EXISTENCE OF FLAT BLOWUPS AT BOUNDARY POINTS OF ANISOTROPIC MINIMIZING HYPERCURRENTS

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Abstract. We consider m-dimensional currents in $\mathbb{R}^{m+1}$ that almost minimize an anisotropic energy whose integrand is continuous in the space variable and $C^{2,\mathrm{Dini}}$ in the normal variable. We prove that if the boundary of such a current is differentiable, then the current admits a flat blowup at every boundary point.

Contents

1. Introduction

In their groundbreaking work [8], Hardt and Simon showed that boundary regularity for codimension-one area-minimizing currents is determined by their blowups. More precisely, let T be an area-minimizing m-current in $\mathbb{R}^{m+1}$ with multiplicity-one boundary Γ of class $C^{1,\alpha}$. They proved that, for every $p \in \Gamma$, T admits a flat blowup at p if, and only if, T is regular near p. This equivalence reduces the boundary regularity problem to the study of blowups. They then proved that

every blowup of T at a boundary point is flat,

and hence that T attaches regularly to Γ at every point.

For anisotropic energies, the problem has remained largely open, the basic obstruction being the absence of a monotonicity formula; see Remarks 1.4 and 1.5. For general boundary points, it was not known whether a flat blowup exists, nor whether the existence of one implies regularity. Indeed, Hardt and Simon [8] observe that for two-dimensional anisotropic minimizers in $\mathbb{R}^3$, "at an arbitrary boundary point a, even the existence of an oriented tangent *cone* is unknown," and we are not aware of any progress on this question. In this paper, we answer the blowup question in every dimension; namely, we prove that every boundary point of an anisotropic minimizer with multiplicity-one boundary supported on a differentiable manifold admits a flat blowup.

The existing regularity results for anisotropic minimizers are conditional on the behavior of the minimizer at the point in question. In the interior, where the theory is well developed, Schoen, Simon, and Almgren [14] proved that the singular set has vanishing $(m-2)$-dimensional Hausdorff measure, a bound which can be improved to $\mathcal{H}^{(m-2-\nu)}$-null for some $\nu > 0$; cf. the discussion in [7, Corollary 2.5]. In the other direction, Morgan [13] produced an example of a three-dimensional minimizing cone in $\mathbb{R}^4$ with a singularity at the origin. At the boundary, Hardt [9] proved regularity at points in which spt T has a supporting hyperplane, and Duzaar and Steffen [5] proved an ε-regularity theorem at points where the lower density is at most $1/2+\varepsilon$. Our theorem, in contrast, gives an unconditional conclusion at the level of blowups: every boundary point admits a flat blowup.

We now state our main result and refer the reader to Section 2 for the precise definitions.

Theorem 1.1 (Existence of a flat blowup at the boundary)**.** *If $m\in\mathbb{N}$, F is a uniformly elliptic integrand of class $C^{2,\mathrm{Dini}}$, T is an integer rectifiable m-current that is $(\mathbf{F},\omega)$-minimizing in $\mathbf{B}_1$, and ∂T is a multiplicity-one current whose support is a differentiable manifold containing 0, then there exist $r_j\to 0$, $Q\in\mathbb{N}$, and a half-hyperplane Π^+ such that*

$$T_{0,r_j}\to Q[\![\Pi^+]\!]-(Q-1)[\![-\Pi^+]\!]\,.$$

Remark 1.2 (Nonautonomous integrands)**.** Theorem 1.1 also applies to energies with nonautonomous integrands, that is, $F:\mathbb{R}^{m+1}\times\mathbb{R}^{m+1}\to\mathbb{R}$ that is continuous in the first variable and uniformly elliptic of class $C^{2,\mathrm{Dini}}$ in the second variable. Indeed, any $(\mathbf{F},\omega)$-minimal current T is $(\mathbf{F}_0,\omega+C\omega_0)$-minimal where $F_0(\nu):=F(0,\nu)$, $\omega_0(r):=\sup_{\nu\in\mathbb{S}^m}\{|F(x,\nu)-F(y,\nu)|:|x-y|\le r\}$, and $C>1$ is a universal constant. Since F is continuous in the space variable, $\omega+C\omega_0$ is an admissible error term, and so the conclusion of Theorem 1.1 holds in this setting.

Remark 1.3 (Higher multiplicity boundaries)**.** If ∂T has multiplicity P for some $P>1$, then an argument of White [17] shows that T decomposes on $\mathbf{B}_1$ into P anisotropic area minimizers $T_1,\dots,T_P$ with multiplicity one boundaries, and so Theorem 1.1 applies separately to each T_i. If one of the T_i's has a two sided blowup, then the blowup of T is a two-sided plane with multiplicities Q and $Q-P$ on each half plane, whereas if all T_i's have one-sided blowups, this blowup will be an open book with sheets that may or may not coincide.

Remark 1.4 (On full boundary regularity)**.** When $Q=1$ in Theorem 1.1, the subsequential blowup is a half-hyperplane with multiplicity one (a "one-sided" blowup) and the result [5] of Duzaar and Steffen applies, and spt T is an anisotropic minimal surface with boundary in a neighborhood of 0. Complete boundary regularity would therefore follow from an ε-regularity theorem when $Q>1$ and a blowup is "two-sided". In the case of the area functional, the argument is rather delicate and relies in a crucial way on the remainder term in the monotonicity formula [8]. Addressing this issue and establishing the full boundary regularity program in the anisotropic case remains open.

1.1. **Outline of the proof.** The existence of nontrivial "classical" blowups, with no need for Preiss blowups, follows from upper mass ratio bounds and the fact that the boundary is induced by a differentiable manifold; see Remark 6.1. Since the set of blowups at zero is closed under further blowups, it therefore suffices to consider the case where, up to a rotation, T is an integer rectifiable m-current with multiplicity one boundary supported in $\mathbb{R}^{m-1}\times\{0\}^2:=L$ that minimizes $\mathbf{F}$ in $\mathbb{R}^{m+1}$. The proof consists of two main steps: a barrier argument that precludes spt T from wrapping around L infinitely often, and a blowup argument that utilizes a parametric Hopf lemma to extract a limiting tangent plane.

1.1.1. *Boundedness of the polar angle function.* Let M be the oriented half-hyperplane $\{x_m>0\}\cap\{x_{m+1}=0\}$ with boundary L. The key quantity is a polar angle function $\boldsymbol{\theta}$ (Definition 5.3), defined on the interior regular set $\mathrm{Reg}_i T\subset\operatorname{spt}T$, which counts how many times T has wound around L in the (x_m,x_{m+1})-variables (relative to a fixed base point). The reader may keep in mind the picture of $T\subset\mathbb{R}^2$ as a spiral winding around L, crossing M once at each rotation and taking values in $2\pi\mathbb{Z}$ on $M\cap\mathrm{Reg}_i T$; see Figure 1. The goal in this step is to show that $\boldsymbol{\theta}$ is locally bounded, cf. Theorem 5.5. This will be necessary in order to identify a candidate tangent plane for T at the origin.

Let us consider bounding $\boldsymbol{\theta}$ from above on $\mathbf{B}_1\cap\mathrm{Reg}_i T$; the lower bound argument is entirely analogous. Such an upper bound would follow from showing that for some large $N\in\mathbb{N}$, the superlevel set $\{\boldsymbol{\theta}(x)>2\pi N\}\cap\mathbf{B}_1$ is contained in the halfspace $\{x_m>0\}$, since this implies that $\boldsymbol{\theta}\le 2\pi N+\pi/2$ on $\mathbf{B}_1\cap\mathrm{Reg}_i T$. In turn, to prove this halfspace containment, we use a barrier set U such that

$$\{\boldsymbol{\theta}(x)>2\pi N\}\cap\mathbf{B}_1\subset U\cap\mathbf{B}_1\subset\{x_m>0\}\qquad\text{for some large } N\in\mathbb{N}\,. \tag{1.1}$$

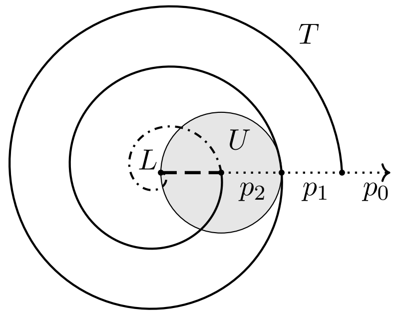

FIGURE 1. At the base point p_0, $\boldsymbol{\theta}(p_0) = 0$. Then $\boldsymbol{\theta}(p_1) = 2\pi$, $\boldsymbol{\theta}(p_2) = 4\pi$. The cross section of the barrier set $U \subset \{x_m > 0\}$ through any two-dimensional slice perpendicular to L over $x \in \mathbf{B}_1 \cap L$ is the gray disk tangent to L shown above; see Figure 2 below to see this slice relative to the whole picture. Since that disk has positive curvature, it acts as a barrier forcing an area minimizing surface to connect p_2 to L through a path contained in U rather than exiting U along the alternating dotted/dashed spiral; in $\mathbb{R}^2$, this path is the bold dashed segment shown above.

We have thus reduced bounding $\boldsymbol{\theta}$ from above to the construction of a barrier set U satisfying (1.1).

To choose N and prove (1.1), let $N \in \mathbb{N}$ and T_N be the current induced by $\mathbf{B}_2 \cap \{\boldsymbol{\theta} > 2\pi N\}$. We first show that for large N, the support of T_N is contained in an arbitrarily small neighborhood of L. This follows from lower density bounds by arguing that, if T had wound around L too many times at positive distance from L, the mass of T would be infinity. We then construct U so that $U \cap \mathbf{B}_1$ is the cylinder $\{|(x_m, x_{m+1}) - (\delta, 0)| < \delta\}$ contained in $\{x_m > 0\}$ and tangent to L, and $U \setminus \mathbf{B}_2$ is a small dilation of this cylinder; see Figure 2. Thus U is $\mathbf{F}$-mean convex. In fact, U is also mean convex with respect to the reflected anisotropy $G(\nu) = F(-\nu)$, which will be necessary in order to address possible mismatches in orientation at touching points. Furthermore, for large enough N, spt T_N is contained in an arbitrarily small neighborhood of U and spt $\partial T_N \subset \overline{U}$, and so a maximum principle (cf. Proposition 4.3) implies that spt T_N must be contained in $\overline{U}$.

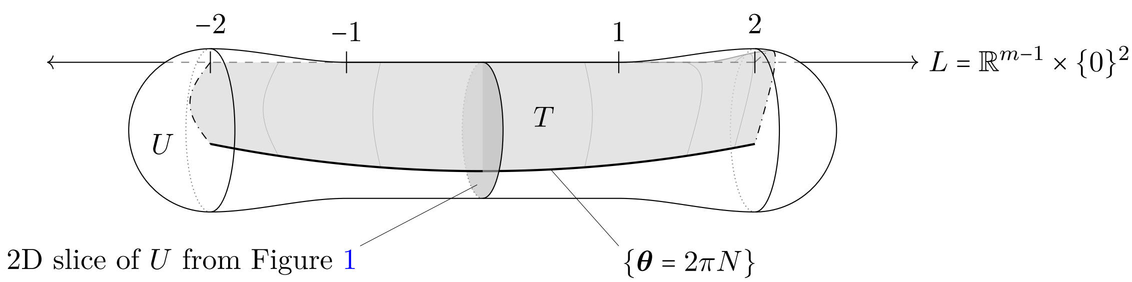

FIGURE 2. Here the plane containing the page is $\{x_m = 0\}$, so that the half plane corresponding to angle 0 and containing the bold level set $\{\boldsymbol{\theta} = 2\pi N\}$ is the portion below L. The anisotropic area minimizer induced by $\mathbf{B}_2 \cap \{\boldsymbol{\theta} > 2\pi N\}$ is the gray surface T_N, and it is bounded by four curves: a portion of L, $\{\boldsymbol{\theta} = 2\pi N\}$, and the alternating dotted/dashed curves $\{\boldsymbol{\theta} > 2\pi N\} \cap \mathbf{B}_2$ coming from slicing T by $\partial \mathbf{B}_2$. The barrier set U is built so that it contains these four curves, and thus the boundary of T_N. This allows for a maximum principle argument yielding spt $T_N \subset U$.

1.1.2. *Parametric Hopf lemma and blowups.* Once we know that $\boldsymbol{\theta}$ is bounded on $\mathbf{B}_1$, we have a natural candidate for the limiting halfplane(s) hopefully contained in a flat blowup: the half plane $M_{\min}$ with boundary L corresponding to the angle

$$\boldsymbol{\theta}_{\min} = \lim_{r \searrow 0} \inf\{\boldsymbol{\theta}(x) : x \in \mathbf{B}_r \cap \mathrm{Reg}_i T\} =: \lim_{r \searrow 0} \boldsymbol{\theta}_r \,.$$

We emphasize that the infimums exist for each $r \in (0,1)$ only because $\boldsymbol{\theta}$ is bounded, and that $\boldsymbol{\theta}_{\min}$ exists since it is the limit of a bounded, monotone sequence. To simplify the notation, let us assume that $\boldsymbol{\theta}_{\min} \in 2\pi\mathbb{Z}$. We also denote by M_r the halfplanes corresponding to angles $\boldsymbol{\theta}_r$.

