Regularity and Variationality of Solutions to Hamilton-Jacobi Equations. part II: variationality, existence, uniqueness

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Abstract

We formulate an Hamilton–Jacobi partial differential equation

$$H(x, Du(x)) = 0$$

on a n dimensional manifold M, with assumptions of convexity of the sets $\{p : H(x,p) \le 0\} \subset T^*_x M$, for all x.

We reduce the above problem to a simpler problem; this shows that u may be built using an asymmetric distance (this is a generalization of the "distance function" in Finsler geometry); this brings forth a 'completeness' condition, and a Hopf–Rinow theorem adapted to Hamilton–Jacobi problems. The 'completeness' condition implies that u is the unique viscosity solution to the above problem.

keywords: Hamilton–Jacobi equation, differentiable manifold, viscosity solution, Kuratowski convergence, asymmetric metric space, Finsler metric, Hopf–Rinow theorem, backward completeness, uniqueness of solution.

1 Introduction

In this article we start from the Dirichlet Hamilton–Jacobi PDE

$$\begin{cases} H(x, Du(x)) = 0 & \text{in } M \setminus K, \\ u(x) = u_0(x) & \text{when } x \in K, \end{cases} \tag{1.1}$$

where

- M is a connected second-countable smooth differentiable manifold of dimension n without boundary,
- H is a (not necessarily continuous) real function defined on the cotangent bundle T^*M, such that

  $$\{p \in T^*_x M \mid H(x,p) \le 0\} \tag{1.2}$$

  is convex for all $x \in M$,

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- K is a closed subset of M
- and u_0 is a continuous real function defined on K.

In the first paper [22] we studied the regularity properties of a generalized solution u. The main aim of this second paper is to prove results on the existence and uniqueness of the solution u.

1.1 Notation

We fix some notations.

We will use the notation $p \cdot v$ to mean that a covector $p \in T_x^* M$ is applied to a vector $v \in T_x M$.

If $g : \mathbb{R} \times M \to \mathbb{R}$ is a regular function, $g = g(t, x)$, we will write $\dot{g}$ for $\frac{\partial g}{\partial t}$.

Definition 1.1 (limit at infinity) *Given a topological space M and a $f : M \to \mathbb{R}$, we define*

$$\liminf_{x \to \infty} f(x) \stackrel{def}{=} \sup_{C \subset\subset M} \inf_{y \notin C} f(x)$$

where $C \subset\subset M$ are compact subsets (indeed this is called "the liminf for x exiting all compact sets").

1.2 The eikonal equation as a general model

1.2.1 The eikonal equation

Throughout this subsection, let (M, g) be a smooth connected finite-dimensional Riemannian manifold without boundary. The **length** of a Lipschitz curve $\gamma : [\alpha, \beta] \to M$ is defined by

$$\operatorname{len}^g \gamma \stackrel{\text{def}}{=} \int_\alpha^\beta |\dot{\gamma}(t)|_g \, dt \quad ;$$

then we can define **distance** $d^g(x, y)$ as the infimum of $\operatorname{len} \gamma$ in the class of all Lipschitz curves connecting x to y; let moreover **minimal geodesic** be a curve that minimize $\operatorname{len}^g \gamma$ given the endpoints.

The following theorem summarizes existence, uniqueness and variationality results for the **eikonal equation**

$$\begin{cases} |\nabla u| - 1 = 0 & \text{in } M \setminus K\,, \\ u = u_0 & \text{on } K. \end{cases} \tag{1.3}$$

Theorem 1.2 *Suppose u_0 is bounded from below.*

(existence) *If*

$$u_0(x) \le d^g(y, x) + u_0(y)$$

*for all $x, y \in K$, then the **value function***

$$V(x) = \inf_{z \in K} \big(u_0(z) + d^g(z, x) \big) \tag{1.4}$$

is a viscosity solution of eqn. (1.3).

(uniqueness) *If (M, g) is complete, and u_0 is bounded from below, then V is the unique solution in the class $\mathfrak{F}$ of continuous functions f that are bounded from below.*

(variationality) *Moreover the problem (1.4) admits a minimizing geodesic curve for each x.*

The above theorem may be proved by the methods found in [19].

Remark 1.3 *The restriction to lower bounded functions is important. Indeed $|x|$ and $-|x|$ are both viscosity solutions of problem (1.3) with $M = \mathbb{R}^n$ and $K = \{0\}, u_0 = 0$. Moreover, the completeness of M plays an important rôle here: if M is the open unit ball of $\mathbb{R}^n$ the same example shows that the uniqueness does not hold.*

Completeness of the Riemannian Manifold may be tested by any of the equivalent relations in the renowned Hopf–Rinow theorem.

Theorem 1.4 (Hopf-Rinow) *The following conditions are equivalent.*

1. *The metric space (M, d^g) is complete;*
2. *bounded closed sets are compact;*
3. *any curve $\gamma(t)$ satisfying the Euler–Lagrange O.D.E. for the action functional*
$$\int_\alpha^\beta |\dot\gamma(t)|_g^2 \, dt \quad ;$$
can be prolonged to $t \to \pm\infty$.

Moreover any of the above implies that minimal geodesics do exist.

The statement (3) is sometimes called **geodesic completeness**.

1.2.2 A general "eikonal equation"

In this paper we will extend the results of the previous section to the problem (1.1), and view it as a sort of *"generalized eikonal equation"*.

We will start by noting that the family of viscosity solutions of (1.1) does not really depend on H, but rather only on its sign (see 2.26). That is, to prove that a function u is a viscosity solution of (1.1), we just need to know the three sets

$$\{H < 0\} \ , \ \{H = 0\} \ , \ \{H > 0\} \tag{1.5}$$

(that are subsets of T^*M). To simplify the matter a bit, we will only consider one set $Z \subset T^*M$, defined as $Z \stackrel{\text{def}}{=} \{H \le 0\}$, and we will state suitable hypotheses on it.

Putting all these remarks together, we will be led to replace problem (1.1) with a new problem

$$\begin{cases} (x, Du(x)) \in Z & \text{if } x \in M \setminus K, \\ u(x) = u_0(x) & \text{when } x \in K, \end{cases} \tag{1.6}$$

where the first equation is the *viscosity differential inclusion*, that will be properly defined in 2.27. The problem (1.6) will then be the first-class problem studied in the rest of the paper.

1.3 On the hypotheses

Riemannian structure

We will not add any Riemannian structure to M. This has many consequences. For example, we will not state any hypothesis of *"(local) Lipschitzianity"* or *"superlinearity of $p \mapsto H(x,p)$"* (or similar) on the function H. Moreover, having expressed the problem (1.1) in the general context of differential manifolds, we will be interested only in results (that is, family of hypotheses and corresponding thesis) that will be *"invariant w.r.t. diffeomorphisms"*.

Surprisingly, the lack of a reference Riemannian metric will be somewhat compensated. We will see in Example 3.5 and Theorem 3.10 that, to build a viscosity solution of (1.6), we will need to assume existence of a strict subsolution $\underline{u}$, and we will use it to define an asymmetric metric b. We will then see in Theorem 3.15 that a sufficient hypothesis to ensure that the viscosity solution to problem (1.6) is unique, is that the asymmetric metric space (M, b) is backward complete. Summarizing, the Hamilton–Jacobi problem will generate a metric structure that we will use to express the hypotheses on the problem itself.

Continuity of the hamiltonian

Another remarkable fact in this paper is that we will not assume that H be continuous. This is based on the aforementioned idea, that the viscosity solution only depends on the sign of H, that is, on the three sets (1.5); and to the general program of substiting the problem (1.1) with the problem (1.6). So, instead of assuming that H be continuous, we will assert the hypotheses 3.1 on the set Z.

However, in section 3.1 we will define a new problem (3.1), that is equivalent to the problem (1.6); by exploting the theory reviewed in the second section, and the hypotheses 3.1, we will obtain that the hamiltonian $\hat{H}$ of the problem (3.1) is continuous. So, in a sense, we could say that the hypotheses 3.1 is a way of defining an "*intrinsic continuity*" of the problem (1.6) (and hence of the original problem (1.1)).

