Emergence of a convex strange term via homogenization of non-local energies at the critical exponent

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Abstract

We derive the Γ-limit of convolution-type and discrete energies subject to Dirichlet boundary conditions on periodically perforated domains at the critical exponent. We assume that the length-scale of the non-local interactions is much smaller than the side-length of the cubic perforations and prove that a separation of scales occurs. Exploiting the analogies between the variational frameworks of our interest, we employ a unified argument to overcome the technical difficulties that arise from the scaling invariance of the energies. Our multiscale analysis yields a novel observation that is not related to the non-local nature of the functionals, but rather to the analysis at the critical exponent: we prove that the energy density of the *strange term* is convex, even in the vector-valued setting.

1 Introduction

In a celebrated paper [23], Cioranescu and Murat investigated the asymptotic behaviour of Dirichlet problems in periodically perforated domains of the form

$$\Omega_\delta := \Omega \setminus \bigcup_{i\in\mathbb{Z}^d} \overline{Q}(i\delta, r_\delta),$$

where Ω is a regular domain of $\mathbb{R}^d$, δ is the period of the array of perforations and the cubic perforation $\overline{Q}(i\delta, r_\delta) = i\delta + [-\frac{r_\delta}{2}, \frac{r_\delta}{2}]^d$ has side-length

$$r_\delta = \begin{cases} \exp(-\delta^{-2}) & \text{if } d = 2, \\ \delta^{\frac{d}{d-2}} & \text{if } d \geq 3; \end{cases}$$

see also the earlier paper by Marchenko and Khruslov [36]. They proved that this choice of the size of the perforations is critical since, for fixed $\phi \in H^{-1}(\Omega)$, the family of solutions $\{u_\delta\}_\delta$ to the problems

$$-\Delta u_\delta = \phi, \qquad u_\delta \in H^1_0(\Omega_\delta),$$

weakly converges to a function $u \in H^1_0(\Omega)$ that solves a different limit problem, namely

$$-\Delta u + C_d u = \phi, \qquad u \in H^1_0(\Omega),$$

where an additional zero-order term, the so-called *strange term*, originates from the presence of the perforations. The constant C_d that appears in the limit equation is characterized by a capacitary problem and, similarly to the scale r_δ, its nature varies in accordance with the dimension d. If $d > 2$,

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the constant is given by the 2-capacity of the unit cube in $\mathbb{R}^d$. If $d = 2$, the same characterization no longer holds since, by scaling invariance, the 2-capacity of every bounded subset in $\mathbb{R}^2$ is zero and the constant is expressed in terms of the capacities of the unit square relative to squares whose side-lengths diverge. These relative 2-capacities exhibit a certain logarithmic decay (with respect to the side-length of the larger square) which also occurs for the relative d-capacities of a d-dimensional cube when $d > 2$, as a general consequence of the scaling invariance. Such a logarithmic behaviour is singular as it is in contrast with the polynomial behaviour of the relative p-capacity of a d-dimensional cube for $p \in (1, d)$ and $d \geq 2$, and for this reason we refer to $p \in (1, d)$ as a subcritical exponent and to $p = d$ as the *critical exponent*.

In [8], Ansini and Braides pursued a variational approach for the study of a non-linear vector-valued variant of the original problem for subcritical exponents $p \in (1, d)$. They computed the Γ-limit of the functionals $F_\delta : L^p(\Omega; \mathbb{R}^m) \to [0, +\infty]$ defined as

$$F_\delta(u) := \begin{cases} \displaystyle\int_\Omega f(\nabla u)\, dx & \text{if } u \in W_0^{1,p}(\Omega_\delta; \mathbb{R}^m), \\ +\infty & \text{otherwise,} \end{cases}$$

for $f : \mathbb{R}^{m \times d} \to [0, +\infty)$ a quasiconvex p-homogeneous integrand satisfying p-growth conditions. Conformally with the work by Cioranescu and Murat, they assumed that $r_\delta = \delta^{d/(d-p)}$ and proved that the Γ-limit of $\{F_\delta\}_\delta$ equals

$$\int_\Omega f(\nabla u)\, dx + \int_\Omega \varphi_p(u)\, dx, \qquad u \in W_0^{1,p}(\Omega; \mathbb{R}^m),$$

where $\varphi_p : \mathbb{R}^m \to [0, +\infty)$ is given by a non-linear capacitary formula:

$$\varphi_p(z) := \min\Big\{ \int_{\mathbb{R}^d} f(\nabla u)\, dx : u - z \in W^{1,p}(\mathbb{R}^d; \mathbb{R}^m), u = 0 \text{ on } Q(1) \Big\}.$$

This result can be regarded as a generalization of the one by Cioranescu and Murat to a non-linear and vectorial setting since, upon assuming that f is convex and differentiable, the property of the convergence of minima of Γ-convergence implies the convergence of the solutions to the corresponding Euler-Lagrange equations (see [14, 24]).

