DOUBLE-ARC REARRANGEMENT AND THE SYMMETRIC ISOPERIMETRIC PROBLEM IN GAUSS SPACE

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Abstract. We provide a new rearrangement in Gauss space transforming symmetric sets while decreasing their Gaussian perimeter. We use this rearrangement to analyze the minimizers of the Gaussian perimeter among symmetric sets. The results are stated for the Gaussian measure but can be generalized to any radial measure which is uniformly log-concave.

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Contents

1. Introduction

The isoperimetric inequality in Gauss space asserts that among all sets with a given Gaussian measure the half-space has the smallest Gaussian perimeter. Since the half-space is not symmetric, a natural question arises: "What is the object minimizing the Gaussian perimeter among symmetric sets?" The question was initially raised in [5] for *coordinate-wise symmetric* sets (also called *unconditional* sets), i.e., invariant by reflections with respect to the coordinate hyperplanes. We shall use the term *n-symmetric* for such sets.

It has been conjectured for the larger class of *centrally symmetric* sets (see [18, Conjecture 1.3]) that the answer is a cylinder $B_r^k \times \mathbb{R}^{n-k}$, or its complement, for some k depending on the volume and on the dimension. Here B_r^k denotes the k-dimensional ball centered at the origin with radius r. In particular, when the volume is one half the conjecture states that the minimizer is the n-dimensional ball B_r^n, while when the volume is close to one the minimizer is the symmetric slab $(-r, r) \times \mathbb{R}^{n-1}$. In dimension one the solution of the conjecture is elementary (Proposition 7). The part of the conjecture asserting that the symmetric slab is the minimizer for volumes close to one has been proved in [4]. In [19], adapting ideas of Colding and Minicozzi from the study of the mean curvature flow, it has been proved that if a minimizer has the form $\Omega \times \mathbb{R}$, then either Ω or its complement is convex.

For the analogous problem on the hypersphere, the corresponding classification has been established by Viana [26] (see also Appendix B). The conjecture resembled also that for the

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relative isoperimetric problem in the hypercube; this one, however, was recently shown to be false in high dimension by Milman [22].

The difficulties in proving the entire conjecture are due to the fact that the approaches developed for the minimization of the Gaussian perimeter cannot be applied. For instance, Ehrhard's rearrangement, which is commonly used in Gauss space, fails to preserve the symmetry constraint. This calls for a new kind of rearrangement, one that decreases the Gaussian perimeter while preserving the symmetry constraint. A possible idea would be to rearrange an n-symmetric set by substituting its one-dimensional sections with the minimal one-dimensional configuration. However, there is an obstruction: the topology of the optimal symmetric set in one dimension depends discontinuously on its mass, switching from two symmetric tails to a centered interval as the Gaussian measure crosses $1/2$. Consequently, applying this rearrangement to a set whose slice measures cross this threshold induces a topological discontinuity, turning the set "inside out" at the critical level. Introducing these additional boundaries orthogonal to the slicing direction strictly increases the total Gaussian perimeter (see Example 1).

We provide a different kind of rearrangement, by adapting a rearrangement introduced by Bonnesen [7] in the Euclidean setting for convex sets in the plane (see also [13, 15]).

To precisely describe this rearrangement we first recall some notation. Given a Borel set $E \subset \mathbb{R}^n$, $\gamma(E)$ denotes its n-dimensional *Gaussian measure*, i.e.,

$$\gamma(E) \coloneqq \frac{1}{(2\pi)^{\frac{n}{2}}} \int_E e^{-\frac{|x|^2}{2}} dx,$$

while $\mathcal{H}^k_\gamma(E)$, $1 \leq k \leq n$, denotes its k-dimensional Hausdorff measure with Gaussian weight, i.e.,

$$\mathcal{H}^k_\gamma(E) \coloneqq \frac{1}{(2\pi)^{\frac{k}{2}}} \int_E e^{-\frac{|x|^2}{2}} d\mathcal{H}^k(x).$$

$P_\gamma(E)$ denotes the *Gaussian perimeter* of E, i.e.,

$$P_\gamma(E) \coloneqq \sqrt{2\pi} \sup \left\{ \int_E (\operatorname{div} \varphi - \varphi \cdot (x,y))\, d\gamma(x,y) : \varphi \in C^1_c(\mathbb{R}^n;\mathbb{R}^n) \text{ and } |\varphi| \leq 1 \right\}.$$

If E is a set of finite Gaussian perimeter, one can define the reduced boundary $\partial^* E$ of E, which is such that $\partial^* E \subset \partial E$ and $P_\gamma(E) = \mathcal{H}^{n-1}_\gamma(\partial^* E)$ (see [1] for more details). If E is an open set with Lipschitz boundary, we also have $P_\gamma(E) = \mathcal{H}^{n-1}_\gamma(\partial E)$.

For points in $\mathbb{R}^n$, $n \geq 2$, we use two notations: the usual $(x_1, \ldots, x_n)$ for a generic point in $\mathbb{R}^n$, but also (x, y), with $x = (x_1, x_2) \in \mathbb{R}^2$ and $y = (y_1, \ldots, y_{n-2}) \in \mathbb{R}^{n-2}$, in order to focus on the first two components. The latter will be our main notation. We denote by $e_1, \ldots, e_n$ the vectors of the canonical basis of $\mathbb{R}^n$, and $|\cdot|$ stands for the norm in $\mathbb{R}, \mathbb{R}^{n-2}$, or $\mathbb{R}^n$, depending on the context. For every $r > 0$, we set $\partial B(r) \coloneqq \{x \in \mathbb{R}^2_0 : |x| = r\}$, where $\mathbb{R}^2_0 \coloneqq \mathbb{R}^2 \setminus \{(0,0)\}$.

Given a Borel set E, for every $(r, y) \in (0, \infty) \times \mathbb{R}^{n-2}$ we define the circular slice $E_{(r,y)}$ as the subset of $\mathbb{R}^2_0$ given by

$$E_{(r,y)} \coloneqq \{x \in \mathbb{R}^2_0 : x \in \partial B(r) \text{ and } (x,y) \in E\},$$

and define the *circular (Gaussian) distribution* of E as the function

$$\mu(r,y) \coloneqq \mathcal{H}^1_\gamma(E_{(r,y)}), \qquad \text{for every } (r,y) \in (0,\infty) \times \mathbb{R}^{n-2}, \tag{1.1}$$

i.e., $\mu(r, y)$ is the 1-dimensional Gaussian length of the slice $E_{(r,y)}$. Additionally, we set

$$\xi_\mu(r,y) \coloneqq \frac{\sqrt{2\pi}}{e^{-\frac{r^2}{2}}} \frac{\mu(r,y)}{r}, \qquad \text{and} \qquad \beta_\mu(r,y) \coloneqq \frac{\xi_\mu(r,y)}{4}, \qquad \text{for every } (r,y) \in (0,\infty) \times \mathbb{R}^{n-2}. \tag{1.2}$$

Thus, $\xi_\mu(r,y) \in [0,2\pi]$ represents the size of an angle corresponding to a connected arc of $\partial B(r)$ whose 1-dimensional Gaussian length is $\mu(r,y)$. The *double-arc rearrangement* of E, denoted F_μ, is defined as

$$F_\mu := \left\{(x,y) \in \mathbb{R}^2_0 \times \mathbb{R}^{n-2} : \arccos(|\hat{x}\cdot e_1|) < \beta_\mu(|x|,y)\right\}, \tag{1.3}$$

where $\hat{x} := x/|x|$ is the radial versor of x. Refer Figures 1 and 2 for examples with $n=2$ and $n=3$.

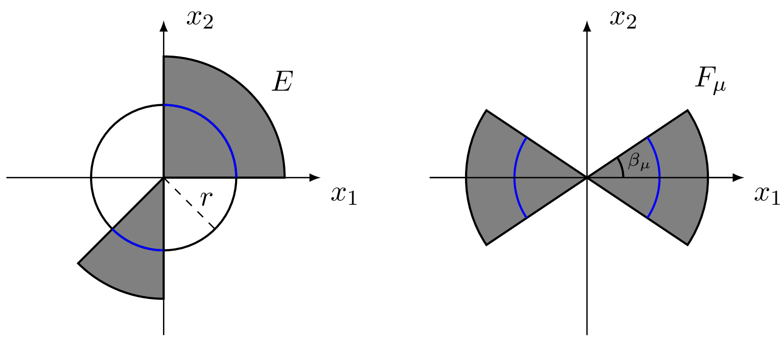

FIGURE 1. The double-arc rearrangement, with $n=2$.

We say that a Borel set $E \subset \mathbb{R}^n$ is μ-distributed if (1.1) holds. By construction, F_μ is μ-distributed. As we said, we are interested in sets that are symmetric with respect to all the coordinate axes. More precisely, we say that E is n-*symmetric* if

$$(x,y) \in E \quad \Longleftrightarrow \quad A_j(x,y) \in E \quad \text{ for every } j = 1,\dots,n,$$

where A_j is the $n\times n$ diagonal matrix in which the j-th element of the diagonal is -1 and all the other ones are 1, i.e., A_j is the reflection across the coordinate hyperplane $\{x \in \mathbb{R}^n : x_j = 0\}$.

Our first result is that *the double-arc rearrangement decreases the Gaussian perimeter of n-symmetric sets.* However, we are also interested in what additional properties the initial set E has when $P_\gamma(E) = P_\gamma(F_\mu)$. We stress the fact that the double-arc rearrangement does not necessarily decrease the Gaussian perimeter if the set is not n-symmetric.

In order to state our result in full, we need some other notation. For every $x \in \mathbb{R}^2_0$, we define the tangential versor $x_\parallel$ as

$$x_\parallel = \left(-\frac{x_2}{|x|}, \frac{x_1}{|x|}\right).$$

For every $(x,y) \in \partial^* E$, we denote by $\nu^E(x,y)$ the measure-theoretic exterior unit normal to E at (x,y), characterized by

$$D\chi_E = -\nu^E\, \mathcal{H}^{n-1} \llcorner \partial^* E.$$

We decompose $\nu^E(x,y)$ as $\nu^E(x,y) = (\nu^E_x(x,y), \nu^E_y(x,y))$, with $\nu^E_x(x,y) \in \mathbb{R}^2$ and $\nu^E_y(x,y) \in \mathbb{R}^{n-2}$. If $x \neq 0$, we further decompose $\nu^E_x(x,y) \in \mathbb{R}^2$ as

$$\nu^E_x(x,y) = \nu^E_{x\perp}(x,y) + \nu^E_{x\parallel}(x,y),$$

where

$$\nu^E_{x\perp}(x,y) := (\nu^E_x(x,y)\cdot\hat{x})\hat{x}, \quad \text{ and } \quad \nu^E_{x\parallel}(x,y) := \nu^E_x(x,y) - \nu^E_{x\perp}(x,y).$$

By symmetry, it follows that for every $(r,y) \in (0,\infty)\times\mathbb{R}^{n-2}$ the functions

$$x \mapsto \hat{x}\cdot\nu^{F_\mu}_x(x,y) \quad \text{ and } \quad x \mapsto |\nu^{F_\mu}_{x\parallel}(x,y)|$$

are constant in $(\partial^* F_\mu)_{(r,y)}$.

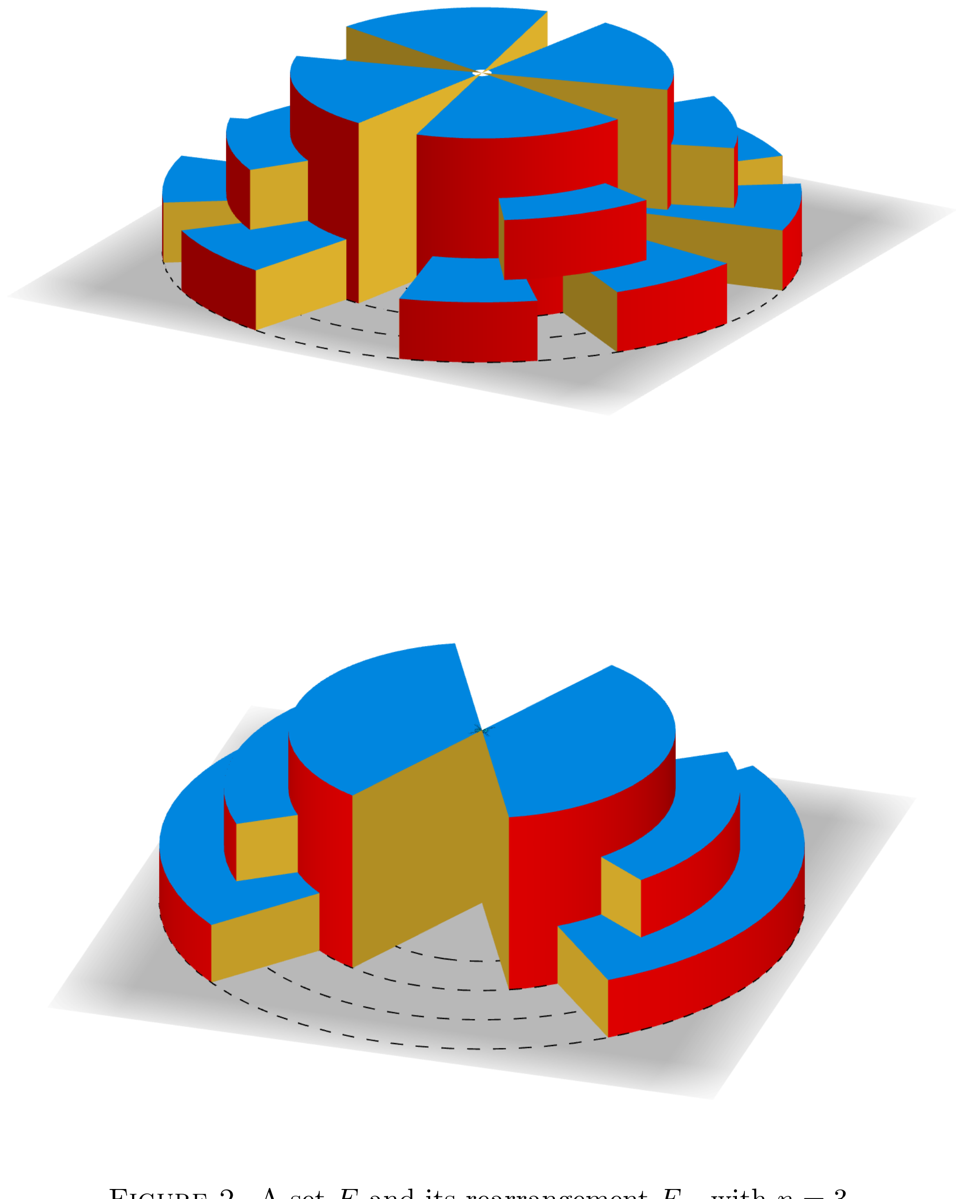

Figure 2. A set E and its rearrangement F_μ, with $n = 3$.

Let $r > 0$, $p \in \mathbb{S}^1$ and $\alpha \in [0, \pi]$ be fixed, where $\mathbb{S}^1$ denotes the unit circle in $\mathbb{R}^2$. The *arc* of centre rp and radius α is the subset of $\partial B(r)$ given by

$$\mathbf{A}_\alpha(rp) := \{x \in \partial B(r) : \arccos(\hat{x} \cdot p) < \alpha\}.$$

Accordingly, if $\alpha \in (0, \pi)$, its *endpoints* are

$$\mathbf{S}_\alpha(rp) := \{x \in \partial B(r) : \arccos(\hat{x} \cdot p) = \alpha\}.$$

For what follows, it is useful to define the diffeomorphism $\Phi : (0,\infty)\times\mathbb{R}^{n-2}\times\mathbb{S}^1 \to \mathbb{R}^2_0\times\mathbb{R}^{n-2}$ given by

$$\Phi(r,y,\omega) := (r\omega, y), \qquad \text{for every } (r,y,\omega)\in(0,\infty)\times\mathbb{R}^{n-2}\times\mathbb{S}^1. \tag{1.4}$$

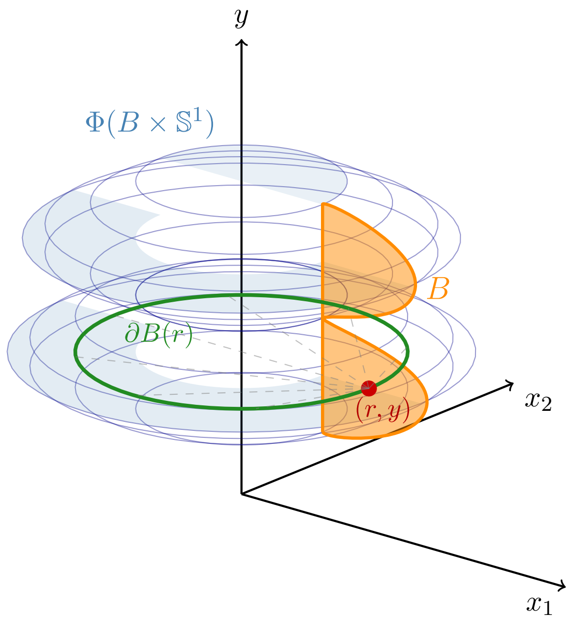

FIGURE 3. The region $\Phi(B\times\mathbb{S}^1)$ for $n=3$. The Borel set $B\subset(0,\infty)\times\mathbb{R}$ (orange) is displayed in the (x_1,y) half-plane. Under the diffeomorphism $\Phi(r,y,\omega) = (r\omega,y)$, each point $(r,y)\in B$ generates a circle $\partial B(r)$ at height y, and the union of these circles gives the region $\Phi(B\times\mathbb{S}^1)$ (in blue).

The first result we prove is that the double-arc Gaussian rearrangement does not increase the Gaussian perimeter. Here and in the following, for $1\le k\le n$ and $A,B\subset\mathbb{R}^n$, we write $A =_{\mathcal{H}^k} B$ whenever $\mathcal{H}^k(A\Delta B)=0$.

Theorem 1 (Perimeter inequality)**.** *Let $E\subset\mathbb{R}^n$ be an n-symmetric set of finite Gaussian perimeter, and let μ be the distribution function of E. Then, $\mu\in BV_{\mathrm{loc}}((0,\infty)\times\mathbb{R}^{n-2})$. Moreover, F_μ is a set of finite Gaussian perimeter, and*

$$P_\gamma(F_\mu;\Phi(B\times\mathbb{S}^1)) \le P_\gamma(E;\Phi(B\times\mathbb{S}^1)), \tag{1.5}$$

for every Borel set $B\subset(0,\infty)\times\mathbb{R}^{n-2}$.

Moreover, if equality in (1.5) holds, then for $\mathcal{H}^{n-1}$-a.e. $(r,y)\in B$ the following two conditions are satisfied:

(i) $E_{(r,y)} =_{\mathcal{H}^1} \mathbf{A}_{\beta_\mu(r,y)}(rp(r,y)) \cup \mathbf{A}_{\beta_\mu(r,y)}(-rp(r,y))$ for some $p(r,y)\in\{(1,0),(0,1)\}$;

(ii) the functions

$$x\mapsto \hat{x}\cdot\nu^E_x(x,y), \quad x\mapsto |\nu^E_{x\|}(x,y)|, \quad x\mapsto \nu^E_y(x,y),$$

are constant in $(\partial^* E)_{(r,y)}$.

Condition (i) means that if the rearrangement does not strictly decrease the Gaussian perimeter, then the initial set E already has circular slices constituted by two symmetric arcs. This is intuitively clear: by looking at Figure 2, if a circular slice consists of more than two arcs, after the rearrangement the yellow part of the boundary is strictly reduced.

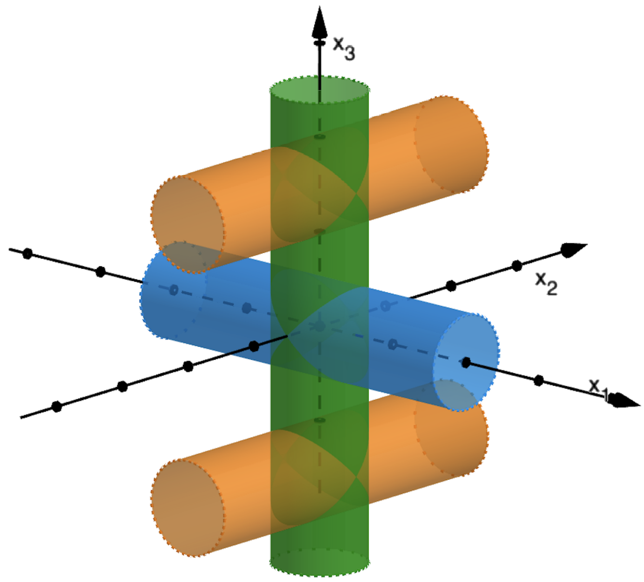

FIGURE 4. The set E and its rearrangement F_μ have the same perimeter, but they are not congruent

It is important to note that even if $P_\gamma(E) = P_\gamma(F_\mu)$, the sets E and F_μ are not necessarily congruent. Consider the following example: let E be the union of the four cylinders (see Figure 4)

$$\{x \in \mathbb{R}^3 \colon x_2^2 + x_3^2 \leq 1\}, \qquad \{x \in \mathbb{R}^3 \colon x_1^2 + (x_3 - 3)^2 \leq 1\},$$
$$\{x \in \mathbb{R}^3 \colon x_1^2 + (x_3 + 3)^2 \leq 1\}, \qquad \{x \in \mathbb{R}^3 \colon x_1^2 + x_2^2 \leq 1\}.$$

The rearrangement reorients the two orange cylinders making them parallel to the blue one. The point is that the set

$$\Omega_\mu := \{(r, y) \in (0, \infty) \times \mathbb{R}^{n-2} : 0 < \beta_\mu(r, y) < \pi/2\} \tag{1.6}$$

is not connected (see Figure 5), and this allows us to rotate each cylinder independently, without creating an additional perimeter.

We now turn our attention to minimizers of the Gaussian isoperimetric problem for n-symmetric sets, and study their regularity and the behavior under the double-arc Gaussian rearrangement. We recall that, for $t \in [0, 1]$, the set $E^{(t)}$ of points of density t of E is given by

$$E^{(t)} := \left\{ x \in \mathbb{R}^n : u_E(x) := \lim_{r \to 0^+} \frac{|E \cap B_r(x)|}{|B_r(x)|} = t \right\}.$$

In what follows, we identify E with its precise representative $E^{(1)}$. We first prove the optimal regularity.

Theorem 2 (Regularity)**.** *Let $E \subset \mathbb{R}^n$ be a minimizer of the Gaussian isoperimetric problem among n-symmetric sets with prescribed Gaussian measure. Then E is open, and ∂E is a real-analytic hypersurface outside the origin, i.e., $\Sigma_E := \partial E \setminus \partial^* E \subset \{0\}$.*

When E is a minimizer, the set Ω_μ defined in (1.6) is connected and we have the rigidity in the double-arc symmetrization.

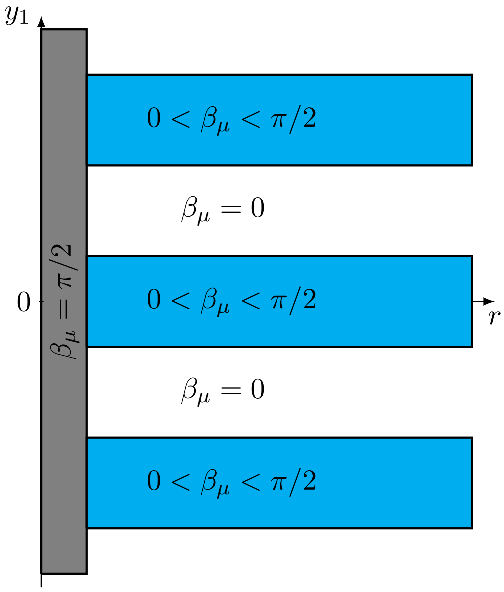

FIGURE 5. Rigidity fails. In this case the set $\{\beta_\mu = 0\} \cup \{\beta_\mu = \pi/2\}$ disconnects the set $\{0 < \beta_\mu < \pi/2\}$.

Theorem 3 (Rigidity). *Let $E \subset \mathbb{R}^n$ be a minimizer of the Gaussian isoperimetric problem among n-symmetric sets with prescribed Gaussian measure, identified with its precise representative $E^{(1)}$, let μ be the circular distribution of E defined in (1.1), and set $F \coloneqq F_\mu^{(1)}$. Then the set Ω_μ defined in (1.6) is open and connected (possibly empty), and*

$$E = F \qquad \text{or} \qquad E = R(F),$$

where R is the clockwise rotation of $\pi/2$ in the plane (x_1, x_2). Moreover, exactly one of the following alternatives holds:

(i) E is rotationally invariant in the (x_1, x_2)-plane, and $\Omega_\mu = \emptyset$;

(ii) $\Omega_\mu \neq \emptyset$, $\Pi\big(\partial E \cap (\mathbb{R}^2_+ \times \mathbb{R}^{n-2})\big) = \Omega_\mu$, and $\nu^E_{x\|}(x, y) \neq 0$ for every $(x, y) \in \partial E \cap (\mathbb{R}^2_+ \times \mathbb{R}^{n-2})$.

Here $\Pi(x, y) \coloneqq (|x|, y)$ and $\mathbb{R}^2_+ \coloneqq \{x \in \mathbb{R}^2 \colon x_1 > 0,\ x_2 > 0\}$.

We believe that the equality $E = F_\mu$ (up to a rotation) should hold for general n-symmetric sets of finite Gaussian perimeter satisfying $P_\gamma(E) = P_\gamma(F_\mu)$, provided that Ω_μ is "connected". The assumption that E is a minimizer guarantees that Ω_μ is an open set and provides the regularity needed for a topological argument. In the general case the connectedness assumption on Ω_μ should be replaced by a suitable notion of *essential connectedness*. The appropriate framework is developed in [8, 9], where essential connectedness is used to establish rigidity for Steiner and Gaussian symmetrization. Adapting their approach to the double-arc rearrangement would require formulating essential connectedness for the set Ω_μ in $(0, \infty) \times \mathbb{R}^{n-2}$ and replacing the topological covering argument with a measure-theoretic one. We do not pursue this generalization here.

An obstruction prevents the generalization of the double-arc rearrangement to higher dimensions, where one might naturally attempt to foliate $\mathbb{R}^n$ ($n \geq 3$) with spheres instead of circles. Indeed, see Proposition 8, the solution to the isoperimetric problem on the sphere $\mathbb{S}^2$ among symmetric sets undergoes a topological phase transition depending on the mass: it switches from the union of two antipodal caps to an equatorial belt as the surface area crosses half the total measure. Consequently, rearranging a set by substituting its spherical slices with the optimal configuration could introduce topological discontinuities, increasing the global Gaussian perimeter rather than decreasing it (see Example 2).

The paper is organized as follows. Section 2 collects preliminary results including the coarea formula and properties of the distribution function. Section 3 proves the perimeter inequality (Theorem 1). Section 4 establishes the regularity theory (Theorem 2). Section 5 proves the rigidity result (Theorem 3). Appendices treat the one-dimensional problem (A), the spherical obstruction (B), and the planar conjecture (C).

2. Preliminary results

To prove Theorem 1 we need some technical results that are stated below. The first two are results of geometric measure theory.