1.4 Outline of the paper

To conclude the introduction, we outline the structure of this paper. The rest of paper is divided in two sections. In Section 2 we will review all methods and tools that we will use in the paper; these are adapted to the need of the rest of the paper from facts well known in the literature (with the addition of some lemmas). In Section 3 we will address the problem (1.6) itself.

2 Preliminary tools

We introduce some auxiliary ideas and methods that will be used in the main section 3 of the paper.

The main tools that we will use will be, Kuratowski upper/lower semicontinuity of set–valued maps in fiber bundles, asymmetric metric spaces and viscosity solutions.

2.1 Kuratowski convergence

The Kuratowski convergence is a tool that in the literature is often successfully associated to the study of optimization, as is shown in Rockafellar-Wets's [25] and in Aubin-Frankowska's [1]. We review definition and results, for convenience of the reader. We will use some concepts of general topology, as defined in Kelley [14].

2.1.1 Kuratowski convergence on maps

Let X, Y be Hausdorff topological spaces. We will consider set–valued maps $A : X \rightsquigarrow Y$, that are maps $A : X \to \mathcal{P}(Y)$. We will usually write A_x instead of $A(x)$.

Definition 2.1 *We define the* ***lower Kuratowski limit*** $\liminf_{x\to\overline{x}} A_x$ *as the set of* $y \in Y$ *such that*

$$\forall V \ni y, \exists U \ni \overline{x}, \forall x \in U \text{ with } x \neq \overline{x} \Longrightarrow A_x \cap V \neq \emptyset,$$

where $V \subset Y$ *and* $U \subset X$ *are open. We define the* ***upper Kuratowski limit*** $\limsup_{x\to\overline{x}} A_x$ *as the set of* $y \in Y$ *such that*

$$\forall V \ni y, \forall U \ni \overline{x}, \exists x \in U \text{ with } x \neq \overline{x} \Longrightarrow A_x \cap V \neq \emptyset,$$

Considering that the inclusion relation is the natural ordering for sets, we define ***lower-semi-continuous maps in the Kuratowski sense*** *when the lower Kuratowski limit* $\liminf_{x\to\overline{x}} A_x$ *contains* $A_{\overline{x}}$*;* ***u.s.c. maps*** *if* $\limsup_{x\to\overline{x}} A_x \subset A_{\overline{x}}$*; and* ***continuous maps*** *when* $\limsup_{x\to\overline{x}} A_x = \liminf_{x\to\overline{x}} A_x = A_{\overline{x}}$*.*

If we choose $X = \mathbb{N} \cup \{\infty\}$, we obtain as a special case the upper and lower Kuratowski limits of sequences of sets $A_n \subset Y$ as $n \to \infty$.

Remark 2.2 *In general* $\liminf_{x\to\overline{x}} A_x \subset \limsup_{x\to\overline{x}} A_x$ *and both these sets are closed. For this reason, in common presentations of this theory it is always assumed that the sets* A_x *be closed – indeed, this is necessary if we consider continuous maps. In the following, though, we will deal mostly with semi-continuous maps, and we will gain some benefit in non assuming that the sets be closed. (Cf. 3.4).*

Example 2.3 *Let* $X = Y = \mathbb{R}$*.*

- *Let, for* $x \neq 0$, $A_x = \{\sin(1/x)\}$ *the singleton; then* $\limsup_{x\to 0} A_x = [-1, 1]$ *while* $\liminf_{x\to 0} A_x = \emptyset$*.*
- *The constant map* $A_x \overset{def}{=} (-1, +1)$ *is lower semi continuous, since at any* $\overline{x}$, $\liminf_{x\to\overline{x}} A_x = \limsup_{x\to\overline{x}} A_x = [-1, 1]$*.*
- *The map*

$$A_x \overset{def}{=} \begin{cases} [-1, +1] & \text{if } x \neq 0 \\ (-2, 2) & \text{if } x = 0 \end{cases}$$

  is upper semi continuous.

Proposition 2.4 *If the topologies of X and Y are characterized by converging sequences then equivalently we say that the map $A : X \rightsquigarrow Y$ is* ***lower-semi-continuous in the Kuratowski sense*** *if, for any sequence $(x_n)_n \subset X$ with $x_n \neq x$ and $x_n \to x$, for any $y \in A$, there is a sequence $y_n \in A_{x_n}$ s.t. $y_n \to y$; resp.* ***upper-semi-continuous in the Kuratowski sense*** *if for any sequence $(x_n)_n \subset X$ with $x_n \neq x$ and $x_n \to x$, for any converging sequence $y_n \in A_{x_n}$ s.t. $y_n \to y$, we have $y \in A$.*

Remark 2.5 *The above is different from the definitions presented in [25] and in 1.4.2 in [1], where (probably involuntarily) it is not assumed that $x_n \neq x$; without this assumption, it would always hold that*

$$\liminf_{x \to \overline{x}} A_x \supset A_{\overline{x}}$$

that is, all maps be would lower semi continuous.

2.1.2 Compact sets

We show a proposition specific for convergence to compact sets; for simplicity, we consider the case of a sequence $\{A_n\}_{n \in \mathbb{N}}$ of subsets of Y.

Proposition 2.6 (Equicompactness.) *Suppose that Y is connected. Choose a sequence $A_n \subset Y$. Let $A \stackrel{def}{=} \limsup_{n\to\infty} A_n$. Suppose A is compact, all A_n are connected, $\liminf_{n\to\infty} A_n$ is non-empty. Then $(A_n)_n$ is* ***eventually equicompact****: for any $K \subset Y$ compact such that $A \subset \mathring{K}$, $A_n \subset K$ eventually in n.*

Proof. A proof may be found in prop. 3 in Salinetti–Wets [26], for the case when Y is a finite dimensional normed vector space. The general proof is similar. □

Remark 2.7 *The above proposition is tricky, it is easy to mistakenly state it using wrong hypotheses. For example, it is not possible to replace the condition* "$\liminf_{n\to\infty} A_n$ is non-empty" *by the condition* "$\limsup_{n\to\infty} A_n$ is non-empty", *(as is done in Corollary. 4.12 in [25]). This is clearly shown by this example: let $A_n \subset \mathbb{R}$ be defined by*

$$A_n \stackrel{def}{=} \begin{cases} \{0\} & \text{for } n \text{ even} \\ \{n\} & \text{for } n \text{ odd} \end{cases}$$

then each A_n is connected, $\liminf_{n\to\infty} A_n = \emptyset$ *while* $\limsup_{n\to\infty} A_n = \{0\}$ *is non-empty and compact; but the sequence A_n is not equicompact.*

The above is a very important remark: basilar proofs in this paper depend on the above proposition.

2.1.3 Kuratowski convergence on fiber bundles

We assume, for sake of ease, that X and Y satisfy *"the first countability axiom"*, so that their topology is characterized by converging sequences.

Suppose that N is a fiber bundle with base X and fiber Y, and let $\pi : N \to X$ be the projection. N is equipped with an atlas of ***fiberwise*** **local coordinates** that trivialize the bundle; those are defined, for any $x \in X$, by a small open $U \subset X$ containing x, and by an homeomorphism

$$\phi : U \times Y \to \pi^{-1}(U) \tag{2.1}$$

that commute with projections on X.

Let $N_x = \pi^{-1}(\{x\})$ be a fiber. We define two useful operations.

Definition 2.8 (Slicing & graph) *If B is a subset of N, we will slice it to define the map*

$$x \mapsto B_x \stackrel{def}{=} N_x \cap B \ .$$

Conversely, if we consider fiberwise maps $x \mapsto A_x$ *such that $A_x \subset N_x$ is not empty, we associate to any such map its graph*

$$A \stackrel{def}{=} \bigcup_x A_x \ .$$

Remark 2.9 *In section 3.2 we will consider maps $x \mapsto A_x$ from M to T^*M, satisfying $A_x \subset T^*_x M$; and the corresponding graphs that are subsets A of T^*M. In this case, for simplicity of notations, we will often define the slicing as*

$$A_x \stackrel{def}{=} \{p \in T^*_x M \mid (x,p) \in A\} \ .$$

We extend the definitions on Kuratowski limits that we presented in the previous sections to maps into the fiber bundle N using the local coordinates. This is justified by the following "locality" result.

