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Ricci curvature, fundamental groups, and the Jordan property

Daniele Semola
Universität Wien

Abstract. We survey recent developments about the structure of fundamental groups of closed manifolds with positive Ricci curvature.

1 Introduction

The study of fundamental groups of complete Riemannian manifolds satisfying suitable restrictions on their curvature has been one of the key lines of research in global Riemannian geometry since its earliest developments. Among the earliest examples we quote Bieberbach's theorem on the structure of flat Riemannian manifolds and their fundamental groups [2, 3], the Bonnet-Myers theorem for compact manifolds with positive Ricci curvature [5, 35], and Synge's theorem for compact manifolds with positive sectional curvature [45].

The focus of this survey is on the family of fundamental groups of closed Riemannian manifolds (M^n, g) of a fixed dimension $n \geq 1$ and with nonnegative Ricci curvature. We will outline some aspects of recent joint work with E. Bruè and A. Naber, [7], whose main result shows that the family of such fundamental groups is not *uniformly Jordan* according to the following:

Definition 1.1 ([41, Definition 1.6])**.** *Let $\mathcal{G}$ be a family of groups. We say that $\mathcal{G}$ is* uniformly Jordan *if there exists a constant $J = J(\mathcal{G})$ such that for every group $\Gamma \in \mathcal{G}$ and every finite subgroup $F \subset \Gamma$, there is a normal abelian subgroup $A \triangleleft F$ satisfying*

$$[F : A] \leq J\,.$$

A group Γ is called Jordan *if the singleton family $\{\Gamma\}$ is uniformly Jordan.*

If we require that all the groups Γ in the family have an abelian subgroup $A < \Gamma$ with $[\Gamma : A] \leq C(\mathcal{G})$, the groups in $\mathcal{G}$ are often said to be (uniformly) $C(\mathcal{G})$-abelian. As we shall discuss the examples in [7] illustrate that this property fails

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for the family of fundamental groups of closed manifolds with $\mathrm{Ric} \geq 0$ and fixed dimension.

Definition 1.1 was introduced by Y. Prokhorov and C. Shramov in [41], generalising the earlier [39, Definition 2.1] proposed by V.-L. Popov in the case of a single group. It is motivated by a classical result due to C. Jordan in 1878:

Theorem 1.2 ([28])**.** *For every integer $n \geq 1$ there exists a constant $J(n)$ such that every finite subgroup $G \subset \mathrm{GL}_n(\mathbb{C})$ contains a normal abelian subgroup $A \triangleleft G$ satisfying*

$$[G:A] \leq J(n)\,.$$

We refer the reader to [6] for a modern exposition of the original, purely algebraic, argument and [15, Chapter V] for a different proof of more geometric flavor of Theorem 1.2.

With the terminology introduced in Definition 1.1, Theorem 1.2 can be rephrased by saying that $\mathrm{GL}_n(\mathbb{C})$ is a Jordan group. Thanks to the Peter-Weyl theorem, Theorem 1.2 implies that any compact Lie group is a Jordan group. Since the isometry group of any compact Riemannian manifold is a compact Lie group [34], every such group is Jordan as well.

Understanding whether the natural *transformation groups* of some family of *mathematical objects* are (uniformly) Jordan is an interesting problem, which has been explicitly raised in various circumstances. Among others:

i) J.-P. Serre asked in [44, Question 6.1] whether the Cremona group $\mathrm{Cr}_n(k)$, i.e., the group of birational transformations of the projective space $\mathbb{P}^n(k)$, is Jordan for fields k of characteristic zero. An affirmative answer for $n = 2$ was obtained in the same paper by Serre [44, Theorem 5.3]. An affirmative answer in arbitrary dimensions follows by combining work of Y. Prokhorov and C. Shramov [41, Theorem 1.8] with work of C. Birkar [4, Theorem 1.1]. The statement holds more generally for the group of birational transformations $\mathrm{Bir}(X)$ of any rationally connected variety X, with the constant only depending on the dimension.

ii) A question popularized by É. Ghys in the mid-Nineties (see [17, Question 13.1]) asked whether the diffeomorphism group $\mathrm{Diff}(M)$ of every compact smooth manifold M is a Jordan group. B. Csikós, L. Pyber, and E. Szabó answered the question in the negative, by showing that $\mathrm{Diff}(T^2 \times S^2)$ is not Jordan, see [13, Theorem 1]. We will see that their construction shares some features with the construction in [7].