Proposition 1 (Coarea Formula)**.** *Let E be a set of finite Gaussian perimeter in $\mathbb{R}^n$, and let $g : \mathbb{R}^n \to [0, \infty]$ be a Borel function. Then,*

$$\begin{aligned}&\int_{\partial^* E} g(x,y)|\nu^E_{x\|}(x,y)|d\mathcal{H}^{n-1}_\gamma(x,y)\\&= \int_{(0,+\infty)\times\mathbb{R}^{n-2}} \frac{e^{-\frac{r^2+|y|^2}{2}}}{(2\pi)^{(n-1)/2}}\left(\int_{(\partial^* E)_{(r,y)}} g(x,y)\, d\mathcal{H}^0(x)\right) drdy.\end{aligned}$$

Proof. We have

$$\begin{aligned}&\int_{\partial^* E} g(x,y)|\nu^E_{x\|}(x,y)|d\mathcal{H}^{n-1}_\gamma(x,y)\\&= \int_{\partial^* E} g(x,y)\frac{e^{-\frac{|x|^2+|y|^2}{2}}}{(2\pi)^{(n-1)/2}}|\nu^E_{x\|}(x,y)|d\mathcal{H}^{n-1}(x,y)\\&= \int_{(0,+\infty)\times\mathbb{R}^{n-2}} \frac{e^{-\frac{r^2+|y|^2}{2}}}{(2\pi)^{(n-1)/2}}\left(\int_{(\partial^* E)_{(r,y)}} g(x,y)\, d\mathcal{H}^0(x)\right) drdy,\end{aligned}$$

where the last equality follows applying coarea formula [1, Remark 2.94] with $N = n-1$, $M = n$, $k = n-1$, and $f(x,y) = (|x|, y)$. $\square$

The following result will also be useful, see [10, Theorem 6.2].

Proposition 2 (Vol'pert)**.** *Let $E \subset \mathbb{R}^n$ be an n-symmetric set of finite Gaussian perimeter and let μ be the distribution function of E. Then, for $\mathcal{H}^{n-1}$-a.e. $(r,y) \in (0,\infty)\times\mathbb{R}^{n-2}$ the following properties hold:*

(i) $E_{(r,y)}$ is a set of finite perimeter in $\partial B(r)$ and $\partial^*\left(E_{(r,y)}\right) = (\partial^* E)_{(r,y)}$;
(ii) for every $\omega \in \mathbb{S}^1$ such that $r\omega \in (\partial^* E)_{(r,y)} \cap \partial^*(E_{(r,y)})$,

$$|\nu^E_{x\|}(r\omega,y)| > 0 \quad \textit{and} \quad \nu^{E_{(r,y)}}(r\omega) = \frac{\nu^E_{x\|}(r\omega,y)}{|\nu^E_{x\|}(r\omega,y)|}.$$

Here $\nu^{E_{(r,y)}}$ denotes the measure-theoretic exterior unit normal to the slice $E_{(r,y)}$ within $\partial B(r)$. In particular, there exists a Borel set $G_E \subset \{0 < \beta_\mu < \pi/2\}$ with $\mathcal{H}^{n-1}(\{0 < \beta_\mu < \pi/2\}\setminus G_E) = 0$ such that (i)-(ii) are fulfilled for all $(r,y) \in G_E$.

Remark 1. We observe that, although [10, Theorem 6.2] was formulated for sets of finite Euclidean perimeter, it still holds for sets of *locally* finite Euclidean perimeter. Therefore the result can be applied to sets of finite Gaussian perimeter.

We also need some technical results about the radial and tangential components of functions and measures. If $v:\mathbb{R}^n\to\mathbb{R}^n$, we decompose v as $v(x,y)=(v_x(x,y),v_y(x,y))$ with $v_x(x,y)\in\mathbb{R}^2$ and $v_y(x,y)\in\mathbb{R}^{n-2}$. Moreover, for every $(x,y)\in\mathbb{R}^2_0\times\mathbb{R}^{n-2}$, we decompose $v_x(x,y)\in\mathbb{R}^2$ as follows:

$$\begin{aligned} v_x(x,y)&=v_{x\perp}(x,y)+v_{x\|}(x,y),\\ \text{where } v_{x\perp}(x,y)&\coloneqq(v_x(x,y)\cdot\hat{x})\hat{x},\\ \text{and } v_{x\|}(x,y)&\coloneqq v_x(x,y)-v_{x\perp}(x,y).\end{aligned}$$

If $\varphi\in C^1_c(\mathbb{R}^2\times\mathbb{R}^{n-2};\mathbb{R}^n)$, we decompose the divergence of φ as

$$\operatorname{div}\varphi(x,y)=\operatorname{div}_x\varphi_x(x,y)+\operatorname{div}_y\varphi_y(x,y), \tag{2.1}$$

where div_x and div_y denote the divergence with respect to the x and y variable, respectively.

If $\varphi\in C^1_c(\mathbb{R}^2_0\times\mathbb{R}^{n-2};\mathbb{R}^n)$, for every $(x,y)\in\mathbb{R}^2_0\times\mathbb{R}^{n-2}$ we define the parallel divergence (with respect to x) as:

$$\operatorname{div}_{x\|}\varphi_x(x,y)\coloneqq\operatorname{div}_x\varphi_x(x,y)-\big(\nabla_x\varphi_x(x,y)\hat{x}\big)\cdot\hat{x}, \tag{2.2}$$

(note that $\nabla_x\varphi_x(x,y)$ is a 2×2 matrix and $\hat{x}\in\mathbb{R}^2$, so $\nabla_x\varphi_x(x,y)\hat{x}\in\mathbb{R}^2$).

The next lemma is a minor variation of [10, Lemma 3.2].

Lemma 1. *Let $\varphi\in C^1_c(\mathbb{R}^2_0\times\mathbb{R}^{n-2};\mathbb{R}^n)$. Then, for every $(x,y)\in\mathbb{R}^2_0\times\mathbb{R}^{n-2}$*

$$(\operatorname{div}_x\varphi_{x\perp})(x,y)=\big(\nabla_x\varphi_x(x,y)\hat{x}\big)\cdot\hat{x}+\frac{1}{|x|}\left(\varphi_x(x,y)\cdot\hat{x}\right), \tag{2.3}$$

$$(\operatorname{div}_x\varphi_{x\|})(x,y)=(\operatorname{div}_{x\|}\varphi_{x\|})(x,y). \tag{2.4}$$

Remark 2. Let $\varphi\in C^1_c(\mathbb{R}^2_0\times\mathbb{R}^{n-2};\mathbb{R}^n)$. Recalling that $\varphi_x=\varphi_{x\perp}+\varphi_{x\|}$, combining (2.3) and (2.4) we have that for every $(x,y)\in\mathbb{R}^2_0\times\mathbb{R}^{n-2}$

$$(\operatorname{div}_x\varphi_x)(x,y)=\big(\nabla_x\varphi_x(x,y)\hat{x}\big)\cdot\hat{x}+\frac{1}{|x|}\left(\varphi_x(x,y)\cdot\hat{x}\right)+\operatorname{div}_{x\|}\varphi_{x\|}(x,y). \tag{2.5}$$

Proof of Lemma 1. First of all, note that

$$\nabla_x\left(\varphi_x(x,y)\cdot\hat{x}\right)=\left(\nabla_x\varphi_x(x,y)\right)^T\hat{x}+\frac{1}{|x|}\varphi_{x\|}(x,y). \tag{2.6}$$

Indeed,

$$\begin{aligned}\nabla_x\left(\varphi_x(x,y)\cdot\hat{x}\right)&=\left(\nabla_x\varphi_x(x,y)\right)^T\hat{x}+\left(\nabla_x\hat{x}\right)^T\varphi_x(x,y)\\ &=\left(\nabla_x\varphi_x(x,y)\right)^T\hat{x}+\frac{I_2-\hat{x}\otimes\hat{x}}{|x|}\varphi_x(x,y)=\left(\nabla_x\varphi_x(x,y)\right)^T\hat{x}+\frac{1}{|x|}\varphi_{x\|}(x,y),\end{aligned} \tag{2.7}$$

where I_2 represents the identity map in $\mathbb{R}^2$, and $\hat{x}\otimes\hat{x}$ is the usual tensor product of $\hat{x}$ with itself (so that $I_2-\hat{x}\otimes\hat{x}$ is the orthogonal projection on the tangent plane to $\mathbb{S}^1$ at $\hat{x}$).

Thanks to (2.6) and (2.7), we have

$$\begin{aligned}\operatorname{div}_x\varphi_{x\perp}(x,y)&=\operatorname{div}_x\left((\varphi_x(x,y)\cdot\hat{x})\hat{x}\right)\\ &=\nabla_x\left(\varphi_x(x,y)\cdot\hat{x}\right)\cdot\hat{x}+\left(\varphi_x(x,y)\cdot\hat{x}\right)\operatorname{div}_x\hat{x}\\ &=\left[\left(\nabla_x\varphi_x(x,y)\right)^T\hat{x}+\frac{1}{|x|}\varphi_{x\|}(x,y)\right]\cdot\hat{x}+\left(\varphi_x(x,y)\cdot\hat{x}\right)\frac{1}{|x|}\\ &=\big(\nabla_x\varphi_x(x,y)\hat{x}\big)\cdot\hat{x}+\left(\varphi_x(x,y)\cdot\hat{x}\right)\frac{1}{|x|},\end{aligned}$$

which proves (2.3).

By (2.2), it follows that

$$\operatorname{div}_x \varphi_x(x,y) = \operatorname{div}_{x\|} \varphi_x(x,y) + \big(\nabla_x \varphi_x(x,y)\hat{x}\big)\cdot \hat{x}. \tag{2.8}$$

On the other hand, recalling that $\varphi_x(x,y) = \varphi_{x\|}(x,y) + \varphi_{x\perp}(x,y)$ and using (2.3)

$$\begin{aligned}\operatorname{div}_x \varphi_x(x,y) &= \operatorname{div}_x \varphi_{x\|}(x,y) + \operatorname{div}_x \varphi_{x\perp}(x,y)\\ &= \operatorname{div}_x \varphi_{x\|}(x,y) + \big(\nabla_x \varphi_x(x,y)\hat{x}\big)\cdot \hat{x} + \frac{1}{|x|}\left(\varphi_x(x,y)\cdot \hat{x}\right).\end{aligned}$$

Comparing last identity with (2.8) we obtain that for every $\varphi \in C^1_c(\mathbb{R}^2_0 \times \mathbb{R}^{n-2};\mathbb{R}^n)$

$$\begin{aligned}\operatorname{div}_{x\|} \varphi_x(x,y) + \big(\nabla_x \varphi_x(x,y)\hat{x}\big)\cdot \hat{x} = \operatorname{div}_x \varphi_{x\|}(x,y) &+ \big(\nabla_x \varphi_x(x,y)\hat{x}\big)\cdot \hat{x}\\ &+ \frac{1}{|x|}\left(\varphi_x(x,y)\cdot \hat{x}\right),\end{aligned}$$

From which it follows that

$$\operatorname{div}_{x\|} \varphi_x(x,y) = \operatorname{div}_x \varphi_{x\|}(x,y) + \frac{1}{|x|}\left(\varphi_x(x,y)\cdot \hat{x}\right).$$

Applying the last identity to the function $\varphi_{x\|}$ we obtain (2.4). $\square$

If μ is an $\mathbb{R}^n$-valued Radon measure on $\mathbb{R}^2_0 \times \mathbb{R}^{n-2}$, we decompose μ as $\mu = (\mu_x, \mu_y)$, where μ_x and μ_y are $\mathbb{R}^2$-valued and $\mathbb{R}^{n-2}$-valued, respectively. Moreover, we decompose μ_x as $\mu_x = \mu_{x\perp} + \mu_{x\|}$, where $\mu_{x\perp}$ and $\mu_{x\|}$ are the $\mathbb{R}^2$-valued Radon measures on $\mathbb{R}^2_0 \times \mathbb{R}^{n-2}$ such that

$$\begin{aligned}\int_{\mathbb{R}^2_0\times\mathbb{R}^{n-2}} \varphi(x,y)\cdot d\mu_{x\perp}(x,y) &= \int_{\mathbb{R}^2_0\times\mathbb{R}^{n-2}} \varphi_{x\perp}(x,y)\cdot d\mu_x(x,y),\\ \int_{\mathbb{R}^2_0\times\mathbb{R}^{n-2}} \varphi(x,y)\cdot d\mu_{x\|}(x,y) &= \int_{\mathbb{R}^2_0\times\mathbb{R}^{n-2}} \varphi_{x\|}(x,y)\cdot d\mu_x(x,y),\end{aligned}$$

for every $\varphi \in C_c(\mathbb{R}^2_0 \times \mathbb{R}^{n-2};\mathbb{R}^n)$.

Note that $\mu_{x\perp}$ and $\mu_{x\|}$ are well defined by Riesz Theorem (see, for instance, [1, Theorem 1.54]). In the special case $\mu = Df$, with $f \in BV_{\mathrm{loc}}(\mathbb{R}^2_0 \times \mathbb{R}^{n-2})$, we shorten the notation writing $D_{x\|}f$ and $D_{x\perp}f$ in place of $(Df)_{x\|}$ and $(Df)_{x\perp}$, respectively. In particular, if $f = \chi_E$ and $E \subset \mathbb{R}^n$ is a set of finite perimeter, by De Giorgi structure theorem we have

$$\begin{aligned}D_{x\perp}\chi_E &= -\nu^E_{x\perp}\, d\mathcal{H}^{n-1} \llcorner \partial^* E,\\ D_{x\|}\chi_E &= -\nu^E_{x\|}\, d\mathcal{H}^{n-1} \llcorner \partial^* E,\\ D_y\chi_E &= -\nu^E_{y}\, d\mathcal{H}^{n-1} \llcorner \partial^* E.\end{aligned} \tag{2.9}$$

The next lemma is a variant of [10, Lemma 3.4].

Lemma 2. *Let $f \in BV_{\mathrm{loc}}(\mathbb{R}^2_0 \times \mathbb{R}^{n-2})$. Then, for every $\varphi \in C^1_c(\mathbb{R}^2_0 \times \mathbb{R}^{n-2};\mathbb{R}^n)$*

$$\int_{\mathbb{R}^2_0\times\mathbb{R}^{n-2}} \varphi(x,y)\cdot dD_{x\|}f = -\int_{\mathbb{R}^2_0\times\mathbb{R}^{n-2}} f(x,y)\, \operatorname{div}_{x\|} \varphi_{x\|}(x,y)\, dx\, dy, \tag{2.10}$$

and

$$\begin{aligned}\int_{\mathbb{R}^2_0\times\mathbb{R}^{n-2}} \varphi(x,y)\cdot dD_{x\perp}f = &-\int_{\mathbb{R}^2_0\times\mathbb{R}^{n-2}} f(x,y)\big(\nabla_x\varphi_x(x,y)\hat{x}\big)\cdot\hat{x}\, dx\\ &-\int_{\mathbb{R}^2_0\times\mathbb{R}^{n-2}} f(x,y)\frac{1}{|x|}\left(\varphi_x(x,y)\cdot\hat{x}\right)\, dx.\end{aligned} \tag{2.11}$$

Proof. Let $\varphi \in C_c^1(\mathbb{R}_0^2 \times \mathbb{R}^{n-2}; \mathbb{R}^n)$. By definition of $D_{x\|}f$ and thanks to (2.4) we have

$$\begin{aligned}\int_{\mathbb{R}_0^2 \times \mathbb{R}^{n-2}} \varphi(x,y) \cdot dD_{x\|} f(x,y) &= \int_{\mathbb{R}_0^2 \times \mathbb{R}^{n-2}} \varphi_{x\|}(x,y) \cdot dD_x f(x,y) \\ &= -\int_{\mathbb{R}_0^2 \times \mathbb{R}^{n-2}} \operatorname{div}_x \varphi_{x\|}(x,y) f(x,y)\, dx\, dy \\ &= -\int_{\mathbb{R}_0^2 \times \mathbb{R}^{n-2}} \operatorname{div}_{x\|} \varphi_{x\|}(x,y) f(x,y)\, dx\, dy,\end{aligned}$$

and this shows (2.10).

Similarly, by definition of $D_{x\perp}f$

$$\begin{aligned}\int_{\mathbb{R}_0^2 \times \mathbb{R}^{n-2}} \varphi(x,y) \cdot dD_{x\perp} f(x,y) &= \int_{\mathbb{R}_0^2 \times \mathbb{R}^{n-2}} \varphi_{x\perp}(x,y) \cdot dD_x f(x,y) \\ &= -\int_{\mathbb{R}_0^2 \times \mathbb{R}^{n-2}} \operatorname{div}_x \varphi_{x\perp}(x,y) f(x,y)\, dx\, dy.\end{aligned}$$

Thanks to (2.3), identity (2.11) follows. □

An immediate consequence of identity (2.10) is the following.

Corollary 1. *Let $f \in BV_{\mathrm{loc}}(\mathbb{R}_0^2 \times \mathbb{R}^{n-2})$ and let $\Omega \subset\subset \mathbb{R}_0^2 \times \mathbb{R}^{n-2}$ be open and bounded. Then,*

$$\left|D_{x\|}f\right|(\Omega) = \sup\left\{\int_{\mathbb{R}_0^2 \times \mathbb{R}^{n-2}} f(x,y)\ \operatorname{div}_{x\|} \varphi_{x\|}(x,y) dx\, dy :\ \varphi \in C_c^1(\Omega; \mathbb{R}^n),\ \|\varphi\|_{L^\infty(\Omega;\mathbb{R}^n)} \le 1\right\}.$$

Lemma 3. *Let $E \subset \mathbb{R}^n$ be an n-symmetric set of finite Gaussian perimeter, let μ be the distribution function of E, and let ξ_μ be defined by (1.2). Then:*

(i) $\mu, \xi_\mu \in BV_{\mathrm{loc}}((0,\infty) \times \mathbb{R}^{n-2})$*;*

(ii) For bounded Borel functions with compact support, the following holds:

$$\int_{(0,\infty)\times\mathbb{R}^{n-2}} \psi(r,y)\, r\, dD_r\xi_\mu(r,y) = \int_{\mathbb{R}_0^2 \times \mathbb{R}^{n-2}} \psi(|x|,y)\, \hat{x} \cdot dD_{x\perp}\chi_E(x,y), \tag{2.12}$$

$$\int_{(0,\infty)\times\mathbb{R}^{n-2}} r\, \phi(r,y)\ \cdot dD_y\xi_\mu(r,y) = \int_{\mathbb{R}_0^2 \times \mathbb{R}^{n-2}} \phi(|x|,y) \cdot dD_y\chi_E(x,y), \tag{2.13}$$

where $\psi : (0,\infty) \times \mathbb{R}^{n-2} \to \mathbb{R}$ and $\phi : (0,\infty) \times \mathbb{R}^{n-2} \to \mathbb{R}^{n-2}$.

(iii) As a consequence:

$$|re^{-\frac{r^2}{2}} e^{-\frac{|y|^2}{2}} D\xi_\mu|(B) \le |e^{-\frac{|x|^2}{2}} e^{-\frac{|y|^2}{2}} D\chi_E|(\Phi(B \times \mathbb{S}^1)), \tag{2.14}$$

for every Borel set $B \subset\subset (0,\infty) \times \mathbb{R}^{n-2}$.

Proof. We divide the proof into several steps.

Step 1: We show that $\mu \in BV_{\mathrm{loc}}((0,\infty) \times \mathbb{R}^{n-2})$.

Let $A \subset\subset (0,\infty) \times \mathbb{R}^{n-2}$ be an open set, and let us show that $\mu \in BV(A)$. Let $r_A, R_A > 0$ be such that

$$r \ge r_A \ \text{ and } \ r^2 + |y|^2 \le R_A^2$$

for every $(r,y) \in A$.

Step 1a: We show that $\mu, \xi_\mu \in L^1(A)$.

We have

$$
\begin{aligned}
1 \geq \gamma(E \cap \Phi(A \times \mathbb{S}^1)) &= \int_{\Phi(A\times\mathbb{S}^1)} \chi_E(x,y)\, d\gamma(x,y) \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\Phi(A\times\mathbb{S}^1)} e^{-\frac{|x|^2+|y|^2}{2}} \chi_E(x,y)\, dxdy \\
&= \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_A e^{-\frac{|y|^2}{2}} \left[\frac{e^{-\frac{r^2}{2}}}{\sqrt{2\pi}} \int_{\partial B(r)} \chi_E(x,y)\, d\mathcal{H}^1(x) \right] drdy \\
&= \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_A e^{-\frac{|y|^2}{2}} \mu(r,y)\, drdy \geq \frac{e^{-\frac{R_A^2}{2}}}{(2\pi)^{\frac{n-1}{2}}} \|\mu\|_{L^1(A)},
\end{aligned}
$$

so that $\|\mu\|_{L^1(A)} < \infty$. Moreover,

$$
\|\xi_\mu\|_{L^1(A)} = \sqrt{2\pi} \int_A \frac{e^{\frac{r^2}{2}}}{r} \mu(r,y)\, drdy \leq \sqrt{2\pi} \frac{e^{\frac{R_A^2}{2}}}{r_A} \|\mu\|_{L^1(A)} < \infty,
$$

which shows that $\|\xi_\mu\|_{L^1(A)} < \infty$.

Step 1b: We show that $|D_r\mu|(A) < \infty$. Let $\psi \in C_c^1(A)$ with $|\psi| \leq 1$. Applying formula (2.3) to the function $\psi(|x|,y)\hat{x}$ and observing that $(\psi(|x|,y)\hat{x})_{x\perp} = \psi(|x|,y)\hat{x}$, we obtain that for every $(x,y) \in \mathbb{R}^2_0 \times \mathbb{R}^{n-2}$

$$
\begin{aligned}
\mathrm{div}_x\left(\psi(|x|,y)\hat{x}\right) &= \left[\nabla_x\left(\psi(|x|,y)\hat{x}\right)\hat{x}\right]\cdot\hat{x} + \frac{1}{|x|}\left[\psi(|x|,y)\hat{x}\cdot\hat{x}\right] \\
&= \left[\left(\frac{\partial\psi}{\partial r}(|x|,y)\,\hat{x}\otimes\hat{x} + \psi(|x|,y)\frac{I_2 - \hat{x}\otimes\hat{x}}{|x|}\right)\hat{x}\right]\cdot\hat{x} + \frac{1}{|x|}\psi(|x|,y) \\
&= \frac{\partial\psi}{\partial r}(|x|,y) + \frac{1}{|x|}\psi(|x|,y),
\end{aligned}
\tag{2.15}
$$

where I_2 denotes the identity matrix in $\mathbb{R}^2$.

Thus,

$$
\begin{aligned}
\mathrm{div}_x\left(e^{-\frac{|x|^2}{2}}\psi(|x|,y)\hat{x}\right) &= \nabla_x\left(e^{-\frac{|x|^2}{2}}\right)\cdot\psi(|x|,y)\hat{x} + e^{-\frac{|x|^2}{2}}\,\mathrm{div}_x\left(\psi(|x|,y)\hat{x}\right) \\
&= e^{-\frac{|x|^2}{2}}\left[\frac{\partial\psi}{\partial r}(|x|,y) + \left(\frac{1-|x|^2}{|x|}\right)\psi(|x|,y)\right].
\end{aligned}
$$

Therefore,

$$\begin{aligned}
\frac{1}{(2\pi)^{\frac{n-1}{2}}} & \int_{(0,\infty)\times\mathbb{R}^{n-2}} \psi(r,y)\, dD_r\mu(r,y) \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^2_0\times\mathbb{R}^{n-2}} \left(\frac{1-|x|^2}{|x|}\right) e^{-\frac{|x|^2}{2}} \psi(|x|,y)\chi_E(x,y)\, dx\, dy \\
&\quad - \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\partial^* E} e^{-\frac{|x|^2}{2}} \psi(|x|,y)\, \hat{x}\cdot \nu^E_{x\perp} d\mathcal{H}^{n-1}(x,y) \qquad (2.16)\\
&\le \int_{\mathbb{R}^2_0\times\mathbb{R}^{n-2}} e^{\frac{|y|^2}{2}} \frac{|1-|x|^2|}{|x|}\, |\psi(|x|,y)|\, \chi_E(x,y)\, d\gamma(x,y) \\
&\quad + \frac{1}{\sqrt{2\pi}} \int_{\partial^* E} e^{\frac{|y|^2}{2}}\, |\psi(|x|,y)|\, d\mathcal{H}^{n-1}_\gamma(x,y) \\
&\le e^{\frac{R_A^2}{2}} \left[M_A + \frac{1}{\sqrt{2\pi}} P_\gamma(E)\right] < \infty,
\end{aligned}$$

where we set

$$M_A = \max_{s\in[r_A,R_A]} \frac{|1-s^2|}{s}.$$

Taking the supremum over ψ we conclude.

Step 1c: We conclude the proof that $\mu \in BV(A)$, showing that

$$|D_y\mu|(A) < \infty.$$

Let $i \in \{1,\dots,n-2\}$ and let $\psi \in C^1_c(A)$ with $|\psi| \le 1$. . We have

$$\begin{aligned}
\frac{1}{(2\pi)^{\frac{n-1}{2}}} & \int_{(0,\infty)\times\mathbb{R}^{n-2}} \psi(r,y)\, dD_{y_i}\mu(r,y) \\
&= -\frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{(0,\infty)\times\mathbb{R}^{n-2}} \frac{\partial\psi}{\partial y_i}(r,y)\, \mu(r,y)\, dr\, dy \\
&= -\frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{(0,\infty)\times\mathbb{R}^{n-2}} \frac{\partial\psi}{\partial y_i}(r,y)\, \frac{e^{-\frac{r^2}{2}}}{\sqrt{2\pi}} \left(\int_{\partial B(r)} \chi_E(x,y)\, d\mathcal{H}^1(x)\right) dr\, dy \\
&= -\frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^2_0\times\mathbb{R}^{n-2}} e^{-\frac{|x|^2}{2}} \frac{\partial\psi}{\partial y_i}(|x|,y)\chi_E(x,y)\, dx\, dy \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^2_0\times\mathbb{R}^{n-2}} e^{-\frac{|x|^2}{2}} \psi(|x|,y)\, dD_{y_i}\chi_E(x,y) \qquad (2.17)\\
&= -\frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\partial^* E} e^{-\frac{|x|^2}{2}} \psi(|x|,y)\, \nu^E_{y_i}(x,y)\, d\mathcal{H}^{n-1}(x,y) \\
&= -\frac{1}{\sqrt{2\pi}} \int_{\partial^* E} e^{\frac{|y|^2}{2}} \psi(|x|,y)\, \nu^E_{y_i}(x,y)\, d\mathcal{H}^{n-1}_\gamma(x,y),
\end{aligned}$$

and therefore

$$\begin{aligned}
\frac{1}{(2\pi)^{\frac{n-1}{2}}}&\left|\int_{(0,\infty)\times\mathbb{R}^{n-2}}\psi(r,y)\,dD_{y_i}\mu(r,y)\right| \\
&= \frac{1}{\sqrt{2\pi}}\left|\int_{\partial^* E} e^{\frac{|y|^2}{2}}\psi(|x|,y)\,\nu^E_{y_i}(x,y)\,d\mathcal{H}^{n-1}_\gamma(x,y)\right| \\
&\le \frac{1}{\sqrt{2\pi}}\int_{\partial^* E} e^{\frac{|y|^2}{2}}|\psi(|x|,y)|\,|\nu^E_{y_i}(x,y)|\,d\mathcal{H}^{n-1}_\gamma(x,y) \\
&\le \frac{e^{\frac{R_A^2}{2}}}{\sqrt{2\pi}}P_\gamma(E)<\infty. \qquad (2.18)
\end{aligned}$$

Taking the supremum over ψ we conclude.

Step 2: We show that $\xi_\mu\in BV_{\mathrm{loc}}((0,\infty)\times\mathbb{R}^{n-2})$.