The first step in the identification of a flat blowup is showing that $M_{\min}$ must be contained in the support of any blowup T_0 of T at 0. Postponing the summary of this argument for a moment, the remainder of the argument is an iterative procedure where we take further blowups, repeating at each stage the argument that yielded $M_{\min}$ to identify further halfplanes contained in these blowups. By upper density bounds on minimizers, this process must terminate after finitely many iterations, resulting in a final blowup given by a union of halfplanes with boundary L. By a two-dimensional argument using the $\mathbf{F}$-minimality of the final blowup and convexity of F (cf. Lemma 3.3), these half hyperplanes must be all contained in the same plane, and so that plane supports a flat blowup of T at 0.

The proof that $M_{\min}$ is contained in the support of any blowup T_0 of T at 0 consists of two steps. First, assuming for contradiction that $M_{\min} \notin \operatorname{spt} T_0$, we may choose $x_0 \in M_{\min} \setminus \operatorname{spt} T_0$ and a small ball $\mathbf{B}_{r_0}(x_0)$ disjoint from $\operatorname{spt} T_0$. We now argue that there exists a connected component of $\operatorname{spt} T_0 \cap \{x_m > 0\}$ contained in the wedge $W = \{x_m > 0, x_{m+1} > 0\}$, which heuristically follows from the definition of $\boldsymbol{\theta}_{\min} \in 2\pi\mathbb{Z}$. Indeed, since $\boldsymbol{\theta}_r$ is the infimum of $\boldsymbol{\theta}$ on $\mathbf{B}_r$, for small $r > 0$, the component of $\operatorname{spt} T_{0,r} \cap \mathbf{B}_1 \cap \{x_m > 0\}$ achieving $\boldsymbol{\theta}_r$ cannot cross the halfplane corresponding to $\boldsymbol{\theta}_r$ (as it would thus take values strictly smaller than $\boldsymbol{\theta}_r$), and so taking a limit in these components yields a component of $\operatorname{spt} T_0 \cap \{x_m > 0\}$ contained in W. Second, using the "separation" between T_0 and $M_{\min}$ given by $\mathbf{B}_{r_0}(x_0)$ and the fact that the support of T is contained in the wedge W, for small $r > 0$ we insert the graph of a barrier function u_r in between M_r and $\operatorname{spt} T_{0,r} \cap \{x_m > 0\} \cap \mathbf{B}_1$ for small r. By a parametric Hopf lemma (Proposition 4.5), the graph of u_r meets L at a small positive angle δ relative to the halfplane corresponding to $\boldsymbol{\theta}_r$, and a compactness argument shows that δ depends on r_0 but not on r; see Figure 3. Thus, for r small enough, the parametric Hopf lemma also gives that the component of $\operatorname{spt} T_{0,r} \cap \mathbf{B}_1 \cap \{x_m > 0\}$ achieving $\boldsymbol{\theta}_r$ lives above the plane determined by $\boldsymbol{\theta}_r$. This contradicts the definition of $\boldsymbol{\theta}_r$ and completes the proof that $M_{\min} \subset \operatorname{spt} T_0$.

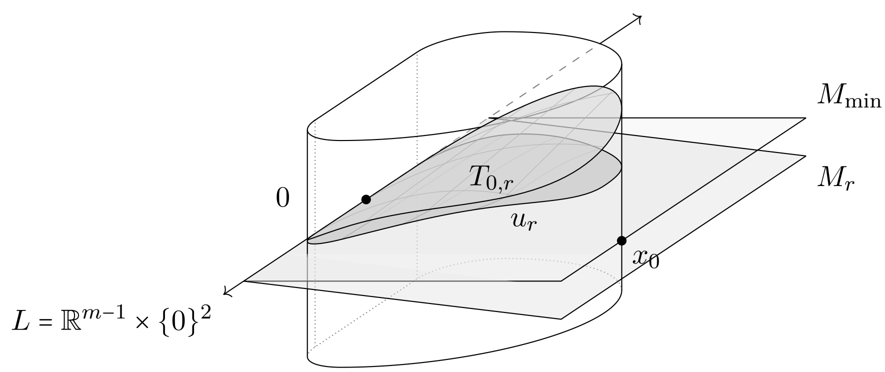

FIGURE 3. The barrier function u_r is constructed on a cylinder contained in $\{x_m > 0\}$, meets L with a positive angle above M_r, and is squeezed in between M_r and $T_{0,r}$. The positive angle δ depends on the distance of x_0 to the blowup.

Remark 1.5 (Comparison with the area case)**.** The idea of using a maximum principle argument nearby a point of minimum (or maximum) angle $\boldsymbol{\theta}$ to deduce flatness is due to Hardt and Simon [8], but they use a completely different argument to control $\boldsymbol{\theta}$. In the isotropic case, the monotonicity formula and resulting *conical* blowups imply that $\boldsymbol{\theta}$ is positively one homogeneous, and in particular is characterized by its values on the link of a blowup cone. By an induction on dimension using their ε-regularity theorem and the conical structure, the link is regular up to the boundary, and so $\boldsymbol{\theta}$ is continuous on the link and thus achieves its maximum. For the anisotropic problem considered here, blowups are not cones a priori and there is not a two-sided ε-regularity theorem, so $\boldsymbol{\theta}$ is not readily bounded by using continuity and compactness on the link. Therefore, we instead introduce the new barrier set to bound the polar angle function and circumvent the lack of a monotonicity formula and ε-regularity theorem.

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2. Notation and background

We use $\mathbf{B}_r(p)$ for the open ball of radius r and centered at p. If $p = 0$, we simply write $\mathbf{B}_r$. We use the notation $p = (p', p_{m+1}) \in \mathbb{R}^m \times \mathbb{R}$ and $B_r(p')$ for the open ball in $\mathbb{R}^m$ of radius r and centered at p'.

For any subset V of $\mathbb{R}^{m+n}$, define its ϵ-neighborhood by $B_\epsilon(V) := \{x \in \mathbb{R}^{m+n} : \operatorname{dist}(x, V) < \epsilon\}$. We denote the k-dimensional Hausdorff measure in $\mathbb{R}^{m+n}$ by $\mathcal{H}^k$.

For background on the theory of currents, we refer the reader to [16, 6].

Fixing $\{e_1, \dots, e_{m+1}\}$ to be the canonical basis of $\mathbb{R}^{m+1}$, the Hodge star operator will always be denoted by $\star$, and it is defined as the linear isometry characterized by $\star(e_1 \wedge \dots \widehat{e_i} \dots \wedge e_{m+1}) = (-1)^{m+1-i} e_i$ for any $i \in \{1, \dots, m+1\}$.

We refer the reader to [12] for background on sets of finite perimeter. Given a set of locally finite perimeter $E \subset \mathbb{R}^{m+1}$, we denote its reduced boundary by $\partial^* E$, the local perimeter by $P(E; A)$ for any measurable set $A \subset \mathbb{R}^{m+1}$, and its outward unit normal at x by $\nu_E(x)$ whenever it exists. We denote by $P_F(E; A)$ the anisotropic perimeter of E relative to A.

Our convention for the mean curvature is such that $\partial \mathbf{B}_r$ has mean curvature $1/r$.

2.1. Anisotropic integrands and $(\mathbf{F}, \omega)$-minimizers.

Let us set the nomenclature and hypothesis on the integrands that will be used in this work. A function $F : \mathbb{R}^{m+1} \to \mathbb{R}_+$ is called a *geometric or parametric integrand*, and the *anisotropic mass of T*, where T is an integer rectifiable m-current in $\mathbb{R}^{m+1}$, is the number

$$\mathbf{F}(T) = \int_{\mathbb{R}^{m+1}} F(\star \vec{T}(x)) d|T|(x).$$

Fixing the direction e_{m+1}, the function $F_\S : \mathbb{R}^m \to \mathbb{R}$ defined by $F_\S(v) = F((-v, 1))$ is called a *non-parametric integrand*. Thus for an open subset $\Omega \subset \mathbb{R}^m$ and the current $\mathbf{G}_u$ induced by the graph of a $W^{1,1}$ function $u : \Omega \to \mathbb{R}$ oriented by its upward pointing normal,

$$\mathbf{F}(\mathbf{G}_u) = \mathscr{F}(u) := \int_\Omega F_\S(\nabla u) \, d\mathcal{H}^m . \tag{2.1}$$

The following is the regularity class of our integrands.

Definition 2.1. We say that F *is uniformly elliptic of class $C^{2,\mathrm{Dini}}$* if F is a geometric integrand that is positively one-homogeneous, C^2 on $\mathbb{R}^{m+1} \setminus \{0\}$, and satisfies

$$a|\tau|^2 \le D^2 F(\nu)[\tau, \tau] \le A|\tau|^2, \qquad \forall\, \nu \in \mathbb{S}^m, \quad \tau \in \nu^\perp .$$

Moreover, setting $\rho_F(r) := \sup\{|D^2F(u) - D^2F(v)| : u, v \in \mathbb{S}^m,\ |u-v| \le r\}$, we require $\int_0^1 \frac{\rho_F(r)}{r} \, dr < +\infty$. We denote the upper and lower bounds of F on $\mathbb{S}^m$ by Λ and λ, respectively.

We define the interior regular set of a current.

Definition 2.2 (Regular sets). Let T be an integer rectifiable m-current in $\mathbb{R}^{m+1}$. The interior regular set $\mathrm{Reg}_i T$ is the set of all points $x \in \operatorname{spt} T \setminus \operatorname{spt} \partial T$ for which there are $r > 0$, $\beta > 0$, and an embedded hypersurface Σ of class $C^{1,\beta}$ satisfying $\Sigma \cap \mathbf{B}_r(x) = \operatorname{spt} T \cap \mathbf{B}_r(x)$.

We now define almost minimality.

Definition 2.3 ($(\mathbf{F},\omega)$-minimizers)**.** Let $U\subset\mathbb{R}^{m+1}$ be open, let B be an $(m-1)$-dimensional integral current, and let $\omega:[0,\infty)\to[0,\infty)$ be nondecreasing with $\lim_{r\to 0}\omega(r)=0$. We say that an m-dimensional integral current T is an *$(\mathbf{F},\omega)$-minimizer in U with boundary B* if $\partial T=B$ and

$$\mathbf{F}(T)\leq\mathbf{F}(T+X)+\omega(r)\mathbf{M}(X)$$

for every $r>0$ and every m-dimensional integer rectifiable current X such that $\partial X=0,\operatorname{spt}X\subset U$, and $\operatorname{diam}(\operatorname{spt}X)\leq r$. If $\omega\equiv 0$, we say that T is an $\mathbf{F}$-minimizer.

3. Preliminary results

3.1. **Small data existence for the Dirichlet problem and a comparison principle.** Here we adapt the argument of [7, Lemma 3.5] to show the existence and $C^{1,\alpha}$-regularity up to the boundary for the Dirichlet problem corresponding to the operator $\nabla u\mapsto\operatorname{div}\nabla F_{\S}(\nabla u)$. We recall for example that in the case of the minimal surface operator, mean convexity of Ω is equivalent to existence of solutions to the Dirichlet problem for arbitrary continuous boundary data, cf. [10, Theorem 1]. Thus some smallness assumption on the boundary data is necessary to have existence for general smooth domains (the sharp condition is essentially smallness in the Lipschitz seminorm [18]).

