

FROM $BV^{\mathcal{A}}$ TO BV : AN ENDPOINT KORN ESTIMATE

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ABSTRACT. In this short note we prove that, given a $\mathbb{C}$ -elliptic operator $\mathcal{A}$ and a map $u \in BV^{\mathcal{A}}(\Omega; V)$ satisfying $\nabla_{\text{ap}} u \in L^1(\Omega; V \otimes \mathbb{R}^d)$, then $u \in BV(\Omega; V)$. The result is quantitative and follows from the Korn-type estimate

$$|Du|(\Omega) \leq C_{d,\mathcal{A}} \left(\|\nabla_{\text{ap}} u\|_{L^1(\Omega)} + |\mathcal{A}u|(\Omega) \right),$$

valid for every such $u \in BV^{\mathcal{A}}(\Omega; V)$. As a consequence, we obtain the characterization

$$BV^{\mathcal{A}}(\Omega; V) \setminus BV(\Omega; V) = \left\{ u \in BV^{\mathcal{A}}(\Omega; V) : \nabla_{\text{ap}} u \notin L^1(\Omega; V \otimes \mathbb{R}^d) \right\}.$$

Generative AI has been used. Detailed workflow in a specific Section.

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1. INTRODUCTION

The classical Korn inequality for $1 < p < \infty$ controls the full gradient of a displacement, up to a constant skew-symmetric matrix, by its symmetric gradient. However, at the endpoint $p = 1$, due to Ornstein's non-inequality [10], this type of estimate generally fails. Several results are available that recover useful Korn-type estimates for $p = 1$, for instance by removing a controlled exceptional set or by allowing different rigid motions on different pieces of the domain, as in [8, 5]. In this paper we prove a Korn-type estimate at the endpoint $p = 1$, valid without removing any exceptional set, but at the price of including the L^1 norm of

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the full approximate gradient. In particular, this result gives a simple characterization of those BD maps which fail to belong to BV, answering the question raised in [7]: they are precisely the ones whose full approximate gradient is not integrable. Since the mechanism behind the argument is not specific to the symmetric gradient, we work in the more general setting of functions of bounded $\mathcal{A}$ -variation, for first-order $\mathbb{C}$ -elliptic operators (see Section 2.2).

Let

$$\mathcal{A}u = \sum_{j=1}^d A_j \partial_j u$$

be a first-order homogeneous differential operator with constant coefficients. Assuming that $\mathcal{A}$ is $\mathbb{C}$ -elliptic, we consider the space $BV^{\mathcal{A}}$ of functions whose $\mathcal{A}$ -gradient is a finite Radon measure (see [4] or Section 2.2 below). Moreover, by [9, Lemma 3.1] (see also Theorem 2.4 below), every $u \in BV^{\mathcal{A}}$ admits an approximate gradient $\nabla_{\text{ap}} u$ almost everywhere. Our main theorem then reads as follows.

Theorem 1.1. *Let $\mathcal{A}$ be a $\mathbb{C}$ -elliptic first-order homogeneous differential operator. Then there exists $C_{d,\mathcal{A}} > 0$ such that, for every open set $\Omega \subset \mathbb{R}^d$ and every $u \in BV^{\mathcal{A}}(\Omega)$ satisfying*

$$\nabla_{\text{ap}} u \in L^1(\Omega; V \otimes \mathbb{R}^d),$$

one has $u \in BV(\Omega; V)$ and

$$|Du|(\Omega) \leq C_{d,\mathcal{A}} \left(|\mathcal{A}u|(\Omega) + \int_{\Omega} |\nabla_{\text{ap}} u| \, dx \right).$$

We emphasize that the constant $C_{d,\mathcal{A}}$ is independent of Ω and that the above does not require any regularity assumption on $\partial\Omega$. For the sake of a cleaner statement, we formulate the result in terms of the full approximate gradient. The essential quantity is however only its component in the kernel of the linear map induced by $\mathcal{A}$ on first derivatives (cf. [9, Lemma 3.1]), the complementary component being already controlled by $\mathcal{A}u$.

The model case is $\mathcal{A} = \mathcal{E}$, the symmetric gradient. In this case, Theorem 1.1 yields

$$BV_{\text{loc}}(\Omega; \mathbb{R}^d) = \{u \in BD_{\text{loc}}(\Omega) : \text{skew } \nabla_{\text{ap}} u \in L^1_{\text{loc}}(\Omega)\}.$$

Thus, the gap between BD and BV is entirely characterized by the integrability of the skew-symmetric part of the approximate gradient, which is not seen by $\mathcal{E}u$. This phenomenon is already visible in the examples constructed in [7]: they produce functions in SBD which fail to belong to GBV by building increasingly large infinitesimal rotations while keeping the symmetric part of the gradient under control. Our result shows that this is the only possible obstruction: the failure of a BD map to belong to BV can be entirely attributed to this uncontrolled part of its absolutely continuous gradient.