We already proved that $\xi_\mu\in L^1_{\mathrm{loc}}((0,\infty)\times\mathbb{R}^{n-2})$. Let us now show that $D\xi_\mu$ is a locally bounded Radon measure in $(0,\infty)\times\mathbb{R}^{n-2}$. We recall that

$$\mu(r,y)=\frac{re^{-\frac{r^2}{2}}}{\sqrt{2\pi}}\xi_\mu(r,y).$$

Since the function $(r,y)\mapsto 1/(re^{-\frac{r^2}{2}})$ is smooth and locally bounded in $(0,\infty)\times\mathbb{R}^{n-2}$, by the chain rule in BV (see [1, Example 3.97]) we have that $\xi_\mu\in BV_{\mathrm{loc}}((0,\infty)\times\mathbb{R}^{n-2})$, with

$$\begin{aligned}
D_r\mu &= \frac{e^{-\frac{r^2}{2}}}{\sqrt{2\pi}}\left[(1-r^2)\xi_\mu\,dr\,dy + rD_r\xi_\mu\right] \\
&= \frac{1-r^2}{r}\mu(r,y)\,dr\,dy+\frac{e^{-\frac{r^2}{2}}}{\sqrt{2\pi}}rD_r\xi_\mu, \qquad (2.19)\\
D_y\mu &= \frac{re^{-\frac{r^2}{2}}}{\sqrt{2\pi}}D_y\xi_\mu(r,y). \qquad (2.20)
\end{aligned}$$

Step 3: We show formulas (2.12) and (2.13).

Let $\psi\in C^1_c((0,\infty)\times\mathbb{R}^{n-2})$. From (2.19) it follows that

$$\begin{aligned}
\int_{(0,\infty)\times\mathbb{R}^{n-2}}\psi(r,y)\,dD_r\mu(r,y) &= \int_{(0,\infty)\times\mathbb{R}^{n-2}}\psi(r,y)\frac{1-r^2}{r}\mu(r,y)\,dr\,dy \\
&\quad+\int_{(0,\infty)\times\mathbb{R}^{n-2}}\psi(r,y)\frac{e^{-\frac{r^2}{2}}}{\sqrt{2\pi}}\,r\,dD_r\xi_\mu(r,y). \qquad (2.21)
\end{aligned}$$

On the other hand, from (2.16) we have

$$\begin{aligned}\int_{(0,\infty)\times\mathbb{R}^{n-2}}\psi(r,y)\,dD_r\mu(r,y) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^2_0\times\mathbb{R}^{n-2}}\Big(\frac{1-|x|^2}{|x|}\Big)e^{-\frac{|x|^2}{2}}\psi(|x|,y)\chi_E(x,y)\,dx\,dy\\ &\quad+\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^2_0\times\mathbb{R}^{n-2}}e^{-\frac{|x|^2}{2}}\psi(|x|,y)\,\hat{x}\cdot dD_{x\perp}\chi_E(x,y)\\ &=\int_{(0,\infty)\times\mathbb{R}^{n-2}}\psi(r,y)\frac{1-r^2}{r}\frac{e^{-\frac{r^2}{2}}}{\sqrt{2\pi}}\int_{\partial B(r)}\chi_E(x,y)d\mathcal{H}^1(x)\,dr\,dy\\ &\quad+\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^2_0\times\mathbb{R}^{n-2}}e^{-\frac{|x|^2}{2}}\psi(|x|,y)\,\hat{x}\cdot dD_{x\perp}\chi_E(x,y)\\ &=\int_{(0,\infty)\times\mathbb{R}^{n-2}}\psi(r,y)\frac{1-r^2}{r}\mu(r,y)\,dr\,dy\\ &\quad+\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^2_0\times\mathbb{R}^{n-2}}e^{-\frac{|x|^2}{2}}\psi(|x|,y)\,\hat{x}\cdot dD_{x\perp}\chi_E(x,y).\end{aligned}$$

Combining the previous identity and (2.21),

$$\int_{(0,\infty)\times\mathbb{R}^{n-2}}e^{-\frac{r^2}{2}}\psi(r,y)\,r\,dD_r\xi_\mu(r,y)=\int_{\mathbb{R}^2_0\times\mathbb{R}^{n-2}}e^{-\frac{|x|^2}{2}}\psi(|x|,y)\,\hat{x}\cdot dD_{x\perp}\chi_E(x,y),$$

for every $\psi\in C^1_c((0,\infty)\times\mathbb{R}^{n-2})$. Applying the identity above to the function $e^{\frac{r^2}{2}}\psi(r,y)$ we obtain that

$$\int_{(0,\infty)\times\mathbb{R}^{n-2}}\psi(r,y)\,r\,dD_r\xi_\mu(r,y)=\int_{\mathbb{R}^2_0\times\mathbb{R}^{n-2}}\psi(|x|,y)\,\hat{x}\cdot dD_{x\perp}\chi_E(x,y),$$

for every $\psi\in C^1_c((0,\infty)\times\mathbb{R}^{n-2})$.

By approximation, the identity above is true also when ψ is a bounded Borel function, and this gives (2.12).

Let us now show (2.13). Let $\phi\in C^1_c((0,\infty)\times\mathbb{R}^{n-2},\mathbb{R}^{n-2})$. Thanks to (2.17) and (2.20),

$$\begin{aligned}\int_{(0,\infty)\times\mathbb{R}^{n-2}}r\,\phi(r,y)\cdot dD_y\xi_\mu(r,y) &= \sqrt{2\pi}\int_{(0,\infty)\times\mathbb{R}^{n-2}}e^{\frac{r^2}{2}}\phi(r,y)\;\cdot dD_y\mu(r,y)\\ &=\int_{\mathbb{R}^2_0\times\mathbb{R}^{n-2}}\phi(|x|,y)\cdot dD_y\chi_E(x,y).\end{aligned}$$

By approximation, the formula also holds whenever ϕ is a bounded Borel function. $\square$

The next result follows combining Lemma 3 and Coarea Formula.

Lemma 4. *Let $E\subset\mathbb{R}^n$ be an n-symmetric set of finite Gaussian perimeter, let μ be the distribution function of E, and let ξ_μ be defined by (1.2). Then*

$$\begin{aligned}(r\,D_r\xi_\mu)(B) &= -\int_{\partial^*E\cap\Phi(B\times\mathbb{S}^1)\cap\{\nu^E_{x\|}=0\}}\hat{x}\cdot\nu^E_x(x,y)\,d\mathcal{H}^{n-1}(x,y)\\ &\quad-\int_B\int_{(\partial^*E)_{(r,y)}\cap\{\nu^E_{x\|}\neq0\}}\frac{\hat{x}\cdot\nu^E_x(x,y)}{|\nu^E_{x\|}(x,y)|}d\mathcal{H}^0(x)\,dr\,dy,\end{aligned}$$

for every Borel set $B\subset\subset(0,\infty)\times\mathbb{R}^{n-2}$. Moreover,

$$r\partial_r\xi_\mu(r,y)=-\int_{(\partial^*E)_{(r,y)}\cap\{\nu^E_{x\|}\neq0\}}\frac{\hat{x}\cdot\nu^E_x(x,y)}{|\nu^E_{x\|}(x,y)|}d\mathcal{H}^0(x),$$

for $\mathcal{H}^{n-1}$-a.e. $(r,y) \in (0,\infty)\times\mathbb{R}^{n-2}$, where $\partial_r\xi_\mu$ denotes the approximate differential of ξ_μ with respect to r. Similarly,

$$(rD_y\xi_\mu)(B) = -\int_{\partial^* E\cap\Phi(B\times\mathbb{S}^1)\cap\{\nu^E_{x\|}=0\}} \nu^E_y(x,y)\,d\mathcal{H}^{n-1}(x,y) - \int_B\int_{(\partial^* E)_{(r,y)}\cap\{\nu^E_{x\|}\neq 0\}} \frac{\nu^E_y(x,y)}{|\nu^E_{x\|}(x,y)|}d\mathcal{H}^0(x)\,dr\,dy,$$

for every Borel set $B\subset\subset(0,\infty)\times\mathbb{R}^{n-2}$, and

$$r\nabla_y\xi_\mu(r,y) = -\int_{(\partial^* E)_{(r,y)}\cap\{\nu^E_{x\|}\neq 0\}} \frac{\nu^E_y(x,y)}{|\nu^E_{x\|}(x,y)|}d\mathcal{H}^0(x),$$

for $\mathcal{H}^{n-1}$-a.e. $(r,y)\in(0,\infty)\times\mathbb{R}^{n-2}$, where $\nabla_y\xi_\mu$ denotes the approximate gradient of ξ_μ with respect to y.

Remark 3. Thanks to Remark 2, applying the previous lemma to F_μ it follows that for every $(r,y)\in G_{F_\mu}$

$$r\partial_r\xi_\mu(r,y) = -4\frac{\hat{x}\cdot\nu^{F_\mu}_x(x,y)}{|\nu^{F_\mu}_{x\|}(x,y)|}$$

$$r\nabla_y\xi_\mu(r,y) = -4\frac{\nu^{F_\mu}_y(x,y)}{|\nu^{F_\mu}_{x\|}(x,y)|}$$

for every x such that $(x,y)\in\partial^* F_\mu$, where we have used the fact that $\mathcal{H}^0((\partial^* F_\mu)_{(r,y)}) = 4$.

Proposition 3. *Let $E\subset\mathbb{R}^n$ be an n-symmetric set of finite Gaussian perimeter, and let μ be the distribution function of E. Then, F_μ is a set of finite Gaussian perimeter in $\mathbb{R}^n$. Moreover, for every Borel set $B\subset\subset(0,\infty)\times\mathbb{R}^{n-2}$*

$$(2\pi)^{\frac{n-1}{2}}P_\gamma(F_\mu;\Phi(B\times\mathbb{S}^1)) \le \left|re^{-\frac{r^2}{2}}e^{-\frac{|y|^2}{2}}D\xi_\mu\right|(B) + \left|e^{-\frac{|x|^2}{2}}e^{-\frac{|y|^2}{2}}D_{x\|}\chi_{F_\mu}\right|(\Phi(B\times\mathbb{S}^1)). \quad (2.22)$$

Proof. The proof is based on the arguments of [12, Lemma 3.5] and [3, Lemma 3.3]. Thanks to Lemma 3, $\mu\in BV_{\rm loc}((0,\infty)\times\mathbb{R}^{n-2})$. Let $\{\mu_j\}_{j\in\mathbb{N}}\subset C^1_c((0,\infty)\times\mathbb{R}^{n-2})$ be a sequence of non-negative functions with $0\le\mu_j(r,y)\le e^{-\frac{r^2}{2}}\sqrt{2\pi}r$ such that $\mu_j\to\mu$ $\mathcal{H}^{n-1}$-a.e. in $((0,\infty)\times\mathbb{R}^{n-2})$ and $|D\mu_j|\overset{*}{\rightharpoonup}|D\mu|$. For every $j\in\mathbb{N}$, we denote by $F_{\mu_j}\subset\mathbb{R}^n$ the set defined by (1.3), with μ_j in place of μ.

Let $A\subset\subset(0,\infty)\times\mathbb{R}^{n-2}$ be open, and let $\varphi\in C^1_c(\Phi(A\times\mathbb{S}^1);\mathbb{R}^n)$ with $\|\varphi\|_{L^\infty(\Phi(A\times\mathbb{S}^1);\mathbb{R}^n)}\le 1$. We have

$$\begin{aligned}&\int_{F_{\mu_j}}((\operatorname{div}\varphi)(x,y)-\varphi(x,y)\cdot(x,y))\,e^{-\frac{|x|^2}{2}}e^{-\frac{|y|^2}{2}}\,dx\,dy\\
&\quad=\int_{\Phi(A\times\mathbb{S}^1)}\chi_{F_{\mu_j}}(x,y)e^{-\frac{|x|^2}{2}}e^{-\frac{|y|^2}{2}}\Big(\operatorname{div}\varphi(x,y)-x\cdot\varphi_x(x,y)-y\cdot\varphi_y(x,y)\Big)dx\,dy\\
&\quad=\int_{\Phi(A\times\mathbb{S}^1)}\chi_{F_{\mu_j}}(x,y)e^{-\frac{|x|^2}{2}}e^{-\frac{|y|^2}{2}}\Big(\operatorname{div}_y\varphi_y(x,y)-y\cdot\varphi_y(x,y)\Big)dx\,dy\\
&\quad\quad+\int_{\Phi(A\times\mathbb{S}^1)}\chi_{F_{\mu_j}}(x,y)e^{-\frac{|x|^2}{2}}e^{-\frac{|y|^2}{2}}\Big(\operatorname{div}_x\varphi_x(x,y)-x\cdot\varphi_x(x,y)\Big)dx\,dy.\end{aligned}$$

Let us now focus on the last integral. Thanks to Remark 2, this can be written as

$$
\begin{aligned}
&\int_{\Phi(A\times\mathbb{S}^1)} \chi_{F_{\mu_j}}(x,y)e^{-\frac{|x|^2}{2}}e^{-\frac{|y|^2}{2}}\,\Big(\operatorname{div}_x\varphi_x(x,y) - x\cdot\varphi_x(x,y)\Big)dx\,dy\\
&= \int_{\Phi(A\times\mathbb{S}^1)} \chi_{F_{\mu_j}}(x,y)e^{-\frac{|x|^2}{2}}e^{-\frac{|y|^2}{2}}\,\operatorname{div}_{x\|}\varphi_{x\|}(x,y)dx\,dy\\
&+ \int_{\Phi(A\times\mathbb{S}^1)} \chi_{F_{\mu_j}}(x,y)e^{-\frac{|x|^2}{2}}e^{-\frac{|y|^2}{2}}\,\big(\nabla_x\varphi_x(x,y)\hat{x}\big)\cdot\hat{x}\,dx\,dy\\
&+ \int_{\Phi(A\times\mathbb{S}^1)} \chi_{F_{\mu_j}}(x,y)e^{-\frac{|x|^2}{2}}e^{-\frac{|y|^2}{2}}\,\varphi_x(x,y)\cdot\left(\frac{\hat{x}}{|x|} - x\right)dx\,dy.
\end{aligned}
\tag{2.23}
$$

In the following, it is convenient to introduce the function $V_j : A \to \mathbb{R}^{n-1}$ given by

$$
\begin{aligned}
V_j(r,y) &\coloneqq e^{-\frac{r^2}{2}}e^{-\frac{|y|^2}{2}}\int_{\mathbf{A}_{\beta_{\mu_j}(r,y)}(re_1)\cup\mathbf{A}_{\beta_{\mu_j}(r,y)}(-re_1)}\big(\varphi_x(x,y)\cdot\hat{x},\varphi_y(x,y)\big)\,d\mathcal{H}^1(x)\\
&= re^{-\frac{r^2}{2}}e^{-\frac{|y|^2}{2}}\int_{\mathbf{A}_{\beta_{\mu_j}(r,y)}(e_1)\cup\mathbf{A}_{\beta_{\mu_j}(r,y)}(-e_1)}\big(\varphi_x(r\omega,y)\cdot\omega,\varphi_y(r\omega,y)\big)\,d\mathcal{H}^1(\omega).
\end{aligned}
$$

We denote the components of V_j by

$$
\begin{aligned}
(V_j)_r(r,y) &= re^{-\frac{r^2}{2}}e^{-\frac{|y|^2}{2}}\int_{\mathbf{A}_{\beta_{\mu_j}(r,y)}(e_1)\cup\mathbf{A}_{\beta_{\mu_j}(r,y)}(-e_1)}\varphi_x(r\omega,y)\cdot\omega\,d\mathcal{H}^1(\omega)\\
(V_j)_y(r,y) &= re^{-\frac{r^2}{2}}e^{-\frac{|y|^2}{2}}\int_{\mathbf{A}_{\beta_{\mu_j}(r,y)}(e_1)\cup\mathbf{A}_{\beta_{\mu_j}(r,y)}(-e_1)}\varphi_y(r\omega,y)\,d\mathcal{H}^1(\omega).
\end{aligned}
$$

We divide the proof into several steps.

Step 1: We show that V_j is Lipschitz continuous with compact support in A.
The fact that V_j has compact support in A follows from the inclusion $\operatorname{supp} V_j \subset \Lambda(\operatorname{supp}\,\varphi)$, where we set

$$
\Lambda(\operatorname{supp}\,\varphi) \coloneqq \left\{(r,y)\in(0,+\infty)\times\mathbb{R}^{n-2} : (\operatorname{supp}\,\varphi)\cap\partial B(r)\neq\emptyset\right\}.
$$

Moreover, for every $(r_1, y_1), (r_2, y_2) \in \Lambda(\mathrm{supp}\ \varphi)$ (here we set $\mathbf{A}_i = \mathbf{A}_{\beta_{\mu_j}(r_i,y_i)}(e_1) \cup \mathbf{A}_{\beta_{\mu_j}(r_i,y_i)}(-e_1)$ for $i = 1, 2$)

$$
\begin{aligned}
&|V_j(r_1, y_1) - V_j(r_2, y_2)| \\
&= \Big| r_1 e^{-\frac{r_1^2}{2}} e^{-\frac{|y_1|^2}{2}} \int_{\mathbf{A}_1} \big(\varphi_x(r_1\omega, y_1) \cdot \omega, \varphi_y(r_1\omega, y_1)\big)\, d\mathcal{H}^1(\omega) \\
&- r_2 e^{-\frac{r_2^2}{2}} e^{-\frac{|y_2|^2}{2}} \int_{\mathbf{A}_2} \big(\varphi_x(r_2\omega, y_2) \cdot \omega, \varphi_y(r_2\omega, y_2)\big)\, d\mathcal{H}^1(\omega) \Big| \\
&\leq \int_{\mathbf{A}_1} \Big| r_1 e^{-\frac{r_1^2}{2}} e^{-\frac{|y_1|^2}{2}} \big(\varphi_x(r_1\omega, y_1) \cdot \omega, \varphi_y(r_1\omega, y_1)\big) \\
&- r_2 e^{-\frac{r_2^2}{2}} e^{-\frac{|y_2|^2}{2}} \big(\varphi_x(r_2\omega, y_2) \cdot \omega, \varphi_y(r_2\omega, y_2)\big) \Big|\, d\mathcal{H}^1(\omega) \\
&+ r_2 e^{-\frac{r_2^2}{2}} e^{-\frac{|y_2|^2}{2}} \int_{\mathbf{A}_1 \Delta \mathbf{A}_2} |\big(\varphi_x(r_2\omega, y_2) \cdot \omega, \varphi_y(r_2\omega, y_2)\big)|\, d\mathcal{H}^1(\omega) \\
&\leq C|(r_1, y_1) - (r_2, y_2)| + r_2 e^{-\frac{r_2^2}{2}} e^{-\frac{|y_2|^2}{2}} \mathcal{H}^1(\mathbf{A}_1 \Delta \mathbf{A}_2) \\
&= C|(r_1, y_1) - (r_2, y_2)| + 2\pi r_2 e^{-\frac{r_2^2}{2}} e^{-\frac{|y_2|^2}{2}} |\xi_{\mu_j}(r_1, y_1) - \xi_{\mu_j}(r_2, y_2)| \\
&\leq C'|(r_1, y_1) - (r_2, y_2)|,
\end{aligned}
$$

for some $C, C' > 0$, where we used the fact that the function

$$(r, y) \mapsto r e^{-\frac{r^2}{2}} e^{-\frac{|y|^2}{2}} \big(\varphi_x(r\omega, y) \cdot \omega, \varphi_y(r\omega, y)\big)$$

is locally Lipschitz continuous.

Step 2: We show that β_{μ_j} is $\mathcal{H}^{n-1}$-a.e. differentiable and that

$$
\begin{aligned}
\partial_r (V_j)_r(r, y) &= e^{-\frac{|y|^2}{2}} e^{-\frac{r^2}{2}} (1 - r^2) \int_{\mathbf{A}_{\beta_{\mu_j}(r,y)}(e_1) \cup \mathbf{A}_{\beta_{\mu_j}(r,y)}(-e_1)} \varphi_x(r\omega, y) \cdot \omega\, d\mathcal{H}^1(\omega) \\
&+ r e^{-\frac{r^2}{2}} e^{-\frac{|y|^2}{2}} \frac{\partial \beta_{\mu_j}}{\partial r}(r, y) \Big(\int_{\mathbf{S}_{\beta_{\mu_j}(r,y)}(e_1) \cup \mathbf{S}_{\beta_{\mu_j}(r,y)}(-e_1)} \varphi_x(r\omega, y) \cdot \omega\, d\mathcal{H}^0(\omega) \Big) \\
&+ r e^{-\frac{r^2}{2}} e^{-\frac{|y|^2}{2}} \int_{\mathbf{A}_{\beta_{\mu_j}(r,y)}(e_1) \cup \mathbf{A}_{\beta_{\mu_j}(r,y)}(-e_1)} \Big((\nabla_x \varphi_x(r\omega, y))\, \omega \Big) \cdot \omega\, d\mathcal{H}^1(\omega),
\end{aligned}
\tag{2.24}
$$

for $\mathcal{H}^{n-1}$-a.e. $(r, y) \in A$.

Let us set $A_j := \{0 < \beta_{\mu_j} < \pi/2\}$. Since $\mu_j \in C^1_c((0,\infty)\times\mathbb{R}^{n-2})$, we have that $\beta_{\mu_j} \in C^1(A_j)$. Moreover, for every $(r,y) \in A_j$

$$
\begin{aligned}
\frac{\partial (V_j)_r}{\partial r}(r,y) &= e^{-\frac{|y|^2}{2}} \frac{\partial}{\partial r}\left(r e^{-\frac{r^2}{2}} \int_0^{\beta_{\mu_j}(r,y)} d\alpha \int_{\mathbf{S}_\alpha(e_1)\cup \mathbf{S}_\alpha(-e_1)} \varphi_x(r\omega,y)\cdot\omega \; d\mathcal{H}^0(\omega)\right) \\
&= e^{-\frac{|y|^2}{2}} e^{-\frac{r^2}{2}}(1-r^2) \int_{\mathbf{A}_{\beta_{\mu_j}(r,y)}(e_1)\cup \mathbf{A}_{\beta_{\mu_j}(r,y)}(-e_1)} \varphi_x(r\omega,y)\cdot\omega \, d\mathcal{H}^1(\omega) \\
&\quad + r e^{-\frac{r^2}{2}} e^{-\frac{|y|^2}{2}} \frac{\partial \beta_{\mu_j}}{\partial r}(r,y) \left(\int_{\mathbf{S}_{\beta_{\mu_j}(r,y)}(e_1)\cup \mathbf{S}_{\beta_{\mu_j}(r,y)}(-e_1)} \varphi_x(r\omega,y)\cdot\omega \, d\mathcal{H}^0(\omega)\right) \\
&\quad + r e^{-\frac{r^2}{2}} e^{-\frac{|y|^2}{2}} \int_0^{\beta_{\mu_j}(r,y)} d\alpha \int_{\mathbf{S}_\alpha(e_1)\cup \mathbf{S}_\alpha(-e_1)} \Big((\nabla_x \varphi_x(r\omega,y))\; \omega\Big)\cdot\omega \, d\mathcal{H}^0(\omega) \\
&= e^{-\frac{|y|^2}{2}} e^{-\frac{r^2}{2}}(1-r^2) \int_{\mathbf{A}_{\beta_{\mu_j}(r,y)}(e_1)\cup \mathbf{A}_{\beta_{\mu_j}(r,y)}(-e_1)} \varphi_x(r\omega,y)\cdot\omega \, d\mathcal{H}^1(\omega) \\
&\quad + r e^{-\frac{r^2}{2}} e^{-\frac{|y|^2}{2}} \frac{\partial \beta_{\mu_j}}{\partial r}(r,y) \left(\int_{\mathbf{S}_{\beta_{\mu_j}(r,y)}(e_1)\cup \mathbf{S}_{\beta_{\mu_j}(r,y)}(-e_1)} \varphi_x(r\omega,y)\cdot\omega \, d\mathcal{H}^0(\omega)\right) \\
&\quad + r e^{-\frac{r^2}{2}} e^{-\frac{|y|^2}{2}} \int_{\mathbf{A}_{\beta_{\mu_j}(r,y)}(e_1)\cup \mathbf{A}_{\beta_{\mu_j}(r,y)}(-e_1)} \Big((\nabla_x \varphi_x(r\omega,y))\; \omega\Big)\cdot\omega \, d\mathcal{H}^1(\omega).
\end{aligned}
$$

This shows (2.24) whenever $(r,y) \in A_j$. Note now that

$$
\begin{aligned}
&(V_j)_r(r,y) = 0 \qquad \text{for every } r \in \mathrm{Int}(\{\beta_{\mu_j} = 0\}),\\
&(V_j)_r(r,y) = r e^{-\frac{r^2}{2}} e^{-\frac{|y|^2}{2}} \int_{\mathbb{S}^1} \varphi_x(r\omega,y)\cdot\omega \, d\mathcal{H}^1(\omega) \quad \text{for every } r \in \mathrm{Int}(\{\beta_{\mu_j} = \pi/2\}),
\end{aligned}
$$

where $\mathrm{Int}(\cdot)$ stands for the interior of a set.

Since

$$
\partial_r \beta_{\mu_j}(r,y) = 0 \text{ for every } (r,y) \in \mathrm{Int}(\{\beta_{\mu_j}=0\}) \cup \mathrm{Int}(\{\beta_{\mu_j}=\pi/2\}),
$$

using the identities above one can see that (2.24) holds true for $\mathcal{H}^{n-1}$-a.e. $(r,y) \in (0,\infty)\times\mathbb{R}^{n-2}$.

Step 3: We show that

$$
\begin{aligned}
\big(\mathrm{div}_y (V_j)_y\big)(r,y) &= e^{-\frac{|y|^2}{2}} e^{-\frac{r^2}{2}} \int_{\mathbf{A}_{\beta_{\mu_j}(r,y)}(re_1)\cup \mathbf{A}_{\beta_{\mu_j}(r,y)}(-re_1)} \Big(\mathrm{div}_y\, \varphi_y(x,y) - y\cdot\varphi_y(x,y)\Big)\, d\mathcal{H}^1(x) \\
&\quad + r e^{-\frac{r^2}{2}} e^{-\frac{|y|^2}{2}} \left(\int_{\mathbf{S}_{\beta_{\mu_j}(r,y)}(e_1)\cup \mathbf{S}_{\beta_{\mu_j}(r,y)}(-e_1)} \nabla_y \beta_{\mu_j}(r,y)\cdot\varphi_y(r\omega,y)\, d\mathcal{H}^0(\omega)\right),
\end{aligned}
$$

for $\mathcal{H}^{n-1}$-a.e. $(r,y) \in A$.

The proof can be done by adapting the arguments used in the previous step.