Proposition 3.1 (Small data existence for the Dirichlet problem)**.** *If $F:\mathbb{R}^{m+1}\to\mathbb{R}_+$ is uniformly elliptic of class $C^{2,\mathrm{Dini}}$, U is a smooth domain, $\eta\in(0,1)$, then there are $\kappa_0>0$, $C>0$, and $\alpha\in(0,1)$ with the following property:*

if $g\in C^\infty(\partial U)$ and $\|g\|_{C^{1,\eta}(\partial U)}\leq\kappa_0$, then there is a unique $u\in(C^{1,\alpha}\cap C^2)(U)$ minimizing $\mathscr{F}$ on U among $v\in W^{1,1}(U)$ with $v=g$ on ∂U and satisfying $\operatorname{div}\nabla F_{\S}(\nabla u)=0$ in the weak sense with

$$\|u\|_{C^{1,\alpha}(U)}\leq C\|g\|_{C^{1,\eta}(\partial U)}\,.$$

Proof. Let $\tilde F$ be a modification of $F_{\S}$ such that $\tilde F=F_{\S}$ on B_{100} and $\tilde F$ satisfies

$$1/c<D^2\tilde F(x)<c\qquad\text{for some } c>1\,.$$

Then $\tilde F$ is uniformly convex with quadratic growth, so the direct method in $W^{1,2}(U)$ yields a unique minimizer of $\int_U\tilde F(\nabla u)$. The minimality of u for this modified functional is enough to apply verbatim the proof of [7, Lemma 3.5] (which is stated for balls but applies to any C^2 domain since the boundary regularity result [15] does), yielding the existence of $\alpha\in(0,1)$ and $C>0$ such that

$$\|u\|_{C^{1,\alpha}(U)}\leq C\|g\|_{C^{1,\eta}(\partial U)}\,.$$

Thus by choosing κ_0 small enough, we may ensure that $|\nabla u|<100$ and $F_{\S}(\nabla u)=\tilde F(\nabla u)$ on U. As a consequence, u in fact solves $\operatorname{div}\nabla F_{\S}(\nabla u)=0$ in the weak sense. The strict convexity of $F_{\S}$ now implies that u minimizes $\mathscr{F}$, and the proof is complete save for the claim that $u\in C^2$. To prove the C^2 regularity, we first observe that by the gradient bound, the Euler Lagrange operator is uniformly elliptic, and so a standard difference quotient argument shows that $u\in W^{2,2}(U)$. Therefore, the partial derivatives of u are weak solutions to the equation $\partial_i[(F_{\S})_{p_ip_j}(\nabla u)\partial_j(\partial_k u)]=0$, which is a uniformly elliptic divergence form equation with Dini coefficients. This allows us to apply the C^1 regularity result of [11] for such equations to deduce that $\partial_k u\in C^1$ for each k, and so $u\in C^2$. $\square$

3.2. **Mass ratio bounds.** The fact that $\mathbf{F}$-minimizers satisfy mass ratio upper bounds is understood to be standard, but we failed to find a reference for it at boundary points. Therefore, we include a proof below.

Lemma 3.2 (Local upper $\mathbf{F}$-bounds)**.** *Under the assumptions of Theorem 1.1, for every $x\in\operatorname{spt}(T)$, there are C_x, Q_x, and $R(x)>0$, depending on T, F, and x, such that $\mathbf{F}(T\llcorner\mathbf{B}_r(x))\leq C_x r^m$ for all $r\in(0,R(x))$ and the multiplicity of T is bounded by Q_x on $\mathbf{B}_{R(x)}(x)$.*

Proof. We denote $\partial T \llcorner \mathbf{B}_1 = [\![\Gamma]\!]$. If $x \in \operatorname{spt} T \setminus \Gamma$, then the result is standard, so we assume $x \in \Gamma$. First we show the multiplicity bound. Decompose $\operatorname{Reg}_i T \cap \mathbf{B}_1$ into its connected components C_i. Note that up to $\mathcal{H}^m$-null sets, the multiplicity function Θ is some constant Q_i on each C_i. Since, by interior regularity, $\mathbf{F}(T \llcorner \mathbf{B}_1) = \sum \mathbf{F}(Q_i [\![C_i]\!] \llcorner \mathbf{B}_1)$, a standard argument using the subadditivity of $\mathbf{F}$ shows that each C_i is an $\mathbf{F}$-minimizer. Now for those C_i such that $\mathbf{B}_1 \cap \partial C_i = \varnothing$, interior lower density bounds (see e.g. [2, Lemma 4]) imply that $\mathbf{F}(Q_i [\![C_i]\!] \llcorner \mathbf{B}_r(y) \gtrsim r^m$ for any $y \in \mathbf{B}_1$ and $r < 1 - |y|$, which implies that only finitely many such C_i have support intersecting $\mathbf{B}_{(1-|x|)/2}$ non-trivially. Thus, choosing $R \in (0, (1-|x|)/2)$ such that $\Gamma \cap \mathbf{B}_R(x)$ is connected, the multiplicities coming from these components are bounded on $\mathbf{B}_{(1-|x|)/2}$. So it remains to bound the multiplicites on the other components.

We next claim that for each C_i with $\mathbf{B}_R(x) \cap \partial C_i \neq \varnothing$, $\partial [\![C_i]\!] \llcorner \mathbf{B}_R(x) = \pm [\![\Gamma]\!] \llcorner \mathbf{B}_R(x)$. To prove this, first we observe that $\partial [\![C_i]\!] \llcorner (\mathbf{B}_R(x) \setminus \Gamma) = 0$. Indeed, $\operatorname{spt}(\partial [\![C_i]\!]) \cap \mathbf{B}_R(x) \subset \operatorname{spt} T \cap \mathbf{B}_R(x) = [\operatorname{Reg}_i T \cup \Gamma \cup \operatorname{Sing}_i T] \cap \mathbf{B}_R(x)$, but for every $x \in \operatorname{Reg}_i T$, there is $\mathbf{B}_{r_x}(x)$ and $j(x)$ such that $T \llcorner \mathbf{B}_{r_x}(x) = Q_{j(x)} [\![C_{j(x)}]\!] \llcorner \mathbf{B}_{r_x}(x)$ (in which case $[\![C_{j'}]\!]$ is 0 on that ball for $j' \neq j(x)$), and $\operatorname{spt} T \setminus (\operatorname{Reg}_i T \cup \Gamma)$ has Hausdorff dimension less than $m-2$ (by interior regularity [7]). So $\partial [\![C_i]\!] \llcorner \mathbf{B}_R(x)$ is supported on Γ, and then by the constancy lemma ([6, 4.1.31]) we conclude that $\partial [\![C_i]\!] \llcorner \mathbf{B}_R(x) = k_i [\![\Gamma]\!] \llcorner \mathbf{B}_R(x)$ for some $k_i \in \mathbb{Z}$. Furthermore, by White's decomposition [17], since each $[\![C_i]\!]$ is an $\mathbf{F}$-minimizer and C_i is connected it must be the case that each $|k_i| = 1$. Indeed, if $|k_i| > 1$, we could decompose $[\![C_i]\!]$ into two nontrivial $\mathbf{F}$-minimizers, $[\![C_i^1]\!] + [\![C_i^2]\!]$, but since C_i is connected, C_i^1 and C_i^2 must coincide, which is impossible since the multiplicity of $[\![C_i]\!]$ is one.

To conclude the proof that the multiplicity is bounded on $\mathbf{B}_R(x)$, since Θ is constant on each C_i which intersects $\mathbf{B}_R(x)$, it suffices to show that there are a finite number of C_i's with $\mathbf{B}_R(x) \cap C_i \neq \varnothing$. Indeed, if there were infinitely many, then, using $\sum \mathbf{F}([\![C_i]\!] \llcorner \mathbf{B}_R(x)) < \infty$ and the properties of F, we could find a subsequential limit T' of $\{[\![C_i]\!] \llcorner \mathbf{B}_R(x)\}_{i \in \mathbb{N}}$ such that $\partial T' \llcorner \mathbf{B}_R(x) = \pm [\![\Gamma]\!] \llcorner \mathbf{B}_R(x)$, and, by lower semicontinuity of the mass, $\mathbf{M}(T') = 0$, which is impossible.

With the local multiplicity bound in hand, the local upper $\mathbf{F}$-bound follows from a standard competitor argument; we refer the reader to [14, (29)-(33)]. □

3.3. **Conical F-minimizers in $\mathbb{R}^2$.** We now show a simple lemma that follows directly from strict convexity of F and shows that conical minimizers in $\mathbb{R}^2$ must be flat. This will be used later on.

Lemma 3.3. *Let C be an integer rectifiable 1-current in $\mathbb{R}^2$ which is a cone, minimizes $\mathbf{F}$, and $\partial C = d[\![0]\!]$, $d \in \mathbb{N} \setminus \{0\}$. Then there are lines $\ell, \ell_1, \dots, \ell_d$ through the origin and an integer $Q \geq d$ such that*

$$C = Q[\![\ell^+]\!] + (Q-d)[\![\ell^-]\!] \text{ if } Q > d \qquad \text{or} \qquad C = \sum_{i=1}^{d} [\![\ell_i^+]\!] \text{ if } Q = d, \tag{3.1}$$

where $\ell^\pm$ and $\ell_i^\pm$ are the two connected components of $\ell \setminus \{0\}$ and $\ell_i \setminus \{0\}$ oriented by a fixed orientation of ℓ and ℓ_i, respectively.

Proof. Since C is a 1-dimensional integer rectifiable cone with locally finite mass, its support is a finite union of rays from the origin. Therefore, we can write

$$C = \sum_{i=1}^{N} k_i [\![r_i]\!] , \tag{3.2}$$

where $k_i \in \mathbb{Z} \setminus \{0\}$ and $r_i = \{t p_i : t \geq 0\}$ for distinct points $p_i \in \mathbb{S}^1$. Since $\partial C = d[\![0]\!]$ and we may take $[\![r_i]\!]$ to be oriented so that $\partial [\![r_i]\!] = [\![0]\!]$, we must have $\sum_{i=1}^N k_i = d$.

We claim that if $k_i > 0$ and $k_j < 0$, then $p_j = -p_i$. Suppose not. Fix $R > 0$. We have $\partial([\![[0, Rp_i]]\!] + [\![[Rp_j, 0]]\!]) = [\![Rp_i]\!] - [\![Rp_j]\!]$. Then we replace it by $[\![[Rp_j, Rp_i]]\!]$, producing an admissible competitor for C in B_R. By strict convexity of F, we derive that

$$\mathbf{F}([\![[Rp_j, Rp_i]]\!]) < \mathbf{F}([\![[0, Rp_i]]\!]) + \mathbf{F}([\![[Rp_j, 0]]\!]).$$

This gives a contradiction and proves the claim.

Now, if all $k_i > 0$, then $\sum_{i=1}^N k_i = d$ and the latter alternative in (3.1) follows from (3.2). Otherwise, there is at least one negative coefficient. Since $\sum_i k_i = d$, there is also at least one positive coefficient. By the claim, every positively oriented ray must be opposite to every negatively oriented ray. Because the directions p_i are distinct, there can be only one positive direction and only one negative direction. This concludes the proof of the lemma. ☐

3.4. **Comparison principle.** The following lemma is an adaptation of [3, Lemma 2.12] (proved under $C^{2,1}$ regularity of F) for $C^{2,\mathrm{Dini}}$ integrands.

Lemma 3.4 (Comparison principle). *If $U \subset \mathbb{R}^m$ is a bounded C^1 domain, F is a uniformly elliptic integrand of class $C^{2,\mathrm{Dini}}$, $\alpha \in (0,1)$, $a < b \in \mathbb{R}$, $u \in C^{1,\alpha}(U)$ is a weak solution to* $\operatorname{div}[\nabla F_{\S}(\nabla u)] = 0$, *$E \subset U \times (a,b)$ minimizes the anisotropic perimeter in $U \times (a,b)$ with $P_F(E) < \infty$, and*

$$E^{(1/2)} \cap \partial[U \times (a,b)] \supset \{(x', x_{m+1}) \in (\partial U) \times (a,b) : x_{m+1} < u(x')\} \cup (U \times \{a\}),$$

then

$$\left|\{(x', x_{m+1}) : x' \in U,\, x_{m+1} \leq u(x')\} \setminus E\right| = 0\,.$$

Proof. The proof of [3, Lemma 2.12], which is stated for $C^{2,1}$ integrands, readily works here as well. The only place this enters in the proof is when applying the divergence theorem to the vector field $X(x) \coloneqq \nabla F(\nabla u(x'), -1)$. Since, as shown in Proposition 3.1, $u \in C^2$, this step and the proof in [3, Lemma 2.12] proceeds without issue. ☐

4. Anisotropic mean convex barriers and a parametric Hopf lemma

In this section, we prove variations of the maximum principle at the level of functions and of **F**-minimizers. These are used to construct anisotropic mean convex barriers to **F**-minimizers, see Proposition 4.3. We also prove a "strong" Hopf's lemma (Proposition 4.5) for currents that will play a crucial role in the proof of our main theorem.

