FROM $\mathrm{BV}^{\mathcal{A}}$ TO BV: AN ENDPOINT KORN ESTIMATE

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ABSTRACT. In this short note we prove that, given a $\mathbb{C}$-elliptic operator $\mathcal{A}$ and a map $u \in \mathrm{BV}^{\mathcal{A}}(\Omega; V)$ satisfying $\nabla_{\mathrm{ap}} u \in L^1(\Omega; V \otimes \mathbb{R}^d)$, then $u \in \mathrm{BV}(\Omega; V)$. The result is quantitative and follows from the Korn-type estimate

$$|Du|(\Omega) \leq C_{d,\mathcal{A}} \left(\|\nabla_{\mathrm{ap}} u\|_{\mathrm{L}^1(\Omega)} + |\mathcal{A}u|(\Omega) \right),$$

valid for every such $u \in \mathrm{BV}^{\mathcal{A}}(\Omega; V)$. As a consequence, we obtain the characterization

$$\mathrm{BV}^{\mathcal{A}}(\Omega; V) \setminus \mathrm{BV}(\Omega; V) = \left\{ u \in \mathrm{BV}^{\mathcal{A}}(\Omega; V) : \nabla_{\mathrm{ap}} u \notin L^1(\Omega; V \otimes \mathbb{R}^d) \right\}.$$

Generative AI has been exploited. The usage is detailed in a specific Section.

CONTENTS

1. INTRODUCTION

The classical Korn inequality for $1 < p < \infty$ controls the full gradient of a displacement, up to a constant skew-symmetric matrix, by its symmetric gradient. However, at the endpoint $p = 1$, due to Ornstein's non-inequality [9], this type of estimate generally fails. Several results are available that recover useful Korn-type estimates for $p = 1$, for instance by removing a controlled exceptional set or by allowing different rigid motions on different pieces of the domain, as in [7, 5]. In this paper we propose a Korn-type estimate at the endpoint $p = 1$, valid without removing any exceptional set, but at the price of including the L^1 norm of the full approximate gradient. In particular, this gives a simple characterization of those BD maps which fail to belong to BV: they are precisely the ones whose full approximate gradient is not integrable. Since the mechanism behind the argument is not specific to the symmetric gradient, we work in the more general setting of functions of bounded $\mathcal{A}$-variation, for first-order $\mathbb{C}$-elliptic operators (see Section 2.2).

Let
$$\mathcal{A}u = \sum_{j=1}^{d} A_j \partial_j u$$
be a first-order homogeneous differential operator with constant coefficients. Assuming that $\mathcal{A}$ is $\mathbb{C}$-elliptic, we consider the space $\mathrm{BV}^{\mathcal{A}}$ of functions whose $\mathcal{A}$-gradient is a finite Radon measure, see [4] or Section 2.2 below. Moreover, by [8, Lemma 3.1] (see also Theorem 2.4 below), every $u \in \mathrm{BV}^{\mathcal{A}}$ admits an approximate gradient $\nabla_{\mathrm{ap}} u$ almost everywhere. Our main theorem then reads as follows.

Theorem 1.1. *Let $\mathcal{A}$ be a $\mathbb{C}$-elliptic first-order homogeneous differential operator. Then there exists $C_{d,\mathcal{A}} > 0$ such that, for every open set $\Omega \subset \mathbb{R}^d$ and every $u \in \mathrm{BV}^{\mathcal{A}}(\Omega)$ satisfying*
$$\nabla_{\mathrm{ap}} u \in L^1(\Omega; V \otimes \mathbb{R}^d),$$
one has $u \in \mathrm{BV}(\Omega; V)$ and
$$|Du|(\Omega) \le C_{d,\mathcal{A}} \left(|\mathcal{A}u|(\Omega) + \int_\Omega |\nabla_{\mathrm{ap}} u| \, dx \right).$$

We emphasize that the constant $C_{d,\mathcal{A}}$ is independent of Ω and that the above does not require any regularity assumption on $\partial\Omega$. For the sake of a cleaner statement, we formulate the result in terms of the full approximate gradient. The essential quantity is however only its component in the kernel of the linear map induced by $\mathcal{A}$ on first derivatives, the complementary component being already controlled by $\mathcal{A}u$.

The model case is $\mathcal{A} = \mathcal{E}$, the symmetric gradient. The gap between BD and BV is closely related to the skew-symmetric part of the approximate gradient, which is not seen by $\mathcal{E}u$. This is already visible in the examples constructed in [6]: they produce functions in SBD which fail to belong to GBV by building increasingly large infinitesimal rotations while keeping the symmetric part of the gradient under control. The natural question is then whether the failure of a BD map to belong to BV can be entirely attributed to its absolutely continuous part. Our result shows that this is exactly the case. In fact,
$$\mathrm{BV}_{\mathrm{loc}}(\Omega; \mathbb{R}^d) = \left\{ u \in \mathrm{BD}_{\mathrm{loc}}(\Omega) : \operatorname{skew} \nabla_{\mathrm{ap}} u \in L^1_{\mathrm{loc}}(\Omega) \right\}.$$

Theorem 1.1 also gives an interesting a posteriori interpretation of $p = 1$ Korn inequalities with exceptional sets. Such results recover coercivity either after discarding a suitable region of the domain or after subtracting suitable rigid motions on different pieces. From the present viewpoint, one may heuristically interpret these procedures as ways of isolating the part of the deformation where the full approximate gradient, or equivalently its uncontrolled null-space component, fails to be integrable.