The same picture holds for general $\mathbb{C}$ -elliptic operators:

$$BV_{\text{loc}}(\Omega; V) = \left\{ u \in BV^{\mathcal{A}}_{\text{loc}}(\Omega; V) : \nabla_{\text{ap}} u \in L^1_{\text{loc}}(\Omega; V \otimes \mathbb{R}^d) \right\}.$$

The measure $\mathcal{A}u$ controls the component of the first derivative seen by the symbol, while the remaining component may lie in its kernel. Requiring $\nabla_{\text{ap}} u \in L^1$ supplies precisely the missing

first-order information. As a consequence of Theorem 1.1, we also obtain the corresponding characterization of special functions:

$$SBV_{\text{loc}}(\Omega; V) = \left\{ u \in SBV_{\text{loc}}^{\mathcal{A}}(\Omega; V) : \nabla_{\text{ap}} u \in L_{\text{loc}}^1(\Omega; V \otimes \mathbb{R}^d) \right\}.$$

Indeed, if

$$u \in SBV^{\mathcal{A}}(\Omega; V) \quad \text{and} \quad \nabla_{\text{ap}} u \in L^1(\Omega; V \otimes \mathbb{R}^d),$$

then Theorem 1.1 gives $u \in BV(\Omega; V)$. Applying the estimate locally on arbitrary open subsets $U \subset \subset \Omega$ yields

$$|Du|(U) \leq C_{d,\mathcal{A}} \left(|\mathcal{A}u|(U) + \int_U |\nabla_{\text{ap}} u| \, dx \right),$$

and hence, by outer regularity,

$$|D^s u| \leq C_{d,\mathcal{A}} |(\mathcal{A}u)^s|.$$

Since $u \in SBV^{\mathcal{A}}(\Omega)$, the singular part $(\mathcal{A}u)^s$ is concentrated on the jump set. Therefore $D^s u$ is concentrated on the same rectifiable set. As the Cantor part of a BV derivative vanishes on every σ -finite $\mathcal{H}^{d-1}$ -set, it follows that $D^c u = 0$, and hence $u \in SBV(\Omega; V)$.

1.1. Sketch of the proof. The proof is based on a localization and mollification argument both occurring at the same scale. Fix $\varepsilon > 0$, mollify u at scale ε and cover the interior of Ω by a grid of cubes of size comparable with ε . On each cube, the Poincaré inequality for $BV^{\mathcal{A}}$ [4, Theorem 3.2] (see also Theorem 3.1) selects a null-space polynomial $a_Q \in \text{Ker}(\mathcal{A})$ such that $u - a_Q$ is controlled in L^1 by $|\mathcal{A}u|$ on a slightly larger ball. We then split locally

$$\nabla u_\varepsilon = \nabla((u - a_Q) * \varrho_\varepsilon) + \nabla(a_Q * \varrho_\varepsilon)$$

and we aim to find an ε -independent L^1 bound for ∇u_ε .

The first term is directly controlled by the Poincaré estimate. The main point is therefore the second term. More precisely, since a_Q is smooth, the crucial point is to estimate $\|\nabla a_Q\|_{L^1}$, since

$$\|\nabla(a_Q * \varrho_\varepsilon)\|_{L^1} \lesssim \|\nabla a_Q\|_{L^1}. \quad (1)$$

This is where the summability of $\nabla_{\text{ap}} u$ enters the argument: the key point is Proposition 4.1, which, for the same a_Q selected above in the Poincaré inequality (Theorem 3.1) provides, by exploiting standard singular integral theory (Proposition 3.4), an estimate of the form

$$\|\nabla_{\text{ap}} u - \nabla a_Q\|_{L^{1,\infty}} \lesssim |\mathcal{A}u|.$$

Since $\text{Ker}(\mathcal{A})$ is finite-dimensional by [4, Theorem 2.6] (see also Theorem 3.2 below), the weak- L^1 quasi-norm and the strong L^1 norm are equivalent on the space of gradients of null-space polynomials. Therefore

$$\|\nabla a_Q\|_{L^1} \lesssim \|\nabla a_Q\|_{L^{1,\infty}} \lesssim \|\nabla_{\text{ap}} u - \nabla a_Q\|_{L^{1,\infty}} + \|\nabla_{\text{ap}} u\|_{L^{1,\infty}} \lesssim |\mathcal{A}u| + \|\nabla_{\text{ap}} u\|_{L^1}. \quad (2)$$

Combining (1), (2) and summing over the grid gives an ε -independent bound

$$\int_{\Omega'} |\nabla u_\varepsilon| \, dx \lesssim |\mathcal{A}u|(\Omega) + \|\nabla_{\text{ap}} u\|_{L^1(\Omega)}$$

for every $\Omega' \subset \subset \Omega$. The conclusion then follows from BV compactness and an exhaustion of Ω .

