overline{x}$ we have $\tilde{u}(\overline{x}) = 2\varepsilon + \tilde{v}(\overline{x})$ with $\varepsilon > 0$.

We apply the Kružhkov transformation to above proposition 2.41 and state that

- for any $x \in M \setminus K$ and any Lipshitz continuous ϕ which is subtangent to $\tilde{v}$ at x there exists $p \in \partial\phi(x)$ such that $h(x, p) + \tilde{v}(x) \ge 0$
- $\tilde{u}$ is locally Lipshitz continuous, and for any $x \in M \setminus K$ and for all $p \in \partial\tilde{u}(x)$ we have $h(x, p) + \tilde{u}(x) \le 0$

[7] The proof in [7] is stated assuming that $\Omega \subset \mathbb{R}^n$, but the result can be generalized to manifolds, using local coordinates

Let $B(x) \stackrel{\text{def}}{=} b(x, \overline{x})$. By 2.19 and 2.35 we know that B is a viscosity solution of $h(x, -p) - 1 = 0$ for $x \neq \overline{x}$: then for all $x \in M$ and all $p \in \partial B(x)$, $h(x, -p) \leq 1$.

Let

$$\Psi(x) = \tilde{u}(x) - \tilde{v}(x) - \varepsilon B(x)$$

This function is bounded from above by C; moreover $\Psi(\overline{x}) = 2\varepsilon$: then $\sup \Psi$ will be positive, and realized in the region $\{x \mid \varepsilon B(x) \leq C\}$ which is a backward closed ball. By the Hopf–Rinow–like theorem 2.34 since the metric space (M, b) is backward complete, then the backward closed balls are compact: so $\Psi(x)$ has a positive maximum in a point $\hat{x}$. This means that the function $\tilde{u}(x) - \varepsilon B(x)$ is a Lipschitz subtangent of $\tilde{v}(x)$ at $\hat{x}$.

We know that $\Psi(\hat{x}) \geq \Psi(\overline{x}) = 2\varepsilon$, while $\Psi(x) \leq 0$ for all $x \in K$: then $\hat{x} \notin K$. Then by (the transformed version of) 2.41, there exists $p \in \partial(\tilde{u}(x) - \varepsilon B(x))$ such that $h(x, p) + \tilde{v}(x) \geq 0$.

At the same time $p = p' + p''$ with $p' \in \partial \tilde{u}(x)$ and $p'' \in \partial(-\varepsilon B(x))$: then (again by 2.41) $h(\hat{x}, p') + \tilde{u}(\hat{x}) \leq 0$; at the same time, as noted above, $h(\hat{x}, p''/\varepsilon) - 1 \leq 0$ that is $h(\hat{x}, p'') \leq \varepsilon$.

Since $h(x, \cdot)$ is convex and 1-homogeneous, summing up we obtain

$$h(\hat{x}, p) \leq h(\hat{x}, p') + h(\hat{x}, p'') \leq -\tilde{u}(\hat{x}) + \varepsilon$$

while $h(x, p) + \tilde{v}(x) \geq 0$: this entails

$$-\tilde{v}(x) \leq h(x, p) \leq -\tilde{u}(\hat{x}) + \varepsilon$$

or

$$\tilde{u}(\hat{x}) - \tilde{v}(\hat{x}) \leq \varepsilon$$

whereas

$$\tilde{u}(\hat{x}) - \tilde{v}(\hat{x}) \geq \Psi(\hat{x}) \geq \Psi(\overline{x}) \geq 2\varepsilon$$

achieving contradiction.

2.7 uniqueness and variationality

We summarize the results in previous sections in this theorem

Theorem 2.43 *Let* $Z \stackrel{\text{def}}{=} \{H \leq 0\}$*; assume hypotheses* `(Z)` *from1.10; assume that* $\partial Z = \{H = 0\}$*. Assume that a choice of the* strict subsolution $\underline{u}'$ *exists so that* $\underline{u}' \leq u_0$ *and*

$$\liminf_{y \to \infty} S(y, x) + \underline{u}'(y) = \infty \tag{1.13}$$

We know from theorem 2.34 that eqn. (1.13) is a completeness hypothesis.

We also assume (2.23.⋆) on u_0*, so* $V = u_0$ *on* K*.*

(variationality) *For each* x*, the value problem (2.20) has a minimum, attained by a Lipschitz curve* ξ *such that* $F(\xi, \dot{\xi})$ *is (almost everywhere) constant.*

In this case we say that the problem (1.1) is <u>*variational*</u>*, since it comes from a variational problem that admits minimum. We symbolize this fact by the symbol* **(**$\exists \min$**V)**.

The proof follows from 2.34.

(uniqueness) *Let $\mathcal{F}$ be the class of all continuous functions $f : M \to \mathbb{R}$ such that there exists a* strict subsolution $\underline{u}$ *with* $\underline{u} \le f$ *on M and satisfying (1.13).*

Then V is the unique viscosity solution to problem (1.1), in the class $\mathcal{F}$.

The proof follows from the comparison theorem 2.40.

Remark 2.44 *Note that, since we supposed that exists a $\underline{u}' \le u_0$, satisfying (1.13), then $V \in \mathcal{F}$ (since, by thm. 2.40, $V \ge \underline{u}'$). But it may be the case that there is no such $\underline{u}'$: see example 5.8, where V is not in $\mathcal{F}$.*

In §5.4 we will compare this theorem to other results in the literature.

There remain an open question: how to better characterize this class $\mathcal{F}$? In the particular cases

- when M is compact and a *strict subsolution* $\underline{u}$ exists, then all continuous functions are in $\mathcal{F}$ (so this uniqueness theorem extends the result in [18]);
- if $H(x,0) < 0$, and $\liminf_{y\to\infty} S(y,x) = \infty$, then all continuous lower bounded functions are in $\mathcal{F}$. So, this theorem extends the uniqueness part in the Riemannian case thm. 1.6
(indeed in that case $d^g \equiv S$, so the condition $\liminf_{y\to\infty} d(y,x) = \infty$ means that (M,g) is complete).

In general, we propose this

Conjecture 2.45 *Suppose that a* strict subsolution $\underline{u}$ *exists; suppose u_0 satisfies (2.23.⋆). Then the class $\mathcal{F}$ contains all f such that*

$$\liminf_{x\to\infty} f(x) + S(y,x) = \infty \tag{2.45.$\star$}$$

for any y (or equivalently for a fixed y).

3 Strictly convex $C^{1,1}$ Hamilton-Jacobi equation

In this section we add other hypotheses: we suppose that H is of class $C^{1,1}$ in a neighborhood of $\{H = 0\}$; and that each set Z_x is strictly convex.

3.1 Characteristics' flow

We consider T^*M as a symplectic manifold: we define the symplectic 2–form

$$\omega\Big((\dot{x},\dot{p}),\ (\dot{y},\dot{q})\Big) \stackrel{\text{def}}{=} \sum_i \dot{q}_i \dot{x}_i - \sum_i \dot{p}_i \dot{y}_i$$

and the duality $\omega^{\#}$ between TT^*M and TTM, given by

$$\omega^{\#}(\nu)\cdot\nu' = \omega(\nu,\nu') \quad \forall \nu' \in T^*M \ . \tag{3.1}$$

We define the *characteristics' flow*

$$(X(\cdot,z,q), P(\cdot,z,q))$$

as the solution of the system of ordinary differential equations

$$\begin{cases} \dot{X}(s) = \frac{\partial H}{\partial p}(X(s), P(s)) \\ X(0) = z \\ \dot{P}(s) = -\frac{\partial H}{\partial x}(X(s), P(s)) \\ P(0) = q \end{cases} \tag{3.2}$$

and we define U by

$$\left\{ \dot{U}(s) = P(s) \cdot \frac{\partial H}{\partial p}(X(s), P(s)), \qquad U(0) = 0 \right.$$

that is, (X, P) is the *Hamiltonian flow* for the symplectic manifold for (T^*M, ω).

If $H(z, q) = 0$, then $H(X, P) = 0$ for all times, so the above O.D.E. is well defined: the solution (X, P) exists and is unique for small times.

We define the *maximal times* of (3.2) to be

$$t^+(z, q) \stackrel{\text{def}}{=} \sup\{t > 0 \quad \text{such that the characteristic } X(t, z, q), P(t, z, q) \text{ exist up to time } t\} \tag{3.3}$$

and

$$t^-(z, q) \stackrel{\text{def}}{=} \inf\{t < 0 \quad \text{such that the characteristic } X(t, z, q), P(t, z, q) \text{ exist down to time } t\} \tag{3.4}$$

note that t^+ is l.s.c, t^- is u.s.c.

3.2 Characteristic–subsolution completeness

We assume `(Z)` from 1.10, and that there exists a subsolution $\underline{u}$. We assume that `(Hnd)` (defined in 1.14 on p. 5) holds. In this case, the characteristic curves (3.2) are *non degenerate* (as explained in remark 2.4).

The hypothesis `(Hnd)` implies some regularity on the set Z (see 3.20); but `(Hnd)` does not imply that Z_x be convex and compact, so we will still need some of the hypotheses 1.10.

We will show that if we assume `(Hnd)` then the problem (1.1) induces a *weaker Finsler Geometry* (M, L). We postpone the precise definition of what (M, L) is to section §3.6, to concentrate on Hamilton–Jacobi theory.

We propose these conditions

(MC$\underline{u}$), (MC$\underline{u}$+), (MC$\underline{u}$−) Suppose `(Hnd)`. The condition (MC$\underline{u}$+) states that there exists a strict subsolution $\underline{u}$ such that

$$\lim_{t \to t^+(z,q)} U(t, z, q) - \underline{u}(X(t, z, q)) = \infty \tag{3.5}$$

while (MC$\underline{u}$−) holds when

$$\lim_{t \to t^-(z,q)} U(t, z, q) - \underline{u}(X(t, z, q)) = -\infty$$

for all (z, q) such that for $H(z, q) = 0$; if both conditions hold for the same $\underline{u}$, we say that (MC$\underline{u}$) holds.

Note that the conditions `(MC`$\underline{u}$`)`, `(MC`$\underline{u}$`+)`, `(MC`$\underline{u}$`−)` are robust wrt a change of dependent and independent variable.

Note also that, if H is $C^{1,1}$ and any of `(MC`$\underline{u}$`+)` or `(MC`$\underline{u}$`−)` hold, then the non degeneracy condition (1.14.$\star$) must hold as well.