The same picture holds for general $\mathbb{C}$-elliptic operators. The measure $\mathcal{A}u$ controls the component of the first derivative seen by the symbol, while the remaining component may lie in its kernel. Requiring $\nabla_{\mathrm{ap}} u \in L^1$ supplies precisely the missing first-order information. As a consequence, we also obtain the corresponding characterization of special functions:
$$\mathrm{SBV}_{\mathrm{loc}}(\Omega; V) = \left\{ u \in \mathrm{SBV}^{\mathcal{A}}_{\mathrm{loc}}(\Omega) : \nabla_{\mathrm{ap}} u \in L^1_{\mathrm{loc}}(\Omega; V \otimes \mathbb{R}^d) \right\}.$$

Indeed, if

$$u \in \mathrm{SBV}^{\mathcal{A}}(\Omega) \qquad \text{and} \qquad \nabla_{\mathrm{ap}} u \in L^1(\Omega; V \otimes \mathbb{R}^d),$$

then Theorem 1.1 gives $u \in \mathrm{BV}(\Omega; V)$. Applying the estimate locally on arbitrary open subsets $U \subset\subset \Omega$ yields

$$|Du|(U) \leq C_{d,\mathcal{A}} \left(|\mathcal{A}u|(U) + \int_U |\nabla_{\mathrm{ap}} u| \, dx \right),$$

and hence, by outer regularity,

$$|D^s u| \leq C_{d,\mathcal{A}} |(\mathcal{A}u)^s|.$$

Since $u \in \mathrm{SBV}^{\mathcal{A}}(\Omega)$, the singular part $(\mathcal{A}u)^s$ is concentrated on the jump set. Therefore $D^s u$ is concentrated on the same rectifiable set. As the Cantor part of a BV derivative vanishes on every σ-finite $\mathcal{H}^{d-1}$-set, it follows that $D^c u = 0$, and hence $u \in \mathrm{SBV}(\Omega; V)$.

Finally, using the characterization $\frac{\mathrm{d}\mathcal{A}u}{\mathrm{d}\mathcal{L}^d} = A(\nabla_{\mathrm{ap}} u)$ for a linear map A, given in [8, Lemma 3.1], we also have

$$\mathrm{SBV}^p_{\mathrm{loc}}(\Omega; V) = \left\{ u \in \mathrm{SBV}^{\mathcal{A}}_{\mathrm{loc}}(\Omega) : \nabla_{\mathrm{ap}} u \in \mathrm{L}^p_{\mathrm{loc}}(\Omega; V \otimes \mathbb{R}^d),\ \mathcal{H}^{d-1}(J_u \cap U) < \infty \text{ for every } U \Subset \Omega \right\}.$$

1.1. **Sketch of the proof.** After interaction with generative AI (see also Section 1.2), the proof turned out to be surprisingly simple. Is based on a localization and mollification argument both occouring at the same scale. Fix $\varepsilon > 0$, mollify u at scale ε and cover the interior of Ω by a grid of cubes of size comparable with ε. On each cube, the Poincaré inequality for $\mathrm{BV}^{\mathcal{A}}$ [4, Theorem 3.2] (see also Theorem 3.1) selects a null-space polynomial $a_Q \in \mathrm{Ker}(\mathcal{A})$ such that $u - a_Q$ is controlled in L^1 by $|\mathcal{A}u|$ on a slightly larger ball. We then split locally

$$\nabla u_\varepsilon = \nabla\big((u - a_Q) * \varrho_\varepsilon\big) + \nabla(a_Q * \varrho_\varepsilon)$$

and we aim to find an ε-independent L^1 bound for ∇u_ε.

The first term is directly controlled by the Poincaré estimate. The main point is therefore the second term. More precisely, since a_Q is smooth, the crucial point is to estimate $\|\nabla a_Q\|_{\mathrm{L}^1}$, since

$$\|\nabla(a_Q \star \varrho_\varepsilon)\|_{\mathrm{L}^1} \lesssim \|\nabla a_Q\|_{\mathrm{L}^1}.$$

This is where the summability of $\nabla_{\mathrm{ap}} u$ enters the argument: the key point is Proposition 4.1, which, for the same a_Q selected above, by exploiting standard singular integral theory (Proposition 3.4) provides an estimate of the form

$$\|\nabla_{\mathrm{ap}} u - \nabla a_Q\|_{\mathrm{L}^{1,\infty}} \lesssim |\mathcal{A}u|.$$

Since $\mathrm{Ker}(\mathcal{A})$ is finite-dimensional, due to [4, Theorem 2.6] (see also Theorem 3.2 below) the weak-L^1 quasi-norm and the strong L^1 norm are equivalent on the space of gradients of null-space polynomials. Therefore

$$\|\nabla a_Q\|_{L^1} \lesssim \|\nabla a_Q\|_{L^{1,\infty}} \lesssim \|\nabla_{\mathrm{ap}} u - \nabla a_Q\|_{L^{1,\infty}} + \|\nabla_{\mathrm{ap}} u\|_{L^{1,\infty}} \lesssim \|\nabla_{\mathrm{ap}} u\|_{L^1} + |\mathcal{A}u|.$$

Combining the two estimates and summing over the grid gives an ε-independent bound

$$\int_{\Omega'} |\nabla u_\varepsilon| \, dx \lesssim |\mathcal{A}u|(\Omega) + \|\nabla_{\mathrm{ap}} u\|_{\mathrm{L}^1(\Omega)}$$

for every $\Omega' \subset\subset \Omega$. The conclusion then follows from BV compactness and an exhaustion of Ω.