The analogous problem at the critical exponent $p = d$ carries the technical complications entailed by the scaling invariance and, for this reason, the corresponding extension of the above analysis required a different proof provided by Sigalotti in [43]. It is shown that, under the assumption $r_\delta = \exp(-\delta^{d/(1-d)})$, the limit energy is given by

$$\int_\Omega f(\nabla u)\, dx + \int_\Omega \varphi_d(u)\, dx, \qquad u \in W_0^{1,d}(\Omega; \mathbb{R}^m),$$

where the density $\varphi_d : \mathbb{R}^m \to [0, +\infty)$ is now described by a *homogenization-type formula*:

$$\varphi_d(z) := \lim_{L \to +\infty} (\log L)^{d-1} \min\Big\{ \int_{Q(L)} f(\nabla u)\, dx : u - z \in W_0^{1,d}(Q(L); \mathbb{R}^m), u = 0 \text{ on } Q(1) \Big\}$$

and the factor $(\log L)^{d-1}$ compensates for the logarithmic decay of the relative non-linear capacity at the critical exponent.

In a similar vein, the aim of the present paper is to perform the variational analysis of Dirichlet problems on periodically perforated domains at the critical exponent in a general non-local framework, providing an extension of the results for subcritical exponents recently obtained by Alicandro, Gelli, and Leone [5] and by the second author [31]. We propose a unified analysis that encompasses both convolution-type functionals, for which a variational theory has been developed by Alicandro, Ansini, Braides, Piatnitski, and Tribuzio [2], and discrete functionals within the framework investigated by Alicandro and Cicalese [4]. Functionals considered in these two settings share many similarities, starting from their structure of pairwise-interactions energies. Given an energy density f that satisfies standard growth conditions of order $p > 1$, convolution-type energies studied in [2] are of the form

$$\frac{1}{\varepsilon^{d+p}} \int_{\Omega \times \Omega} f\Big(\frac{y - x}{\varepsilon}, u(y) - u(x)\Big)\, d\mathcal{L}^d(x)\, d\mathcal{L}^d(y)$$

and can be regarded as a continuous counterpart of the discrete energies considered in [4] given by

$$\varepsilon^{d-p}\sum_{\substack{\alpha,\beta\in\Omega\cap\varepsilon\mathbb{Z}^d\\ [\alpha,\beta]\subset\Omega}} f\Big(\frac{\beta-\alpha}{\varepsilon}, u(\beta)-u(\alpha)\Big),$$

where $[\alpha,\beta] := \{(1-t)\alpha + t\beta : t\in[0,1]\}$, up to taking into account all possible interactions and not only those involving points connected by a segment that lies in the domain. In both instances, ε is a small positive length-scale (that in the discrete case corresponds to the lattice spacing) responsible for a non-local-to-local passage, provided that the density f satisfies suitable decay assumptions in the first variable. As $\varepsilon\to 0$, the limit energy is given by the *local* functional

$$\int_\Omega f_{\rm hom}(\nabla u)\,dx, \qquad u\in W^{1,p}(\Omega;\mathbb{R}^m),$$

where $f_{\rm hom}$ is expressed by an asymptotic formula having the same structure in both the continuous and discrete case. As for convolution-type energies, such limit behaviour can be interpreted as a 'concentration' phenomenon since relevant pairwise interactions are close to the diagonal of the domain of integration $\Omega\times\Omega$ as $\varepsilon\to 0$ (see the seminal papers by Bourgain, Brezis, and Mironescu [11] and by Ponce [38, 39]), while the discrete-to-continuum passage is usually referred to as a homogenization process. For the sake of notation, we let $f_{\rm hom}$ denote the 'homogenized' energy density in both circumstances.

