Regularity and Variationality of Solutions to Hamilton-Jacobi Equations. part II: variationality, existence, uniqueness

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Abstract

We formulate an Hamilton–Jacobi partial differential equation

$$H(x, Du(x)) = 0$$

on a n dimensional manifold M, with assumptions of convexity of the sets $\{p : H(x,p) \le 0\} \subset T^*_x M$, for all x.

In this paper we reduce the above problem to a simpler problem: this shows that u may be built using an asymmetric distance (this is a generalization of the "distance function" in Finsler Geometry): this brings forth a 'completeness' condition, and a Hopf–Rinow theorem adapted to Hamilton–Jacobi problems. The 'completeness' condition implies that u is the unique viscosity solution to the above problem.

When H is moreover of class $C^{1,1}$, we show how the completeness condition is equivalent to a condition expressed using the characteristics equations.

7 Introduction

In this article [1] we will study the Dirichlet Hamilton–Jacobi PDE

$$\begin{cases} H(x, Du(x)) = 0 & \text{in } M \setminus K \\ u(x) = u_0(x) & \text{when } x \in K. \end{cases} \tag{7.1}$$

where M is a borderless smooth manifold, $K \subset M$ is a closed subset, u_0 is a continuous real function on K, H is a continuous real function on T^*M such that

$$Z_x \doteq \{p \in T^*_x M \mid H(x,p) \le 0\}$$

is convex for all $x \in M$.

In the first part [26] we studied the regularity properties of a generalized solution u. The main aim of this second part is to prove results on the existence and uniqueness of the solution u.

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[1] This is a longer version of the paper, with all detailed proofs and more remarks and examples.

7.1 Problem environment

We have stated the problem (7.1) in a quite general environment: Hamilton–Jacobi equations commonly have $\mathbb{R}^n$ (with the usual norm $|\cdot|$) as the ambient space M. We simbolically represent this common setting as

$$((\mathbb{R}^n, |\cdot|),\ H, K, u_0) \tag{7.2}$$

It may be interesting to generalize the problem setting: for example, we may assume that the Euclidean space $(\mathbb{R}^n, |\cdot|)$ is substituted by a Riemannian manifold (M, g); we may then reasonably think that the class of problems

$$((M, g),\ H, K, u_0) \tag{7.3}$$

share all the properties and results of the common problems (7.2).

If we wish to further generalize the problem (7.3), we are faced by an obstacle: most results that are found in common literature regarding the problem (7.1) are stated using the distance and the distance–related properties of M; that is, they use some geometrical structure of the manifold M.

This comment is reversable: isn't it possible that most results are strongly influenced by the geometrical structure lying under them? that is, *what can be said of the problem (7.1), "as is"?*

Se we decided to look for general hypotheses on problem (7.1), to provide existence and uniqueness of viscosity solution. In the course of our study, while we where searching these hypotheses, we have found that the problem (7.1) itself imposes a *"geometric structure"* on the manifold M: [2] this structure takes the form of a Finsler asymmetric distance b; moreover when $H \in C^{1,1}$, the distance b is the generated by the metric of a Finsler Geometry (M, L) [3] where L is Legendre–Fenchel dual (see (12.25)) to $\hat{H} : T^*M \to \mathbb{R}$; whereas $\hat{H}$ is easily derived from the Hamiltonian H (see (12.3)).

The studied problem takes then the form

$$((M, b),\ H, K, u_0) \tag{7.4}$$

The (asymmetric) metric structure (M, b) is powerful enough to express hypotheses for uniqueness of viscosity solutions to (7.1). For example we will present in theorem 9.5 a result on uniqueness of viscosity solutions [4] to problem (7.1), which basically states that ≪given a strict subsolution $\underline{u}$, there exists an unique viscosity solution in the class of all continuous functions bigger than $\underline{u}$≫. We will then show in examples in §10 that we need a "completeness hypotheses" on the metric space (M, b) to state the above result.

7.2 Equivalent problems

We further discuss on the quality of hypotheses that are usally asked when studying problem (7.1).

In the equation (7.1) we may perform independent and dependent change of variables, as shown in §2 in [21], and in 12.3 in this paper. As an example of change of independent variable, we state this simple proposition

[2] and this is mainly due to the fact that we will assume that $\{p \mid H(x, p) \le 0\}$ is convex

[3] See 13.1 for a definition; but we anticipate that this Finsler geometry $L : TM \to \mathbb{R}$ is C^1 and strictly convex in the second variable: this hypotheses are weaker than what is commonly found in books on Finsler geometry, so we provide in appendix 13 the needed results.

[4] viscosity solutions are properly defined in 8.4

Proposition 7.5 *Consider the two problems (7.1) and*

$$\begin{cases} \tilde H(x,Du(x)) = 0 & \text{in } M\setminus K \\ u(x) = u_0(x) & \text{when } x\in K. \end{cases} \tag{7.5.$\star$}$$

if

$$\textit{sign}H(x,p) = \textit{sign}\tilde H(x,p) \quad \forall x,p\in T^*M$$

then the definition of viscosity solution immmediatly implies that (7.1) and (7.5.⋆) have the same viscosity solutions

This fact is trivially proved; at the same time, it is not widely exploited in the literature (with some exceptions; to cite some examples, [24], [5], [18]).

For example, consider a Riemannian manifold M; the *eikonal equation* [5]

$$\begin{cases} |du(x)|^2 - 1 = 0 & \text{on } M\setminus K \\ u = 0 & \text{on } K \end{cases} \tag{7.6}$$

has the same viscosity solutions of the problem

$$\begin{cases} (|du(x)|^4 - 1)^3 = 0 & \text{on } M\setminus K \\ u = 0 & \text{on } K \end{cases} \text{, or of } \begin{cases} 2^{|du(x)|} - 2 = 0 & \text{on } M\setminus K \\ u = 0 & \text{on } K \end{cases}$$

For this reason, in sec.12.1 we will reduce the problem (7.1) to a canonical problem.

Actually, we may generalize the definition of viscosity solution and simply require that u be a *viscosity solution by inclusion* (see 8.34) of

$$\begin{cases} (x,du(x))\in Z & \text{for } x\in M\setminus K \\ u = 0 & \text{on } K \end{cases}$$

where

$$Z \doteq \left\{(x,p) \mid H(x,p)\le 0\right\} \subset T^*M$$

This does not, though, provide a more general setting than (7.1). (See also 8.35).

Given the above behavior, we will seek results on the problem (7.1) that use hypotheses that are robust w.r.t a change to an equivalent problem; that is, *what can be said of the problem (7.1), "up to equivalences"?*

Remark 7.7 *Intuitively, the remark 7.5 should hold also for solutions defined using the method of characteristics (such as* min *solution, that was introduced in the first part); but the situation is slightly more complicated; indeed, if for example*

$$\tilde H(x,p) = H(x,p)^3$$

then the problem (7.5.⋆) is degenerate, $\tilde H(x,p) = 0 \Rightarrow \frac{\partial}{\partial p}\tilde H = \frac{\partial}{\partial x}\tilde H = 0$ *so that the characteristics (9.9) are constant in* t*: it is impossible to use them to define a solution.*

For this reason we will introduce a condition `(Hnd)`*: see equation (9.12.⋆), lemma 12.10 and 12.11.*

[5] for a more detailed description, see [21], or in §3.v in [26]

Remark 7.8 *The reader will also notice that there is no hypothesis, in this paper, about the behaviour of $H(x,p)$ for generical x,p* [6]*. The theorems in this paper depend on H only through the set Z, and the regularity of H in a neighbourhood of the zero set*

$$\{H=0\}\doteq\{(x,p) \mid H(x,p)=0\}$$

and indeed we remarked above that the fact that u be a viscosity solution to (7.1) depends only on the sign of H and not on H.

We may go as far as saying that, whenever we focus on the Hamilton-Jacobi problem (7.1),[7] *then* ***it is not a natural approach to the problem*** *to ask any hypothesis about H outside a neigbourhood of $\{H=0\}$.*

8 Prelims

We now properly define all notations and objects. We then reintroduce briefly some concepts from the first part [26] and from [23]. And we proceed to integrate this whole in a study of solutions to the Hamilton-Jacobi equation (7.1).

8.1 Notation

We fix some notations.

- M will be a connected boundaryless differentiable manifold of class C^∞ and of dimension n;
- K will be a closed subset of M
- H will be a real function defined on the cotangent bundle T^*M;
- and u_0 will be a continuous real function defined on K.

We will use the notation $p \cdot v$ to mean that a covector $p \in T^*_x M$ is applied to a vector $v \in T_x M$.

If $f : M \to \mathbb{R}$ is a regular function, we will write $df(x)$ or $Df(x)$ for its differential in the point x; if $g : \mathbb{R} \times M \to \mathbb{R}$ is a regular function, $g = g(t,x)$, we will write $\dot g$ for $\frac{\partial g}{\partial t}$ (and not g', which will be a different function).

We conclude the introduction with a remark on definitions

Definition 8.1 *Let $f : \Omega \to \mathbb{R}$ where $\Omega \subset \mathbb{R}^n$ is convex. We define that*

1. ***"f is strongly convex"*** *when $f \in C^2$ and the Hessian $D^2 f(x) = \frac{\partial^2 f}{\partial x x}(x)$ is positive definite* [8] *$\forall x$; whereas*
2. ***"f is strictly convex"*** *when*

$$f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda) f(y)$$

for all $x, y \in \Omega$, $0 < \lambda < 1$

We must warn the reader that some authors use different definitions (and call the first definition "strictly convex").[9]

[6] like, superlinearity of $p \mapsto H(x,p)$, see condition (57) in chapter 2 in [19]; or condition 3.2 in [31] (see (11.1.$\star$) here)

[7] and not on a variational problem like (1.12) or (1.13) of which (7.1) is the dual problem; or on a class of problems such as (9.18)

[8] that is, $\langle \alpha \cdot D^2 f(x) \alpha \rangle > 0$ for all $\alpha \in \mathbb{R}^n$, $\alpha \neq 0$

[9] This unfortunately happens also in some papers referenced from this paper.

8.2 Asymmetric metric spaces

We recall some concepts from [23].

Definition 8.2 *Let M be a generic set.*
$b : M \times M \to \mathbb{R}^+$ *is an* asymmetric distance[10] *if b satisfies*

- $b \geq 0$ *and* $b(x,y) = 0$ *iff* $x = y$
- $b(x,y) \leq b(x,z) + b(z,y)\ \forall x,y,z \in M.$

We call the pair (M,b) an asymmetric metric space.

We agree that b defines a topology τ on M, generated by the symmetric distance

$$d(x,y) \doteq b(x,y) \vee b(y,x) \tag{8.3}$$

This is the topology that we will always associate to (M,b).

We define that

Definition 8.4 *A sequence $(x_n) \subset M$ is called* "forward Cauchy" *if*

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } \forall n,m,\ \ m \geq n \geq N,\ \ b(x_n, x_m) < \varepsilon \tag{8.4.$\star$}$$

We say that (M,b) is "forward complete" *if any* forward Cauchy *sequence (x_n) converges to a point x.* [11]

The above definitions agree with those used in Finsler Geometry (as defined in ch. VI in [3]).

Remark 8.5 *For any* forward *definition above there is a corresponding* backward *definition, obtained by using the* conjugate distance $\bar{b}$ *defined by*

$$\bar{b}(x,y) = b(y,x) \quad .$$

We induce from b the length $\operatorname{len}^b \gamma$ of a continuous curve $\gamma : [\alpha,\beta] \to M$, by using the total variation

$$\operatorname{len}^b \gamma \doteq \sup_T \sum_{i=1}^n b\big(\gamma(t_{i-1}), \gamma(t_i)\big) \tag{8.6}$$

where the sup is carried out over all finite subsets $T = \{t_0, \cdots, t_n\}$ of $[\alpha,\beta]$ and $t_0 \leq \cdots \leq t_n$.

We define b^g

$$b^g(x,y) = \inf \operatorname{len}^b \gamma \tag{8.7}$$

where the inf is taken in the class of all Lipschitz curves γ connecting x to y. If the $\inf$ is a minimum, the curve providing the minimum is called a *geodesic*.

If the space (M,b) is Lipschitz-arcwise connected, then it is easily proved that b^g is an asymmetric distance.

We say that the (asymmetric) metric space (M,b) is *a path-metric space* if $b = b^g$. (Note that this property depends on the pair (M,b): if we choose $N \subset M$ and define the space $(N, b_{|N})$ by restricting b to $N \times N$, then we can only say that $(b^g)_{|N} \leq (b_{|N})^g$; but it may happen that (M,b) is path-metric while $(N, b_{|N})$ is not, or viceversa.)

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[10] b is also known as a *quasi metric*; the differences between these two theories are in the choice of the topology and of the definition of "Cauchy sequences" and "completeness", as is explained in sec. §2.v in [23].

[11] idem est, $x_n \to x$ according to the topology τ: cf. 2.3 and §2.v.1 in [23] on the notion of convergence

8.2.1 Tilting the distance

The realm of asymmetric distances admits a nice operation which is not allowed when dealing with symmetric metrics. Let b be an asymmetric distance on M, and let $\varphi : M \to \mathbb{R}$ be a function; we define that

$$\tilde{b}(x,y) = b(x,y) - \varphi(x) + \varphi(y) \tag{8.8}$$

is a *tilted version* of b by means of φ.

It is readily seen that $\tilde{b}$ satisfies the triangular inequality; if moreover $\tilde{b}(x,y) \ge 0$ (with equality only for $x = y$), then $\tilde{b}$ is an asymmetric distance; we have then proved

Proposition 8.9 *Suppose φ is strictly-1-Lipschitz with respect to b, that is,*

$$b(x,y) \ge \varphi(x) - \varphi(y) \quad \forall x,y, \text{ with equality only for } x = y \tag{8.9.$\star$}$$

then

$$\tilde{b}(x,y) \doteq b(x,y) - \varphi(x) + \varphi(y)$$

is an asymmetric distance.

Note that by (8.9.$\star$) follows that

$$d(x,y) \ge |\varphi(x) - \varphi(y)|, \text{ with equality only for } x = y \tag{8.10}$$

that is, φ must be strictly-1-Lipschitz w.r.t. to d : then φ must be continuous.

The tilting relation is invertible: b is a *tilted version* of $\tilde{b}$ by means of $-\varphi$: then φ must be strictly-1-Lipschitz w.r.t. to

$$\tilde{d}(x,y) \doteq \tilde{b}(x,y) \vee \tilde{b}(y,x) \tag{8.11}$$

as well.[12]

The tilting relation is also transitive: if $\tilde{b}$ is a *tilted version* of b by means of φ and $\tilde{\tilde{b}}$ is a *tilted version* of $\tilde{b}$ by means of $\tilde{\varphi}$, then $\tilde{\tilde{b}}$ is a *tilted version* of b by means of $\tilde{\varphi} + \varphi$.

As aforementioned, the *tilting operation* is not useful in the realm of symmetric distances: if both b and $\tilde{b}$ are symmetric distances, then φ must be constant, that is, $b = \tilde{b}$. It is instead possible to tilt a symmetric distance to obtain an asymmetric distance: for example, if b derives from a Riemannian geometry, then its tilted $\tilde{b}$ would be a *Randers metric*.[13]

Tilted metrics are equivalent in many respects

Theorem 8.12 *Let b and $\tilde{b}$ as in proposition above: then*

- *the topologies generated by d and $\tilde{d}$ (cf. (8.3) and (8.11)) are the same.*
- *Suppose that (M,b) is forward locally compact. Then the forward topology generated by b is finer than the forward topology generated by $\tilde{b}$.*
- *let $\xi : [0,1] \to M$: then*

  $$\operatorname{len}^{\tilde{b}} \xi = \operatorname{len}^{b} \xi - \varphi(\xi(0)) + \varphi(\xi(1))$$

  As a consequence, b and $\tilde{b}$ share the same minimal geodesics; and (M,b) is path–metric iff $(M,\tilde{b})$ is path–metric.

[12]Note that there is no easy algebraic relation relating d and $\tilde{d}$

[13]Randers metrics are a particular case of a Finsler metrics: they have the form $F(x,v) = \sqrt{\alpha_{i,j}(x)v^i v^j} + \beta_i(x)v^i$ where α is a Riemannian metric and β is a 1-form; see §1.3C in [3]; if β is exact then the distance corresponding to F is a tilted version of the Riemannian distance induced by α.

Proof.