1.2. **AI content disclosure.** The problem addressed in this note had been a long-term project of the author. The author's initial approach was based on a nonlocal approximation of the gradient by spherical averages. In the model case of BD, the resulting quantity splits into a part controlled by the symmetric gradient and a rotational part, which can be estimated by Calderón–Zygmund theory for $p > 1$. The endpoint strategy was to seek a direct L^1 estimate of this rotational term, initially near regular jump sets, using the geometry of tubular neighbourhoods. Successive interaction with generative AI brought this strategy to a final complete working form but not in the generality here presented.

Generative AI was then used again as an interactive research assistant to discuss, test, and simplify the argument. Through an extended dialogue, the working strategy was progressively dismantled: several of its technical components turned out not to be essential, and the argument was reduced to a much simpler local mechanism. In particular, the relevant issue was identified as the control of the gradient of the null-space polynomial selected by the Poincaré projection Proposition 4.1. The local weak endpoint estimate and its combination with finite-dimensional norm equivalence emerged from this iterative discussion rather than being part of the author's initial proof strategy. This observation led to the localization and mollification argument used in the present paper.

The AI was also used to test intermediate arguments, locate relevant references, and assist with the English presentation. The author takes full responsibility for the contents of the paper.

The specific AI provider used is not relevant to the mathematical discussion or to the account of the interaction given above.

2. Preliminaries and main result

2.1. **General notation.** Throughout the paper we assume $d \geq 2$. The one-dimensional case is immediate. The notation $\mathcal{L}^d$ stands for the d-dimensional Lebesgue measure on $\mathbb{R}^d$. Throughout, constants denoted by $C_{d,\mathcal{A}}$ may change from line to line and depend only on the dimension and the operator $\mathcal{A}$. $B_r(x) := x + rB$ where B stands for the unit Euclidean ball of $\mathbb{R}^d$, while $Q_r(x) := x + rQ$ where $Q := [-1/2, 1/2]^d$ is the d-dimensional unit cube with barycenter at 0. For a given measurable set A, we denote by $\mathbb{1}_A(x)$ the characteristic function of A, which takes the value 1 whenever $x \in A$ and 0 elsewhere. For $v \in V$ and $\xi \in \mathbb{R}^d$, we identify the tensor $v \otimes \xi \in V \otimes \mathbb{R}^d$ with the linear map $\mathbb{R}^d \to V$ defined by

$$(v \otimes \xi)h = (\xi \cdot h)v.$$

2.2. **Maps of bounded $\mathcal{A}$-variation.** Let $\mathcal{A} : C^\infty(\mathbb{R}^d; V) \to C^\infty(\mathbb{R}^d; W)$ be, for some Euclidean spaces V, W, a first-order homogeneous differential operator with constant coefficients

$$\mathcal{A}u := \sum_{j=1}^{d} A_j \partial_j u, \qquad A_j \in \mathrm{Lin}(V, W). \tag{1}$$

The space $\mathrm{Lin}(V; W)$ denotes the family of all linear maps from V to W. The space of functions with *bounded $\mathcal{A}$-variation* (see also [4]) is defined by

$$\mathrm{BV}^{\mathcal{A}}(\Omega) := \left\{ u \in L^1(\Omega; V) \;\middle|\; \mathcal{A}u \in \mathcal{M}(\Omega; W) \right\},$$

where $\mathcal{M}(\Omega; W)$ denotes the space of W-valued finite Radon measures on Ω, and the distributional $\mathcal{A}$-gradient is defined by:

$$\int_\Omega \varphi \cdot \mathrm{d}\mathcal{A}u := \int_\Omega \mathcal{A}^*\varphi \cdot u\mathrm{d}x, \ \forall\ \varphi \in C_c^\infty(\Omega; W), \tag{2}$$

where the L^2-adjoint operator $\mathcal{A}^* : C^1(\mathbb{R}^d; W) \to C^0(\mathbb{R}^d; V)$ is defined in terms of $\{A_j^*\}_{j=1}^n \subset \mathrm{Lin}(W; V)$, satisfying $A_j z \cdot \eta = z \cdot A_j^* \eta$ for all $z \in V, \eta \in W$, by

$$\mathcal{A}^* v = -\sum_{j=1}^n A_j^* \partial_j v.$$

The classical examples are $\mathcal{A} = \nabla$, for which $\mathrm{BV}^{\mathcal{A}} = \mathrm{BV}$ (see [3]), and the symmetric gradient $\mathcal{A} = \mathcal{E} = \frac{\nabla + \nabla^t}{2}$, for which $\mathrm{BV}^{\mathcal{A}} = \mathrm{BD}$ (see [2]).

For $\xi \in \mathbb{C}^d$ and $v \in V + iV$, the *complexified symbol* is defined as

$$\mathbb{A}[\xi]v := \sum_{j=1}^d \xi_j A_j v. \tag{3}$$

Definition 2.1. *The operator $\mathcal{A}$ is said to be* $\mathbb{R}$-***elliptic*** *if $\mathbb{A}[\xi] : V \to W$ is injective for every $\xi \in \mathbb{R}^d \setminus \{0\}$. The operator $\mathcal{A}$ is said to be* $\mathbb{C}$-***elliptic*** *if its complexified symbol $\mathbb{A}[\xi] : V + iV \to W + iW$ is injective for every $\xi \in \mathbb{C}^d \setminus \{0\}$.*

Remark 2.2. Clearly $\mathbb{C}$-ellipticity implies $\mathbb{R}$-ellipticity.