We refer to the above `(MC`$\underline{u}$`)`, `(MC`$\underline{u}$`+)`, `(MC`$\underline{u}$`−)` as *bilateral/forward/backward characteristics–subsolution completeness*: the reason is in this theorem (that adds a fifth condition to 2.34):

Theorem 3.6 *We assume* `(Z)` *from 1.10, and* `(Hnd)`*. We suppose that Z_x is strictly convex, for all x.*

For any choice of a strict subsolution $\underline{u}$ (see defn. 1.12), we define the metric space (M, b): then

1. *the metric space (M, b) is backward-complete (resp. forward), if and only*

6. *the condition* `(MC`$\underline{u}$`−)` *(resp.* `(MC`$\underline{u}$`+)`*) holds.*

The remanining part of this section 3 will develop all arguments needed fot the proof; the proof itself will be a consequence of 3.27, of 3.31 and of the Hopf-Rinow thm.3.44.

3.3 Reduction to a simpler problem

The main tool to prove the results is to reduce the model problem (1.1) to an equivalent simpler problem.

We assume that `(`$\exists\underline{u}$`)` holds (see 1.12).

We substitute the problem (1.1) that we are studying, with the problem

$$\begin{cases} \hat{H}(x, Du(x)) - 1 = 0 & \text{in } M \setminus K \\ u(x) = \hat{u_0}(x) & \text{when } x \in K. \end{cases} \tag{3.7}$$

We then study how the conditions and hypotheses are affected. To this end, we construct $\hat{H}$ in two steps:

1. We define
$$\widetilde{H}(x, p) \stackrel{\text{def}}{=} H(x, p + d\underline{u}(x)) \quad , \quad \hat{u_0} \stackrel{\text{def}}{=} u_0 - \underline{u} \ . \tag{3.8}$$
As shown in 3.16, if V is the value of 1.1, $\tilde{V} = V - \underline{u}$ is the value solution of
$$\begin{cases} \widetilde{H}(x, D\widetilde{u}(x)) = 0 & \text{in } M \setminus K \\ \widetilde{u}(x) = \widetilde{u_0}(x) & \text{when } x \in K. \end{cases}$$
and similarly for viscosity solutions.
The above transformation implies that $\widetilde{H}(x, 0) < 0$, and basilarly say that we can assume that $\underline{u} = 0$ in our proofs, with no loss of generality.

2. Assume `(Z)` from 1.10. We now define
$$\tilde{Z}_x \stackrel{\text{def}}{=} \left\{ p \mid \widetilde{H}\,(x, p) \le 0 \right\}$$
$\tilde{Z}_x$ is a convex set; we define then the gauge function j_z of $\tilde{Z}_x$ as
$$j_x(p) \stackrel{\text{def}}{=} \inf \left\{ t > 0 \mid p/t \in \tilde{Z}_x \right\}$$
and eventually we define
$$\hat{H}(x, p) \stackrel{\text{def}}{=} \big(j_x(p)\big)^2$$

Summarizing, $\hat{H}$ is built from H (or from Z) by

$$\hat{H}(x,p) \stackrel{\text{def}}{=} \inf\left\{t^2 \,\middle|\, t>0 \text{ s.t. } H\left(x, \frac{p}{t} + d\underline{u}(x)\right) \le 0\right\} = \\ = \inf\left\{t^2 \,\middle|\, t>0 \text{ s.t. } \left(\frac{p}{t} + d\underline{u}(x)\right) \in Z_x\right\} \tag{3.9}$$

We call $\hat{H}$ *the gauge Hamiltonian*.

By the definition, $\hat{H}$ is positively 2-homogeneous, ie

$$\forall \lambda \ge 0 \qquad \hat{H}(x, \lambda p) = \lambda^2 \hat{H}(x,p) \tag{3.10}$$

It is easily proved (by the definition of $\hat{H}$) that $(\hat{H}(x,p) - 1)$ has the same sign of $\tilde{H}(x,p)$: then, u is a viscosity solution for $\hat{H}$ iff it is a viscosity solution for $\tilde{H}$.

Suppose that $H \in C^{1,1}$ in a neighbourhood of $\{H = 0\}$, and $\hat{H}$ as well. We do not know a priori if the characteristics curves of H and the characteristics curves of $\hat{H}$ are related: the idea being that, they are related if `(Hnd)` holds for H. We explore this idea in sections 3.4 and 3.5.

Remark 3.11 *We easily see that, if $p \neq 0$, there is only one value of t such that*

$$\tilde{H}\left(x, \frac{p}{t}\right) = 0 \quad ,$$

and it is $t = \sqrt{\hat{H}(x,p)}$.

3.4 Equivalent problems

We remarked in 2.1 that the viscosity solution does not depend on the values of H, but only on its sign. We now need to find a relationship between characteristics curves of equivalent problems:

Lemma 3.12 *Suppose that*

$$\tilde{H}(x,p) = \tilde{\rho}(x,p) H(x,p) \tag{3.12.$\star$}$$

where $\tilde{\rho} : T^*M \to \mathbb{R}$ is a positive function; suppose that ρ is locally Lipschitz in a neighbourhood of $\{H = 0\}$; and $\tilde{H}(x,p)$ is $C^{1,1}_{loc}$ in a neighbourhood of $\{H = 0\}$.

The value function $\tilde{V}$ of

$$\begin{cases} \tilde{H}(x, d\tilde{u}(x)) = 0 & \textit{on } M \\ \tilde{u} = u_0 & \textit{on } K \end{cases} , \tag{3.12.$\star\star$}$$

coincides with the value function V of (1.1); the same holds also (since $\tilde{\rho} > 0$, see remark 2.1) for viscosity solutions.

We indeed note that the characteristics $\tilde{X}, \tilde{P}, \tilde{U}$ of $\tilde{H}$ are related to the characteristics X, P, U of H through a reparameterization of the t variable: fix $y = (z,q)$ s.t. $H(z,q) = 0$, and let $a, b, \tilde{a}, \tilde{b} > 0$ be such that $(-a,b)$ is the maximal interval of definition for X, P, U, and $(-\tilde{a}, \tilde{b})$ is the maximal interval of definition for $\tilde{X}, \tilde{P}, \tilde{U}$; then there is a diffeomorphism $\varphi_y : (-\tilde{a}, \tilde{b}) \to (-a, b)$ s.t.

$$(\tilde{X}(t,y), \tilde{P}(t,y), \tilde{U}(t,y)) = (X(\varphi_y(t), y), P(\varphi_y(t), y), U(\varphi_y(t), y)) \tag{3.12.$\diamond$}$$

that is defined by

$$\varphi_y(t) = \int_0^t \tilde{\rho}(\tilde{X}(s,y), \tilde{P}(s,y))\, ds = \qquad (3.12.\diamond\diamond)$$
$$= \int_0^t \tilde{\rho}(X(\varphi_y(s),y), P(\varphi_y(s),y))\, ds$$

A similar result is found in lemma 3.1 in §3.5 in [28].

Remark 3.13 *Suppose that $\tilde{H}$ satisfies* `(Hnd)` *(in p. 5) and similarly for $\widetilde{H}$, and suppose that $H, \tilde{H} \in C^{1,1}$, and*

$$\textit{sign} H(x,p) = \textit{sign} \tilde{H}(x,p) \quad \forall x, p \in T^*M$$

Then there exists a $\tilde{\rho} > 0$ locally Lipschitz such that (3.12.$\star$) holds. So, in a sense, (3.12.$\star$) is the most general change of equation that preserves the viscosity solutions and the characteristics flow.,

Every time we will be able to transform the problem (1.1) in another problem (3.12.$\star\star$), so that the solutions are related by a simple algebraic relationship, we will say that the two problems are *equivalent*.

Definition 3.14 (Invariant conditions) *Consider a condition $\mathcal{P}$ on a problem such as (1.1); suppose that (1.1) satisfies $\mathcal{P}$ iff (3.12.$\star\star$) satisfies $\mathcal{P}$: then we say that this condition $\mathcal{P}$ is* invariant *under the equivalence of the two problems.*

Proposition 3.15 *The conditions* $(\exists \underline{u})$, $(MC\underline{u})$, $(MC\underline{u}\pm)$, *and* $(\exists\, \min V)$ *are invariant: indeed, the variable t does not play an explicit role in them, so that by (3.12.$\diamond$) we just need to change the time variable with $\varphi_y(t)$ in the definitions.*

Lemma 3.16 (calibration) *If ψ is any regular function on M, we can define*

$$\widetilde{H}(x,p) \stackrel{def}{=} H(x, p + d\psi(x)) \quad , \quad \tilde{u_0} \stackrel{def}{=} u_0 - \psi \ ;$$

then, if V is the value solution of 1.1, $\widetilde{V} = V - \psi$ is the value solution of

$$\begin{cases} \widetilde{H}(x, D\widetilde{u}(x)) = 0 & \text{in } M \setminus K \\ \widetilde{u}(x) = \tilde{u_0}(x) & \text{when } x \in K. \end{cases} \qquad (3.16.\star)$$

to prove this, it is sufficient to note that the characteristics of $\tilde{H}$ are related to the characteristics X, P, U of H through the relation

$$(\tilde{X}, \tilde{P}, \tilde{U}) = (X, P - D\psi(X), U - \psi(X) + \psi(X(0))) \ ;$$

and a similar statement holds for viscosity solutions and min *solutions.*

Proposition 3.17 *If we relate a strict subsolution $\underline{u}$ of (1.1) to a strict subsolution $\widetilde{\underline{u}}$ of (3.16.$\star$) using the formula $\widetilde{\underline{u}} = \underline{u} - \psi$, then* $(\exists \underline{u})$, $(MC\underline{u}\pm)$ *are invariant, by construction.*

Another way to relate two problems is in proposition 2.10 in [27].

3.5 Equivalence of characteristics of $\hat{H}$ and H

We again define

$$\tilde{Z} \stackrel{\text{def}}{=} \{(x,p) \mid \tilde{H} \le 0\} = \{(x,p) \mid \hat{H} \le 1\} \tag{3.18}$$

Note that Z and $\tilde{Z}$ are easily related by

$$Z_x = \tilde{Z}_x + d\underline{u}(x) \tag{3.19}$$

As aforementioned, the hypothesis `(Hnd)` (that was defined in (1.14.⋆)) directly implies some of the hypotheses were used previously (e.g. in theorem 2.43):

Proposition 3.20 *Assume* `(Hnd)`. *Then $\partial Z = \{H = 0\}$; moreover $\{H = 0\}$ is a regular $C^{1,1}$ submanifold of T^*M; and $Z_x = \overline{A_x}$, where $A = \mathring{Z}$. Assume moreover that Z_x is convex, choose a strict subsulution $\underline{u}$ (cf. 1.12), and define $\hat{H}$: then $\hat{H}$ is of class $C^{1,1}$ on $T^*M \cap \{p \ne 0\}$.*

The proof is by implicit function theorem.