Proposition 2.10 *Let $A, A_n \subset Y$. For any $V \subset Y$ open, we have*

$$\begin{aligned} V \cap \limsup_{n\to\infty}(A_n \cap V) &= V \cap (\limsup_{n\to\infty} A_n) \\ V \cap \liminf_{n\to\infty}(A_n \cap V) &= V \cap (\liminf_{n\to\infty} A_n) \end{aligned} \tag{2.2}$$

(The proof being a direct consequence of the definitions).

This following result is a slight generalization of Prop. 1.4.8 in [1].

Lemma 2.11 *Choose $A \subset N$, let A_x be the slicing of A. If A is closed, then $x \mapsto A_x$ is upper-semi-continuous. Viceversa if every slice A_x is closed, and the map $x \mapsto A_x$ is upper-semi-continuous in the Kuratowski sense, then A is closed.*

Proof. Fix $x \in X$. Pull back $x \mapsto A_x$ to a map $U \to Y$ (using local coordinates ϕ around x), that we call $\tilde{A}_x$.

Choose $x_n \to x$ in X. Choose any $y \in \limsup_{n\to\infty} \tilde{A}_{x_n}$, then there is a sequence y_n with $y_n \in \tilde{A}_{x_{m(n)}}$ for some $m(n) \geq n$, such that $y_n \to y$. Since $\phi(x_{m(n)}, y_n) \in A$ then $\phi(x,y) \in A$, so $y \in \tilde{A}_x$.

Viceversa, let z_n be a sequence in A converging to a $z \in N$; we write it in local coordinates as $(x_n, y_n) \to (x,y)$; if $x_n = x$ eventually, we use the fact that A_x is closed; otherwise by $x_n \to x$ we know that $y \in \limsup_{n\to\infty} \tilde{A}_{x_n} \subset \tilde{A}_x$, that is, $z \in A$. □

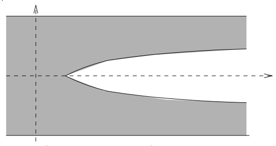
Lemma 2.12 *Suppose $A \subset N$ is open; let A_x be the slicing of A: then the map $x \mapsto A_x$ is lower-semi-continuous in the Kuratowski sense.*

Proof. Fix $x \in X$. Pull back $x \mapsto A_x$ to a map $U \to Y$ (using local coordinates ϕ around x), that we call $\tilde{A}_x$. Choose $y \in \tilde{A}_x$, choose any $x_n \to x$, then eventually $\phi(x_n, y) \in A$, that is $y \in \tilde{A}_{x_n}$. Then $y \in \liminf_{n\to\infty} \tilde{A}_{x_n}$. □

Note that there is no "viceversa" part in this lemma, even assuming that slices A_x are open, as a simple example can show.

Example 2.13 *Let* $X = Y = \mathbb{R}$*, let* $N = X \times Y = \mathbb{R}^2$*, and let*

$$A_x \stackrel{def}{=} \begin{cases} (-2, +2) & \text{if } x \le 1 \\ (-2, -1 + 1/x) \cup (1 - 1/x, +2) & \text{if } x > 1 \end{cases} .$$



Then any slice is open, and $x \mapsto A_x$ *is lower semi continuous, since at any* $\overline{x}$*,* $\liminf_{x\to\overline{x}} A_x = \overline{A_{\overline{x}}}$*; but the graph is not open, since* $x = 1, y = 0$ *is in the graph but not in the interior part* $\mathring{A}$*.*

2.1.4 Dual of a set map

When we moreover suppose that N is a *finite dimensional vector bundle*, we have that Y is a finite dimensional normable vector space and (for all $x \in U$) the local chart $\phi(x, \cdot)$ is a linear isomorphism between $\{x\} \times Y$ and $N_{\phi(x)}$.

Let N^* be the dual bundle of N, that is, $\pi^* : N^* \to X$ and $N^*_x \stackrel{\text{def}}{=} (\pi^*)^{-1}(\{x\})$ is isomorphic to Y^*.

We consider *fiberwise maps* $x \mapsto A_x$ with $A_x \subset N^*_x$; we suppose that any A_x is non empty. We define the *fiberwise support function*

$$F : N \to (-\infty, \infty]$$

so that $F(x, \cdot)$ is the support function of the set A_x: in local coordinates,

$$F(x, v) \stackrel{\text{def}}{=} \sup\left\{p \cdot v \mid p \in A_x\right\} . \tag{2.3}$$

We will need to relate the Kuratowski convergence to the convergence of support functions; Mosco [23] proves indeed that Kuratowski convergence $A_n \to A$ is equivalent to convergence $F_{A_n} \to F_A$ (both the "pointwise" convergence and the "locally uniform" convergence). We split this kind of result in two, and we state an upper and a lower semicontinuity result.

Lemma 2.14 *If the map* $x \mapsto A_x$ *is lower-semi-continuous in the Kuratowski sense, then* F *is lower-semi-continuous.*

Proof. Fix $x \in X$. Pull back $x \mapsto A_x$ to a map $U \to Y^*$ (using fiberwise local coordinates ϕ in a neighborhood U around x). Similarly, pull back F in local coordinates.

Fix x, v, and choose any $r < F(x, v)$. Choose sequences such that $v_n \to v$ and $x_n \to x$. Choose p such that $p \in A_x$ and $r < p \cdot v \le F(x, v)$.

Since $x \mapsto A_x$ is l.s.c., there is a sequence $p_n \in A_{x_n}$ such that $p_n \to p$. So

$$F(x_n, v_n) \ge p_n \cdot v_n \to p \cdot v \ge r$$

and then

$$\liminf_{n\to\infty} F(x_n, v_n) \ge F(x, v)$$

by arbitrariness of r.

□

Lemma 2.15 *Fix $\overline{x} \in X$. If the map $x \mapsto A_x$ is upper-semi-continuous in the Kuratowski sense at $\overline{x}$, any A_x is connected, $\liminf_{x\to\overline{x}} A_x$ is non-empty, and $A_{\overline{x}}$ is compact, then for any fixed $\overline{z}$ with $\pi^*(\overline{z}) = \overline{x}$, F is locally bounded and upper-semi-continuous at $\overline{z}$.*

Proof. We again work in local coordinates. Suppose $A_{\overline{x}}$ is compact: by 2.6 there is a neighborhood U of $\overline{x}$ and a K compact such that $A_x \subset K$ for all $x \in U$. Fix $\overline{v}$, and V a compact neighborhood of $\overline{v}$ in Y: then $|F(x,v)|$ is bounded in a neighborhood $U \times V$ of $\overline{z} = (\overline{x}, \overline{v})$ (by $\sup\{|p \cdot v|, p \in K, v \in V\}$).

Choose sequences in V and U, such that $v_n \to \overline{v}$ and $x_n \to \overline{x}$, and suppose, without loss of generality, that the sequence $F(x_n, v_n)$ is increasing.

Choose $p_n \in A_{x_n}$ such that

$$\lim_n F(x_n, v_n) = \lim_n p_n \cdot v_n$$

On the other hand, $p_n \in K$ so we may extract a subsequence p_{n_m} converging to a limit point q: we have that $q \in \limsup_m A_{n_m} \subset A_x$ and then

$$F(x,v) \geq q \cdot v = \lim_m p_{n_m} \cdot v_{n_m} = \lim_n F(x_n, v_n)$$

□

2.2 Asymmetric metric spaces

We now provide a brief compendium of the theory of asymmetric metric spaces. Throughout this subsection, M may be a generic set.

Definition 2.16 *$b : M \times M \to \mathbb{R}^+$ is an* ***asymmetric distance*** *if b satisfies*

- *$b \geq 0$ and $b(x,y) = 0$ iff $x = y$*
- *$b(x,y) \leq b(x,z) + b(z,y) \quad \forall x,y,z \in M$.*

We call the pair (M, b) an **asymmetric metric space**. We agree that b defines a topology τ on M, generated by the families of **forward** and **backward open balls**

$$B^+(x,\varepsilon) \stackrel{\text{def}}{=} \{y \mid b(x,y) < \varepsilon\}, \quad B^-(x,\varepsilon) \stackrel{\text{def}}{=} \{y \mid b(y,x) < \varepsilon\} \ ,$$

that is, the topology is generated by the symmetric distance

$$d(x,y) \stackrel{\text{def}}{=} b(x,y) \vee b(y,x) \quad . \tag{2.4}$$

More details on the notion of convergence may be found in in [21].