Step 4: We show that

$$\int_{\Phi(A\times\mathbb{S}^1)} \chi_{F_{\mu_j}}(x,y)e^{-\frac{|x|^2}{2}}e^{-\frac{|y|^2}{2}} \left(\nabla_x\varphi_x(x,y)\hat{x}\right)\cdot\hat{x}\,dx\,dy$$
$$+\int_{\Phi(A\times\mathbb{S}^1)} \chi_{F_{\mu_j}}(x,y)e^{-\frac{|x|^2}{2}}e^{-\frac{|y|^2}{2}}\,\varphi_x(x,y)\cdot\left(\frac{\hat{x}}{|x|}-x\right)dx\,dy$$
$$+\int_{\Phi(A\times\mathbb{S}^1)} \chi_{F_{\mu_j}}(x,y)e^{-\frac{|x|^2}{2}}e^{-\frac{|y|^2}{2}}\,\Big(\operatorname{div}_y\varphi_y(x,y)-y\cdot\varphi_y(x,y)\Big)dx\,dy$$
$$=-\int_A re^{-\frac{r^2}{2}}e^{-\frac{|y|^2}{2}}\left(\int_{\mathbf{S}_{\beta_{\mu_j}(r,y)}(e_1)\cup\mathbf{S}_{\beta_{\mu_j}(r,y)}(-e_1)}\nabla_{(r,y)}\beta_{\mu_j}(r,y)\cdot\big(\varphi_x(r\omega,y)\cdot\omega,\varphi_y(r\omega,y)\big)\,d\mathcal{H}^0(\omega)\right)dr\,dy.$$

Integrating (2.24), thanks to the classical divergence theorem applied in A, and recalling that V_j has compact support in A, we obtain

$$0=\int_A \operatorname{div}_{(r,y)}V_j(r,y)\,dr\,dy$$
$$=\int_A \left(\partial_r(V_j)_r(r,y)+\operatorname{div}_y(V_j)_y(r,y)\right)\,dr\,dy$$
$$=\int_A\left(e^{-\frac{|y|^2}{2}}e^{-\frac{r^2}{2}}(1-r^2)\int_{\mathbf{A}_{\beta_{\mu_j}(r,y)}(e_1)\cup\mathbf{A}_{\beta_{\mu_j}(r,y)}(-e_1)}\varphi_x(r\omega,y)\cdot\omega\,d\mathcal{H}^1(\omega)\right)dr\,dy$$
$$+\int_A re^{-\frac{r^2}{2}}e^{-\frac{|y|^2}{2}}\left(\int_{\mathbf{S}_{\beta_{\mu_j}(r,y)}(e_1)\cup\mathbf{S}_{\beta_{\mu_j}(r,y)}(-e_1)}\nabla_{(r,y)}\beta_{\mu_j}(r,y)\cdot\big(\varphi_x(r\omega,y)\cdot\omega,\varphi_y(r\omega,y)\big)\,d\mathcal{H}^0(\omega)\right)dr\,dy$$
$$+\int_A re^{-\frac{r^2}{2}}e^{-\frac{|y|^2}{2}}\left(\int_{\mathbf{A}_{\beta_{\mu_j}(r,y)}(e_1)\cup\mathbf{A}_{\beta_{\mu_j}(r,y)}(-e_1)}(\nabla_x\varphi_x(r\omega,y)\,\omega)\cdot\omega\,d\mathcal{H}^1(\omega)\right)dr\,dy$$
$$+\int_A e^{-\frac{|y|^2}{2}}e^{-\frac{r^2}{2}}\int_{\mathbf{A}_{\beta_{\mu_j}(r,y)}(re_1)\cup\mathbf{A}_{\beta_{\mu_j}(r,y)}(-re_1)}\Big(\operatorname{div}_y\varphi_y(x,y)-y\cdot\varphi_y(x,y)\Big)\,d\mathcal{H}^1(x)\,dr\,dy$$
$$=\int_{\Phi(A\times\mathbb{S}^1)} \chi_{F_{\mu_j}}(x,y)e^{-\frac{|x|^2}{2}}e^{-\frac{|y|^2}{2}}\,\varphi_x(x,y)\cdot\left(\frac{\hat{x}}{|x|}-x\right)dx\,dy$$
$$+\int_A re^{-\frac{r^2}{2}}e^{-\frac{|y|^2}{2}}\left(\int_{\mathbf{S}_{\beta_{\mu_j}(r,y)}(e_1)\cup\mathbf{S}_{\beta_{\mu_j}(r,y)}(-e_1)}\nabla_{(r,y)}\beta_{\mu_j}(r,y)\cdot\big(\varphi_x(r\omega,y)\cdot\omega,\varphi_y(r\omega,y)\big)\,d\mathcal{H}^0(\omega)\right)dr\,dy$$
$$+\int_{\Phi(A\times\mathbb{S}^1)} \chi_{F_{\mu_j}}(x,y)e^{-\frac{|x|^2}{2}}e^{-\frac{|y|^2}{2}}\left(\nabla_x\varphi_x(x,y)\hat{x}\right)\cdot\hat{x}\,dx\,dy$$
$$+\int_{\Phi(A\times\mathbb{S}^1)} \chi_{F_{\mu_j}}(x,y)e^{-\frac{|x|^2}{2}}e^{-\frac{|y|^2}{2}}\,\Big(\operatorname{div}_y\varphi_y(x,y)-y\cdot\varphi_y(x,y)\Big)dx\,dy,$$

which gives the claim.

Step 5: We prove that

$$\int_{\Phi(A\times\mathbb{S}^1)}\chi_{F_{\mu_j}}(x,y)((\operatorname{div}\varphi)(x,y)-\varphi(x,y)\cdot(x,y))e^{-\frac{|x|^2}{2}}e^{-\frac{|y|^2}{2}}\,dx\,dy$$
$$\leq\left|re^{-\frac{r^2}{2}}e^{-\frac{|y|^2}{2}}D\xi_{\mu_j}\right|(\Lambda(\operatorname{supp}\varphi)) \tag{2.25}$$
$$+\int_A e^{-\frac{r^2}{2}}e^{-\frac{|y|^2}{2}}\mathcal{H}^0(\mathbf{S}_{\beta_{\mu_j}(r,y)}(re_1)\cup\mathbf{S}_{\beta_{\mu_j}(r,y)}(-re_1))\,dr\,dy,$$

where $\Lambda(\operatorname{supp}\varphi)\subset(0,\infty)\times\mathbb{R}^{n-2}$ is the compact set defined in Step 1.

Thanks to (2.23) and Step 4

$$
\begin{aligned}
&\int_{\Phi(A\times\mathbb{S}^1)}\chi_{F_{\mu_j}}(x,y)((\operatorname{div}\varphi)(x,y)-\varphi(x,y)\cdot(x,y))e^{-\frac{|x|^2}{2}}e^{-\frac{|y|^2}{2}}\,dx\,dy\\
&=\int_{\Phi(A\times\mathbb{S}^1)}\chi_{F_{\mu_j}}(x,y)e^{-\frac{|x|^2}{2}}e^{-\frac{|y|^2}{2}}\ \operatorname{div}_{x\|}\varphi_{x\|}(x,y)dx\,dy\\
&-\int_A re^{-\frac{r^2}{2}}e^{-\frac{|y|^2}{2}}\left(\int_{\mathbf{S}_{\beta_{\mu_j}(r,y)}(e_1)\cup\mathbf{S}_{\beta_{\mu_j}(r,y)}(-e_1)}\nabla_{(r,y)}\beta_{\mu_j}(r,y)\cdot\left(\varphi_x(r\omega,y)\cdot\omega,\varphi_y(r\omega,y)\right)d\mathcal{H}^0(\omega)\right)dr\,dy.
\end{aligned}
\tag{2.26}
$$

We now estimate the right hand side of the expression above. Thanks to (1.2) and arguing as in Step 2 we have that

$$
\nabla_{(r,y)}\xi_{\mu_j}(r,y)=4\nabla_{(r,y)}\beta_{\mu_j}(r,y)=\nabla_{(r,y)}\beta_{\mu_j}(r,y)\mathcal{H}^0(\mathbf{S}_{\beta_{\mu_j(r)}}(e_1)\cup\mathbf{S}_{\beta_{\mu_j(r)}}(-e_1)),
$$

for $\mathcal{H}^{n-1}$-a.e. $(r,y)\in(0,\infty)\times\mathbb{R}^{n-2}$.

Therefore,

$$
\begin{aligned}
&-\int_A re^{-\frac{r^2}{2}}e^{-\frac{|y|^2}{2}}\left(\int_{\mathbf{S}_{\beta_{\mu_j}(r,y)}(e_1)\cup\mathbf{S}_{\beta_{\mu_j}(r,y)}(-e_1)}\nabla_{(r,y)}\beta_{\mu_j}(r,y)\cdot\left(\varphi_x(r\omega,y)\cdot\omega,\varphi_y(r\omega,y)\right)d\mathcal{H}^0(\omega)\right)dr\,dy\\
&\le\int_{\Lambda(\operatorname{supp}\varphi)}re^{-\frac{r^2}{2}}e^{-\frac{|y|^2}{2}}|\nabla_{(r,y)}\beta_{\mu_j}(r,y)|\mathcal{H}^0\left(\mathbf{S}_{\beta_{\mu_j}(r,y)}(e_1)\cup\mathbf{S}_{\beta_{\mu_j}(r,y)}(-e_1)\right)dr\,dy\\
&=\int_{\Lambda(\operatorname{supp}\varphi)}re^{-\frac{r^2}{2}}e^{-\frac{|y|^2}{2}}|\nabla_{(r,y)}\xi_{\mu_j}(r,y)|\,dr\,dy\\
&=\left|re^{-\frac{r^2}{2}}e^{-\frac{|y|^2}{2}}D\xi_{\mu_j}\right|(\Lambda(\operatorname{supp}\varphi)).
\end{aligned}
\tag{2.27}
$$

Let us now focus on the term in (2.26). Applying the divergence theorem in the circle and denoting by $\nu_*(x,y)$ the exterior unit conormal in $\partial B(r)$ to $\mathbf{A}_{\beta_{\mu_j}(r,y)}(re_1)\cup\mathbf{A}_{\beta_{\mu_j}(r,y)}(-re_1)$ at its endpoints, we have

$$
\begin{aligned}
&\int_{\Phi(A\times\mathbb{S}^1)}\chi_{F_{\mu_j}}(x,y)e^{-\frac{|x|^2}{2}}e^{-\frac{|y|^2}{2}}\ \operatorname{div}_{x\|}\varphi_{x\|}(x,y)dx\,dy\\
&=\int_A e^{-\frac{r^2}{2}}e^{-\frac{|y|^2}{2}}\left[\int_{\mathbf{A}_{\beta_{\mu_j}(r,y)}(re_1)\cup\mathbf{A}_{\beta_{\mu_j}(r,y)}(-re_1)}\operatorname{div}_{x\|}\varphi_{x\|}(x,y)\,d\mathcal{H}^1(x)\right]dr\,dy\\
&=\int_A e^{-\frac{r^2}{2}}e^{-\frac{|y|^2}{2}}\left[\int_{\mathbf{S}_{\beta_{\mu_j}(r,y)}(re_1)\cup\mathbf{S}_{\beta_{\mu_j}(r,y)}(-re_1)}\varphi_{x\|}(x,y)\cdot\nu_*(x,y)\,d\mathcal{H}^0(x)\right]dr\,dy\\
&\le\int_A e^{-\frac{r^2}{2}}e^{-\frac{|y|^2}{2}}\mathcal{H}^0(\mathbf{S}_{\beta_{\mu_j}(r,y)}(re_1)\cup\mathbf{S}_{\beta_{\mu_j}(r,y)}(-re_1))\,dr\,dy.
\end{aligned}
\tag{2.28}
$$

Combining (2.26), (2.27), and (2.28), we obtain (2.25).

Step 6: We show that F_μ is a set of finite Gaussian perimeter.

Note that $\chi_{F_{\mu_j}} \to \chi_{F_\mu}$ $\mathcal{H}^n$-a.e. in $\mathbb{R}^n$, and $\beta_{\mu_j} \to \beta_\mu$ $\mathcal{H}^{n-1}$-a.e. in $(0,\infty)\times\mathbb{R}^{n-2}$. Note also that, from our choice of the sequence $\{\mu_j\}_{j\in\mathbb{N}}$ and thanks to (2.19), it follows that

$$\left|re^{-\frac{r^2}{2}}e^{-\frac{|y|^2}{2}}D\xi_{\mu_j}\right| \overset{*}{\rightharpoonup} \left|re^{-\frac{r^2}{2}}e^{-\frac{|y|^2}{2}}D\xi_{\mu}\right| \quad \text{as } j\to\infty.$$

Therefore, taking the limsup as $j\to\infty$ in (2.25), and using the fact that $\Lambda(\text{supp }\varphi)$ is compact,

$$\begin{aligned}
&\int_{\Phi(A\times\mathbb{S}^1)} \chi_{F_\mu}(x,y)\left((\operatorname{div}\varphi)(x,y)-\varphi(x,y)\cdot(x,y)\right)e^{-\frac{|x|^2}{2}}e^{-\frac{|y|^2}{2}}\,dx\,dy\\
&=\limsup_{j\to\infty}\int_{\Phi(A\times\mathbb{S}^1)} \chi_{F_{\mu_j}}(x,y)\left((\operatorname{div}\varphi)(x,y)-\varphi(x,y)\cdot(x,y)\right)e^{-\frac{|x|^2}{2}}e^{-\frac{|y|^2}{2}}\,dx\,dy\\
&\le \limsup_{j\to\infty}\left|re^{-\frac{r^2}{2}}e^{-\frac{|y|^2}{2}}D\xi_{\mu_j}\right|(\Lambda(\text{supp }\varphi))\\
&+\limsup_{j\to\infty}\int_A e^{-\frac{r^2}{2}}e^{-\frac{|y|^2}{2}}\mathcal{H}^0(\mathbf{S}_{\beta_{\mu_j}(r,y)}(re_1)\cup\mathbf{S}_{\beta_{\mu_j}(r,y)}(-re_1))\,dr\,dy\\
&\le \left|re^{-\frac{r^2}{2}}e^{-\frac{|y|^2}{2}}D\xi_{\mu}\right|(\Lambda(\text{supp }\varphi))\\
&+\int_A e^{-\frac{r^2}{2}}e^{-\frac{|y|^2}{2}}\mathcal{H}^0(\mathbf{S}_{\beta_{\mu}(r,y)}(re_1)\cup\mathbf{S}_{\beta_{\mu}(r,y)}(-re_1))\,dr\,dy\\
&\le \left|re^{-\frac{r^2}{2}}e^{-\frac{|y|^2}{2}}D\xi_{\mu}\right|(A)+\int_A e^{-\frac{r^2}{2}}e^{-\frac{|y|^2}{2}}\mathcal{H}^0(\partial^*E_{(r,y)})\,dr\,dy\\
&\le \left|re^{-\frac{r^2}{2}}e^{-\frac{|y|^2}{2}}D\xi_{\mu}\right|(A)+P_\gamma(E;\Phi(A\times\mathbb{S}^1)),
\end{aligned}$$

where we used the Coarea formula and the fact that, since E is symmetric with respect to the origin, we have

$$\mathcal{H}^0(\mathbf{S}_{\beta_{\mu}(r,y)}(re_1)\cup\mathbf{S}_{\beta_{\mu}(r,y)}(-re_1))\le\mathcal{H}^0(\partial^*E_{(r,y)}),$$

for $\mathcal{H}^{n-1}$-a.e. $(r,y)\in(0,\infty)\times\mathbb{R}^{n-2}$. Taking the supremum of the above inequality over all functions $\varphi\in C^1_c(\Phi(A\times\mathbb{S}^1);\mathbb{R}^n)$ with $\|\varphi\|_{L^\infty(\Phi(A\times\mathbb{S}^1);\mathbb{R}^n)}\le 1$, we obtain

$$P_\gamma(F_\mu;\Phi(A\times\mathbb{S}^1))\le\frac{1}{(2\pi)^{\frac{n-1}{2}}}\left|re^{-\frac{r^2}{2}}e^{-\frac{|y|^2}{2}}D\xi_{\mu}\right|(A)+P_\gamma(E;\Phi(A\times\mathbb{S}^1)).$$

Thanks to (2.14), using the fact that E is a set of finite Gaussian perimeter, we have

$$P_\gamma(F_\mu;\Phi(A\times\mathbb{S}^1))\le 2P_\gamma(E;\Phi(A\times\mathbb{S}^1))<\infty.$$

Since A was arbitrary, this shows that F_μ is a set of locally finite perimeter.

Step 7: We conclude. Let $A\subset\subset(0,\infty)\times\mathbb{R}^{n-2}$ be open, and let $\varphi\in C^1_c(\Phi(A\times\mathbb{S}^1);\mathbb{R}^n)$ with $\|\varphi\|_{L^\infty(\Phi(A\times\mathbb{S}^1);\mathbb{R}^n)}\le 1$. Combining (2.26) and (2.27), we have that for every $j\in\mathbb{N}$

$$\begin{aligned}
&\int_{\Phi(A\times\mathbb{S}^1)}\chi_{F_{\mu_j}}(x,y)e^{-\frac{|x|^2}{2}}e^{-\frac{|y|^2}{2}}\Big(\operatorname{div}_x\varphi_x(x,y)-x\cdot\varphi_x(x,y)\Big)dx\,dy\\
&\le\left|re^{-\frac{r^2}{2}}e^{-\frac{|y|^2}{2}}D\xi_{\mu_j}\right|(\Lambda(\text{supp }\varphi))\\
&+\int_{\Phi(A\times\mathbb{S}^1)}\chi_{F_{\mu_j}}(x,y)e^{-\frac{|x|^2}{2}}e^{-\frac{|y|^2}{2}}\operatorname{div}_{x\|}\varphi_{x\|}(x,y)dx\,dy.
\end{aligned}$$

Taking the limsup as $j \to \infty$, since $\Lambda(\text{supp } \varphi)$ is compact, and thanks to Corollary 1, we have

$$\begin{aligned}
&\int_{\Phi(A\times\mathbb{S}^1)} \chi_{F_\mu}(x,y) e^{-\frac{|x|^2}{2}} e^{-\frac{|y|^2}{2}} \Big(\operatorname{div}\varphi(x,y) - (x,y)\cdot\varphi(x,y)\Big) dx\, dy \\
&\le \left| r e^{-\frac{r^2}{2}} e^{-\frac{|y|^2}{2}} D\xi_\mu \right| (\Lambda(\text{supp } \varphi)) \\
&+ \int_{\Phi(A\times\mathbb{S}^1)} \chi_{F_\mu}(x,y) e^{-\frac{|x|^2}{2}} e^{-\frac{|y|^2}{2}} \operatorname{div}_{x\|} \varphi_{x\|}(x,y) dx\, dy \\
&\le \left| r e^{-\frac{r^2}{2}} e^{-\frac{|y|^2}{2}} D\xi_\mu \right| (\Lambda(\text{supp } \varphi)) + |D_{x\|}(e^{-\frac{|x|^2}{2}} e^{-\frac{|y|^2}{2}} \chi_{F_\mu})|(\Phi(A\times\mathbb{S}^1)) \\
&= \left| r e^{-\frac{r^2}{2}} e^{-\frac{|y|^2}{2}} D\xi_\mu \right| (\Lambda(\text{supp } \varphi)) + |e^{-\frac{|x|^2}{2}} e^{-\frac{|y|^2}{2}} D_{x\|}\chi_{F_\mu}|(\Phi(A\times\mathbb{S}^1)),
\end{aligned}$$

where we also used the fact that $D_{x\|}(e^{-\frac{|x|^2}{2}} e^{-\frac{|y|^2}{2}} \chi_{F_\mu}) = e^{-\frac{|x|^2}{2}} e^{-\frac{|y|^2}{2}} D_{x\|}\chi_{F_\mu}$. Taking the supremum over all $\varphi \in C^1_c(\Phi(A\times\mathbb{S}^1);\mathbb{R}^n)$ with $\|\varphi\|_{L^\infty(\Phi(A\times\mathbb{S}^1);\mathbb{R}^n)} \le 1$,

$$(2\pi)^{\frac{n-1}{2}} P_\gamma(F_\mu;\Phi(A\times\mathbb{S}^1)) \le \left| r e^{-\frac{r^2}{2}} e^{-\frac{|y|^2}{2}} D\xi_\mu \right|(A) + \left| e^{-\frac{|x|^2}{2}} e^{-\frac{|y|^2}{2}} D_{x\|}\chi_{F_\mu} \right| (\Phi(A\times\mathbb{S}^1)), \quad (2.29)$$

which shows (2.22) when B is an open set included in $(0,\infty)\times\mathbb{R}^{n-2}$. The general case with B a Borel set follows by approximation. □

3. Perimeter inequality

The following easy result is the key in the proof of Theorem 1.

Lemma 5 (Isoperimetric inequality for 2-symmetric sets in the circle)**.** *Let $r > 0$ be fixed, and let $0 < \alpha < \pi/2$. Then, for every 2-symmetric Borel subset D of $\partial B(r)$ with $\mathcal{H}^1(D) = 4\alpha r$, we have*

$$4 = \mathcal{H}^0\left(\partial^*\Big(\mathbf{A}_\alpha(re_1) \cup \mathbf{A}_\alpha(-re_1)\Big)\right) \le \mathcal{H}^0(\partial^* D).$$

Moreover, equality holds if and only if

$$D =_{\mathcal{H}^1} \mathbf{A}_\alpha(rp) \cup \mathbf{A}_\alpha(-rp),$$

for some $p \in \{(1,0),(0,1)\}$.

Proof. Since $0 < \alpha < \pi/2$, D cannot be empty, nor the full circle. By 2-symmetry, D must have at least one portion in each quadrant. In each quadrant, there is at least one point of the boundary (indeed, two, unless D crosses the axes). A minimizer is a union of two arcs centered on an axis (crossing 2 axes), yielding exactly 4 boundary points. □

In the following, we denote by $\beta_\mu^\vee$ and $\beta_\mu^\wedge$ the approximate limsup and the approximate liminf of β_μ, respectively.

Proof of Theorem 1. We adapt the arguments of the proof of [3, Theorem 1.1]. We can assume that $B \subset\subset (0,\infty)\times\mathbb{R}^{n-2}$, since the general case $B \subset (0,\infty)\times\mathbb{R}^{n-2}$ can be obtained by approximation.

Let $A \subset\subset (0,\infty)\times\mathbb{R}^{n-2}$ be a Borel set such that $B \subset\subset A \subset\subset (0,\infty)\times\mathbb{R}^{n-2}$, and let G_E and G_{F_μ} be the sets relative to A given by Proposition 2.

We start by proving (1.5). We first prove the inequality when $B \subset ((0,\infty)\times\mathbb{R}^{n-2})\setminus(G_{F_\mu}\cap G_E)$, and then in the case $B \subset G_{F_\mu}\cap G_E$. The case of a general Borel set $B \subset\subset (0,\infty)\times\mathbb{R}^{n-2}$ then follows by decomposing B as $B = (B\setminus(G_{F_\mu}\cap G_E)) \cup (B\cap G_{F_\mu}\cap G_E)$.

Step 1: We show (1.5) when $B \subset ((0,\infty)\times\mathbb{R}^{n-2}) \setminus (G_{F_\mu}\cap G_E)$.

The set $(0,\infty)\times\mathbb{R}^{n-2}\setminus(G_{F_\mu}\cap G_E)$ can be decomposed as

$$\{\beta_\mu = 0\} \cup \{\beta_\mu = \pi/2\} \cup \underbrace{\big(\{0<\beta_\mu<\pi/2\}\setminus(G_{F_\mu}\cap G_E)\big)}_{=:\mathcal{N}}, \tag{3.1}$$

where $\mathcal{H}^{n-1}(\mathcal{N}) = 0$ by the definition of G_{F_μ} and G_E in Proposition 2. We treat each piece separately, establishing the following auxiliary results first.

Step 1a: We show that for every μ-distributed set $W \subset \mathbb{R}^n$ we have

$$\mathcal{H}^{n-1}\left(\partial^* W \cap \Phi(\mathcal{N}\times\mathbb{S}^1) \cap \{\nu^W_{x\|}\neq 0\}\right) = 0 \tag{3.2}$$

for every Borel set $\mathcal{N} \subset\subset (0,\infty)\times\mathbb{R}^{n-2}$ such that $\mathcal{H}^{n-1}(\mathcal{N}) = 0$.

Indeed, by Proposition 1,

$$\begin{aligned}&\mathcal{H}^{n-1}\left(\partial^* W \cap \Phi(\mathcal{N}\times\mathbb{S}^1) \cap \{\nu^W_{x\|}\neq 0\}\right)\\ &= \int_R \int_{(\partial^* W)_{(r,y)}} \frac{\chi_{\{\nu^W_{x\|}\neq 0\}}(x,y)}{|\nu^W_{x\|}|}\, d\mathcal{H}^0(x)\, dr\, dy = 0,\end{aligned}$$

where in the last equality we used the fact that $\mathcal{H}^{n-1}(\mathcal{N}) = 0$.

Step 1b: We show that for every μ-distributed set $W \subset \mathbb{R}^n$ we have

$$\mathcal{H}^{n-1}\left(\partial^* W \cap \Phi(\{\beta_\mu = \pi/2\}\times\mathbb{S}^1) \cap \{\nu^W_{x\|}\neq 0\}\right) = 0. \tag{3.3}$$

Indeed, by Proposition 1,

$$\begin{aligned}&\mathcal{H}^{n-1}\left(\partial^* W \cap \Phi(\{\beta_\mu = \pi/2\}\times\mathbb{S}^1) \cap \{\nu^W_{x\|}\neq 0\}\right)\\ &= \int_{\{\beta_\mu=\pi/2\}} \int_{(\partial^* W)_{(r,y)}} \frac{\chi_{\{\nu^W_{x\|}\neq 0\}}(x,y)}{|\nu^W_{x\|}(x,y)|}\, d\mathcal{H}^0(x)\, dr\, dy\\ &= \int_{\{\beta_\mu=\pi/2\}} \int_{\partial^* (W_{(r,y)})} \frac{\chi_{\{\nu^W_{x\|}\neq 0\}}(x,y)}{|\nu^W_{x\|}(x,y)|}\, d\mathcal{H}^0(x)\, dr\, dy = 0,\end{aligned}$$

where in the last equality we used the fact that, thanks to Vol'pert Theorem, for $\mathcal{H}^{n-1}$-a.e. $(r,y)\in\{\beta_\mu = \pi/2\}$ we have $(\partial^* W)_{(r,y)} = \partial^*(W_{(r,y)}) = \partial^*(\partial B(r)) = \emptyset$. Similarly,

$$\mathcal{H}^{n-1}\left(\partial^* W \cap \Phi(\{\beta_\mu = 0\}\times\mathbb{S}^1) \cap \{\nu^W_{x\|}\neq 0\}\right) = 0.$$

Step 1c: Using the decomposition (3.1), we can write $B = (B\cap\{\beta_\mu = 0\}) \cup (B\cap\{\beta_\mu = \pi/2\}) \cup (B\cap\mathcal{N})$ and treat each piece separately. Therefore, by applying **Step 1a** and **Step 1b** to $W = F_\mu$, we have

$$P_\gamma(F_\mu; \Phi(B\times\mathbb{S}^1)\cap\{\nu^{F_\mu}_{x\|}\neq 0\}) = 0.$$

In particular, by (2.9),

$$\left| e^{-\frac{|x|^2}{2}} e^{-\frac{|y|^2}{2}} D_{x\|}\chi_{F_\mu}\right| (\Phi(B\times\mathbb{S}^1)) = \int_{\partial^* F_\mu\cap\Phi(B\times\mathbb{S}^1)} e^{-\frac{|x|^2}{2}} e^{-\frac{|y|^2}{2}} |\nu^{F_\mu}_{x\|}(x)|\, d\mathcal{H}^{n-1}(x) = 0.$$

Then, by Proposition 3,

$$(2\pi)^{\frac{n-1}{2}} P_\gamma(F_\mu;\Phi(B\times\mathbb{S}^1)) \le \left| re^{-\frac{r^2}{2}} e^{-\frac{|y|^2}{2}} D\xi_\mu \right| (B) + \left| e^{-\frac{|x|^2}{2}} e^{-\frac{|y|^2}{2}} D_{x\|}\chi_{F_\mu} \right| (\Phi(B\times\mathbb{S}^1))$$
$$= \left| re^{-\frac{r^2}{2}} e^{-\frac{|y|^2}{2}} D\xi_\mu \right| (B) \le (2\pi)^{\frac{n-1}{2}} P_\gamma(E;\Phi(B\times\mathbb{S}^1)),$$

where in the last inequality we used (2.14). Notice moreover that for the set F_μ and a Borel set $B\subset((0,\infty)\times\mathbb{R}^{n-2})\setminus G_{F_\mu}$, inequality (2.14) holds as an equality.