The focus of this note is on an analogous problem posed by V. Kapovitch and B. Wilking at the end of [31]:

Problem 1.3. *Given an integer $n \geq 1$, is the family of fundamental groups of complete Riemannian manifolds (M^n, g) with* $\mathrm{Ric} \geq 0$ *uniformly Jordan?*[1]

The problem was partly motivated by an analogous conjecture raised by K. Fukaya and T. Yamaguchi in [18, Conjecture 0.16] in the context of manifolds with non-negative sectional curvature:

Conjecture 1.4. *Let $n \geq 1$ be an integer. There exists a constant $C(n)$ such that for every complete manifold (M^n, g) with nonnegative sectional curvature, $\pi_1(M)$ has an abelian subgroup with index less than $C(n)$.*

We refer the reader also to the work of J. Pan and X. Rong, see in particular [38, Conjecture 2.22], where a variant of Problem 1.3 was stated as a conjecture. While the Fukaya-Yamaguchi conjecture remains open as of now, despite partial progress under additional restrictions (see, for instance, [42, 33, 30, 8]), in [7] we resolved Problem 1.3 in the negative.

2 Fundamental groups, virtual nilpotency, and curvature bounds

Further motivation for Problem 1.3 and Conjecture 1.4, apart from the broader relevance of the Jordan property discussed before, came from two distinct results:

i) Fundamental groups of closed manifolds with $\mathrm{Ric} \geq 0$ are virtually abelian, i.e., they have a finite index, normal, abelian subgroup, as proved by J. Cheeger and D. Gromoll in 1971, see [9, Theorem 3].

ii) Fundamental groups of complete manifolds with $\mathrm{Ric} \geq 0$ and dimension $n \geq 1$ are $C(n)$-nilpotent, i.e., they have a nilpotent subgroup with index less than $C(n)$, as proved by Kapovitch and Wilking in [31, Theorem 1, Corollary 4]. In the closed case, this subgroup can be chosen to be normal, see [31, Theorem 6].

In a sense, the goal with Problem 1.3 was to sharpen the Cheeger-Gromoll theorem in item i), by controlling the index of the abelian subgroup effectively in terms of the dimension, as well as the Kapovitch-Wilking theorem in item ii), by upgrading the nilpotency conclusion to commutativity.

There were some crucial antecedents to Kapovitch and Wilking's result. M. Gromov initiated this whole line of inquiry in [20], with his *almost-flat manifolds* theorem:

[1] The original formulation of the problem in [31] was slightly different, but equivalent in the present context of closed manifolds with $\mathrm{Ric} \geq 0$.

Theorem 2.1 ([20]). *For every $n \geq 1$ there exist $\varepsilon(n) > 0$ and $C(n) \geq 1$ such that the following holds. Let (M^n, g) be a closed Riemannian manifold such that*

$$\left(\sup_M |\mathrm{Sect}_g|\right) \mathrm{diam}_g^2(M) \leq \varepsilon(n)\,. \tag{1}$$

Then M admits a finite covering with covering degree less than $C(n)$ which is diffeomorphic to a nilmanifold.

Manifolds satisfying (1) are said to be almost-flat, for the obvious reason that in case one replaces $\varepsilon(n)$ with 0 at the right-hand side, one is simply left with flat manifolds. Note also that the left-hand side in (1) is invariant under rescaling of the metric.