- If $d(x,y) < r$ then
  $$\tilde d(x,y) \doteq \tilde b(x,y) \vee \tilde b(y,x) = (b(x,y) - \varphi(x) + \varphi(y)) \vee (b(y,x) - \varphi(y) + \varphi(x)) \le 2r$$
  by (8.10): so
  $$B(x,r) \doteq \{y \mid d(x,y) < r\} \subset \tilde B(x,2r) \doteq \{y \mid \tilde d(x,y) < 2r\}$$
- Let $\delta > 0$ be small so that $D \doteq \{y \mid b(x,y) \le \delta\}$ is compact. By lemma 2.16 in [23], there exists a continuous monotonically (weakly) increasing function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$, with $\omega(0) = 0$, such that
  $$\forall x, y \in D, \quad b(x,y) \le \omega(b(y,x))$$
  (ω is called a *modulus of symmetrization*).
  Then choose any $0 < \varepsilon < \delta$: if $b(x,y) < \varepsilon$
  $$\tilde b(x,y) = b(x,y) - \varphi(x) + \varphi(y) \le$$
  (by (8.9.$\star$))
  $$\le b(x,y) + b(y,x) \le \varepsilon + \omega(\varepsilon)$$
  so
  $$B^+(x,\varepsilon) \doteq \{y \mid b(x,y) < \varepsilon\} \subset \tilde B^+(x, \varepsilon + \omega(\varepsilon)) \doteq \{y \mid \tilde b(x,y) < \varepsilon + \omega(\varepsilon)\}$$
- by directly substituting (8.8) in (8.6)

□

But tilted metrics are not equivalent w.r.t. completeness

Example 8.13 *Consider $M = \mathbb{R}$ and $b(x,y) = |x-y|$: then (M,b) is complete. Let*

$$\varphi(x) \doteq \begin{cases} -x^2/(1+x) & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases}$$

φ satisfies (8.9.$\star$) since $-1 < \varphi' \le 0$. Define $\tilde b$ as before: $(M, \tilde b)$ is not forward complete since the sequence $x_n = n$ is forward Cauchy.

(The example 10.2 shows the same phenomenon.)

So we understand that the class of φ that preserve completeness is smaller than the class allowed by (8.9.$\star$).

Definition 8.14 (limit at infinity) *Given a $f : M \to \mathbb{R}$, we define*[14]

$$\liminf_{x\to\infty} f(x) \doteq \sup_{C \subset\subset M} \inf_{y \notin C} f(x)$$

This is equivalent to using Alexandroff compactification to add the point ∞ to M. We agree that, whenever M is compact, then $\liminf_{x\to\infty} f(x) = \infty$. We remark that $\liminf_{x\to\infty} f(x) = \infty$ if and only if there exists a sequence of compact sets K_n such that $M = \bigcup_n K_n$ and for any sequence $(x_n)_n$ with $x_n \notin K_n$ we have $\lim_n f(x_n) = \infty$

[14] $\liminf_{x\to\infty} f(x)$ is called "the liminf for x exiting all compact sets".

We can then for example prove that

Proposition 8.15 *Suppose that φ satisfies (8.9.$\star$), and moreover*

$$\forall x,\ \liminf_{y\to\infty} b(x,y)+\varphi(y)=\infty$$

or equivalently

$$\exists x,\ \liminf_{y\to\infty} b(x,y)+\varphi(y)=\infty$$

There follows

$$\liminf_{y\to\infty} \tilde b(x,y)=\infty$$

this is equivalent to asking that any forward disc

$$\tilde D^+(x,r)\doteq\{y \mid \tilde b(x,y)\le r\}$$

is compact, which in turn implies that $(M,\tilde b)$ is forward complete.

This proposition is generalized in 8.31. The proof is easy, it follows from the triangular inequality and the definition of the topology.

8.3 Finsler Metric

We would like to derive an asymmetric distance b from the problem (7.1). To this end we define

$$Z\doteq\{(x,p)\in T^*M \mid H(x,p)\le 0\} \tag{8.16}$$

We then define, for any $x\in M$, the *figuratrix* set Z_x by slicing Z along the fibers of T^*M:

$$Z_x\doteq Z\cap T^*_xM=\{p\in T^*_xM \mid (x,p)\in Z\}=\{p\in T^*_xM \mid H(x,p)\le 0\} \tag{8.17}$$

Z_x can be seen as a set-valued map $x\mapsto Z_x$ from M to the fibers of T^*M (see the discussion in §A.ii).

We will then actually induce an asymmetric metric F from Z_x, and a distance b from F.

8.3.1 Finsler Metric from Z

We always assume that Z is closed (then any slice Z_x is closed). As done in other papers (e.g. [5]), we define the *slicewise support function*

$$\sigma : TM\to[-\infty,\infty]$$

to be the support function of the set Z_x:

$$\sigma(x,v)\doteq\sup\left\{p\cdot v \mid p\in Z_x\right\} \tag{8.18}$$

Lemma 8.19 *Let $A=\mathring{Z}$ and let A_x be the slicing of A.*

Suppose that A_x is non-empty and $Z_x=\overline{A_x}$ for all x. Then σ is the support of A_x, namely

$$\sigma(x,v)=\sup\left\{p\cdot v \mid p\in A_x\right\} \tag{8.19.$\star$}$$

and σ is lower–semi–continuous.

Suppose moreover that Z_x is convex and compact for all x: then σ is continuous and locally bounded.

Proof. Since $Z_x = \overline{A_x}$ then (8.19.$\star$) holds.

We use the definition of Kuratowski convergence, and the results, exposited in appendix A. By A.13 we know that $x \mapsto A_x$ is l.s.c.: then by A.15 and (8.19.$\star$), σ is l.s.c.

If moreover every Z_x is convex and compact, then, by A.12, $x \mapsto Z_x$ is u.s.c.; since $\liminf_{x\to\overline{x}} Z_x \supset \liminf_{x\to\overline{x}} A_x \supset A_{\overline{x}} \neq \emptyset$ then by A.16 we know that σ is u.s.c. and locally bounded. □

We (almost always) assume in this paper that

Hypothesis 8.20 ((∃u̲)) *There exists a smooth function* $\underline{u}$ *on* M *such that*

$$d\underline{u}(x) \in A_x \ \ \forall x \ ; \qquad (8.20.\star)$$

or equivalently (see though 8.35)

$$H(x, d\underline{u}(x)) < 0$$

This $\underline{u}$ is called a *strict subsolution* [15] of problem (7.1). A discussion of this condition is in section 9.5.1.

Given a choice of a strict subsolution $\underline{u}$, we define then

$$F : TM \to [0, \infty]$$

so that $F(x, \cdot)$ is the support function of the set Z_x corrected by $d\underline{u}(x)$,

$$F(x, v) \doteq \sigma(x, v) - v \cdot d\underline{u}(x) = \qquad (8.21)$$

$$= \sup \left\{ p \cdot v \mid (p + d\underline{u}(x)) \in Z_x \right\} \qquad (8.22)$$

By the definition it is clear that $F \geq 0$, and $F(x, v) = 0$ iff $v = 0$: indeed, we know that $d\underline{u}(x) \in \mathring{Z}_x$ (and so $\mathring{Z}_x \neq \emptyset$).

We summarize all the above in a set of hypotheses on Z:

Hypotheses 8.23 *Suppose* Z *is closed. Let* $A = \mathring{Z}$ *and let* A_x *be the slicing of* A*: we suppose that, for all* x*,* Z_x *is nonempty, convex and compact and* $Z_x = \overline{A_x}$*.*

and in a theorem:

Theorem 8.24 *Assume 8.23. Choose a strict subsolution* $\underline{u}$*, and define* F*. Then* F *is a* Finsler metric, *satisfying*

- $F \geq 0$*, and* $F(x, v) = 0$ *iff* $v = 0$*,*
- F *is continuous and locally bounded.*

Note that the thesis above implies the hypotheses 3.2 in [23] (and then it implies all results there).

We conclude this section with two remarks on the condition $Z_x = \overline{A_x}$:

[15] note that we do need to speak of "strict subsolution in the viscosity sense", since such a solution may be mollified, as shown in Lemma 6.3 in [12]

Example 8.25 *Let $M = \mathbb{R}$ and*

$$Z_x \doteq \begin{cases} [-1,1] & \text{if } x \le 0 \\ [-1+1/x, 1+1/x] & \text{if } x > 0 \end{cases}$$

Then Z is closed, and any slice Z_x is convex and compact, but σ is not locally bounded and continuous.

Remark 8.26 *The condition $Z_x = \overline{A_x}$ implies that $Z = \overline{\mathring{Z}}$.*

In general, $A_x \subset \mathring{Z}_x$ and $\overline{A_x} \subset Z_x$. If $A_x \ne \emptyset$ and Z_x is convex [16] for all x, the conditions

$$\begin{array}{rl} (i) & Z_x = \overline{A_x} \\ (ii) & A_x = \mathring{Z}_x \\ (iii) & (\partial Z)_x = \partial(Z_x) \end{array}$$

are equivalent.

Proof. Being Z_x convex, if (ii) then $Z_x = \overline{\mathring{Z}_x} = \overline{A_x}$, so (i) holds.

Suppose (ii) is false, there is a $p \in \mathring{Z}_x, p \notin A_x$, (that is $(x,p) \notin \mathring{Z}$); this means that there exists a sequence $(x_n, p_n) \notin Z$ converging to (x,p). We work in local coordinates, identifying $T^*_{x_n}M$ to $\mathbb{R}^k$, $k = \dim M$. Since Z_{x_n} is convex, there are closed semispaces α_n containing Z_{x_n} but not p_n; up to a subsequence, α_n converges to a semispace α. We know by A.13 that $x \to A_x$ is l.s.c. in the Kuratowski sense, so that $A_x \subset \liminf A_{x_n}$: then $A_x \subset \alpha$; at the same time $p \notin \mathring{\alpha}$, but $p \in \mathring{Z}_x$ so there is a $q \in \mathring{Z}_x$ such that $q \notin \alpha \supset \overline{A_x}$, and this contradicts with (i).

We know that

$$(\partial Z)_x = (Z \setminus A)_x = Z_x \setminus A_x$$

so (ii) implies (iii); viceversa if (iii) then

$$Z_x = \partial(Z_x) \cup \mathring{Z}_x = (\partial Z)_x \cup \mathring{Z}_x = (Z_x \setminus A_x) \cup \mathring{Z}_x$$

since the union is disjoint, this implies (ii). □

8.3.2 Finsler distance from Z

We define the *length* $\operatorname{len}^L \gamma$ of a locally Lipschitz curve $\xi : [0,1] \to M$ as

$$\operatorname{len}^L \gamma = \int_0^1 F(\xi(s), \dot{\xi}(s))\, ds \tag{8.27}$$

As in §2.ix in [23], we then define the asymmetric distance

$$b(x,y) = \operatorname{len}^L \gamma \tag{8.28}$$

where the infimum is computed in the class of all locally Lipschitz ξ with given extrema $\xi(0) = x, \xi(1) = y$.

Under the hypotheses 8.23 above, b is an asymmetric distance. Further properties are listed in the proposition below.

b is itself also called a *Finsler metric*,[17] and is naturally associated to Hamilton-Jacobi equations; see [31] and references therein.

We recall 3.6 and 3.7 from [23]:

---

[16] if we drop the assumption that Z_x be convex, then it is easy to find examples where (i) holds but (ii) does not, and so on

[17] whereas in this paper we usually prefer to call F “the metric” and b “the distance”

Proposition 8.29 *Assume 8.23. Assume $\underline{u}$ of eq.(8.20.$\star$) exists. Define F and b.*

1. *The topology τ induced by d (cf. (8.3)) coincides with the topology of the manifold M*
2. *(M, b) is path-metric, that is, $b = b^g$*
3. *for any Lipschitz γ, $\operatorname{len}^L \gamma$ coincides with the $\operatorname{len}^b \gamma$ defined in (8.6).*
4. *For any $x \in M$ there exists a neighbourhood U where, $\forall y \in U$, there exists a geodesic γ connecting x to y with*

$$F(\xi(s), \dot{\xi}(s)) = 1 \quad \forall s. \tag{8.29.$\star$}$$

Remark 8.30 *We define (as in [12] and many other papers)*

$$S(x, y) = \inf \int_0^1 \sigma(\xi(s), \dot{\xi}(s)) ds \tag{8.30.$\star$}$$

where the infimum is computed in the class of all locally Lipschitz ξ with given extrema $\xi(0) = x, \xi(1) = y$. This quantity $S(x, y)$ does not depend on $\underline{u}$, but it may fail to be positive (and hence to be an asymmetric distance).

By (8.21),(8.27),(8.28),(8.30.$\star$) and direct calculation

$$S(x, y) = b(x, y) + \underline{u}(x) - \underline{u}(y)$$

that is, b is obtained from S by a tilting operation.

This means that the family of all possible asymmetric metrics that we may associate to (7.1) using strict subsolutions, are equivalent up to tilting.

Moreover (by reasoning as in 8.12) we see that there is Lipschitz curve that provides the minimum $S(x, y)$ in (8.30.$\star$), if and only if there is a geodesic in (M, b) connecting x to y.

We can then state this version of the *Hopf-Rinow* theorem that is adapted to Hamilton-Jacobi problems

Theorem 8.31 (Hopf-Rinow for Hamilton-Jacobi) *Assume all hypotheses 8.23 on Z. Choose a strict subsolution $\underline{u}$ (as in eq.(8.20.$\star$)), and define F and b with it, as above. Then the conditions 1-5 here following are equivalent.* [18]

1. *(M, b) is backward-complete,*
2. *backward bounded closed sets are compact,*
3. *backward closed balls are compact, that is,*

$$\forall r > 0, \quad D^-(x, r) \doteq \{y \mid b(y, x) \le r\} \text{ is compact} \tag{8.31.$\star$}$$

4.
$$\liminf_{y \to \infty} b(y, x) = \infty, \tag{8.31.$\star\star$}$$

[18] As remarked in 8.5, corresponding *forward* conditions may be obtained by using the conjugate distance $\overline{b}(x, y) = b(y, x)$ and the conjugate $\overline{S}(x, y) = S(y, x)$: these *forward* conditions would be equivalent as well.

5.

$$\liminf_{y\to\infty} S(y,x) + \underline{u}(y) = \infty \tag{8.31.$\diamond$}$$

We remark that: $\forall x$ (8.31.$\star\star$) iff $\exists x$ (8.31.$\star\star$); and similarly for (8.31.$\diamond$) and (8.31.$\star$).

If the above conditions hold, then, for any fixed $x, y \in M$, there is a Lipschitz geodesic ξ connecting them that minimizes the length $\operatorname{len}^L \xi$ defined in (8.27), and that satisfies (8.29.$\star$); this curve ξ is also a minimum for $S(x,y)$ in (8.30.$\star$).

Theorem 9.14 will add a sixth condition to the above, and Theorem 13.7 will add two more.

Proof. The equivalence $1 \iff 2$ of the first two statements, and the existence of geodesics, follows from the more general Hopf-Rinow theorems in 2.37 and 3.9 in [23]. The other equivalences are easy (and does not need any special hypotheses on (M,b)) they may be proven using the triangular inequality and the definition of the topology (as in 8.15). Moreover (8.31.$\diamond$) is just a rewriting of (8.31.$\star\star$). $\square$

8.4 Viscosity solutions

Now we briefly reintroduce the definition of viscosity solutions of PDE on manifolds. As in the standard case $M = \mathbb{R}^n$, we begin with the definition of the following generalized differentials. Let Ω be an open subset of M.

Definition 8.32 *Given a continuous function $u : \Omega \to \mathbb{R}$ and a point $x \in M$, the* superdifferential *of u at x is the subset of $T^*_x M$ defined by*

$$\partial^+ u(x) = \left\{ d\psi(x) \,|\, \psi \in C^1(M), \psi(x) - u(x) = \min_M(\psi - u) \right\}.$$

Similarly, the set

$$\partial^- u(x) = \left\{ d\psi(x) \,|\, \psi \in C^1(M), \psi(x) - u(x) = \max_M(\psi - u) \right\}$$

is called the subdifferential *of u at y.*

Definition 8.33 *We say that a continuous function u is a* viscosity solution *of equation*

$$H(x, Du(x)) = 0$$

in Ω if for every $x \in \Omega$,

$$\begin{cases} H(x,p) \le 0 & \forall p \in \partial^+ u(x), \\ H(x,p) \ge 0 & \forall p \in \partial^- u(x). \end{cases} \tag{8.33.$\star$}$$

If only the first condition is satisfied (resp. the second), u is called a viscosity subsolution *(resp. a* viscosity supersolution*).*

We equivalently define here the viscosity solutions to the Hamilton-Jacobi problem using the set Z:

Definition 8.34 (Viscosity solutions by inclusion) *Let $Z \subset T^*M$, define $Z_x = Z \cap T_x^*M$.*

We say that a continuous function u is a viscosity solution *of differential inclusion*

$$(x, Du(x)) \in Z \tag{8.34.$\star$}$$

in the open set $\Omega \subset M$ if for every $x \in \Omega$,

$$\begin{cases} p \in \overline{Z}_x & \forall p \in \partial^+ u(x), \\ p \notin \mathring{Z}_x & \forall p \in \partial^- u(x). \end{cases} \tag{8.34.$\star\star$}$$

If only the first condition is satisfied (resp. the second), u is called a viscosity subsolution *(resp. a* viscosity supersolution*).*

Remark 8.35 *Assume that $Z = \{H \le 0\}$ and Z is closed: then any solution of (8.34.$\star\star$) will be also a viscosity solution of (7.1), but not viceversa, as shown in this simple example: $u \equiv 0$ is a viscosity solution to $H(x, du(x)) = 0$ when*

$$H(x,p) \doteq \max\{0, (|p| - 1)\}$$

but 0 is not a solution to (8.34.$\star\star$), since

$$0 \in \mathring{Z}_x = \{p \mid |p| < 1\}$$

(contradicting the second condition in (8.34.$\star\star$)).

Obviously, the two problems coincide if we further assume that

$$\mathring{Z} = \{H < 0\} \tag{8.35.$\star$}$$

(or equivalently $\partial Z = \{H = 0\}$).