Step 2: We prove inequality (1.5) when $B\subset G_{F_\mu}\cap G_E$. In this step, we set

$$p_E(r,y)=\mathcal{H}^0(\partial^*(E_{(r,y)})),$$

and we use a similar notation for the set F_μ.

First of all note that, thanks to Proposition 2, we have $\nu^{F_\mu}_{x\|}(x,y)\neq 0$ whenever $(|x|,y)\in G_{F_\mu}$. Therefore,

$$\begin{aligned} P_\gamma(F_\mu;\Phi(B\times\mathbb{S}^1)) &= P_\gamma(F_\mu;\Phi(B\times\mathbb{S}^1)\cap\{\nu^{F_\mu}_{x\|}\neq 0\}) \\ &= \int_{\partial^* F_\mu\cap\Phi(B\times\mathbb{S}^1)\cap\{\nu^{F_\mu}_{x\|}\neq 0\}} d\mathcal{H}^{n-1}_\gamma(x,y) \\ &= \int_B \frac{e^{-\frac{r^2+|y|^2}{2}}}{(2\pi)^{(n-1)/2}} \int_{(\partial^* F_\mu)_{(r,y)}} \frac{1}{|\nu^{F_\mu}_{x\|}(x,y)|} d\mathcal{H}^0(x)\,dr\,dy. \end{aligned}$$

Now, using the fact that

$$1=|\nu^{F_\mu}_{x\perp}(x,y)|^2+|\nu^{F_\mu}_{x\|}(x,y)|^2+|\nu^{F_\mu}_{y}(x,y)|^2=(\hat{x}\cdot\nu^{F_\mu}_{x}(x,y))^2+|\nu^{F_\mu}_{x\|}(x,y)|^2+|\nu^{F_\mu}_{y}(x,y)|^2,$$

thanks to Remark 3, for $\mathcal{H}^{n-1}$-a.e. $(r,y)\in B$ we have

$$\begin{aligned} &\int_{(\partial^* F_\mu)_{(r,y)}} \frac{1}{|\nu^{F_\mu}_{x\|}(x,y)|} d\mathcal{H}^0(x) \\ &= \int_{(\partial^* F_\mu)_{(r,y)}} \sqrt{1+\frac{(\hat{x}\cdot\nu^{F_\mu}_{x}(x,y))^2}{|\nu^{F_\mu}_{x\|}(x,y)|^2}+\frac{|\nu^{F_\mu}_{y}(x,y)|^2}{|\nu^{F_\mu}_{x\|}(x,y)|^2}} d\mathcal{H}^0(x) \\ &= p_{F_\mu}(r,y)\sqrt{1+\frac{|r\nabla_{(r,y)}\xi_\mu(r,y)|^2}{p^2_{F_\mu}(r,y)}} = \sqrt{p^2_{F_\mu}(r,y)+|r\nabla_{(r,y)}\xi_\mu(r,y)|^2} \\ &\le \sqrt{p^2_E(r,y)+|r\nabla_{(r,y)}\xi_\mu(r,y)|^2} = p_E(r,y)\sqrt{1+\frac{|r\nabla_{(r,y)}\xi_\mu(r,y)|^2}{p^2_E(r,y)}} \\ &= p_E(r,y)\sqrt{1+\left(\fint_{(\partial^* E)_{(r,y)}} \frac{\hat{x}\cdot\nu^E_x(x,y)}{|\nu^E_{x\|}(x,y)|} d\mathcal{H}^0(x)\right)^2+\left|\fint_{(\partial^* E)_{(r,y)}} \frac{\nu^E_y(x,y)}{|\nu^E_{x\|}(x,y)|} d\mathcal{H}^0(x)\right|^2}, \end{aligned} \tag{3.4}$$

thanks to Lemma 5 and Lemma 4.

Observe now that the function $f:\mathbb{R}\times\mathbb{R}^{n-2}\to[0,\infty)$ given by

$$f(t,z):=\sqrt{1+t^2+|z|^2},$$

is strictly convex. Therefore, $\mathcal{H}^{n-1}$-a.e. $(r,y) \in B$ we can apply Jensen's inequality and

$$\begin{aligned}\int_{(\partial^* F_\mu)_{(r,y)}} \frac{1}{|\nu^{F_\mu}_{x\|}(x,y)|} d\mathcal{H}^0(x) \\ \leq \int_{(\partial^* E)_{(r,y)}} \sqrt{1 + \frac{(\hat{x} \cdot \nu^E_x(x,y))^2}{|\nu^E_{x\|}(x,y)|^2} + \frac{|\nu^E_y(x,y)|^2}{|\nu^E_{x\|}(x,y)|^2}} d\mathcal{H}^0(x) \\ = \int_{(\partial^* E)_{(r,y)}} \frac{1}{|\nu^E_{x\|}(x,y)|} d\mathcal{H}^0(x).\end{aligned} \tag{3.5}$$

From this it follows that

$$(2\pi)^{(n-1)/2} P_\gamma(F_\mu; \Phi(B \times \mathbb{S}^1)) = \int_B e^{-\frac{r^2+|y|^2}{2}} \int_{(\partial^* F_\mu)_{(r,y)}} \frac{1}{|\nu^{F_\mu}_{x\|}(x,y)|} d\mathcal{H}^0(x)\, dr\, dy$$

$$\leq \int_B e^{-\frac{r^2+|y|^2}{2}} \int_{(\partial^* E)_{(r,y)}} \frac{1}{|\nu^E_{x\|}(x,y)|} d\mathcal{H}^0(x)\, dr\, dy = (2\pi)^{(n-1)/2} P_\gamma(E; \Phi(B \times \mathbb{S}^1)).$$

Step 3: We conclude the proof of the theorem. Suppose equality holds in (1.5). Then, equality holds also in (3.4) and in (3.5). From the equality in (3.4), thanks to Lemma 5, we have (i). Instead, the equality in (3.5) gives (ii). □

4. Regularity

In this section we prove Theorem 2. We will proceed in two steps, where we first prove the regularity up to a small singular set (Proposition 4). The singular set is small enough for us to use the second variation and to deduce that the boundary $\partial E \cap \{x_i > 0\}$ for every $i = 1, \dots, n$ is a graph of a BV-function. We then apply an argument similar to the one used in [6], to deduce that the singular set is empty.

The strategy for the first step is to remove the symmetry constraint at small scales and reduce the problem to the standard regularity theory for almost-minimizers of the perimeter. More precisely, starting from penalized minimality with respect to n-symmetric competitors, Lemma 6 yields an unconstrained almost-minimality property by replicating a local perturbation along the orbit of the reflection group $\mathcal{G}$. This argument is close in spirit to standard symmetry-reduction principles for invariant variational problems; for related ideas see [23, 17]. Since we did not find the precise local estimate needed here in the literature, we include a self-contained proof.

Once the symmetry constraint has been removed, the classical theory of almost-minimizers yields the $C^{1,\alpha}$ regularity of $\partial^* E$ and the estimate $\dim_{\mathcal{H}}(\Sigma_E) \leq n-8$, where $\dim_{\mathcal{H}}$ denotes the Hausdorff dimension. We then use symmetric criticality (see [25]) to derive the Euler–Lagrange equation away from the coordinate hyperplanes, and finally extend the analyticity across the hyperplanes by reflection argument.

The first lemma deals with the symmetry of the problem.

Lemma 6. *Let $E \subset \mathbb{R}^n$ be an n-symmetric set of finite Gaussian perimeter. Let $p \in \partial E$. Define the index set $I := \{i \in \{1, \dots, n\} : p \in \Pi_i\}$, where $\Pi_i := \{x \in \mathbb{R}^n : x_i = 0\}$ are the coordinate hyperplanes. Let $\mathcal{G}$ be the reflection group of order 2^n generated by the reflections $\{A_1, \dots, A_n\}$ across the coordinate hyperplanes. Let $r > 0$ be small enough such that $B_r(p) \cap \Pi_j = \emptyset$ for all $j \notin I$. Assume that for every n-symmetric set W such that $W \triangle E \subset\subset \bigcup_{g \in \mathcal{G}} g(B_r(p))$ the following control holds: there exists $\Lambda > 0$ such that*

$$P_\gamma(E; B_r(p)) \leq P_\gamma(W; B_r(p)) + \Lambda \gamma(W \triangle E). \tag{4.1}$$

Then, for every perturbation F of E in $B_r(p)$, i.e., another set such that $F \triangle E \subset\subset B_r(p)$, one has

$$P_\gamma(E; B_r(p)) \le P_\gamma(F; B_r(p)) + 2^n \Lambda\, \gamma(F \triangle E). \tag{4.2}$$

Proof. Let $B = B_r(p)$ and let F be any set of finite Gaussian perimeter with $F \triangle E \subset\subset B$. We have to construct a globally n-symmetric variant W of F, keeping a certain control inside B.

If $I = \emptyset$, i.e., if p does not belong to any coordinate hyperplane, B is disjoint from its reflections. The modification can be easily done by substituting F with its symmetral inside the reflections of B.

We assume henceforth that $I \neq \emptyset$. We follow an orbit averaging strategy. We decompose $\mathcal{G}$ based on the local geometry at p:

(i) *The Stabilizer:* $\mathcal{G}_{\text{stab}} \coloneqq \{g \in \mathcal{G} : g(p) = p\}$. This subgroup is generated by $\{A_i\}_{i\in I}$ and has order $|\mathcal{G}_{\text{stab}}| = 2^{|I|}$.

(ii) *The Orbit Representatives:* Let $\{g_1, \dots, g_N\}$ be a set of representatives for the cosets $\mathcal{G}/\mathcal{G}_{\text{stab}}$, where $N = 2^{n-|I|}$.

Note that the images $\{g_m(B)\}_{m=1}^N$ are pairwise disjoint, as they lie in distinct orthants not separated by any Π_j with $j \in I$.

Step 1: Local Symmetrization. The hyperplanes $\{\Pi_i\}_{i\in I}$ partition the ball B into $|\mathcal{G}_{\text{stab}}|$ disjoint sectors. Let $\mathcal{S}$ denote this collection of sectors. For each sector $S \in \mathcal{S}$, we construct a locally symmetric candidate F_S on B by reflecting the restriction of F to S:

$$F_S \coloneqq \bigcup_{h \in \mathcal{G}_{\text{stab}}} h(F \cap S).$$

By construction, F_S is symmetric with respect to the hyperplanes $\{\Pi_i\}_{i\in I}$. Using the $\mathcal{G}_{\text{stab}}$-invariance of the Gaussian weight and the sub-additivity of the perimeter, we obtain:

$$\sum_{S\in\mathcal{S}} P_\gamma(F_S; B) \le |\mathcal{G}_{\text{stab}}|\, P_\gamma(F; B). \tag{4.3}$$

For each sector S, we define a set W_S by replacing E with F_S inside B, and replicating this modification to all other balls in the orbit:

$$W_S \coloneqq \left(E \setminus \bigcup_{m=1}^N g_m(B)\right) \cup \bigcup_{m=1}^N g_m(F_S).$$

Since E is globally n-symmetric and F_S is symmetric under $\mathcal{G}_{\text{stab}}$, the set W_S is invariant under the full group $\mathcal{G}$, i.e., it is an n-symmetric set.

Step 2: Perimeter control. By the hypothesis (4.1), applied to the competitor W_S, we have

$$P_\gamma(E; B) \le P_\gamma(W_S; B) + \Lambda\gamma(W_S \triangle E). \tag{4.4}$$

Note that $P_\gamma(W_S; B) = P_\gamma(F_S; B)$. We now explicitly compute the symmetric difference term. Since the modification is replicated N times on pairwise disjoint balls and E is $\mathcal{G}$-invariant,

$$\gamma(W_S \triangle E) = \sum_{m=1}^N \gamma(g_m(F_S) \triangle E) = N\gamma(F_S \triangle (E \cap B)).$$

Inside B, F_S consists of $|\mathcal{G}_{stab}|$ copies of $F \cap S$ reflected by elements of $\mathcal{G}_{stab}$. Since $E \cap B$ is also symmetric with respect to $\mathcal{G}_{stab}$, the symmetric difference on each reflected sector is identical to the difference on the generating sector S. Thus:

$$\gamma(F_S \triangle (E \cap B)) = |\mathcal{G}_{stab}|\gamma((F \triangle E) \cap S).$$

Inserting this into (4.4), we obtain the local estimate for the sector S:

$$P_\gamma(E;B) \le P_\gamma(F_S;B) + N\Lambda|\mathcal{G}_{stab}|\gamma((F\triangle E)\cap S).$$

Summing this inequality over all sectors $S\in\mathcal{S}$:

$$\begin{aligned}|\mathcal{G}_{\rm stab}|\, P_\gamma(E;B) &\le \sum_{S\in\mathcal{S}} P_\gamma(F_S;B) + N\Lambda \sum_{S\in\mathcal{S}} |\mathcal{G}_{\rm stab}|\,\gamma\big((F\triangle E)\cap S\big)\\ &\le |\mathcal{G}_{\rm stab}|\, P_\gamma(F;B) + |\mathcal{G}_{\rm stab}|\, N\Lambda\,\gamma(F\triangle E),\end{aligned}$$

where we used (4.3) and the fact that $\sum_{S\in\mathcal{S}}\gamma((F\triangle E)\cap S) = \gamma(F\triangle E)$. Dividing by $|\mathcal{G}_{\rm stab}|$ we get (4.2). □

We may proceed to prove the partial regularity.

Proposition 4. *Let E be a minimizer of the Gaussian isoperimetric problem among n-symmetric sets with prescribed Gaussian measure. Then the reduced boundary ∂^*E is a relatively open, real analytic hypersurface, and the singular set $\Sigma_E = \partial E\setminus\partial^*E$ satisfies* $\dim_{\mathcal{H}}(\Sigma_E)\le n-8$.

Proof. We divide the proof into four main steps, and we will use the notation introduced in Lemma 6 (and in its proof) for the reflection group $\mathcal{G}$.

Step 1: $C^{1,\alpha}$ regularity and dimension of the singular set. Let $p\in\partial E$ and let $I \coloneqq \{i\in\{1,\dots,n\} : p\in\Pi_i\}$. Set $r_p>0$ as the distance between p and $\bigcup_{j\notin I}\Pi_j$.

Since E is a minimizer of the Gaussian perimeter among n-symmetric sets subject to a volume constraint, we first observe that E satisfies a penalized minimality condition within the class of symmetric sets. By performing symmetric volume-fixing variations supported in $\bigcup_{g\in\mathcal{G}} g(B_r(p))$ (adapting standard arguments, see e.g. [21, Part III]), we deduce that there exists a constant $\Lambda>0$ such that for every $r<r_p$ and for every n-symmetric competitor W with $W\Delta E\subset\subset \bigcup_{g\in\mathcal{G}} g(B_r(p))$,

$$P_\gamma(E;B_r(p)) \le P_\gamma(W;B_r(p)) + \Lambda\gamma(W\triangle E).$$

Applying Lemma 6, we remove the symmetry constraint: for any set F (not necessarily symmetric) such that $F\Delta E\subset\subset B_r(p)$,

$$P_\gamma(E;B_r(p)) \le P_\gamma(F;B_r(p)) + 2^n\Lambda\gamma(F\triangle E). \tag{4.5}$$

Following [2, Proposition 2], from (4.5) we get that E is an almost-minimizer of the *Euclidean* perimeter: there exists $\tilde\Lambda>0$, depending on $|p|$, such that

$$P(E;B_r(p)) \le P(F;B_r(p)) + \tilde\Lambda r^n.$$

We can now apply the classical regularity theory for almost-minimizers (see [21, Theorem 21.8 and Theorem 28.1]). Let us define, for every $p\in\mathbb{R}^n$,

$$\overline{u}_E(p) \coloneqq \limsup_{r\to 0}\frac{|E\cap B_r(p)|}{|B_r(p)|}, \quad \text{and } \underline{u}_E(p) \coloneqq \liminf_{r\to 0}\frac{|E\cap B_r(p)|}{|B_r(p)|},$$

and let us use the notation $u_E(p)$ if the liminf and the limsup above coincide. We assume that E coincides with the precise representative

$$\{p\in\mathbb{R}^n : u_E(p) = 1\}. \tag{4.6}$$

Then, thanks to the density estimate on ∂E for almost-minimizers (see [21, Theorem 21.11]), we have that the precise representative $E=\{u_E=1\}$ and its essential complement $\{u_E=0\}$ are both open, and $\partial E\subset\mathbb{R}^n\setminus(\{u_E=1\}\cup\{u_E=0\})$. The *topological* boundary ∂E coincides with the *essential* boundary $\{p\in\mathbb{R}^n : u_E(p)\notin\{0,1\}\}$ and it is partitioned in two parts:

(i) the *reduced* boundary ∂^*E, that is a subset of $\{p\in\mathbb{R}^n : u_E(p)=1/2\}$, is a $C^{1,\alpha}$ hypersurface for every $\alpha<1/2$, and is relatively open in ∂E.

(ii) the singular set $\Sigma_E = \partial E \setminus \partial^* E$; it satisfies $\dim_{\mathcal{H}}(\Sigma_E) \le n-8$.

Step 2: Lagrange multiplier for symmetric variations. Let $X \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$ be a vector field. Let $0<\delta\ll 1$, and let $\Phi : \mathbb{R}^n \times (-\delta,\delta) \to \mathbb{R}^n$ be the flow associated with X, i.e.,

$$\frac{\partial}{\partial t}\Phi(x,t) = X(\Phi(x,t)), \qquad \Phi(x,0) = x.$$

We perturb E through the flow Φ by defining $E_t \coloneqq \Phi(E,t)$ for $t \in (-\delta,\delta)$. The first variations of the perimeter and the volume are given by

$$\begin{aligned} \frac{d}{dt}P_\gamma(E_t)\Big|_{t=0} &= \int_{\partial^* E} (\mathscr{H}_E - x\cdot\nu^E)\,(X\cdot\nu^E)\,d\mathcal{H}^{n-1}_\gamma \\ \frac{d}{dt}\gamma(E_t)\Big|_{t=0} &= \frac{1}{\sqrt{2\pi}}\int_{\partial^* E}(X\cdot\nu^E)\,d\mathcal{H}^{n-1}_\gamma. \end{aligned}$$

If Φ preserves n-symmetry, since E minimizes the Gaussian perimeter among n-symmetric sets of finite perimeter subject to the volume constraint $\gamma(E)=m$, by the method of Lagrange multipliers there exists $\lambda\in\mathbb{R}$ such that

$$\frac{d}{dt}P_\gamma(E_t)\Big|_{t=0} = \lambda\frac{d}{dt}\gamma(E_t)\Big|_{t=0}. \tag{4.7}$$

Step 3: Extension to general variations via symmetric criticality. Recall that $|\mathcal{G}| = 2^n$. Both the Gaussian perimeter and the Gaussian volume are $\mathcal{G}$-invariant: for every $g\in\mathcal{G}$ and every set F of finite perimeter,

$$P_\gamma(g\cdot F) = P_\gamma(F), \qquad \gamma(g\cdot F) = \gamma(F).$$

Since E and the Gaussian weight $e^{-|x|^2/2}$ are $\mathcal{G}$-invariant, a change of variables gives that the first variation is invariant with respect to $\mathcal{G}$.

Now let $X \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$ be an arbitrary vector field supported away from the coordinate hyperplanes. Define the symmetrized field

$$\hat{X}(x) \coloneqq \frac{1}{|\mathcal{G}|}\sum_{g\in\mathcal{G}} g^{-1}\cdot X(g\cdot x).$$

By construction, $\hat{X}$ is $\mathcal{G}$-equivariant and generates a flow preserving n-symmetry. Let $\hat{E}_t$ be the perturbation of E through the flow associated to $\hat{X}$.

Using the fact that X is supported away from the coordinate hyperplanes and the $\mathcal{G}$-invariance,

$$\frac{d}{dt}P_\gamma(\hat{E}_t)\Big|_{t=0} = \frac{d}{dt}P_\gamma(E_t)\Big|_{t=0},$$

and similarly for the volume term. Since $\hat{X}$ preserves n-symmetry, equation (4.7) holds for $\hat{X}$ and therefore also for X. Since this is true for any X supported away from the coordinate hyperplanes, we conclude that pointwise

$$\mathscr{H}_E - \nu^E\cdot x = \frac{\lambda}{\sqrt{2\pi}} \qquad \text{on } \partial^* E \setminus \bigcup_{i=1}^n \Pi_i, \tag{4.8}$$

where $\mathscr{H}_E$ denotes the mean curvature.

Step 4: Upgrade to analyticity. Away from the coordinate hyperplanes, since $\partial^* E$ satisfies the Euler-Lagrange equation (4.8), standard elliptic regularity (see [24, Chapter 6]) upgrades $C^{1,\alpha}$ regularity to real analyticity, i.e., $\partial^* E \setminus \bigcup_{i=1}^n \Pi_i$ is a real analytic hypersurface.

We now extend the analyticity across the coordinate hyperplanes by a standard multi-fold reflection argument. Let $p \in \partial^* E \cap \bigcup_{i=1}^n \Pi_i$ and let $I = \{i : p \in \Pi_i\}$, $k \coloneqq |I|$. By n-symmetry, $\nu^E(p)\cdot e_i = 0$ for every $i\in I$, so $k \le n-1$ and, after relabeling, rotating and translating within

$\mathrm{span}(e_{k+1},\dots,e_n)$, we may write $\partial^* E$ near p as the graph $z_n = u(z')$ with $u \in C^{1,\alpha}(B'_\rho(0))$ even in each of $z_1,\dots,z_k$, where ρ is sufficiently small and $B'_\rho(0)$ denotes the open ball of $\mathbb{R}^{n-1}$ centered in 0 with radius ρ. The even extension $\tilde u(z') := u(|z_1|,\dots,|z_k|,z_{k+1},\dots,z_{n-1})$ belongs to $C^{1,\alpha}(B'_\rho(0))$, because evenness forces homogeneous Neumann conditions $\partial_{z_i}u|_{z_i=0} = 0$ for $i \le k$. Therefore we deduce from (4.8) that the function u is a $C^{1,\alpha}$-regular weak solution of the equation

$$\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) - \frac{(z',u)\cdot(\nabla u,-1)}{\sqrt{1+|\nabla u|^2}} = \frac{\lambda}{\sqrt{2\pi}} \tag{4.9}$$

in $B'_\rho(0)$ in the sense of integration by parts against a smooth test function. By standard elliptic regularity results (see e.g. [24, Theorem 6.7.6]) it is analytic. $\square$

Remark 4. From now on, we will always identify a minimizer E with the open set $\{u_E = 1\}$. Therefore, we have that $\{u_E = 0\} = \mathbb{R}^n \setminus \overline{E}$ is also open and

$$\partial E = \partial^* E \cup \Sigma_E,$$

where $\partial^* E$ is (analytic and) relatively open in ∂E, while Σ_E is relatively closed in ∂E, and thus also closed in $\mathbb{R}^n$.

We introduce some notation for calculus on smooth hypersurfaces. Let M be a C^∞ hypersurface. We may extend a given vector field $X \in C^\infty(M;\mathbb{R}^n)$ to $\mathbb{R}^n$ and denote by $D_M X$ the tangential differential of X on M and for a real valued function $u \in C^\infty(M)$ we denote the tangential gradient as $\nabla_M u$. The tangential divergence of X on M is defined by $\mathrm{div}_M X = \mathrm{tr}(D_M X)$ and the Laplace-Beltrami of operator u on M is

$$\Delta_M u := \mathrm{div}_M(D_M u).$$

When M is the boundary of a set E, the exterior unit normal ν^E fixed above is a smooth vector field in $C^\infty(M;\mathbb{R}^n)$. Then the second fundamental form is $B_E = D_{\partial E}\nu^E$ and the mean curvature $\mathscr{H}_E$ is

$$\mathscr{H}_E = \mathrm{div}_{\partial E}(\nu^E).$$

Finally, the divergence theorem on hypersurfaces in the Gaussian setting states that for every Lipschitz continuous vector field $X : \partial E \to \mathbb{R}^n$ with compact support, using the notation $X_{\partial E} = X - (\nu^E \otimes \nu^E)X$, it holds

$$\int_{\partial E} (\mathrm{div}_{\partial E} X - X_{\partial E}\cdot x)\, d\mathcal{H}^{n-1}_\gamma(x) = \int_{\partial E} \mathscr{H}_E X \cdot \nu^E \, d\mathcal{H}^{n-1}_\gamma(x).$$

The drift Laplacian (or the surface Ornstein–Uhlenbeck operator) on M associated with the Gaussian measure is defined as

$$\Delta_M \varphi - \nabla_M \varphi \cdot x, \quad \varphi \in C^2(M).$$

Integration by parts with respect to the Gaussian measure $\mathcal{H}^{n-1}_\gamma$ yields

$$\int_M |D_M\varphi|^2 \, d\mathcal{H}^{n-1}_\gamma = -\int_M \varphi(\Delta_M\varphi - \nabla_M\varphi\cdot x)\, d\mathcal{H}^{n-1}_\gamma, \quad \varphi \in C^2_c(M). \tag{4.10}$$

The following is the standard stability inequality for the Gaussian isoperimetric problem (see [2, Proposition 3]), adapted to the n-symmetric class via the principle of symmetric criticality.

Proposition 5. *Let $E \subset \mathbb{R}^n$ be a minimizer of the Gaussian isoperimetric problem among n-symmetric sets with prescribed Gaussian measure. Set $M = \partial^* E$. Define the quadratic form $Q : C_c^\infty(M) \to \mathbb{R}$ by*

$$Q(\varphi) := \int_M \Big(|\nabla_M \varphi|^2 \ - \ \big(|B_E|^2 + 1\big)\varphi^2 \Big)\, d\mathcal{H}_\gamma^{n-1}.$$

Then

$$Q(\varphi) \geq 0 \tag{4.11}$$

for every n-symmetric $\varphi \in C_c^\infty(M)$ satisfying $\int_M \varphi \, d\mathcal{H}_\gamma^{n-1} = 0$.

The following result is a direct adaptation of [2, Lemma 1]. Indeed, the cutoff construction used there relies only on the finiteness of $P_\gamma(E)$, the local perimeter-growth estimate, and

$$\mathcal{H}^{n-3}(\partial E \setminus \partial^* E) = 0,$$

all of which hold in the present setting. Averaging over the finite reflection group $\mathcal{G}$, in order to make the cutoffs n-symmetric, yields the following lemma.

Lemma 7. *Let E be as in Proposition 4, and set $M = \partial^* E$. There exists a sequence of n-symmetric functions*

$$\zeta_h \in C_c^\infty(M), \qquad 0 \leq \zeta_h \leq 1,$$

such that

$$\int_M \big((1-\zeta_h)^2 + |\nabla_M \zeta_h|^2\big)\, d\mathcal{H}_\gamma^{n-1} \longrightarrow 0.$$

Remark 5. Let $H^1_\gamma(M)$ be the Sobolev space of functions which are square-integrable with respect to the measure $\mathcal{H}_\gamma^{n-1}$ in M, together with their weak first derivatives. Lemma 7, combined with truncation and local smooth approximation, implies that

$$H^1_\gamma(M) = H^1_{\gamma,0}(M) := \overline{C_c^\infty(M)}^{H^1_\gamma(M)}.$$

Moreover, Proposition 5 can be extended to all n-symmetric $H^1_\gamma(M)$ functions with zero average.

We may use the second variation condition to deduce some information about the global structure of the minimizers. The first result deals with the connected components of $\partial^* E$.