The following useful Leibniz rule applies (see also [4]).

Lemma 2.3 (Cut-off product rule [4])**.** *Let $U \subset \mathbb{R}^d$ be open, $v \in \mathrm{BV}^{\mathcal{A}}(U)$, and $\eta \in C_c^1(U)$. Extend ηv by zero outside U. Then $\eta v \in \mathrm{BV}^{\mathcal{A}}(\mathbb{R}^d)$ and*

$$\mathcal{A}(\eta v) = \eta\, \mathcal{A}v + \mathbb{A}[\nabla \eta] v\, \mathcal{L}^d \qquad \text{in } \mathbb{R}^d. \tag{4}$$

Consequently,

$$|\mathcal{A}(\eta v)|(\mathbb{R}^d) \le \|\eta\|_{L^\infty(U)} |\mathcal{A}v|(U) + C_{\mathcal{A}} \|\nabla \eta\|_{L^\infty(U)} \|v\|_{\mathrm{L}^1(U)}. \tag{5}$$

Proof. For $\varphi \in C_c^1(\mathbb{R}^d; W)$, using the distributional adjoint $\mathcal{A}^*$ in (2) and the Leibniz rule for $\mathcal{A}^*(\eta\varphi)$, a simple computation immediately gives

$$\langle \mathcal{A}(\eta v), \varphi \rangle = \langle \eta \mathcal{A} v, \varphi \rangle + \int_U \varphi \cdot \mathbb{A}[\nabla \eta] v\, dx.$$

This is (4). Since the finite-dimensional bilinear map $(\xi, v) \mapsto \mathbb{A}[\xi]v$ satisfies $|\mathbb{A}[\xi]v| \le C_{\mathcal{A}} |\xi||v|$, by $\mathbb{C}$-ellipticity, estimate (5) follows. □

We recall from [8] that, whenever $\mathcal{A}$ is $\mathbb{C}$-elliptic, every $u \in \mathrm{BV}^{\mathcal{A}}(\Omega)$ is approximately differentiable a.e. in Ω. The cited result is stated on $\mathbb{R}^d$; the local version below follows easily by multiplication with a compactly supported cut-off and the Leibniz rule for $\mathcal{A}$ in Lemma 2.3.

Theorem 2.4 ([8, Lemma 3.1])**.** *Let $\Omega \subset \mathbb{R}^d$ be open, let $\mathcal{A}$ be an $\mathbb{R}$-elliptic first-order homogeneous differential operator, and let $u \in \mathrm{BV}^{\mathcal{A}}_{\mathrm{loc}}(\Omega)$. Then u is L^1-differentiable at $\mathcal{L}^d$-almost every point of Ω. In particular, there exists $\nabla_{\mathrm{ap}} u(x) \in V \otimes \mathbb{R}^d$ such that*

$$\fint_{B_r(x)} |u(y) - u(x) - \nabla_{\mathrm{ap}} u(x)(y - x)|\, dy = o(r)$$

for $\mathcal{L}^d$-almost every $x \in \Omega$.

3. Technical tools

3.1. **Poincarè inequality and characterization of $\mathbb{C}$-ellipticity.** For a set $U \subset \mathbb{R}^d$, let Π_U denote the $L^2(U;V)$-orthogonal projection onto the finite-dimensional space

$$\operatorname{Ker}(\mathcal{A}) := \{a \in (C_c^\infty(\mathbb{R}^d;V))' \ : \ \mathcal{A}a = 0\}.$$

As shown in [4, Section 3.1], Π_U extends boundedly to $\mathrm{L}^1(U;V)$; this extension is the one used below.

Theorem 3.1 (Poincaré inequality in $\mathrm{BV}^{\mathcal{A}}$; [4, Theorem 3.2])**.** *Assume that $\mathcal{A}$ is $\mathbb{C}$-elliptic. There exists $C_{d,\mathcal{A}} > 0$ such that for every ball $B_r(x) \subset \mathbb{R}^d$ and every $u \in \mathrm{BV}^{\mathcal{A}}(B_r(x))$,*

$$\inf_{q \in \operatorname{Ker}(\mathcal{A})} \|u - q\|_{\mathrm{L}^1(B_r(x))} \le \|u - \Pi_{B_r(x)} u\|_{\mathrm{L}^1(B_r(x))} \le C_{d,\mathcal{A}}\, r\, |\mathcal{A}u|(B_r(x)). \tag{6}$$

Finally, the following useful characterization will be used to transform the $L^{1,\infty}$ control on rotations into an L^1-type control.