A nilmanifold is a differentiable manifold which has a transitive nilpotent group of diffeomorphisms acting on it. By a theorem of A. Mal'cev, see [32], every compact nilmanifold is the quotient of a simply connected nilpotent Lie group by a cocompact discrete subgroup. An explicit example to keep in mind is the so-called Heisenberg nilmanifold. One starts with the Heisenberg group

$$H_3(\mathbb{R}) := \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a\,,b\,,c \in \mathbb{R} \right\} < \mathrm{GL}_3(\mathbb{R})\,, \tag{2}$$

and then considers the quotient $H_3(\mathbb{R})/H_3(\mathbb{Z})$ where $H_3(\mathbb{Z})$ denotes the lattice

$$H_3(\mathbb{Z}) := \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a\,,b\,,c \in \mathbb{Z} \right\} < \mathrm{SL}_3(\mathbb{Z})\,. \tag{3}$$

Any compact nilmanifold admits Riemannian metrics such that the left-hand side in (1) is arbitrarily small. Gromov's Theorem 2.1 provides a converse to such statement, up to finite covering. The result was subsequently refined by E. Ruh in [43], where he proved that every almost-flat manifold M is diffeomorphic to an *infranilmanifold*, i.e., to a quotient N/Γ where N is a simply connected nilpotent Lie group and Γ is a finite extension of a lattice $L \subset N$ by a finite group $H = \Gamma/L$ with $|H| \leq C(n)$.

The characterization of almost-flat manifolds up to diffeomorphism implies that their fundamental groups have a finite index nilpotent subgroup, with index bounded by a dimensional constant. In his ICM lecture in 1978 [21], Gromov conjectured that this property should hold under the weaker (with respect to (1)) assumption that

$$\left(\inf_M \mathrm{Sect}_g\right) \mathrm{diam}^2(M) \geq -\varepsilon\,, \tag{4}$$

for some $0 < \varepsilon < \varepsilon(n)$. Note that no clean, complete characterization of such manifolds up to diffeomorphism can be expected in general. Closed Riemannian manifolds satisfying (4) were named *near elliptic* in Gromov's survey. Compact differentiable manifolds admitting Riemannian metrics g_ε satisfying (4) for every $\varepsilon > 0$ are referred to as *almost nonnegatively curved* in the literature, see [29, Definition 1.0.1]. Fukaya and Yamaguchi made substantial progress towards the resolution of Gromov's conjecture in [18], by establishing the existence of a finite index nilpotent subgroup without uniform control on the index. The conjecture was established subsequently by Kapovitch, A. Petrunin and W. Tuschmann in [29]. The uniform (for fixed dimension) virtual nilpotency of fundamental groups was original even for manifolds of nonnegative sectional curvature.

As mentioned before, a corollary of the Cheeger-Gromoll splitting theorem, obtained in [9] for closed manifolds with $\mathrm{Ric} \geq 0$ and earlier in [10] for closed manifolds with nonnegative sectional curvature, was that their fundamental groups are virtually abelian. In the case of nonnegative sectional curvature the closedness assumption is not essential, by the Cheeger-Gromoll soul theorem from [10]. For manifolds with nonnegative Ricci curvature on the other hand, the closedness assumption turns out to be essential for such statement. This is illustrated by a family of examples constructed by G. Wei in [46]. She proved that any finitely generated torsion free nilpotent group can be realized as the fundamental group of a complete Riemannian manifold with $\mathrm{Ric} > 0$. Wilking subsequently generalized Wei's construction in [47], by showing that any finitely generated virtually nilpotent group can be realized as the fundamental group of a complete Riemannian manifold with $\mathrm{Ric} > 0$.