Proposition 6. *Let $E \subset \mathbb{R}^n$ be a minimizer of the Gaussian isoperimetric problem among n-symmetric sets with prescribed Gaussian measure. Then one of the following holds:*

(i) $\partial^* E$ is connected.

(ii) $\partial^* E$ consists of exactly two parallel hyperplanes, one being the reflection of the other one across a coordinate hyperplane (in which case E is a symmetric slab or its complement).

Proof. Let $\mathcal{G}$ be the reflection group of order 2^n generated by the reflections $\{A_1, \ldots, A_n\}$ across the coordinate hyperplanes. We first show that $\mathcal{G}$ acts transitively on the set of connected components $\mathscr{C}$ of $\partial^* E$.

Assume, by contradiction, that $\mathcal{G}$ does not act transitively on the set $\mathscr{C}$ of connected components of $M = \partial^* E$. Then $\mathscr{C}$ can be partitioned into two nonempty $\mathcal{G}$-invariant subcollections, whose unions we denote by Γ_A and Γ_B. Define the test function

$$\varphi := \mathcal{H}_\gamma^{n-1}(\Gamma_B)\chi_{\Gamma_A} - \mathcal{H}_\gamma^{n-1}(\Gamma_A)\chi_{\Gamma_B}.$$

We have $\varphi \in H^1_\gamma(M)$ and $\nabla_M \varphi = 0$ a.e. on M. Moreover, φ is n-symmetric and $\int_M \varphi \, d\mathcal{H}_\gamma^{n-1} = 0$. Thus Remark 5 allows us to use φ in the stability inequality. We obtain

$$\int_M (|B_E|^2 + 1)\varphi^2 \, d\mathcal{H}_\gamma^{n-1} \leq \int_M |\nabla_M \varphi|^2 \, d\mathcal{H}_\gamma^{n-1} = 0.$$

On the other hand,

$$\int_M (|B_E|^2 + 1)\varphi^2 \, d\mathcal{H}^{n-1}_\gamma \geq \int_M \varphi^2 \, d\mathcal{H}^{n-1}_\gamma > 0,$$

which is a contradiction. Hence $\mathcal{G}$ acts transitively on $\mathscr{C}$.

Let N be the number of connected components. By transitivity, $N = 2^k$ for some $k \in \{0, \dots, n\}$. If $N = 1$, we are in case (i).

Assume $N \geq 2$. The stabilizer $\mathcal{G}_\Gamma$ of any component $\Gamma \in \mathscr{C}$ is a proper subgroup of $\mathcal{G}$. Thus, there exists an index $j \in \{1, \dots, n\}$ such that the reflection $A_j \notin \mathcal{G}_\Gamma$. We can assume without loss of generality that $j = 1$. This implies A_1 moves every component to a disjoint component, so no component can intersect its fixed-point space $\{x_1 = 0\}$. Consequently, $\partial^* E \cap \{x_1 = 0\} = \emptyset$. By Remark 4, E is open. Therefore, recalling that $\partial E = \partial^* E \cup \Sigma_E$, we have

$$\{x_1 = 0\} \setminus \Sigma_E \subset E \cup (\mathbb{R}^n \setminus \overline{E}).$$

Since $\dim_{\mathcal{H}}(\Sigma_E) \leq n - 8$, the punctured hyperplane $\{x_1 = 0\} \setminus \Sigma_E$ is connected. Because E and $\mathbb{R}^n \setminus \overline{E}$ are open in $\mathbb{R}^n$, their intersections with $\{x_1 = 0\}$ form a partition of a connected topological space by two disjoint, relatively open sets. Thus,

$$\text{either} \quad \{x_1 = 0\} \setminus \Sigma_E \subset E \quad \text{or} \quad \{x_1 = 0\} \setminus \Sigma_E \subset \mathbb{R}^n \setminus \overline{E}.$$

Assume $\{x_1 = 0\} \setminus \Sigma_E \subset \mathbb{R}^n \setminus \overline{E}$ (the other case follows from similar arguments). Then $S := E \cap \{x_1 > 0\}$ is a set of finite perimeter with no reduced boundary on $\{x_1 = 0\}$. By the n-symmetry of E and the fact that $\partial^* E \cap \{x_1 = 0\} = \emptyset$, both the Gaussian measure and perimeter split evenly: $\gamma(S) = \frac{1}{2}\gamma(E)$ and $P_\gamma(S) = \frac{1}{2}P_\gamma(E)$.

Let $I(m)$ be the Gaussian perimeter of a half-space of Gaussian mass m. By the standard Gaussian isoperimetric inequality, $P_\gamma(S) \geq I(\gamma(S))$, with equality if and only if S is a half-space.

We claim this must hold with equality. Suppose, for contradiction, that the inequality is strict: $P_\gamma(S) > I(\gamma(S))$. Consider the half-space $H_a := \{x \in \mathbb{R}^n : x_1 > a\}$, with $a > 0$ be the chosen so that $\gamma(H_a) = \gamma(S)$. Define the competitor set $W := H_a \cup A_1(H_a)$, which is the complement of a symmetric slab. Since $a > 0$, the half-spaces H_a and $A_1(H_a)$ are essentially disjoint. Therefore, W preserves the mass of E:

$$\gamma(W) = 2\gamma(H_a) = 2\gamma(S) = \gamma(E).$$

Since W is n-symmetric, it is an admissible competitor. This contradicts the assumption that E is a global minimizer of the Gaussian perimeter among n-symmetric sets, since

$$P_\gamma(W) = 2P_\gamma(H_a) = 2I(\gamma(S)) < 2P_\gamma(S) = P_\gamma(E).$$

Therefore, $P_\gamma(S) = I(\gamma(S))$, and S must coincide with H_a. By symmetry, E coincides with W, and its reduced boundary consists of exactly two hyperplanes $\{x_1 = \pm a\}$, corresponding to case (ii). □

The second result on the global structure of the minimizers is a dichotomy related to the components of the normal ν^E. For the rest of this section $x = (x_1, \dots, x_n)$ is a generic point in $\mathbb{R}^n$ (we pause the notation (x, y) with $x \in \mathbb{R}^2$ and $y \in \mathbb{R}^{n-2}$).

Lemma 8. *Let $E \subset \mathbb{R}^n$ be a minimizer of the Gaussian isoperimetric problem among n-symmetric sets with prescribed Gaussian measure. Given $i \in \{1, \dots, n\}$, set $\nu_i := \nu^E \cdot e_i$, and $M_i^+ := \partial^* E \cap \{x \in \mathbb{R}^n : x_i > 0\}$. Then one of the following holds:*

(i) $\nu_i \equiv 0$ on $\partial^* E$. In this case E is a cylinder in the direction e_i: there exists a Borel set $E' \subset \mathbb{R}^{n-1}$ such that $E = \{x \in \mathbb{R}^n : (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in E'\}$.

(ii) ν_i does not change sign on M_i^+ (either $\nu_i(x) > 0$ for every $x \in M_i^+$, or $\nu_i(x) < 0$ for every $x \in M_i^+$).

Proof. Without loss of generality, let us assume $i = n$. Set $M := \partial^* E$ and $M^+ := M_n^+$.

Step 1: Slab case. By Proposition 6, either M is connected or E is a symmetric slab (or its complement). In the slab case, $E = \{|x_j| < a\}$ for some $j \in \{1, \dots, n\}$ and some $a > 0$. If $j \neq n$ one has the alternative (i), while if $j = n$ one has the alternative (ii). Henceforth for the remainder of the proof we can assume that M is connected.

Step 2: M^+ is connected. By Proposition 4, M is real-analytic and, by symmetry, it meets each coordinate hyperplane Π_k orthogonally; hence $M \cap \{x \in \mathbb{R}^n : x_n \geq 0\}$ is a topological manifold with corners. The continuous folding map $x \mapsto (x_1, \dots, x_{n-1}, |x_n|)$ from M onto $M \cap \{x_n \geq 0\}$ is surjective, so the latter is connected. By symmetry, it then follows that also its relative interior $M^+ = M \cap \{x_n > 0\}$ is connected.

Step 3: ν_n does not change sign on M^+. Suppose, for contradiction, that both sets

$$\{\nu_n > 0\} \cap M^+, \qquad \{\nu_n < 0\} \cap M^+$$

are nonempty. Define on M^+

$$u_+ := (\nu_n)^+, \qquad u_- := (\nu_n)^- = \max\{-\nu_n, 0\}.$$

Since E is invariant under reflection across $\{x_n = 0\}$, and M meets this hyperplane orthogonally, we have

$$\nu_n = 0 \qquad \text{on } M \cap \{x_n = 0\}.$$

We may therefore extend u_+ and u_- evenly across $\{x_n = 0\}$, retaining the same notation for the extended functions. They are locally Lipschitz, n-symmetric, nonzero, and satisfy

$$0 \leq u_\pm \leq 1, \qquad u_+ u_- = 0.$$

Let

$$e_n^T := e_n - \nu_n \nu^E, \qquad x^T := x - (x \cdot \nu^E)\nu^E$$

denote the tangential projections of e_n and x, respectively. Differentiating the Euler–Lagrange equation in the direction e_n^T gives

$$\nabla_M \mathscr{H}_E \cdot e_n = D_{e_n^T}(x \cdot \nu^E) = x \cdot D_{e_n^T} \nu^E,$$

because $e_n^T \cdot \nu^E = 0$. By the symmetry of the second fundamental form,

$$x \cdot D_{e_n^T} \nu^E = B_E(e_n^T, x^T) = B_E(x^T, e_n^T) = D_{x^T} \nu^E \cdot e_n = x \cdot \nabla_M \nu_n.$$

Combining the well-known identity $\Delta_M \nu^E + |B_E|^2 \nu^E = \nabla_M \mathscr{H}_E$ (see, e.g., [16, formula (10.17)]) with the previous ones, yields

$$\Delta_M \nu_n - x \cdot \nabla_M \nu_n + |B_E|^2 \nu_n = 0 \ \text{ on } \ M. \tag{4.12}$$

Let ζ_h be given by Lemma 7. Since $u_\pm \in H^1_{\mathrm{loc}}(M^+)$ and have zero trace on $M \cap \{x_n = 0\}$, the functions

$$\zeta_h^2 u_+, \qquad -\zeta_h^2 u_-$$

belong to $H^1_0(M^+)$. Therefore, by density of $C_c^\infty(M^+)$ in $H^1_0(M^+)$, the integration-by-parts identity associated with (4.10) applies to these test functions. Testing the weak formulation of (4.12) with $\psi = \zeta_h^2 u_+$ and $\psi = -\zeta_h^2 u_-$, respectively, we obtain

$$\int_{M^+} \left(|\nabla_M(\zeta_h u_\pm)|^2 - |B_E|^2 \zeta_h^2 u_\pm^2 \right) d\mathcal{H}^{n-1}_\gamma = \int_{M^+} u_\pm^2 |\nabla_M \zeta_h|^2 \, d\mathcal{H}^{n-1}_\gamma.$$

All the quantities involved are supported in the compact set $\operatorname{supp} \zeta_h \subset M$, where the second fundamental form is bounded. After even reflection across $\{x_n = 0\}$, both sides are multiplied by 2, and hence

$$\int_{M} \left(|\nabla_M(\zeta_h u_\pm)|^2 - |B_E|^2 \zeta_h^2 u_\pm^2 \right) d\mathcal{H}^{n-1}_\gamma = \int_{M} u_\pm^2 |\nabla_M \zeta_h|^2 \, d\mathcal{H}^{n-1}_\gamma. \tag{4.13}$$

Set

$$a_h := \int_M \zeta_h u_+ \, d\mathcal{H}^{n-1}_\gamma, \qquad b_h := \int_M \zeta_h u_- \, d\mathcal{H}^{n-1}_\gamma.$$

Since u_+ and u_- are nonzero, $a_h, b_h > 0$ for all sufficiently large h. Define

$$c_h := \frac{a_h}{b_h}, \qquad \varphi_h := \zeta_h (u_+ - c_h u_-).$$

Then φ_h is n-symmetric, has compact support in M, and

$$\int_M \varphi_h \, d\mathcal{H}^{n-1}_\gamma = 0.$$

By remark 5 we can apply the stability inequality (4.11) to φ_h. By the Sobolev chain rule, on M^+ one has

$$\nabla_M u_+ = \chi_{\{\nu_n>0\}} \nabla_M \nu_n, \qquad \nabla_M u_- = -\chi_{\{\nu_n<0\}} \nabla_M \nu_n \quad \text{a.e.}$$

The same orthogonality is preserved by even reflection. Consequently,

$$u_+ u_- = 0, \qquad \nabla_M u_+ \cdot \nabla_M u_- = 0 \quad \text{a.e. on } M,$$

and all the cross terms vanish. Using (4.13), we obtain

$$Q(\varphi_h) = \int_M (u_+^2 + c_h^2 u_-^2) |\nabla_M \zeta_h|^2 \, d\mathcal{H}^{n-1}_\gamma - \int_M \zeta_h^2 (u_+^2 + c_h^2 u_-^2) \, d\mathcal{H}^{n-1}_\gamma. \tag{4.14}$$

Since $\zeta_h \to 1$ in $L^2_\gamma(M)$ and the sequence (c_h) is bounded,

$$\int_M (u_+^2 + c_h^2 u_-^2) |\nabla_M \zeta_h|^2 \, d\mathcal{H}^{n-1}_\gamma \le (1 + c_h^2) \int_M |\nabla_M \zeta_h|^2 \, d\mathcal{H}^{n-1}_\gamma \longrightarrow 0,$$

while

$$\int_M \zeta_h^2 (u_+^2 + c_h^2 u_-^2) \, d\mathcal{H}^{n-1}_\gamma \longrightarrow \int_M (u_+^2 + c^2 u_-^2) \, d\mathcal{H}^{n-1}_\gamma > 0.$$

It follows from (4.14) that $Q(\varphi_h) < 0$ for all sufficiently large h, contradicting the stability inequality. Therefore, ν_n cannot change sign on M^+.

Step 4: Strong maximum principle. Let $\sigma \in \{-1, 1\}$ be such that $v := \sigma \nu_n \ge 0$ on M^+. Since v satisfies the same linear equation

$$\Delta_M v - x \cdot \nabla_M v + |B_E|^2 v = 0 \qquad \text{on } M^+,$$

the strong maximum principle implies that either $v > 0$ on M^+ or $v \equiv 0$ there. Thus either ν_n has a strict sign on M^+ or $\nu_n \equiv 0$ on M^+. De Giorgi's structure theorem (see [21, Proposition II.4.9]) yields

$$D\chi_E = -\nu^E \mathcal{H}^{n-1} \llcorner M.$$

Therefore, in the case $\nu_n \equiv 0$, the distributional partial derivative $D_n \chi_E$ is zero and necessarily $E = E' \times \mathbb{R}$ for some set E' of locally finite perimeter in $\mathbb{R}^{n-1}$. □

We may now proceed to the proof of Theorem 2.

Proof of Theorem 2. If $n \le 7$, then Proposition 4 implies that E is analytic. So we can assume $n \ge 8$. After relabeling the coordinates and iterating the cylindrical alternative in Lemma 8, exactly one of the following occurs:

(i) *Cylindrical case.* There exist $k \in \{1, \dots, n-1\}$ and $E' \subset \mathbb{R}^{n-k}$ such that

$$E = E' \times \mathbb{R}^k;$$

(ii) *Non-cylindrical case.* For every $i \in \{1, \dots, n\}$ the component ν_i does not change sign on $\partial^* E \cap \{x \in \mathbb{R}^n : x_i > 0\}$.

We first consider the cylindrical case. The set E' is a minimizer of the Gaussian isoperimetric problem among $(n-k)$-symmetric sets in $\mathbb{R}^{n-k}$. By adapting to the n-symmetric case Heilman's strategy, either E' or $(E')^c$ is convex. If an almost minimizer of the perimeter E', or its complement, is convex then it is well known that the singular set is empty (see e.g. [14, Lemma 3]).

We henceforth assume that E is not cylindrical in any coordinate direction. We want initially to show that the set $\partial E \cap \{x_i > 0\}$ is the graph, in the e_i-direction, of an extended function u_i. Consider, for instance, the direction $i = n$. We define

$$U_n \coloneqq \Pi_n(E) \subset \mathbb{R}^{n-1},$$

which is open because E is open. By studying E or its complement, we may assume that $\nu_n > 0$ on $\partial^* E \cap \{x_n > 0\}$. This means that the partial derivative $\partial_{x_n}\chi_E$ is a non-positive Radon measure. Therefore, for a.e. $x' \in \mathbb{R}^{n-1}$ the function $t \mapsto \chi_E(x', t)$ is monotone for $t \geq 0$. In particular, for a.e. $x' \in U_n$ the section $\{t \geq 0 : (x', t) \in E\}$ is an interval of the form

$$[0, u_n(x')).$$

We then define

$$\begin{aligned} u_n(x') &\coloneqq \sup\{t \geq 0 : (x', t) \in \partial E\}, \qquad \text{and} \\ v_n(x') &\coloneqq \inf\{t \geq 0 : (x', t) \in \partial E\} \qquad x' \in U_n. \end{aligned} \tag{4.15}$$

It is immediate that u_n is upper semicontinuous and v_n lower semincontinuous.

For $\eta > 0$ we denote $U_\eta = \{x' \in U_n : \eta < v_n(x') \leq u_n(x') < \frac{1}{\eta}\}$. Since u_n is upper semicontinuous and v_n lower semicontinuous U_η is an open set. Our aim is to show that u_n is locally Lipschitz continuous in U_η for every η. To this aim we fix $\hat{x} \in U_\eta$ and choose $0 < \delta < \eta/8$ such that $B_\delta(\hat{x}) \subset U_\eta$.

We denote further the set

$$M_\delta \coloneqq \{(x', x_n) \in \partial^* E : x' \in B_{\frac{\delta}{4}}(\hat{x}), x_n > 0\}.$$

Then by construction it holds for $x = (x', x_n) \in M_\delta$ that $8\delta < v_n(x') \leq x_n \leq u_n(x') < \frac{1}{8\delta}$. In particular, $M_\delta \subset \{x_n > 0\}$. We prove that for all $x \in M_\delta$ it holds

$$\nu_n(x) \geq c(\delta) > 0. \tag{4.16}$$

The argument is analogous to the classical result [6] and we follow the presentation in [16] highlighting the differences in the argument.

Define

$$w(x) = \log\left(\frac{1+\varepsilon}{\nu_n + \varepsilon}\right)$$

for small $\varepsilon > 0$. Then by (4.12) and by straightforward calculation we deduce that w satisfies

$$\Delta_{\partial E} w - (\nabla_{\partial E} w) \cdot x = |B_E|^2 \frac{\nu_n}{\nu_n + \varepsilon} + |\nabla_{\partial E} w|^2.$$

By Young's inequality we have

$$\Delta_{\partial E} w \geq -\frac{|x|^2}{2} + \frac{|\nabla_{\partial E} w|^2}{2}. \tag{4.17}$$

Step 1: Let us fix $y \in M_\delta$. We claim that for $r_0 = \frac{\delta}{4}$ it holds

$$w(y) \leq \frac{C}{r_0^{n-1}} \int_{B_{r_0}(y) \cap \partial E} (w+1)\, d\mathcal{H}^{n-1}. \tag{4.18}$$

The argument is similar to [16, Theorem 13.2]. For $0 < \sigma < r \leq r_0$ we choose

$$\varphi_\sigma(x) = \begin{cases} \frac{1}{2(n-3)}(\sigma^{3-n} - r^{3-n}) + \frac{1}{2(n-1)}(r^{1-n} - \sigma^{1-n})|x-y|^2, & \text{for } |x-y| \leq \sigma \\ \frac{1}{(n-1)(n-3)}|x-y|^{3-n} + \frac{r^{1-n}}{2(n-1)}|x-y|^2 - \frac{r^{3-n}}{2(n-3)}, & \text{for } \sigma < |x-y| < r \end{cases}$$

and $\varphi_\sigma(x) = 0$ for $|x-y| \geq r$. Recall that $n \geq 8$. Then $\varphi_\sigma \leq C\sigma^{3-n}$ and $|\nabla \varphi_\sigma| \leq C\sigma^{2-n}$. Moreover, using

$$\Delta_{\partial E}\varphi_\sigma = \Delta\varphi - \left(\nabla^2\varphi_\sigma \nu_E\right)\cdot \nu_E - H_E \partial_\nu \varphi_\sigma$$

and the fact that H_E is locally bounded, we have

$$\Delta_{\partial E}\varphi_\sigma \leq \frac{1}{r^{n-1}}\chi_{B_r(y)} - \frac{1}{\sigma^{n-1}}\chi_{B_\sigma(y)} + C\sigma^{2-n} \qquad \text{on } \partial^* E.$$

Using the cutoffs ζ_h provided by Lemma 7, localized to $\operatorname{supp}\varphi_\sigma$, and the local energy estimate following from (4.17), we may pass to the limit in Green's identity and obtain

$$\int_{\partial^* E} \Delta_{\partial E}\varphi_\sigma\, w d\mathcal{H}^{n-1} = \int_{\partial^* E} \varphi_\sigma\, \Delta_{\partial E} w d\mathcal{H}^{n-1} \geq -C\int_{\partial^* E}\varphi_\sigma\, d\mathcal{H}^{n-1}.$$

Therefore using the estimate $\mathcal{H}^{n-1}(\partial^* E \cap B_r(y)) \leq Cr^{n-1}$, which follows from the minimality, we deduce

$$\frac{1}{\sigma^{n-1}}\int_{B_\sigma(y)\cap\partial E} w\, d\mathcal{H}^{n-1} \leq \frac{1}{r^{n-1}}\int_{B_r(y)\cap\partial E} w\, d\mathcal{H}^{n-1} + \frac{C}{\sigma^{n-2}}\int_{B_r(y)\cap\partial E} w\, d\mathcal{H}^{n-1} + C\frac{r^{n-1}}{\sigma^{n-3}}. \tag{4.19}$$

We choose $r_k = 2^{-k}r_0$ and denote $X_k = \frac{1}{r_k^{n-1}}\int_{B_{r_k}(y)\cap\partial E} w\, d\mathcal{H}^{n-1}$. Applying (4.19) for $\sigma = r_{k+1}$ and $r = r_k$ yields

$$X_{k+1} \leq (1 + Cr_k)X_k + Cr_k^2.$$

Then for $Z_k = \max\{X_k, 1\}$ the above implies

$$Z_{k+1} \leq (1 + Cr_k)Z_k.$$

By iterating this we deduce

$$X_k \leq Z_k \leq C\prod_{i=0}^{k-1}(1 + C2^{-i}r_0)Z_0 \leq C(\delta)Z_0 \leq \frac{C(\delta)}{r_0^{n-1}}\int_{B_{r_0}(y)\cap\partial E}(w+1)\, d\mathcal{H}^{n-1}.$$

The estimate (4.18) follows by letting $k \to \infty$.

Step 2: Let us denote $x = (x', x_n)$ and the cylinder $C_\rho(y) = \{x \in \mathbb{R}^n : |x' - y'| < \rho\}$. We claim that it holds

$$\int_{\partial E\cap\{x_n>0\}\cap C_{\frac{\delta}{4}}(y)} w\, d\mathcal{H}^{n-1}_\gamma \leq C. \tag{4.20}$$

The argument is similar to [16, Theorem 13.4]. Note that the estimates (4.18) and (4.20) imply the claim (4.16) by letting $\varepsilon \to 0$.

Let $\eta : \mathbb{R} \to \mathbb{R}$ be a smooth cut-off function such that $0 \leq \eta \leq 1$, $\eta(t) = 1$ for $|t| \leq \frac{\delta}{4}$ and $\eta(t) = 0$ for $|t| \geq \frac{\delta}{2}$. By the Euler-Lagrange equation (4.8) it holds for $M^+ := \partial^* E \cap \{x_n > 0\}$

$$\int_{M^+} \nabla_{\partial E}\big(x_n\eta(|x'-y'|)\, w\big)\cdot e_n\, d\mathcal{H}^{n-1} = \int_{M^+}\big(\nu^E\cdot x + \lambda\big)\, x_n\eta(|x'-y'|)\, w\, \nu_n\, d\mathcal{H}^{n-1}.$$

Since $\nabla_{\partial E} x_n \cdot e_n = 1 - \nu_n^2$ and $\nabla_{\partial E}\eta(|x'-y'|)\cdot e_n = -(\nabla\eta\cdot\nu)\nu_n$, where the latter follows from the fact that the function is independent of x_n, we have by the above

$$\begin{aligned}
\int_{M^+\cap C_{\frac{\delta}{4}}(y)} w\, d\mathcal{H}^{n-1} &\le \int_{M^+} \eta(|x'-y'|) w\, d\mathcal{H}^{n-1}\\
&\le -\int_{M^+} x_n \eta(|x'-y'|)\,(\nabla_{\partial E} w\cdot e_n)\, d\mathcal{H}^{n-1} + \int_{M^+} x_n(\nabla\eta\cdot\nu) w\nu_n\, d\mathcal{H}^{n-1}\\
&\quad + \int_{M^+} \big((\nu^E\cdot x+\lambda)x_n + \nu_n\big)\,\eta(|x'-y'|)\, w\,\nu_n\, d\mathcal{H}^{n-1}.
\end{aligned} \tag{4.21}$$

In order to estimate the last three terms in (4.21), we recall that for $(x', x_n) \in M_+ \cap C_{\delta/2}(y)$ it holds $x' \in U_\eta$. Therefore it holds $\eta \le x_n \le \frac{1}{\eta}$. In particular, the set $M_+ \cap C_{\delta/2}(y)$ is bounded, where the bounds depend on δ and η. Recall that $w(x) = \log\left(\frac{1+\varepsilon}{\nu_n+\varepsilon}\right)$ and therefore $w\nu_n \le 1$ when ε is small. We may thus estimate the last two terms in (4.21) as

$$\begin{aligned}
C\int_{M^+} x_n(\nabla\eta\cdot\nu) w\nu_n\, d\mathcal{H}^{n-1} + \int_{M^+} \big((\nu^E\cdot x+\lambda)x_n + \nu_n\big)\,\eta(|x'-y'|)\, w\,\nu_n\, d\mathcal{H}^{n-1}&\\
\le C\mathcal{H}^{n-1}\big(M_+\cap C_{\delta/2}(y)\big) \le C.&
\end{aligned} \tag{4.22}$$

We proceed by estimating the second last term by denoting $\zeta(x) = x_n\eta(|x'-y'|)$ and have

$$-\int_{M^+} x_n\eta(|x'-y'|)\,(\nabla_{\partial E} w\cdot e_n)\, d\mathcal{H}^{n-1} \le C + \int_{M^+} \zeta^2 |\nabla_{\partial E} w|^2\, d\mathcal{H}^{n-1}.$$

We use (4.17), and use the cut-off function to avoid the singular set as in step 1, to estimate

$$\begin{aligned}
\int_{M^+} \zeta^2 |\nabla_{\partial E} w|^2\, d\mathcal{H}^{n-1} &\le C + \int_{M^+} \zeta^2 \Delta_{\partial E} w\, d\mathcal{H}^{n-1}\\
&= C - \int_{M^+} 2\zeta\,(\nabla_{\partial E}\zeta\cdot\nabla_{\partial E} w)\, d\mathcal{H}^{n-1}\\
&\le C + 2\int_{M^+} |\nabla_{\partial E}\zeta|^2\, d\mathcal{H}^{n-1} + \frac{1}{2}\int_{M^+} \zeta^2|\nabla_{\partial E} w|^2\, d\mathcal{H}^{n-1}.
\end{aligned}$$

Therefore we have

$$\int_{M^+} \zeta^2 |\nabla_{\partial E} w|^2\, d\mathcal{H}^{n-1} \le C + 4\int_{M^+} |\nabla_{\partial E}\zeta|^2\, d\mathcal{H}^{n-1} \le C. \tag{4.23}$$

The inequalities (4.21), (4.22) and (4.23) imply the estimate (4.20). This concludes the proof of (4.16).