9 Solutions to Hamilton-Jacobi equations

9.1 Metric case

We assume in this section that

Hypotheses 9.1 *Let $Z \doteq \{H \le 0\}$; we assume 8.23 on Z. Supposing that a strict subsolution $\underline{u}$ exists (see eq.(8.20.$\star$)), we associate to the problem (7.1) a metric structure (M, b), as was explained in §8.3.1. We suppose moreover that $\partial Z = \{H = 0\}$.* [19]

We know that

Proposition 9.2 *Fix $a \in M$. The function*

$$u(x) \doteq S(a,x) = b(a,x) + \underline{u}(x) - \underline{u}(a) \tag{9.2.$\star$}$$

is a viscosity solution of $H(x, Du(x)) = 0$ if $x \ne a$, and a viscosity subsolution of $H(x, Du(x)) = 0$ for all x.

[19] otherwise in the following we would obtain solutions of (8.34.$\star$) but possibly not of (7.1).

the proof being standard [20].

We define the *value function*

$$V(x) \doteq \inf \Big(u_0(\xi(0)) + \int_0^1 \sigma(\xi(s), \dot{\xi}(s)) ds \Big) \tag{9.3}$$

where the infinimum is in the class of Lipschitz paths ξ with $\xi(1) = x$ and $\xi(0) \in K$. By combining all identities we can rewrite V in many different ways: for example,

$$\begin{aligned} V(x) &= \inf_{z \in K} (u_0(z) + S(z, x)) = \\ &= \underline{u}(x) + \inf_{z \in K} \big(u_0(z) - \underline{u}(z) + b(z, x) \big) \end{aligned}$$

This formula is the inf-convolution of the solution (9.2.$\star$): then this builds a solution to (7.1):

Theorem 9.4 *Assume 9.1. Then V is a viscosity solution to $H(x, du(x)) = 0$ on $M \setminus K$.*

If moreover

$$u_0(x) \le S(y, x) + u_0(y) \tag{9.4.$\star$}$$

for all $x, y \in K$, then $V = u_0$ on K: so V is the viscosity solution to (7.1).

Proof. Again, the proof is standard, and may be carried on in many different fashions; for example: V solves a “minimum time problem”

$$V(x) = \inf_{\gamma, \gamma(T) = x} (u_0(\gamma(0)) + T)$$

where the infimum is computed in the class of locally Lipschitz curves $\gamma : \mathbb{R}^+ \to M$ s.t. $\gamma(T) = x$ for a $T > 0$ and s.t. $F(\gamma(s), \dot{\gamma}(s)) \le 1$ for almost all s; then the proof follows from prop. 2.3 in ch. IV in [4]. □

If we add a *completeness hypothesis*, we get more interesting results

Theorem 9.5 *We add these hypotheses to to 9.1: we assume (9.4.$\star$) on u_0; we suppose there is a subsolution $\underline{u}$ (as in eq.(8.20.$\star$)) such that $\underline{u} < u_0$, and the (asymmetric) metric space (M, b) is backward-complete (or any other equivalent condition from 8.31).*

(uniqueness) *Let $\mathcal{A}$ be the class of all continuous functions $f : M \to \mathbb{R}$ such that $f = u_0$ on K and $\inf_M (f - \underline{u}) > -\infty$*

Then V is the unique viscosity solution to problem (7.1), in the class $\mathcal{A}$.

The proof follows from the comparison theorem 12.7.

(variationality) *For each x, the value problem (9.3) has a minimum, attained by a Lipschitz curve ξ such that $F(\xi, \dot{\xi})$ is (almost everywhere) constant.*

In this case we say that the problem (7.1) is variational, since it comes from a variational problem that admits minimum. We symbolize this fact by the symbol **($\exists \min$V)**.

The proof follows from 8.31.

[20] in the $\mathbb{R}^n$ case, see for example Theorem 2.1 in [31]; or see Prop.4.2 in [12]

We discuss other results in sec.11 and examples in sec.10.

We conjecture that some of our sufficient conditions are also necessary

Conjecture 9.6 *Let $\underline{u}$ be a strict subsolution, but suppose that (M, b) is not backward complete with this choice of $\underline{u}$. Then there is a choice of $K \subset M$ closed and $u_0 \geq \underline{u}$ such that the value problem V does not admit minimum.*

(the proof of this proposition should be simply a generalization of the idea in example 10.1).

Remark 9.7 (Conjugate problems) *If we define a problem conjugate to (7.1) by using the Hamiltonian $\overline{H}(x,p) \doteq H(x,-p)$, then we may restate all above theorems by using* forward *conditions, and using the conjugate distance $\overline{b}(x,y) = b(y,x)$ and the conjugate $\overline{S}(x,y) = S(y,x)$. (cf. 8.5)*

9.2 Characteristics' flow

We consider T^*M as a symplectic manifold: [21] we define the symplectic 2–form

$$\omega\Big((\dot{x},\dot{p}),\ (\dot{y},\dot{q})\Big) \doteq \sum_i \dot{q}_i \dot{x}_i - \sum_i \dot{p}_i \dot{y}_i$$

and the duality $\omega^\#$ between TT^*M and TTM, given by

$$\omega^\#(\nu) \cdot \nu' = \omega(\nu, \nu') \quad \forall \nu' \in T^*M \quad . \tag{9.8}$$

Suppose that H is of class $C^{1,1}$ in a neighborhood of $\{H = 0\}$.
We define the *characteristics' flow*

$$(X(\cdot, z, q), P(\cdot, z, q))$$

as the solution of the system of ordinary differential equations

$$\begin{cases} \dot{X}(s) = \frac{\partial H}{\partial p}(X(s), P(s)) \\ X(0) = z \\ \dot{P}(s) = -\frac{\partial H}{\partial x}(X(s), P(s)) \\ P(0) = q \end{cases} \tag{9.9}$$

and we define U by

$$\Big\{ \dot{U}(s) = P(s) \cdot \frac{\partial H}{\partial p}(X(s), P(s)), \qquad U(0) = 0$$

that is, (X, P) is the *Hamiltonian flow* for the symplectic manifold for (T^*M, ω).
If $H(z, q) = 0$, then $H(X, P) = 0$ for all times, so the above O.D.E. is well defined: the solution (X, P) exists and is unique for small times.
We define the *maximal times* of (9.9) to be

$$t^+(z,q) \doteq \sup\{t > 0 \quad \text{such that the characteristic } X(t,z,q), P(t,z,q) \text{ exist up to time } t\} \tag{9.10}$$

and

$$t^-(z,q) \doteq \inf\{t < 0 \quad \text{such that the characteristic } X(t,z,q), P(t,z,q) \text{ exist down to time } t\} \tag{9.11}$$

note that t^+ is l.s.c, t^- is u.s.c.

[21] More details are in appendix in the preprint version [25] of [26].

9.3 Weak Finsler case

We assume 8.23, and that there exists a subsolution $\underline{u}$. We assume that

Hypothesis 9.12 (`(Hnd)`) *$H \in C^{1,1}$ in a neighbourhood of $\{H = 0\}$, and*

$$\forall (x,p) \in T^*M, \;\; H(x,p) = 0 \Longrightarrow \frac{\partial}{\partial p} H(x,p) \neq 0 \qquad (9.12.\star)$$

In this case, the characteristic curves (9.9) are *non degenerate* (as explained in 7.7; see also 12.20).

The hypothesis `(Hnd)` (on p. 16) implies some regularity on the set Z (see 12.19); but `(Hnd)` does not imply that Z_x be convex and compact, so we will still need some of the hypotheses 8.23.

We will show in sec.13.1 that if we assume `(Hnd)` then the problem (7.1) induces a *weaker Finsler Geometry* (M, L). We postpone the precise definition of what (M, L) is to section §12.5 and appendix 13.2, to concentrate on Hamilton–Jacobi theory.

We moreover propose these conditions

`(MCu)`,`(MCu+)`,`(MCu-)` Suppose `(Hnd)`. The condition `(MCu+)` states that there exists a strict subsolution $\underline{u}$ such that

$$\lim_{t \to t^+(z,q)} U(t,z,q) - \underline{u}(X(t,z,q)) = \infty \qquad (9.13)$$

while `(MCu-)` holds when

$$\lim_{t \to t^-(z,q)} U(t,z,q) - \underline{u}(X(t,z,q)) = -\infty$$

for all (z, q) such that for $H(z, q) = 0$; if both conditions hold for the same $\underline{u}$, we say that `(MCu)` holds.

Note that the conditions `(MCu)`,`(MCu+)`,`(MCu-)` are robust wrt a change of dependent and independent variable, as requested in the introduction (see §12.3 for details).

We refer to the above `(MCu)`,`(MCu+)`,`(MCu-)` as *bilateral/forward/backward subsolution completeness*: the reason is in this thm (that adds a sixth condition to 8.31):

Theorem 9.14 *We assume 8.23, and `(Hnd)`. We suppose that Z_x is strictly convex, for all x.*

For any choice of a strict subsolution $\underline{u}$ (see (8.20.⋆)), we define the metric space (M, b): then

1. *the metric space (M, b) is backward-complete (resp. forward), if and only*

6. *the condition `(MCu-)` (resp. `(MCu+)`) holds.* [22].

The proof is a consequence of 12.24, of 12.28 and of the Hopf-Rinow thm.13.7.

[22]Note also that, if H is $C^{1,1}$, then stating each one of `(MCu+)` and `(MCu-)` implicitely implies (9.12.⋆)

9.3.1 Relation to [26].

We use 9.5 to reconnect to the theory developed in [26].

For convenience of the reader, we redefine some quantities from [26].

Suppose `(u`$_0$`K1)` holds, namely

`(u`$_0$`K1)` K is a C^1-regular closed embedded submanifold of M of dimension k with $0 \le k \le \dim(M) - 1$, and u_0 is a C^1 real function defined on K. (from 3.2).

We define the set O as

$$\begin{aligned} O \doteq & TK^{\perp u_0} \cap \{H = 0\} = \\ = & \{(x,p) \in T^*M \mid \ x \in K, \quad H(x,p) = 0, \quad \forall v \in T_x K, p \cdot v = du_0(x) \cdot v\} \end{aligned}$$

(from (3.4)).

We assume that O is a submanifold of T^*M (see 3.13).

We define the *reachable set*

$$\Omega \doteq \{x \in M \mid x = X(s,z,q) \text{ for } s \ge 0, (z,q) \in O\}$$

(from (3.5)).

We define the *min* solution $u : \Omega \to \mathbb{R}$ by using the method of characteristics

$$u(x) \doteq \inf_{\begin{cases} t \ge 0, \ (z,q) \in O \\ \text{s.t. } X(t,z,q) = x \end{cases}} U(t,z,q) + u_0(z) \tag{9.15}$$

(from (3.6)).

Proposition 9.16 *Suppose 8.23 and Z_x is strictly convex; suppose there exists a strict subsolution such that $\underline{u} < u_0$ and* `(MC`$\underline{\texttt{u}}$`-)` *holds (or any other equivalent condition from 8.31 and 9.14). Then*

- *the reachable set Ω covers all M; that is, given $x \in M$, there always exists a characteristic X such that $x = X(t,z,q)$ and $(z,q) \in O$*
- *the value function V coincides with the* min *solution u*
- *this condition (from (3.11.⋆)) holds:*

  `(OXUp)` if C is a compact subset of M, $a \in \mathbb{R}$, then

  $$\{(t,y) \in \mathbb{R}^+ \times O \mid \ X(t,y) \in C, \ (U(t,y) + u_0(z)) \le a\} \text{ is compact.} \tag{9.16.$\star$}$$

  Then the infimum in (9.15) is a minimum.

Proof. By 9.5, let z be the minimum for $V(x)$ in (9.3) and ξ^* be the minimum curve for $S(z,x)$; from 13.18, we know that ξ is a minimum of the energy $E(\xi)$ defined in (13.4.⋆), and then $b(z,x) = \sqrt{E(\xi^*)}$.

From standard results of calculus of variation, the optimal curve ξ^* must satisfy a constraint: if $\xi(t,\varepsilon)$ is any deformation of ξ^* having $\xi(t,0) = \xi^*(t)$ and $\xi(0,\varepsilon) \in K, \xi(1,\varepsilon) = x$, we must have

$$\frac{d}{d\varepsilon}\big(2\sqrt{E(\xi(\cdot,\varepsilon))} + u_0(\xi(0,\varepsilon))\big)|_{\varepsilon=0} = 0$$

which implies

$$\frac{\int_0^1 \frac{\partial L}{\partial x}\frac{d}{d\varepsilon}\xi(t,0)+\frac{\partial L}{\partial v}\frac{d}{d\varepsilon}\dot{\xi}(t,0)\,dt}{\sqrt{E(\xi(\cdot,\varepsilon))}}+du_0(z)\frac{d}{d\varepsilon}\xi(0,0)=0$$

$$0=\int_0^1 \frac{\partial L}{\partial x}\frac{d}{d\varepsilon}\xi(t,0)-\frac{d}{dt}\frac{\partial L}{\partial v}\frac{d}{d\varepsilon}\xi(t,0)\,dt-$$

$$-\frac{\partial L}{\partial v}(z,\dot{\xi}(0,0))\frac{d}{d\varepsilon}\xi(0,0)+\sqrt{E(\xi(\cdot,\varepsilon))}du_0(z)\frac{d}{d\varepsilon}\xi(0,0)$$

which by the Euler equation becomes

$$-\frac{\partial L}{\partial v}(z,\dot{\xi}(0,0))\frac{d}{d\varepsilon}\xi(0,0)+du_0(z)\cdot\left(\frac{d}{d\varepsilon}\xi(0,0)\right)\sqrt{E(\xi(\cdot,0))}=0$$

which, by arbitrariness of $\frac{d}{d\varepsilon}\xi(0,0)\in T_zK$, implies

$$p^*|_{T_zK}=du_0(z)\sqrt{H(x,p^*)}$$

(where $p^*=\frac{\partial L}{\partial v}(z,\dot{\xi}(0,0))$), and then, for $p=p^*/\sqrt{H(x,p^*)}$,

$$p|_{T_zK}=du_0(z)$$

We have then that $(z,p)\in O$. By the reasonings in 12.23 on the reparameterization of the geodesics, we have proved that there exists $(t,x,p)\in\mathbb{R}^+\times O$ such that $x=X(t,z,p)$. □

9.4 Finsler case

Assume (Hnd) and moreover that

(H2) $H\in C^2$ and H is strongly convex in the p variable (cf. 8.1) in a neighbourhood of $\{H=0\}$,

and assume 8.23: then Z_x is always strongly convex, and ∂Z_x is a C^2 submanifold of T_x^*M; note that this implies most of the hypotheses in 8.23.

Assume moreover that a strict subsolution $\underline{u}$ does exist: in these hypotheses

Theorem 9.17 *The geometry (M,L) defined in sec.13.1 is a regular Finsler Geometry, and*

1. *V is locally semiconcave in $M\setminus K$ (as defined in 2.3 in [26]), and $\Sigma=\{x\mid\not\exists du\}$*
2. *the closure of Σ coincides with $\Sigma\cup\Gamma$;*

where the sets Σ and Γ are defined in 4.1 and 4.2 in part I.

Proof. 1. the proof uses the theorem 5.3 in [19], by expressing the PDE (12.1) in local coordinates (it is the same as the proof of prop. 8 in [21]).

2. it is enough to prove that there are no conjugate points $x\in\Gamma$ outside of $\overline{\Sigma}$. If B is any open set containing no points of Σ, V is C^1 in B, by the semiconcavity of V (see prop. 5,9 and 10 in [21]), and the flow is a local homeomorphism. If there would be a neighbourhood B of $x\in\Gamma$ containing no points of Σ, then,

then there would a point $y \in B$ and an optimal geodesic ξ for $V(y)$ that would pass through x; the initial segment of this geodesic would be the optimal curve for $V(x)$: but this is impossible, since a geodesic ceases to be minimal after it meets a conjugate point; see the first statement in prop. 14 in [21] for details.

□

"Semiconcavity of V" implies some interesting properties of Σ, like:

- the set Σ may be covered by a countable number of C^2 manifolds (see [1])
- moreover the singularities of du (i.e. the set Σ) propagate in a way that is related to the shape of $\partial^+ u(x)$. See [2].

9.5 Some remarks

9.5.1 on the hypothesis $(\exists \underline{u})$

We comment on the hypothesis $(\exists \underline{u})$ of existence of a strict subsolution (see (8.20.$\star$)).

For any choice of $c \in \mathbb{R}$, consider the problem

$$\{H(x, du(x)) = c \quad x \in M \tag{9.18}$$

These family of problems has been studied in many papers; the result most relevant to our current interest is

Theorem 9.19 (Weak KAM) *Suppose that $H \in C^2$ and $H(x,\cdot)$ is strongly convex (cf. 8.1); suppose that there is a reference Riemannian Geometry such that $\forall k \in \mathbb{R}$ $\exists a \in \mathbb{R}$*

$$H(x,p) \geq k|p| - a, \quad \forall x, p \tag{9.19.$\star$}$$

and $\forall r \in \mathbb{R}$

$$\sup\{H(x,p) \mid |p| \leq r\} < \infty \tag{9.19.$\star\star$}$$

Then there exists $c(H) \in \mathbb{R}$, such that the Hamilton-Jacobi equation above admits a global viscosity solution $u : M \to \mathbb{R}$ for $c \geq c(H)$ and does not admit any such solution for $c < c(H)$. $c(H)$ is called the critical value.