3 Fundamental groups of compact manifolds with positive Ricci curvature and failure of the Jordan property

The main result of [7] resolves Problem 1.3 in the negative. To state it, we need to introduce some notation. The 3-dimensional Heisenberg group over a commutative ring $(A, +, \cdot)$ is defined as

$$H_3(A) := \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in A \right\} < \mathrm{GL}_3(A)\,. \tag{5}$$

In the case that $(A,+)$ is generated by the identity $1 \in A$, we have that $H_3(A)$ is generated as a group by the two elements

$$\gamma_1 := \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \gamma_2 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6}$$

If we set

$$\gamma_3 := \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{7}$$

then we have the relations

$$\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1} = \gamma_3 \quad \gamma_1\gamma_3 = \gamma_3\gamma_1, \quad \gamma_2\gamma_3 = \gamma_3\gamma_2. \tag{8}$$

We already discussed the real Heisenberg group $H_3(\mathbb{R})$, and its lattice $H_3(\mathbb{Z})$. For the forthcoming discussion the finite groups $H_3(\mathbb{Z}/k\mathbb{Z})$ with $k \geq 1$ integer play a key role. They are nilpotent of step two. One can show that

$$[H_3(\mathbb{Z}/k\mathbb{Z}) : [H_3(\mathbb{Z}/k\mathbb{Z}), H_3(\mathbb{Z}/k\mathbb{Z})]] = k^2, \quad \text{for all } k \geq 1. \tag{9}$$

More importantly for us, there is no sequence of abelian subgroups $A_k < H_3(\mathbb{Z}/k\mathbb{Z})$ such that

$$\sup_{k\geq 1} [H_3(\mathbb{Z}/k\mathbb{Z}) : A_k] < \infty, \tag{10}$$

see [48, Section 3] for a detailed proof. In particular, such family of groups is not uniformly Jordan.

The main result from [7] reads as follows:

Theorem 3.1. *There exists a sequence of closed smooth 9-dimensional Riemannian manifolds (N_k^9, g_k) such that:*

i) $\mathrm{Ric}_{g_k} \geq 8$*;*

ii) The fundamental group $\pi_1(N_k)$ is a degree-two extension of the finite Heisenberg group $H_3(\mathbb{Z}/k\mathbb{Z})$ for each $k \in \mathbb{N}$.

The fundamental groups $\pi_1(N_k)$ cannot admit abelian subgroups $\overline{A}_k < \pi_1(N_k)$ such that

$$\sup_{k\geq 1} [\pi_1(N_k) : \overline{A}_k] < \infty, \tag{11}$$

since they are degree-two extensions of the finite Heisenberg groups $H_3(\mathbb{Z}/k\mathbb{Z})$ which do not admit such family of subgroups. In particular, Theorem 3.1 settles Problem 1.3 in the negative. A minor variant of the construction yields sequences of closed manifolds with positive Ricci curvature whose fundamental groups are not uniformly virtually abelian in any dimension $n \geq 9$. In dimensions $n = 2$ and $n = 3$, fundamental groups of closed manifolds with $\mathrm{Ric} \geq 0$ are uniformly Jordan. This follows for instance from R. Hamilton's work [24] in the case of closed universal covers, and from the arguments in [8] in the case of noncompact universal covers. Problem 1.3 remains open in dimensions $4 \leq n < 9$.

Theorem 3.1 is a consequence of the following result, also obtained in [7]:

Theorem 3.2. *For every $k \geq 1$ there exists a closed, simply connected 4-manifold (M^4_k, g_k) with $\mathrm{Ric}_{g_k} \geq 3$, admitting an effective isometric action of the finite Heisenberg group $H_3(\mathbb{Z}/k\mathbb{Z})$.*

Theorem 3.2 is sharp in two regards:

i) A similar result fails under a uniform lower bound of the sectional curvature, see [8].

ii) No such sequence of examples can exist in lower dimensions, by one of the main results in [16] combined with Jordan's Theorem 1.2.