Theorem 3.2 ([4, Theorem 2.6])**.** *Let $\mathcal{A}$ be a first-order homogeneous differential operator with constant coefficients. The following are equivalent:*

(a) *$\mathcal{A}$ has finite-dimensional null-space* $\operatorname{Ker}(\mathcal{A})$*;*
(b) *$\mathcal{A}$ is $\mathbb{C}$-elliptic;*
(c) *there exists $\ell \in \mathbb{N}$ such that* $\operatorname{Ker}(\mathcal{A})$ *is contained in the space of V-valued polynomials of degree at most ℓ.*

3.2. **Mollified gradients at differentiability points.** Fix a standard mollifier $\varrho \in C_c^\infty(B_1)$, $\varrho \ge 0$, $\int \varrho = 1$, and set $\varrho_\delta(x) = \delta^{-d}\varrho(x/\delta)$. Note that, in our notation, for a map $v : \mathbb{R}^d \to V$,

$$\nabla(v * \varrho_\delta)(x) = \int_{\mathbb{R}^d} v(y) \otimes \nabla\varrho_\delta(x-y)\, dy,$$

since, when applied to e_j, the right-hand side equals $\int_{\mathbb{R}^d} v(y)\partial_j\varrho_\delta(x-y)\, dy = \partial_j(v * \varrho_\delta)(x)$.

Lemma 3.3 (Mollified approximate gradients)**.** *Let $v \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d;V)$. Suppose that v is L^1-differentiable at x, namely*

$$\fint_{B_r(x)} |v(y) - v(x) - \nabla_{\mathrm{ap}}v(x)(y-x)|\mathrm{d}y = o(r).$$

Then

$$\nabla(v * \varrho_\delta)(x) \longrightarrow \nabla_{\mathrm{ap}}v(x) \qquad \text{as } \delta \downarrow 0. \tag{7}$$

Proof. Set

$$R(y) := v(y) - v(x) - \nabla_{\mathrm{ap}}v(x)(y-x).$$

Then

$$\begin{aligned}
\nabla(v * \varrho_\delta)(x) &= \int_{\mathbb{R}^d} v(y) \otimes \nabla\varrho_\delta(x-y)\, dy \\
&= \int_{\mathbb{R}^d} \Big(v(x) + \nabla_{\mathrm{ap}}v(x)(y-x) + R(y)\Big) \otimes \nabla\varrho_\delta(x-y)\, dy \\
&= v(x) \otimes \int_{\mathbb{R}^d} \nabla\varrho_\delta(x-y)\, dy
\end{aligned}$$

$$
\begin{aligned}
&+ \nabla_{\mathrm{ap}} v(x) \int_{\mathbb{R}^d} (y-x) \otimes \nabla \varrho_\delta(x-y)\, dy \\
&+ \int_{\mathbb{R}^d} R(y) \otimes \nabla \varrho_\delta(x-y)\, dy \\
&= \nabla_{\mathrm{ap}} v(x) + \int_{B_\delta(x)} R(y) \otimes \nabla \varrho_\delta(x-y)\, dy.
\end{aligned}
$$

The last equality follows from integration by parts and the properties of the mollifier. Therefore

$$|\nabla(v * \varrho_\delta)(x) - \nabla_{\mathrm{ap}} v(x)| \le C_\varrho \delta^{-d-1} \int_{B_\delta(x)} |R(y)|\, dy = o(1).$$

□

3.3. $\mathrm{L}^{1,\infty}$**-control on the approximate gradient.** For a measurable function f defined on an open set $U \subset \mathbb{R}^d$, we recall that the weak-L^1 quasi-norm is defined by:

$$\|f\|_{\mathrm{L}^{1,\infty}(U)} := \sup_{s>0} s\, \mathcal{L}^d\big(\{x \in U : |f(x)| > s\}\big). \tag{8}$$

In particular, it is immediate that

$$\|f\|_{\mathrm{L}^{1,\infty}(U)} \le \|f\|_{\mathrm{L}^1(U)}. \tag{9}$$

The next proposition is a standard consequence of the potential representation for elliptic operators and the weak $(1,1)$ boundedness of Calderón–Zygmund operators. See also [2] for the BD case.

Proposition 3.4 (Global weak endpoint estimate)**.** *Assume that $\mathcal{A}$ is elliptic. Then there exists $C_{d,\mathcal{A}} > 0$ such that every compactly supported $v \in BV^{\mathcal{A}}(\mathbb{R}^d)$ satisfies*

$$\|\nabla_{\mathrm{ap}} v\|_{L^{1,\infty}(\mathbb{R}^d)} \le C_{d,\mathcal{A}} |\mathcal{A}v|(\mathbb{R}^d). \tag{10}$$

Proof. By the potential representation for elliptic operators [8, Lemma 2.1], one can write $v = K_{\mathcal{A}} * \mathcal{A}v$, where $K_{\mathcal{A}}$ is smooth away from the origin and $(1-d)$-homogeneous. By [1, Section 3.3, Theorem 3.4], $\nabla_{\mathrm{ap}} v$ is the sum of the singular integral associated with $\nabla K_{\mathcal{A}}$ and a bounded linear image of $\frac{d\mathcal{A}v}{d\mathcal{L}^d}$. The former satisfies the classical weak-$(1,1)$ estimate, while the latter belongs to L^1 with norm bounded by $C_{\mathcal{A}}|\mathcal{A}v|(\mathbb{R}^d)$. Hence

$$\|\nabla_{\mathrm{ap}} v\|_{L^{1,\infty}(\mathbb{R}^d)} \le C_{d,\mathcal{A}} |\mathcal{A}v|(\mathbb{R}^d).$$

□

4. Weak Korn-Poincaré estimate and control of the null-space gradient

We now combine Poincaré with Proposition 3.4. The key point is that the weak estimate is obtained with the same null-space polynomial selected by the Poincaré projection.