In the case where M is the n-dimensional torus T^n, this theorem is due to P.L. Lions, G. Papanicolaou & S.R.S. Varadhan [28]; for M an arbitrary compact connected manifold it is due to A. Fathi [10]. A more complete discussion of history and applications of this result may be found in [11].

Remark 9.20 *It may be interesting also to study whether the hypotheses (9.19.$\star$) and (9.19.$\star\star$) may be replaced by more intrinsic hypotheses: indeed they imply that the Finsler metric spaces associated to (9.18) when $c > c[h]$ are complete; so, maybe, it should be enough to assume directly that those Finsler metric spaces are backward complete, without resorting to an auxiliary Riemannian structure.*

More recently, in the case where M is the n-dimensional torus T^n, some results on the regularity of critical solutions have appeared in [12]

Theorem 9.21 (6.2 in [12]) *Suppose $M = T^n$. Suppose that H is continuous, and for all $x \in T^n$, $a \in \mathbb{R}$,*

- *$\{p \mid H(x,p) \leq a\}$ is convex*

- $$\liminf_{p\to\infty} H(x,p) = \infty$$

  uniformly in x (see also 8.14 and (11.1.$\star$))

- $\partial\{p \mid H(x,p) \le a\} = \{p \mid H(x,p) = a\}$ [23]

Let $\mathcal{A}$ be the generalized projected Aubry set.
There exists a C^1 critical subsolution, which is strict in every open subset whose closure is disjoint from $\mathcal{A}$.

In view of these theories, we understand that our assumption $(\exists\underline{\mathtt{u}})$ is roughly saying that for our problem (7.1) [24] the critical value $c(H) < 0$ is negative.

Note that the hypotheses in above theorems are quite stringent; since our problem (7.1) is obtained from (9.18) fixing $c = 0$, we are asking hypotheses (such as 8.23) only on Z, or on H in a neighbourhood of $\{H = 0\}$: this allows for much generalized hypotheses.

9.5.2 . . . on completeness, and the time parameter:

In the case of the eikonal problem (7.6) on Riemannian manifolds, if we set $\underline{u} \equiv 0$, since $U(t, z, q) = 2t$ for geodesics, we obtain that the condition $(\mathtt{MC}\underline{\mathtt{u}})$ defined in 9.3 is exactly saying that the Riemannian space M is (geodesically) complete.

Indeed, it was shown in thm 3.1 in [21] that

Theorem 9.22 *problem (7.6) has an unique viscosity solution u in the class of continuous functions that are bounded from below, and $u(x) = d(x, K)$ where $d(x, K)$ is the Riemannian distance from x to K*

all this under the assumption that the space M be complete.

In the more general case that we are studying here, the completeness hypothesis is substituted by the hypothesis $(\mathtt{MC}\underline{\mathtt{u}}\mathtt{-})$ (in the weak Finsler case); this suggests that

$$U(t, z, q) - \underline{u}(X(t, z, q))$$

is a *natural choice of time parameter* for the characteristics (a sort of "arc parameter").

9.5.3 Boundary value problems

Let Ω be an open subset of a differentiable manifold. Consider the problem

$$\begin{cases} H(x, Du(x)) = 0 & \text{in } \Omega \\ u(x) = u_0(x) & \text{when } x \in \partial\Omega. \end{cases} \tag{9.23}$$

The theory in this paper does not directly include boundary value problems. To include it, we may set $M = \overline{\Omega}$, $K = \partial\Omega$ and consider M to be a manifold with boundary. This needs many adjustments in the hypotheses.

---

[23] as noted in 8.26, this last hypothesis is equivalent to the hypothesis $\{p \mid H(x,p) \le a\} = \overline{\{p \mid H(x,p) < a\}}$; this is as asking $Z_x = \overline{A_x}$ (that was required in 8.23) for all problems $H - a = 0$.

[24] or maybe for the equivalent problem (12.1)

10 Examples

The other uniqueness results that we know of do not explicitely state a "completeness" hypothesis; [25] we now provide some examples where this hypothesis is relevant.

Example 10.1 *If we state the simple problem $M = (-1,1) \subset \mathbb{R}$, $K = \{0\}$, $u_0(0) = 0$, $H(x,p) = |p|^2 - 1$, then [26] we are giving to M the usual euclidean structure $\hat{H} = |\cdot|^2$; we readily see that this problem has two viscosity solutions bigger than $\underline{u} \equiv -2$: but this problem violates the hypothesis on completeness.*

Another (graphically much nicer) example is:

Example 10.2 $M = \mathbb{R}^2$, $H(x,p) = |p|^2 - 1$, $K = \{x \mid x_1 = -x_2^2/4\}$, $u_0(x) = x_1$.

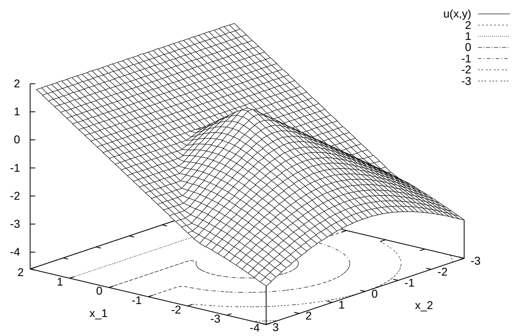

Figure 1: $u^{**}(x)$ in example 10.2

This example satisfies condition (∃u̲), *and we may choose $\underline{u}(x) = -\sqrt{|x|^2+1}$ (as in (3.18) in [26]), but there is no strict subsolution $\underline{u}$ such that $\underline{u} \le u_0$ on K, and which satisfies* (MCu̲-). *This problem has two viscosity solutions:*

$$u^*(x) \doteq x_1$$
$$u^{**}(x) \doteq \max\left\{x_1,\ \left(1 - \sqrt{{x_2}^2 + (x_1+1)^2}\right)\right\}$$

the second solution is a patchwork of a cone and a plane, which intersect in the parabola $\{(x_1,x_2,x_3) \mid x_1 = x_3 = -{x_2}^2/4\}$. See fig. 1.

10.1 Asymmetric eikonal example

Let $M = \mathbb{R}$, let

$$\hat{H}(x,p) = \begin{cases} p^2 e^{-2x} & \text{if } p \ge 0 \\ p^2 e^{2x} & \text{if } p < 0 \end{cases}$$

[25] sometimes the "backward completeness" is implicitely assumed: this is the case when the ambient space M is compact

[26] note that we assumed that M must be a borderless manifold: then, if $M \subset \mathbb{R}$, M must be an open subset

We formulate the Hamilton-Jacobi eikonal problem

$$\begin{cases} \hat{H}(x, Du(x)) - 1 = 0 & \text{for } x \neq 0 \\ u(0) = 1 \end{cases} \tag{10.3}$$

The above has two regular solutions,

$$u_1(x) = e^x, \quad u_2(x) = e^{-x}$$

We set $\underline{u} = 0$.

We can easily solve for the characteristic functions (9.9) satisfying $\hat{H}(X, P) = 1$: for the case $P(0) = q > 0$ we define $z = \log q, X(0) = z$

$$\dot{X} = 2Pe^{-2X}, \ \ \dot{P} = 2P^2 e^{-2X}, \ \ \dot{U} = 2P^2 e^{-2X}$$

If we choose $\hat{H}(z, q) = 1$ then $\hat{H}(X, P) = 1$ for all t, and these become

$$\dot{X} = 2P^{-1}, \ \ \dot{P} = 2, \ \ \dot{U} = 2$$

and since $du(X(t)) = P(t) = e^{X(t)} = P(0) + 2t$, we obtain

$$X(t) = \log(q + 2t), \ \ P(t) = q + 2t, \ \ U(t) = 2t$$

that is defined for $t \in (t^-, t^+)$ with $t^- = -q/2$, $t^+ = \infty$. Similarly if $P(0) = q < 0$, we define $z = -\log(-q)$, $X(0) = z$, then

$$X(t) = -\log(-q + 2t), \ \ P(t) = q - 2t, \ \ U(t) = 2t$$

that is defined for $t \in (t^-, t^+)$ with $t^- = q/2$, $t^+ = \infty$.

Coupling them with the initial conditions in (10.3), they become

$$X(t) = \log(1 + 2t), \ \ P(t) = 1 + 2t, \ \ U(t) = 1 + 2t$$

and

$$X(t) = -\log(1 + 2t), \ \ P(t) = -1 - 2t, \ \ U(t) = 1 + 2t$$

Let $\hat{L}$ be a Lagrange dual to $\hat{H}$, as per (12.25):

$$\hat{L}(x, p) = \begin{cases} \frac{1}{4} v^2 e^{2x} & \text{if } v \geq 0 \\ \frac{1}{4} v^2 e^{-2x} & \text{if } v < 0 \end{cases}$$

and $(\mathbb{R}, \hat{L})$ is a Finsler Geometry [27]. We then define the Finsler distance $b(x, y)$ as in (8.28); in this specific case,

$$2b(x, y) = \begin{cases} e^y - e^x & \text{if } y > x \\ e^{-y} - e^{-x} & \text{if } y < x \end{cases} \tag{10.4}$$

From the above equations and theorem 9.14, we derive that the space $(\mathbb{R}, \hat{L})$ is forward complete, but not backward complete; and indeed there are 4 viscosity solutions that are bounded from below, namely

$$u_1, \quad u_2, \quad u_1 \wedge u_2, \quad u_1 \vee u_2$$

[27] according to our definition in sec.13.1

We reformulate (10.3) with different conditions

$$\begin{cases} \hat{H}(x, Du(x)) - 1 = 0 & \text{for } x \notin \mathbb{N} \\ u(x) = u_0(x) \doteq 2e^{-x} & \text{for } x \in \mathbb{N} \end{cases} \tag{10.5}$$

In this case, the problem

$$V(x) = \inf_{z \in \mathbb{N}} u_0(z) + 2b(z, x)$$

has no minimum: indeed

$$V(0) = \inf_{y \in \mathbb{N}} e^{-y} + 1$$

11 Comparison with previous results

11.1 Comparison with [31]

We compare our approach to Hamilton-Jacobi problems with this result, found as Proposition 3.1 in Siconolfi's [31]

Proposition 11.1 *Let $M = \mathbb{R}^n$.* [28] *Suppose that $H(x, 0) < 0$.* [29] *Suppose H is continuous, and*

$$\partial(\mathring{Z}_x) = \{p \mid H(x, p) = 0\} \quad \forall x$$

and

$$\liminf_{|p| \to \infty} H(x, p) > 0 \tag{11.1.$\star$}$$

locally uniformly in x.

Then Z_x is compact (uniformly w.r.t x), and F is locally bounded (by (11.1.$\star$)). Moreover the maps $x \mapsto Z_x$ and $x \mapsto \partial Z_x$ are Hausdorff–continuous. Consequently F is continuous. [30]

Following the discussion in 7.8, we state that we prefer to avoid an hypotheses as (11.1.$\star$), which is not local around $\{H = 0\}$; the above theorem may be compared then to theorem 9.4.

We point out that [31] addresses also the case when Z_x is not convex: it then builds a solution of (7.1) by a sup-inf formula, which provides a viscosity solution of the convexified of (7.1). We do not address the case where Z_x is not convex in this paper.

Again, in the case $M = \mathbb{R}^n$, Proposition 2.2 in [31] proves that

Theorem 11.2 *Suppose $M = \mathbb{R}^n$. Suppose that $H(x, 0) < 0$. Suppose that H is continuous and (11.1.$\star$) and Z_x is convex and*

$$\partial(\mathring{Z}_x) = \{p \mid H(x, p) = 0\} \ \forall x$$

Assume the condition 2.9 in [31], namely there exist $a, b > 0$

$$H(x, p) < 0 \ \forall x, p \text{ with } |p| < \frac{a}{|x| + b} \tag{11.2.$\star$}$$

[28] The original proposition assumes $M = \mathbb{R}^n$; but the adaptation to having M a manifold would be straightforward.

[29] We may generalize the condition $H(x, 0) < 0$ to (8.20.$\star$), by using 12.15

[30] Since Z_x is compact (uniformly w.r.t x) then $x \mapsto Z_x$ is Hausdorff–continuous iff it is Kuratowski–continuous.

Then

$$u(x) \doteq S(a,x)$$

is the unique viscosity solution of $H(x, du(x)) = 0$ *(for* $x \neq 0$*) in the class of continuous functions* $v : M \to \mathbb{R}$ *such that*

$$\liminf_{|x|\to\infty} v(x) = \infty \qquad (11.2.\star\star)$$

and $v(0) = 0$.

We now understand that (11.2.$\star$) is a "completeness condition", since it implies that the space (M, b) (defined with $\underline{u} = 0$) is complete. Indeed the condition (11.2.$\star$) is equivalent to: there exist $a, b > 0$

$$L(x,v) \geq \frac{a|v|}{|x| + b} \qquad (11.3)$$

(that is the condition 1.7 in [31]).

In this paper we have shown that, indeed, a completeness assumption is fundamental to achieve uniqueness of solution; we have also remarked that, due to the asymmetry of the equation and of the metric, we may distinguish a backward and a forward completeness hypothesis, and that the correct one is the "backward completeness assumption". So our theorem 9.5 clearly generalizes the above theorem.

11.2 Comparison with [24]

We show how this paper naturally follows and generalizes the paper [24]. We assume `(Hnd)` and 8.23, and that Z_x is strictly convex for all x.

Consider the conditions that were introduced in [24]:

`(MC)` *((M, H) is characteristically complete)* we can solve (9.9) for all times, that is, we have a flow $(X, P) : \mathbb{R} \times T^*M \to T^*M$ whenever the initial conditions $X(0) = z, P(0) = q$ satisfy $H(z, q) = 0$.

`(Gu̲)` there exists a strict subsolution $\underline{u}$ and constants $c' > c > 0$ such that

$$c' \geq \frac{d}{dt}U - \frac{d}{dt}(\underline{u}(X)) \geq c$$

for any characteristic (X, P, U) (such that $H(X, P) = 0$), that is,

$$c' \geq (p - d\underline{u}(x)) \cdot \frac{\partial}{\partial p} H(x,p) \geq c$$

for all $(x, p) \in T^*M$ such that $H(x, p) = 0$.

These conditions were used in [24], were it was proven that they imply existence and uniqueness of the viscosity solution.

We show a sequence of simple implications that involve these conditions.

Proposition 11.4
- *if M is compact, then the maximal times (defined in (9.10) and (9.11)) are $t^+ \equiv \infty$, $t^- \equiv -\infty$, and this is equivalent to saying that* `(MC)` *holds;*

suppose that Z_x is strictly convex; take any $x \in M$, and p in the border of Z_x, that is, $H(x,p) = 0$; let $T_p Z_x \subset T^*_x M$ be the hyperplane tangent to Z_x in the point p: this hyperplane intersects Z_x only in the boundary, and it is perpendicular to $\frac{\partial}{\partial p} H(x,p)$ (see fig. 2); whereas the point $(x, d\underline{u}(x))$ is contained in the internal part of Z_x; then, we obtain that

$$(p - d\underline{u}(x)) \cdot \frac{\partial}{\partial p} H(x,p) > 0 \tag{11.4.$\star$}$$

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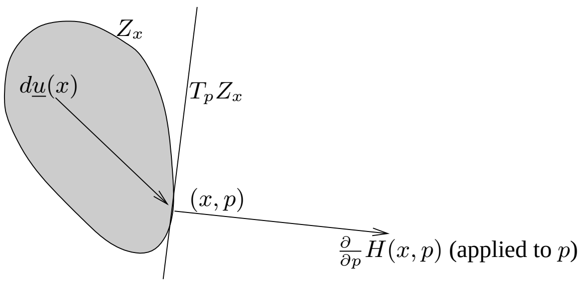


Figure 2: condition (G$\underline{u}$)

The above reasoning shows that, if M is compact and if Z_x is strictly convex and C^1, we obtain that any strict subsolution $\underline{u}$ satisfies (G$\underline{u}$)

- *the conditions* (G$\underline{u}$,MC) *readily imply* (MC$\underline{u}$)*: indeed,* (MC) *says that $t^+ = \infty, t^- = -\infty$, and* (G$\underline{u}$) *implies,*

$$c' t \geq U(t,z,q) - \underline{u}(X(t,z,q)) - \big(U(0,z,q) - \underline{u}(X(0,z,q))\big) =$$
$$= \int_0^t (P(t,z,q) - d\underline{u}(X(t,z,q))) \cdot \frac{\partial}{\partial p} H(X,P) \geq ct$$

so that

$$c' t \geq U(t,z,q) - \underline{u}(X(t,z,q)) \geq ct$$

The last point shows that the hypotheses (MC$\underline{u}$) generalizes the conditions (G$\underline{u}$,MC): then the results in this paper generalize the results in [24].

12 Proofs

12.1 Reduction to a simpler problem

The main tool to prove the results is to reduce the model problem (7.1) to an equivalent simpler problem.

We assume that ($\exists\underline{u}$) holds (see (8.20.$\star$)).