Since the functions $\tilde{H}$ and $\hat{H} - 1$ have the same zero set $\partial\tilde{Z}$, their derivatives $D\tilde{H}(x,p)$ and $D\hat{H}(x,p)$ are parallel when $(x,p) \in \partial\tilde{Z}$; it is easily proved, (by using homogeneity, see (3.24.⋆⋆)), that $D\hat{H} \ne 0$ on $\partial\tilde{Z}$; then the function

$$\tilde{\rho}(x,p) \stackrel{\text{def}}{=} \frac{\tilde{H}(x,p)}{\hat{H}(x,p) - 1}$$

is positive and locally Lipschitz in a neighborhood of $\partial\tilde{Z}$: applying the lemma 3.12 we are sure that the problem (3.7) and the problem (1.1) are equivalent, that is, they share the same properties and the same solutions.

Using the canonical coordinates of T^*M, we can define the Hamiltonian flow $(\hat{X},\hat{P}) : \mathbb{R} \times T^*M \to T^*M$ of $\hat{H}$, as the solution of the *system of characteristics*

$$\begin{cases} \dot{\hat{X}}_i(s) = \frac{\partial \hat{H}}{\partial p_i}(\hat{X}(s),\hat{P}(s)) \\ \dot{\hat{P}}_i(s) = -\frac{\partial \hat{H}}{\partial x_i}(\hat{X}(s),\hat{P}(s)) \end{cases} \tag{3.21}$$

with initial conditions

$$\begin{cases} \hat{X}(0) = z \\ \hat{P}(0) = q \ . \end{cases} \tag{3.22}$$

Lets call $\hat{U}$ the solution of

$$\begin{cases} \dot{\hat{U}}(s) = \hat{P}(s) \cdot \dfrac{\partial \hat{H}}{\partial p}(\hat{X}(s),\hat{P}(s)), \qquad \hat{U}(0) = 0 \end{cases}$$

Lemma 3.23 *Suppose that H satisfies* `(Hnd)` *and Z satisfies 1.10. If we want to compute the reparameterization φ_y between the characteristics of $\tilde{H}$ and the characteristics of $\hat{H}$, then by the lemma 3.12, we only need the value of $\tilde{\rho}$ where $\tilde{H}(x,p) = 0$; there, by Höpital's theorem,*

$$\tilde{\rho}(x,p) = \frac{\nu \cdot D\tilde{H}(x,p)}{\nu \cdot D\hat{H}(x,p)}$$

for almost any $\nu \in TT^*M$; in particular,

$$\tilde{\rho}(x,p) = \frac{p \cdot \frac{\partial}{\partial p}\tilde{H}(x,p)}{p \cdot \frac{\partial}{\partial p}\hat{H}(x,p)} = \frac{1}{2} p \cdot \frac{\partial}{\partial p}\tilde{H}(x,p) \tag{3.23.⋆}$$

Using the above reduction we can easily prove that

Proposition 3.24 *let* $y = (z, q) \in T^*M$ *then*

$$\frac{d}{dt}\hat{U}(t, y) = 2\hat{H}(y) \tag{3.24.$\star$}$$

Proof. By deriving (3.10) wrt λ, we get the Euler identity

$$2\hat{H}(x, p) = \frac{\partial \hat{H}}{\partial p}(x, p) \cdot p \tag{3.24.$\star\star$}$$

since $\hat{H}(\hat{X}(t, y), \hat{P}(t, y))$ is constantly equal to $\hat{H}(y)$, we get

$$2\hat{H}(y) = \frac{\partial \hat{H}}{\partial p}(\hat{X}, \hat{P}) \cdot \hat{P} = \frac{d}{dt}\hat{U}(t, y)$$

□

3.6 Towards a weaker Finsler Geometry

Suppose Z satisfies 1.10. Choose a strict subsolution $\underline{u}$. Let $\hat{H}$ be as in (3.9).

Let (M, b) be the asymmetric metric space associated to (1.1) in sec. 2.2. Because of the relation (3.19) between $\{H \le 0\}$ and $\{\hat{H} \le 0\}$, we associate (M, b) to $\hat{H}$ as well.

To prove thm. 3.6 we need to prove that: $(M, \hat{H})$ is "*characteristically backward complete*" (resp.forward) iff the metric space (M, b) associated to it is backward complete (resp. forward). This suggests that we need a kind of Geometry, and a Hopf-Rinow theorem.

To this end, we define $L : TM \to [0, \infty)$ using the Legendre-Fenchel transform,

Definition 3.25 *the Legendre–Fenchel transform* L *of* $\hat{H}$ *is*

$$L(x, v) \stackrel{def}{=} \max_{p \in T^*_x M} \big(p \cdot v - \hat{H}(x, p)\big) \tag{3.25.$\star$}$$

which can be inverted by the dual formula

$$\hat{H}(x, p) \stackrel{def}{=} \max_{v \in T_x M} \big(p \cdot v - L(x, v)\big) \tag{3.25.$\star\star$}$$

Remark 3.26 *Note by homogeneity, we may use the Legendre–Fenchel formulas*

$$L(x, v) \stackrel{def}{=} \max_{p \neq 0} \frac{(p \cdot v)^2}{4\hat{H}(x, p)} = \max_{p \text{ s.t. } \hat{H}(x,p) \le 1} \frac{(p \cdot v)^2}{4} \tag{3.26.$\star$}$$

$$\hat{H}(x, p) \stackrel{def}{=} \max_{v \neq 0} \frac{(p \cdot v)^2}{4L(x, v)} = \max_{v \text{ s.t. } L(x,v) \le 1} \frac{(p \cdot v)^2}{4} \tag{3.26.$\star\star$}$$

which show that $4L = F^2$*, where* F *is the support function of* $\{p \mid \hat{H}(x, p) \le 1\}$*; but then, by eqn.(3.18) and eqn.(3.19),* F *coincides with what was defined in (2.11).*

Remark 3.27 *By the discussion in § 3.4, the original problem (1.1) satisfies* `(MCu)` *if and only if the problem (3.7) satisfies* `(MCu)`. *We have just seen in prop.3.24 that, for any (z, q) with $\hat{H}(z, q) = 1$, we have*

$$\hat{U}(t, z, q) = 2t$$

this means that, for the problem (3.7), the condition `(MCu)` *is simply saying that $(M, \hat{H})$ is "*characteristically complete*"* [8]*: that is, the characteristics $(\hat{X}, \hat{P})$ of $\hat{H}$ can be evolved for all times.*

We will show in the following that we can view (M, L) as "weak Finsler space", with an exponential map (see §3.8), and a Hopf–Rinow theorem (see §3.9); since characteristics of $\hat{H}$ are geodesics of (M, L), then the condition `(MCu)` *is equivalent to saying that (M, L) is "*geodesically complete*".*

A similar statement holds for forward-only and backward-only completeness.

Example 3.28 *In the case of the eikonal equation (1.5) on a Riemannian Manifold, we have that $H(x, p) = |p|^2 - 1$, $\hat{H}(x, p) = |p|^2$, $L(x, v) = |v|^2/4$ and $F(x, v) = |v|$; and $b(x, y)$ is the Riemannian distance.*

Remark 3.29 *We remark that* "EL-geodesic" *of Riemannian manifolds (that we have defined in 1.8) have $|\dot{\gamma}(t)|_g$ constant.*

In the case of characteristic of (1.1), instead, the discussion above suggests that

$$U(t, z, q) - \underline{u}(X(t, z, q))$$

is a natural choice of time parameter *(a sort of "arc parameter").*

We would like to view (3.7) as the *eikonal problem* for the Geometry (M, L); but, in the general case that we consider here, (M, L) is not a regular Finsler Geometry.

We collect a reasonable set of hypotheses

Hypotheses 3.30 **($\hat{H}$1)** *$\hat{H}(x, p)$ is locally a $C^{1,1}$ map on $T^*M \cap \{p \neq 0\}$.*

($\hat{H}$3!) *the* figuratrix *set of $\hat{H}$*

$$\{p \in T^*_x M \mid \hat{H}(x, p) \le 1\}$$

is compact and strictly convex, or equivalently, $\hat{H}$ is strictly convex in p;

($\hat{H}$4) *$\hat{H}(x, \cdot)$ is positively 2-homogeneous, that is, $\hat{H}(x, lp) = l^2 \hat{H}(x, p)$ for $l \ge 0$.*

This proposition is the link between the theorem 3.6, that we still need to prove, and the Hopf–Rinow theorem 3.44:

Proposition 3.31 *Suppose that H satisfies* `(Hnd)`, *Z satisfies 1.10 and Z_x is strictly convex. Then $\hat{H}$ satisfies all these properties 3.30.*

Proof. Since Z_x is strictly convex, then $\hat{H}$ is strictly convex in p: this is property ($\hat{H}$3!) in 3.30. By 3.20, $\hat{H}$ is $C^{1,1}$ when $p \neq 0$: this is property ($\hat{H}$1) in 3.30. By its own definition (3.9), $\hat{H}$ is positively 2-homogeneus in p: this is property ($\hat{H}$4) in 3.30. □

This means that $\hat{H}$ induces a *weaker Finsler Geometry* (M, L).

We will indeed prove that in this case $L \in C^1$ but in general $L \notin C^2$ (see 3.36 and 3.37). We will be able nonetheless to define an exponential map (M, L), and use it to formulate an Hopf–Rinow theorem, in §3.9.

[8] that is condition `(MC)` in page 35

3.7 On the Legendre-Fenchel transform

When we assume hypotheses 3.30 on $\hat{H}$, and then compute the Legendre-Fenchel transform, we need to understand the regularity and properties of L; this case is not covered in the most common books about the so we outline how some basic facts do hold true.

This subsection is more of a comment on Calculus of Variations than on Finsler spaces: so $\hat{H}$ will not be necessarily homogeneous in p; we will though suppose, for convenience, that $\hat{H}(x,\cdot)$ is *superlinear*, that is,

$$\lim_{p\to\infty} \frac{\hat{H}(x,p)}{|p|} = \infty \tag{3.32}$$

where $|p|$ is any chosen norm on T_x^*M; we know indeed that $\hat{H}(x,\cdot)$ is *superlinear* iff L is defined on the whole TM, and proper on each T_xM (see thm. 1.4.11 in [10]).