4.1. **Maximum principles and anisotropic mean convex barriers.**

Lemma 4.1 (Strong maximum principle for graphs). *Suppose that G is uniformly elliptic of class $C^{2,\mathrm{Dini}}$, $\Omega \subset \mathbb{R}^m$ is a bounded, connected open set, $\beta \in (0,1]$, $u, v \in C^{1,\beta}(\Omega)$ are weak solutions to*

$$-\operatorname{div}\left[\nabla G_{\S}(\nabla v)\right] = 0, \qquad -\operatorname{div}\left[\nabla G_{\S}(\nabla u)\right] = g \in L^\infty(\Omega), \tag{4.1}$$

where $\operatorname{ess\,inf}_\Omega g \geq 0$, *and $u \geq v$ on Ω. If in addition*

(i) $\operatorname{ess\,inf}_\Omega g \geq \kappa > 0$ *for some $\kappa > 0$, then $u > v$ on Ω; whereas*
(ii) *if $g \equiv 0$, then either $u \equiv v$ on Ω or $u > v$ on Ω.*

Proof. For every $x \in \Omega$, we have $\nabla G_{\S}(\nabla u) - \nabla G_{\S}(\nabla v) = \mathrm{M}(x)(\nabla u - \nabla v)$ where

$$\mathrm{M}(x) \coloneqq \int_0^1 D^2 G_{\S}\big(\nabla v(x) + t(\nabla u(x) - \nabla v(x))\big)dt.$$

Subtracting the two equations in (4.1), we find

$$-\operatorname{div}\left[\mathrm{M}\nabla(u-v)\right] = g \geq 0 \quad \text{ weakly in } \Omega.$$

The matrix M is clearly locally bounded and its uniform ellipticity follows from uniform ellipticity of G under L^∞ gradient bounds on u and v. Note that the coefficient matrix M is Dini continuous since G is $C^{2,\mathrm{Dini}}$.

Suppose now for contradiction that $\{x \in \Omega : u(x) = v(x)\} \neq \emptyset$ but that $\{u = v\} \neq \Omega$. Then we can find $B_r(x_0) \subset\subset \Omega$ such that $u > v$ on $B_r(x_0)$ and $u(y_0) = v(y_0)$ for some $y_0 \in \partial B_r(x_0)$. By the Hopf lemma for divergence form equations with Dini coefficients [1, Theorem 2.1], we find that $\nabla(u-v)(y_0) \neq 0$. But together with $u(y_0) = v(y_0)$, this contradicts $u \geq v$ on Ω. So we either have $u > v$ on Ω or $u \equiv v$ on Ω, and (i)-(ii) follow immediately. ☐

The next item is a strong maximum principle for $\mathbf{F}$-minimizers, similar to the statement in [3, Lemma 2.13]. The proof is nearly identical; since the integrands considered in [3] are $C^{2,1}$ rather than $C^{2,\mathrm{Dini}}$, it only requires a couple minor technical modifications that we describe below.

Lemma 4.2 (Strong maximum principle for $\mathbf{F}$-minimizers)**.** *If F is a uniformly elliptic integrand of class $C^{2,\mathrm{Dini}}$, T is an integer rectifiable m-current in $\mathbb{R}^{m+1}$ that minimizes $\mathbf{F}$ in $\mathbf{B}_1$, $\partial T \llcorner \mathbf{B}_1 = 0$, $v \in \mathbb{S}^m$, $\operatorname{spt} T \cap \mathbf{B}_1 \subset \{x \cdot v \geq 0\} \cap \mathbf{B}_1$, and $0 \in \operatorname{spt} T$, then there is $q \in \mathbb{Z}$ and T' with $0 \notin \operatorname{spt} T'$ such that, setting $H = [\![\{x \cdot v > 0\}]\!]$ oriented by $e_1 \wedge \cdots \wedge e_{m+1}$,*

$$T \llcorner \mathbf{B}_1 = q \partial H \llcorner \mathbf{B}_1 + T' .$$

Proof. By a change of coordinates, it suffices to prove the lemma when $v = e_{m+1}$. In a slight abuse of notation, we call $H = \{x_{m+1} > 0\}$. We decompose $T \llcorner \mathbf{B}_1 = \sum_i \partial [\![E_i]\!]$, where each E_i locally minimizes the anisotropic perimeter in $\mathbf{B}_1$. We normalize each E_i so that $\partial E_i \cap \mathbf{B}_1 = \overline{\partial^* E_i} \cap \mathbf{B}_1$. By interior regularity, i.e., $\mathcal{H}^{m-2}(\mathrm{Sing}_i\, \partial [\![E_i]\!]) = 0$ [7], we have $\mathcal{H}^m(\mathbf{B}_1 \cap \partial E_i \setminus \partial^* E_i) = 0$ and may assume that $E_i = E_i^{(1)}$ is open, cf. [4, Lemma 6.2]. Note that by lower volume density estimates (cf. [3, Lemma 2.8], where the argument does not require any regularity on $D^2 F$, and so applies here), only finitely many E_i can have boundary in $\mathbf{B}_{1/2}$. Thus, to prove the lemma, it suffices to show that for each E_i,

$$\text{either } 0 \notin \partial E_i \text{ or } \partial H \cap \mathbf{B}_1 \subset \partial E_i .$$

Fix i. We assume that ∂E_i does not contain $\partial H \cap \mathbf{B}_1$ and aim to prove

$$0 \notin \partial E_i .$$

Depending on the orientation of $[\![E_i]\!]$ and since $\operatorname{spt} T \cap \mathbf{B}_1 \subset H$, we have either $E_i \subset H$ or $E_i^c \subset H$. We set $E = E_i \cap \mathbf{B}_1$ and $G(x) = F(x)$ if $E_i \subset H$ and $E = E_i^c \cap \mathbf{B}_1$ and $G(x) = F(-x)$ if $E_i^c \subset H$. We need to show that

$$\text{if } E \text{ minimizes } \textstyle\int G(\nu_E) \text{ in } \mathbf{B}_1 \text{ and } \partial E_i \text{ does not contain } \partial H \cap \mathbf{B}_1, \text{ then } 0 \notin \partial E . \tag{4.2}$$

Equation (4.2) is now exactly what is proved in [3, Lemma 2.13] for $C^{2,1}$ integrands. With this reduction in place, there is only one difference compared to [3, Lemma 2.13]: their use of $F \in C^{2,1}$ to use classical elliptic regularity theory/maximum principles to derive $u(0) > 0$ ([3, Eq. (2.100)]). Here, we can directly apply our Lemma 4.1 for G of class $C^{2,\mathrm{Dini}}$.

Once the above is settled, the proof in [3, Lemma 2.13] is finished by an application of [3, Lemma 2.12] which is our Lemma 3.4. □

Next, we formulate the barrier principle.

Proposition 4.3 (Anisotropic mean convex barriers)**.** *Let F be a uniformly elliptic integrand of class $C^{2,\mathrm{Dini}}$. If $U \subset \mathbb{R}^{m+1}$ is a C^2 bounded open set such that the anisotropic mean curvatures $H^F_{\partial U}, H^G_{\partial U} \in L^\infty(\partial U)$ corresponding to the anisotropies $\nu \mapsto F(\nu)$ and $\nu \mapsto F(-\nu) =: G(\nu)$ satisfy*

$$\min_{\partial U} H^F_{\partial U} \geq \kappa , \qquad \min_{\partial U} H^G_{\partial U} \geq \kappa$$

for some $\kappa > 0$, then there is $\delta_0 = \delta_0(U) > 0$ with the following property:

if S is an $\mathbf{F}$-minimizer in $\mathbb{R}^{m+1}$ with $\operatorname{spt}(\partial S) \subset \overline{U}$ and $\operatorname{spt} S \subset \overline{B_{\delta_0}(U)}$, then $\operatorname{spt} S \subset \overline{U}$.

Remark 4.4 (On the assumption $\operatorname{spt} S \subset \overline{B_{\delta_0}(U)}$)**.** Without this assumption the result is false, as is seen by the example where S is an equatorial disk in $\mathbb{R}^3$ with boundary $\mathbb{S}^1 \times \{0\}$ and U is a small tubular neighborhood containing $\operatorname{spt} \partial S$ but not $\operatorname{spt} S$.

Proof. Let U_δ denote the bounded open sets $\{x : \operatorname{dist}(x, U) < \delta\}$. We choose δ_0 small enough so that for all $0 \leq \delta \leq \delta_0$,

$$\min_{\partial U_\delta} H^F_{\partial U_\delta} \geq \kappa/2 , \qquad \min_{\partial U_\delta} H^G_{\partial U_\delta} \geq \kappa/2.$$

Now let S be an $\mathbf{F}$-minimizer as in the statement. Assume for contradiction that $\operatorname{spt} S \setminus \overline{U} \neq \varnothing$. Then there is $\delta_1 \geq 0$ which is the smallest $\delta \in [0, \delta_0]$ such that $\operatorname{spt} S \subset \overline{U_\delta}$. We must show δ_1 cannot be positive. Since $\operatorname{spt} S$ is compact, there must be $x \in \partial U_{\delta_1} \cap \operatorname{spt} S$. We blow up S at x by some $r_j \to 0$ such that $S_{x,r_j} \to S_0$ for some S_0, recall that S_0 exists thanks to the mass ratio upper bound in Lemma 3.2. Note that the blowup satisfies the assumptions of Lemma 4.2. In particular, S_0 decomposes as q copies of an oriented hyperplane plus some current S' such that $\operatorname{dist}(\operatorname{spt} S' \cap \mathbf{B}_1, 0) > 0$. If the orientation of S_0 at zero coincides with $T_x(\partial U_{\delta_1})$ oriented by the outward normal $\nu_{U_{\delta_1}}$ to ∂U_{δ_1}, we define $S' = S$, where if the orientation is opposite, we set $S' = -S$. Note in this latter case that S' is $\mathbf{G}$-minimizer.

We decompose $S'_{x,r_j} \llcorner \mathbf{B}_1$ into finitely many boundaries $\sum_i \partial [\![E_i^j]\!]$ of some anisotropic perimeter minimizers in $\mathbf{B}_1$. Let us assume without loss of generality that $0 \in \partial E_1^j$ and ∂E_1^j blows up to a plane for all j. By applying the interior regularity of [7], we find that for large j, ∂E_1^j is locally a graph of some $f_j \in C^{1,\beta}, \beta > 0$, over $T_x(\partial U_{\delta_1})$, as is $(\partial U_{\delta_1} - x)/r_j$ with some g_j. Up to a rotation, let us assume that the tangent plane is $\mathbb{R}^m \times \{0\}$ and that E_1^j and $(U_{\delta_1} - x)/r_j$ correspond to the epigraphs of f_j and g_j, so that $g_j \geq f_j$ with equality at 0. Then f_j and g_j are $C^{1,\beta}$ weak solutions to

$$-\operatorname{div}\left[\nabla F_{\S}(\nabla f_j)\right] = 0\,, \qquad -\operatorname{div}\left[\nabla F_{\S}(\nabla g_j)\right] \geq r_j \kappa/2\,,$$

in the case $S' = S$, with $F_{\S}$ replaced by $G_{\S}$ if $S' = -S$. By Lemma 4.1, $g_j > f_j$ on a ball containing 0 for all large j, which contradicts $g_j(0) = f_j(0)$. Thus $\delta_1 > 0$ is impossible and proof is complete. □

4.2. **Parametric Hopf lemma.** The Hopf lemma for graphs can be used to prove an analogous statement for an anisotropic area-minimizing hypercurrents sitting above a hyperplane by "squeezing" a minimizing graph in between them, an idea which is due to De Philippis and Maggi [3, Lemma 2.13].

Proposition 4.5 (Hopf-type lemma for $\mathbf{F}$-minimizers)**.** *If F is a uniformly elliptic integrand of class $C^{2,\mathrm{Dini}}$, $\alpha \in (0,1)$ is as in Proposition 3.1, $\Omega \subset \mathbb{R}^m$ is an open, star-shaped, bounded set with smooth boundary, and $r_0 \in (0, 1/2)$ is fixed, then there is $\eta_0 > 0$ with the following property:*

if $L : \mathbb{R}^m \to \mathbb{R}$ given by $L(x') = a \cdot x'$ is a linear function with $|a| \leq \eta_0$, T is an $\mathbf{F}$-minimizer in $\Omega \times (-1,1)$ with $\partial T \llcorner (\Omega \times (-1,1)) = 0$ and

$$\operatorname{spt} T \subset \{(x', x_{m+1}) \in \overline{\Omega} \times [-1,1] : x_{m+1} \geq L(x')\}\,, \tag{4.3}$$

and there exists $x_0 = (x_0', x_{m+1}^0) \in \operatorname{graph}_{\partial\Omega} L$ such that

$$\operatorname{dist}(x_0, \operatorname{spt} T) \geq r_0\,, \tag{4.4}$$

then there are $\tau_0 > 0$ and $t_0 > 0$, both depending on $\mathbf{F}$, m, and r_0, and $u \in \operatorname{Lip}(\Omega)$ such that

$$u(x') \geq L(x') \qquad \forall x' \in \overline{\Omega}\,, \tag{4.5}$$

$$u(x') = L(x') \qquad \forall x' \in \partial\Omega \setminus B_{r_0}(x_0')\,, \tag{4.6}$$

$$u(x' + t\nu_\Omega) > u(x') + t a \cdot \nu_\Omega + |t|\tau_0 \qquad \forall x' \in \partial\Omega \setminus B_{r_0}(x_0'),\ t \in [-t_0, 0)\,, \qquad \textit{and} \tag{4.7}$$

$$\operatorname{spt} T \subset \{(x', x_{m+1}) \in \overline{\Omega} \times [-1,1] : x_{m+1} \geq u(x')\}\,. \tag{4.8}$$

Proof. We choose $\eta_0 = \kappa_0/2$, where κ_0 is as in Proposition 3.1. Let us fix $r_0 \in (0, 1/2)$.