Given the sequence of 4-manifolds in Theorem 3.2, the proof of Theorem 3.1 is fairly direct. If the isometric $H_3(\mathbb{Z}/k\mathbb{Z})$-actions on the manifolds M^4_k were not only effective but also free, it would suffice to consider the corresponding quotient manifolds. The actions we are dealing with are not free. To fix this, we lift them to free isometric actions over suitable 9-dimensional fiber bundles over the M^4_k's which are still positively Ricci curved and simply connected, and only at that stage we consider the corresponding quotients. Such lifting construction is by now quite standard and relies on some ideas introduced at the end of the Seventies in [1, 37].

In the rest of the section we outline some of the ideas involved in the proof of Theorem 3.2. Observe that the finite Heisenberg group $H_3(\mathbb{Z}/k\mathbb{Z})$ admits a faithful representation

$$H_3(\mathbb{Z}/k\mathbb{Z}) \longrightarrow \mathrm{GL}_k(\mathbb{C}). \tag{12}$$

One can describe it using two generators (γ_1, γ_2) introduced in (6). The first generator γ_1 acts by cyclically permuting the k components:

$$\gamma_1(e_j) = e_{j+1}\,, \quad j = 1\ldots, k-1\,, \quad \gamma_1(e_k) = e_1\,. \tag{13}$$

The second generator γ_2 acts diagonally by

$$\gamma_2(e_j) = e^{2\pi i j/k} \cdot e_j\,, \quad j = 1\ldots, k. \tag{14}$$

Then one can easily compute that the commutator acts by a scalar multiplication of factor $e^{-2\pi i/k}$. This representation is especially useful to build intuition, but its clear drawback is that the dimension of the vector space grows in k, as we expect from Jordan's Theorem 1.2. Our construction replaces this high-dimensional linear representation by actions on low-dimensional pieces, which must be carefully assembled together.

The first key piece in the construction of the manifolds M_k^4 is the so-called Kodaira fiber I_k. Consider a torus $T^2 = S^1 \times S^1$ and pinch one of the two circle factors at k points. Topologically, the result is a Kodaira fiber of type I_k, namely a cyclic chain of k copies of S^2.

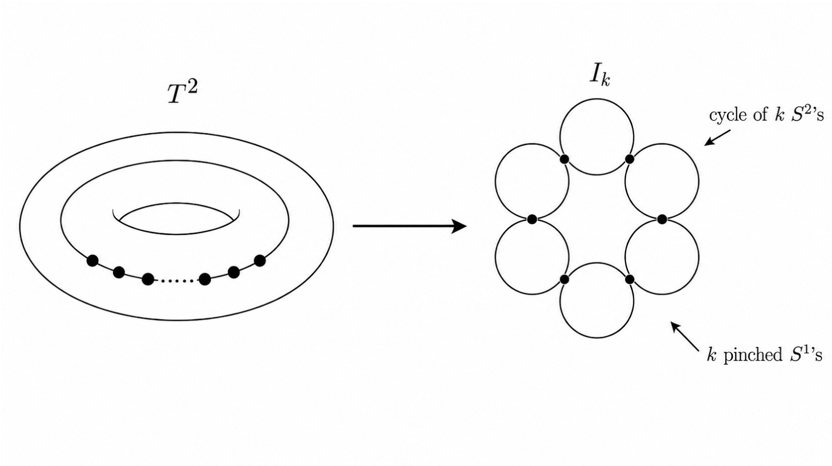

Figure 1: Kodaira fiber I_k.

The group $H_3(\mathbb{Z}/k\mathbb{Z})$ acts effectively by homeomorphisms on the I_k fiber in a way analogous to the linear representation in (12):

i) The generator γ_1 cyclically permutes the k spherical components;

ii) The generator γ_2 rotates the spherical components asynchronously around the axes joining the pinching points.

Similarly as before, the commutator γ_3 simply rotates all the spheres synchronously around the axes joining the pinching points by angle $-2\pi/k$.