We define L using the Legendre–Fenchel transform 3.25

Proposition 3.33 *Since $\hat{H}(x,\cdot)$ is C^1 and strictly convex, then, by duality, $v \mapsto L(x,v)$ is C^1 and strictly convex; then the points p^0 and v^0, where the above maxima are realized, are related by the Legendre reciprocity formula*

$$p^0 = \frac{\partial L}{\partial v}(x,v^0), \quad v^0 = \frac{\partial \hat{H}}{\partial p}(x,p^0) \tag{3.33.$\star$}$$

Remark 3.34 *The relationship (3.33.$\star$) is obviously a homeomorphism.*

This would bring to the classical approach [9]:

Proposition 3.35 *if $\hat{H}$ is C^2 and strongly convex in p, then L is C^2 and strongly convex in v, and viceversa; and in this case $\hat{H}$ is C^r iff L is C^r ($r \geq 2$), and the relation (3.33.$\star$) is a diffeomorphism for any fixed x.*

If $\hat{H}$ is only strictly convex, then, in general, we cannot expect that L be regular:

Example 3.36 *if we take $M = \mathbb{R}$, $H(x,p) = p^{2n}/2n$, then $L(x,v) = \frac{2n+1}{2n} v^{\frac{2n}{2n-1}}$: so, if we just say that $H \in C^\infty$ and strictly convex in p, we don't have any bound on the Hölder exponent of $\frac{\partial}{\partial v}L$.*

Moreover the relation (3.33.$\star$) is *not* in general a diffeomorphism for fixed x; but we show in the following that we may use all the common methods in Calculus of Variations nonetheless.

Proposition 3.37 *If $\hat{H}(x,p)$ is locally a $C^{1,1}$ map on T^*M, strictly convex in p, then it is easily proved that $L \in C^1$ as a function on TM*

Proof. We have noted that $v \mapsto L(x,v)$ is, by duality, C^1.

We will prove that $x \mapsto L(x,p)$ is derivable and that

$$\frac{\partial L}{\partial x}(x^0,v^0) = \frac{\partial \hat{H}}{\partial x}(x^0,p^0) \tag{3.37.$\star$}$$

[9] see §1.8 in [14], or §2.3 in [10].... and many other text in Calculus of Variations, or Hamilton–Jacobi problems

(where x^0, v^0, p^0 are related by (3.33.$\star$)). We can assume without loss of generality that M is substituted by an open subset of $\mathbb{R}^n$, and $\frac{d}{dx,p}\hat{H}(0,0) = 0$, $\hat{H}(x,0) = 0$, $x^0 = 0, v^0 = 0, p^0 = 0$, so that $L(0,0) = 0$: we will then prove that

$$\frac{L(x,0)}{|x|} = \frac{\max_p -\hat{H}(x,p)}{|x|} \xrightarrow[x\to 0]{} 0$$

that is, that $\frac{\partial}{\partial x}L(0,0) = 0$. Obviously

$$0 \leq \frac{\max_p -\hat{H}(x,p)}{|x|}$$

Lets call $p^* = p^*(x)$ the point where the maximum is attained; then, by (3.32), $p^*(x)$ is bounded; and by strict convexity, $p^*(x)$ is continuous, and $p^*(x) \to 0$ for $x \to 0$; then

$$\max_p -\hat{H}(x,p) = -\hat{H}(x,p^*(x)) \leq \frac{d}{dp}\hat{H}(x,0)\cdot p^*(x) \leq C|x||p^*(x)|$$

□

Proposition 3.38 *Suppose again that $\hat{H}(x,p)$ is locally a $C^{1,1}$ map on T^*M, strictly convex in p.*

Suppose that ξ is a critical curve of the energy functional *(or,* action functional*)*

$$E(\xi) \stackrel{def}{=} \int_0^1 L(\xi(s),\dot{\xi}(s))\,ds \qquad (3.38.\star)$$

then, by associating

$$p(s) \stackrel{def}{=} \frac{\partial L}{\partial v}(\xi(s),\dot{\xi}(s)) \qquad (3.38.\star\star)$$

we obtain that $\big(\xi(s), p(s)\big)$ is a solution of the Cauchy problem (3.21); and then ξ is C^1 in the s variable.

This shows that an extremal is determined uniquely (and continuously) by the inital values $(\xi(0),\dot{\xi}(0))$, even if L is not $C^{1,1}$.

Proof. Suppose that ξ is an extremal of (3.38.$\star$): then ξ satisfies the Euler condition (in integral form)

$$\frac{\partial L}{\partial v}(\xi(t),\dot{\xi}(t)) = \int_0^t \frac{\partial L}{\partial x}(\xi(s),\dot{\xi}(s))\,ds + c$$

By what we proved above, we substitute

$$p(s) \stackrel{\text{def}}{=} \frac{\partial L}{\partial v}(\xi(s),\dot{\xi}(s))$$

and then, by the above proposition, the Euler condition becomes

$$p(s) = \int_0^t \frac{\partial H}{\partial x}(\xi(s),p(s))\,ds + c$$

then $\xi(s), p(s)$ is a solution of the Cauchy problem (3.2); and then it is C^1 in the s variable. □

3.8 Exponential map

We briefly review how we define the *exponential map* in our weaker Finsler geometry; we cannot use the common definition, since $L \notin C^2$ in general.

We now jump back to the setting in §3.5. We suppose that $\hat{H}$ satisfies all of 3.30.

The (weaker) Finsler structure (M, L) is related to $\hat{H}$ as shown in the previous section; in this case, moreover, $v \mapsto L(x, v)$ and $p \mapsto \hat{H}(x, p)$ are positively 2-homogeneous.

We then define

Definition 3.39 (Forward exponential map) *For fixed x we define the* forward exponential map [10]

$$\exp_x : T_x M \to M$$

as

$$\exp_x(v) \stackrel{def}{=} \hat{X}\left(1, x, \frac{\partial L}{\partial v}(x, v)\right)$$

and we will prove that

$$\hat{X}(t, x, p) = \hat{X}(1, x, tp) = \exp_x(tv) \tag{3.40}$$

when p, v are related by the Legendre relation (3.33.$\star$).

From this we obtain that

$$L(x, v) = v \cdot p - \hat{H}(x, p) = \frac{\partial \hat{H}}{\partial p}(x, p) \cdot p - \hat{H}(x, p) = \hat{H}(x, p) \tag{3.41}$$

whenever x, v, p are related by the duality (3.33.$\star$). Moreover

Proposition 3.42 *If ξ is an extremal, then $L(\xi, \dot{\xi})$ =const; indeed,*

$$L(\xi, \dot{\xi}) - \dot{\xi} \cdot \frac{\partial L}{\partial v}(\xi, \dot{\xi}) = \mathit{constant}$$

(by direct derivation and the Euler equation, see [16, pag. 76]); but,

$$v \cdot \frac{\partial L}{\partial v}(x, v) = 2L(x, v) \ .$$

To conclude the proof of (3.40), we then need this simple lemma

Lemma 3.43 *for any $\lambda \geq 0$ we have*

$$\begin{aligned} X(t\lambda, z, q) &= X(t, z, \lambda q), \\ \lambda P(t\lambda, z, q) &= P(t, z, \lambda q) \end{aligned} \tag{3.43.$\star$}$$

The proof follows from deriving both sides of the equation.

[10] We will in the following drop the attribute of "forward" from the definition, for simplicity

3.9 Hopf–Rinow theorem in weaker Finsler space

Suppose Z satisfies 1.10. Choose a strict subsolution $\underline{u}$. Let $\hat{H}$ be as in (3.9).

We can prove this addition to the previous (purely metric) Hopf-Rinow theorem 2.31

Theorem 3.44 *The following assertions are equivalent:*

1. (M, b) *is* forward-complete

5. *for a certain x, the forward exponential map* $\exp_x(v)$ *is defined for all $v \in T_x M$ (in this case we say that (M, L) is* forward geodesically complete*); or equivalently for all x;*

6. *for a certain x, the forward exponential map is defined on all $T_x M$ and is surjective, that is,* $\exp_x(T_x M)$ *covers M; or equivalently for all x;*

Using the purely-metric Hopf-Rinow theorem 2.31 the proof to this theorem just needs simple extra arguments that are identical to those used for the Hopf–Rinow theorem in [9], or in [3].

We recall that, following the discussion in 3.27 and in §3.6, (M, L) is "*forward geodesically complete*" iff the original problem (1.1) satisfies `(MCu+)` (from eq. (3.5)): so the above Theorem immediatly proves 3.6.

4 Strongly convex C^2 Hamilton-Jacobi equation

Assume `(Hnd)` and moreover that

`(H2)` $H \in C^2$ and H is strongly convex in the p variable (cf. 1.4) in a neighbourhood of $\{H = 0\}$,

(this condition `(H2)` was used in paper [31]); then Z_x is always strongly convex, and ∂Z_x is a C^2 submanifold of $T_x^* M$; note that this implies most of the hypotheses in 1.10.

Suppose K is an embedded submanifold in M and suppose $K, u_0 \in C^r$, $r \geq 2$. We recollect some definitions from [31]. Let

$$O \stackrel{\text{def}}{=} \{(x,p) \in T^*M \mid \ x \in K, \quad H(x,p) = 0, \quad \forall v \in T_x K, p \cdot v = du_0(x) \cdot v\} \ .$$

If $K, u_0 \in C^r$,$H \in C^{r-1}$ then O is a $(n-1)$-dim. embedded submanifold of T^*M of class C^{r-1}.

Let

$$\Sigma_u \stackrel{\text{def}}{=} \{x \in M : \nexists Du(x)\}$$

be the set of **singular points**; let Σ be the set of x such that the minimization problem in eqn.(2.22) defining $V(x)$ has two different minima $z, z' \in K$ [11]; and let

$$\Gamma \stackrel{\text{def}}{=} \left\{ x \in \Omega \mid \begin{array}{c} \exists s \geq 0, \exists y = (z,q) \in O, \ \ x = X(s,y), \\ u(x) = U(s,y) + u_0(z), \frac{\partial X}{\partial(s,y)}(s,y) \text{ has not rank } n \end{array} \right\} \tag{4.1}$$

be the set of **conjugate points** (as defined in 4.2 in [31]).

Assume moreover that a strict subsolution $\underline{u}$ does exist: in these hypotheses

[11] compare the defition in 4.1 in [31]

Theorem 4.2 *The geometry (M, L) defined before is a regular Finsler Geometry, and*

1. *V is locally semiconcave in $M \setminus K$ (this means that around any point $x \in M \setminus K$ there is a neighbourhood $\mathcal{U}$ and a smooth ϕ on $\mathcal{U}$ s.t. $V + \phi$ is concave);*
2. *$\Sigma = \Sigma_u$*
3. *the closure of Σ_u coincides with $\Sigma_u \cup \Gamma$;*

Proof. 1. the proof uses the theorem 5.3 in [25], by expressing the PDE (3.7) in local coordinates (it is the same as the proof of prop. 8 in [27]);

2. this is a known consequence of semiconcavity;
3. it is enough to prove that there are no conjugate points $x \in \Gamma$ outside of $\overline{\Sigma}$. If B is any open set containing no points of Σ, V is C^1 in B, by the semiconcavity of V (see prop. 5,9 and 10 in [27]), and the flow is a local homeomorphism. If there would be a neighbourhood B of $x \in \Gamma$ containing no points of Σ, then, then there would a point $y \in B$ and an optimal geodesic ξ for $V(y)$ that would pass through x; the initial segment of this geodesic would be the optimal curve for $V(x)$: but this is impossible, since a geodesic ceases to be minimal after it meets a conjugate point; see the first statement in prop. 14 in [27] for details.