Putting aside temporarily the difference between the admissible pairs of interaction (that, in fact, coincide if Ω is convex), the comparable structure of convolution-type and discrete energies suggests rewriting both functionals as the same double integral that, at the critical exponent $p=d$, reads as

$$\int_{\Omega\times\Omega} f\Big(\frac{y-x}{\varepsilon}, u(y)-u(x)\Big)\,d\mu_\varepsilon(x)\,d\mu_\varepsilon(y),$$

upon introducing the reference measures

$$\mu_\varepsilon := \begin{cases} \dfrac{1}{\varepsilon^d}\mathcal{L}^d & \textit{(continuous case)},\\ \displaystyle\sum_{\alpha\in\varepsilon\mathbb{Z}^d}\delta_\alpha & \textit{(discrete case)}, \end{cases}$$

for every $\varepsilon>0$. With this approach, admissible configurations are μ_ε-measurable functions; i.e., either Lebesgue-measurable functions (continuous case) or functions defined on $\varepsilon\mathbb{Z}^d$ (discrete case) that we identify with their piecewise-constant interpolations on the cells of the lattice. This identification allows us to regard both continuous and discrete configurations as elements of $L^d(\Omega;\mathbb{R}^m)$ and to study their asymptotic behaviour through Γ-convergence with respect to the same strong topology. At this point, boundary conditions can be imposed μ_ε-a.e. on the set of perforations obtaining the constrained non-local energies

$$F_{\varepsilon,\delta}(u) := \int_{\Omega\times\Omega} f\Big(\frac{y-x}{\varepsilon}, u(y)-u(x)\Big)\,d\mu_\varepsilon(x)\,d\mu_\varepsilon(y), \qquad \text{if } u(x)=0 \text{ for } \mu_\varepsilon\text{-a.e. } x\in\Omega\setminus\Omega_\delta. \tag{1.1}$$

In this work we assume that the density f is d-homogeneous and satisfies d-growth conditions in the second variable and that the vanishing parameters δ, r_δ are ruled by ε; i.e,

$$\delta=\delta_\varepsilon, \qquad r_\varepsilon = r_{\delta_\varepsilon}\sim \exp(-\delta_\varepsilon^{\frac{d}{1-d}}).$$

Up to fixing the same kind of reference measures for every ε, we prove that if the length-scale ε is small compared to the side-length of a perforation r_ε, then a separation of scales occurs and the Γ-limit of $\{F_{\varepsilon,\delta_\varepsilon}\}_\varepsilon$ is the same that is obtained letting first $\varepsilon\to 0$ and then $\delta\to 0$ in (1.1); that is,

$$\int_\Omega f_{\rm hom}(\nabla u)\,dx + \int_\Omega \varphi(u)\,dx, \qquad u\in W^{1,d}(\Omega;\mathbb{R}^m), \tag{1.2}$$

where $f_{\rm hom}$ is determined by the continuous or discrete setting and, accordingly, the density of the strange term is given by the homogenization-type formula

$$\varphi(z) := \lim_{L\to+\infty}(\log L)^{d-1}\min\Big\{\int_{Q(L)} f_{\rm hom}(\nabla u)\,dx : u - z \in W_0^{1,d}(Q(L);\mathbb{R}^m),\ u = 0 \text{ on } Q(1)\Big\}. \quad (1.3)$$

The problem of perforated domains is a classical topic. Several results have been obtained concerning its many variants and employing different techniques such as the work by Dal Maso and Garroni [25] and by Dal Maso and Murat [26] for general domains, the extension theorem by Acerbi, Chiadò Piat, Dal Maso, and Percivale [1], and the approach based on the unfolding method pursued by Cioranescu, Damlamian, Donato, Griso, and Zaki [21] (see also [22]). Among the results obtained over the last years in various frameworks, we mention [19, 20, 34] in the case of different boundary conditions, [29] in the fractional setting, and [13, 16] in the setting of convolution-type energies. Very recently, the problem has been treated in the context of stochastic homogenization by Scardia, Zemas, and Zeppieri [42] (see also [9]) and, for fractional energies, by Deangelis, Focardi, and Zeppieri [27] (see also the earlier paper [28]). In the fractional setting, Palatucci has thoroughly discussed the phenomenon of capacitary screening in periodically perforated domains for Gagliardo seminorms defined on the whole $\mathbb{R}^d$ [37] and a description of fractional capacitary potentials has been given in [30]. Moreover, Braides, Noselli, and Vincini have studied vectorial problems in the context of hyperelasticity [15]. These works, as well as many others in the literature, mainly deal with the case of subcritical exponents (that, in a fractional setting, are characterized by the inequality $p < d/s$, where s denotes the fractional order). The necessity of treating a scaling-invariant problem leads us to carry out a delicate study, that substantially departs from the ones performed in the subcritical cases. Our strategy is partly inspired by the works [12, 18] where, in the scalar case, an extension to the critical exponent for the problem of perforated domains in heterogeneous media originally studied in [7] is obtained.