We substitute the problem (7.1) that we are studying, with the problem

$$\begin{cases} \hat{H}(x, Du(x)) - 1 = 0 & \text{in } M \setminus K \\ u(x) = \hat{u_0}(x) & \text{when } x \in K. \end{cases} \tag{12.1}$$

We then study how the conditions and hypotheses are affected. To this end, we construct $\hat{H}$ in two steps:

1. We define
$$\widetilde{H}(x,p)\doteq H(x,p+d\underline{u}(x)) \quad , \quad \hat{u_0}\doteq u_0-\underline{u} \ . \tag{12.2}$$
As shown in 8.12 and 12.15, if V is the value of 7.1, $\tilde{V} = V - \underline{u}$ is the value solution of
$$\begin{cases} \widetilde{H}(x, D\widetilde{u}(x)) = 0 & \text{in } M \setminus K \\ \widetilde{u}(x) = \tilde{u_0}(x) & \text{when } x \in K. \end{cases}$$
and similarly for viscosity solutions.
The above transformation implies that $\widetilde{H}(x,0) < 0$, and basilarly say that we can assume that $\underline{u} = 0$ in our proofs, with no loss of generality.

2. Assume 8.23. We now define
$$\tilde{Z}_x \doteq \left\{ p \mid \widetilde{H}\,(x,p) \le 0 \right\}$$
$\tilde{Z}_x$ is a convex set; we define then the gauge function j_z of $\tilde{Z}_x$ as
$$j_x(p) \doteq \inf \left\{ t > 0 \mid p/t \in \tilde{Z}_x \right\}$$
and eventually we define
$$\hat{H}(x,p) \doteq \big(j_x(p) \big)^2$$

Summarizing, $\hat{H}$ is built from H (or from Z) by
$$\begin{aligned} \hat{H}(x,p) \doteq \inf \left\{ t^2 \;\middle|\; t > 0 \text{ s.t. } H\left(x, \frac{p}{t} + d\underline{u}(x)\right) \le 0 \right\} = \\ = \inf \left\{ t^2 \;\middle|\; t > 0 \text{ s.t. } \left(\frac{p}{t} + d\underline{u}(x)\right) \in Z_x \right\} \end{aligned} \tag{12.3}$$

We call $\hat{H}$ *the gauge Hamiltonian*.

Remark 12.4 *A similar transformation is used in studying the existence of periodic solutions to (9.9) in symplectic manifolds; see thm 3.4 in §3.5 in [22], or chap. I sec. 7 ex. 2 in [8]; the main difference being that in those theorems it is assumed that* $\{(x,p) \mid H(x,p) \le 0\}$ *is convex, and* $\hat{H}$ *is built from* $\{H \le 0\}$, *and not sectionwise as above.*

By the definition, $\hat{H}$ is positively 2-homogeneous, ie
$$\forall \lambda \ge 0 \qquad \hat{H}(x, \lambda p) = \lambda^2 \hat{H}(x,p) \tag{12.5}$$

It is easily proved (by the definition of $\hat{H}$) that $(\hat{H}(x,p) - 1)$ has the same sign of $\tilde{H}(x,p)$: then, u is a viscosity solution for $\hat{H}$ iff it is a viscosity solution for $\tilde{H}$.

Suppose that $H \in C^{1,1}$ in a neighbourhood of $\{H = 0\}$, and $\hat{H}$ as well. We do not know a priori if the characteristics curves of H and the characteristics curves of $\hat{H}$ are related: the idea being that, they are related if `(Hnd)` holds for H. We explore this idea in sections 12.3 and 12.4.

Remark 12.6 *We easily see that, if* $p \neq 0$, *there is only one value of* t *such that*
$$\tilde{H}\left(x, \frac{p}{t}\right) = 0 \quad ,$$
and it is $t = \sqrt{\hat{H}(x,p)}$.

12.2 Comparison Theorem

This section is devoted to the proof of this result.

Theorem 12.7 *Assume 8.23 on Z, that $\underline{u}$ exists, and that the asymmetric metric space (M, b) is backward complete. Suppose that u is a viscosity subsolution and v is a supersolution of $H(x, du(x)) = 0$ for $x \in M \setminus K$ whereas $u \le v$ on K, and suppose that $v \ge \underline{u} + c$ for a constant $c \in \mathbb{R}$: then $u \le v$.*

If M is a complete Riemannian manifold and $H(x,p) = |p|^2 - 1$, we may choose $\underline{u} \equiv 0$: then b is equal to the Riemannian distance d; when moreover $u_0 = 0$, problem (7.1) reduces to the eikonal equation (7.6), whose viscosity solution is the distance $d_K(x) \doteq d(x, K)$ from K; Thm. 3.1 in [21] (see 9.22 here) proves that d_K is the unique solution in the class of continuous functions bounded from below.[31]

To prove the above theorem, we will use the following result from prop. 4.3 in Camilli and Siconolfi [5] [32]

Proposition 12.8 (prop. 4.3 in [5]) *Let $\Omega \subset M$ be open. For any $f : \Omega \to \mathbb{R}$ continuous, we define the Clarke generalized differential as*

$$\partial f(x) \doteq co\{p \in T^*_x M \mid \exists (x_n) \subset \Omega, \exists df(x_n) \doteq p_n, (x_n, p_n) \to (x,p) \text{ in } T^*M\}$$

where $co(A)$ is the convex envelope of a set $A \subset T^*_x M$. Consider a problem $H(x, du(x)) = 0$ such that 8.23 holds on $Z \doteq \{H \le 0\}$.

- *v is a supersolution of $H(x, dv(x)) = 0$ in Ω iff for any $x \in \Omega$ and any Lipshitz continuous ϕ which is subtangent to v at x there exists $p \in \partial\phi(x)$ such that $H(x,p) \ge 0$*
- *Any subsolution of $H(x, du(x)) = 0$ is Lipshitz continuous, and for all $p \in \partial u(x)$ we have $H(x,p) \le 0$*

Using this result we can simplify the proof in [21], and yet prove the general theorem 12.7 here proposed.

By means of the transformation in (12.2), we assume without loss of generality that $\underline{u} \equiv 0$, and we replace the problem at hand with the problem (12.1). Let

$$h(x,p) \doteq \sqrt{\hat{H}(x,p)}$$

in the following.

As in the work of Kružhkov [16], we consider the transformed functions $\tilde{u} = -e^{-u}$ and $\tilde{v} = -e^{-v}$, which are respectively a viscosity subsolution and a supersolution of

$$\begin{cases} h(x, dv) + v = 0 & \text{in } M \setminus K, \\ v = -e^{-u_0} & \text{on } K \end{cases} \tag{12.9}$$

(see proposition 6 in [21]) moreover, $0 > \tilde{v} \ge -e^{-\inf v} \ge -e^{-c}$ and $\tilde{u} < 0$.

We establish a comparison result for this last problem (12.9): this clearly implies the above theorem. We fix $C \doteq e^{-c}$. We argue by contradiction, and suppose that $\tilde{u}$ and $\tilde{v}$ are resp. a subsolution and a supersolution of (12.9), $0 > \tilde{v} \ge -C$, $\tilde{u} < 0$, and that at a point $\overline{x}$ we have $\tilde{u}(\overline{x}) = 2\varepsilon + \tilde{v}(\overline{x})$ with $\varepsilon > 0$.

We apply the Kružhkov transformation to above proposition 12.8 and state that

[31] Moreover the proof of Thm. 3.1 contains a comparison result between super and sub solution.

[32] The proof in [5] is stated assuming that $\Omega \subset \mathbb{R}^n$, but the result can be generalized to manifolds, using local coordinates

- for any $x \in M \setminus K$ and any Lipshitz continuous ϕ which is subtangent to $\tilde{v}$ at x there exists $p \in \partial\phi(x)$ such that $h(x,p) + \tilde{v}(x) \geq 0$

- $\tilde{u}$ is locally Lipshitz continuous, and for any $x \in M \setminus K$ and for all $p \in \partial\tilde{u}(x)$ we have $h(x,p) + \tilde{u}(x) \leq 0$

Let $B(x) \doteq b(x, \overline{x})$. By 9.2 and 9.7 we know that B is a viscosity solution of $h(x,-p) - 1 = 0$ for $x \neq \overline{x}$: then for all $x \in M$ and all $p \in \partial B(x)$, $h(x,-p) \leq 1$.

Let

$$\Psi(x) = \tilde{u}(x) - \tilde{v}(x) - \varepsilon B(x)$$

This function is bounded from above by C; moreover $\Psi(\overline{x}) = 2\varepsilon$: then $\sup \Psi$ will be positive, and realized in the region $\{x \mid \varepsilon B(x) \leq C\}$ which is a backward closed ball. By the Hopf–Rinow–like theorem 8.31 since the metric space (M,b) is backward complete, then the backward closed balls are compact: so $\Psi(x)$ has a positive maximum in a point $\hat{x}$. This means that the function $\tilde{u}(x) - \varepsilon B(x)$ is a Lipschitz subtangent of $\tilde{v}(x)$ at $\hat{x}$.

We know that $\Psi(\hat{x}) \geq \Psi(\overline{x}) = 2\varepsilon$, while $\Psi(x) \leq 0$ for all $x \in K$: then $\hat{x} \notin K$. Then by (the transformed version of) 12.8, there exists $p \in \partial(\tilde{u}(x) - \varepsilon B(x))$ such that $h(x,p) + \tilde{v}(x) \geq 0$.

At the same time $p = p' + p''$ with $p' \in \partial\tilde{u}(x)$ and $p'' \in \partial(-\varepsilon B(x))$: then (again by 12.8) $h(\hat{x}, p') + \tilde{u}(\hat{x}) \leq 0$; at the same time, as noted above, $h(\hat{x}, p''/\varepsilon) - 1 \leq 0$ that is $h(\hat{x}, p'') \leq \varepsilon$.

Since $h(x, \cdot)$ is convex and 1-homogeneous, summing up we obtain

$$h(\hat{x}, p) \leq h(\hat{x}, p') + h(\hat{x}, p'') \leq -\tilde{u}(\hat{x}) + \varepsilon$$

while $h(x,p) + \tilde{v}(x) \geq 0$: this entails

$$-\tilde{v}(x) \leq h(x,p) \leq -\tilde{u}(\hat{x}) + \varepsilon$$

or

$$\tilde{u}(\hat{x}) - \tilde{v}(\hat{x}) \leq \varepsilon$$

whereas

$$\tilde{u}(\hat{x}) - \tilde{v}(\hat{x}) \geq \Psi(\hat{x}) \geq \Psi(\overline{x}) \geq 2\varepsilon$$

achieving contradiction.

12.3 Equivalent problems

We remarked in 7.5 that the viscosity solution does not depend on the values of H, but only on its sign. We also foretold in 7.7 that this holds (roughly speaking) for the *min* solution, that is defined in (9.15) by means of characteristics; but we need to find a relationship between characteristics of different problems:

Lemma 12.10 *Suppose that*

$$\tilde{H}(x,p) = \tilde{\rho}(x,p) H(x,p) \tag{12.10.$\star$}$$

where $\tilde{\rho} : T^*M \to \mathbb{R}$ is a positive function; suppose that ρ is locally Lipschitz in a neighbourhood of $\{H = 0\}$; and $\tilde{H}(x,p)$ is $C^{1,1}_{loc}$ in a neighbourhood of $\{H = 0\}$.

The value function $\tilde V$ *of*

$$\begin{cases}\tilde H(x, d\tilde u(x)) = 0 & \text{on } M\\ \tilde u = u_0 & \text{on } K\end{cases}, \qquad (12.10.\star\star)$$

coincides with the value function V *of (7.1); the same holds also for the* min *solution, and (since* $\tilde\rho > 0$*, see 7.5) for viscosity solutions.*

We indeed note that the characteristics $\tilde X, \tilde P, \tilde U$ *of* $\tilde H$ *are related to the characteristics* X, P, U *of* H *through a reparameterization of the* t *variable: fix* $y = (z, q)$ *s.t.* $H(z, q) = 0$*, and let* $a, b, \tilde a, \tilde b > 0$ *be such that* $(-a, b)$ *is the maximal interval of definition for* X, P, U*, and* $(-\tilde a, \tilde b)$ *is the maximal interval of definition for* $\tilde X, \tilde P, \tilde U$*; then there is a diffeomorphism* $\varphi_y : (-\tilde a, \tilde b) \to (-a, b)$ *s.t.*

$$(\tilde X(t, y), \tilde P(t, y), \tilde U(t, y)) = (X(\varphi_y(t), y), P(\varphi_y(t), y), U(\varphi_y(t), y)) \qquad (12.10.\diamond)$$

that is defined by

$$\begin{aligned}\varphi_y(t) &= \int_0^t \tilde\rho(\tilde X(s, y), \tilde P(s, y))\, ds = \qquad (12.10.\diamond\diamond)\\ &= \int_0^t \tilde\rho(X(\varphi_y(s), y), P(\varphi_y(s), y))\, ds\end{aligned}$$

indeed deriving X

$$\dot X(\varphi_y(t), y)\dot\varphi_y(t) = \dot{\tilde X}(t, y) = \frac{\partial \tilde H}{\partial p}(\tilde X, \tilde P) =$$
$$\tilde\rho(\tilde X, \tilde P)\frac{\partial H}{\partial p}(\tilde X, \tilde P) = \tilde\rho(\tilde X, \tilde P)\frac{\partial H}{\partial p}(X(\varphi_y(t), y), P(\varphi_y(t), y))$$

since $\tilde H(\tilde X, \tilde P) = 0$ *and deriving* U

$$\dot U(\varphi_y(t), y)\dot\varphi_y(t) = \dot{\tilde U}(t, y) = \tilde P \cdot \frac{\partial \tilde H}{\partial p}(\tilde X, \tilde P) =$$
$$P(\varphi_y(t), y) \cdot \tilde\rho(\tilde X, \tilde P)\frac{\partial H}{\partial p}(\tilde X, \tilde P) =$$
$$= P(\varphi_y(t), y) \cdot \tilde\rho(\tilde X, \tilde P)\frac{\partial H}{\partial p}(X(\varphi_y(t), y), P(\varphi_y(t), y))$$

A similar result is found in lemma 3.1 in §3.5 in [22].

Remark 12.11 *Suppose that* $\tilde H$ *satisfies* `(Hnd)` *(in p. 16) and similarly for* $\widetilde H$*, and suppose that* $H, \tilde H \in C^{1,1}$*, and*

$$\text{sign} H(x, p) = \text{sign}\tilde H(x, p) \quad \forall x, p \in T^*M$$

Then there exists a $\tilde\rho > 0$ *locally Lipschitz such that (12.10.⋆) holds. So, in a sense, (12.10.⋆) is the most general change of equation that preserves the viscosity solutions, the characteristics flow, and the* min *solution.*

Every time we will be able to transform the problem (7.1) in another problem (12.10.⋆⋆), so that the solutions are related by a simple algebraic relationship, we will say that the two problems are *equivalent*.

Definition 12.12 (Invariant conditions) *Consider a condition $\mathcal{P}$ on a problem such as (7.1); suppose that (7.1) satisfies $\mathcal{P}$ iff (12.10.$\star\star$) satisfies $\mathcal{P}$: then we say that this condition $\mathcal{P}$ is* invariant *under the equivalence of the two problems.*

Proposition 12.13 *The conditions* `(∃u)`, `(MCu)`, `(MCu±)`, *and* `(∃minV)` *are invariant: indeed, the variable t does not play an explicit role in them, so that by (12.10.$\diamond$) we just need to change the time variable with $\varphi_y(t)$ in the definitions.*

Remark 12.14 *The conditions* `(MC)`, `(Gu)` *(that were proposed in [24] and are discussed here in §11.2) are not in general invariant. If the problem (12.10.$\star\star$) satisfies the condition* `(MC)` *and $\forall y$ s.t. $H(y) = 0$,*

$$\int_0^\infty \tilde\rho(\tilde X(s,y), \tilde P(s,y))\, ds = \infty,$$

$$\int_{-\infty}^0 \tilde\rho(\tilde X(s,y), \tilde P(s,y))\, ds = -\infty,$$

then φ_y is a diffeomorphism from $\mathbb{R}$ to $\mathbb{R}$ and the problem (7.1) satisfies `(MC)`*; this happens, e.g., when $\inf \tilde\rho > 0$ on the set $\{(z,q) \mid H(z,q) = 0\}$; and vice versa. A particular case is when there are constants $c' > c > 0$ such that $c' > \tilde\rho > c$ on the set $\{H = 0\}$: then these conditions* `(MC)`, `(Gu)` *are invariant. This problem gave the starting impulse to the study that eventually generated this whole paper.*

Lemma 12.15 *If ψ is any regular function on M, we can define*

$$\widetilde H(x,p) \doteq H(x, p + d\psi(x)) \quad , \quad \tilde{u_0} \doteq u_0 - \psi \ ;$$

then, if V is the value solution of 7.1, $\widetilde V = V - \psi$ is the value solution of

$$\begin{cases} \widetilde H(x, D\widetilde u(x)) = 0 & \text{in } M \setminus K \\ \widetilde u(x) = \tilde{u_0}(x) & \text{when } x \in K. \end{cases} \tag{12.15.$\star$}$$

to prove this, it is sufficient to note that the characteristics of $\tilde H$ are related to the characteristics X, P, U of H through the relation

$$(\tilde X, \tilde P, \tilde U) = (X, P - D\psi(X), U - \psi(X) + \psi(X(0))) \ ;$$

and a similar statement holds for viscosity solutions and min *solutions.*

Proposition 12.16 *By the relation before mentioned between the characteristics, the conditions* `(MC)`, `(∃u)`, `(OXUp)`, `(∃V)`, `(KXUp)`, `(CC)` *are invariant; if we relate a strict subsolution $\underline{u}$ of (7.1) to a strict subsolution $\underline{\widetilde u}$ of (12.15.$\star$) using the formula $\underline{\widetilde u} = \underline{u} - \psi$, then* `(∃u)`, `(MCu±)` *are invariant, by construction.*

Another way to relate two problems is in proposition 2.10 in [21].