Proposition 4.1 (Local weak Korn–Poincaré estimate)**.** *Assume that $\mathcal{A}$ is $\mathbb{C}$-elliptic. Let $u \in \mathrm{BV}^{\mathcal{A}}(B_r(x))$. Set*

$$a_{r,x} := \Pi_{B_r(x)} u \in \mathrm{Ker}(\mathcal{A}).$$

Then

$$\int_{B_r(x)} |u - a_{r,x}|\, dx \le C_{d,\mathcal{A}} r |\mathcal{A}u|(B_r(x)), \tag{11}$$

and

$$\|\nabla_{\mathrm{ap}} u - \nabla a_{r,x}\|_{\mathrm{L}^{1,\infty}(B_{r/2}(x))} \le C_{d,\mathcal{A}} |\mathcal{A}u|(B_r(x)). \tag{12}$$

Proof. Estimate (11) follows directly from Theorem 3.1. Now choose a cutoff function $\eta \in C_c^\infty(B_r(x))$ such that

$$0 \le \eta \le 1, \qquad \eta \equiv 1 \text{ on } B_{r/2}(x), \qquad |\nabla \eta| \le C_d r^{-1}.$$

Set

$$w := \mathbb{1}_{B_r(x)} \eta (u - a_{r,x}) \in \mathrm{BV}^{\mathcal{A}}(\mathbb{R}^d). \tag{13}$$

Since $a_{r,x} \in \mathrm{Ker}(\mathcal{A})$, one has $\mathcal{A}a_{r,x} = 0$. By Lemma 2.3 and (11),

$$|\mathcal{A}w|(\mathbb{R}^d) \le |\mathcal{A}u|(B_r(x)) + C_{\mathcal{A}} r^{-1} \int_{B_r(x)} |u - a_{r,x}|\, dx \le C_{d,\mathcal{A}} |\mathcal{A}u|(B_r(x)). \tag{14}$$

Applying Proposition 3.4 to w gives

$$\|\nabla_{\mathrm{ap}} w\|_{\mathrm{L}^{1,\infty}(\mathbb{R}^d)} \le C_{d,\mathcal{A}} |\mathcal{A}u|(B_r(x)).$$

On $B_{r/2}(x)$, $w = u - a_{r,x}$ and hence

$$\nabla_{\mathrm{ap}} w = \nabla_{\mathrm{ap}} u - \nabla a_{r,x} \qquad \text{a.e.}$$

Restricting the weak estimate proves (12). □

Unlike the BV or BD case, $a_{r,x}$ need not be affine; however, by Theorem 3.2, it is a polynomial of degree bounded solely in terms of $\mathcal{A}$, and thus we can recover an L^1-type control on the rotational part from the $L^{1,\infty}$ control given by Proposition 4.1.

Lemma 4.2. *Assume that $\mathcal{A}$ is $\mathbb{C}$-elliptic. There exists $C_{d,\mathcal{A}} > 0$ such that for every ball $B_r(x_0)$ and every $a \in \mathrm{Ker}(\mathcal{A})$,*

$$\int_{B_r(x_0)} |\nabla a|\, dx \le C_{d,\mathcal{A}} \|\nabla a\|_{\mathrm{L}^{1,\infty}(B_r(x_0))}. \tag{15}$$

The constant is independent of x_0 and r.

Proof. By translation and scaling it is enough to prove the estimate on the unit ball. Indeed, if $\widetilde{a}(y) := a(x_0 + ry)$, then $\widetilde{a} \in \mathrm{Ker}(\mathcal{A})$ (since $\mathcal{A}$ is a first-order homogeneous differential operator with constant coefficients) and $\nabla \widetilde{a}(y) = r \nabla a(x_0 + ry)$. By Theorem 3.2, the space

$$X := \{\nabla a|_B : a \in \mathrm{Ker}(\mathcal{A})\}$$

is finite-dimensional and consists of polynomial tensor fields. Since $\|\cdot\|_{\mathrm{L}^{1,\infty}(B)}$ is a quasi-norm and $\|\cdot\|_{\mathrm{L}^1(B)}$ is a norm (hence, in particular, a quasi-norm), and since all quasi-norms are equivalent on finite-dimensional spaces, we conclude. □

Lemma 4.3 (L^1 control of the null-space gradient)**.** *Under the assumptions of Proposition 4.1,*

$$\int_{B_{r/2}(x)} |\nabla a_{r,x}|\, dx \le C_{d,\mathcal{A}} \left(\int_{B_{r/2}(x)} |\nabla_{\mathrm{ap}} u|\, dx + |\mathcal{A}u|(B_r(x)) \right). \tag{16}$$

Proof. Observe that, for two arbitrary measurable functions f, g, we trivially have

$$\{|f + g| > s\} \subset \{|f| > s/2\} \cup \{|g| > s/2\}.$$

Thus

$$\|f + g\|_{\mathrm{L}^{1,\infty}(U)} \le 2\big(\|f\|_{\mathrm{L}^{1,\infty}(U)} + \|g\|_{\mathrm{L}^{1,\infty}(U)}\big), \tag{17}$$