□

"Semiconcavity of V" implies some interesting properties of Σ, like:

- the set Σ may be covered by a countable number of C^2 manifolds (see [1])
- moreover the singularities of du (i.e. the set Σ) propagate in a way that is related to the shape of $\partial^+ u(x)$. See [2].

5 Some remarks

5.1 the Problem environment

We have stated the problem (1.1) in a quite general environment: Hamilton–Jacobi equations commonly have $\mathbb{R}^n$ (with the usual norm $|\cdot|$) as the ambient space M. We simbolically represent this common setting as

$$((\mathbb{R}^n, |\cdot|),\ H, K, u_0) \tag{5.1}$$

Suppose, for example, that we may generalize: the Euclidean space $(\mathbb{R}^n, |\cdot|)$ is substituted by a Riemannian manifold (M, g); we may then reasonably think that the class of problems

$$((M, g),\ H, K, u_0) \tag{5.2}$$

share all the properties and results of the common problems (5.1).

If we wish to further generalize the problem (5.2), we are faced by an obstacle: most results that are found in common literature regarding the problem (1.1) are stated using the distance and the distance–related properties of M; that is, they use some geometrical structure of the manifold M.

This comment is reversable: isn't it possible that most results are strongly influenced by the geometrical structure lying under them? that is, *what can be said of the problem (1.1), "as is"?*

We also noted that in the equation (1.1) we may perform independent and dependent change of variables: compare prop. 2.1, or §3.4, or §2 in [27]. By those changes, we obtain equivalent problems. It is then reasonable to ask that all hypotheses that are formulated in theorems on (1.1) should be robust w.r.t a change to an equivalent problem.

We saw (in particular in §3.4) that we obtained that goal.

5.2 on the hypothesis $(\exists \underline{u})$

We comment on the hypothesis $(\exists \underline{u})$ of existence of a strict subsolution (see defn. 1.12).

For any choice of $c \in \mathbb{R}$, consider the problem

$$\{H(x, du(x)) = c \quad x \in M \tag{5.3}$$

These family of problems has been studied in many papers; the result most relevant to our current interest is

Theorem 5.4 (Weak KAM) *Suppose that $H \in C^2$ and $H(x, \cdot)$ is strongly convex (cf. 1.4); suppose that there is a reference Riemannian Geometry such that $\forall k \in \mathbb{R}\ \exists a \in \mathbb{R}$*

$$H(x,p) \geq k|p| - a, \quad \forall x, p \tag{5.4.$\star$}$$

and $\forall r \in \mathbb{R}$

$$\sup\{H(x,p) \mid |p| \leq r\} < \infty \tag{5.4.$\star\star$}$$

Then there exists $c(H) \in \mathbb{R}$, such that the Hamilton-Jacobi equation above admits a global viscosity solution $u : M \to \mathbb{R}$ for $c = c(H)$ and does not admit any such solution for $c \neq c(H)$. $c(H)$ is called the critical value.

In the case where M is the n-dimensional torus T^n, this theorem is due to P.L. Lions, G. Papanicolaou & S.R.S. Varadhan [35]; for M an arbitrary compact connected manifold it is due to A. Fathi [11]. A more complete discussion of history and applications of this result may be found in [12].

Remark 5.5 *It may be interesting also to study whether the hypotheses (5.4.$\star$) and (5.4.$\star\star$) may be replaced by more intrinsic hypotheses: indeed they imply that the Finsler metric spaces associated to (5.3) when $c > c[h]$ are complete; so, maybe, it should be enough to assume directly that those Finsler metric spaces are backward complete, without resorting to an auxiliary Riemannian structure.*

More recently, in the case where M is the n-dimensional torus T^n, some results on the regularity of critical solutions have appeared in [13]

Theorem 5.6 (6.2 in [13]) *Suppose $M = T^n$. Suppose that H is continuous, and for all $x \in T^n$, $a \in \mathbb{R}$,*

- *$\{p \mid H(x,p) \leq a\}$ is convex*
- $$\liminf_{p \to \infty} H(x,p) = \infty$$

  uniformly in x (see also 1.3 and (5.12.$\star$))

- $\partial\{p \mid H(x,p) \le a\} = \{p \mid H(x,p) = a\}$ [12]

Let $\mathcal{A}$ be the generalized projected Aubry set.
There exists a C^1 critical subsolution, which is strict in every open subset whose closure is disjoint from $\mathcal{A}$.

Concluding, when M is compact, by the methods in [10], it is possible to prove that if $c(H) < 0$ then the strict smooth subsolution $\underline{u}$ (from hyp.1.12) will exists; whereas if $c(H) \ge 0$, then the strict subsolution will not exists.

Note that the hypotheses in above theorems are quite stringent; since our problem (1.1) is obtained from (5.3) fixing $c = 0$, we are asking hypotheses (such as 1.10) only on Z, or on H in a neighbourhood of $\{H = 0\}$: this allows for much generalized hypotheses.

5.2.1 Boundary value problems

Let Ω be an open subset of a differentiable manifold. Consider the problem

$$\begin{cases} H(x, Du(x)) = 0 & \text{in } \Omega \\ u(x) = u_0(x) & \text{when } x \in \partial\Omega. \end{cases} \tag{5.7}$$

The theory in this paper does not directly include boundary value problems. To include it, we may set $M = \overline{\Omega}$, $K = \partial\Omega$ and consider M to be a manifold with boundary. This needs many adjustments in the hypotheses.

5.3 Examples

The other uniqueness results that we know of do not explicitely state a "completeness" hypothesis; [13] we now provide some examples where this hypothesis is relevant.

Example 5.8 $M = \mathbb{R}^2$, $H(x,p) = |p|^2 - 1$, $K = \{x \mid x_1 = -x_2^2/4\}$, $u_0(x) = x_1$.
This example satisfies condition (∃$\underline{\text{u}}$)*, and we may choose $\underline{u}(x) = -\sqrt{|x|^2+1}$ (as in (3.8) in [31]), but there is no strict subsolution $\underline{u}$ such that $\underline{u} \le u_0$ on K, and which satisfies* (MC$\underline{\text{u}}$-)*. This problem has two viscosity solutions:*

$$u^*(x) \stackrel{def}{=} x_1$$
$$u^{**}(x) \stackrel{def}{=} \max\left\{x_1,\ \left(1 - \sqrt{{x_2}^2 + (x_1+1)^2}\right)\right\}$$

the second solution is the value function V, it is a patchwork of a cone and a plane, which intersect in the parabola $\{(x_1, x_2, x_3) \mid x_1 = x_3 = -{x_2}^2/4\}$. See fig. 2 on the following page.

5.3.1 Asymmetric eikonal example

Let $M = \mathbb{R}$, let

$$\hat{H}(x,p) = \begin{cases} p^2 e^{-2x} & \text{if } p \ge 0 \\ p^2 e^{2x} & \text{if } p < 0 \end{cases}$$

[12] this last hypothesis is equivalent to the hypothesis $\{p \mid H(x,p) \le a\} = \overline{\{p \mid H(x,p) < a\}}$; this is as asking $Z_x = \overline{A_x}$ (that was required in 1.10) for all problems $H - a = 0$.

[13] sometimes the "backward completeness" is implicitely assumed: this is the case when the ambient space M is compact

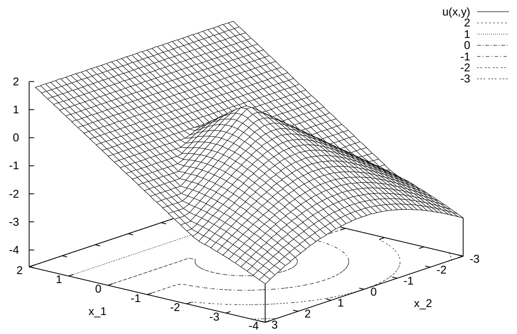

Figure 2: $u^{**}(x)$ in example 5.8

We formulate the Hamilton-Jacobi eikonal problem

$$\begin{cases} \hat{H}(x, Du(x)) - 1 = 0 & \text{for } x \neq 0 \\ u(0) = 1 \end{cases} \tag{5.9}$$

The above has two regular solutions,

$$u_1(x) = e^x, \quad u_2(x) = e^{-x}$$

We set $\underline{u} = 0$.

We can easily solve for the characteristic functions (3.2) satisfying $\hat{H}(X, P) = 1$: for the case $P(0) = q > 0$ we define $z = \log q$, $X(0) = z$

$$X(t) = \log(q + 2t), \ \ P(t) = q + 2t, \ \ U(t) = 2t$$

that is defined for $t \in (t^-, t^+)$ with $t^- = -q/2$, $t^+ = \infty$. Similarly if $P(0) = q < 0$, we define $z = -\log(-q)$, $X(0) = z$, then

$$X(t) = -\log(-q + 2t), \ \ P(t) = q - 2t, \ \ U(t) = 2t$$

that is defined for $t \in (t^-, t^+)$ with $t^- = q/2$, $t^+ = \infty$.