12.4 Equivalence of characteristics of $\hat{H}$ and H

We again define

$$\tilde{Z} \doteq \{(x,p) \mid \tilde{H} \le 0\} = \{(x,p) \mid \hat{H} \le 1\} \tag{12.17}$$

Note that Z and $\tilde{Z}$ are easily related by

$$Z_x = \tilde{Z}_x + d\underline{u}(x) \tag{12.18}$$

As aforementioned, the hypothesis `(Hnd)` (that was defined in (9.12.⋆)) directly implies some of the hypotheses we stated in 8.23 and in 9.1:

Proposition 12.19 *Assume* `(Hnd)`*. Then $\partial Z = \{H = 0\}$; moreover $\{H = 0\}$ is a regular $C^{1,1}$ submanifold of T^*M; and $Z_x = \overline{A_x}$, where $A = \mathring{Z}$. Assume moreover that Z_x is convex, choose a $\underline{u}$ satisfying (8.20.⋆), and define $\hat{H}$: then $\hat{H}$ is of class $C^{1,1}$ on $T^*M \cap \{p \neq 0\}$.*

The proof is by implicit function theorem.

Proof. We work in local coordinates $(x,p) \in \mathbb{R}^{2k}$ where $k \doteq \dim(M)$. By implicit function theorem and `(Hnd)`, we can write $\{H = 0\}$ locally as the graph of a function $p_1 = f(x_1, \ldots, x_k, p_2, \ldots, p_k)$ where $f \in C^{1,1}$; this implies that $\{H = 0\}$ is (locally) of class $C^{1,1}$. In local coordinates Z_x is the slice of the closed epigraph of f:

$$\{p \mid p_1 \le f(x_1, \ldots, x_k, p_2, \ldots, p_k)\}$$

whereas A_x is the slice of the open epigraph of f:

$$\{p \mid p_1 < f(x_1, \ldots, x_k, p_2, \ldots, p_k)\}$$

this proves that $Z_x = \overline{A_x}$.

Assume moreover that Z_x is convex; we locally use spherical coordinates r, θ for $T^*_x M$ (with $r \in \mathbb{R}^+$ and $\theta \in S^{k-1}$), and $x \in \mathbb{R}^k$ for M. We know that $\tilde{Z}_x$ is convex and $0 \in \mathring{\tilde{Z}}_x$; fix a point $(\overline{x}, \overline{r}, \overline{\theta})$ where $\tilde{H} = 0$; by `(Hnd)` we know that $(\frac{\partial}{\partial r}\tilde{H}, \frac{\partial}{\partial \theta}\tilde{H}) \neq 0$, but moreover we can prove that $\frac{\partial}{\partial r}\tilde{H} \neq 0$, otherwise the radious connecting 0 to $(\overline{r}, \overline{\theta})$ in $T^*_{\overline{x}}M$ would be tangent to $\tilde{Z}_{\overline{x}}$. Apply again the implicit function theorem, to express $\{\tilde{H} = 0\}$ as the graph of a function $r = g(x, \theta)$, for $r \in I \subset \mathbb{R}$, $\theta \in V \subset S^{k-1}$, $x \in U \subset \mathbb{R}^k$ where I, U, V are open sets: then in this coordinates, for $r \in \mathbb{R}^+$ and $\theta \in V$, $x \in U$,

$$\tilde{Z}_x = \{r \le g(x, \theta)\}$$

and

$$\hat{H}(x, r, \theta) = \inf\left\{t^2 \mid r > 0, \frac{r}{t} \le g(x, \theta)\right\} = \frac{r^2}{g(x,\theta)^2}$$

that is of class $C^{1,1}$. □

Remark 12.20 *We may consider a more general condition than* `(Hnd)`*, that is*

$$\forall (x,p) \in T^*M, \ \ H(x,p) = 0 \Longrightarrow \left(\frac{\partial}{\partial p}H(x,p), \frac{\partial}{\partial x}H(x,p)\right) \neq 0 \tag{12.20.⋆}$$

This last condition avoids the problem of degeneracies of geodesics, but does not guarantee that $\hat{H}$ is $C^{1,1}$, as shown in the following example.

Example 12.21 *Let $M = \mathbb{R}$, and suppose that, inside the region $(-1,1) \times [0,\infty)$ of $T^*M \equiv \mathbb{R}^2$, the set Z coincides with the set*

$$\{(x,p) \mid x > (p-2)^3\}$$

This set Z has a smooth boundary, so it is possible to build a smooth Hamiltonian H satisying (12.20.⋆), and such that $Z = \{H \le 0\}$; but, at the same time, if we choose $\underline{u} \equiv 0$, when $x \in (-1,1)$ and $p > 0$, $\hat{H}$ takes the value

$$\hat{H}(x,p) = \frac{p^2}{(\sqrt[3]{x}+2)^2}$$

and this is not differentiable at $(x,p) = (0,1)$.

Since the functions $\tilde{H}$ and $\hat{H} - 1$ have the same zero set $\partial\tilde{Z}$, their derivatives $D\tilde{H}(x,p)$ and $D\hat{H}(x,p)$ are parallel when $(x,p) \in \partial\tilde{Z}$; it is easily proved, (by using homogeneity, see (12.23.⋆⋆)), that $D\hat{H} \neq 0$ on $\partial\tilde{Z}$; then the function

$$\tilde{\rho}(x,p) \doteq \frac{\tilde{H}(x,p)}{\hat{H}(x,p)-1}$$

is positive and locally Lipschitz in a neighborhood of $\partial\tilde{Z}$: applying the lemma 12.10 we are sure that the problem (12.1) and the problem (7.1) are equivalent, that is, they share the same properties and the same solutions.

Lemma 12.22 *Suppose that H satisfies* `(Hnd)` *and Z satisfies 8.23. If we want to compute the reparameterization φ_y between the characteristics of $\tilde{H}$ and the characteristics of $\hat{H}$, then by the lemma 12.10, we only need the value of $\tilde{\rho}$ where $\tilde{H}(x,p) = 0$; there, by Höpital's theorem,*

$$\tilde{\rho}(x,p) = \frac{\nu \cdot D\tilde{H}(x,p)}{\nu \cdot D\hat{H}(x,p)}$$

for almost any $\nu \in TT^*M$; in particular,

$$\tilde{\rho}(x,p) = \frac{p \cdot \frac{\partial}{\partial p}\tilde{H}(x,p)}{p \cdot \frac{\partial}{\partial p}\hat{H}(x,p)} = \frac{1}{2} p \cdot \frac{\partial}{\partial p}\tilde{H}(x,p) \tag{12.22.$\star$}$$

Lets call $\hat{X}, \hat{P}, \hat{U}$ the characteristics of $\hat{H}$.

Using the above reduction we can easily prove that

Proposition 12.23 *let $y = (z,q) \in T^*M$ then*

$$\frac{d}{dt}\hat{U}(t,y) = 2\hat{H}(y) \tag{12.23.$\star$}$$

Proof. By deriving (12.5) wrt λ, we get the Euler identity

$$2\hat{H}(x,p) = \frac{\partial \hat{H}}{\partial p}(x,p) \cdot p \tag{12.23.$\star\star$}$$

since $\hat{H}(\hat{X}(t,y), \hat{P}(t,y))$ is constantly equal to $\hat{H}(y)$, we get

$$2\hat{H}(y) = \frac{\partial \hat{H}}{\partial p}(\hat{X}, \hat{P}) \cdot \hat{P} = \frac{d}{dt}\hat{U}(t,y)$$

□

Remark 12.24 *If the original problem (7.1) satisfies* `(MCu)`*, then the problem (12.1) satisfies* `(MCu)`*. We have just seen that, for any (z,q) with $\hat{H}(z,q)=1$, we have*

$$\hat{U}(t,z,q)=2t$$

this means that, for the problem (12.1), the condition `(MCu)` *is simply saying that $(M,\hat{H})$ is* "characteristically complete"[33]*: that is, the characteristics $(\hat{X},\hat{P})$ of $\hat{H}$ can be evolved for all times.*

As we will show in (13.6), characteristics of $\hat{H}$ are geodesics of the weak Finsler space (M,L) that will be defined in appendix: then the condition `(MCu)` *is equivalent to saying that (M,L) is* "geodesically complete"*.*

A similar statement holds for forward-only and backward-only completeness.

12.5 Towards a weaker Finsler Geometry

Suppose Z satisfies 8.23. Choose a strict subsolution $\underline{u}$. Let $\hat{H}$ be as in (12.3).

Let (M,b) be the asymmetric metric space associated to (7.1) in sec. 8.3.1. Because of the relation (12.18) between $\{H\le 0\}$ and $\{\hat{H}\le 0\}$, we associate (M,b) to $\hat{H}$ as well.

To prove thm. 9.14 we need to prove that: $(M,\hat{H})$ is "*characteristically backward complete*" (resp.forward) iff the metric space (M,b) associated to it is backward complete (resp. forward). This suggests that we need a kind of Geometry, and a Hopf-Rinow theorem.

To this end, we define $L:TM\to[0,\infty)$ using the Legendre-Fenchel transform, namely

$$L(x,v)\doteq \sup_{p\in T_x^*M}\big(p\cdot v-\hat{H}(x,p)\big) \tag{12.25}$$

Note that by 13.16

$$F=2\sqrt{L} \tag{12.26}$$

where F was defined in (8.21).

Example 12.27 *In the case of the eikonal equation (7.6) on a Riemannian Manifold, we have that $H(x,p)=|p|^2-1$, $\hat{H}(x,p)=|p|^2$, $L(x,v)=|v|^2/4$ and $F(x,v)=|v|$; and $b(x,y)$ is the Riemannian distance.*

We would like to view (12.1) as the *eikonal problem* for the Geometry (M,L); but, in the general case that we consider here, (M,L) is not a regular Finsler Geometry.

This proposition is the link between the above theory, and in particular 9.14, and the theory in 13.2.

Proposition 12.28 *Suppose that H satisfies* `(Hnd)`*, Z satisfies 8.23 and Z_x is strictly convex. Then $\hat{H}$ satisfies all properties in 13.1: this means that $\hat{H}$ induces a* weaker Finsler Geometry *(M,L).*

Proof. Since Z_x is strictly convex, then $\hat{H}$ is strictly convex in p: this is property $(\hat{H}3!)$ in 13.1. By 12.19, $\hat{H}$ is $C^{1,1}$ when $p\neq 0$: this is property $(\hat{H}1)$ in 13.1. By its own definition (12.3), $\hat{H}$ is positively 2-homogeneus in p: this is property $(\hat{H}4)$ in 13.1. □

[33] that is condition `(MC)` in page 24

We will indeed prove that in this case $L \in C^1$ but in general $L \notin C^2$ (see 13.12 and 13.13). We will be able nonetheless to define an exponential map (M, L), and use it to formulate an Hopf–Rinow theorem. The discussion of this ideas is postponed to the section 13.3.

13 Weaker Finsler space

As an application of the theory that was shown in [23], we review here the approach to the theory of Finsler spaces that we need to prove theorem 9.14.

13.1 Finsler Geometry

We present now a definition of *Finsler Geometry*.

A *Finsler Geometry* is a pair (M, L) where

$$L : TM \to [0, \infty)$$

satisfies

(L1') $L(x, v)$ is C^2 on the *slit tangent bundle* $TM \cap \{v \neq 0\}$, and C^1 globally;

(L2') for any fixed x, the function $v \mapsto L(x, v)$ is strongly convex at any $v \neq 0$;

(L4') L is positively homogeneous, of degree 2, in the v variable, ie

$$\forall \lambda \geq 0 \qquad L(x, \lambda v) = \lambda^2 L(x, v) \quad .$$

In the following, we will try to associate a "Finsler Geometry" (M, L) to the problem (7.1); but the function L that we will get will not be, in general, be of class C^2; for this reason, we will talk of *a weaker Finsler structure*.

Nonetheless, some results of Finsler Geometry hold true in the hypotheses that we will assume on (7.1).

13.2 Weaker Finsler Geometry

We collect all needed hypotheses on $\hat{H}$ and Z.

Hypotheses 13.1 *We suppose that* $\hat{H} : T^*M \to \mathbb{R}^+$ *satisfies*

($\hat{H}$1) $\hat{H}(x, p)$ *is locally a* $C^{1,1}$ *map on* $T^*M \cap \{p \neq 0\}$.

($\hat{H}$3!) *the* figuratrix *set of* $\hat{H}$

$$\{p \in T^*_x M \mid \hat{H}(x, p) \leq 1\}$$

is compact and strictly convex, or equivalently, $\hat{H}$ *is strictly convex in* p*;*

($\hat{H}$4) $\hat{H}(x, \cdot)$ *is positively 2-homogeneous, that is,* $\hat{H}(x, lp) = l^2 \hat{H}(x, p)$ *for* $l \geq 0$.

We recall that, in the hypotheses of theorem 9.14, the *gauge Hamiltonian* $\hat{H}$ defined in (12.3) satisfies all the above (see the discussion in prop. 12.28).

13.2.1 Hamiltonian flow of $\hat{H}$

Using the canonical coordinates of T^*M, we can define the Hamiltonian flow (X,P) : $\mathbb{R} \times T^*M \to T^*M$ of $\hat{H}$, as the solution of the *system of characteristics*

$$\begin{cases} \dot{X}_i(s) = \frac{\partial \hat{H}}{\partial p_i}(X(s),P(s)) \\ \dot{P}_i(s) = -\frac{\partial \hat{H}}{\partial x_i}(X(s),P(s)) \end{cases} \tag{13.2}$$

with initial conditions

$$\begin{cases} X(0) = z \\ P(0) = q \ . \end{cases} \tag{13.3}$$

that is, for the sake of this appendix, we agree that we denote by (X,P) the Hamiltonian flow of $\hat{H}$ (and not the flow of H).

13.3 Summary

We now briefly express the main results, leaving details to the following sections.

We define $L : TM \to \mathbb{R}$ as the Legendre–Fenchel transform of $\hat{H}$, as defined in (12.25). If we assume hypotheses 13.1 on $\hat{H}$, then L is not $C^{1,1}$ regular in general; since this case is not covered in the most common books, we outline how some basic facts do hold true:

Proposition 13.4 *L is C^1. Suppose that ξ is a critical curve of the* energy functional *(or,* action functional*)*

$$E(\xi) \doteq \int_0^1 L(\xi(s),\dot{\xi}(s))\, ds \tag{13.4.$\star$}$$

then, by associating

$$p(s) \doteq \frac{\partial L}{\partial v}(\xi(s),\dot{\xi}(s)) \tag{13.4.$\star\star$}$$

we obtain that $\big(\xi(s),p(s)\big)$ is a solution of the Cauchy problem (13.2); and then ξ is C^1 in the s variable.

We leave the proof and the discussion to the next section.

We then define

Definition 13.5 (Forward exponential map) *For fixed x we define the* forward exponential map [34]

$$\exp_x : T_xM \to M$$

as

$$\exp_x(v) \doteq X\left(1, x, \frac{\partial L}{\partial v}(x,v)\right)$$

and we will prove that

$$X(t,x,p) = X(1,x,tp) = \exp_x(tv) \tag{13.6}$$

when p,v are related by the Legendre relation (13.9.$\diamond$). We postpone the proof to section 13.5 (and in particular to (13.17.$\star$)).

We can prove this addition to the previous (purely metric) Hopf-Rinow theorem 9.14

[34] We will in the following drop the attribute of "forward" from the definition, for simplicity

Theorem 13.7 *The following assertions are equivalent:*

1. (M, b) *is* forward-complete

6. *for a certain x, the forward exponential map $\exp_x(v)$ is defined for all $v \in T_x M$ (in this case we say that (M, L) is* forward geodesically complete*); or equivalently for all x;*

7. *for a certain x, the forward exponential map is defined on all $T_x M$ and is surjective, that is, $\exp_x(T_x M)$ covers M; or equivalently for all x;*

Using the purely-metric Hopf-Rinow theorem in 2.37 in [23], the proof to this theorem just needs simple extra arguments that are identical to those used for the Hopf–Rinow theorem in [7], or in [3]. The proof is in section 13.6.

We recall that, following the discussion in 12.24 and in §12.5, (M, L) is "*forward geodesically complete*" iff the original problem (7.1) satisfies (MC$\underline{\text{u}}$+) (from eq. (9.13)): so the above Theorem immediatly proves 9.14.