Remark 2.17 *b is also known as a* "quasi metric" *or* "ostensible metric"*; see for example Kelly [15], Reilly, Subrahmanyam and Vamanamurthy [12]* [1] *Fletcher and Lindgren [11, (pp 176-181)], Künzi [17].*

The differences between what we present here and what is discussed in those reference is mainly in the choice of the topology and of the definition of "Cauchy sequences" and "completeness" (a detailed explanation is in [21]).

[1] [12] provides also a wide discussion of the references on *quasi metrics*

Remark 2.18 *If we would add to the definition 2.16 this additional statement*

$$\forall (x_n) \subset M, x \in M \quad , \quad b(x_n, x) \to 0 \iff b(x, x_n) \to 0 \tag{2.5}$$

then the space (M, b) *would be a* "general metric spaces", *as defined by Busemann [3], [4]. This extra hypothesis (2.5) is equivalent to saying that the topology* τ *generated by the symmetric distance (2.4) may be generated by forward balls only (or backwards balls only). If the topology* τ *is locally compact, then the above hypothesis is satisfied.*

We define that

Definition 2.19 *A sequence* $(x_n) \subset M$ *is called "**forward Cauchy**" if*

$$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N} \text{ such that } \forall n, m, \quad m \geq n \geq N, \quad b(x_n, x_m) < \varepsilon \ . \tag{2.6}$$

We say that (M, b) *is "**forward complete**" if any forward Cauchy sequence* (x_n) *converges to a point* x *(according to the topology* τ *previously defined).*

The above definitions agree with those used in Finsler geometry (as defined in ch. VI in [2]).

Remark 2.20 *For any* forward *definition above there is a corresponding* backward *definition, obtained by using the **conjugate distance*** $\overline{b}$ *defined by*

$$\overline{b}(x, y) = b(y, x) \quad .$$

We induce from b the length $\operatorname{len}^b \gamma$ of a continuous curve $\gamma : [\alpha, \beta] \to M$, by using the total variation

$$\operatorname{len}^b \gamma \stackrel{\text{def}}{=} \sup_T \sum_{i=1}^n b\big(\gamma(t_{i-1}), \gamma(t_i)\big) \tag{2.7}$$

where the sup is carried out over all finite subsets $T = \{t_0, \cdots, t_n\}$ of $[\alpha, \beta]$ and $t_0 \leq \cdots \leq t_n$.

We define the **geodesically induced distance** b^g by

$$b^g(x, y) \stackrel{\text{def}}{=} \inf \operatorname{len}^b \gamma \tag{2.8}$$

where the infimum is taken in the class of all continuous curves γ connecting x to y. If the inf is a minimum, the curve providing the minimum is called a **geodesic**. More in general, a curve $\gamma : I \to M$ (with $I \subset \mathbb{R}$ an interval) is called a geodesic if, for any $a, b \in I$ with $a < b$, setting $x = \gamma(a), y = \gamma(b)$, we have that γ restricted to $[a, b]$ provides the minimum in (2.8).

If the space (M, b) is Lipschitz-arcwise connected, then it is easily proved that b^g is an asymmetric distance.

Definition 2.21 *We say that the (asymmetric) metric space* (M, b) *is **a path-metric space**, or that* b *is **intrinsic**, if* $b = b^g$.

We now state a Hopf–Rinow-like theorem.

Theorem 2.22 *Suppose that (M, b) is path-metric, and forward-locally compact (that is, for any x there exists $\varepsilon > 0$ s.t. $\{y \mid b(x, y) \le \varepsilon\}$ is compact); then the following are equivalent*

- *(M, b) is forward complete,*
- *forward-bounded and closed sets are compact,*
- *any geodesic $\gamma : [0, l) \to M$ may be completed;*

and all imply that (M, b) admits geodesics.

A similar statement holds for "backward" conditions, by using the conjugate distance $\bar{b}(x, y) = b(y, x)$ (cf. remark 2.20). This theorem is proved in [21]; with the stronger hypothesis that (M, b) be locally compact, it is proved in I.8 in [4].

Remark 2.23 *The* forward-local compactness *is called* weak local compactness *in [4]; on this hypothesis, see the notes at the end of section I of [4]; or page 7 in Zaustinsky's [28]; or Phadke's [24] – where though a* weak global compactness *is addressed.*

2.3 Viscosity solutions on manifolds

The standard definitions of the *superdifferential* $\partial^+ u(x)$ and the *subdifferential* $\partial^- u(x)$ in $\mathbb{R}^n$ can be easily generalized to maps on a manifold, by means of local charts, or as follows. Let Ω be an open subset of the manifold M.

Definition 2.24 *Given a continuous function $u : \Omega \to \mathbb{R}$ and a point $x \in M$, the* ***superdifferential*** *of u at x is the subset of $T_x^* M$ defined by*

$$\partial^+ u(x) = \left\{ d\varphi(x) \,|\, \varphi \in C^1(M), \varphi(x) - u(x) = \min_M (\varphi - u) \right\}.$$

Similarly, the set

$$\partial^- u(x) = \left\{ d\psi(x) \,|\, \psi \in C^1(M), \psi(x) - u(x) = \max_M (\psi - u) \right\}$$

is called the ***subdifferential*** *of u at y.*

Now we briefly reintroduce the definition of viscosity solutions of PDE on manifolds (as in [19] or [22]).

Definition 2.25 *We say that a continuous function u is a* ***generalized viscosity solution*** *of equation*

$$H(x, Du(x)) = 0$$

in Ω if for every $x \in \Omega$,

$$\begin{cases} \forall p \in \partial^+ u(x), & H(x, p) \le 0 \\ \forall p \in \partial^- u(x), & H(x, p) \ge 0 \end{cases} \tag{2.9}$$

If only the first condition is satisfied (resp. the second), u is called a ***generalized viscosity subsolution*** *(resp. a* ***generalized viscosity supersolution****).*

As we noted in the introduction, the viscosity solution depends only on the sign of H. We use the terminology "generalized viscosity solution" since continuity of H is not assumed at this point.

Proposition 2.26 *Consider the two problems (1.1) and*

$$\begin{cases} \tilde{H}(x, Du(x)) = 0 & \text{in } M \setminus K \\ u(x) = u_0(x) & \text{when } x \in K. \end{cases} \tag{2.10}$$

If

$$signH(x,p) = sign\tilde{H}(x,p) \qquad \forall x, p \in T^*M$$

then the definition of viscosity solution immediately implies that (1.1) and (2.10) have the same viscosity solutions.

This fact is trivially proved; at the same time, it is not widely exploited in the literature (with some exceptions; to cite some examples, [20], [5], [18]).

2.3.1 Viscosity solution by inclusion

So we naturally come to the idea of defining the viscosity solutions to the Hamilton-Jacobi problem using a set $Z \subset T^*M$.

Definition 2.27 (Viscosity solutions by inclusion) *Let $Z \subset T^*M$, define Z_x to be the slicing*

$$Z_x \stackrel{def}{=} Z \cap T^*_x M = \{p \in T^*_x M \mid (x,p) \in Z\} \ .$$

*We say that a continuous function u is a **viscosity solution** of the differential inclusion*

$$(x, Du(x)) \in Z \tag{2.11}$$

in the open set $\Omega \subset M$ if for every $x \in \Omega$,

$$\begin{cases} \partial^+ u(x) \subset \overline{Z_x} \\ \partial^- u(x) \subset \overline{T^*_x M \setminus Z_x} \end{cases} \tag{2.12}$$

*If only the first condition is satisfied (resp. the second), u is called a **viscosity subsolution** (resp. a **viscosity supersolution**).*

The second condition in (2.12) can be equivalently stated as $\partial^- u(x) \cap \mathring{Z}_x = \emptyset$.

Since we started by stating the problem (1.1), but we will be studying the problem (1.6) in the rest of the paper, we now establish this result.

Proposition 2.28 *Suppose that $\{H \le 0\}$ is closed; let $Z = \{H \le 0\}$, and suppose that, for all $x \in M$,*

$$\mathring{Z}_x = \{p \mid H(x,p) < 0\} \tag{2.13}$$

(where the internal part $\mathring{Z}_x$ of Z_x is computed w.r.t. $T^*_x M$); then the two problems are equivalent, that is, that they have the same family of solutions.