Let $\hat{L}$ be a Lagrange dual to $\hat{H}$, as per 3.25:

$$\hat{L}(x, p) = \begin{cases} \frac{1}{4} v^2 e^{2x} & \text{if } v \geq 0 \\ \frac{1}{4} v^2 e^{-2x} & \text{if } v < 0 \end{cases}$$

and $(\mathbb{R}, \hat{L})$ is a Finsler Geometry. We then define the Finsler distance $b(x, y)$ as in (2.16); in this specific case,

$$2b(x, y) = \begin{cases} e^y - e^x & \text{if } y > x \\ e^{-y} - e^{-x} & \text{if } y < x \end{cases} \tag{5.10}$$

From the above equations and theorem 3.6, we derive that the space $(\mathbb{R}, \hat{L})$ is forward complete, but not backward complete; and indeed there are 4 viscosity solutions that are bounded from below, namely

$$u_1, \quad u_2, \quad u_1 \wedge u_2, \quad u_1 \vee u_2$$

We reformulate (5.9) with different conditions

$$\begin{cases} \hat{H}(x, Du(x)) - 1 = 0 & \text{for } x \notin \mathbb{N} \\ u(x) = u_0(x) \stackrel{\text{def}}{=} 2e^{-x} & \text{for } x \in \mathbb{N} \end{cases} \tag{5.11}$$

In this case, the problem

$$V(x) = \inf_{z \in \mathbb{N}} u_0(z) + 2b(z, x)$$

has no minimum: indeed

$$V(0) = \inf_{y \in \mathbb{N}} e^{-y} + 1$$

5.4 Comparison with previous results

5.4.1 Comparison with [38]

We compare our approach to Hamilton-Jacobi problems with this result, found as Proposition 3.1 in Siconolfi's [38]

Proposition 5.12 *Let* $M = \mathbb{R}^n$. [14] *Suppose that* $H(x, 0) < 0$. [15] *Suppose* H *is continuous, and*

$$\partial(\mathring{Z}_x) = \{p \mid H(x,p) = 0\} \quad \forall x$$

and

$$\liminf_{|p| \to \infty} H(x,p) > 0 \tag{5.12.$\star$}$$

locally uniformly in x.

Then Z_x *is compact (uniformly w.r.t* x*), and* F *is locally bounded (by (5.12.*$\star$*)). Moreover the maps* $x \mapsto Z_x$ *and* $x \mapsto \partial Z_x$ *are Hausdorff–continuous. Consequently* F *is continuous.* [16]

the above theorem may be compared then to theorem 2.23.

We point out that [38] addresses also the case when Z_x is not convex: it then builds a solution of (1.1) by a sup-inf formula, which provides a viscosity solution of the convexified of (1.1). We do not address the case where Z_x is not convex in this paper.

Again, in the case $M = \mathbb{R}^n$, Proposition 2.2 in [38] proves that

Theorem 5.13 *Suppose* $M = \mathbb{R}^n$. *Suppose that* $H(x, 0) < 0$. *Suppose that* H *is continuous and (5.12.*$\star$*) and* Z_x *is convex and*

$$\partial(\mathring{Z}_x) = \{p \mid H(x,p) = 0\} \ \forall x$$

Assume the condition 2.9 in [38], namely there exist $a, b > 0$

$$H(x,p) < 0 \ \forall x, p \text{ with } |p| < \frac{a}{|x| + b} \tag{5.13.$\star$}$$

Then

$$u(x) \stackrel{\text{def}}{=} S(a, x)$$

[14] The original proposition assumes $M = \mathbb{R}^n$; but the adaptation to having M a manifold would be straightforward.

[15] We may generalize the condition $H(x, 0) < 0$ to defn. 1.12, by using 3.16

[16] Since Z_x is compact (uniformly w.r.t x) then $x \mapsto Z_x$ is Hausdorff–continuous iff it is Kuratowski–continuous.

is the unique viscosity solution of $H(x, du(x)) = 0$ (for $x \neq 0$) in the class of continuous functions $v : M \to \mathbb{R}$ such that

$$\liminf_{|x|\to\infty} v(x) = \infty \tag{5.13.$\star\star$}$$

and $v(0) = 0$.

We now understand that (5.13.$\star$) is a "completeness condition", since it implies that the space (M, b) (defined with $\underline{u} = 0$) is complete. Indeed the condition (5.13.$\star$) is equivalent to: there exist $a, b > 0$

$$L(x, v) \geq \frac{a|v|}{|x| + b} \tag{5.14}$$

(that is the condition 1.7 in [38]).

In this paper we have shown that, indeed, a completeness assumption is fundamental to achieve uniqueness of solution; we have also remarked that, due to the asymmetry of the equation and of the metric, we may distinguish a backward and a forward completeness hypothesis, and that the correct one is the "backward completeness assumption". So our theorem 2.43 clearly generalizes the above theorem.

5.4.2 Relation to [31].

We use the variational theorem 2.43 to reconnect to the theory developed in [31].

For convenience of the reader, we redefine some quantities from [31].

Suppose `(u`$_0$`K1)` holds, namely

`(u`$_0$`K1)` K is a C^1-regular closed embedded submanifold of M of dimension k with $0 \leq k \leq \dim(M) - 1$, and u_0 is a C^1 real function defined on K. (from 3.1).

We define the set O as

$$\begin{aligned} O &\stackrel{\text{def}}{=} TK^{\perp u_0} \cap \{H = 0\} = \\ &= \{(x,p) \in T^*M \mid \ x \in K, \quad H(x,p) = 0, \quad \forall v \in T_xK, p \cdot v = du_0(x) \cdot v\} \end{aligned}$$

(from (3.3)).

We assume that O is a submanifold of T^*M (see 3.7).

We define the *reachable set*

$$\Omega \stackrel{\text{def}}{=} \{x \in M \mid x = X(s, z, q) \text{ for } s \geq 0, (z, q) \in O\}$$

(from (3.4)).

We define the *min* solution $u : \Omega \to \mathbb{R}$ by using the method of characteristics

$$u(x) \stackrel{\text{def}}{=} \inf_{\begin{cases} t \geq 0, \ (z,q) \in O \\ \text{s.t. } X(t,z,q) = x \end{cases}} U(t, z, q) + u_0(z) \tag{5.15}$$

(from (3.5)).

Proposition 5.16 *Suppose 1.10 and Z_x is strictly convex; suppose there exists a strict subsolution such that $\underline{u} < u_0$ and* `(MC`$\underline{\text{u}}$`-)` *holds (or any other equivalent condition from 2.34 and 3.6). Then*

- *the reachable set Ω covers all M; that is, given $x \in M$, there always exists a characteristic X such that $x = X(t, z, q)$ and $(z, q) \in O$*
- *the value function V coincides with the* min *solution u*
- *this condition (from (3.7)) holds:*

  (OXUp) if C is a compact subset of M, $a \in \mathbb{R}$, then

  $$\{(t, y) \in \mathbb{R}^+ \times O \mid X(t, y) \in C, (U(t, y) + u_0(z)) \leq a\} \text{ is compact.} \quad (5.16.\star)$$

Then the infimum in (5.15) is a minimum.

5.4.3 Comparison with [30]

We show how this paper naturally follows and generalizes the paper [30]. We assume `(Hnd)` and 1.10, and that Z_x is strictly convex for all x.

Consider the conditions that were introduced in [30]:

(MC) *((M, H) is characteristically complete)* we can solve (3.2) for all times, that is, we have a flow $(X, P) : \mathbb{R} \times T^*M \to T^*M$ whenever the initial conditions $X(0) = z, P(0) = q$ satisfy $H(z, q) = 0$.

(G$\underline{u}$) there exists a strict subsolution $\underline{u}$ and constants $c' > c > 0$ such that

$$c' \geq \frac{d}{dt}U - \frac{d}{dt}(\underline{u}(X)) \geq c$$

for any characteristic (X, P, U) (such that $H(X, P) = 0$), that is,

$$c' \geq (p - d\underline{u}(x)) \cdot \frac{\partial}{\partial p} H(x, p) \geq c$$

for all $(x, p) \in T^*M$ such that $H(x, p) = 0$.

These conditions were used in [30], were it was proven that they imply existence and uniqueness of the viscosity solution.

We show a sequence of simple implications that involve these conditions.

Proposition 5.17 • *suppose that Z_x is strictly convex; take any $x \in M$, and p in the border of Z_x, that is, $H(x, p) = 0$; let $T_p Z_x \subset T_x^* M$ be the hyperplane tangent to Z_x in the point p: this hyperplane intersects Z_x only in the boundary, and it is perpendicular to $\frac{\partial}{\partial p} H(x, p)$ (see fig. 3); whereas the point $(x, d\underline{u}(x))$ is contained in the internal part of Z_x; then, we obtain that*

$$(p - d\underline{u}(x)) \cdot \frac{\partial}{\partial p} H(x, p) > 0 \quad (5.17.\star)$$

The above reasoning shows that, if M is compact and if Z_x is strictly convex and C^1, we obtain that any strict subsolution $\underline{u}$ satisfies `(G`$\underline{\texttt{u}}$`)`

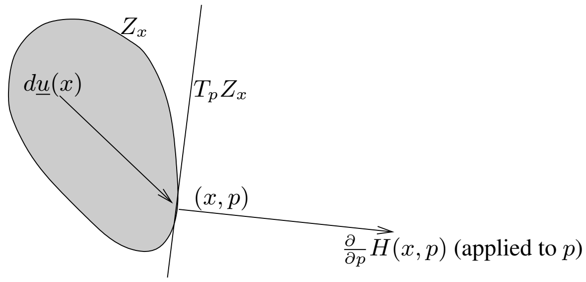

Figure 3: condition `(G`$\underline{\text{u}}$`)`

- *the conditions* `(G`$\underline{\text{u}}$`,MC)` *readily imply* `(MC`$\underline{\text{u}}$`)`*: indeed,* `(MC)` *says that* $t^+ = \infty, t^- = -\infty$*, and* `(G`$\underline{\text{u}}$`)` *implies,*

$$c't \geq U(t,z,q) - \underline{u}(X(t,z,q)) - \big(U(0,z,q) - \underline{u}(X(0,z,q))\big) =$$
$$= \int_0^t (P(t,z,q) - d\underline{u}(X(t,z,q))) \cdot \frac{\partial}{\partial p} H(X,P) \geq ct$$

so that

$$c't \geq U(t,z,q) - \underline{u}(X(t,z,q)) \geq ct$$

The last point shows that the hypotheses `(MC`$\underline{\text{u}}$`)` generalizes the conditions `(G`$\underline{\text{u}}$`,MC)`: then the results in this paper generalize the results in [30].

A Kuratowski convergence

We review definition and results regarding the Kuratowski convergence, for convenience of the reader. We will use many concepts of general topology, as defined in Kelley [19]. Let in the following Y be a Hausdorff[17] topological space.