13.4 On the Legendre-Fenchel transform

This subsection is more of a comment on Calculus of Variations than on Finsler spaces: so $\hat{H}$ will not be necessarily homogeneous in p; we will though suppose, for convenience, that $\hat{H}(x, \cdot)$ is *superlinear*, that is,

$$\lim_{p \to \infty} \frac{\hat{H}(x,p)}{|p|} = \infty \tag{13.8}$$

where $|p|$ is any chosen norm on $T_x^* M$; we know indeed that $\hat{H}(x, \cdot)$ is *superlinear* iff L is defined on the whole TM, and proper on each $T_x M$ (see thm. 1.4.11 in [9]).

Definition 13.9 *the Legendre–Fenchel transform L of $\hat{H}$ is defined as in (12.25), that is:*

$$L(x,v) \doteq \max_{p \in T_x^* M} \big(p \cdot v - \hat{H}(x,p) \big) \tag{13.9.$\star$}$$

which can be inverted by the dual formula

$$\hat{H}(x,p) \doteq \max_{v \in T_x M} \big(p \cdot v - L(x,v) \big) \tag{13.9.$\star\star$}$$

Since $\hat{H}(x, \cdot)$ is C^1 and strictly convex, then, by duality, $v \mapsto L(x,v)$ is C^1 and strictly convex; then the points p^0 and v^0, where the above maxima are realized, are related by the Legendre reciprocity formula

$$p^0 = \frac{\partial L}{\partial v}(x, v^0), \qquad v^0 = \frac{\partial \hat{H}}{\partial p}(x, p^0) \tag{13.9.$\diamond$}$$

Remark 13.10 *The relationship (13.9.$\diamond$) is obviously a homeomorphism.*

This would bring to the classical approach [35]:

[35] see §1.8 in [13], or §2.3 in [9].... and many other text in Calculus of Variations, or Hamilton–Jacobi problems

Proposition 13.11 *if $\hat{H}$ is C^2 and strongly convex in p, then L is C^2 and strongly convex in v, and viceversa; and in this case $\hat{H}$ is C^r iff L is C^r ($r \geq 2$), and the relation (13.9.$\diamond$) is a diffeomorphism for any fixed x.*

If $\hat{H}$ is only strictly convex, then, in general, we cannot expect that L be regular:

Example 13.12 *if we take $M = \mathbb{R}$, $H(x,p) = p^{2n}/2n$, then $L(x,v) = \frac{2n+1}{2n} v^{\frac{2n}{2n-1}}$: so, if we just say that $H \in C^\infty$ and strictly convex in p, we don't have any bound on the Hölder exponent of $\frac{\partial}{\partial v} L$.*

Moreover the relation (13.9.$\diamond$) is *not* in general a diffeomorphism for fixed x; [36] but we show in the following that we may use all the common methods in Calculus of Variations nonetheless.

Proposition 13.13 *If $\hat{H}(x,p)$ is locally a $C^{1,1}$ map on T^*M, strictly convex in p, then it is easily proved that $L \in C^1$ as a function on TM*

Proof. We have noted that $v \mapsto L(x,v)$ is, by duality, C^1.

We will prove that $x \mapsto L(x,p)$ is derivable and that

$$\frac{\partial L}{\partial x}(x^0, v^0) = \frac{\partial \hat{H}}{\partial x}(x^0, p^0) \qquad (13.13.\star)$$

(where x^0, v^0, p^0 are related by (13.9.$\diamond$)). We can assume without loss of generality that M is substituted by an open subset of $\mathbb{R}^n$, and $\frac{d}{dx,p}\hat{H}(0,0) = 0$, $\hat{H}(x,0) = 0$, $x^0 = 0, v^0 = 0, p^0 = 0$, so that $L(0,0) = 0$: we will then prove that

$$\frac{L(x,0)}{|x|} = \frac{\max_p -\hat{H}(x,p)}{|x|} \xrightarrow[x \to 0]{} 0$$

that is, that $\frac{\partial}{\partial x} L(0,0) = 0$. Obviously

$$0 \leq \frac{\max_p -\hat{H}(x,p)}{|x|}$$

Lets call $p^* = p^*(x)$ the point where the maximum is attained; then, by (13.8), $p^*(x)$ is bounded; and by strict convexity, $p^*(x)$ is continuous, and $p^*(x) \to 0$ for $x \to 0$; then

$$\max_p -\hat{H}(x,p) = -\hat{H}(x, p^*(x)) \leq \frac{d}{dp}\hat{H}(x,0) \cdot p^*(x) \leq C|x||p^*(x)|$$

□

Suppose now that ξ is an extremal of (13.4.$\star$): then ξ satisfies the Euler condition (in integral form)

$$\frac{\partial L}{\partial v}(\xi(t), \dot{\xi}(t)) = \int_0^t \frac{\partial L}{\partial x}(\xi(s), \dot{\xi}(s))\, ds + c$$

By what we proved above, we substitute

$$p(s) \doteq \frac{\partial L}{\partial v}(\xi(s), \dot{\xi}(s))$$

[36] but it is an homeomorphism: see previous note

and then, by the above proposition, the Euler condition becomes

$$p(s) = \int_0^t \frac{\partial H}{\partial x}(\xi(s), p(s))\, ds + c$$

then $\xi(s), p(s)$ is a solution of the Cauchy problem (9.9); and then it is C^1 in the s variable.

This shows that an extremal is determined uniquely (and continuously) by the inital values $(\xi(0), \dot\xi(0))$, even if L is not $C^{1,1}$. This ends the proof of 13.4.

13.5 Exponential map

We briefly review how we define the *exponential map* in our weaker Finsler geometry; we cannot use the common definition, since $L \notin C^2$ in general.

We suppose that $\hat H$ satisfies all of 13.1.

The Finsler structure (M, L) is related to $\hat H$ as shown in the previous section; in this case, moreover, $v \mapsto L(x, v)$ and $p \mapsto \hat H(x, p)$ are positively 2-homogeneous. From this we obtain that

$$L(x, v) = v \cdot p - \hat H(x, p) = \frac{\partial \hat H}{\partial p}(x, p) \cdot p - \hat H(x, p) = \hat H(x, p) \qquad (13.14)$$

whenever x, v, p are related by the duality (13.9.$\diamond$). Moreover

Proposition 13.15 *If ξ is an extremal, then $L(\xi, \dot\xi)$ =const; indeed,*

$$L(\xi, \dot\xi) - \dot\xi \cdot \frac{\partial L}{\partial v}(\xi, \dot\xi) = \textit{constant}$$

(by direct derivation and the Euler equation, see [14, pag. 76]); but, by `(L4')`*,*

$$v \cdot \frac{\partial L}{\partial v}(x, v) = 2L(x, v) \ .$$

Remark 13.16 *Note by homogeneity, we may use the Legendre–Fenchel formulas*

$$L(x, v) \doteq \max_{p \neq 0} \frac{(p \cdot v)^2}{4\hat H(x, p)} = \max_{p \text{ s.t. } \hat H(x,p) \leq 1} \frac{(p \cdot v)^2}{4} \qquad (13.16.\star)$$

$$\hat H(x, p) \doteq \max_{v \neq 0} \frac{(p \cdot v)^2}{4L(x, v)} = \max_{v \text{ s.t. } L(x,v) \leq 1} \frac{(p \cdot v)^2}{4} \qquad (13.16.\star\star)$$

which show that $4L = F^2$, where F is the support function of $\{p \mid \hat H(x, p) \leq 1\}$

To conclude the proof of (13.6), we then need this simple lemma

Lemma 13.17 *for any $\lambda \geq 0$ we have*

$$X(t\lambda, z, q) = X(t, z, \lambda q), \qquad (13.17.\star)$$
$$\lambda P(t\lambda, z, q) = P(t, z, \lambda q)$$

Proof. Note that, by `(H4′)`, $\frac{\partial H}{\partial x}$ is positively homogeneous in p of degree 2, and $\frac{\partial H}{\partial p}$ of degree 1.

Lets call

$$\tilde{X}(t,z,q) \doteq X(\lambda t,z,q),$$
$$\tilde{P}(t,z,q) \doteq \lambda P(\lambda t,z,q)$$

the LHS of (13.17.⋆); then

$$\lambda \dot{\tilde{X}}(t,z,q) = \lambda \dot{X}(\lambda t,z,q), \dot{\tilde{P}}(t,z,q) = \lambda^2 \dot{P}(t,z,q),$$

Then these solve the equation

$$\begin{cases} \frac{\partial \tilde{X}}{\partial t}(t,z,q) = \lambda \frac{\partial X}{\partial t}(\lambda t,z,q) = \lambda \frac{\partial H}{\partial p}(X,P) = \frac{\partial H}{\partial p}(\tilde{X}(t,z,q),\tilde{P}(t,z,q)) \\ \frac{\partial \tilde{P}}{\partial t}(t,z,q) = \lambda^2 \dot{P}(t,z,q) = -\lambda^2 \frac{\partial H}{\partial x}(X,P) = -\frac{\partial H}{\partial x}(\tilde{X},\tilde{P}) \\ \tilde{X}(0,z,q) = z, \quad \tilde{P}(0,z,q) = q\lambda \quad ; \end{cases}$$

and the solution of this system coincides with the RHS of (13.17.⋆). □

13.6 Proof of 13.7

We outline here the proof of 13.7.

The implication $1 \implies 6$ is straightforward: indeed (by seeking contradiction) if $t \to \exp_x(tv)$ were defined only on an interval $[0,T)$, then the sequence

$$t_n \doteq \exp_x\big((T-1/n)v\big)$$

would be forward–Cauchy, so it would have a limit point, from where we can prolong the geodesic.

To prove $6 \implies 7$ (for a fixed x), we will need some preliminary results.

Proposition 13.18 *Fix $x,y \in M$ and suppose that there exists a geodesic γ that gives the minimum of $b(x,y)$ in the class of paths connecting x to y. As shown in sec. 3 in [23], we can reparameterize*[37] *γ to obtain another geodesic ξ such that $F(\xi,\dot{\xi})$ is constant. By the Cauchy–Schwartz inequality, we have in general that*

$$\sqrt{\int_0^1 L(\xi(s),\dot{\xi}(s))\,ds} \geq \int_0^1 \sqrt{L(\xi(s),\dot{\xi}(s))}\,ds$$

and there is equality iff $L(\xi,\dot{\xi})$ is (almost everywhere) constant: then this geodesic ξ is also a minimum of the energy functional *(13.4.⋆) and $\gamma(t) = \exp_x(tv)$ for the choice of $v = \dot{\gamma}(0) \in T_xM$.*

Proposition 13.19 *Fix x. Let $\varepsilon > 0$ s.t. $\{y \mid b(x,y) \leq \varepsilon\}$ is compact.* [38] *Let*

$$B^+(x,\varepsilon) \doteq \{y \mid b(x,y) < \varepsilon\} \quad, \mathcal{B}^+(x,\varepsilon) \doteq \{v \in T_xM \mid F(x,v) < \varepsilon\}$$

Then for any $y \in B^+(x,\varepsilon)$, there is a minimal geodesic connecting x to y (see lemma 2.34 in [23]); once reparameterized to constant parameterization, this geodesic is a

[37] see also 2.33 in [23]

[38] We know that (M,b) is path–metric, by 3.7 in [23]; then $\{y \mid b(x,y) \leq \varepsilon\} = \overline{B^+(x,\varepsilon)}$ (by 2.20 in [23])

minimum of the energy functional (13.4.⋆), so it is a characteristic: we have proved that

$$\exp_x(\mathcal{B}^+(x,\varepsilon)) \supset B^+(x,\varepsilon)$$

conversely, any such characteristic has length less than ε*, so*

$$\exp_x(\mathcal{B}^+(x,\varepsilon)) = B^+(x,\varepsilon)$$

Remark 13.20 *From the above, we understand that* $\exp_x$ *is locally surjective for* ε *small; we do not know, though, if it is injective (the usual proof relies on the local inversion theorem, that needs the hypothesis* (H2)*).*

In spite of the possible lack of local injectivity, we can anyway prove

Proposition 13.21 *Suppose* $\exp_x(v)$ *is defined for all* v*: then for any* $y \in M$ *there exists a geodesic connecting* x *to* y*.*

indeed, the proof goes exactly as in prop. 6.5.1 [3]. Combining the above propositions, we have obtained that, if for a certain x the condition 6 of 13.7 holds, then 7 holds.

To conclude, we prove that 7 (for a fixed x) implies that forward–bounded closed sets are compact: indeed, if a set A is forward–bounded, then it is contained in $B^+(x,r)$ for r large; [39] by hypothesis 7 and 13.21, we know that each point y in $B^+(x,r)$ can be reached by a geodesic, so again

$$\exp_x(\mathcal{B}^+(x,r)) \supset B^+(x,r) \supset A$$

but then

$$\exp_x(\mathcal{D}^+(x,r)) \supset A$$

where

$$\mathcal{D}^+(x,r) \doteq \{v \in T_xM \mid F(x,v) \le r\}$$

is compact: so A is compact. So 7 implies condition 2 in 8.31 that implies 1 (in their forward form).

Since condition 1 does not involve a specific point x, then conditions 6 and 7 can be equivalently stated *"for a certain x"* or *"for all x"*.

A Kuratowski convergence

We review definition and results regarding the Kuratowski convergence, for convenience of the reader. We will use many concepts of general topology, as defined in Kelley [15]. Let in the following Y be a Hausdorff[40] topological space.

Definition A.1 (Kuratowski convergence [17]) *Let* $(A_n)_{n\in\mathbb{N}}$ *be a sequence of subsets of* Y*. We define the* ***lower Kuratowski limit*** $\liminf_{n\to\infty} A_n$ *of* $(A_n)_n$

- $\liminf_{n\to\infty} A_n$ *is the set of* $y \in Y$ *such that any neighborhood of* y *meets eventually all of the* A_n*.*

[39] the condition that A is forward–bounded can be stated using any point x at will, as remarked in §2.iv in [23]

[40] also known as *"T2"* or *"separated"* space; see [15], ch. 2

- *(If the topology of Y is characterized by converging sequences:*[41]*) Equivalently* $\liminf_{n\to\infty} A_n$ *is the set of all possible limits of sequences* y_n *with* $y_n \in A_n$*.*
- *(If* (Y, d') *is a metric space:) Equivalently* $\liminf_{n\to\infty} A_n$ *is*

$$\liminf_{n\to\infty} A_n \doteq \bigcap_{\delta>0} \bigcup_{m\in\mathbb{N}} \bigcap_{n\geq m} (A_n^\delta)$$

where

$$A_n^\delta \doteq \{x \mid d'(x, A) < \delta\}$$

is the fattened of A_n*.*

If $(Y, |\cdot|)$ *is a normed vector space, then in particular*

$$A_n^\delta = A_n + B_\delta \doteq \{y + z \mid y \in A_n, z \in B_\delta\} = \{x \mid \exists y \in A_n, |x - y| < \delta\}$$

where $B_\delta \doteq \{z \mid |z| < \delta\}$ *is the ball centered in 0 of radious* $\delta > 0$*.*

The ***upper Kuratowski limit*** $\limsup_{n\to\infty} A_n$ *of* $(A_n)_n$ *is*

- *the set of all points* $y \in Y$ *such that any neighborhood of* y *meets frequently all of the* A_n*.*
- *Equivalently it is the set of all possible limits of sequences* y_m *with* $y_m \in A_n$ *for some* $n \geq m$*.*
- *Equivalently it is*

$$\limsup_{n\to\infty} A_n \doteq \bigcap_{\delta>0} \bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} (A_n^\delta)$$

In general,

$$\liminf_{n\to\infty} A_n \subset \limsup_{n\to\infty} A_n$$

We will say that $\lim_{n\to\infty} A_n = A$ ***in the Kuratowski sense*** *if*

$$\liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n = A$$

Other equivalent conditions for the case when $Y = \mathbb{R}^n$ may be found in ch. 4 in [29].

The Kuratowski convergence enjoys many useful properties

Proposition A.2 *Let* $A, A_n \subset Y$.

- $\liminf_{n\to\infty} A_n$ *and* $\limsup_{n\to\infty} A_n$ *are closed.*
- ***(Locality).*** *For any* $V \subset Y$ *open, we have*

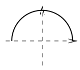
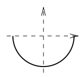
$$\begin{aligned} V \cap \limsup_{n\to\infty}(A_n \cap V) = V \cap (\limsup_{n\to\infty} A_n) \\ V \cap \liminf_{n\to\infty}(A_n \cap V) = V \cap (\liminf_{n\to\infty} A_n) \end{aligned} \qquad \text{(A.2.}\star\text{)}$$

- ***(Convexity).*** *If* Y *is a topological vector space and all* A_n *are convex, then* $\liminf A_n$ *is convex.*

[41] such is the case when Y satisfies *"the first countability axiom"*: see [15], ch. 2

Note that this last one does not hold if we replace convexity *by* connectedness. *Let indeed* $S^1 \doteq \{x : |x| = 1\} \subset \mathbb{R}^2$ *be the circle, and let*

$$A_n = S^1 \cap \{(-1)^n x_2 \geq 0\}$$

$A_0, A_2, A_4, \ldots =$  $A_1, A_3, A_5, \ldots =$ 

then $\liminf_{n\to\infty} A_n = \{(-1, 0), (1, 0)\}$ *is not connected.*

Proof.