3 Convex Hamilton-Jacobi equation

We now proceed to integrate all the tools presented in the previous section in a study of solutions to the Hamilton-Jacobi problem (1.6).

We will assume the following hypotheses.

Hypotheses 3.1 *Let $Z \subset T^*M$ be closed. We will assume that the following hold.*

(Z) *Let $A = \mathring{Z}$ (where the "internal part" is computed w.r.t. T^*M) and let A_x be the* fiberwise slicing *of A,*

$$A_x \overset{def}{=} \{p \in T^*_x M \mid (x,p) \in A\}.$$

We suppose that, for all x, Z_x is nonempty, convex and compact and $Z_x = \overline{A_x}$.

($\exists$u) *There exists a smooth function $\underline{u}$ on M such that $D\underline{u}(x) \in \mathring{Z}_x$, $\forall x \in M$. (Here the "internal part" $\mathring{Z}_x$ of Z_x is computed w.r.t. $T^*_x M$).*

3.1 Gauge Transform

One main tool to prove many of the results in this paper is to transform the problem (1.6) into this equivalent simpler problem

$$\begin{cases} \hat{H}(x, Du(x)) - 1 = 0 & \text{in } M \setminus K \\ u(x) = \hat{u_0}(x) & \text{when } x \in K. \end{cases} \tag{3.1}$$

To this end, we define $\hat{u_0} \overset{\text{def}}{=} u_0 - \underline{u}$ and we construct $\hat{H}$ in two steps.

1. $\tilde{Z}_x$ is obtained from Z_x by translation,

$$\tilde{Z}_x = Z_x - D\underline{u}(x) \quad . \tag{3.2}$$

The above transformation implies that $0 \in \mathring{\tilde{Z}}_x$, (and basically says that we can assume that $\underline{u} = 0$ in our proofs, with no loss of generality).

2. Since Z_x is convex, then $\tilde{Z}_x$ is a convex set. We define then the gauge function j_z of $\tilde{Z}_x$ as

$$j_x(p) \overset{\text{def}}{=} \inf \left\{ t > 0 \mid p/t \in \tilde{Z}_x \right\}$$

and eventually we define

$$\hat{H}(x,p) \overset{\text{def}}{=} \big(j_x(p)\big)^2$$

We call $\hat{H}$ **the gauge Hamiltonian**.

Summarizing all above steps in one, we can say that $\hat{H}$ is built from Z_x as

$$\hat{H}(x,p) \overset{\text{def}}{=} \inf \left\{ t^2 \;\middle|\; t > 0 \text{ s.t. } \left(\frac{p}{t} + D\underline{u}(x)\right) \in Z_x \right\} \quad . \tag{3.3}$$

By the definition, $\hat{H}$ is positively 2-homogeneous, ie

$$\forall \lambda \geq 0 \qquad \hat{H}(x, \lambda p) = \lambda^2 \hat{H}(x,p) \quad ; \tag{3.4}$$

this simplifies some arguments (in particular in the comparison theorem 3.13).

The above two steps have a clear and simple effect on solutions.

Proposition 3.2 *Assume that Z_x is convex and closed. $(u - \underline{u})$ is a solution of the problem (3.1) if and only u is a solution of (1.6).*

Proof. We follow the two steps that we used to build $\hat{H}$.

1. In the first step, we simply translate all cotangent planes by $D\underline{u}$, so (obviously) u is a viscosity solution of $(x, Du(x)) \in Z_x$ iff $(u - \underline{u})$ is a viscosity solution of $(x, Du(x)) \in \tilde{Z}_x$.

2. Since $\tilde{Z}_x$ is convex and closed, it is easily proved (by the definition of $\hat{H}$) that

   - $(\hat{H}(x, p) - 1) \leq 0$ iff $(x, p) \in \tilde{Z}_x$
   - $(\hat{H}(x, p) - 1) \geq 0$ iff $(x, p) \in \overline{T_x M \setminus \tilde{Z}_x}$

   Then, u is a viscosity solution for $\hat{H}(x, Du(x)) - 1 = 0$ iff it is a viscosity solution for the inclusion $(x, Du(x)) \in \tilde{Z}_x$.

□

Remark 3.3 *It is also possible to define $\hat{H}$ starting from H; this does link the original problem (1.1) to the problem (3.1); the two problems are equivalent if the condition (2.13) holds.*

3.2 Finsler metrics

Let Z be as in hypotheses 3.1. We then define, for any $x \in M$, the **figuratrix** set Z_x by slicing Z along the fibers of T^*M, exactly as we did in eqn. (1.2) in introduction. Z_x can be seen as a set-valued map $x \mapsto Z_x$ from M to the fibers of T^*M (we discussed these maps in section 2.1.3).

We define the *slicewise support function* $\sigma : TM \to [-\infty, \infty]$ by

$$\sigma(x, v) \stackrel{\text{def}}{=} \sup \left\{ p \cdot v \mid p \in Z_x \right\} \tag{3.5}$$

similarly to what is done in other papers (e.g. [5]).

Lemma 3.4 *Let $A = \mathring{Z}$ and let A_x be the slicing of A:*

$$A_x \stackrel{\text{def}}{=} \{ p \in T^*_x M \mid (x, p) \in A \}$$

Suppose that A_x is non-empty and $Z_x = \overline{A_x}$ for all x. Then σ is the support of A_x, namely

$$\sigma(x, v) = \sup \left\{ p \cdot v \mid p \in A_x \right\} \tag{3.6}$$

and σ is lower–semi–continuous.

Suppose moreover that Z_x is convex and compact for all x; then σ is continuous and locally bounded.

Proof. Since $Z_x = \overline{A_x}$ then eqn. (3.6) holds. We use the definition of Kuratowski convergence, and the results, exposited in section 2.1. By 2.12 we know that $x \mapsto A_x$ is l.s.c.: then by 2.14 and eqn. (3.6), σ is l.s.c. If moreover every Z_x is convex and compact, then, by 2.11, $x \mapsto Z_x$ is u.s.c.; since $\liminf_{x \to \overline{x}} Z_x \supset \liminf_{x \to \overline{x}} A_x \supset A_{\overline{x}} \neq \emptyset$ then by 2.15 we know that σ is u.s.c. and locally bounded. □

We conclude that when the hypotheses `(Z)` (that were expressed in 3.1) hold then σ is continuous and locally bounded from above.

We then define

$$S(x,y) = \inf \int_0^1 \sigma(\xi(s), \dot\xi(s))ds \tag{3.7}$$

where the infimum is computed in the class of all locally Lipschitz ξ with given extrema $\xi(0) = x, \xi(1) = y$. This quantity $S(x,y)$ does not depend on $\underline{u}$, but it may fail to be positive (and hence to be an asymmetric distance, as is defined in section 2.2). Unfortunately, it may even be the case that $S \equiv -\infty$, as in this simple example.

Example 3.5 *Let $M = S^1 = \mathbb{R}/\mathbb{Z}$ and let $Z_x = [1,2]$ for all x; then $\sigma(x,v) = 2v$ for $v > 0$ and $\sigma(x,v) = v$ for $v < 0$. Given any $x, y \in M$, the winding curves $\gamma_n(t) = x + t(-n + y - x)$ prove that $S(x,y) = -\infty$. See fig.1.*

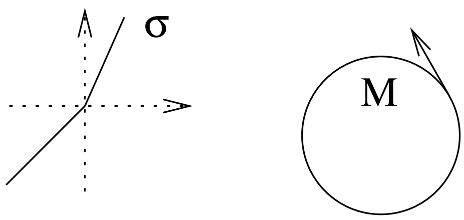


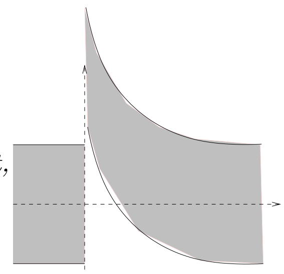
Figure 1: Example 3.5

We will see that the hypothesis `(∃u)` is specifically needed to avoid such pathological examples.