Definition A.1 (Kuratowski convergence [23]) *Let* $(A_n)_{n\in\mathbb{N}}$ *be a sequence of subsets of* Y*. We define the* ***lower Kuratowski limit*** $\liminf_{n\to\infty} A_n$ *of* $(A_n)_n$

- $\liminf_{n\to\infty} A_n$ *is the set of* $y \in Y$ *such that any neighborhood of* y *meets eventually all of the* A_n*.*
- *(If the topology of* Y *is characterized by converging sequences:*[18]*) Equivalently* $\liminf_{n\to\infty} A_n$ *is the set of all possible limits of sequences* y_n *with* $y_n \in A_n$*.*
- *(If* (Y, d') *is a metric space:) Equivalently* $\liminf_{n\to\infty} A_n$ *is*

$$\liminf_{n\to\infty} A_n \stackrel{def}{=} \bigcap_{\delta>0} \bigcup_{m\in\mathbb{N}} \bigcap_{n\geq m} (A_n^\delta)$$

where

$$A_n^\delta \stackrel{def}{=} \{x \mid d'(x,A) < \delta\}$$

is the fattened of A_n*.*

[17] also known as *"T2"* or *"separated"* space; see [19], ch. 2
[18] such is the case when Y satisfies *"the first countability axiom"*: see [19], ch. 2

If $(Y, |\cdot|)$ *is a normed vector space, then in particular*

$$A_n^\delta = A_n + B_\delta \overset{def}{=} \{y + z \mid y \in A_n, z \in B_\delta\} = \{x \mid \exists y \in A_n, |x - y| < \delta\}$$

where $B_\delta \overset{def}{=} \{z \mid |z| < \delta\}$ *is the ball centered in 0 of radious* $\delta > 0$.

The ***upper Kuratowski limit*** $\limsup_{n\to\infty} A_n$ *of* $(A_n)_n$ *is*

- *the set of all points* $y \in Y$ *such that any neighborhood of* y *meets frequently all of the* A_n.
- *Equivalently it is the set of all possible limits of sequences* y_m *with* $y_m \in A_n$ *for some* $n \geq m$.
- *Equivalently it is*

$$\limsup_{n\to\infty} A_n \overset{def}{=} \bigcap_{\delta>0} \bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} (A_n^\delta)$$

In general,

$$\liminf_{n\to\infty} A_n \subset \limsup_{n\to\infty} A_n$$

We will say that $\lim_{n\to\infty} A_n = A$ ***in the Kuratowski sense*** *if*

$$\liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n = A$$

Other equivalent conditions for the case when $Y = \mathbb{R}^n$ may be found in ch. 4 in Rockafellar-Wets's [36].

The Kuratowski convergence enjoys many useful properties

Proposition A.2 *Let* $A, A_n \subset Y$.

- $\liminf_{n\to\infty} A_n$ *and* $\limsup_{n\to\infty} A_n$ *are closed.*
- ***(Locality).*** *For any* $V \subset Y$ *open, we have*

$$V \cap \limsup_{n\to\infty}(A_n \cap V) = V \cap (\limsup_{n\to\infty} A_n)$$
$$V \cap \liminf_{n\to\infty}(A_n \cap V) = V \cap (\liminf_{n\to\infty} A_n) \qquad (A.2.\star)$$

- ***(Convexity).*** *If* Y *is a topological vector space and all* A_n *are convex, then* $\liminf A_n$ *is convex.*

We moreover prove

Proposition A.3 (Equicompactness.) *Suppose that* Y *is connected. Choose a sequence* $A_n \subset Y$. *Suppose all* A_n *are connected,* $\liminf_{n\to\infty} A_n$ *is non-empty, and*

$$A \overset{def}{=} \limsup_{n\to\infty} A_n$$

is compact.

Then $(A_n)_n$ *is* eventually equicompact*: for any* $K \subset Y$ *compact such that* $A \subset \mathring{K}$, *(*$\mathring{K}$ *is the internal part of* K*) then* $A_n \subset K$ *eventually in* n.[19]

[19] that is, there exists $N \in \mathbb{N}$, such that $A_n \subset K\ \forall n \geq N$.

Proof. A proof may be found in prop. 3 in Salinetti–Wets [37], for the case when Y is a finite dimensional normed vector space. The general proof is similar. □

Remark A.4 *The above proposition is tricky*[20]*: it is easy to mistakenly state it using wrong hypotheses. For example, it is not possible to replace the condition* "$\liminf_{n\to\infty} A_n$ is non-empty" *by the condition* "$\limsup_{n\to\infty} A_n$ is non-empty", *(as is done in cor. 4.12 in [36]). This is clearly shown by this example: let $A_n \subset \mathbb{R}$ be defined by*

$$A_n \overset{def}{=} \begin{cases} \{0\} & \text{for n even} \\ \{n\} & \text{for n odd} \end{cases}$$

then each A_n is connected, $\liminf_{n\to\infty} A_n = \emptyset$ *while* $\limsup_{n\to\infty} A_n = \{0\}$ *is non-empty and compact; but the sequence A_n is not equicompact. (This is a very important remark: basilar proofs in this paper depend on the above proposition.)*

We will moreover need to relate the Kuratowski convergence to the convergence of support functions; we state a well known result

Proposition A.5 *Suppose Y is a finite dimensional normed vector space and $A, A_n \subset Y$ are convex and A is compact and non-empty; let $F_A : Y^* \to \overline{\mathbb{R}}$,*

$$F_A(v) \overset{def}{=} \sup\{p \cdot v \mid p \in A\}$$

be the support function of A, and similarly F_{A_n} of A_n; then the following are equivalent

i. $A_n \to A$ *in the Kuratowski sense*

ii. $F_{A_n} \to F_A$ *locally uniformly*

iii. $F_{A_n} \to F_A$ *pointwise.*[21]

The proof is based on adapting Theorem. 3.1 in Mosco [33] (that is stated in a generical reflexive Banach space) to a finite dimensional case.

In this paper, things will be slightly more complicated, since we will use *fiberwise support functions*; we discuss the result in in the following section, where we will moreover split the above in a l.s.c. statement A.15 and a u.s.c. statement A.16.

The compactness hypothesis in the above is important: we present a counterexample (derived from the similar example 5.2 in Löhne–Zălinescu [26])

Example A.6 *Let $A : \mathbb{R} \to \mathcal{P}\mathbb{R}^2$ be defined by*

$$A_\theta \overset{def}{=} \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \cos\theta = x_2 \sin\theta\}$$

(A_θ is a line passing through 0 with angle θ w.r.t the vertical axis). Then for the support function we have

$$F_{A_\theta}(v_1, v_2) \overset{def}{=} \begin{cases} 0 & \text{if } v_2 \cos\theta = -v_1 \sin\theta \\ +\infty & \text{if not} \end{cases}$$

The function

$$(\theta, v_1, v_2) \mapsto F_{A_\theta}(v_1, v_2)$$

is l.s.c. in $\mathbb{R}^3$ but is not continuous.

[20] and indeed it fouled the author in a preliminary draft version of this paper

[21] pointwise convergence of support functions is known as *scalar convergence* of A_n to A.

§A.i .. on maps

Let then X be a Hausdorff topological space; we extend the above notions to maps $A : X \to \mathcal{P}Y$:

Definition A.7 *We define the **lower Kuratowski limit** $\liminf_{x\to\overline{x}} A_x$ as the set of $y \in Y$ such that*

$$\forall V \ni y, \exists U \ni \overline{x}, \forall x \in U, x \neq \overline{x}, A_x \cap V \neq \emptyset,$$

*where $V \subset Y$ and $U \subset X$ are open. We define the **upper Kuratowski limit** $\limsup_{x\to\overline{x}} A_x$ as the set of $y \in Y$ such that*

$$\forall V \ni y, \forall U \ni \overline{x}, \exists x \in U, x \neq \overline{x}, A_x \cap V \neq \emptyset,$$

*We define **lower-semi-continuous (and u.s.c.) maps in the Kuratowski sense** when the lower Kuratowski limit $\liminf_{x\to\overline{x}} A_x$ contains $A_{\overline{x}}$ (resp. if $\limsup_{x\to\overline{x}} A_x \subset A_{\overline{x}}$); as usual.*

In the following section we further extend the above notion to fiber bundles.

Proposition A.8 *If the topology of X and Y are characterized by converging sequences (cf. note 18), then equivalently we say that the map $A : X \to \mathcal{P}Y$ is **lower-semi-continuous in the Kuratowski sense** if, for any sequence $(x_n)_n \subset X$ with $x_n \neq x$ and $x_n \to x$, for any $y \in A$, there is a sequence $y_n \in A_{x_n}$ s.t. $y_n \to y$; resp. **upper-semi-continuous in the Kuratowski sense** if for any sequence $(x_n)_n \subset X$ with $x_n \neq x$ and $x_n \to x$, for any converging sequence $y_n \in A_{x_n}$ s.t. $y_n \to y$, we have $y \in A$.*

Remark A.9 *The above is different from the definition presented in [36], where it is not assumed that $x_n \neq x$; using the definition in [36], it would always hold that*

$$\limsup_{x\to\overline{x}} A_x \supset A_{\overline{x}}$$

and then some following results would have a different statement.

§A.ii .. on fiber bundles

We assume, for sake of ease, that X satisfies *"the first countability axiom"*. Suppose that Y is a finite dimensional normable vector space; suppose that N is a fiber bundle with fiber Y, and $\pi : N \to X$ be the projection; let $N_x = \pi^{-1}(x)$ be the fiber. N is equipped with an atlas of *fiberwise* local coordinates that are defined, for any $x \in U$, by a small open $U \subset X$ containing x, and by an homeomorphism

$$\phi : U \times Y \to \pi^{-1}(U) \tag{A.10}$$

such that (for all $x \in U$) $\phi(x, \cdot)$ is a linear isomorphism between $\{x\} \times Y$ and $N_{\phi(x)}$.

We extend the above definitions on Kuratowski limits, to maps into the fiber bundle N, using these local coordinates (this is OK in view of (A.2.$\star$)).

Definition A.11 (Slicing & graph) *If B is a subset of N, we will slice it to define the map*

$$x \mapsto B_x \overset{def}{=} N_x \cap B$$

Conversely, if we consider fiberwise maps *$x \mapsto A_x$ such that $A_x \subset N_x$ is not empty; we associate to any such map its graph*

$$A \overset{def}{=} \bigcup_x A_x$$

For example in section 2.2, we considered maps $x \mapsto A_x$ from M to $N = T^*M$, with $A_x \subset T^*_x M$, then we will for simplicity define the graph as

$$A \stackrel{\text{def}}{=} \{(x,p) \in T^*M \mid p \in A_x\} \ .$$

Lemma A.12 *Choose $A \subset N$, let A_x be the slicing of A. If A is closed, then $x \mapsto A_x$ is upper-semi-continuous. Viceversa if every slice A_x is closed, and the map $x \mapsto A_x$ is upper-semi-continuous in the Kuratowski sense, then A is closed.*

Proof. Fix $x \in X$. Pull back $x \mapsto A_x$ to a map $U \to Y$ (using local coordinates ϕ around x), that we call $\tilde{A}_x$.

Choose $x_n \to x$ in X. Choose any $y \in \limsup_{n\to\infty} \tilde{A}_{x_n}$: then there is a sequence y_n with $y_n \in \tilde{A}_{x_{m(n)}}$ for some $m(n) \geq n$, such that $y_n \to y$. From $\phi(x_{m(n)}, y_n) \in A$ then $\phi(x, y) \in A$: so $y \in \tilde{A}_x$.