An outcome of our analysis is that the energy density φ in (1.3) is convex, a feature that is relevant (and non-trivial) in the vector-valued case $m > 1$. Besides the technical aspect, the convexity of the strange term reflects the emergence of a multiscale microstructure since optimal configurations are obtained distributing energy equally on every scale from r_ε (the side-length of the perforations) to the much larger scale δ_ε (the period of the perforations). This behaviour displays an analogy with the Ginzburg-Landau model (see [10, 33, 41]). In our argument, such uniform energy distribution is obtained through a dyadic construction around each perforation similarly to [3], where the formation of topological singularities is studied in presence of inhomogeneities. The convexity of the energy density φ is not related to the non-local nature of the problem and actually holds for the original Dirichlet problem on perforated domains at the critical exponent. In fact, this property has not been the starting point of our investigation; from our perspective, it is more appropriate to regard it as an outcome of the multiscale analysis as we shall explain in a forthcoming section. We believe that this observation, together with the flexibility of the techniques employed in the present work, may be of use for the study of other problems concerning perforated domains at the critical exponent.

The approach that we pursue highlights the correspondence between convolution-type and discrete settings. As already mentioned, the main difference lies in the family of pairwise interactions taken into account: on the one hand, in the continuous case every pair of points in Ω interacts; on the other hand, discrete energies only allow for interactions of points connected by a segment inside the domain. At a technical level, it is reasonable to expect that the asymptotic analysis yields the same outcome for both kinds of interaction, upon assuming that Ω has Lipschitz boundary. Indeed, short-range interactions are favored in the passage from non-local to local and, therefore, the global geometry of Ω should not affect the limit energy. In terms of discrete energies, this means that we should be able to consider equivalently all possible pairwise interactions; however, we stick to the above technical constraint in order to resort to the results in [4].

To overcome this discrepancy, it is enough to rewrite the starting energies through a change of variables: letting $\xi := (y - x)/\varepsilon$, convolution-type energies equal

$$\frac{1}{\varepsilon^d}\int_{\mathbb{R}^d}\int_{\{x\in\Omega : x+\varepsilon\xi\in\Omega\}} f(\xi, u(x+\varepsilon\xi) - u(x))\,d\mathcal{L}^d(x)\,d\mathcal{L}^d(\xi)$$

and analogously, letting $\xi := (\beta - \alpha)/\varepsilon$, discrete energies read as

$$\sum_{\xi \in \mathbb{Z}^d} \sum_{\{\alpha \in \Omega \cap \varepsilon\mathbb{Z}^d : [\alpha, \alpha + \varepsilon\xi] \subset \Omega\}} f(\xi, u(\alpha + \varepsilon\xi) - u(\alpha)),$$

in such a way that the unconstrained energies can be written as

$$\int_{\mathbb{R}^d} \int_{\Omega_{\mu_\varepsilon}(\xi)} f(\xi, u(x + \varepsilon\xi) - u(x))\, d\mu_\varepsilon(x)\, d\mu_1(\xi), \tag{1.4}$$

where we set

$$\Omega_{\mu_\varepsilon}(\xi) := \begin{cases} \{x \in \Omega : x + \varepsilon\xi \in \Omega\} & \text{if } \mu_\varepsilon = \dfrac{1}{\varepsilon^d}\mathcal{L}^d \qquad \textit{(continuous case),} \\ \{x \in \Omega : [x, x + \varepsilon\xi] \subset \Omega\} & \text{if } \mu_\varepsilon = \displaystyle\sum_{\alpha \in \varepsilon\mathbb{Z}^d} \delta_\alpha \qquad \textit{(discrete case).} \end{cases}$$

In practice, we investigate the asymptotic behaviour of energies (1.4) subject to Dirichlet boundary conditions, where the energy density $f : \mathbb{R}^d \times \mathbb{R}^m \to [0, +\infty)$ is assumed to be d-homogeneous and locally Lipschitz-continuous in the second variable, coercive on short-range interactions and to satisfy suitable decay assumptions as $|\xi| \to +\infty$ (see (H), (G), and (L) in Section 3). A simultaneous treatment is made possible by the common variational structure of these frameworks that allows us to employ a common argument for the derivation of the Γ-limit drawing from the general results contained in [2] and [4]. Nevertheless, it is worth mentioning that some useful tools such as Poincaré and Poincaré-Wirtinger inequalities in discrete form seem to be lacking in the literature. These constitute a substantial part of our technical analysis and they are proved in Appendix A using some genuinely discrete arguments such as the construction of paths involving nearest-neighbours, uniform estimates on their multiplicity, and the use of suitable piecewise-affine interpolations.

Finally, we note that energies (1.1) are ruled by two small parameters ε, δ; hence, their asymptotic behaviour is affected by the interplay between these scales and other significant regimes may be investigated, such as $\varepsilon \sim r_\delta$. This regime has been already analyzed at the critical exponent in the discrete case (see [44]) and at subcritical exponents for convolution-type energies (see [5]). As for the latter, the corresponding extension to the critical exponent is not dealt with in this work; however, we expect that this could be achieved by adapting (and possibly simplifying) some of the arguments illustrated herein and that the limit energy is of the same form as (1.2), with a density φ given now in terms of some non-linear *non-local* capacitary problems.