The subsequent argument is divided into two main steps. In the first step, we prove the desired conclusions with additional L and x_0' dependence. More precisely, we fix a linear function L with gradient modulus at most η_0 and $(x_0', x_{m+1}^0) \in \operatorname{graph}_{\partial\Omega} L$, and prove the existence of $\tau_{L,x_0'}$, $t_{L,x_0'}$, and $u_{L,x_0'}$ such that (4.5)-(4.8) hold for any T as in the assumptions of the lemma. Then in the second step, we prove that $\tau_{L,x_0'}$ and $t_{L,x_0'}$ can be made uniform among x_0' and L.

For later use, we let $\varphi \in C^\infty(B_{r_0/2};[0,\infty))$ be such that $\varphi = 0$ on $\mathbb{R}^m \setminus B_{r_0/2}$, $0 < \varphi < r_0/4$ on $B_{r_0/2}$, and, for every $x' \in \partial\Omega$ and $L(x') = a \cdot x'$ with $|a| \le \eta_0$,

$$\operatorname{dist}\big((x_0', L(x_0') + \varphi(0)), \operatorname{spt} T\big) > r_0/4 \quad \text{and} \quad \|L(\cdot) + \varphi(\cdot - x')\|_{C^{1,\eta}(\Omega)} \le 3\kappa_0/4\,; \tag{4.9}$$

such a choice of φ is possible since $\eta_0 = \kappa_0/2$.

Step one: (4.5)-(4.8) *hold with L and x_0' dependence.* Fix L and x_0 as in the statement of the lemma. Let G be the anisotropy characterized by $\nu \mapsto F(-\nu) =: G(\nu)$. By Proposition 3.1 and (4.9), let $u_{L,x_0',F}, u_{L,x_0',G} \in C^{1,\alpha}(\Omega)$ be the unique minimizers of $\mathscr{F}$ and $\mathscr{G}$, respectively (see (2.1) for the definition of $\mathscr{F}$), on Ω with boundary data

$$\varphi(\cdot - x_0') + L(\cdot).$$

By Proposition 3.1 and the second estimate in (4.9), we have

$$\max\{\|u_{L,x_0',F}\|_{C^{1,\alpha}(\Omega)}, \|u_{L,x_0',G}\|_{C^{1,\alpha}(\Omega)}\} \le C3\kappa_0/4\,. \tag{4.10}$$

Setting $u_{L,x_0'} = \min\{u_{L,x_0',F}, u_{L,x_0',G}\}$, we point out that (4.5) follows by the maximum principle. Also, (4.6) holds since $\operatorname{spt}\varphi \subset B_{r_0/2}$ implies that $\varphi(x' - x_0') + L(x') = L(x')$ if $x' \in \partial\Omega \setminus B_{r_0}(x_0')$.

Next, notice that by the definitions of r_0 and the choice φ (its support is contained in $B_{r_0/2}$), we have

$$\operatorname{spt} T \subset \{(x', x_{m+1}) \in \overline{\Omega} \times [-1,1] : x_{m+1} \ge L(x') + \varphi(x' - x_0')\}\,. \tag{4.11}$$

We define

$$\mathrm{M}_F(x) = \int_0^1 D^2 F_\S((1-t)a + t\nabla u_{L,x_0',F}(x))\,dt \quad \text{and} \quad \mathrm{M}_G(x) = \int_0^1 D^2 G_\S((1-t)a + t\nabla u_{L,x_0',G}(x))\,dt\,.$$

Then $u_{L,x_0',F} - L$ satisfies the equation $-\operatorname{div}[\mathrm{M}_F \nabla(u_{L,x_0',F} - L)] = 0$, which is a uniformly elliptic equation with Dini coefficients (the uniform ellipticity is due to the L^∞ gradient bounds on $u_{L,x_0',F}$ from Proposition 3.1). Also $u_{L,x_0',G}$ satisfies the analogous equation with G instead of F. By the Hopf lemma [1, Theorem 2.1], the compactness of $\overline{\Omega}$, and the Hölder continuity of $\nabla u_{L,x_0',F}$ and $\nabla u_{L,x_0',G}$, there are $\tau_{L,x_0'}$ and $t_{L,x_0'}$ such that

$$u_{L,x_0',F}(x' + t\nu_\Omega) - u_{L,x_0',F}(x') - tL(\nu_\Omega) > |t|\tau_{L,x_0'} \quad \forall x' \in \partial\Omega \setminus B_{r_0}(x_0'),\ t \in [-t_{L,x_0'}, 0) \quad \text{and} \tag{4.12}$$

$$u_{L,x_0',G}(x' + t\nu_\Omega) - u_{L,x_0',G}(x') - tL(\nu_\Omega) > |t|\tau_{L,x_0'} \quad \forall x' \in \partial\Omega \setminus B_{r_0}(x_0'),\ t \in [-t_{L,x_0'}, 0)\,. \tag{4.13}$$

Recalling that $u_{L,x_0'} = \min\{u_{L,x_0',F}, u_{L,x_0',G}\}$, we thus obtain (4.7) with L and x_0' dependence. To finish this step, it remains to prove (4.8) with L and x_0' dependence.

Since Ω (and thus $\Omega \times (-1,1)$) is star-shaped, we can write T as a sum of boundaries of anisotropic relative perimeter minimizers (that is, of P_F)

$$T \llcorner (\Omega \times (-1,1)) = \sum [\![\partial E_i \cap (\Omega \times (-1,1))]\!]\,.$$

By (4.11), for each E_i, we have either

$$[(\Omega\times(-1,1))\setminus E_i]^{(1/2)} \cap \partial[\Omega\times(-1,1)] \supset \{(x', x_{m+1}) \in \partial\Omega\times(-1,1) : x_{m+1} < \varphi(x'-x_0') + L(x')\} \cup (\Omega \times \{-1\}) \tag{4.14}$$

or

$$E_i^{(1/2)} \cap \partial[\Omega \times (-1,1)] \supset \{(x', x_{m+1}) \in \partial\Omega \times (-1,1) : x_{m+1} < \varphi(x' - x_0') + L(x')\} \cup (\Omega \times \{-1\})\,. \tag{4.15}$$

For those E_i such that (4.14) holds, we apply Lemma 3.4 with $u_{L,x_0',G}$, $(\Omega \times (-1,1)) \setminus E_i$, and anisotropy G to conclude that

$$\big|\{(x', x_{m+1}) : x' \in \Omega,\ x_{m+1} \le u_{L,x_0',G}(x')\} \cap E_i\big| = 0\,, \tag{4.16}$$

whereas for those E_i such that (4.15) holds, we apply Lemma 3.4 with $u_{L,x_0',F}$, E_i, and anisotropy F to conclude that

$$\big|\{(x', x_{m+1}) : x' \in \Omega,\ x_{m+1} \le u_{L,x_0',F}(x')\} \setminus E_i\big| = 0\,. \tag{4.17}$$

As a consequence of (4.16) and (4.17), if $\operatorname{spt} T$ touches the graph of u inside $\Omega \times \mathbb{R}$, it can only do so when one of these ∂E_i's touches its corresponding graph but does not cross it. Thus to finish proving (4.8), we only need to preclude this possibility. This is impossible because if there were such a touching by some E_i, then the blowup of ∂E_i at that point would sit on one side of a plane, and therefore coincide with that plane by Lemma 4.2. By the interior regularity of [7] and Lemma 4.1, $\operatorname{spt} \partial E_i$ coincides with either the graph of $u_{L,x_0',F}$ or $u_{L,x_0',G}$ in a neighborhood of that touching point, depending on the orientation of ∂E_i. This shows that the set of such coincidence points is open relative to the graph, since it is also closed and $\Omega \times \mathbb{R}$ is connected, we conclude that $\operatorname{spt} T$ contains either the graph of $u_{L,x_0',F}$ or $u_{L,x_0',G}$. We obtain a contradiction by recalling (4.9) and that these functions have $\varphi(x'-x_0') + L(x')$ as boundary data.

Step two: Removing the L and x_0' dependence. This is a compactness argument. Assume for contradiction that for some sequence $\{(L_j, x_j')\}_j$ satisfying $\|\nabla L_j\|_{L^\infty} \le \eta_0$ and $x_j \in \operatorname{graph}_{\partial\Omega} L_j$, at least one of the sequences of constants $\tau_{L_j,x_j'}$ and $t_{L_j,x_j'}$ converge to 0 no matter the choices of them for each j. Recalling that $u_{L_j,x_j'} = \min\{u_{L_j,x_j',F}, u_{L_j,x_j',G}\}$, up to a subsequence, we can use (4.10) and the Arzela-Ascoli theorem to obtain $x_0' \in \partial\Omega$, L_0 with $\|\nabla L_0\|_{L^\infty} \le \eta_0$, $u_{x_0',L_0,F}$, and $u_{x_0',L_0,G}$ such that

$$x_j' \to x_0' \,, \quad \nabla L_j \to \nabla L_0 \,, \quad u_{x_j',L_j,F} \to u_{x_0',L_0,F} \text{ and } u_{x_j',L_j,G} \to u_{x_0',L_0,G} \text{ in } C^{1,\beta} \text{ for all } 0<\beta<\alpha.$$

Furthermore, $u_{x_0',L_0,F}$ and $u_{x_0',L_0,G}$ minimize $\mathscr{F}$ and $\mathscr{G}$, respectively, for the boundary data $L_0+\varphi(\cdot - x_0')$. Therefore, the same argument as leading to (4.12)-(4.13) yields $\tau_{L_0,x_0'}$ and $t_{L_0,x_0'}$ such that, for any $t \in [-t_{L_0,x_0'}, 0)$,

$$u_{L_0,x_0',F}(x'+t\nu_\Omega) - u_{L_0,x_0',F}(x') - tL_0(\nu_\Omega) > |t|\tau_{L_0,x_0'}, \ \forall x' \in \partial\Omega \setminus B_{r_0}(x_0'), \text{ and} \tag{4.18}$$

$$u_{L_0,x_0',G}(x'+t\nu_\Omega) - u_{L_0,x_0',G}(x') - tL_0(\nu_\Omega) > |t|\tau_{L_0,x_0'}, \ \forall x' \in \partial\Omega \setminus B_{r_0}(x_0')\,. \tag{4.19}$$

Since we have β-Hölder convergence of gradients and $\nabla L_j \to \nabla L_0$ uniformly, (4.18)-(4.19) imply that, for j large enough and $t \in [-t_{L_0,x_0'}/2, 0)$,

$$u_{L_j,x_j',F}(x'+t\nu_\Omega) - u_{L_j,x_j',F}(x') - tL_j(\nu_\Omega) > |t|\tau_{L_0,x_0'}/2, \ \forall x' \in \partial\Omega \cap \{u_{L_j,x_j',F} = L_j\}, \text{ and}$$
$$u_{L_j,x_j',G}(x'+t\nu_\Omega) - u_{L_j,x_j',G}(x') - tL_j(\nu_\Omega) > |t|\tau_{L_0,x_0'}/2, \ \forall x' \in \partial\Omega \cap \{u_{L_j,x_j',G} = L_j\}\,.$$

But this is a contradiction of our assumption that it was impossible to choose $\tau_{L_j,x_j'}$ and $t_{L_j,x_j'}$ so that neither limit is zero. □

5. Boundedness of the polar angle function

The goal of this section is to bound the polar angle function on T.

5.1. **Setup and notation.** Throughout this section, we will always be working under the following conditions.

Assumption 1. We assume that F is uniformly elliptic of class $C^{2,\mathrm{Dini}}$ and T is a nontrivial integer multiplicity rectifiable m-current that minimizes $\mathbf{F}$ in $\mathbb{R}^{m+1}$ with boundary $[\![\mathbb{R}^{m-1} \times \{(0,0)\}]\!]$.

Definition 5.1 (Rays, half hyperplanes, and halfspaces)**.** Given $s \in \mathbb{S}^1$, we let $R_s \subset \mathbb{R}^2$ be the ray through the origin containing $s \in \mathbb{S}^1$ and $M_s = \mathbb{R}^{m-1} \times R_s$ be the corresponding half hyperplane. We consider its oriented counterpart

$$[\![M_s]\!] := [\![\mathbb{R}^{m-1} \times R_s]\!]\,,$$

where the orientation is chosen so that $\partial[\![M_s]\!] = [\![\mathbb{R}^{m-1} \times \{(0,0)\}]\!]$. We will also denote by H_s the open halfspace

$$\mathbb{R}^{m-1} \times \{r\sigma : r>0,\, \sigma \in \mathbb{S}^1,\, \operatorname{dist}_{\mathbb{S}^1}(\sigma, s) < \pi/2\}\,,$$

so that M_s is perpendicular to ∂H_s and $M_s \subset H_s$. To streamline notation, we will often view $\mathbb{S}^1$ as a subset of $\mathbb{C}$ rather than $\mathbb{R}^2$, so that we can describe its elements by e^{it} for $t \in \mathbb{R}$.