Conclusion: The claim follows from the inequality (4.16) by the following maximum principle argument. Recall the function $u_n : U_n \to \mathbb{R}$ defined as in (4.15) and the definition of the set U_η and $B_\delta(\hat{x}) \subset U_\eta$. Let us prove that u_n is Lipschitz continuous in $B_{\delta/8}(\hat{x})$.

Fix $z' \in B_{\delta/8}(\hat{x})$ and let $c(\delta)$ be the constant from (4.16). Define $\varphi : \mathbb{R}^n \to \mathbb{R}$ as $\varphi(x') = \Lambda|x'-z'| + u_n(z')$ for $\Lambda = \frac{1}{\delta^2} + \frac{1}{c(\delta)}$. We claim that

$$\varphi(x') \ge u_n(x') \qquad \text{for all } x' \in B_{\frac{\delta}{8}}(z'). \tag{4.24}$$

Note that if (4.24) holds for every $z' \in B_{\delta/8}(\hat{x})$, it implies that u_n is Lipschitz continuous in $B_{\delta/8}(\hat{x})$. We argue by contradiction and assume that

$$\sup_{x'\in B_{\delta/8}(z')} (u_n(x') - \varphi(x')) = c_1 > 0.$$

Recall that $u_n < \frac{1}{8\delta}$ in $B_\delta(\hat{x})$. Therefore by the choice of Λ it holds $\varphi(x') \geq u_n(x')$ on $x' \in \partial B_{\delta/8}(z')$. Since u_n is upper semicontinuous there exists a point $y' \in B_{\delta/8}(z')$, $y' \neq z'$, such that $u_n(y') - \varphi(y') = c_1$. Then $u_n(x') \leq \varphi(x') + c_1$ for all $x' \in B_{\delta/8}(z')$ and $u_n(y') = \varphi(y') + c_1$. In other words, the function $\varphi + c_1$ touches u_n from above at y'. Since the subgraph of φ in a neighborhood of $(y', \varphi(y') + c_1)$ is smooth, then the point $(y', u_n(y'))$ belongs to $\partial^* E$ (see [14, Lemma 3]) and thus u_n is differentiable at y'. Since $u_n - \varphi$ obtains a local maximum at y' we have

$$\nabla u_n(y') = \nabla \varphi(y') = \Lambda \frac{y' - z'}{|y' - z'|}.$$

But since $\Lambda > \frac{1}{c(\delta)}$ we have

$$\nu_n\big(y', u_n(y')\big) = \frac{1}{\sqrt{1 + |\nabla u_n(y')|^2}} = \frac{1}{\sqrt{1 + \Lambda^2}} < c(\delta),$$

which contradicts (4.16). Hence we have (4.24).

Since u_n is Lipschitz continuous in $B_{\frac{\delta}{8}}(\hat{x})$, then the Euler-Lagrange equation and standard elliptic regularity imply that the set

$$\tilde{M}_\delta := \{(x', x_n) \in \partial E : x' \in B_{\frac{\delta}{10}}(\hat{x}), x_n > 0\}$$

is analytic. Since $\hat{x}$ was arbitrary, this proves that u_n is locally Lipschitz continuous in U_η. This also implies that $u_n = v_n$ in U_η. Since this is true for all $\eta > 0$ we deduce, together with the fact that for a.e. $x' \in U_n$ the section $\{t \geq 0 : (x', t) \in E\}$ is an interval, that the set $E \cap \{(x', x_n) : x' \in U_n, 0 < v_n(x'), u_n(x') < \infty\}$ is a subgraph of a locally Lipschitz function, and thus the associated boundary is analytic. By repeating the argument in every direction $i = 1, \ldots, n$ we obtain the regularity of the boundary ∂E outside the origin. Here we note that for any point $x \in \partial E \setminus \{0\}$ there is a coordinate direction e_i such that for the associated functions $u_i, v_i : U_i \to \mathbb{R}$ it holds $v_i(x') > 0$ and $u_i(x') < \infty$. $\square$

5. Rigidity

In this section we resume the notation $(x, y) \in \mathbb{R}^2 \times \mathbb{R}^{n-2}$, and E is a minimizer of the Gaussian isoperimetric problem among n-symmetric sets with prescribed Gaussian measure. Recall that E is identified with its precise representative (4.6), which is an open set, and that by Theorem 2 the set $\partial E \setminus \{0\} = \partial^* E \setminus \{0\}$ is a real-analytic hypersurface. Since E^c is also a minimizer (for the complementary volume $1 - \gamma(E)$), Theorem 2 applies to E^c as well: its precise representative $\{u_{E^c} = 1\} = \{u_E = 0\} = \mathbb{R}^n \setminus \overline{E}$ is open.

Define the "arc-projection" $\Pi : \mathbb{R}^2 \times \mathbb{R}^{n-2} \to [0, \infty) \times \mathbb{R}^{n-2}$ as $\Pi(x, y) := (|x|, y)$. If $B \subset (0, \infty) \times \mathbb{R}^{n-2}$, then $\Pi^{-1}(B) = \Phi(B \times \mathbb{S}^1)$, where Φ is the diffeomorphism defined in (1.4). Set $\mathbb{R}^2_+ := \{x \in \mathbb{R}^2 \colon x_1 > 0,\ x_2 > 0\}$ and $\omega_\theta := (\cos\theta, \sin\theta)$. Recalling (1.1) and (1.2), for every $(r, y) \in (0, \infty) \times \mathbb{R}^{n-2}$

$$\beta_\mu(r, y) = \frac{1}{4} \int_0^{2\pi} \chi_E(r\omega_\theta, y)\, d\theta. \tag{5.1}$$

Lemma 9. *Let $E \subset \mathbb{R}^n$ be a minimizer of the Gaussian isoperimetric problem among n-symmetric sets with prescribed Gaussian measure. Then, for every $(r, y) \in (0, \infty) \times \mathbb{R}^{n-2}$,*

$$\widetilde{E}_{(r,y)} := E_{(r,y)} \cap \{x_1 \geq 0,\ x_2 \geq 0\}$$

is connected. Moreover, if $\widetilde{E}_{(r,y)}$ is non-empty, then it contains at least one of the two points $(r, 0)$ and $(0, r)$.

Proof. Set $G := \mathbb{R}^n \setminus \overline{E}$, which is open. Every $p \in \partial E$ is a limit of points of G: otherwise some ball $B_\rho(p)$ would be contained in $E \cup \partial E$, hence in E up to the $\mathcal{H}^n$-null set ∂E, and then in $E = E^{(1)}$, contradicting $p \notin E$.

Almost-everywhere structure of the fibers. By minimality, (1.5) holds with equality for every Borel set $B \subset (0,\infty)\times\mathbb{R}^{n-2}$. By Theorem 1(i), for $\mathcal{H}^{n-1}$-a.e. $(\rho,\eta) \in (0,\infty)\times\mathbb{R}^{n-2}$ the angular section

$$I(\rho,\eta) := \{\theta \in [0,\pi/2] : (\rho\omega_\theta,\eta) \in E\}$$

agrees, up to $\mathcal{H}^1$-null sets, with the empty set or with an interval attached to one of the endpoints of $[0,\pi/2]$.

Consequently, *no circular fiber can contain three points, in strictly increasing angular order in* $[0,\pi/2]$*, belonging respectively to* E, G, E *or to* G, E, G. Indeed, since E and G are open, either configuration persists for all parameters (ρ,η) in a non-empty open neighborhood of the given fiber, and each of its three points has a relative angular neighborhood of positive length in the corresponding phase. This contradicts the preceding almost-everywhere description.

Connectedness. Suppose that $\widetilde{E}_{(r,y)}$ is disconnected. Since E is open, there are angles $0 < \theta_1 < \theta_2 < \theta_3 < \pi/2$ such that

$$(r\omega_{\theta_1},y),\ (r\omega_{\theta_3},y) \in E, \qquad p := (r\omega_{\theta_2},y) \notin E.$$

If $p \in G$, this is already a forbidden configuration. Otherwise $p \in \partial E$, and p can be approached by points of G. Choose $(\rho\omega_\theta,\eta) \in G$ so close to p that $\theta_1 < \theta < \theta_3$ and $(\rho\omega_{\theta_1},\eta), (\rho\omega_{\theta_3},\eta) \in E$. This gives the forbidden configuration E, G, E on the fiber (ρ,η). Thus $\widetilde{E}_{(r,y)}$ is connected.

The endpoint assertion. Write $\nu_i := \nu^E \cdot e_i$, $i = 1,2$. By Lemma 8, set $\sigma_i := 0$ in the cylindrical alternative, and otherwise let $\sigma_i \in \{-1,1\}$ be such that $\sigma_i\nu_i > 0$ on $\partial^*E \cap \{x_i > 0\}$. By (2.9), $D_i\chi_E = -\nu_i\,\mathcal{H}^{n-1} \llcorner \partial^*E$, so that

$$\sigma_i D_i\chi_E \le 0 \ \text{ in } \{x_i > 0\} \quad \text{if } \sigma_i \in \{-1,1\}, \qquad D_i\chi_E = 0 \ \text{ in } \mathbb{R}^n \quad \text{if } \sigma_i = 0. \tag{5.2}$$

By the one-dimensional slicing theorem for BV functions (see [1]), writing $x = (x', x_i)$ up to a permutation of the coordinates, for a.e. $x' \in \mathbb{R}^{n-1}$ the function $t \mapsto \chi_E(x',t)$ is a.e. equal on $(0,\infty)$ to a nonincreasing function if $\sigma_i = 1$, to a nondecreasing function if $\sigma_i = -1$, and to a constant if $\sigma_i = 0$. We claim that the corresponding statements hold pointwise for $E = E^{(1)}$. First, membership in E is preserved when $|x_i|$ is decreased if $\sigma_i = 1$, when $|x_i|$ is increased if $\sigma_i = -1$, and is independent of x_i if $\sigma_i = 0$. Indeed, let $q \in E$ with $q_i \ge 0$ (the case $q_i < 0$ follows by symmetry) and let $U' \times (q_i - \delta, q_i + \delta) \subset E$ be an open product neighborhood of q. If $\sigma_i = 1$, the a.e. monotonicity of the sections gives $|(U' \times (0, q_i + \delta)) \setminus E| = 0$, and the reflection symmetry of E gives $|(U' \times (-q_i - \delta, q_i + \delta)) \setminus E| = 0$. Every point of this open set has density one, so it is contained in $E = E^{(1)}$. The cases $\sigma_i = -1$ (yielding $U' \times (q_i - \delta, \infty) \subset E$) and $\sigma_i = 0$ (yielding $U' \times \mathbb{R} \subset E$) are analogous.

Suppose now that $\widetilde{E}_{(r,y)}$ is non-empty but that

$$p_1 := (r,0,y) \notin E, \qquad p_2 := (0,r,y) \notin E.$$

Choose $q = (x_1,x_2,y) \in E$ on this fiber with $x_1, x_2 > 0$. If $\sigma_1 \ge 0 \ge \sigma_2$, first decreasing x_1 to zero and then increasing x_2 to r gives $p_2 \in E$. If $\sigma_2 \ge 0 \ge \sigma_1$, the analogous changes give $p_1 \in E$. Both are contradictions, and these cases include all cylindrical alternatives and all opposite-sign alternatives. We are therefore left with $\sigma_1 = \sigma_2 = \sigma \in \{-1,1\}$.

For $i = 1,2$, if $p_i \in G$, then $p_i + \sigma t e_i \in G$ for all sufficiently small $t > 0$, by openness. If instead $p_i \in \partial E$, then $p_i \ne 0$ and $(p_i)_i = r > 0$; hence $p_i \in \partial^*E \cap \{x_i > 0\}$ by Theorem 2, and Lemma 8, whose strict-sign conclusion holds on the whole set $\partial^*E \cap \{x_i > 0\}$ (including points where other coordinates vanish), gives

$$\nu^E(p_i)\cdot(\sigma e_i) = \sigma\nu_i(p_i) > 0.$$

Since ∂E is a C^1 hypersurface near p_i, with E on the side of $-\nu^E$, the same conclusion $p_i+\sigma t e_i \in G$ follows by transversality. Consequently, for all sufficiently small $t>0$, with $r_t := r+\sigma t>0$,

$$(r_t,0,y),\ (0,r_t,y)\in G, \qquad q_t := \Big(\frac{r_t}{r}x_1, \frac{r_t}{r}x_2, y\Big)\in E,$$

where the last inclusion follows from the openness of E. These three points lie on the fiber (r_t, y), at the angles $0<\arctan(x_2/x_1)<\pi/2$, and give the forbidden configuration G, E, G. This contradiction proves the endpoint assertion. □

Proof of Theorem 3. Recall that $E = E^{(1)}$ and $F = F_\mu^{(1)}$. By Theorem 1 and minimality, F is also a minimizer. Hence Theorem 2, Proposition 6, Lemma 8 and Lemma 9 apply to both E and F; in particular F is open and $\partial F\setminus\partial^* F\subset\{0\}$. If E is cylindrical in both directions x_1 and x_2 (Lemma 8(i)), then $E=\mathbb{R}^2\times A$ for the precise representative $A\subset\mathbb{R}^{n-2}$ of the corresponding section, every circular fiber of E is empty or full, $F=E$, $\Omega_\mu=\emptyset$, and (i) holds. We therefore assume henceforth that at most one of x_1, x_2 is a cylindrical direction for E.

Step 1: the precise representative has the same pointwise circular distribution. We claim that

$$\mathcal{H}^1_\gamma(F_{(r,y)}) = \mu(r,y) \qquad \text{for every } (r,y)\in(0,\infty)\times\mathbb{R}^{n-2}. \tag{5.3}$$

Fix (r,y) and write $C_{r,y} := \partial B(r)\times\{y\}$. The circle $C_{r,y}$ stays away from the origin, so ∂E is analytic in a neighborhood of it by Theorem 2; hence either $C_{r,y}\cap\partial E$ is finite or $C_{r,y}\subset\partial E$. Indeed, if $C_{r,y}\cap\partial E$ has an accumulation point p, a local analytic defining function of ∂E near p vanishes at infinitely many points of the analytic curve $\theta\mapsto(r\omega_\theta, y)$ accumulating at p, hence identically along an arc; therefore the set of $\theta\in\mathbb{S}^1$ such that an arc of $C_{r,y}$ around $(r\omega_\theta,y)$ is contained in ∂E is non-empty, open, and closed by the same argument, so it is all of $\mathbb{S}^1$.

Case 1: $C_{r,y}\cap\partial E$ is finite. Then, for all but finitely many $\theta\in[0,2\pi)$, the point $(r\omega_\theta,y)$ belongs to one of the two open phases E and $\mathbb{R}^n\setminus\overline{E}$, so that $\chi_E(\rho\omega_\theta,\eta)\to\chi_E(r\omega_\theta,y)$ as $(\rho,\eta)\to(r,y)$. By (5.1) and dominated convergence, β_μ is continuous at (r,y). Therefore, for every $\theta\in(0,\pi/2)$ with $\theta\neq\beta_\mu(r,y)$, by (1.3) the point $(r\omega_\theta,y)$ has a neighborhood contained in F_μ if $\theta<\beta_\mu(r,y)$, and a neighborhood disjoint from F_μ if $\theta>\beta_\mu(r,y)$: in the first case it belongs to F, in the second it does not. By symmetry, $F_{(r,y)}$ and $(F_\mu)_{(r,y)}$ differ at most at the four angular endpoints $\pm\beta_\mu(r,y)$, $\pi\pm\beta_\mu(r,y)$ and at the four axis points, and (5.3) follows since F_μ is μ-distributed.

Case 2: $C_{r,y}\subset\partial E$. Then $\mu(r,y)=0$, since E is open. The circle is tangent to ∂E, so

$$\nu^E_x(p) = a(p)\,\widehat{x}, \qquad p=(x,y)\in C_{r,y},$$

for the continuous function $a := \nu^E_x\cdot\widehat{x}$. Neither x_1 nor x_2 is a cylindrical direction for E: if, say, $\nu_1\equiv 0$ on ∂^*E, then $a=0$ at every point of $C_{r,y}$ with $x_1\neq 0$, hence $\nu_2 = a\,\widehat{x}_2 = 0$ at a point of $\partial^*E\cap\{x_2>0\}$, and Lemma 8 makes x_2 cylindrical too, contrary to our reduction. Let then $\sigma_1,\sigma_2\in\{-1,1\}$ be the strict signs of ν_1 on $\partial^*E\cap\{x_1>0\}$ and of ν_2 on $\partial^*E\cap\{x_2>0\}$ given by Lemma 8. At an open-quadrant point of $C_{r,y}$ both $\nu_1 = a\widehat{x}_1$ and $\nu_2 = a\widehat{x}_2$ have the sign of $a\neq 0$, so $\sigma_1=\sigma_2 =: \sigma$; at the axis points $(\pm r,0,y)$, $(0,\pm r,y)$ one has $\sigma a>0$ by the same lemma; and $a(A_j p) = a(p)$ for $j=1,2$ by the reflection symmetry of E. Hence, by continuity and compactness,

$$\inf_{p\in C_{r,y}} \sigma a(p) > 0.$$

Since ∂E is C^1 in a neighborhood of the compact set $C_{r,y}$, with E on the side of $-\nu^E$, there exists $t_0>0$ such that for every $t\in(0,t_0)$

$$C_{r-\sigma t,y}\subset E, \qquad C_{r+\sigma t,y}\subset\mathbb{R}^n\setminus\overline{E}.$$

Each inclusion persists on a neighborhood $N_t^{\mp}$ of $(r \mp \sigma t, y)$ in $(0,\infty) \times \mathbb{R}^{n-2}$, on which β_μ is identically $\pi/2$, respectively 0. Thus the open set $\Phi(N_t^- \times \mathbb{S}^1)$ is contained in F_μ up to an $\mathcal{H}^n$-null set, hence in F, while the open set $\Phi(N_t^+ \times \mathbb{S}^1)$ is disjoint from F_μ, hence from $\overline{F}$. Letting $t \downarrow 0$, every point of $C_{r,y}$ is approached by both F and $\mathbb{R}^n \setminus \overline{F}$, so $C_{r,y} \subset \partial F$. Since F is open, $F_{(r,y)} = \emptyset$, proving (5.3) also in this case.

In particular, F is μ-distributed, and Ω_μ is the same set whether computed from E or from F.

Step 2: the rotational Jacobi field. Set

$$M := \partial^* F, \qquad M^+ := M \cap (\mathbb{R}^2_+ \times \mathbb{R}^{n-2}), \qquad X(x,y) := (-x_2, x_1, 0).$$

Since $\partial F \setminus M \subset \{0\}$, we have $M^+ = \partial F \cap (\mathbb{R}^2_+ \times \mathbb{R}^{n-2})$.

Slab cases. If M is disconnected, Proposition 6 implies that $F = \{|z_j| < a\}$ or $F = \{|z_j| > a\}$ for some $a > 0$ and $j \in \{1, \dots, n\}$, where $z = (x_1, x_2, y)$. If $j \geq 3$, then F is rotationally invariant in the (x_1, x_2)-plane, every fiber of F is empty or full, and $\Omega_\mu = \emptyset$ by (5.3). If $j \in \{1, 2\}$ and $F = \{|z_j| < a\}$, the fiber $F_{(r,y)}$ is full (up to two points) for $r \leq a$ and a non-empty proper union of arcs for $r > a$, so (5.3) gives $\Omega_\mu = (a, \infty) \times \mathbb{R}^{n-2}$, which is open and connected; moreover $M^+ = \{z_j = a\} \cap (\mathbb{R}^2_+ \times \mathbb{R}^{n-2})$, so that $\Pi(M^+) = (a,\infty) \times \mathbb{R}^{n-2} = \Omega_\mu$, and $\nu^F = \pm e_j$ gives $\nu^F_{x\|} = \pm\big(e_j - (e_j \cdot \widehat{x})\widehat{x}\big) \neq 0$ on M^+, since $\widehat{x} \neq \pm e_j$ there. Similarly if $F = \{|z_j| > a\}$. Thus, all the conclusions of the theorem hold for F in place of E.

Connectedness of M^+. Assume henceforth that M is connected. The continuous folding map $(x_1, x_2, y) \mapsto (|x_1|, |x_2|, y)$ maps M onto $M \cap \{x_1 \geq 0,\, x_2 \geq 0\}$, so this latter set is connected. For $i = 1, 2$ and $p \in M \cap \{x_i = 0\}$, the reflection symmetry $\nu^F(A_i p) = A_i \nu^F(p)$ gives $\nu^F_i(p) = 0$, hence

$$\nabla_M x_i(p) = e_i - \nu^F_i(p)\, \nu^F(p) = e_i.$$

Thus x_1 and x_2 are submersions on M near $\{x_1 = 0\}$ and $\{x_2 = 0\}$ respectively, and their gradients are linearly independent at the points of $M \cap \{x_1 = x_2 = 0\}$. Therefore $M \cap \{x_1 \geq 0,\, x_2 \geq 0\}$ is a manifold with corners, whose interior is M^+. The interior of a connected manifold with corners is connected: every point has a corner chart whose interior is connected and non-empty, so it is adjacent to exactly one component of the interior, and the closures of two distinct components would disconnect the manifold. Hence M^+ is connected, and non-empty because $M \neq \emptyset$.

Sign of u. Define

$$u := X \cdot \nu^F \quad \text{on } M.$$

In angular coordinates $x = |x|\,\omega_\theta$, $\theta \in (0, \pi/2)$, on the open quadrant $\mathbb{R}^2_+ \times \mathbb{R}^{n-2}$, one has $X = \partial_\theta$ and, by (1.3), χ_{F_μ} is nonincreasing in θ. Since $\operatorname{div} X = 0$ and $F =_{\mathcal{H}^n} F_\mu$, for every $\varphi \in C^\infty_c(\mathbb{R}^2_+ \times \mathbb{R}^{n-2})$ with $\varphi \geq 0$

$$\langle X \cdot D\chi_F, \varphi\rangle = -\int_{\mathbb{R}^n} \chi_{F_\mu}\, X \cdot \nabla\varphi \, dz = -\int_{(0,\infty)\times\mathbb{R}^{n-2}} \int_0^{\pi/2} \chi_{F_\mu}(r\omega_\theta, y)\, \partial_\theta\big(\varphi(r\omega_\theta, y)\big)\, d\theta\, r\, dr\, dy \leq 0,$$

i.e. $X \cdot D\chi_F \leq 0$ as a Radon measure in $\mathbb{R}^2_+ \times \mathbb{R}^{n-2}$. On the other hand $X \cdot D\chi_F = -u\, \mathcal{H}^{n-1} \llcorner M$. Therefore $u \geq 0$ $\mathcal{H}^{n-1}$-a.e. on M^+, and hence everywhere on M^+ by continuity.

The Jacobi equation. Let J be the skew-symmetric endomorphism of $\mathbb{R}^n$ with $Je_1 = e_2$, $Je_2 = -e_1$, $Je_k = 0$ for $k \geq 3$, so that $X(z) = Jz$, and for $z \in M$ write $z^T := z - (z \cdot \nu^F)\nu^F$, $X^T := X - u\,\nu^F$. By Proposition 4, $\mathscr{H}_F - z \cdot \nu^F = \lambda/\sqrt{2\pi}$ on M. For $\tau \in T_z M$, since $D_\tau X = J\tau$ and $D_\tau \nu^F = B_F(\tau, \cdot)$,

$$D_\tau u = J\tau \cdot \nu^F + B_F(\tau, X^T), \qquad D_\tau \mathscr{H}_F = D_\tau(z \cdot \nu^F) = B_F(\tau, z^T).$$

Choosing $\tau = z^T$ in the first identity, where $Jz^T \cdot \nu^F = Jz \cdot \nu^F - (z \cdot \nu^F)\, J\nu^F \cdot \nu^F = u$, and $\tau = X^T$ in the second, the symmetry of B_F gives

$$z \cdot \nabla_M u = u + B_F(z^T, X^T) = u + X^T \cdot \nabla_M \mathscr{H}_F.$$

Moreover $\Delta_M(Jz) = J\Delta_M z = -\mathscr{H}_F\, J\nu^F$ is orthogonal to ν^F, and, for an orthonormal tangent frame $\{\tau_i\}$, $\sum_i D_{\tau_i}(Jz) \cdot D_{\tau_i}\nu^F = \sum_{i,k} B_F(\tau_i, \tau_k)\, J\tau_i \cdot \tau_k = 0$ by the symmetry of B_F and the skew-symmetry of J. Hence, by the identity $\Delta_M \nu^F + |B_F|^2 \nu^F = \nabla_M \mathscr{H}_F$ used in the proof of Lemma 8,

$$\Delta_M u = X \cdot \Delta_M \nu^F = X^T \cdot \nabla_M \mathscr{H}_F - |B_F|^2 u = z \cdot \nabla_M u - \big(|B_F|^2 + 1\big)u,$$

i.e.,

$$\Delta_M u - z \cdot \nabla_M u + \big(|B_F|^2 + 1\big)u = 0 \qquad \text{on } M. \tag{5.4}$$

Equivalently, (5.4) is the linearization of the Gaussian constant-mean-curvature equation along the rotations $R_t := e^{tJ}$, which preserve the Gaussian weight and hence the equation and its constant right-hand side; no admissibility of the rotated sets in the symmetric variational problem is used. The term u, absent in (4.12), comes from $D_\tau X = J\tau \neq 0$.

Minimum principle. By (5.4) and $u \geq 0$,

$$\Delta_M u - z \cdot \nabla_M u = -\big(|B_F|^2 + 1\big)u \leq 0 \qquad \text{on } M^+.$$

The operator $\Delta_M - z \cdot \nabla_M$ has no zeroth-order term, so the strong minimum principle applies locally to the nonnegative supersolution u: an interior zero forces u to vanish in a neighborhood, and the connectedness of M^+ yields

$$u \equiv 0 \quad \text{on } M^+ \qquad \text{or} \qquad u > 0 \quad \text{on } M^+.$$

Step 3: openness and connectedness of Ω_μ. Suppose first that $u \equiv 0$ on M^+. Since $u(A_i p) = -u(p)$ for $i = 1, 2$, and $M \cap \{x_1 x_2 = 0\}$ is $\mathcal{H}^{n-1}$-negligible (it is a proper submanifold of M by the transversality proved in Step 2), we get $u = 0$ $\mathcal{H}^{n-1}$-a.e. on M, hence $X \cdot D\chi_F = -u\, \mathcal{H}^{n-1} \llcorner M = 0$ in $\mathbb{R}^n$. Let $R_t = e^{tJ}$ be the flow of X. For $\varphi \in C_c^\infty(\mathbb{R}^n)$, since R_t commutes with J and $\operatorname{div} X = 0$,

$$\frac{d}{dt} \int_{\mathbb{R}^n} \chi_F(z)\, \varphi(R_t z)\, dz = \int_{\mathbb{R}^n} \chi_F(z)\, X(z) \cdot \nabla(\varphi \circ R_t)(z)\, dz = -\langle X \cdot D\chi_F, \varphi \circ R_t \rangle = 0.$$

Hence $R_t F =_{\mathcal{H}^n} F$ for every t. Orthogonal maps commute with the passage to density-one representatives, and $F = F^{(1)}$; thus $R_t F = F$ pointwise for every t, i.e. F is rotationally invariant in the (x_1, x_2)-plane. Every circular fiber of F is then empty or full, and (5.3) gives $\Omega_\mu = \emptyset$.