The second key piece for the construction is the Heisenberg nilmanifold of degree k, here denoted by $\mathrm{Nil}(k)$, i.e., the degree-k principal S^1-bundle over T^2. One can alternatively view $\mathrm{Nil}(k)$ as a T^2-bundle over S^1 with monodromy

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) . \tag{15}$$

The nilmanifold $\mathrm{Nil}(k)$ admits an effective action of $H_3(\mathbb{Z}/k\mathbb{Z})$ by diffeomorphisms lifting the natural action of

$$H_3(\mathbb{Z}/k\mathbb{Z})/[H_3(\mathbb{Z}/k\mathbb{Z}),H_3(\mathbb{Z}/k\mathbb{Z})] \cong \mathbb{Z}/k\mathbb{Z} \times \mathbb{Z}/k\mathbb{Z} \tag{16}$$

by rotations on the base T^2. We refer the reader to [7, Section 2.2] for the explicit form of the action, and note that its existence has been known for quite some time, see for instance [36]. It plays a key role in the construction of the counterexamples to Ghys' conjecture in [13] as well.

The pieces above are assembled as follows. First, build a compact 4-manifold with boundary $\overline{M}_k^4$ which admits a *singular fibration*

$$T^2 \longrightarrow \overline{M}_k^4 \xrightarrow{\pi_k} D^2 \tag{17}$$

with a unique *singular fiber* homeomorphic to the Kodaira fiber I_k.

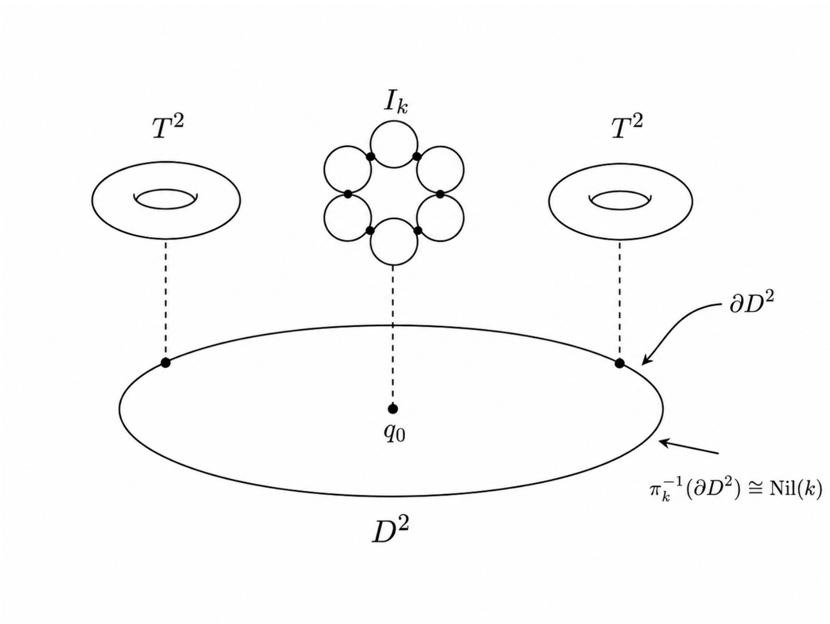

Figure 2: Singular T^2-fibration over the disk.

We avoid discussing the precise meaning of the terminology *singular fibration* in this context and just observe that the projection map π_k is smooth, and it is a submersion away from what we called the singular fiber. In particular, the map induces a torus bundle structure over the boundary circle of D^2 where we have

$$\pi_k^{-1}(\partial D^2) \cong \mathrm{Nil}(k). \tag{18}$$

One can also show that $\overline{M}^4_k$ deformation retracts onto the singular fiber. Thus it has fundamental group $\mathbb{Z}$.