We also provide an example where the hypothesis $Z_x = \overline{A_x}$ does not hold.

Example 3.6 *Let $M = \mathbb{R}$ and*

$$Z_x \stackrel{def}{=} \begin{cases} [-1,1] & \text{if } x \le 0 \text{ and} \\ [-1+1/x, 1+1/x] & \text{if } x > 0 \end{cases} .$$



Then Z is closed, any slice Z_x is convex and compact, but σ is not locally bounded and continuous.

If we set $A = \mathring{Z}$, then

$$A_x = \begin{cases} (-1,1) & \text{if } x < 0 \\ \emptyset & \text{if } x = 0 \\ (-1+1/x, 1+1/x) & \text{if } x > 0 \end{cases} ,$$

3.2.1 Finsler Metric from Z and $\underline{u}$

As aforementioned, we assume in this paper that `(∃u)` (see 3.1). This $\underline{u}$ is called a *strict subsolution.* A discussion of this condition is in §3.6. Note that we do need to speak of "strict subsolution in the viscosity sense", since such a solution may be mollified, as shown in Lemma 6.3 in [10]. Note also that $\underline{u} + c$ is again a *strict subsolution*, for any constant c.

Given a choice of a strict subsolution $\underline{u}$, we define then $F : TM \to [0,\infty]$ so that $F(x,\cdot)$ is the support function σ of the set Z_x corrected by $D\underline{u}(x)$,

$$F(x,v) \stackrel{\text{def}}{=} \sigma(x,v) - v \cdot D\underline{u}(x) = \tag{3.8}$$

$$= \sup\left\{p \cdot v \mid (p + D\underline{u}(x)) \in Z_x\right\} . \tag{3.9}$$

By the definition it is clear that $F \geq 0$, and $F(x,v) = 0$ iff $v = 0$: indeed, we know that $D\underline{u}(x) \in \mathring{Z}_x$ (and so $\mathring{Z}_x \neq \emptyset$).

The hypotheses that were used in all lemmas up to now are exactly those that were anticipated in 3.1; so we summarize all the results in a theorem.

Theorem 3.7 *Assume* `(Z)` *in 3.1. Choose a strict subsolution $\underline{u}$, and define F. Then F is a* ***Finsler metric****, satisfying*

- *$F \geq 0$, and $F(x,v) = 0$ iff $v = 0$,*
- *$v \mapsto F(x,v)$ is positively 1-homogeneous,*
- *F is continuous and locally bounded.*

3.2.2 Asymmetric distance from Z and $\underline{u}$

We define the **length** $\text{len}^F \gamma$ of a locally Lipschitz curve $\xi : [0,1] \to M$ as

$$\text{len}^F \gamma = \int_0^1 F(\xi(s), \dot{\xi}(s))\, ds \tag{3.10}$$

As in [21], we then define

$$b(x,y) = \inf \text{len}^F \gamma \tag{3.11}$$

where the infimum is computed in the class of all locally Lipschitz ξ with given extrema $\xi(0) = x, \xi(1) = y$.

Under the hypotheses 3.1, b is an asymmetric distance.

b is itself also sometimes called a *Finsler metric*[2] in the current literature, and is naturally associated to Hamilton-Jacobi equations; see [27] and references therein.

Some useful properties b are listed in the following proposition 3.8.

Proposition 3.8 *Assume* `(Z)` *and* `(∃u)`*. Define F and b as above.*

1. *The topology τ induced by d (cf. (2.4)) coincides with the topology of the manifold M; so the* asymmetric metric space *(M,b) is indeed locally compact. So (M,b) is a* general metric space*, as defined by Busemann (see 2.18).*
2. *For any Lipschitz γ,* $\text{len}^F \gamma$ *(defined in eqn. (3.10)) coincides with* $\text{len}^b \gamma$ *(defined in eqn. (2.7));*
3. *consequently, (M,b) is path-metric, that is, $b = b^g$.*

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[2] whereas in this paper we usually prefer to call F "the metric" and b "the distance"

4. *For any $x \in M$ there exists a neighborhood U where, $\forall y \in U$, there exists a minimal geodesic γ connecting x to y with*

$$F(\xi(s), \dot{\xi}(s)) = 1 \qquad \forall s. \tag{3.12}$$

Proof. The first point is proved as the equivalent proposition in Finsler or Riemann geometry. The second point is due to the lower semi continuity of $\mathrm{len}^F \gamma$ and $\mathrm{len}^b \gamma$ w.r.t. uniform convergence, and so third point follows. The fourth point is proved by using a reparametrization lemma. A detailed proof of all of the above is in in [21]. □

By equations (3.8),(3.10),(3.11),(3.7) and direct calculation

$$S(x, y) = b(x, y) + \underline{u}(y) - \underline{u}(x) \quad . \tag{3.13}$$

This means that, when the strict subsolution exists, then S will also be locally bounded from below. (This is not a necessary condition, though — the existence of a locally Lipschitz continuous almost everywhere solution $\underline{u}$ would still imply that S is locally bounded from below).

3.3 Viscosity solutions of the HJ equation

In this section we will study existence of solutions of problem (1.6); if $Z = \{H \leq 0\}$ and we add the hypothesis $\partial(Z_x) = \{p \mid H(x, p) = 0\}$, then the problem (1.6) coincides with the problem (1.1).

We know that

Proposition 3.9 *Assume 3.1. Fix $a \in M$. The function*

$$u(x) \stackrel{def}{=} S(a, x) \tag{3.14}$$

is a viscosity solution of $Du(x) \in Z_x$ if $x \neq a$, and a viscosity subsolution for all x.

The proof is nowadays standard (in the $\mathbb{R}^n$ case, see for example Theorem 2.1 in [27]; or see Prop.4.2 in [10]).

We define the *value function*

$$V(x) \stackrel{\text{def}}{=} \inf \Big(u_0(\xi(0)) + \int_0^1 \sigma(\xi(s), \dot{\xi}(s)) ds \Big) \tag{3.15}$$

where the infinimum is in the class of Lipschitz paths ξ with $\xi(1) = x$ and $\xi(0) \in K$. By combining all identities (3.10) and (3.9) we can rewrite V also as

$$V(x) \quad = \quad \inf_{z \in K} (u_0(z) + S(z, x)) \tag{3.16}$$

When $u_0 = 0$, this last formula is known as the **inf-convolution**.

This formula builds a solution to (1.1).

Theorem 3.10 *Assume 3.1. Then V is a viscosity solution to $du(x) \in Z_x$ on $M \setminus K$. If moreover*

$$u_0(x) \leq S(y, x) + u_0(y) \tag{3.17}$$

for all $x, y \in K$, then $V = u_0$ on K: so V is a viscosity solution to (1.6).

The proof is standard, and can be obtained by combining proposition 3.9 and prop. 8.2.1 in [7].

3.3.1 Hopf-Rinow for Hamilton-Jacobi

We now use the theory presented in the preliminary sections to state this version of the *Hopf-Rinow* theorem that is adapted to Hamilton-Jacobi problems

Theorem 3.11 (Hopf-Rinow for Hamilton-Jacobi) *Assume* `(Z)` *and* `(∃u)`. *Choose a strict subsolution $\underline{u}$, and define b with it (as done in equation (3.11)). Then the conditions 1-4 here following are equivalent.*

1. *(M,b) is backward-complete,*
2. *backward bounded closed sets are compact,*
3. $$\forall x, \quad \liminf_{y\to\infty} b(y,x) = \infty, \tag{3.18}$$
4. $$\forall x, \quad \liminf_{y\to\infty} S(y,x) + \underline{u}(y) = \infty \ . \tag{3.19}$$

If the above conditions 1-4 hold, then, for any fixed $x, y \in M$, there is a Lipschitz curve ξ connecting x to y that minimizes the length $\mathrm{len}^F \xi$ and such that $F(\xi,\dot\xi)$ is constant. This curve ξ is also a minimum for $S(x,y)$.

Note that in eqn. (3.18), we may replace "$\forall x$" with "$\exists x$", and obtain an equivalent statement; and similarly for (3.19).