1.2. AI content disclosure. The problem addressed in this note had been a long-term project of the author. The mathematical starting point and the original proof strategy were developed by the author independently of generative AI. This approach was based on a nonlocal approximation of the gradient by spherical averages. In the model case of BD, the resulting quantity splits into a part controlled by the symmetric gradient and a rotational part, which can be estimated by Calderón–Zygmund theory for $p > 1$. The endpoint strategy was to seek a direct estimate of this rotational term by treating separately regions near regular jump sets and far from regular jump sets, using the geometry of tubular neighbourhoods. During subsequent interaction with generative AI, this strategy was extensively tested and refined, eventually leading to a complete working argument, although not in the generality presented in the present paper.

At that stage, generative AI was used mainly as an interactive research assistant and mathematical sounding board to discuss, test, and simplify the proof. Through an extended dialogue, the already working strategy was progressively dismantled: several of its technical components turned out not to be essential, and the argument was reduced to a much simpler local mechanism. In particular, the relevant issue was isolated as the control of the gradient of the null-space polynomial selected by the Poincaré projection in Proposition 4.1. The local weak endpoint estimate and its combination with finite-dimensional norm equivalence emerged from this iterative discussion rather than being part of the author’s original proof strategy. Recognizing this mechanism allowed the original approach to be replaced by the substantially simpler localization and mollification argument developed in the present paper.

The AI was also used to locate relevant references and to assist with the English presentation. The author takes full responsibility for the contents of the paper.

The specific AI provider used is not relevant to the mathematical discussion or to the account of the interaction given above.

2. PRELIMINARIES

2.1. General notation. Throughout the paper we assume $d \geq 2$. The notation $\mathcal{L}^d$ stands for the d -dimensional Lebesgue measure on $\mathbb{R}^d$. Throughout, constants denoted by $C_{d,\mathcal{A}}$ may change from line to line and depend only on the dimension and the operator $\mathcal{A}$. $B_r(x) := x + rB$ where B stands for the unit Euclidean ball of $\mathbb{R}^d$, while $Q_r(x) := x + rQ$ where $Q := [-1/2, 1/2]^d$ is the d -dimensional unit cube with barycenter at 0. For a given measurable set A , we denote by $\mathbb{1}_A(x)$ the characteristic function of A , which takes the value 1 whenever $x \in A$ and 0 elsewhere. Given an open set A , the notation $U \subset\subset A$ means that $\overline{U} \subset A$. For $v \in V$ and $\xi \in \mathbb{R}^d$, we identify the tensor $v \otimes \xi \in V \otimes \mathbb{R}^d$ with the linear map $\mathbb{R}^d \rightarrow V$ defined by

$$(v \otimes \xi)h := (\xi \cdot h)v.$$

2.2. Maps of bounded $\mathcal{A}$ -variation. Let $\mathcal{A} : C^\infty(\mathbb{R}^d; V) \rightarrow C^\infty(\mathbb{R}^d; W)$ be, for some Euclidean spaces V, W , a first-order homogeneous differential operator with constant coefficients

$$\mathcal{A}u := \sum_{j=1}^d A_j \partial_j u, \quad A_j \in \text{Lin}(V, W). \quad (3)$$

The space $\text{Lin}(V; W)$ denotes the family of all linear maps from V to W . The space of functions with *bounded $\mathcal{A}$ -variation* (see also [4]) is defined by

$$BV^{\mathcal{A}}(\Omega) := \{u \in L^1(\Omega; V) \mid \mathcal{A}u \in \mathcal{M}(\Omega; W)\},$$

where $\mathcal{M}(\Omega; W)$ denotes the space of W -valued finite Radon measures on Ω , and the distributional $\mathcal{A}$ -gradient is defined by

$$\int_{\Omega} \varphi \cdot d\mathcal{A}u := \int_{\Omega} \mathcal{A}^* \varphi \cdot u dx, \quad \forall \varphi \in C_c^\infty(\Omega; W), \quad (4)$$

where the L^2 -adjoint operator $\mathcal{A}^* : C^1(\mathbb{R}^d; W) \rightarrow C^0(\mathbb{R}^d; V)$ is defined in terms of $\{A_j^*\}_{j=1}^d \subset \text{Lin}(W; V)$, satisfying $A_j z \cdot \eta = z \cdot A_j^* \eta$ for all $z \in V, \eta \in W$, by

$$\mathcal{A}^* v = - \sum_{j=1}^d A_j^* \partial_j v.$$

The classical examples are $\mathcal{A} = \nabla$, for which $BV^{\mathcal{A}} = BV$ (see [3]), the symmetric gradient $\mathcal{A} = \mathcal{E} = \frac{\nabla + \nabla^t}{2}$, for which $BV^{\mathcal{A}} = BD$ (see [2]) and the deviatoric operator $\mathcal{A} = \mathcal{E} - \frac{\text{div}(\cdot)}{d} I_d$ (see [6]), for which $BV^{\mathcal{A}} = BD_{\text{dev}}$.