5.2. **The polar angle function and its super-/sublevel sets.** In this subsection we define the polar angle function $\boldsymbol{\theta}$ following [8, Section 11].

First, we need some notation. Following [8, Section 11], we define $\mathbf{q} : \mathbb{R}^{m+1} \to \mathbb{C}$ by

$$\mathbf{q}(x_1, \ldots, x_{m+1}) = x_m + i x_{m+1} \,.$$

For any piecewise C^1 curve $\gamma : [0,1] \to \mathbb{R}^{m+1} \setminus \mathbf{q}^{-1}(0)$, we set

$$\theta(\gamma) = \operatorname{Im}\left(\int_{\mathbf{q}\circ\gamma} \frac{dz}{z} \right) .$$

The statement and proof of the following lemma are the same as in [8, Section 11], except there the curves in question are confined to the link of the blowup cone; we include it for convenience.

Lemma 5.2. *If T satisfies Assumption 1 and $\gamma : [0,1] \to \operatorname{Reg}_i T$ is piecewise C^1 with $\gamma(0) = \gamma(1)$, then $\theta(\gamma) = 0$.*

Proof. First, we note that for $\rho > 0$ small enough, the modified curve $\beta := \gamma + \rho \nu_{\operatorname{Reg}_i T} \circ \gamma$ satisfies

$$\theta(\gamma) = \theta(\beta) \,, \qquad \beta([0,1]) \cap \operatorname{spt} T = \varnothing \,, \tag{5.1}$$

where $\theta(\gamma) = \theta(\beta)$ follows from the homotopy invariance of winding number. As observed in [8, Section 11], $\theta(\beta)$ corresponds to algebraically counting the number of crossings (modulo a multiplicative factor of 2π) of γ with any oriented half-hyperplane $P = (-1)^{m-1}[\![\mathbb{R}^{m-1} \times L]\!]$, where $L \subset \mathbb{R}^2$ is any ray chosen so that β intersects $\operatorname{spt}(P)$ transversally; thus

$$\theta(\gamma) = \theta(\beta) = 2\pi (P \cap \beta_\#[0,1])(1) \,, \tag{5.2}$$

where the current in the right-most term is the *intersection between two currents* as defined in [6, 4.3.20, page 460]. Note that since $\beta([0,1])$ is compact, 1 is an admissible test function for the current $P \cap \beta_\#[0,1]$.

Next, fix a cube $[-R,R]^{m+1}$ large enough so that $\beta([0,1]) \subset\subset [-R,R]^{m+1}$. Since $\partial P - \partial T = 0$, we can find an integer rectifiable current J on $\mathbb{R}^{m+1}$ such that

$$(P - T) \llcorner (-R,R)^{m+1} = (\partial J) \llcorner (-R,R)^{m+1} \,. \tag{5.3}$$

Using in order (5.2), (5.3), and the second assertion in (5.1), we compute

$$(2\pi)^{-1}\theta(\gamma) = \big([T + \partial J] \cap \beta_\#[0,1]\big)(1) = 0 + \big(\partial J \cap \beta_\#[0,1]\big)(1) \,.$$

To simplify the right hand side we will use the following formula from [6, bottom of page 460] for the boundary of the intersection current applied to $J \cap \beta_\#[0,1]$:

$$\partial(J \cap \beta_\#[0,1]) - J \cap (\partial \beta_\#[0,1]) = (-1)^m (\partial J) \cap \beta_\#[0,1].$$

Recall that $\partial S(1) = S(d1) = 0$ for any compactly supported 1-current S. We apply this formula with $S = J \cap \beta_\#[0,1]$, use that $\partial\beta_\#[0,1] = 0$ (since β is closed), and put together the last two displayed equations to obtain

$$(2\pi)^{-1}\theta(\gamma) = (-1)^m \partial(J \cap \beta_\#[0,1])(1) - (-1)^m J \cap (\partial\beta_\#[0,1])(1) = 0 + 0 = 0. \qquad \square$$

The preceding lemma ensures that the following definition is well-posed and depends only on the initial choice of base points a_j; cf. [8, page 478].

Definition 5.3 (Polar angle function on $\mathbf{B}_{10} \cap \operatorname{Reg}_i T$ and its super-/sublevel sets)**.** Suppose T satisfies Assumption 1 and $\{C_j\}$ are the path connected components of $\mathbf{B}_{10} \cap \operatorname{Reg}_i T$. For each C_j, we choose base point $a_j \in C_j$ and $\theta_j \in [0, 2\pi)$ such that θ_j is the polar angle of $\mathbf{q}(a_j)$. Then we define the continuous function $\boldsymbol{\theta} : \mathbf{B}_{10} \cap \operatorname{Reg}_i T \to \mathbb{R}$ by

$$\boldsymbol{\theta}(x) = \theta_j + \theta(\gamma)$$

if $x \in C_j$ and $\gamma : [0,1] \to C_j$ is any C^1 curve with $\gamma(0) = a_j$ and $\gamma(1) = x$. For $N \in \mathbb{N}$, we set

$$T_N := [\![\{x \in \mathbf{B}_{10} \cap \mathrm{Reg}_i T : \boldsymbol{\theta}(x) > 2N\pi\}]\!]\,,$$
$$T_{-N} := [\![\{x \in \mathbf{B}_{10} \cap \mathrm{Reg}_i T : \boldsymbol{\theta}(x) < -2N\pi\}]\!]\,.$$

Lemma 5.4 (Super-/sublevel sets)**.** *Let F and T be as in Assumption 1. Then for all $N \in \mathbb{N}$:*

(i) *T_N and T_{-N} are integer rectifiable m-currents, and*
(ii) *$\partial T_{\pm N}$ is supported in $\overline{M_1}$.*

Proof. Item (i) holds since the *sets* defining $T_{\pm N}$ are $\mathcal{H}^m$-measurable sets contained in a C^1 submanifold of dimension m, namely, $\mathrm{Reg}_i T$, and are therefore locally m-rectifiable. For item (ii), fix $x \in \mathrm{spt}\, T_{\pm N} \setminus \overline{M_1}$. Then we can choose $r > 0$ small enough so that $\mathrm{dist}(\mathbf{B}_r(x), M_1) =: d > 0$. Since $\{\boldsymbol{\theta} = \pm 2N\pi\} \subset \overline{M_1}$, there is $\delta > 0$ depending on d such that $\boldsymbol{\theta} \geq 2N\pi + \delta$ on $\mathbf{B}_r(x) \cap \mathrm{Reg}_i T$ if $x \in \mathrm{spt}\, T_N \setminus \overline{M_1}$ and $\boldsymbol{\theta}(x) \geq 2N\pi$ or $\boldsymbol{\theta} \leq -2N\pi - \delta$ on $\mathbf{B}_r(x) \cap \mathrm{Reg}_i T$ if $x \in \mathrm{spt}\, T_{-N} \setminus \overline{M_1}$ and $\boldsymbol{\theta}(x) \leq -2N\pi$. Thus $T_{\pm N} \llcorner \mathbf{B}_r(x)$ is concentrated on a union of connected components of $\mathrm{Reg}_i T \cap \mathbf{B}_r(x)$. Since T is boundaryless on that ball and thus each connected component of $\mathrm{Reg}_i T$ is, $T_{\pm N}$ is boundaryless there as well and the proof is complete. □

5.3. **Barrier construction.** In this subsection, we build an anisotropic mean convex barrier U to trap the support of $T_{\pm N} \llcorner \mathbf{B}_5$ inside $\overline{U}$ for large N, from which we will conclude that $\boldsymbol{\theta}$ is bounded.

Theorem 5.5 ($\boldsymbol{\theta} \in L^\infty(\mathbf{B}_5)$)**.** *If T satisfies Assumption 1, then there is an anisotropic mean convex smooth open set $U \subset \mathbb{R}^{m+1}$ and $N_0 = N_0(U, T) > 0$ such that for any $N \geq N_0$,*

$$\mathrm{spt}(T_{\pm N} \llcorner \mathbf{B}_5) \subset \overline{U \cap \mathbf{B}_5} \subset \overline{H_1}. \tag{5.4}$$

As a consequence,

$$\boldsymbol{\theta} \in L^\infty(\mathbf{B}_5)\,.$$

Proof. The proof is divided into steps. We prove it for T_N only since the argument for T_{-N} is entirely analogous.

In Step 1, we will show that the supports of T_N and T_{-N} are trapped in a small neighborhood of the boundary $\mathbb{R}^{m-1} \times \{0\}^2$. More precisely, we show that for any $0 < \delta < 1/2$, there is $N_0 \in \mathbb{N}$ such that

$$\mathrm{spt}(T_N \cup T_{-N}) \cap \overline{\mathbf{B}_8} \subset B_\delta(\mathbb{R}^{m-1} \times \{0\}^2) \qquad \forall N \geq N_0. \tag{5.5}$$

In Step 2, we construct an anisotropic mean convex smooth open set U such that $U \cap \mathbf{B}_5 \subset H_1$ and, for any $N > N_0$, we use Step 1 to show

$$\mathrm{spt}(\partial(T_N \llcorner \mathbf{B}_5)) \subset \overline{U} \text{ and } \mathrm{spt}(T_N \llcorner \mathbf{B}_5) \subset \overline{B_\delta(U)}\,.$$

For δ small enough depending on U and $N > N_0$, the last displayed equation allows us to apply Proposition 4.3 to $S = T_N \llcorner \mathbf{B}_5$, which yields $\mathrm{spt}(T_N \llcorner \mathbf{B}_5) \subset \overline{U}$. Since $\overline{U} \cap \mathbf{B}_5 \subset \overline{H_1}$, this implies that $\boldsymbol{\theta} \leq 2\pi N + \pi/2$ on $\mathbf{B}_5$. This will conclude the proof of the theorem.

Step 1: It is enough to prove the claim for T_N; the proof for T_{-N} is the same. By way of contradiction, assume (5.5) fails, that is, there is $0 < \delta < 1/2$ such that for any N large enough there exists

$$x_N \in \mathrm{spt}(T_N) \cap \left(\overline{\mathbf{B}_8} \setminus B_\delta(\mathbb{R}^{m-1} \times \{0\}^2)\right). \tag{5.6}$$

Since the set in (5.6) is contained in the compact set $\mathrm{spt}(T) \cap \left(\overline{\mathbf{B}_8} \setminus B_\delta(\mathbb{R}^{m-1} \times \{0\}^2)\right)$, then up to a subsequence, there is a limit $x_\infty \in \mathrm{spt}(T) \cap \left(\overline{\mathbf{B}_8} \setminus B_\delta(\mathbb{R}^{m-1} \times \{0\}^2)\right)$ of the sequence $\{x_N\}_N$. Up to a rotation, we may assume that $x_\infty \in M_1$. Observe that, since $\mathbf{B}_{\delta/2}(x_\infty) \subset H_1$, we get

$$\boldsymbol{\theta}(\mathbf{B}_{\delta/2}(x_\infty)) \subset \cup_{z \in \mathbb{Z}} \left(-\pi/2 + 2\pi z, \pi/2 + 2\pi z\right).$$

Letting $S_z = [\![\{x \in \mathbf{B}_{\delta/2}(x_\infty) \cap \mathrm{Reg}_i\, T : \boldsymbol{\theta}(x) \in (-\pi/2 + 2\pi z, \pi/2 + 2\pi z)\}]\!]$, note that by definition, $\mathrm{Reg}_i S_z \cap \mathrm{Reg}_i S_{z'} = \varnothing$ if $z \neq z'$. In particular, $\sum_z \mathbf{F}(S_z) = \mathbf{F}(T \llcorner \mathbf{B}_{\delta/2}(x_\infty))$, and each S_z is $\mathbf{F}$-minimizing on $\mathbf{B}_{\delta/2}(x_\infty)$.

Up to relabelling, let $\{S_{z_j}\}_{j=1}^J$ be those currents such that $\operatorname{spt} S_{z_j} \cap \mathbf{B}_{\delta/4}(x_\infty) \neq \emptyset$. We claim that J is finite. Indeed, by the lower density bounds (see e.g. [2, Lemma 4]) applied to each S_{z_j}, we may estimate

$$+\infty > \Lambda|T|(\mathbf{B}_{\delta/2}(x_\infty)) \geq \mathbf{F}(T \llcorner \mathbf{B}_{\delta/2}(x_\infty)) \geq \sum_{j=1}^J \mathbf{F}(S_{z_j}; \mathbf{B}_{\delta/4}(x_\infty)) \geq c_0 \frac{\delta^m}{4^m} \sum_{j=1}^J 1\,.$$

This implies that $J < +\infty$, and as a consequence that $\boldsymbol{\theta} \leq \pi/2 + 2\pi \max_j z_j$ on $\mathbf{B}_{\delta/4}(x_\infty)$. But this contradicts $x_N \in \operatorname{spt} T_N$ for large enough N.