Viceversa, let z_n be a sequence in A converging to a $z \in N$; we write it in local coordinates as $(x_n, y_n) \to (x, y)$; if $x_n = x$ eventually, we use the fact that A_x is closed; otherwise by $x_n \to x$ we know that $y \in \limsup_{n\to\infty} \tilde{A}_{x_n} \subset \tilde{A}_x$, that is, $z \in A$. □

Lemma A.13 *Suppose $A \subset N$ is open; let A_x be the slicing of A: then the map $x \mapsto A_x$ is lower-semi-continuous in the Kuratowski sense.*

Proof. Fix $x \in X$. Pull back $x \mapsto A_x$ to a map $U \to Y$ (using local coordinates ϕ around x), that we call $\tilde{A}_x$. Choose $y \in \tilde{A}_x$. Choose any $x_n \to x$: then eventually $\phi(x_n, y) \in A$, that is $y \in \tilde{A}_{x_n}$. Then $y \in \liminf_{n\to\infty} \tilde{A}_{x_n}$. □

(Note that there is no "viceversa" part in this lemma — as simple examples can show).

§A.iii Dual of a set map

Suppose moreover that N^* is the dual bundle of N, that is, $\pi^* : N^* \to X$ and $N^*_x \stackrel{\text{def}}{=} (\pi^*)^{-1}(\{x\})$ is isomorphic to Y^*.

We consider *fiberwise maps* $x \mapsto A_x$ with $A_x \subset N^*_x$; we suppose that any A_x is non empty. We define the *fiberwise support function*

$$F : N \to (-\infty, \infty]$$

so that $F(x, \cdot)$ is the support function of the set A_x: in local coordinates,

$$F(x, v) \stackrel{\text{def}}{=} \sup \left\{ p \cdot v \mid p \in A_x \right\} \ . \tag{A.14}$$

Lemma A.15 *If the map $x \mapsto A_x$ is lower-semi-continuous in the Kuratowski sense, then F is lower-semi-continuous.*

Proof. Fix $x \in X$. Pull back $x \mapsto A_x$ to a map $U \to Y^*$ (using fiberwise local coordinates ϕ in a neighbourhood U around x). Similarly, pull back F in local coordinates.

Fix x, v, and choose any $r < F(x, v)$. Choose sequences such that $v_n \to v$ and $x_n \to x$. Choose p such that $p \in A_x$ and $r < p \cdot v \leq F(x, v)$.

Since $x \mapsto A_x$ is l.s.c., there is a sequence $p_n \in A_{x_n}$ such that $p_n \to p$. So

$$F(x_n, v_n) \geq p_n \cdot v_n \to p \cdot v \geq r$$

and then

$$\liminf_{n\to\infty} F(x_n, v_n) \geq F(x, v)$$

by arbitrariness of r.

□

Lemma A.16 *Fix $\overline{x} \in X$. If the map $x \mapsto A_x$ is upper-semi-continuous in the Kuratowski sense at $\overline{x}$, any A_x is connected, $\liminf_{x\to\overline{x}} A_x$ is non-empty, and $A_{\overline{x}}$ is compact, then for any fixed $\overline{z}$ with $\pi^*(\overline{z}) = \overline{x}$, F is locally bounded and upper-semi-continuous at $\overline{z}$.*

Proof. We again work in local coordinates. Suppose $A_{\overline{x}}$ is compact: by A.3 there is a neighbourhood U of $\overline{x}$ and a K compact such that $A_x \subset K$ for all $x \in U$. Fix $\overline{v}$, and V a compact neighbourhood of $\overline{v}$ in Y: then $|F(x, v)|$ is bounded in a neighbourhood $U \times V$ of $\overline{z} = (\overline{x}, \overline{v})$ (by $\sup\{|p \cdot v|, p \in K, v \in V\}$).

Choose sequences in V and U, such that $v_n \to \overline{v}$ and $x_n \to \overline{x}$, and suppose, without loss of generality, that the sequence $F(x_n, v_n)$ is increasing.

Choose $p_n \in A_{x_n}$ such that

$$\lim_n F(x_n, v_n) = \lim_n p_n \cdot v_n$$

On the other hand, $p_n \in K$ so we may extract a subsequence p_{n_m} converging to a limit point q: we have that $q \in \limsup_m A_{n_m} \subset A_x$ and then

$$F(x, v) \geq q \cdot v = \lim_m p_{n_m} \cdot v_{n_m} = \lim_n F(x_n, v_n)$$

□

Contents

References

[1] G. Alberti. On the structure of singular sets of convex functions. *Calc. Var.*, 2:17–27, 1994.

[2] L. Ambrosio, P. Cannarsa, and H. M. Soner. On the propagation of singularities of semi–convex functions. *Ann. Sc. Norm. Sup. Pisa*, XX (4):597–616, 1993.

[3] D. Bao, S. S. Chern, and Z. Shen. An introduction to Riemann-Finsler geometry. (October 1, 1999 version).

[4] M. Bardi and I. Capuzzo Dolcetta. *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*. Birkhäuser,Boston., 1997.

[5] H. Busemann. Local metric geometry. *Trans. Amer. Math. Soc.*, 56:200–274, 1944.

[6] H. Busemann. *Recent syntetic differential geometry*, volume 54 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer Verlag, 1970.

[7] Fabio Camilli and Antonio Siconolfi. Maximal subsolutions for a class of degenerate Hamilton-Jacobi problems. *Indiana University Math. J.*, 48, 1999.

[8] Theodore Djaferis and Irvin Schick, editors. *Advances in System Theory*, Boston, October 1999. Kluwer Academic Publishers.

[9] Manfredo Perdigão do Carmo. *Riemannian Geometry*. Birkhäuser, 1992.

[10] A. Fathi. Weak KAM theorem in Lagrangian dynamics. (Preliminary version, 2 Feb 2001).

[11] A. Fathi. Théorème KAM faible et théorie de Mather sur les systèmes lagrangiens. *C. R. Acad. Sci. Paris*, (324):1043 1046, 1997.

[12] A. Fathi and Ezequiel Maderna. Weak KAM theorem on non compact manifolds.

[13] A. Fathi and A. Siconolfi. PDE aspects of Aubry–Mather theory for quasiconvex Hamiltonians. *Calc. Var.*, 2004.

[14] W. H. Fleming and H. M. Soner. *Controlled Markov processes and viscosity solutions*. Springer–Verlag, 1993.

[15] P. Fletcher and W.F.Lindgren. *Quasi-uniform spaces*, volume 77 of *Lecture notes in pure and applied mathematics*. Marcel Dekker, 1982.

[16] M.R. Hestenes. *Calculus of variations and optimal control theory*. Wiley, New York, 1966.

[17] M.K.Vamanamurthy I.L.Reilly, P.V.Subrahmanyam. Cauchy sequences in quasi-pseudo-metric spaces. *Monat.Math.*, 93:127–140, 1982.

[18] H. Ishii. A simple, direct proof of uniqueness for solutions of the Hamilton–Jacobi equations of eikonal type. *Proc. of the Am. Math. Soc.*, 100(2):247–249, June 1987.

[19] J. L. Kelley. *General topology*. The university series in higher mathematics. D. Van Nostrand, 1955.

[20] J. C. Kelly. Bitopological spaces. *Proc. London Math. Soc.*, 13(3):71–89, 1963.

[21] S. N. Kružhkov. Generalized solutions of the Hamilton–Jacoby equations of eikonal type. *I. Math. USSR Sb.*, 27:406–446, 1975.

[22] H. P. Künzi. complete quasi-pseudo-metric spaces. *Acta Math. Hung.*, 59(1-2):121–146, 1992.

[23] K. Kuratowski. *Topologie*. Panstwowe Wydawnictwo Naukowe, Warszawa, 1958.

[24] YanYan Li and Louis Nirenberg. The distance function to the boundary, Finsler geometry and the singular set of viscosity solutions of some Hamilton-Jacobi equations. *Comm. Pure Appl. Math.*, LVIII, 2005. (first received as a personal communication in June 2003).

[25] P. L. Lions. *Generalized Solutions of Hamilton–Jacobi Equations*. Pitman, Boston, 1982.

[26] Andreas Löhne and Constantin Zălinescu. On convergence of closed convex sets. http://www.mathematik.uni-halle.de/reports/sources/2005/05-02report.pdf.

[27] C. Mantegazza and A. C. Mennucci. Hamilton–Jacobi equations and distance functions on Riemannian manifolds. *Applied Math. and Optim.*, 47(1):1–25, 2002.

[28] Jean L. Mawhin and M. Willem. *Critical point theory and hamiltonian systems*. Applied mathematical sciences 74. Springer–Verlag, New York, 1989.

[29] A. C. G. Mennucci. On asymmetric distances. preprint, `http://cvgmt.sns.it/papers/and04/`.

[30] A. C. G. Mennucci. Regularity of solutions to Hamilton-Jacobi equations. In Djaferis and Schick [8]. (abridged version).

[31] A. C. G. Mennucci. Regularity and variationality of solutions to Hamilton-Jacobi equations. part i: regularity. *ESAIM Control Optim. Calc. Var*, 10(3):426–451, 2004.

[32] A. C. G. Mennucci. Regularity and variationality of solutions to Hamilton-Jacobi equations. part ii: variationality, existence, uniqueness. 1st (draft) version, online since 2005 at `http://cvgmt.sns.it/papers/men05/`, 2005.

[33] U. Mosco. On the continuity of the Young–Fenchel transform. *J. Math. Anal. Appl*, 35:518–539, 1971.

[34] B. B. Phadke. Nonsymmetric weakly complete g-spaces. *Fundamenta Mathematicae*, 1974.

[35] G. Papanicolaou P.L. Lions and S.R.S. Varadhan. Homogenization of Hamilton-Jacobi equations. (unpublished), 1987.

[36] R. T. Rockafellar and R. J-B. Wets. *Variational analysis*, volume 317 of *A series of comprehensive studies in mathematics*. Springer–Verlag, 1998.

[37] G. Salinetti and R. J. B. Wets. On the convergence of convex sets in finite dimensions. *SIAM Review*, 21:18–33, 1979.

[38] Antonio Siconolfi. Metric character of Hamilton-Jacobi equations. *Transactions of the american mathematical society*, 355(5):1987–2009, 2003. electronically published on January 8, 2003.

[39] E. M. Zaustinsky. *Spaces with non-symmetric distances*. Number 34 in Mem. Amer. Math. Soc. AMS, 1959.