For $\xi \in \mathbb{C}^d$ and $v \in V + iV$, the *complexified symbol* is defined as

$$\mathbb{A}[\xi]v := \sum_{j=1}^d \xi_j A_j v. \quad (5)$$

Definition 2.1. *The operator $\mathcal{A}$ is said to be $\mathbb{R}$ -elliptic if $\mathbb{A}[\xi] : V \rightarrow W$ is injective for every $\xi \in \mathbb{R}^d \setminus \{0\}$. The operator $\mathcal{A}$ is said to be $\mathbb{C}$ -elliptic if its complexified symbol $\mathbb{A}[\xi] : V + iV \rightarrow W + iW$ is injective for every $\xi \in \mathbb{C}^d \setminus \{0\}$.*

Remark 2.2. Clearly $\mathbb{C}$ -ellipticity implies $\mathbb{R}$ -ellipticity.

The following useful Leibniz rule applies (see also [4]).

Lemma 2.3 (Cut-off product rule [4]). *Let $U \subset \mathbb{R}^d$ be open, $v \in BV^{\mathcal{A}}(U)$, and $\eta \in C_c^1(U)$. Extend ηv by zero outside U . Then $\eta v \in BV^{\mathcal{A}}(\mathbb{R}^d)$ and*

$$\mathcal{A}(\eta v) = \eta \mathcal{A}v + \mathbb{A}[\nabla \eta]v \mathcal{L}^d \quad \text{in } \mathbb{R}^d. \quad (6)$$

Consequently,

$$|\mathcal{A}(\eta v)|(\mathbb{R}^d) \leq \|\eta\|_{L^\infty(U)} |\mathcal{A}v|(U) + C_{\mathcal{A}} \|\nabla \eta\|_{L^\infty(U)} \|v\|_{L^1(U)}. \quad (7)$$

Proof. For $\varphi \in C_c^1(\mathbb{R}^d; W)$, using the distributional adjoint $\mathcal{A}^*$ in (4) and the Leibniz rule for $\mathcal{A}^*(\eta \varphi)$, a simple computation immediately gives

$$\langle \mathcal{A}(\eta v), \varphi \rangle = \langle \eta \mathcal{A}v, \varphi \rangle + \int_U \varphi \cdot \mathbb{A}[\nabla \eta]v dx.$$

This is (6). Since the finite-dimensional bilinear map $(\xi, v) \mapsto \mathbb{A}[\xi]v$ satisfies $|\mathbb{A}[\xi]v| \leq C_{\mathcal{A}} |\xi| |v|$ estimate (7) follows. $\square$

We recall from [9] that, whenever $\mathcal{A}$ is $\mathbb{C}$ -elliptic, every $u \in BV^{\mathcal{A}}(\Omega)$ is approximately differentiable a.e. in Ω . The cited result is stated on $\mathbb{R}^d$; the local version below follows easily by multiplication with a compactly supported cut-off and the Leibniz rule for $\mathcal{A}$ in Lemma 2.3.

Theorem 2.4 ([9, Lemma 3.1]). *Let $\Omega \subset \mathbb{R}^d$ be open, let $\mathcal{A}$ be an $\mathbb{R}$ -elliptic first-order homogeneous differential operator, and let $u \in \text{BV}_{\text{loc}}^{\mathcal{A}}(\Omega)$. Then u is L^1 -differentiable at $\mathcal{L}^d$ -almost every point of Ω . In particular, there exists $\nabla_{\text{ap}} u(x) \in V \otimes \mathbb{R}^d$ such that*

$$\int_{B_r(x)} |u(y) - u(x) - \nabla_{\text{ap}} u(x)(y - x)| dy = o(r)$$

for $\mathcal{L}^d$ -almost every $x \in \Omega$.

3. TECHNICAL TOOLS

3.1. Poincaré inequality and characterization of $\mathbb{C}$ -ellipticity. For a set $U \subset \mathbb{R}^d$, let Π_U denote the $L^2(U; V)$ -orthogonal projection onto the finite-dimensional space

$$\text{Ker}(\mathcal{A}) := \{a \in (C_c^\infty(\mathbb{R}^d; V))' : \mathcal{A}a = 0\}.$$

As shown in [4, Section 3.1], Π_U extends boundedly to $L^1(U; V)$; this extension is the one used in Theorem 3.1 below.

Theorem 3.1 (Poincaré inequality in $\text{BV}^{\mathcal{A}}$; [4, Theorem 3.2]). *Assume that $\mathcal{A}$ is $\mathbb{C}$ -elliptic. There exists $C_{d,\mathcal{A}} > 0$ such that for every ball $B_r(x) \subset \mathbb{R}^d$ and every $u \in \text{BV}^{\mathcal{A}}(B_r(x))$,*

$$\inf_{q \in \text{Ker}(\mathcal{A})} \|u - q\|_{L^1(B_r(x))} \leq \|u - \Pi_{B_r(x)} u\|_{L^1(B_r(x))} \leq C_{d,\mathcal{A}} r |\mathcal{A}u|(B_r(x)). \quad (8)$$

Finally, the following useful characterization will be used to transform the $L^{1,\infty}$ control on $\Pi_{B_r(x)} u$ into an L^1 -type control.