Step two: Fix small $\epsilon \in (0,1)$ to be chosen later. Consider the set

$$C_\epsilon = \left(B_7^{m-1} \times B_\epsilon^2\right) \cup D_\epsilon,$$

where B_7^{m-1} and B_ϵ^2 denote the balls of radius 7 and ϵ in $\mathbb{R}^{m-1}$ and $\mathbb{R}^2$, respectively, and we choose the ends D_ϵ so that C_ϵ is C^∞, convex, and the anisotropic mean curvatures $H^F_{\partial C_\epsilon}$ and $H^G_{\partial C_\epsilon}$ (where $G(\nu) = F(-\nu)$) are of order $1/\epsilon$ for all small ϵ.

Now we compose C_ϵ with a dilation map that stretches the cylinder in the (x_m, x_{m+1}) directions for $|(x_1, \ldots, x_{m-1})| > 6$. Specifically, let $\varphi \in C^\infty(\mathbb{R}; [0,1])$ be such that $\varphi \equiv 0$ on $[0,6]$ and $\varphi = 1$ on $[6.5, \infty)$. For $x = (x', y) \in \mathbb{R}^{m-1} \times \mathbb{R}^2$, we define a dilation map and the mean convex barrier set as

$$d_\epsilon(x) := (x', (1 + \epsilon^2 \varphi(|x'|))y) \qquad \text{and} \qquad S_\epsilon := d_\epsilon(C_\epsilon) + (0_{m-1}, \epsilon, 0).$$

For all ϵ small enough, S_ϵ is anisotropic mean convex and $\mathbf{G}$-mean convex since C_ϵ is convex with anisotropic mean curvature order $1/\epsilon$ and d_ϵ is ϵ^2-close to the identity. Choosing such an ϵ, from the definitions, we see that

$$(\mathbb{R}^{m-1} \times \{0\}^2) \cap \mathbf{B}_6 \subset \partial S_\epsilon\,.$$

Furthermore, we claim that due to the dilation (which fattened S_ϵ near its "ends"), S_ϵ satisfies

$$\partial B_r^{m-1} \times \{(0,0)\} \subset\subset S_\epsilon \qquad \forall r \in (6.5, 7)\,. \tag{5.7}$$

Indeed, given $z' \in \partial B_r^{m-1}$, let $x := (z', -\frac{\epsilon}{1+\epsilon^2}, 0) \in C_\epsilon$, and note that $\operatorname{dist}(x, \partial C_\epsilon) > 0$. Since $\epsilon^2 \varphi(|z'|) = \epsilon^2$, we thus have $(z', 0, 0) = d_\epsilon(x) + (0_{m-1}, \epsilon, 0)$ and $\operatorname{dist}((z', 0, 0), \partial S_\epsilon) > 0$. Lastly, before moving on to the barrier argument, we fix $\delta_0(S_\epsilon)$ as in Proposition 4.3.

Moving on to the barrier argument, by slicing theory, for almost every $r > 0$, we know that $\partial(T_N \llcorner \mathbf{B}_r) = \langle T_N, |\cdot|, r\rangle + (\partial T_N) \llcorner \mathbf{B}_r$ for every N. We fix such an $r \in (6.5, 7)$. By Step 1 applied with any $\delta < \min\{\epsilon^2, \operatorname{dist}(\partial B_r^{m-1} \times \{0\}^2, \partial S_\epsilon)/2, \delta_0\}$ and (5.7), it follows that there is N_0 such that for all $N \geq N_0$,

$$\operatorname{spt}(\langle T_N, |\cdot|, r\rangle) \subset \partial \mathbf{B}_r \cap B_\delta(\mathbb{R}^{m-1} \times \{0\}^2) \subset S_\epsilon\,.$$

Since $\operatorname{spt}(\partial T_N) \subset \overline{M_1} \cap B_\delta(\mathbb{R}^{m-1} \times \{0\}^2)$, according to Lemma 5.4, and by Step 1, we obtain

$$\operatorname{spt}((\partial T_N) \llcorner \mathbf{B}_r) \subset \overline{M_1} \cap B_\delta(\mathbb{R}^{m-1} \times \{0\}^2) \cap \mathbf{B}_r \subset \overline{S_\epsilon}$$

since $\delta \leq \epsilon^2 \leq 2\epsilon$, where 2ϵ is the "width" of $S_\epsilon \cap \mathbf{B}_r$ in the (x_m, x_{m+1}) directions. We have then proved

$$\operatorname{spt}(\partial(T_N \llcorner \mathbf{B}_r)) \subset \overline{S_\epsilon} \qquad \text{for all } N \geq N_0. \tag{5.8}$$

We also notice that since $(\mathbb{R}^{m-1} \times \{0\}^2) \cap \mathbf{B}_r \subset \overline{S_\epsilon}$, the application of Step 1 directly implies that

$$\operatorname{spt}(T_N \llcorner \mathbf{B}_r) \subset \overline{B_{\delta_0}(S_\epsilon)}\,. \tag{5.9}$$

Equations (5.8)-(5.9) allow for the application of Proposition 4.3 to $T_N \llcorner B_r$, yielding

$$\operatorname{spt}(T_N \llcorner \mathbf{B}_r) \subset \overline{S_\epsilon} =: \overline{U}\,.$$

Since $r > 5$ and $\overline{S_\epsilon} \cap \mathbf{B}_5 \subset \overline{H_1}$, this concludes the proof of (5.4). $\square$

6. Existence of a flat blowup

The goal of this section is to prove Theorem 1.1.

Remark 6.1 (Reduction of Theorem 1.1 to the flat case)**.** We observe that a simple argument shows that up to a rotation, any blowup of T is a non-trivial $\mathbf{F}$-minimizer on $\mathbb{R}^{m+1}$ with flat boundary $[\![\mathbb{R}^{m-1} \times \{0\}^2]\!]$. Indeed, if a blowup T_0 of T is of mass zero, then we must have $\partial T_0 = 0$ contradicting that the blowup limit of the boundary is flat, which follows from the fact that the boundary is differentiable at 0. Therefore, we can work under Assumption 1.

We will need one more definition, which describes a family of cylinders that are rotations of each other around the boundary $\mathbb{R}^{m-1}$ and all of which contains a portion of the boundary.

Definition 6.2 (Cylinders tangent to the boundary)**.** Let $C \subset \mathbb{R}^m$ be a smooth, convex, open set such that ∂C is diffeomorphic to $\mathbb{S}^{m-1}$, and

$$\{x \in \mathbb{R}^m : |x| < 1/2,\, x_m > 0\} \subset C \subset \{x \in \mathbb{R}^m : |x| < 2,\, x_m > 0\} \qquad \text{and}$$
$$B_1^{m-1}(0) \times \{0\} \subset \partial C\,.$$

For $s \in \mathbb{S}^1$ and $t > 0$, we define

$$C_{s,t} = t\left[R_s\left(C \times (-1,1)\right)\right]\,,$$

where R_s is the rotation in $\mathbb{R}^{m+1}$ which fixes $(x_1,\dots,x_{m-1})$ and sends $(x_m, x_{m+1}) \sim re^{i\theta}$ to $sre^{i\theta}$.

Theorem 6.3 (Flat blowups)**.** *If T satisfies Assumption 1, then there exists a half-hyperplane M_s and $Q \in \mathbb{N}$ such that a blowup of T at 0 is given by $Q[\![M_s]\!] - (Q-1)[\![M_{-s}]\!]$.*

Proof. The proof is divided into steps.

Set-up. By Theorem 5.5, we know that $\boldsymbol{\theta}$ is bounded on $\mathbf{B}_5 \cap \mathrm{Reg}_i\, T$. We set

$$\boldsymbol{\theta}_r := \inf\{\boldsymbol{\theta}(x) : x \in \mathbf{B}_r \cap \mathrm{Reg}_i T\} \qquad \text{and} \qquad \boldsymbol{\theta}_{\min} = \lim_{r \searrow 0} \boldsymbol{\theta}_r;$$

note that the limit exists since $\boldsymbol{\theta}$ is bounded and $\boldsymbol{\theta}_r$ is a decreasing function of r. Up to a rotation, we may assume that $\boldsymbol{\theta}_{\min} \in 2\pi\mathbb{Z}$. Our candidate tangent currents are thus

$$Q[\![M_1]\!] - (Q-1)[\![M_{-1}]\!] \qquad \text{and} \qquad Q[\![M_{-1}]\!] - (Q-1)[\![M_1]\!] \qquad\qquad Q \in \mathbb{N}\,,$$

that is, those flat cones which contain $M_1 = M_{e^{i\boldsymbol{\theta}_{\min}}}$ in their supports.

Alternative characterization of $\boldsymbol{\theta}_{\min}$. The goal of this step is to obtain $\boldsymbol{\theta}_{\min}$ as a limit over nested cylinders tangent to the boundary at the origin rather than nested balls containing the origin.

We begin by observing that for r small enough such that

$$\inf\{\boldsymbol{\theta}(x) : x \in \mathbf{B}_r \cap \mathrm{Reg}_i T\} > \boldsymbol{\theta}_{\min} - \pi/4\,,$$

we may assume that a sequence $\{x_j^r\}_j$ with polar angles converging to the infimum on $\mathbf{B}_r \cap \mathrm{Reg}_i T$ is chosen such that

$$\{\boldsymbol{\theta}(x_j^r)\}_j \subset (\boldsymbol{\theta}_{\min} - \pi/4, \boldsymbol{\theta}_{\min}]\,;$$

in other words, for all small r, the optimal sequences are contained in the halfspace H_1. With this and recalling the definition of the cylinders $C_{s,t}$ (Definition 6.2), we obtain

$$\begin{aligned}\inf\{\boldsymbol{\theta}(x) : x \in \mathbf{B}_{2r} \cap \mathrm{Reg}_i T\} &\leq \inf\{\boldsymbol{\theta}(x) : x \in C_{1,r} \cap \mathrm{Reg}_i T\}\\ &\leq \inf\{\boldsymbol{\theta}(x) : x \in H_1 \cap \mathbf{B}_{r/2} \cap \mathrm{Reg}_i T\}\\ &= \inf\{\boldsymbol{\theta}(x) : x \in \mathbf{B}_{r/2} \cap \mathrm{Reg}_i T\}\,.\end{aligned} \tag{6.1}$$

Since the quantity

$$\boldsymbol{\theta}_{\text{cylindrical}\min} := \lim_{r \searrow 0}\Big[\inf\{\boldsymbol{\theta}(x) : x \in C_{1,r} \cap \mathrm{Reg}_i T\}\Big]$$

is also determined by an infimum which is decreasing in r, (6.1) implies that

$$\boldsymbol{\theta}_{\mathrm{cylindrical\,min}} = \boldsymbol{\theta}_{\min} \in 2\pi\mathbb{Z}\,. \tag{6.2}$$

Every blowup contains M_1. Here we prove that for any sequence $r_j \searrow 0$ and blowup T_0 such that $T_{0,r_j} \to T_0$, there is $z \in \mathbb{N}$ such that

$$M_1 \subset \operatorname{spt} T_0 \qquad \text{and} \qquad \Theta^m_{T_0} \equiv z \text{ on } M_1\,. \tag{6.3}$$

First, note that every point $x \in \mathrm{Reg}_i T \cap C_{1,1}$ satisfies $\boldsymbol{\theta}(x) \in (-\pi/2, \pi/2) + 2\pi\mathbb{Z}$. Therefore, recalling that $\boldsymbol{\theta}_{\min} \in 2\pi\mathbb{Z}$, we can write $\mathrm{Reg}_i T \cap C_{1,1} = A_1 \cup A_2$, where

$$A_1 = \{x \in \mathrm{Reg}_i T \cap C_{1,1} : \boldsymbol{\theta}(x) < \boldsymbol{\theta}_{\min} + \pi/2\}\,, \qquad A_2 = \{x \in \mathrm{Reg}_i T \cap C_{1,1} : \boldsymbol{\theta}(x) > \boldsymbol{\theta}_{\min} + 3\pi/2\}\,.$$

Define the $\mathbf{F}$-minimizers $S_k := T_k \llcorner A_k$ for $k = 1, 2$ whose support is contained in $\overline{C_{1,1}}$. Up to an unrelabelled subsequence, let us assume that $(S_1)_{0,r_j}$ converges to some $\mathbf{F}$-minimizer S_0. Since the outward normal vector to $\partial C_{1,1}$ at 0 is $-e_m$, then the blowup of $C_{1,1}$ at the origin is the halfspace H_1. Thus S_0 is an $\mathbf{F}$-minimizer in H_1.