The paper is organized as follows. In Section 2 we introduce the notation and recall the notions of capacity and Γ-convergence used throughout the paper. In Section 3 we present the unified continuous-discrete framework, recall the homogenization result for the unconstrained energies (see Theorem 3.2), and state the main result of the paper (see Theorem 3.4). In Section 4 we recall some useful results such as estimates for long-range interactions and Poincaré-type inequalities. Here, we also prove the convexity of the energy density of the strange term. In Section 5 we study the asymptotics of the capacitary-type energy densities and, finally, in Section 6 we carry out the asymptotic analysis: first, we prove a joining lemma on varying domains and an auxiliary result that allows one to reconstruct the strange term as the limit contribution of the energy near the perforations; then, we prove the lower and the upper bounds for the Γ-limit. The appendix contains the proofs of the Poincaré and Poincaré-Wirtinger inequalities, with particular emphasis on the discrete setting.

2 Notation and basic definitions

In this section we introduce the main notation and recall the definitions of capacity and Γ-convergence.

Notation. In what follows we let $d, m \in \mathbb{N}$ denote two fixed positive integers corresponding to the dimension of the reference and the target space of functions we consider, respectively. We assume that $d \geq 2$ and let Ω denote a bounded open subset of $\mathbb{R}^d$ with Lipschitz boundary. We let $\{e_1, \ldots, e_d\}$ denote the standard orthonormal basis in $\mathbb{R}^d$. Given any positive integer k, we let $|x|$ denote the

Euclidean norm of $x \in \mathbb{R}^k$. We denote by $\mathbb{S}^{d-1}$ and $\mathbb{S}^{m-1}$ the unit sphere in $\mathbb{R}^d$ and $\mathbb{R}^m$, respectively. For $x \in \mathbb{R}^d$ and $r > 0$, we denote the open cube in $\mathbb{R}^d$ of center x and side length r as

$$Q(x,r) := x + \Big(-\frac{r}{2}, \frac{r}{2}\Big)^d,$$

and we denote the open ball in $\mathbb{R}^d$ of center x and radius r as $B(x,r)$. When the center x coincides with 0, we simply write $Q(r)$ or $B(r)$. Given points $x, y \in \mathbb{R}^d$, we set $[x,y] := \{(1-t)x+ty : t \in [0,1]\}$. If A is a subset of $\mathbb{R}^d$, we let $\overline{A}$ denote its closure, and set $A^c := \mathbb{R}^d \backslash A$ and $\operatorname{dist}_\infty(x,A) := \inf\{|y-x|_\infty : y \in A\}$ for all $x \in \mathbb{R}^d$, where we let $|x|_\infty := \max\{|x_i| : i \in \{1,\dots,d\}\}$. Given $t \in \mathbb{R}$, we let $\lfloor t \rfloor$ and $\lceil t \rceil$ denote the lower and the upper integer part of t, respectively. Unless otherwise stated, C will always denote a generic strictly positive constant that may change from line to line.

Capacity. We recall the definition of p-capacity for $p > 1$. We refer to [32] for further details. Given open sets $U \subset V \subseteq \mathbb{R}^d$ and $p > 1$, we define the p-capacity of U relative to V as

$$\operatorname{Cap}_p(U,V) := \min\Big\{\int_V |\nabla v|^p \,dx : v \in W_0^{1,p}(V) : v = 1 \text{ on } U\Big\}.$$

If $U = B(r)$ and $V = B(R)$ for some $0 < r < R$, the explicit computation of the corresponding relative capacity can be performed by the reduction to a 1-dimensional problem. In particular, when $p = d$, it holds that

$$\operatorname{Cap}_d(B(r), B(R)) = \mathcal{H}^{d-1}(\mathbb{S}^{d-1})\Big(\log\Big(\frac{R}{r}\Big)\Big)^{1-d}. \tag{2.1}$$

This logarithmic behaviour differs from that exhibited in the case $p < d$, where the relative capacity depends on the radii r and R with polynomial growth. For this reason, we address to $p = d$ as the *critical exponent* for the capacity. By (2.1), we have that $\operatorname{Cap}_d(B(1), B(L)) = \mathcal{H}^{d-1}(\mathbb{S}^{d-1})(\log L)^{1-d}$ and, through a comparison argument, it is possible to infer that

$$\lim_{L\to+\infty} (\log L)^{d-1}\operatorname{Cap}_d(Q(1), Q(L)) = \mathcal{H}^{d-1}(\mathbb{S}^{d-1}),$$

so that, in particular, there exists a positive constant C such that

$$\frac{1}{C} \le (\log L)^{d-1}\operatorname{Cap}_d(Q(1), Q(L)) \le C \tag{2.2}$$

for every $L > 4$.