- $x \notin \liminf_n A_n$ iff there is an open neighbourhood V of x s.t. $V \cap A_n = \emptyset$ frequently; but V is an open neighbourhood of each $z \in V$, so $V \subset Y \setminus \liminf_n A_n$. Similarly for $\limsup_n A_n$.
- Let $x \in V$; $x \in \limsup_{n\to\infty}(A_n \cap V)$ iff for any neighbourhood U of x, $(A_n \cap V) \cap U \neq \emptyset$ frequently in n; whereas $x \in (\limsup_{n\to\infty} A_n)$ iff for any neighbourhood U of x, $(A_n \cap U) \neq \emptyset$ frequently in n. The first implies the second; the second implies the first if we replace V by $U \cap V$. Similarly for $\liminf_n A_n$.
- if $x, y \in \liminf_{n\to\infty} A_n$ and $\tau \in [0, 1]$ then there are $x_n \to x$ and $y_n \to y$ with $x_n, y_n \in A_n$ and $(x_n\tau + y_n(1 - \tau)) \to (x\tau + y(1 - \tau))$.

□

We moreover prove

Proposition A.3 (Equicompactness.) *Suppose that* Y *is connected. Choose a sequence* $A_n \subset Y$. *Suppose all* A_n *are connected,* $\liminf_{n\to\infty} A_n$ *is non-empty, and* $A \doteq \limsup_{n\to\infty} A_n$ *is compact.*

Then $(A_n)_n$ *is* eventually equicompact*; and moreover for any* $K \subset Y$ *compact such that* $A \subset \mathring{K}$*, then* $A_n \subset K$ *eventually in* n.[42]

Proof. A proof may be found in prop. 3 in Salinetti–Wets [30], for the case when Y is a finite dimensional normed vector space. We present here the more general proof. We seek contradiction: let K be contradicting the thesis, that is, $A \subset \mathring{K}$ but there is a (n_k) such that $A_{n_k} \setminus K \neq \emptyset$. Then $K \neq Y$: since Y is connected, then ∂K is non-empty. ∂K divides Y in two disconnected open sets, $\mathring{K}$ and $Y \setminus K$. Since $\liminf_{n\to\infty} A_n$ is non empty, then there is a sequence $y_n \in A_n$ s.t. y_n converges to an y; then $y \in \limsup A_n \subset \mathring{K}$; for n large, $y_n \in \mathring{K}$, so $A_n \cap \mathring{K} \neq \emptyset$; at the same time,

$$A_{n_k} \cap (Y \setminus K) = A_{n_k} \setminus K \neq \emptyset \ :$$

since A_{n_k} are connected, and they intersect both sets, then they must intersect also ∂K, so there exists $z_k \in A_{n_k} \cap \partial K$. Since ∂K is compact, then $\{z_k\}$ has a cluster point $z \in \partial K$, (by thm. 5 in ch. 5 in [15]); by definition of $\limsup$, though, $z \in A$ contradicting the fact that $A \subset \mathring{K}$. □

Remark A.4 *The above proposition is tricky*[43]*: it is easy to mistakenly state it using wrong hypotheses. For example, it is not possible to replace the condition* "$\liminf_{n\to\infty} A_n$

[42] that is, there exists $N \in \mathbb{N}$, such that $A_n \subset K \ \forall n \geq N$.

[43] and indeed it fouled the author in a preliminary draft version of this paper

is non-empty" *by the condition* "$\limsup_{n\to\infty} A_n$ is non-empty", *(as is done in cor. 4.12 in [29]). This is clearly shown by this example: let* $A_n \subset \mathbb{R}$ *be defined by*

$$A_n \doteq \begin{cases} \{0\} & \textit{for n even} \\ \{n\} & \textit{for n odd} \end{cases}$$

then each A_n *is connected,* $\liminf_{n\to\infty} A_n = \emptyset$ *while* $\limsup_{n\to\infty} A_n = \{0\}$ *is non-empty and compact; but the sequence* A_n *is not equicompact. (This is a very important remark: basilar proofs in this paper depend on the above proposition.)*

Here we show moreover some other examples and remarks on sequences A_n *that are not* eventually equicompact*:*

- *Let* $Y = \mathbb{R}$*, and let* $A_n = \{0, n\}$ *be disconnected:* $\lim_n A_n = \{0\}$ *is compact and non-empty*
- *Let* $Y = \mathbb{R}$*, and let* $A_n = \{n\}$ *so that* $\lim_n A_n = \emptyset$ *(that is compact)*
- *Moreover in the thesis we assume that, given a compact* A*, it is possible to find a compact neighbourhood* K *of* A*. We recall that: every point in* Y *has a compact neighbourhood, iff any compact set* A *in* Y *has an open neighbourhood* V *with compact closure* $K = \overline{V}$*. So this hypothesis means that the property holds only in finite dimensional cases: indeed, let* Y *be an infinite dimensional Hilbert space, and let* $A_n \doteq \{y \mid |y| < 1/n\}$ *be decreasing open balls: then* $\lim_n A_n = \{0\}$ *is compact and non-empty.*

We will moreover need to relate the Kuratowski convergence to the convergence of support functions; we state a well known result

Proposition A.5 *Suppose* Y *is a finite dimensional normed vector space and* $A, A_n \subset Y$ *are convex and* A *is compact and non-empty; let* $F_A : Y^* \to \overline{\mathbb{R}}$*,*

$$F_A(v) \doteq \sup\{p \cdot v \mid p \in A\}$$

be the support function of A*, and similarly* F_{A_n} *of* A_n*; then the following are equivalent*

i. $A_n \to A$ *in the Kuratowski sense*

ii. $F_{A_n} \to F_A$ *locally uniformly*

iii. $F_{A_n} \to F_A$ *pointwise.*[44]

Proof. When Y is a generic reflexive Banach space and $f_n, f : Y \to (-\infty, \infty]$ (and $f, f_n \not\equiv \infty$), Mosco [27] defines that $f_n \to f$ when[45]

$$\inf_{\{(z_n), z_n \to y\}} \limsup_{n\to\infty} f_n(z_n) \le f(y) \le \inf_{\{(n_k)\}} \inf_{\{(y_n), y_n \rightharpoonup y\}} \liminf_k f_{n_k}(y_n) \qquad (A.5.\star)$$

for all $y \in Y$; in the above, (z_n) are sequences strongly converging to y, (y_n) are sequences weakly converging to y,and (n_k) are subsequences. Then Mosco proves in Theorem. 3.1 in [27] that $A_n \to A$ iff $F_{A_n} \to F_A$ in the above sense (A.5.$\star$) in Y^*.

[44] pointwise convergence of support functions is known as *scalar convergence* of A_n to A.

[45] Indeed (A.5.$\star$) is now known as *Mosco convergence of functions*.

Suppose now that Y is finite dimensional: it is easy to prove that local uniform convergence implies Mosco convergence (A.5.$\star$), that in turn implies pointwise convergence: that is, $(ii) \implies (i) \implies (iii)$.

Suppose now that (i) holds (and then (iii) holds): since A is compact and non-empty, then, by the above proposition, (A_n) is (eventually) equicompact: this implies that (F_n) is equi-Lipschitz; by the Ascoli–Arzelà theorem, (ii) holds. (Summarizing: $(i) \implies ((iii) \wedge (i)) \implies (ii)$.)

See also cor. P4A in Salinetti–Wets [30]. □

In this paper, things will be slightly more complicated, since we will use *fiberwise support functions*; we discuss the result in in the following section, where we will moreover split the above in a l.s.c. statement A.15 and a u.s.c. statement A.16.

The compactness hypothesis in the above is important: we present a counterexample (derived from the similar example 5.2 in Löhne–Zălinescu [20])

Example A.6 *Let $A : \mathbb{R} \to \mathcal{P}\mathbb{R}^2$ be defined by*

$$A_\theta \doteq \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \cos\theta = x_2 \sin\theta\}$$

(A_θ is a line passing through 0 with angle θ w.r.t the vertical axis). Then for the support function we have

$$F_{A_\theta}(v_1, v_2) \doteq \begin{cases} 0 & \text{if } v_2 \cos\theta = -v_1 \sin\theta \\ +\infty & \text{if not} \end{cases}$$

The function

$$(\theta, v_1, v_2) \mapsto F_{A_\theta}(v_1, v_2)$$

is l.s.c. in $\mathbb{R}^3$ but is not continuous.

§A.i .. on maps

Let then X be a Hausdorff topological space; we extend the above notions to maps $A : X \to \mathcal{P}Y$:

Definition A.7 *We define the **lower Kuratowski limit** $\liminf_{x\to\overline{x}} A_x$ as the set of $y \in Y$ such that*

$$\forall V \ni y, \exists U \ni \overline{x}, \forall x \in U, x \neq \overline{x}, A_x \cap V \neq \emptyset,$$

*where $V \subset Y$ and $U \subset X$ are open. We define the **upper Kuratowski limit** $\limsup_{x\to\overline{x}} A_x$ as the set of $y \in Y$ such that*

$$\forall V \ni y, \forall U \ni \overline{x}, \exists x \in U, x \neq \overline{x}, A_x \cap V \neq \emptyset,$$

*We define **lower-semi-continuous (and u.s.c.) maps in the Kuratowski sense** when the lower Kuratowski limit $\liminf_{x\to\overline{x}} A_x$ contains $A_{\overline{x}}$ (resp. if $\limsup_{x\to\overline{x}} A_x \subset A_{\overline{x}}$); as usual.*

In the following section we further extend the above notion to fiber bundles.

Proposition A.8 *If the topology of X and Y are characterized by converging sequences (cf. note 41), then equivalently we say that the map $A : X \to \mathcal{P}Y$ is **lower-semi-continuous in the Kuratowski sense** if, for any sequence $(x_n)_n \subset X$ with $x_n \neq x$ and $x_n \to x$, for any $y \in A$, there is a sequence $y_n \in A_{x_n}$ s.t. $y_n \to y$; resp. **upper-semi-continuous in the Kuratowski sense** if for any sequence $(x_n)_n \subset X$ with $x_n \neq x$ and $x_n \to x$, for any converging sequence $y_n \in A_{x_n}$ s.t. $y_n \to y$, we have $y \in A$.*

Remark A.9 *The above is different from the definition presented in [29], where it is not assumed that $x_n \neq x$; using the definition in [29], it would always hold that*

$$\limsup_{x\to\overline{x}} A_x \supset A_{\overline{x}}$$

and then some following results would have a different statement.

§A.ii .. on fiber bundles

We assume, for sake of ease, that X satisfies *"the first countability axiom"*. Suppose that Y is a finite dimensional normable vector space; suppose that N is a fiber bundle with fiber Y, and $\pi : N \to X$ be the projection; let $N_x = \pi^{-1}(x)$ be the fiber. N is equipped with an atlas of *fiberwise* local coordinates that are defined, for any $x \in U$, by a small open $U \subset X$ containing x, and by an homeomorphism

$$\phi : U \times Y \to \pi^{-1}(U) \tag{A.10}$$

such that (for all $x \in U$) $\phi(x, \cdot)$ is a linear isomorphism between $\{x\} \times Y$ and $N_{\phi(x)}$.

We extend the above definitions on Kuratowski limits, to maps into the fiber bundle N, using these local coordinates (this is OK in view of (A.2.$\star$)).

Definition A.11 (Slicing & graph) *If B is a subset of N, we will slice it to define the map*

$$x \mapsto B_x \doteq N_x \cap B$$

Conversely, if we consider fiberwise maps *$x \mapsto A_x$ such that $A_x \subset N_x$ is not empty; we associate to any such map its graph*

$$A \doteq \bigcup_x A_x$$

For example in section 8.3, we considered maps $x \mapsto A_x$ from M to $N = T^*M$, with $A_x \subset T^*_x M$, then we will for simplicity define the graph as

$$A \doteq \{(x, p) \in T^*M \mid p \in A_x\} \ .$$

Lemma A.12 *Choose $A \subset N$, let A_x be the slicing of A. If A is closed, then $x \mapsto A_x$ is upper-semi-continuous. Viceversa if every slice A_x is closed, and the map $x \mapsto A_x$ is upper-semi-continuous in the Kuratowski sense, then A is closed.*

Proof. Fix $x \in X$. Pull back $x \mapsto A_x$ to a map $U \to Y$ (using local coordinates ϕ around x), that we call $\tilde{A}_x$.

Choose $x_n \to x$ in X. Choose any $y \in \limsup_{n\to\infty} \tilde{A}_{x_n}$: then there is a sequence y_n with $y_n \in \tilde{A}_{x_{m(n)}}$ for some $m(n) \geq n$, such that $y_n \to y$. From $\phi(x_{m(n)}, y_n) \in A$ then $\phi(x, y) \in A$: so $y \in \tilde{A}_x$.

Viceversa, let z_n be a sequence in A converging to a $z \in N$; we write it in local coordinates as $(x_n, y_n) \to (x, y)$; if $x_n = x$ eventually, we use the fact that A_x is closed; otherwise by $x_n \to x$ we know that $y \in \limsup_{n\to\infty} \tilde{A}_{x_n} \subset \tilde{A}_x$, that is, $z \in A$. □

Lemma A.13 *Suppose $A \subset N$ is open; let A_x be the slicing of A: then the map $x \mapsto A_x$ is lower-semi-continuous in the Kuratowski sense.*

Proof. Fix $x \in X$. Pull back $x \mapsto A_x$ to a map $U \to Y$ (using local coordinates ϕ around x), that we call $\tilde{A}_x$. Choose $y \in \tilde{A}_x$. Choose any $x_n \to x$: then eventually $\phi(x_n, y) \in A$, that is $y \in \tilde{A}_{x_n}$. Then $y \in \liminf_{n\to\infty} \tilde{A}_{x_n}$. □

§A.iii Dual of a set map

Suppose moreover that N^* is the dual bundle of N, that is, $\pi^* : N^* \to X$ and $N^*_x \doteq (\pi^*)^{-1}(\{x\})$ is isomorphic to Y^*.

We consider *fiberwise maps* $x \mapsto A_x$ with $A_x \subset N^*_x$; we suppose that any A_x is non empty. We define the *fiberwise support function*

$$F : N \to (-\infty, \infty]$$

so that $F(x, \cdot)$ is the support function of the set A_x: in local coordinates,

$$F(x,v) \doteq \sup\left\{p \cdot v \mid p \in A_x\right\} . \qquad (A.14)$$

Lemma A.15 *If the map $x \mapsto A_x$ is lower-semi-continuous in the Kuratowski sense, then F is lower-semi-continuous.*

Proof. Fix $x \in X$. Pull back $x \mapsto A_x$ to a map $U \to Y^*$ (using fiberwise local coordinates ϕ in a neighbourhood U around x). Similarly, pull back F in local coordinates.

- Fix x, v, and choose any $r < F(x,v)$. Choose sequences such that $v_n \to v$ and $x_n \to x$. Choose p such that $p \in A_x$ and $r < p \cdot v \le F(x,v)$.

  Since $x \mapsto A_x$ is l.s.c., there is a sequence $p_n \in A_{x_n}$ such that $p_n \to p$. So

  $$F(x_n, v_n) \ge p_n \cdot v_n \to p \cdot v \ge r$$

  and then

  $$\liminf_{n\to\infty} F(x_n, v_n) \ge F(x,v)$$

  by arbitrariness of r.

- We also show the proof in case A is open (which is redundant, in view of A.13): indeed we may see $F(x,v)$ as the supremum for $p \in Y^*$ of the l.s.c. functions

  $$(x,v) \mapsto \begin{cases} p \cdot v & \text{if } (x,p) \in A \\ -\infty & \text{if not} \end{cases}$$

□

Lemma A.16 *Fix $\overline{x} \in X$. If the map $x \mapsto A_x$ is upper-semi-continuous in the Kuratowski sense at $\overline{x}$, any A_x is connected, $\liminf_{x\to\overline{x}} A_x$ is non-empty, and $A_{\overline{x}}$ is compact, then for any fixed $\overline{z}$ with $\pi^*(\overline{z}) = \overline{x}$, F is locally bounded and upper-semi-continuous at $\overline{z}$.*

Proof. We again work in local coordinates. Suppose $A_{\overline{x}}$ is compact: by A.3 there is a neighbourhood U of $\overline{x}$ and a K compact such that $A_x \subset K$ for all $x \in U$. Fix $\overline{v}$, and V a compact neighbourhood of $\overline{v}$ in Y: then $|F(x,v)|$ is bounded in a neighbourhood $U \times V$ of $\overline{z} = (\overline{x}, \overline{v})$ (by $\sup\{|p \cdot v|, p \in K, v \in V\}$).

Choose sequences in V and U, such that $v_n \to \overline{v}$ and $x_n \to \overline{x}$, and suppose, without loss of generality, that the sequence $F(x_n, v_n)$ is increasing.

Choose $p_n \in A_{x_n}$ such that

$$\lim_n F(x_n, v_n) = \lim_n p_n \cdot v_n$$

On the other hand, $p_n \in K$ so we may extract a subsequence p_{n_m} converging to a limit point q: we have that $q \in \limsup_m A_{n_m} \subset A_x$ and then

$$F(x,v) \ge q \cdot v = \lim_m p_{n_m} \cdot v_{n_m} = \lim_n F(x_n, v_n)$$

□

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