Theorem 3.2 ([4, Theorem 2.6]). *Let $\mathcal{A}$ be a first-order homogeneous differential operator with constant coefficients. The following are equivalent:*

- (a) $\mathcal{A}$ has finite-dimensional null-space $\text{Ker}(\mathcal{A})$;
- (b) $\mathcal{A}$ is $\mathbb{C}$ -elliptic;
- (c) there exists $\ell \in \mathbb{N}$ such that $\text{Ker}(\mathcal{A})$ is contained in the space of V -valued polynomials of degree at most ℓ .

3.2. Mollified gradients at differentiability points. Fix a standard mollifier $\varrho \in C_c^\infty(B_1)$, $\varrho \geq 0$, $\int \varrho = 1$, and set $\varrho_\delta(x) = \delta^{-d} \varrho(x/\delta)$. Note that, in our notation, for a map $v : \mathbb{R}^d \rightarrow V$,

$$\nabla(v * \varrho_\delta)(x) = \int_{\mathbb{R}^d} v(y) \otimes \nabla \varrho_\delta(x - y) dy,$$

since, when applied to e_j , the right-hand side equals $\int_{\mathbb{R}^d} v(y) \partial_j \varrho_\delta(x - y) dy = \partial_j(v * \varrho_\delta)(x)$.

Lemma 3.3 (Mollified approximate gradients). *Let $v \in L_{\text{loc}}^1(\mathbb{R}^d; V)$. Suppose that v is L^1 -differentiable at x , namely*

$$\int_{B_r(x)} |v(y) - v(x) - \nabla_{\text{ap}} v(x)(y - x)| dy = o(r).$$

Then

$$\nabla(v * \varrho_\delta)(x) \longrightarrow \nabla_{\text{ap}} v(x) \quad \text{as } \delta \downarrow 0. \quad (9)$$

Proof. Set

$$R(y) := v(y) - v(x) - \nabla_{\text{ap}} v(x)(y - x).$$

Then

$$\begin{aligned}
\nabla(v * \varrho_\delta)(x) &= \int_{\mathbb{R}^d} v(y) \otimes \nabla \varrho_\delta(x - y) dy \\
&= \int_{\mathbb{R}^d} \left(v(x) + \nabla_{\text{ap}} v(x)(y - x) + R(y) \right) \otimes \nabla \varrho_\delta(x - y) dy \\
&= v(x) \otimes \int_{\mathbb{R}^d} \nabla \varrho_\delta(x - y) dy \\
&\quad + \nabla_{\text{ap}} v(x) \int_{\mathbb{R}^d} (y - x) \otimes \nabla \varrho_\delta(x - y) dy \\
&\quad + \int_{\mathbb{R}^d} R(y) \otimes \nabla \varrho_\delta(x - y) dy \\
&= \nabla_{\text{ap}} v(x) + \int_{B_\delta(x)} R(y) \otimes \nabla \varrho_\delta(x - y) dy.
\end{aligned}$$

The last equality follows from integration by parts and the properties of the mollifier. Therefore

$$|\nabla(v * \varrho_\delta)(x) - \nabla_{\text{ap}} v(x)| \leq C_\varrho \delta^{-d-1} \int_{B_\delta(x)} |R(y)| dy = o(1).$$

□

3.3. $L^{1,\infty}$ -control on the approximate gradient. For a measurable function f defined on an open set $U \subset \mathbb{R}^d$, we recall that the weak- L^1 quasi-norm is defined by

$$\|f\|_{L^{1,\infty}(U)} := \sup_{s>0} s \mathcal{L}^d(\{x \in U : |f(x)| > s\}). \quad (10)$$

In particular, it is immediate that

$$\|f\|_{L^{1,\infty}(U)} \leq \|f\|_{L^1(U)}. \quad (11)$$

The next proposition is a standard consequence of the potential representation for elliptic operators and the weak $(1, 1)$ boundedness of Calderón–Zygmund operators. See also [2] for the BD case.

Proposition 3.4 (Global weak endpoint estimate). *Assume that $\mathcal{A}$ is elliptic. Then there exists $C_{d,\mathcal{A}} > 0$ such that every compactly supported $v \in BV^{\mathcal{A}}(\mathbb{R}^d)$ satisfies*

$$\|\nabla_{\text{ap}} v\|_{L^{1,\infty}(\mathbb{R}^d)} \leq C_{d,\mathcal{A}} |\mathcal{A}v|(\mathbb{R}^d). \quad (12)$$