Proof. The equivalence $1 \iff 2$ of the first two statements, and the existence of geodesics, follows from the more general Hopf-Rinow theorem in 2.22. The other equivalences are easy (and do not need any special hypotheses on (M,b)); they may be proven using the triangular inequality and the definition of the topology. Moreover (3.19) is just a rewriting of (3.18). □

Remark 3.12 (Conjugate problems) *If we define a problem conjugate to (1.1) by using the Hamiltonian $\overline{H}(x,p) \stackrel{def}{=} H(x,-p)$, then we may restate all above theorems by using* forward *conditions, and using the conjugate distance $\overline{b}(x,y) = b(y,x)$ and the conjugate $\overline{S}(x,y) = S(y,x)$. (Cf. 2.20)*

3.4 Comparison Theorem

This section is devoted to the proof of this result.

Theorem 3.13 *Assume* `(Z)` *from 3.1; suppose that a choice of strict subsolution $\underline{u}$ exists so that the asymmetric metric space (M,b) is backward complete.*

Suppose that u is a viscosity subsolution and v is a supersolution of

$$(x, Du(x)) \in Z$$

for $x \in M \setminus K$ whereas $u \le v$ on K, and suppose that $v \ge \underline{u} + c$ for a constant $c \in \mathbb{R}$: then $u \le v$.

To prove the above theorem, we will use the following result from prop. 4.3 in Camilli and Siconolfi [5]. [3]

Proposition 3.14 (prop. 4.3 in [5]) *Let $\Omega \subset M$ be open. For any $f : \Omega \to \mathbb{R}$ continuous, we define the Clarke generalized differential as*

$$\partial f(x) \stackrel{def}{=} co\{p \in T_x^*M \mid \exists (x_n) \subset \Omega, \exists df(x_n) \stackrel{def}{=} p_n, (x_n, p_n) \to (x,p) \text{ in } T^*M\}$$

where $co(A)$ is the convex envelope of a set $A \subset T_x^*M$. Consider a generic problem $H(x, Du(x)) = 0$.

- *v is a supersolution of $H(x, Dv(x)) = 0$ in Ω iff for any $x \in \Omega$ and any Lipschitz continuous ϕ which is subtangent to v at x there exists $p \in \partial\phi(x)$ such that $H(x,p) \geq 0$.*
- *Any subsolution of $H(x, Du(x)) = 0$ is Lipschitz continuous, and for all $p \in \partial u(x)$ we have $H(x,p) \leq 0$.*

Using this result we can simplify the proof in [19], and yet prove the general theorem 3.13 here proposed.

Proof. By means of the translation (3.2), we assume without loss of generality that $\underline{u} \equiv 0$, and we replace the problem at hand with the problem (3.1). Let

$$h(x,p) \stackrel{\text{def}}{=} \sqrt{\hat{H}(x,p)}$$

in the following.

As in the work of Kružhkov [16], we consider the transformed functions $\tilde{u} = -e^{-u}$ and $\tilde{v} = -e^{-v}$, which are respectively a viscosity subsolution and a supersolution of

$$\begin{cases} h(x, Dv) + v = 0 & \text{in } M \setminus K, \\ v = -e^{-u_0} & \text{on } K \end{cases} \tag{3.20}$$

(see proposition 6 in [19]) moreover, $0 > \tilde{v} \geq -e^{-\inf v} \geq -e^{-c}$ and $\tilde{u} < 0$.

We establish a comparison result for this last problem (3.20): this clearly implies the above theorem. We fix $C \stackrel{\text{def}}{=} e^{-c}$. We argue by contradiction, and suppose that $\tilde{u}$ and $\tilde{v}$ are resp. a subsolution and a supersolution of (3.20), $0 > \tilde{v} \geq -C$, $\tilde{u} < 0$, and that at a point $\overline{x}$ we have $\tilde{u}(\overline{x}) = 2\varepsilon + \tilde{v}(\overline{x})$ with $\varepsilon > 0$.

We apply the Kružhkov transformation to above proposition 3.14 and state that

- for any $x \in M \setminus K$ and any Lipschitz continuous ϕ which is subtangent to $\tilde{v}$ at x there exists $p \in \partial\phi(x)$ such that $h(x,p) + \tilde{v}(x) \geq 0$
- $\tilde{u}$ is locally Lipschitz continuous, and for any $x \in M \setminus K$ and for all $p \in \partial\tilde{u}(x)$ we have $h(x,p) + \tilde{u}(x) \leq 0$

Let $B(x) \stackrel{\text{def}}{=} b(x, \overline{x})$. By 3.9 and 3.12 we know that B is a viscosity solution of $h(x, -p) - 1 = 0$ for $x \neq \overline{x}$: then for all $x \in M$ and all $p \in \partial B(x)$, $h(x, -p) \leq 1$.

Let

$$\Psi(x) = \tilde{u}(x) - \tilde{v}(x) - \varepsilon B(x)$$

[3]The proof in [5] is stated assuming that $\Omega \subset \mathbb{R}^n$, but the result can be generalized to manifolds, using local charts.

This function is bounded from above by C; moreover $\Psi(\overline{x}) = 2\varepsilon$: then $\sup \Psi$ will be positive, and realized in the region $\{x \mid \varepsilon B(x) \leq C\}$ which is a backward closed ball. By the Hopf–Rinow–like theorem 3.11 since the metric space (M, b) is backward complete, then the backward closed balls are compact: so $\Psi(x)$ has a positive maximum in a point $\hat{x}$. This means that the function $\tilde{u}(x) - \varepsilon B(x)$ is a Lipschitz subtangent of $\tilde{v}(x)$ at $\hat{x}$.

We know that $\Psi(\hat{x}) \geq \Psi(\overline{x}) = 2\varepsilon$, while $\Psi(x) \leq 0$ for all $x \in K$: then $\hat{x} \notin K$. Then by (the transformed version of) 3.14, there exists $p \in \partial(\tilde{u}(x) - \varepsilon B(x))$ such that $h(x, p) + \tilde{v}(x) \geq 0$.

At the same time $p = p' + p''$ with $p' \in \partial\tilde{u}(x)$ and $p'' \in \partial(-\varepsilon B(x))$: then (again by 3.14) $h(\hat{x}, p') + \tilde{u}(\hat{x}) \leq 0$; at the same time, as noted above, $h(\hat{x}, p''/\varepsilon) - 1 \leq 0$ that is $h(\hat{x}, p'') \leq \varepsilon$.

Since $h(x, \cdot)$ is convex and 1-homogeneous, summing up we obtain

$$h(\hat{x}, p) \leq h(\hat{x}, p') + h(\hat{x}, p'') \leq -\tilde{u}(\hat{x}) + \varepsilon$$

while $h(x, p) + \tilde{v}(x) \geq 0$: this entails

$$-\tilde{v}(x) \leq h(x, p) \leq -\tilde{u}(\hat{x}) + \varepsilon$$

or

$$\tilde{u}(\hat{x}) - \tilde{v}(\hat{x}) \leq \varepsilon$$

whereas

$$\tilde{u}(\hat{x}) - \tilde{v}(\hat{x}) \geq \Psi(\hat{x}) \geq \Psi(\overline{x}) \geq 2\varepsilon$$

achieving contradiction. □

3.5 Uniqueness and variationality

We summarize the results in previous sections in this theorem.

Theorem 3.15 *We assume hypotheses* (Z). *We assume that a choice of the* strict subsolution $\underline{u}'$ *exists so that* $\underline{u}' \leq u_0$ *and* [4]

$$\liminf_{y \to \infty} S(y, x) + \underline{u}'(y) = \infty \quad . \tag{3.21}$$

We also assume (3.17) on u_0*, so* $V = u_0$ *on* K*.*

(variationality) *For each* x*, the value problem (3.15) has a minimum, attained by a Lipschitz curve* ξ *such that* $F(\xi, \dot{\xi})$ *is (almost everywhere) constant.*

In this case we say that the problem (1.1) is ***variational****, since it comes from a variational problem that admits minimum.*

(uniqueness) *Let* $\mathcal{F}$ *be the class of all continuous functions* $f : M \to \mathbb{R}$ *such that there exists a* strict subsolution $\underline{u}$ *with* $\underline{u} \leq f$ *on* M *and satisfying (3.19).*

Then V *is the unique viscosity solution to problem (1.1), in the class* $\mathcal{F}$*.*

The proof of the first point follows from 3.11; the proof of the last point follows from the comparison theorem 3.13.