The manifold M^4_k in Theorem 3.2 is obtained by gluing two copies of $\overline{M}^4_k$ along their common boundary with a suitable gluing map. The effect of this gluing is to kill the fundamental group of the two components, in analogy with the standard decomposition of the 3-sphere as union of two solid tori

$$S^3 \cong \left(D^2 \times S^1\right) \cup \left(S^1 \times D^2\right) ,$$

thus yielding a closed simply connected 4-manifold M^4_k. The manifold M^4_k can alternatively be viewed as a singular circle bundle

$$S^1 \longrightarrow M^4_k \longrightarrow S^3 \tag{19}$$

with $2k$ singular fibers corresponding to pinched S^1's. With respect to the Hopf fibration $S^3 \to S^2$, the singular fibers in (19) correspond to k points in the S^1-fiber of the Hopf fibration projecting to the north pole in S^2 and to k points in the S^1-fiber of the Hopf fibration projecting to the south pole.

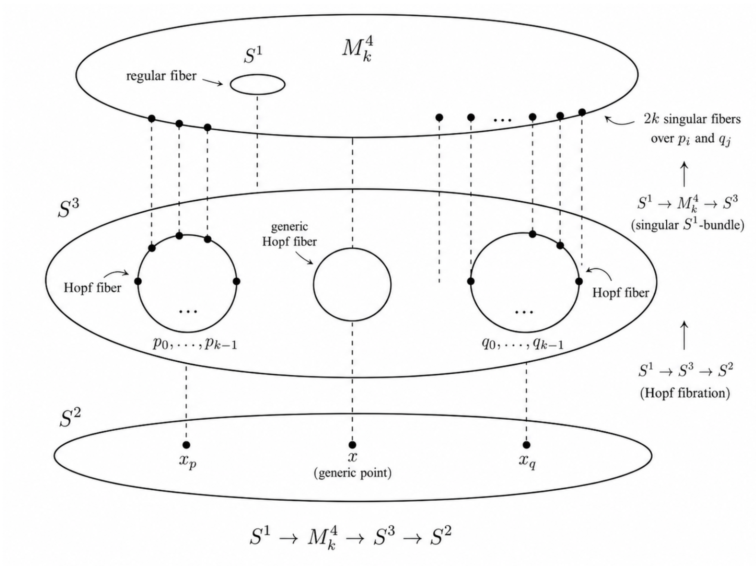

Figure 3: Singular S^1-fibration over S^3.

Eventually one can view M^4_k as a singular Nil(k)-bundle over a closed interval I, with two singular fibers corresponding to the endpoints of the interval, each homeomorphic to a Kodaira fiber I_k.

Although this is not evident at all from any of the descriptions discussed so far, it follows from [12, Theorem 1.6] that M^4_k is diffeomorphic to a connected sum of $(k-1)$ copies of $S^2 \times S^2$.

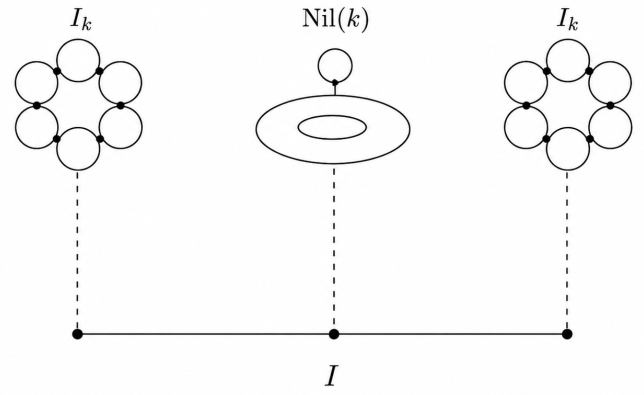

Figure 4: Singular Nil(k)-fibration over the interval.