Analogous capacitary problems can be set in a vector-valued framework. Given $z \in \mathbb{R}^m$ and U, V as above, we may consider the minimum problem

$$\min\Big\{\int_V |\nabla v|^d \,dx : v \in W_0^{1,d}(V;\mathbb{R}^m) : v = z \text{ on } U\Big\}.$$

Due to the scaling property and rotational invariance, it is easily seen that the above minimum coincides with

$$\begin{aligned}\min\Big\{\int_V |\nabla v|^d \,dx &: v \in W_0^{1,d}(V;\mathbb{R}^m) : v = (1,0,...,0) \text{ on } U\Big\}|z|^d \\ &= \min\Big\{\int_V |\nabla v|^d \,dx : v \in W_0^{1,d}(V) : v = 1 \text{ on } U\Big\}|z|^d = \operatorname{Cap}_d(U,V)|z|^d.\end{aligned}$$

Γ-convergence. For our purposes, it suffices to recall the definition of Γ-convergence with respect to the L^d-topology. We refer to [14] and [24]. Given functionals $F_\varepsilon : L^d(\Omega;\mathbb{R}^m) \to [0,+\infty]$, $\varepsilon > 0$, and $F : L^d(\Omega;\mathbb{R}^m) \to [0,+\infty]$, we say that F is the Γ-limit of $\{F_\varepsilon\}_\varepsilon$ with respect to the strong topology of $L^d(\Omega;\mathbb{R}^m)$ as $\varepsilon \to 0$ if, given any sequence $\{\varepsilon_j\}_j$ converging to 0, the following hold:

(i) for every $u \in L^d(\Omega;\mathbb{R}^m)$ and every $\{u_j\}_j \subset L^d(\Omega;\mathbb{R}^m)$ such that $u_j \to u$ in $L^d(\Omega;\mathbb{R}^m)$ as $j \to +\infty$, it holds

$$\liminf_{j\to+\infty} F_{\varepsilon_j}(u_j) \ge F(u) \qquad \text{(liminf inequality)},$$

(ii) for every $u \in L^d(\Omega;\mathbb{R}^m)$ there exists $\{v_j\}_j \subset L^d(\Omega;\mathbb{R}^m)$ such that $v_j \to u$ in $L^d(\Omega;\mathbb{R}^m)$ as $j \to +\infty$ and

$$\limsup_{j\to+\infty} F_{\varepsilon_j}(v_j) \le F(u) \qquad \text{(limsup inequality)}.$$

3 Setting of the problem and the main result

In this section we introduce further notation, we describe the setting of our problem, and we state our main result.

To make our notation more flexible for later use, in this first part we let σ denote a positive parameter. We let

$$\mu_\sigma := \begin{cases} \dfrac{1}{\sigma^d}\mathcal{L}^d & \textit{(continuous case)}, \\ \displaystyle\sum_{\alpha\in\sigma\mathbb{Z}^d} \delta_\alpha & \textit{(discrete case)}, \end{cases}$$

and, given $\xi \in \mathbb{R}^d$ and $A \subseteq \mathbb{R}^d$ a μ_σ-measurable set, we define

$$A_{\mu_\sigma}(\xi) := \begin{cases} \{x \in A : x + \sigma\xi \in A\} & \text{if } \mu_\sigma = \dfrac{1}{\sigma^d}\mathcal{L}^d \quad \textit{(continuous case)}, \\ \{x \in A : [x, x+\sigma\xi] \subset A\} & \text{if } \mu_\sigma = \displaystyle\sum_{\alpha\in\sigma\mathbb{Z}^d} \delta_\alpha \quad \textit{(discrete case)}. \end{cases} \tag{3.1}$$

In the continuous case, $A_{\mu_\sigma}(\xi) = A \cap (A - \sigma\xi)$. In the discrete case, $A_{\mu_\sigma}(\xi)$ is only contained in $A \cap (A - \sigma\xi)$ and these two sets coincide if A is convex.