Proof. By the potential representation for elliptic operators [9, Lemma 2.1], one can write $v = K_{\mathcal{A}} * \mathcal{A}v$, where $K_{\mathcal{A}}$ is smooth away from the origin and $(1 - d)$ -homogeneous. By [1, Section 3.3, Theorem 3.4], $\nabla_{\text{ap}} v$ is the sum of the singular integral associated with $\nabla K_{\mathcal{A}}$ and a bounded linear image of $\frac{d\mathcal{A}v}{d\mathcal{L}^d}$. The former satisfies the classical weak- $(1, 1)$ estimate, while the latter belongs to L^1 with norm bounded by $C_{\mathcal{A}} |\mathcal{A}v|(\mathbb{R}^d)$. Hence

$$\|\nabla_{\text{ap}} v\|_{L^{1,\infty}(\mathbb{R}^d)} \leq C_{d,\mathcal{A}} |\mathcal{A}v|(\mathbb{R}^d).$$

□

4. WEAK KORN-POINCARÉ ESTIMATE AND CONTROL OF THE NULL-SPACE GRADIENT

We now combine Poincaré with Proposition 3.4. The key point is that the weak estimate is obtained with the same null-space polynomial selected by the Poincaré projection.

Proposition 4.1 (Local weak Korn–Poincaré estimate). *Assume that $\mathcal{A}$ is $\mathbb{C}$ -elliptic. Let $u \in \text{BV}^{\mathcal{A}}(B_r(x))$. Set*

$$a_{r,x} := \Pi_{B_r(x)} u \in \text{Ker}(\mathcal{A}).$$

Then

$$\int_{B_r(x)} |u - a_{r,x}| dx \leq C_{d,\mathcal{A}} r |\mathcal{A}u|(B_r(x)), \quad (13)$$

and

$$\|\nabla_{\text{ap}} u - \nabla a_{r,x}\|_{L^{1,\infty}(B_{r/2}(x))} \leq C_{d,\mathcal{A}} |\mathcal{A}u|(B_r(x)). \quad (14)$$

Proof. Estimate (13) follows directly from Theorem 3.1. Now choose a cutoff function $\eta \in C_c^\infty(B_r(x))$ such that

$$0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ on } B_{r/2}(x), \quad |\nabla \eta| \leq C_d r^{-1}.$$

Set

$$w := \mathbb{1}_{B_r(x)} \eta(u - a_{r,x}) \in \text{BV}^{\mathcal{A}}(\mathbb{R}^d). \quad (15)$$

Since $a_{r,x} \in \text{Ker}(\mathcal{A})$, one has $\mathcal{A}a_{r,x} = 0$. By Lemma 2.3 and (13),

$$|\mathcal{A}w|(\mathbb{R}^d) \leq |\mathcal{A}u|(B_r(x)) + C_{\mathcal{A}} r^{-1} \int_{B_r(x)} |u - a_{r,x}| dx \leq C_{d,\mathcal{A}} |\mathcal{A}u|(B_r(x)). \quad (16)$$

Applying Proposition 3.4 to w gives

$$\|\nabla_{\text{ap}} w\|_{L^{1,\infty}(\mathbb{R}^d)} \leq C_{d,\mathcal{A}} |\mathcal{A}u|(B_r(x)).$$

On $B_{r/2}(x)$, $w = u - a_{r,x}$ and hence

$$\nabla_{\text{ap}} w = \nabla_{\text{ap}} u - \nabla a_{r,x} \quad \text{a.e.}$$

Restricting the weak estimate proves (14). $\square$

Unlike in the BV or BD case, $a_{r,x}$ need not be affine; however, by Theorem 3.2, it is a polynomial of degree bounded solely in terms of $\mathcal{A}$, and thus we can recover an L^1 -type control on the null-space gradient from the $L^{1,\infty}$ control given by Proposition 4.1.

Lemma 4.2. *Assume that $\mathcal{A}$ is $\mathbb{C}$ -elliptic. There exists $C_{d,\mathcal{A}} > 0$ such that for every ball $B_r(x)$ and every $a \in \text{Ker}(\mathcal{A})$,*

$$\int_{B_r(x)} |\nabla a| dx \leq C_{d,\mathcal{A}} \|\nabla a\|_{L^{1,\infty}(B_r(x))}. \quad (17)$$

The constant is independent of x and r .

Proof. By translation and scaling it is enough to prove the estimate on the unit ball. Indeed, if $\tilde{a}(y) := a(x + ry)$, then $\tilde{a} \in \text{Ker}(\mathcal{A})$ (since $\mathcal{A}$ is a first-order homogeneous differential operator with constant coefficients) and $\nabla \tilde{a}(y) = r \nabla a(x + ry)$. By Theorem 3.2, the space

$$X := \{\nabla a|_B : a \in \text{Ker}(\mathcal{A})\}$$

is finite-dimensional and consists of polynomial tensor fields. Since $\|\cdot\|_{L^{1,\infty}(B)}$ is a quasi-norm and $\|\cdot\|_{L^1(B)}$ is a norm (hence, in particular, a quasi-norm), and since all quasi-norms are equivalent on finite-dimensional spaces, we conclude. $\square$