Using Lemma 4.2 on $B_{r/2}(x)$, then (17), (9), and (12), we obtain

$$\begin{aligned}\int_{B_{r/2}(x)} |\nabla a_{r,x}|\,dx &\le C_{d,\mathcal{A}} \|\nabla a_{r,x}\|_{\mathrm{L}^{1,\infty}(B_{r/2}(x))} \\ &\le C_{d,\mathcal{A}} \left(\|\nabla_{\mathrm{ap}} u\|_{\mathrm{L}^{1,\infty}(B_{r/2}(x))} + \|\nabla_{\mathrm{ap}} u - \nabla a_{r,x}\|_{\mathrm{L}^{1,\infty}(B_{r/2}(x))} \right) \\ &\le C_{d,\mathcal{A}} \left(\int_{B_{r/2}(x)} |\nabla_{\mathrm{ap}} u|\,dx + |\mathcal{A}u|(B_r(x)) \right).\end{aligned}$$

□

5. Proof of the main theorem

Proof of Theorem 1.1. Fix $\Omega' \subset\subset \Omega$ and set

$$\delta := \mathrm{dist}(\Omega', \partial\Omega) > 0.$$

Let $\varepsilon > 0$ be small and consider the standard grid of pairwise disjoint cubes $Q_{4\varepsilon}(x_Q)$. Let $\mathcal{Q}_\varepsilon(\Omega')$ be the family of grid cubes meeting Ω'. For each $Q_{4\varepsilon}(x_Q) \in \mathcal{Q}_\varepsilon(\Omega')$ consider the concentric balls $B_{6\varepsilon\sqrt{d}}(x_Q) \supset B_{3\varepsilon\sqrt{d}}(x_Q) \supset Q_{6\varepsilon}(x_Q)$. Note that for $\varepsilon < \varepsilon_0(\delta)$ small enough we can guarantee that $B_{6\varepsilon\sqrt{d}}(x_Q) \subset\subset \Omega$ for every $Q_{4\varepsilon}(x_Q) \in \mathcal{Q}_\varepsilon(\Omega')$.

Now, for each $Q_{4\varepsilon}(x_Q)$, apply Proposition 4.1 on $B_{6\varepsilon\sqrt{d}}(x_Q)$. Thus, setting

$$a_{\varepsilon,x_Q} := \Pi_{B_{6\varepsilon\sqrt{d}}(x_Q)} u \in \mathrm{Ker}(\mathcal{A}),$$

we have

$$\int_{B_{6\varepsilon\sqrt{d}}(x_Q)} |u - a_{\varepsilon,x_Q}|\,dx \le C_{d,\mathcal{A}}\,\varepsilon\,|\mathcal{A}u|\big(B_{6\varepsilon\sqrt{d}}(x_Q)\big), \tag{18}$$

and

$$\|\nabla_{\mathrm{ap}} u - \nabla a_{\varepsilon,x_Q}\|_{\mathrm{L}^{1,\infty}(B_{3\varepsilon\sqrt{d}}(x_Q))} \le C_{d,\mathcal{A}}\,|\mathcal{A}u|\big(B_{6\varepsilon\sqrt{d}}(x_Q)\big). \tag{19}$$

Take a mollifier $\{\varrho_\varepsilon\}_{\varepsilon>0}$ as in Subsection 3.2. For $x \in \Omega_\varepsilon := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \varepsilon\}$, set

$$u_\varepsilon(x) := \int_{B_\varepsilon} u(x-z)\varrho_\varepsilon(z)\,dz.$$

On $Q_{4\varepsilon}(x_Q)$ we decompose

$$\nabla u_\varepsilon = \nabla\big((u - a_{\varepsilon,x_Q}) * \varrho_\varepsilon\big) + \nabla(a_{\varepsilon,x_Q} * \varrho_\varepsilon). \tag{20}$$

For the oscillation term, Fubini's theorem gives

$$\begin{aligned}\int_{Q_{4\varepsilon}(x_Q)} \big|\nabla\big((u - a_{\varepsilon,x_Q}) * \varrho_\varepsilon\big)(x)\big|\,dx &\le \int_{B_\varepsilon} |\nabla\varrho_\varepsilon(z)| \int_{Q_{4\varepsilon}(x_Q)} |u(x-z) - a_{\varepsilon,x_Q}(x-z)|\,dx\,dz \\ &\le \|\nabla\varrho_\varepsilon\|_{\mathrm{L}^1} \int_{Q_{6\varepsilon}(x_Q)} |u - a_{\varepsilon,x_Q}|\,dx \\ &\le \frac{C_\varrho}{\varepsilon} \int_{B_{6\varepsilon\sqrt{d}}(x_Q)} |u - a_{\varepsilon,x_Q}|\,dx \\ &\le C_{d,\mathcal{A}} |\mathcal{A}u|(B_{6\varepsilon\sqrt{d}}(x_Q)).\end{aligned} \tag{21}$$

The second inequality follows from the fact that, since $\operatorname{spt} \varrho_\varepsilon \subset B_\varepsilon$, for every $x \in Q_{4\varepsilon}(x_Q)$ and $z \in \operatorname{spt} \varrho_\varepsilon$ one has $x - z \in Q_{6\varepsilon}(x_Q)$. In the last inequality, we applied (18).