In the discrete case, a μ_σ-measurable function u on A is a function defined on the restriction of the lattice $u : A \cap \sigma\mathbb{Z}^d \to \mathbb{R}^m$. It is not restrictive to extend such a function to the whole lattice $\sigma\mathbb{Z}^d$ and to identify it with its piecewise constant interpolation on the cells that intersect $A\cap\sigma\mathbb{Z}^d$; that is,

$$u : \mathbb{R}^d \to \mathbb{R}^m, \qquad u(x) = u(\alpha) \text{ if } x \in \alpha + [0,\sigma)^d, \alpha \in A \cap \sigma\mathbb{Z}^d.$$

The piecewise constant identification of u is a $\mathcal{L}^d$-measurable function; hence, in the discrete setting, we define

$$\mathcal{A}_{\mu_\sigma}(A;\mathbb{R}^m) := \{u : \mathbb{R}^d \to \mathbb{R}^m : u(x) = u(\alpha) \text{ if } x \in \alpha + [0,\sigma)^d, \alpha \in A\cap\sigma\mathbb{Z}^d\} \cap L^d(A;\mathbb{R}^m).$$

In the continuous case, a μ_σ-measurable function on A is simply a $\mathcal{L}^d$-measurable function on A. For mere notational consistency, we then let

$$\mathcal{A}_{\mu_\sigma}(A;\mathbb{R}^m) := \{u : \mathbb{R}^d \to \mathbb{R}^m : u \text{ is } \mathcal{L}^d\text{-measurable}\} \cap L^d(A;\mathbb{R}^m).$$

With these choices, we regard functions in $\mathcal{A}_{\mu_\sigma}(A;\mathbb{R}^m)$ as functions in $L^d(A;\mathbb{R}^m)$ for every $\sigma > 0$ in both continuous and discrete case, making then possible the asymptotic analysis via Γ-convergence with respect to the strong L^d-topology.

We let $f : \mathbb{R}^d \times \mathbb{R}^m \to [0,+\infty)$ be a function satisfying the following assumptions:

(H) $f(\xi,\cdot)$ is d-homogeneous for every $\xi \in \mathbb{R}^d$; i.e., $f(\xi, tz) = t^d f(\xi,z)$ for every $t \geq 0, \xi \in \mathbb{R}^d$, and $z \in \mathbb{R}^m$;

(G) the functions $m(\xi) := \inf_{z\in\mathbb{S}^{m-1}} f(\xi,z)$ and $M(\xi) := \sup_{z\in\mathbb{S}^{m-1}} f(\xi,z)$ defined for every $\xi \in \mathbb{R}^d$ satisfy:

- (G0) there exist $C > 0$ and $r_0 > 1$ such that $m(\xi) \geq C$ for μ_1-a.e. ξ such that $|\xi| \leq r_0$,
- (G1) $\displaystyle\int_{\mathbb{R}^d} M(\xi)(1 + |\xi|^d)\, d\mu_1(\xi) < +\infty$;

(L) there exists a positive constant C such that

$$|f(\xi,z) - f(\xi,z')| \leq CM(\xi)(|z|^{d-1} + |z'|^{d-1})|z - z'|$$

for every $\xi \in \mathbb{R}^d$ and for every $z, z' \in \mathbb{R}^m$.

Remark 3.1. We comment on the above hypotheses:

(i) assumption (H) yields $m(\xi)|z|^d \leq f(\xi,z) \leq M(\xi)|z|^d$ for every $(\xi,z) \in \mathbb{R}^d \times \mathbb{R}^m$;

(ii) the constant r_0 in (G0) is assumed to be strictly larger than 1 to deal with the discrete case. In such a case, the relevant values of the energy density f are attained at $\mathbb{Z}^d \times \mathbb{R}^m$; therefore, it is needed to require that $r_0 > 1$ in order to have a meaningful condition on the function m defined in (G). In the continuous case, it would suffice to suppose $r_0 > 0$; we choose to assume that $r_0 > 1$ in order to keep our argument consistent in both frameworks;

(iii) in the discrete case, assuming without loss of generality that $M(0) < +\infty$, we have that (G1) is equivalent to

$$\sum_{\xi \in \mathbb{Z}^d} M(\xi)|\xi|^d < +\infty,$$

which corresponds to the hypothesis (G1) in [31], up to adapting the notation;

(iv) hypothesis (L) is satisfied if f fulfills (H), (G), and $f(\xi, \cdot)$ is convex for every $\xi \in \mathbb{R}^d$.