Lemma 4.3 (L¹ control of the null-space gradient). *Under the assumptions of Proposition 4.1,*

$$\int_{B_{r/2}(x)} |\nabla a_{r,x}| dx \leq C_{d,A} \left(\int_{B_{r/2}(x)} |\nabla_{\text{ap}} u| dx + |\mathcal{A}u|(B_r(x)) \right). \quad (18)$$

Proof. Observe that, for two arbitrary measurable functions f, g , we trivially have

$$\{|f + g| > s\} \subset \{|f| > s/2\} \cup \{|g| > s/2\}.$$

Thus

$$\|f + g\|_{L^{1,\infty}(U)} \leq 2(\|f\|_{L^{1,\infty}(U)} + \|g\|_{L^{1,\infty}(U)}), \quad (19)$$

Using Lemma 4.2 on $B_{r/2}(x)$, then (19), (11), and (14), we obtain

$$\begin{aligned} \int_{B_{r/2}(x)} |\nabla a_{r,x}| dx &\leq C_{d,A} \|\nabla a_{r,x}\|_{L^{1,\infty}(B_{r/2}(x))} \\ &\leq C_{d,A} \left(\|\nabla_{\text{ap}} u\|_{L^{1,\infty}(B_{r/2}(x))} + \|\nabla_{\text{ap}} u - \nabla a_{r,x}\|_{L^{1,\infty}(B_{r/2}(x))} \right) \\ &\leq C_{d,A} \left(\int_{B_{r/2}(x)} |\nabla_{\text{ap}} u| dx + |\mathcal{A}u|(B_r(x)) \right). \end{aligned}$$

□

5. PROOF OF THE MAIN THEOREM

Proof of Theorem 1.1. Fix $\Omega' \subset \subset \Omega$ and set

$$\delta := \text{dist}(\Omega', \partial\Omega) > 0.$$

Let $\varepsilon > 0$ be small and consider the standard dyadic grid of pairwise disjoint cubes $Q_{4\varepsilon}(x_Q)$. Let $\mathcal{Q}_\varepsilon(\Omega')$ be the family of grid cubes intersecting Ω' in a set of positive measure. For each $Q_{4\varepsilon}(x_Q) \in \mathcal{Q}_\varepsilon(\Omega')$ consider the concentric balls $B_{6\varepsilon\sqrt{d}}(x_Q) \supset B_{3\varepsilon\sqrt{d}}(x_Q) \supset Q_{6\varepsilon}(x_Q)$. Note that for $\varepsilon < \varepsilon_0(\delta)$ small enough we can guarantee that $B_{6\varepsilon\sqrt{d}}(x_Q) \subset \subset \Omega$ for every $Q_{4\varepsilon}(x_Q) \in \mathcal{Q}_\varepsilon(\Omega')$.

Now, for each $Q_{4\varepsilon}(x_Q)$, apply Proposition 4.1 on $B_{6\varepsilon\sqrt{d}}(x_Q)$. Thus, setting

$$a_{\varepsilon,x_Q} := \Pi_{B_{6\varepsilon\sqrt{d}}(x_Q)} u \in \text{Ker}(\mathcal{A}),$$

we have

$$\int_{B_{6\varepsilon\sqrt{d}}(x_Q)} |u - a_{\varepsilon,x_Q}| dx \leq C_{d,A} \varepsilon |\mathcal{A}u|(B_{6\varepsilon\sqrt{d}}(x_Q)), \quad (20)$$

and

$$\|\nabla_{\text{ap}} u - \nabla a_{\varepsilon,x_Q}\|_{L^{1,\infty}(B_{3\varepsilon\sqrt{d}}(x_Q))} \leq C_{d,A} |\mathcal{A}u|(B_{6\varepsilon\sqrt{d}}(x_Q)). \quad (21)$$

Take a mollifier $\{\varrho_\varepsilon\}_{\varepsilon>0}$ as in Subsection 3.2. For $x \in \Omega_\varepsilon := \{x \in \Omega : \text{dist}(x, \partial\Omega) > \varepsilon\}$, set

$$u_\varepsilon(x) := \int_{B_\varepsilon} u(x-z) \varrho_\varepsilon(z) dz.$$

On $Q_{4\varepsilon}(x_Q)$ we decompose

$$\nabla u_\varepsilon = \nabla((u - a_{\varepsilon,x_Q}) * \varrho_\varepsilon) + \nabla(a_{\varepsilon,x_Q} * \varrho_\varepsilon). \quad (22)$$