Suppose instead that $u > 0$ on M^+. For every $p \in M^+$, the circular fiber through p is tangent to $X(p)$, hence transversal to M at p, and thus contains positive-length portions of both open phases F and $\mathbb{R}^n \setminus \overline{F}$. By (5.3), $0 < \mu(\Pi(p)) < \mathcal{H}^1_\gamma(\partial B(|x|))$, i.e. $\Pi(M^+) \subset \Omega_\mu$. Conversely, let $(r, y) \in \Omega_\mu$. By (5.3), $0 < \mathcal{H}^1(F_{(r,y)}) < 2\pi r$; by Lemma 9 applied to the minimizer F, and by symmetry, the quadrant section $\widetilde{F}_{(r,y)}$ is a non-empty, connected, relatively open proper subset of the closed quarter-circle containing one of its endpoints, hence it is $\{\theta \in [0, \theta^*)\}$ or $\{\theta \in (\theta^*, \pi/2]\}$ for some $\theta^* \in (0, \pi/2)$. The point $(r\omega_{\theta^*}, y)$ lies in $\overline{F} \setminus F = \partial F$ and in the open quadrant, hence in M^+ since $r > 0$. Consequently

$$\Omega_\mu = \Pi(M^+). \tag{5.5}$$

In particular $\Omega_\mu \neq \emptyset$ is connected, as the continuous image of the connected set M^+. Moreover, for $p = (x, y) \in M^+$,

$$X(p) \cdot \nu^F(p) = u(p) > 0,$$

so $\Pi|_{M^+}$ is a local diffeomorphism onto its image; therefore Ω_μ is open. Finally, $u(p) = |x|\,\nu^F_x(p)\cdot x_\parallel$ and $\nu^F_{x\parallel} = (\nu^F_x \cdot x_\parallel)\,x_\parallel$ show that $\nu^F_{x\parallel}(p) \neq 0$ for every $p \in M^+$.

Together with the slab cases, we have proved: either F is rotationally invariant in the (x_1, x_2)-plane and $\Omega_\mu = \emptyset$, or $\Omega_\mu \neq \emptyset$, (5.5) holds, and $\nu^F_{x\parallel} \neq 0$ on $M^+ = \partial F \cap (\mathbb{R}^2_+ \times \mathbb{R}^{n-2})$. In both cases Ω_μ is open and connected. In the first case we clearly have also $E = F$ since the circular sections of F are empty or full.

Step 4: rigidity of the original minimizer in the case $\Omega_\mu \neq \emptyset$. Set

$$\Omega^1_\mu := \{(r, y) \in (0,\infty) \times \mathbb{R}^{n-2} : (r, 0, y) \in E\}, \qquad \Omega^2_\mu := \{(r, y) \in (0,\infty) \times \mathbb{R}^{n-2} : (0, r, y) \in E\}.$$

These sets are open, being inverse images of the open set E under continuous maps. Let $(r, y) \in \Omega_\mu$. Then $\widetilde{E}_{(r,y)}$ is non-empty and is not the whole closed quarter-circle, so by Lemma 9 it contains exactly one of the two axis points (containing both would force, by connectedness, the whole quarter-circle). Therefore

$$\Omega_\mu = (\Omega^1_\mu \cap \Omega_\mu) \,\dot\cup\, (\Omega^2_\mu \cap \Omega_\mu).$$

This is a covering of the connected set Ω_μ by two disjoint relatively open sets, so one of them is empty. In the first case $E = F$, while in the second case $E = R(F)$. □

Corollary 2. *Let $E \subset \mathbb{R}^n$ be a minimizer of the Gaussian perimeter among n-symmetric sets with prescribed Gaussian measure. For $i \neq j$, let $\mathcal{R}_{i,j}$ denote the double-arc rearrangement in the coordinate plane (x_i, x_j), with arcs centered on the x_i-axis. When applied to a minimizer, $\mathcal{R}_{i,j}(E)$ is understood as the density-one representative of the corresponding double-arc rearrangement, as in Theorem 3. Then, after a permutation of the coordinate axes, one has*

$$E = \mathcal{R}_{i,j}(E) \qquad \text{for every } i < j. \tag{5.6}$$

Moreover, if there exists a point $p \in \partial E \cap \mathbb{R}^n_+$ such that

$$\nu^E(p) = \frac{p}{|p|} \qquad \left(\text{respectively, } \nu^E(p) = -\frac{p}{|p|}\right),$$

then E (respectively, $\mathbb{R}^n \setminus \overline{E}$) is the open n-dimensional ball of radius $|p|$ centred at the origin.

Proof. Let $M := \partial^* E$ and $M^+ := M \cap \mathbb{R}^n_+$. For $i \neq j$, consider the infinitesimal rotation in the plane (x_i, x_j),

$$X_{i,j}(x) := -x_j e_i + x_i e_j,$$

and set

$$u_{i,j} := X_{i,j} \cdot \nu^E = x_i \nu^E_j - x_j \nu^E_i \qquad \text{on } M.$$

Step 1. By Theorem 3, applied in the plane (x_i, x_j), either $E = \mathcal{R}_{i,j}(E)$ or $E = \mathcal{R}_{j,i}(E)$. Moreover, the sign argument in the proof of Theorem 3 gives the following alternative on the positive orthant of the plane (x_i, x_j):

$$E = \mathcal{R}_{i,j}(E) \quad \Longrightarrow \quad u_{i,j} \geq 0,$$

whereas

$$E = \mathcal{R}_{j,i}(E) \quad \Longrightarrow \quad u_{i,j} \leq 0.$$

If $u_{i,j} \equiv 0$, then E is rotationally invariant in the plane (x_i, x_j), and the two choices are equivalent.

Given $p \in M^+$, for every i define

$$\rho_i := \frac{\nu^E_i(p)}{p_i}.$$

Since $p_i > 0$ for all i, we have

$$u_{i,j}(p) = p_i p_j\,(\rho_j - \rho_i)\,.$$

Hence the choice of rearrangement axis in the plane (x_i, x_j) is determined by the ordering of ρ_i and ρ_j. Indeed, if $\rho_i < \rho_j$, then $u_{i,j}(p) > 0$, so the preceding sign implications rule out $E = \mathcal{R}_{j,i}(E)$ and yield $E = \mathcal{R}_{i,j}(E)$. Similarly, if $\rho_j < \rho_i$, then $E = \mathcal{R}_{j,i}(E)$.

If $\rho_i = \rho_j$, then $u_{i,j}(p) = 0$. Since $p_i, p_j > 0$, this means that the angular component of $\nu^E(p)$ in the (x_i, x_j)-plane vanishes, contradicting alternative (ii) of Theorem 3, applied in that plane. Thus alternative (i) holds: E is rotationally invariant in the (x_i, x_j)-plane, and consequently

$$E = \mathcal{R}_{i,j}(E) = \mathcal{R}_{j,i}(E).$$

We have therefore proved that

$$\rho_i \leq \rho_j \implies E = \mathcal{R}_{i,j}(E), \qquad \rho_j \leq \rho_i \implies E = \mathcal{R}_{j,i}(E),$$

with both rearrangements leaving E unchanged when $\rho_i = \rho_j$.

After relabelling the coordinate axes so that

$$\rho_1 \leq \rho_2 \leq \cdots \leq \rho_n,$$

we obtain $E = \mathcal{R}_{i,j}(E)$ for every $i < j$, which proves (5.6).

Step 2. If M is not connected, then by Proposition 6 the set E is a symmetric slab or its complement. This is incompatible with the existence of a point $p \in M^+$ such that $\nu^E(p) = p/|p|$. Hence we may assume that M is connected. As in the proof of Theorem 3, the folded positive orthant part M^+ is connected.

By the previous step we can assume that $E = \mathcal{R}_{i,j}(E)$ for $i < j$. Since $u_{i,j}(p) = 0$, using the Jacobi equation and the strong maximum principle as in the proof of Theorem 3, we get that $u_{i,j} \equiv 0$ on M^+. By analyticity and symmetry, this identity extends to the whole of M. This means that the infinitesimal rotation field $X_{i,j}$ is tangent to M.

Since this holds true for any pair $\{i, j\}$ with $i < j$, M has to be a hypersphere. □

Appendix A. The problem in one dimension

The conjecture about the shape of the symmetric Gaussian isoperimetric problem can be easily verified in one dimension.

Proposition 7 (Isoperimetric problem on the real line with symmetry)**.** *Let $m \in (0, 1)$. On the real line $\mathbb{R}$, the Gaussian perimeter has a minimum among the sets of Gaussian length m, which are symmetric with respect to the origin. More precisely,*

- *If $m < 1/2$, the unique perimeter minimizer is the symmetric tails $(-\infty, -a) \cup (a, \infty)$ for a suitable $a > 0$.*
- *If $m > 1/2$, the unique perimeter minimizer is the centered interval $(-a, a)$ for a suitable $a > 0$.*
- *If $m = 1/2$, both configurations have equal perimeter.*

Proof. The existence of minimizers is standard and follows by the direct method in the calculus of variations in the BV setting. Let $E \subset \mathbb{R}$ be a minimizer of the Gaussian perimeter. Since χ_E belongs to $SBV_{\mathrm{loc}}(\mathbb{R})$, the set E is equivalent, up to a set of measure zero, to a countable and locally finite union of intervals (a_h, b_h) with disjoint closure. We have two cases.

The origin does not belong to any of the intervals (a_h, b_h). If we write $E^- := (-\infty, 0) \cap E$ and $E^+ := (0, \infty) \cap E$, we have $P_\gamma(E) = P_\gamma(E^-) + P_\gamma(E^+)$. Necessarily, E^- must be the interval $(-\infty, -a)$, with $a > 0$ such that $\gamma((-\infty, -a)) = m/2$, because this is the minimal configuration of the Gaussian isoperimetric problem. By symmetry, E^+ must be the interval (a, ∞).

The origin belongs to one of the intervals (a_h, b_h). Instead of E we consider its complement, reducing to the previous case (with mass $(1 - m)$).

Having established that the only candidates are the symmetric tails and the centered interval, a direct computation completes the classification.

□

Because of this dichotomy, we cannot rearrange an n-symmetric set E by substituting its one-dimensional slices with the minimal one-dimensional configuration. The following example in dimension 2 illustrates this failure.

Example 1. *Let $a > 0$ be the threshold such that the centered interval $(-a, a)$ has Gaussian measure $1/2$. Consider the 2-symmetric set $E \subset \mathbb{R}^2$ defined by*

$$E := ((-a-\varepsilon, a+\varepsilon) \times [-1,1]) \cup \big((-a+\varepsilon, a-\varepsilon) \times \big((-\infty,-1) \cup (1,\infty)\big)\big),$$

where $\varepsilon > 0$ is sufficiently small.

For $|y| \leq 1$, the horizontal slice is $E_y = (-a-\varepsilon, a+\varepsilon)$, which has measure $m_+ > 1/2$. According to Proposition 7, this is the unique minimizer.

For $|y| > 1$, the horizontal slice is $E_y = (-a+\varepsilon, a-\varepsilon)$, which has measure $m_- < 1/2$. According to Proposition 7, the unique minimizer for this mass is the union of symmetric tails $(-\infty, -b) \cup (b, \infty)$, with $b > a+\varepsilon$.

If we construct a rearranged set F by replacing each slice with its unique 1-dimensional minimizer, we obtain

$$F_y = \begin{cases} (-a-\varepsilon, a+\varepsilon) & \text{for } |y| \leq 1, \\ (-\infty, -b) \cup (b, \infty) & \text{for } |y| > 1. \end{cases}$$

At the interfaces $y = \pm 1$, the set F undergoes a topological change, effectively turning "inside out", see Figure 6. This discontinuity introduces horizontal boundary components with significant Gaussian weight, whereas the original set E has only small horizontal shoulders of total length 4ε at each interface. Moreover, as $\varepsilon \to 0$, we have $b \to a$, so the difference between the vertical contributions to the perimeters tends to zero, while the excess horizontal contribution tends to $2e^{-1/2}$. Consequently, $P_\gamma(F) > P_\gamma(E)$ for $\varepsilon > 0$ sufficiently small, showing that optimizing slices individually does not minimize the total perimeter.

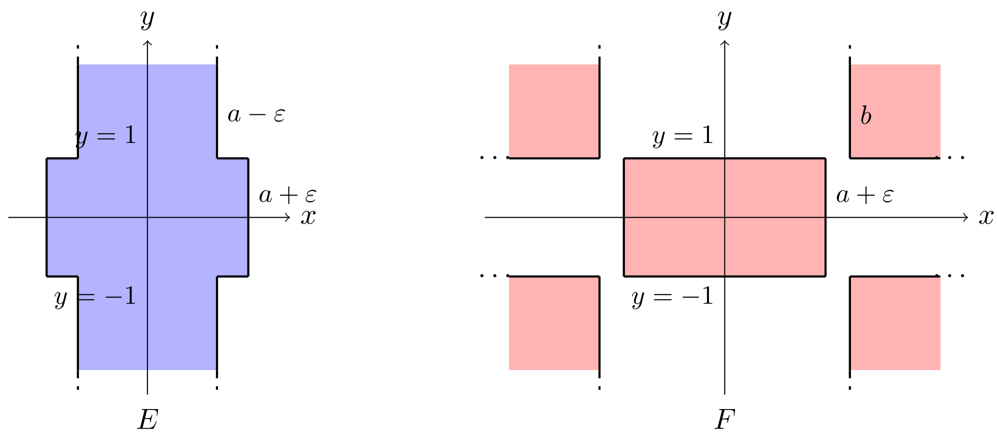

FIGURE 6. The set E (left) has only short horizontal boundary at $y = \pm 1$. Slice-by-slice minimization produces F (right), which switches from a centered interval to symmetric tails at both interfaces and therefore creates horizontal boundary of order one in Gaussian weight.

Remark 6. This obstruction arises because the optimal configuration in Proposition 7 depends discontinuously on the mass m at the threshold $m = 1/2$. In contrast, the double-arc rearrangement of Theorem 1 uses circular slices, where the optimal symmetric configuration (two antipodal arcs) is independent of the mass, cf. Lemma 5.

Appendix B. Obstruction

It would be useful to generalize the double-arc rearrangement, by foliating a n-symmetric set E using spheres, or more generally using k-dimensional spheres, instead of circles. More precisely, for every point in $\mathbb{R}^n$, $n \geq 3$, we write (x, y), with $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and $y = (y_1, \ldots, y_{n-3}) \in \mathbb{R}^{n-3}$. For every $r > 0$, we set $\partial B(r) := \{x \in \mathbb{R}^3 \colon |x| = r\}$.

Given a Borel set E, for every $(r, y) \in (0, \infty) \times \mathbb{R}^{n-3}$ we define the spherical slice $E_{(r,y)}$ as the subset of $\mathbb{R}^3$ given by

$$E_{(r,y)} := \{x \in \mathbb{R}^3 : x \in \partial B(r) \text{ and } (x, y) \in E\},$$

and define the *spherical distribution* of E as the function

$$\mu(r, y) := \mathcal{H}^2_\gamma(E_{(r,y)}), \qquad \text{for every } (r, y) \in (0, \infty) \times \mathbb{R}^{n-3}.$$

The idea would be to substitute $E_{(r,y)}$ with the minimizer of the symmetric isoperimetric problem on the sphere $\partial B(r)$, under the constraint of having $\mathcal{H}^2_\gamma$ measure equal to μ.

Since the Gaussian measure is constant on spheres centered at the origin, it is enough to analyse the problem on the unit sphere $\mathbb{S}^2$, endowed with the standard Hausdorff measure.

The needed classification follows from the work of Viana [26] on the isoperimetric problem in real projective spaces $\mathbb{RP}^{n+1}$. Equivalently, Viana's result implies that, among antipodal-invariant regions of $\mathbb{S}^{n+1}$ with fixed volume, the least perimeter is attained by sets of the form

$$\left\{ x \in \mathbb{S}^{n+1} : \sum_{i=1}^{r} x_i^2 > a \right\},$$

for some $r \in \{1, \ldots, n+1\}$ and $a \in \mathbb{R}_+$. The class of sets which are symmetric with respect to the coordinate hyperplanes is contained in the class of antipodal-invariant sets. Moreover, the minimizers above are themselves symmetric with respect to all coordinate hyperplanes. Hence Viana's classification immediately yields the corresponding classification in the smaller n-symmetric class.

We record the statement in the form needed below.

Proposition 8 (Isoperimetric problem on the sphere with symmetry)**.** *Let $m \in (0, 4\pi)$. On the sphere $\mathbb{S}^2$, the perimeter has a minimum among the sets of area $\mathcal{H}^2(E) = m$, which are 3-symmetric (symmetric with respect to the three coordinate planes). More precisely,*

- *If $m < 2\pi$, the unique perimeter minimizer is the union of two antipodal geodesic caps.*
- *If $m > 2\pi$, the unique perimeter minimizer is a symmetric equatorial belt (the complement of two antipodal caps).*
- *If $m = 2\pi$, both configurations have equal perimeter.*

The fact that the shape of the symmetric isoperimetric problem on the sphere depends on the mass provides an obstruction to our idea: the rearrangement proposed above is not going to decrease the global Gaussian perimeter in general.

Example 2. *Let $E := (-a, a) \times \mathbb{R}^2$ be a symmetric slab. Taking a large enough, E is the only minimizer of the 3-symmetric Gaussian isoperimetric problem (see [4, Corollary 1]). If we foliate E by spheres $\partial B(r)$, the spherical slice E_r is a full sphere for $r < a$ and an equatorial belt for*

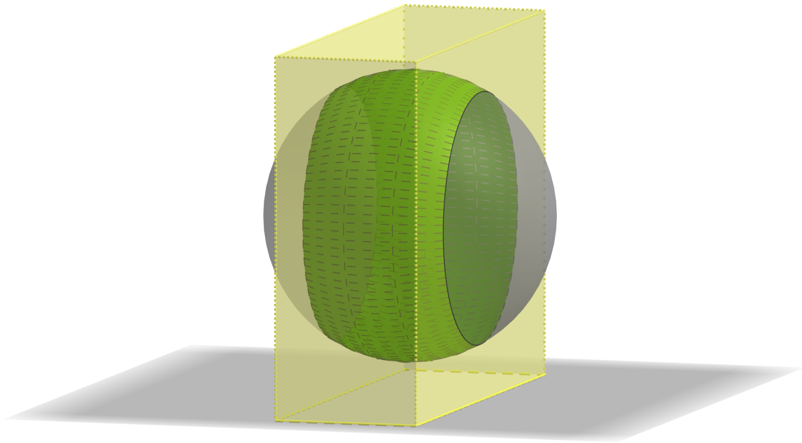

FIGURE 7. The intersection of a symmetric slab with a sphere is an equatorial belt.

$r > a$ (see Figure 7). However, for r large enough, the equatorial belt is no longer the minimizer on $\partial B(r)$ under the constraint to have measure $\mu(r)$. If we substitute the slice E_r with the two antipodal caps (or any other shape), we change the global shape of E and we strictly increase the Gaussian perimeter.

Remark 7. Viana's classification also bears on a different strategy: rather than substituting the spherical slices of a fixed $\mathbb{R}^n$ as above, one lifts the problem to spheres of growing dimension, in the spirit of the classical spherical derivation of the Gaussian isoperimetric inequality (see, e.g., [20]). Let $\mathbb{S}^N_{\sqrt{N}} \subset \mathbb{R}^{N+1}$ be the sphere of radius $\sqrt{N}$ and let $\Pi_{N+1,n} \colon \mathbb{R}^{N+1} \to \mathbb{R}^n$ be the projection onto the first n coordinates. An n-symmetric (hence centrally symmetric) set $E \subset \mathbb{R}^n$ lifts to the antipodally invariant set $\Pi^{-1}_{N+1,n}(E) \cap \mathbb{S}^N_{\sqrt{N}}$, whose normalized measure converges to $\gamma(E)$ as $N \to \infty$. Since the n-symmetric class lies inside the antipodally invariant one and Viana's minimizers $\{x : \sum_{i \le r} x_i^2 > a\}$ are themselves n-symmetric, one could hope to recover the symmetric Gaussian minimizers in the Poincaré limit $N \to \infty$.

This may fail because a minimizing rank $r = r_N$ need not stay bounded. For $\gamma(E) \in (0, 1/2)$ this boundedness agrees with Heilman's conjecture [19], underlying the a priori assumption $E = E' \times \mathbb{R}$. At mass $1/2$, there is always an escaping branch of minimizing ranks. Indeed, if r is a minimizing rank on $S^N_{\sqrt{N}} \subset \mathbb{R}^{N+1}$, then so is $N + 1 - r$, since the corresponding regions are complementary. Choosing the larger rank in each complementary pair gives a sequence $r_N \ge (N + 1)/2$, and hence $r_N \to \infty$.

APPENDIX C. CLASSIFICATION IN DIMENSION TWO

In this section, we address the symmetric Gaussian isoperimetric problem in the planar case $n = 2$. Unlike the one-dimensional setting, where the classification is straightforward, the two-dimensional problem presents a richer geometric structure among the possible competitors. A set E that is stationary for the Gaussian perimeter subject to a volume constraint must have a boundary ∂E satisfying the constant mean curvature equation:

$$\kappa(x) - x \cdot \nu^E(x) = \lambda, \tag{C.1}$$

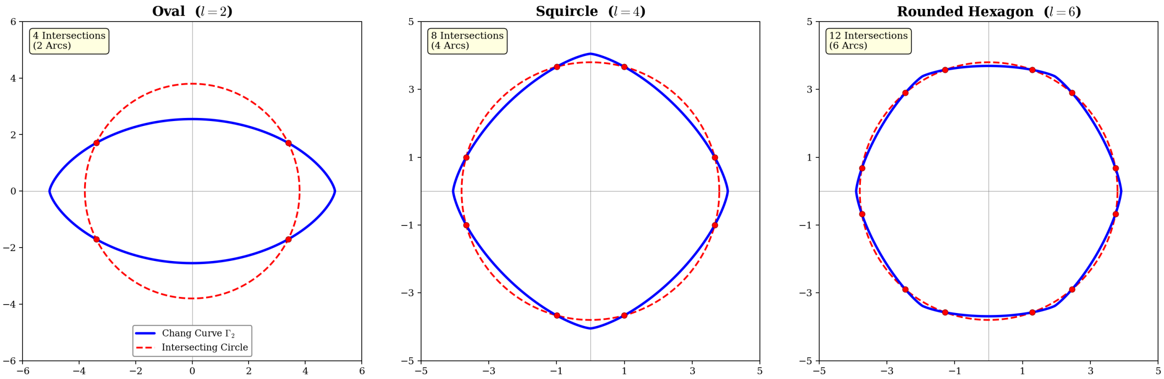

FIGURE 8. Chang curves Γ_l for $l = 2, 4, 6$, generated via the support function $h(\theta) = r_0 + \varepsilon \cos(l\theta)$. A generic circle $\partial B(r)$ with $r_{\min} < r < r_{\max}$ intersects Γ_l in exactly $2l$ points (marked in red). For $l \geq 4$, this yields at least 8 intersection points.

where ν^E is the exterior unit normal and $\kappa = \mathrm{div}_{\partial E}\, \nu^E$ is the corresponding signed curvature. While circles centered at the origin and straight lines (forming strips) are the most immediate solutions to (C.1), they are not the only ones. A notable class of non-trivial solutions is given by the so-called *Chang curves* [11]. These are closed compact curves that satisfy (C.1) with $\lambda < 0$ and exhibit non-trivial rotational symmetry (see Figure 8). The support function h satisfies the ODE

$$h'' + h = \frac{1}{h + \lambda_l}. \tag{C.2}$$

Chang curves with l-fold symmetry arise as solutions bifurcating from the trivial circle solution $h \equiv r_0$ at critical values of the radius. Linearizing (C.2) leads to the equation $\phi'' + (1 + r_0^2)\phi = 0$. The kernel of the linearized operator is non-trivial if and only if $\sqrt{1 + r_0^2} = k$ for some integer $k \geq 2$; for l-fold symmetry, one considers the branch corresponding to the eigenfunction $\cos(l\theta)$.

It is known that Chang curves Γ_l with l-fold symmetry, $l \geq 6$, are unstable critical points for the unconstrained Gaussian isoperimetric problem. However, instability in the general case does not automatically rule them out for the *symmetric* problem, as the directions of instability might break the symmetry constraint. We can prove via Theorem 1 that for the specific class of 2-symmetric sets, a Gaussian isoperimetric set cannot have a Chang curve of parameter $l \geq 4$ as boundary.

Lemma 10. *Let $l \geq 4$ be an even integer, and let $\Gamma_l \subset \mathbb{R}^2$ be the Chang curve with l-fold rotational symmetry. Since Γ_l is strictly convex and encloses the origin, for each $\theta \in [0, 2\pi)$ there exists a unique $r(\theta) > 0$ such that $r(\theta)e^{i\theta} \in \Gamma_l$. Define $r_{min} := \min r(\theta)$ and $r_{max} := \max r(\theta)$. Then, for every $r \in (r_{min}, r_{max})$, the circle $\partial B(r)$ intersects Γ_l in at least $2l$ points. In particular, the number of intersection points is at least 8. Consequently, by Theorem 1, the convex region E_l enclosed by Γ_l cannot be a minimizer of the symmetric Gaussian isoperimetric problem.*

Proposition 9. *Let $E \subset \mathbb{R}^2$ be a 2-symmetric minimizer of the Gaussian perimeter under fixed Gaussian area. Then, up to replacing E by E^c, the possible configurations for E reduce to only three configurations:*

(i) a symmetric strip;
(ii) a circle;
(iii) a compact convex set such that $\nu^E(p) \neq p/|p|$ for any $p \in \partial E \cap \mathbb{R}^2_+$.

Proof. Let $M := \partial E$. By Proposition 6, if M is disconnected, then E is a symmetric strip or its complement. We may therefore assume that M is connected.

After replacing E by E^c if necessary, we may assume that E is the bounded component enclosed by M. By regularity, M is analytic and satisfies (C.1), with ν^E the exterior unit normal.

Orient M counterclockwise by arclength s, and denote by τ its unit tangent. With our convention,

$$\partial_s \nu^E = \kappa\tau.$$

Differentiating (C.1) along M, we obtain

$$\partial_s \kappa = \partial_s(x \cdot \nu^E) = \kappa\, x \cdot \tau.$$

Consequently,

$$\partial_s \left(\kappa e^{-\frac{|x|^2}{2}} \right) = e^{-\frac{|x|^2}{2}} \left(\partial_s \kappa - \kappa x \cdot \tau \right) = 0.$$

Since M is connected, there exists $c \in \mathbb{R}$ such that

$$\kappa e^{-\frac{|x|^2}{2}} = c \qquad \text{on } M.$$

The turning-number theorem gives

$$2\pi = \int_M \kappa\, ds = c \int_M e^{\frac{|x|^2}{2}}\, ds,$$

and hence $c > 0$. Therefore $\kappa > 0$ everywhere on M, so E is strictly convex.

Set $M_+ := M \cap \mathbb{R}^2_+$. If there exists $p \in M_+$ such that $\nu^E(p) = \frac{p}{|p|}$, then Corollary 2 implies that M is a circle centered at the origin. Otherwise, alternative (iii) holds. □

Remark 8. Let $\Gamma_2 \subset \mathbb{R}^2$ be the Chang oval, i.e., the Chang curve with 2-fold rotational symmetry bifurcating from the circle of radius $r_0 = \sqrt{3}$. In [11], the existence of the Chang oval as a solution of the constant mean curvature equation (C.1) has been established for every $\lambda \in (\lambda_{\text{bif}}, 0)$, where $\lambda_{\text{bif}} = -2/\sqrt{3}$. As possible minimizer, it falls in the third case of the previous proposition.

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