Next, recalling that H_i is the open halfspace with boundary $\{x_{m+1} = 0\}$, we claim that

$$\operatorname{spt} S_0 \subset \overline{H_1} \cap \overline{H_i}\,. \tag{6.4}$$

The containment in $\overline{H}_1$ is already established above. For the containment in $\overline{H}_i$, we observe that, by definition,

$$\boldsymbol{\theta}_{\min} + \pi/2 > \boldsymbol{\theta}(y) \geq \inf\{\boldsymbol{\theta}(x) : x \in C_{1,r} \cap \mathrm{Reg}_i S_1\} =: \boldsymbol{\theta}^{\mathrm{cyl}}_r \geq \boldsymbol{\theta}_{\min} - \pi/2, \qquad \forall y \in C_{1,r} \cap \mathrm{Reg}_i S_1\,. \tag{6.5}$$

Phrased in terms of halfspaces, (6.5) says that

$$C_{1,r} \cap \mathrm{Reg}_i S_1 \subset C_{1,r} \cap \overline{H_{e^{i(\boldsymbol{\theta}^{\mathrm{cyl}}_r + \pi/2)}}}\,. \tag{6.6}$$

Recalling from (6.2) that $\boldsymbol{\theta}^{\mathrm{cyl}}_r \to \boldsymbol{\theta}_{\min} \in 2\pi\mathbb{Z}$ as $r \to 0$, taking $r \to 0$ yields

$$\operatorname{spt}\, S_0 \subset \overline{H_{e^{i(\boldsymbol{\theta}_{\min} + \pi/2)}}} = \overline{H_i}\,.$$

Now we assume for contradiction that

$$M_1 \not\subset \operatorname{spt} T_0\,. \tag{6.7}$$

Since $\operatorname{spt} S_0 \subset \operatorname{spt} T_0$, then there exists $(x_0', 0) \in M_1 \setminus \operatorname{spt} S_0$ with

$$\operatorname{dist}((x_0', 0), \operatorname{spt} S_0) =: r_0 > 0. \tag{6.8}$$

Let r_1 be such that $(x_0', 0) \in \partial C_{1,r_1}$. We claim that for all large j, we may apply Proposition 4.5 to $(S_1)_{0,r_j}$ on C_{1,r_1}. Indeed, by (6.6), for j large, we have

$$\operatorname{spt}\left[(S_1)_{0,r_j} \llcorner C_{1,r_1}\right] \subset \{(x', x_{m+1}) \in \overline{C_{1,r_1}} : x_{m+1} \geq L_j(x')\}\,, \tag{6.9}$$

for some linear functions L_j's which converge uniformly to zero. Moreover, since $\operatorname{spt}(S_1)_{0,r_j}$ converge in the local Hausdorff sense to $\operatorname{spt} S_0$, (6.8) and the uniform convergence of L_j to 0 imply that, for large j,

$$\operatorname{dist}((x_0', L_j(x_0')), \operatorname{spt}(S_1)_{0,r_j}) \geq r_0/2\,. \tag{6.10}$$

Together with the convergence of L_j to 0, (6.9) and (6.10) allow for the application of Proposition 4.5. Then the conclusions (4.7)-(4.8) of the proposition imply that there is $\theta_0 > \pi/2$ and $0 < r_2 < r_1$ such that

$$\operatorname{spt}(S_1)_{0,r_j} \cap C_{1,r_2} \subset \overline{H_{e^{i\theta_0}}}\,;$$

in words, $\operatorname{spt}(S_1)_{0,r_j} \cap C_{1,r_2}$ is contained in a wedge with boundary coming in to the origin with angle strictly greater than 0. Using that $\boldsymbol{\theta}_{\min} \in 2\pi\mathbb{Z}$, we get existence of $\delta > 0$ such that

$$\boldsymbol{\theta}(y) > \boldsymbol{\theta}_{\min} + \delta \qquad \forall y \in \operatorname{spt} S_1 \cap C_{1,r_2 r_j}\,.$$

By definition of A_1 and S_1, the last displayed inequality contradicts (6.2). Thus (6.7) is false, that is $M_1 \subset \operatorname{spt} T_0$. By Lemma 4.2 (which applies due to (6.4)), we deduce that the multiplicity of T_0 is constant on M_1, which finishes the proof of (6.3).

Blowups containing M_1 in their support have flat blowups. Here, we prove the following. Let T_0 be a blowup of T at 0 such that $M_1 \subset \operatorname{spt} T_0$, then any blowup T_{00} of T_0 at the origin satisfies

$$\operatorname{spt} T_{00} \subset \partial H_i\,.$$

First, we observe that (6.3) implies that M_1 is a connected component of $\mathrm{Reg}_i T_0$, and so no other connected component of $\mathrm{Reg}_i T_0$ can intersect it by Lemma 4.2 and interior regularity. As a consequence, if we define as in Definition 5.3 a polar angle function $\boldsymbol{\theta}_{T_0}$ for the blowup with domain in $\mathrm{Reg}_i T_0$, it can never take a value in $2\pi\mathbb{Z}$. Therefore, $\boldsymbol{\theta}_{T_0}$ is globally bounded, and we may as well assume that it takes values off of M_1 in the open interval $(0, 2\pi)$.

We claim that there is $\delta \in (0, \pi/2)$ such that

$$\boldsymbol{\theta}_{T_0}(\mathbf{B}_1 \cap \mathrm{Reg}_i T_0 \setminus M_1) \subset (\delta, 2\pi - \delta)\,. \tag{6.11}$$

To see this, let R be chosen large enough so that $C_{1,R}$ contains $\mathbf{B}_1 \cap \{x_m > 0\}$. Letting z be as in (6.3), we split $(T_0 - z[\![M_1]\!]) \llcorner C_{1,R}$ into two pieces: $T_1 \coloneqq (T_0 - z[\![M_1]\!]) \llcorner C_{1,R} \cap \{\boldsymbol{\theta}_{T_0} \in (0, \pi/2)\}$, that is, the part of $(T_0 - z[\![M_1]\!]) \llcorner C_{1,R}$ living above M_1, and T_2 is the restriction to those points with $\boldsymbol{\theta}_{T_0} \in (3\pi/2, 2\pi)$. Since $T_0 - z[\![M_1]\!]$ can never take polar angle value in $2\pi\mathbb{Z}$, the conditions (4.3) and (4.4) are satisfied by T_1 with $L = 0$. Thus (4.7)-(4.8) yield $\delta_1 > 0$ such that $\boldsymbol{\theta}_{T_0} > \delta_1$ on $\mathrm{Reg}_i T_1$, and a reflection argument and (4.7)-(4.8) yield the corresponding upper bound $\boldsymbol{\theta}_{T_0} < 2\pi - \delta_2$ on $\mathrm{Reg}_i T_2$ for some $\delta_2 > 0$. Letting $\delta = \min\{\delta_1, \delta_2\}$ yields (6.11).

Next, we perform a similar blowup procedure as in the previous step. For $r < 1$, we define

$$\boldsymbol{\theta}_r^{\inf} = \inf\{\boldsymbol{\theta}_{T_0}(y) : y \in \mathrm{Reg}_i(T_0 - z[\![M_1]\!]) \cap \mathbf{B}_r\} \geq \delta \qquad \text{and}$$
$$\boldsymbol{\theta}_r^{\sup} = \sup\{\boldsymbol{\theta}_{T_0}(y) : y \in \mathrm{Reg}_i(T_0 - z[\![M_1]\!] \cap \mathbf{B}_r\} \leq 2\pi - \delta\,.$$

Note that $\boldsymbol{\theta}_r^{\inf}$ is a decreasing function of r and $\boldsymbol{\theta}_r^{\sup}$ is increasing, and so there are limits

$$\delta \leq \lim_{r\to 0} \boldsymbol{\theta}_r^{\inf} =: \boldsymbol{\theta}_{T_0}^{\min} \leq \boldsymbol{\theta}_{T_0}^{\max} \coloneqq \lim_{r\to 0} \boldsymbol{\theta}_r^{\sup} \leq 2\pi - \delta\,.$$

Arguing as in the previous step using Proposition 4.5, for any blowup T_{00} of T_0 at 0, it holds

$$M_1 \cup M_{e^{i\boldsymbol{\theta}_{T_0}^{\min}}} \cup M_{e^{i\boldsymbol{\theta}_{T_0}^{\max}}} \subset \operatorname{spt} T_{00}\,.$$

We continue on iteratively, blowing up and identifying more and more halfplanes contained in further blowups. Since T, T_0, T_{00}, and so on all satisfy uniform upper mass bounds (Lemma 3.2), this procedure must terminate after a finite number of blowups. The end result is a current which is supported on a finite number of oriented half hyperplanes and has constant multiplicity on each. By Lemma 3.3, this last blowup must be supported on the plane containing M_1 and this finishes the proof. $\square$

Remark 6.4. We notice that, if 0 is a one-sided point for T, i.e., the density of T at 0 is close to 1/2, we could have stopped the iteration in the very first step.

References

[1] D. E. Apushkinskaya and A. I. Nazarov. On the boundary point principle for divergence-type equations. *Rendiconti Lincei*, 30(4):677–699, 2019. 4.1, 4.2

[2] E. Bombieri. Regularity theory for almost minimal currents. *Arch. Ration. Mech. Anal.*, 78(2):99–130, 1982. 3.2, 5.3

[3] G. De Philippis and F. Maggi. Regularity of free boundaries in anisotropic capillarity problems and the validity of Young's law. *Arch. Ration. Mech. Anal.*, 216(2):473–568, 2015. 3.4, 3.4, 4.1, 4.1, 4.1, 4.2

[4] A. De Rosa, S. Kolasiński, and M. Santilli. Uniqueness of critical points of the anisotropic isoperimetric problem for finite perimeter sets. *Arch. Ration. Mech. Anal.*, 238(3):1157–1198, 2020. 4.1

[5] F. Duzaar and K. G. Steffen. Optimal interior and boundary regularity for almost minimizers to elliptic variational integrals. *Journal für die reine und angewandte Mathematik*, 2002(546):73–138, 2002. 1, 1.4

[6] H. Federer. *Geometric measure theory*. Springer, 2014. 2, 3.2, 5.2, 5.2

[7] A. Figalli. Regularity of codimension 1 minimizing currents under minimal assumptions on the integrand. *Journal of Differential Geometry*, 106(3):371–391, 2017. 1, 3.1, 3.1, 3.2, 4.1, 4.1, 4.2

[8] R. Hardt and L. Simon. Boundary regularity and embedded solutions for the oriented Plateau problem. *Annals of Mathematics*, 110(3):439–486, 1979. 1, 1.4, 1.5, 5.2, 5.2, 5.2

[9] R. M. Hardt. On boundary regularity for integral currents or flat chains modulo two minimizing the integral of an elliptic integrand. *Communications in Partial Differential Equations*, 2(12):1163–1232, 1977. 1

[10] H. Jenkins and J. Serrin. The Dirichlet problem for the minimal surface equation in higher dimensions. *Journal für die reine und angewandte Mathematik*, 229:170–187, 1968. 3.1

[11] Y. Li. On the C^1 regularity of solutions to divergence form elliptic systems with Dini-continuous coefficients. *Chinese Annals of Mathematics, Series B*, 38(2):489–496, 2017. 3.1

[12] F. Maggi. *Sets of finite perimeter and geometric variational problems*, volume 135 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2012. 2

[13] F. Morgan. The cone over the Clifford torus in $\mathbb{R}^4$ is ϕ-minimizing. *Mathematische Annalen*, 289(1):341–354, 1991. 1

[14] R. Schoen, L. Simon, and F. J. Almgren, Jr. Regularity and singularity estimates on hypersurfaces minimizing parametric elliptic variational integrals. I, II. *Acta Math.*, 139(3-4):217–265, 1977. 1, 3.2

[15] L. Silvestre and B. Sirakov. Boundary regularity for viscosity solutions of fully nonlinear elliptic equations. *Comm. Partial Differential Equations*, 39(9):1694–1717, 2014. 3.1

[16] L. Simon. *Lectures on geometric measure theory*, volume 3 of *Proceedings of the Centre for Mathematical Analysis, Australian National University*. Australian National University, Centre for Mathematical Analysis, Canberra, 1983. 2

[17] B. White. Regularity of area-minimizing hypersurfaces at boundaries with multiplicity. In *Seminar on minimal submanifolds*, volume 103, pages 293–301. Princeton University Press Princeton, NJ, 1983. 1.3, 3.2

[18] G. H. Williams. The Dirichlet problem for the minimal surface equation with Lipschitz continuous boundary data. *J. Reine Angew. Math.*, 354:123–140, 1984. 3.1

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