For the oscillation term, Fubini's theorem gives

$$\begin{aligned}
\int_{Q_{4\varepsilon}(x_Q)} |\nabla((u - a_{\varepsilon, x_Q}) * \varrho_\varepsilon)(x)| \, dx &\leq \int_{B_\varepsilon} |\nabla \varrho_\varepsilon(z)| \int_{Q_{4\varepsilon}(x_Q)} |u(x - z) - a_{\varepsilon, x_Q}(x - z)| \, dx \, dz \\
&\leq \|\nabla \varrho_\varepsilon\|_{L^1} \int_{Q_{6\varepsilon}(x_Q)} |u - a_{\varepsilon, x_Q}| \, dx \\
&\leq \frac{C_\varrho}{\varepsilon} \int_{B_{6\varepsilon\sqrt{d}}(x_Q)} |u - a_{\varepsilon, x_Q}| \, dx \\
&\leq C_{d, \mathcal{A}} |\mathcal{A}u|(B_{6\varepsilon\sqrt{d}}(x_Q)). \tag{23}
\end{aligned}$$

The second inequality follows from the fact that, since $\text{spt } \varrho_\varepsilon \subset B_\varepsilon$, for every $x \in Q_{4\varepsilon}(x_Q)$ and $z \in \text{spt } \varrho_\varepsilon$ one has $x - z \in Q_{6\varepsilon}(x_Q)$. In the last inequality, we applied (20).

For the null-space term we now use that a_{ε, x_Q} is smooth. Hence

$$\nabla(a_{\varepsilon, x_Q} * \varrho_\varepsilon) = (\nabla a_{\varepsilon, x_Q}) * \varrho_\varepsilon,$$

and, again by Fubini,

$$\begin{aligned}
\int_{Q_{4\varepsilon}(x_Q)} |\nabla(a_{\varepsilon, x_Q} * \varrho_\varepsilon)(x)| \, dx &\leq \int_{B_\varepsilon} \varrho_\varepsilon(z) \int_{Q_{4\varepsilon}(x_Q)} |\nabla a_{\varepsilon, x_Q}(x - z)| \, dx \, dz \\
&\leq \int_{Q_{6\varepsilon}(x_Q)} |\nabla a_{\varepsilon, x_Q}| \, dx \leq \int_{B_{3\varepsilon\sqrt{d}}(x_Q)} |\nabla a_{\varepsilon, x_Q}| \, dx. \tag{24}
\end{aligned}$$

Applying Lemma 4.3 on $B_{6\varepsilon\sqrt{d}}(x_Q)$ and using (24) gives

$$\int_{Q_{4\varepsilon}(x_Q)} |\nabla(a_{\varepsilon, x_Q} * \varrho_\varepsilon)| \, dx \leq C_{d, \mathcal{A}} \left(\int_{B_{3\varepsilon\sqrt{d}}(x_Q)} |\nabla_{\text{ap}} u| \, dx + |\mathcal{A}u|(B_{6\varepsilon\sqrt{d}}(x_Q)) \right). \tag{25}$$

Combining (22), (23), and (25) yields

$$\int_{Q_{4\varepsilon}(x_Q)} |\nabla u_\varepsilon| \, dx \leq C_{d, \mathcal{A}} \left(\int_{B_{3\varepsilon\sqrt{d}}(x_Q)} |\nabla_{\text{ap}} u| \, dx + |\mathcal{A}u|(B_{6\varepsilon\sqrt{d}}(x_Q)) \right). \tag{26}$$

The family $\{B_{6\varepsilon\sqrt{d}}(x_Q) : Q_{4\varepsilon}(x_Q) \in \mathcal{Q}_\varepsilon(\Omega')\}$ has overlap bounded by a constant depending only on d . The same is true for the half-balls, while the cubes are essentially disjoint. Summing (26) over $Q_{4\varepsilon}(x_Q) \in \mathcal{Q}_\varepsilon(\Omega')$ yields

$$\int_{\Omega'} |\nabla u_\varepsilon| \, dx \leq C_{d, \mathcal{A}} \left(\int_{\Omega} |\nabla_{\text{ap}} u| \, dx + |\mathcal{A}u|(\Omega) \right) \tag{27}$$

for every $\varepsilon < \varepsilon_0(\delta)$ small enough.

By the standard properties of mollifiers, $u_\varepsilon \rightarrow u$ in $L^1(\Omega'; V)$. At the same time, the compactness theorem in $BV(\Omega'; V)$, together with (27), yields, up to a subsequence, $u_\varepsilon \rightarrow v$ in $L^1(\Omega'; V)$ for some $v \in BV(\Omega'; V)$. Hence $u = v$ and therefore $u \in BV(\Omega'; V)$. Moreover, by lower semicontinuity,

$$|Du|(\Omega') \leq C_{d, \mathcal{A}} \left(\int_{\Omega} |\nabla_{\text{ap}} u| \, dx + |\mathcal{A}u|(\Omega) \right).$$

Since $\Omega' \subset\subset \Omega$ is arbitrary and the right-hand side is independent of Ω' , an exhaustion of Ω by relatively compact open subsets yields $u \in BV(\Omega; V)$ and

$$|Du|(\Omega) \leq C_{d,A} \left(\int_{\Omega} |\nabla_{\text{ap}} u| \, dx + |\mathcal{A}u|(\Omega) \right).$$

□

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