[4] we know from theorem 3.11 that this equation (3.21) is a *completeness hypothesis*.

Remark 3.16 *Note that, since we supposed that there exists a $\underline{u}' \le u_0$, satisfying (3.19), then $V \in \mathcal{F}$ (since, by thm. 3.13, $V \ge \underline{u}'$). But it may be the case that there is no such $\underline{u}'$: see example 3.21, where V is not in $\mathcal{F}$.*

In §3.8 we will compare this theorem to other results in the literature.

There remain an open question: how to better characterize this class $\mathcal{F}$? In the particular cases

- when M is compact and a *strict subsolution* $\underline{u}$ exists, then all continuous functions are in $\mathcal{F}$ (so this uniqueness theorem extends the result in [13]);
- if $H(x,0) < 0$, and $\liminf_{y\to\infty} S(y,x) = \infty$, then all continuous lower bounded functions are in $\mathcal{F}$. So, this theorem extends the uniqueness part in the Riemannian case thm. 1.2; indeed in that case $d^g \equiv S$, so the condition $\liminf_{y\to\infty} d(y,x) = \infty$ means that (M,g) is complete.

3.6 A remark on the hypothesis (∃$\underline{\mathbf{u}}$)

We comment on the hypothesis (∃$\underline{u}$) of existence of a smooth strict subsolution (see defn. 3.1).

For any choice of $c \in \mathbb{R}$, we consider the problem

$$H(x, Du(x)) = c \quad x \in M \quad . \tag{3.22}$$

We define the *critical value* $c(H)$ to be the minimum value $c \in \mathbb{R}$ for which the equation (3.22) admits global viscosity subsolutions.

The family of problems (3.22) have been studied in many papers; the results most relevant to our current interest are summarized in the two following theorems. The first theorem is Theorem 1.1 in [8].

Theorem 3.17 (Weak KAM) *Suppose that the manifold M is compact. Suppose that $H \in C^2$ and $H(x,\cdot)$ is **strongly convex**, that is, the Hessian matrix of H w.r.t. p is positive definite. Suppose that there is a complete Riemannian geometry on M such that $\forall k \in \mathbb{R}\ \exists a \in \mathbb{R}$*

$$H(x,p) \ge k|p| - a, \qquad \forall x, p \tag{3.23}$$

and $\forall r \in \mathbb{R}$

$$\sup\{H(x,p) \mid |p| \le r\} < \infty \ . \tag{3.24}$$

Then the Hamilton-Jacobi equation (3.22) admits a global viscosity solution $u : M \to \mathbb{R}$ for $c = c(H)$ and does not admit any such solution for $c < c(H)$.

A more complete discussion of history and applications of this result may be found in [8].

The second theorem is Theorem 1.3 in [9].

Theorem 3.18 *In the same hypotheses as the theorem above, when $c = c(H)$, there exists a C^1 subsolution u of (3.22), and moreover $H(x, Du(x)) < c(H)$ iff x is not in the projected Aubry set.*

As a corollary of the previous two theorems (recalling also the mollification arguments in Lemma 6.3 in [10]) we obtain that,

Corollary 3.19 *When $c < c(H)$, there cannot be any strict subsolution; when $c = c(H)$, there exists a C^1 strict smooth subsolutions iff the Aubry set is empty; when $c > c(H)$ strict smooth subsolutions do exist.*

Remark 3.20 *It may be interesting also to study whether the hypotheses (3.23) and (3.24) may be replaced by more intrinsic hypotheses: indeed they imply that the Finsler metric spaces associated to (3.22) when $c > c(H)$ are complete; so, maybe, it should be enough to assume directly that those Finsler metric spaces are backward complete, without resorting to an auxiliary Riemannian structure.*

To conclude this section, we simply remark that the problem (1.1) that started this paper is is obtained from (3.22) by choosing $c = 0$.

3.7 Examples

The other uniqueness results that we know of do not explicitly state a "completeness" hypothesis; [5] we now provide an example where this hypothesis is relevant.

Example 3.21 $M = \mathbb{R}^2$, $H(x,p) = |p|^2 - 1$, $K = \{x \mid x_1 = -x_2^2/4\}$, $u_0(x) = x_1$.

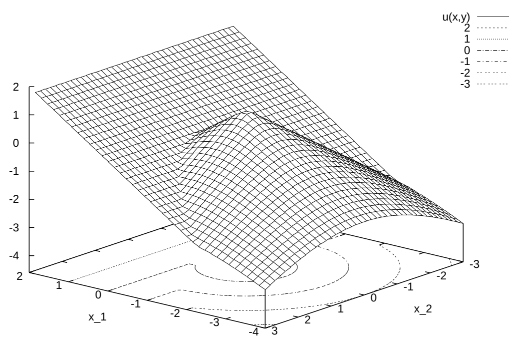

Figure 2: $u^{**}(x)$ in example 3.21

This example satisfies condition $(\exists \underline{u})$ *, and we may choose* $\underline{u}(x) = -\sqrt{|x|^2+1}$ *(as in eqn. (3.8) in [22]), but there is no strict subsolution* $\underline{u}$ *such that* $\underline{u} \leq u_0$ *on K, and which satisfies the completeness conditions. This problem has two viscosity solutions:*

$$u^*(x) \stackrel{def}{=} x_1$$
$$u^{**}(x) \stackrel{def}{=} \max\left\{x_1,\ \left(1 - \sqrt{{x_2}^2 + (x_1+1)^2}\right)\right\}$$

the second solution is the value function V, it is a patchwork of a cone and a plane, which intersect in the parabola $\{(x_1,x_2,x_3) \mid x_1 = x_3 = -{x_2}^2/4\}$. See fig. 2.

[5] sometimes the "backward completeness" is implicitly assumed: this is the case when the ambient space M is compact.

3.8 Comparison with previous results

We compare our approach to Hamilton-Jacobi problems with Siconolfi's [27].

We point out that [27] addresses also the case when Z_x is not convex: it then builds a solution of (1.1) by a sup-inf formula, which provides a viscosity solution of the convexified of (1.1). We do not address the case where Z_x is not convex in this paper.

In the case $M = \mathbb{R}^n$, Proposition 2.2 in [27] proves that

Theorem 3.22 *Suppose $M = \mathbb{R}^n$. Suppose that $H(x, 0) < 0$. Suppose that H is continuous and*

$$\liminf_{|p|\to\infty} H(x, p) > 0$$

and Z_x is convex and

$$\partial(\mathring{Z}_x) = \{p \mid H(x, p) = 0\} \quad \forall x$$

Assume the condition 2.9 in [27], namely there exist $a, b > 0$

$$H(x, p) < 0 \quad \forall x, p \text{ with } |p| < \frac{a}{|x| + b} \tag{3.25}$$

Then

$$u(x) \stackrel{def}{=} S(0, x)$$

is the unique viscosity solution of $H(x, du(x)) = 0$ (for $x \neq 0$) in the class of continuous functions $v : M \to \mathbb{R}$ such that

$$\liminf_{|x|\to\infty} v(x) = \infty \tag{3.26}$$

and $v(0) = 0$.

We now understand that (3.25) is a "completeness condition", since it implies that the space (M, b) (defined with $\underline{u} = 0$) is complete. Indeed the condition (3.25) is equivalent to: there exist $a, b > 0$

$$L(x, v) \geq \frac{a|v|}{|x| + b} \tag{3.27}$$

(that is the condition 1.7 in [27]).

In this paper we have shown that, indeed, a completeness assumption is fundamental to achieve uniqueness of solution; we have also remarked that, due to the asymmetry of the equation and of the metric, we may distinguish a backward and a forward completeness hypothesis, and that the correct one is the "backward completeness assumption". So our theorem 3.15 clearly generalizes the above theorem.

Conclusions

We obtained a family of results of existence,uniqueness and variationality that extend to the general Hamilton–Jacobi problem (1.6) the "purely metric part" of the results of Theorem 1.2 that addresses the Riemannian eikonal equation.

In stating the *Hopf–Rinow theorem for Hamilton–Jacobi* 3.11, we did not provide a condition that would be analogous to the *geodesic completeness* condition that is in the classical theorem 1.4. This result will be the argument of a forthcoming paper.

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