The manifold M_k^4 admits an effective action of $H_3(\mathbb{Z}/k\mathbb{Z})$ by diffeomorphisms. In the last description as singular Nil(k)-bundle over the interval, the action leaves the fibers invariant and it restricts to the previously discussed actions on the Nil(k) and I_k fibers. The center $[H_3(\mathbb{Z}/k\mathbb{Z}), H_3(\mathbb{Z}/k\mathbb{Z})] \cong \mathbb{Z}/k\mathbb{Z}$ of $H_3(\mathbb{Z}/k\mathbb{Z})$ leaves the fibers of the singular fibration $S^1 \to M_k^4 \to S^3$ invariant. One can also note that M_k^4 admits an action of S^1 by diffeomorphisms, which is free away from the singular fibers of $S^1 \to M_k^4 \to S^3$. Of course the existence of a smooth action of $H_3(\mathbb{Z}/k\mathbb{Z})$ is not quite sufficient to finish the proof of Theorem 3.2. We still need to endow each M_k^4 with a metric with $\mathrm{Ric} > 0$ which is invariant under the action of $H_3(\mathbb{Z}/k\mathbb{Z})$.

For the construction of the invariant (with respect to the above-described $H_3(\mathbb{Z}/k\mathbb{Z})$ action) metric g_k with $\mathrm{Ric} > 0$ on M_k^4 it is convenient to view M_k^4 as a singular S^1-bundle over S^3. Near the singular fibers, the metric is modeled by a variant of the so-called Gibbons-Hawking ansatz introduced in [19, 25] to produce complete S^1-invariant Ricci-flat metrics on 4-manifolds. Variants of the Gibbons-Hawking ansatz have played a pivotal role in a number of constructions of Ricci-flat metrics over the past fifty years. We mention here [23, 26, 27, 11] without the aim of being exhaustive. We avoid discussing further details about the metrics g_k in Theorem 3.2 and refer the interested reader to [7] for the precise construction.

4 Some open questions and future directions

We close with a discussion about some natural open questions which arise in the context of the survey.

There is of course the obvious question about the optimality of the dimension in Theorem 3.1:

Problem 4.1. *Let* $4 \leq n < 9$. *Is the family of fundamental groups of closed manifolds* (M^n, g) *with* $\mathrm{Ric} > 0$ *uniformly Jordan (or uniformly virtually abelian)?*

No restriction that would help settle Problem 4.1 in the positive sense is known to the author. On the other hand, it does not seem that the constructions in [7] can be easily modified to yield a negative answer.

The next problem is a variant of Problem 1.3 in the case of Einstein manifolds.

Problem 4.2. *Let* $n \geq 4$ *be an integer. Is the family of fundamental groups (or even isometry groups) of closed manifolds* (M^n, g) *with* $\mathrm{Ric} = 0$ *uniformly Jordan? Is the family of fundamental groups (or even isometry groups) of closed manifolds* (M^n, g) *with* $\mathrm{Ric} = (n-1)g$ *uniformly Jordan?*

As before, no restriction is known to the author for fundamental groups of Einstein manifolds with nonnegative Einstein constant, besides those available more broadly for manifolds with nonnegative Ricci curvature. On the other hand, perturbing the metrics in Theorem 3.2 so that they become Einstein does not seem easy, although they are locally modeled through the Gibbons-Hawking ansatz.

The last problem we would like to put forward is partly motivated by a result due to Csikós, Pyber, and Szabó, see [14, Theorem 1.4]. They proved that every finite subgroup of the homeomorphism group of a topological manifold with finitely generated homology (with integer coefficients) has a nilpotent subgroup whose index only depends on the dimension and on the integral homology of the manifold.

Problem 4.3. *Let* M^n *be a compact simply connected manifold. Is* $\mathrm{Homeo}(M)$ *Jordan? Does the Jordan property hold uniformly with a constant only depending on the dimension of* M *and on the sum of the Betti numbers of* M*, maximized over all coefficient fields?*

If one could fully solve Problem 4.3 in the positive sense, an affirmative resolution of the Fukaya-Yamaguchi conjecture (Conjecture 1.4) would follow immediately from Gromov's Betti numbers estimate [22].

5 Acknowledgments

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Author's address:
University of Vienna, Faculty of Mathematics, Oskar-Morgenstern-Platz 1, 1090 Wien, AUSTRIA
email daniele.semola@univie.ac.at