For the null-space term we now use that a_{ε,x_Q} is smooth. Hence

$$\nabla(a_{\varepsilon,x_Q} * \varrho_\varepsilon) = (\nabla a_{\varepsilon,x_Q}) * \varrho_\varepsilon,$$

and, again by Fubini,

$$\begin{aligned}\int_{Q_{4\varepsilon}(x_Q)} |\nabla(a_{\varepsilon,x_Q} * \varrho_\varepsilon)(x)|\,dx &\le \int_{B_\varepsilon} \varrho_\varepsilon(z) \int_{Q_{4\varepsilon}(x_Q)} |\nabla a_{\varepsilon,x_Q}(x-z)|\,dx\,dz \\ &\le \int_{Q_{6\varepsilon}(x_Q)} |\nabla a_{\varepsilon,x_Q}|\,dx \le \int_{B_{3\varepsilon\sqrt{d}}(x_Q)} |\nabla a_{\varepsilon,x_Q}|\,dx. \end{aligned} \tag{22}$$

Applying Lemma 4.3 on $B_{6\varepsilon\sqrt{d}}(x_Q)$ and using (22) gives

$$\int_{Q_{4\varepsilon}(x_Q)} |\nabla(a_{\varepsilon,x_Q} * \varrho_\varepsilon)|\,dx \le C_{d,\mathcal{A}} \left(\int_{B_{3\varepsilon\sqrt{d}}(x_Q)} |\nabla_{\mathrm{ap}} u|\,dx + |\mathcal{A}u|(B_{6\varepsilon\sqrt{d}}(x_Q)) \right). \tag{23}$$

Combining (20), (21), and (23) yields

$$\int_{Q_{4\varepsilon}(x_Q)} |\nabla u_\varepsilon|\,dx \le C_{d,\mathcal{A}} \left(\int_{B_{3\varepsilon\sqrt{d}}(x_Q)} |\nabla_{\mathrm{ap}} u|\,dx + |\mathcal{A}u|(B_{6\varepsilon\sqrt{d}}(x_Q)) \right). \tag{24}$$

The family $\{B_{6\varepsilon\sqrt{d}}(x_Q) : Q_{4\varepsilon}(x_Q) \in \mathcal{Q}_\varepsilon(\Omega')\}$ has overlap bounded by a constant depending only on d. The same is true for the half-balls, while the cubes are essentially disjoint. Summing (24) over $Q_{4\varepsilon}(x_Q) \in \mathcal{Q}_\varepsilon(\Omega')$ yields

$$\int_{\Omega'} |\nabla u_\varepsilon|\,dx \le C_{d,\mathcal{A}} \left(\int_{\Omega} |\nabla_{\mathrm{ap}} u|\,dx + |\mathcal{A}u|(\Omega) \right) \tag{25}$$

for every $\varepsilon < \varepsilon_0(\delta)$ small enough.

By the standard properties of mollifiers, $u_\varepsilon \to u$ in $\mathrm{L}^1(\Omega'; V)$. At the same time, the compactness theorem in $\mathrm{BV}(\Omega'; V)$, together with (25), yields, up to a subsequence, $u_\varepsilon \to v$ in $\mathrm{L}^1(\Omega'; V)$ for some $v \in \mathrm{BV}(\Omega'; V)$. Hence $u = v$ and therefore $u \in \mathrm{BV}(\Omega'; V)$. Moreover, by lower semicontinuity,

$$|Du|(\Omega') \le C_{d,\mathcal{A}} \left(\int_{\Omega} |\nabla_{\mathrm{ap}} u|\,dx + |\mathcal{A}u|(\Omega) \right).$$

Since $\Omega' \subset\subset \Omega$ is arbitrary and the right-hand side is independent of Ω', an exhaustion of Ω by relatively compact open subsets yields $u \in \mathrm{BV}(\Omega; V)$ and

$$|Du|(\Omega) \le C_{d,\mathcal{A}} \left(\int_{\Omega} |\nabla_{\mathrm{ap}} u|\,dx + |\mathcal{A}u|(\Omega) \right).$$

□

References

[1] Giovanni Alberti, Stefano Bianchini, and Gianluca Crippa. On the l^p-differentiability of certain classes of functions. *Revista Matemática Iberoamericana*, 30(1):349–367, 2014.

[2] Luigi Ambrosio, Alessandra Coscia, and Gianni Dal Maso. Fine properties of functions with bounded deformation. *Archive for Rational Mechanics and Analysis*, 139(3):201–238, 1997.

[3] Luigi Ambrosio, Nicola Fusco, and Diego Pallara. *Functions of bounded variation and free discontinuity problems*, volume 254. Clarendon Press Oxford, 2000.

[4] Dominic Breit, Lars Diening, and Franz Gmeineder. On the trace operator for functions of bounded $\mathbb{A}$-variation. *Analysis e PDE*, 13(2):559 – 594, 2020.

[5] Filippo Cagnetti, Antonin Chambolle, and Lucia Scardia. Korn and poincaré-korn inequalities for functions with a small jump set. *Mathematische Annalen*, 383(3–4):1179–1216, 2022.

[6] Sergio Conti, Matteo Focardi, and Flaviana Iurlano. Which special functions of bounded deformation have bounded variation? *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, 148(1):33–50, 2018.

[7] Manuel Friedrich. A piecewise korn inequality in sbd and applications to embedding and density results. *SIAM Journal on Mathematical Analysis*, 50(4):3842–3918, 2018.

[8] Franz Gmeineder and Bogdan Raiţă. On critical l^p-differentiability of *bd*-maps. *Revista Matemática Iberoamericana*, 35(7):2071–2078, 2019.

[9] Donald Ornstein. A non-inequality for differential operators in the l^1 norm. *Archive for Rational Mechanics and Analysis*, 11(1):40–49, 1962.

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