We introduce the family of unconstrained functionals and, for later convenience, we also highlight the dependence on the domain. For fixed $\sigma > 0$ and $A \subseteq \mathbb{R}^d$ a μ_σ-measurable set, we let $\mathcal{F}_\sigma(\cdot, A) : L^d(A; \mathbb{R}^m) \to [0, +\infty]$ be defined as

$$\mathcal{F}_\sigma(u, A) := \begin{cases} \displaystyle\int_{\mathbb{R}^d} \int_{A_{\mu_\sigma}(\xi)} f(\xi, u(x + \sigma\xi) - u(x))\, d\mu_\sigma(x)\, d\mu_1(\xi) & \text{if } u \in \mathcal{A}_{\mu_\sigma}(A; \mathbb{R}^m), \\ +\infty & \text{otherwise.} \end{cases} \tag{3.2}$$

For the sake of notation, if $A = \Omega$ we simply write $\mathcal{F}_\sigma(u)$ in place of $\mathcal{F}_\sigma(u, \Omega)$.

The asymptotic analysis of the above energies and the convergence of minimum problems are obtained in [2, Theorem 6.1, Proposition 6.4] for the continuous case and in [4, Theorem 4.1, Corollary 4.6] for the discrete case. We reformulate these results in a unified statement suited to our purposes.

Theorem 3.2. *Let A be a bounded open set with Lipschitz boundary and let*

$$\mu_\sigma = \frac{1}{\sigma^d}\mathcal{L}^d \qquad \text{or} \qquad \mu_\sigma = \sum_{\alpha \in \sigma\mathbb{Z}^d} \delta_\alpha$$

for every $\sigma > 0$. Let $\{\mathcal{F}_\sigma(\cdot, A)\}_\sigma$ be defined by (3.2) with f satisfying assumptions (H) *and* (G). *Then, $\{\mathcal{F}_\sigma(\cdot, A)\}_\sigma$ Γ-converges with respect to the $L^d(A; \mathbb{R}^m)$-topology as $\sigma \to 0$ to the functional $\mathcal{F}(\cdot, A) : L^d(A; \mathbb{R}^m) \to [0, +\infty]$ defined as*

$$\mathcal{F}(u, A) := \begin{cases} \displaystyle\int_A f_{\mathrm{hom}}(\nabla u)\, dx & \text{if } u \in W^{1,d}(A; \mathbb{R}^m), \\ +\infty & \text{otherwise,} \end{cases}$$

where $f_{\mathrm{hom}} : \mathbb{R}^{m\times d} \to [0, +\infty)$ is given by the following homogenization formula

$$f_{\mathrm{hom}}(M) := \lim_{h\to+\infty} \frac{1}{h^d} \inf\Big\{ \int_{Q(h)} \int_{Q(h)} f(y - x, v(y) - v(x))\, d\mu_1(x)\, d\mu_1(y) : v \in \mathcal{A}^M_{\mu_1}(Q(h)) \Big\}, \tag{3.3}$$

with

$$\mathcal{A}^M_{\mu_1}(Q(h)) := \{u \in \mathcal{A}_{\mu_1}(Q(h); \mathbb{R}^m) : u(x) = Mx \text{ for } \mu_1\text{-a.e. } x \in Q(h), \mathrm{dist}_\infty(x, Q(h)^c) \le 1\} \tag{3.4}$$

for every $M \in \mathbb{R}^{m\times d}$. In particular, f_{hom} is a d-homogeneous quasiconvex function and there exists a positive constant C such that, for every $M \in \mathbb{R}^{m\times d}$,

$$\frac{1}{C}|M|^d \le f_{\mathrm{hom}}(M) \le C|M|^d. \tag{3.5}$$

Moreover, for any $g : \mathbb{R}^d \to \mathbb{R}^m$ Lipschitz continuous function and $T > 0$ natural, there holds

$$\lim_{\sigma\to 0} \inf\{\mathcal{F}_\sigma(u, A) : u \in \mathcal{A}^{\sigma T, g}_{\mu_\sigma}(A)\} = \min\Big\{ \int_A f_{\mathrm{hom}}(\nabla u(x))\, dx : u - g \in W^{1,d}_0(A; \mathbb{R}^m)\Big\},$$

where

$$\mathcal{A}^{\sigma T, g}_{\mu_\sigma}(A) := \{u \in \mathcal{A}_{\mu_\sigma}(A; \mathbb{R}^m) : u(x) = g(x) \text{ for } \mu_\sigma\text{-a.e. } x \in A, \mathrm{dist}_\infty(x, A^c) \le \sigma T\}. \tag{3.6}$$

Remark 3.3. If $f(\xi,\cdot)$ is convex for every $\xi\in\mathbb{R}^d$, the asymptotic formula (3.3) reduces to

$$f_{\mathrm{hom}}(M)=\int_{\mathbb{R}^d} f(\xi,M\xi)\,d\mu_1(\xi);$$

see [2, Theorem 6.2].

Now we define the functionals of our interest by imposing Dirichlet boundary conditions on an array of perforations. We let ε be a positive reference parameter (that shall tend to 0) and we introduce two further positive vanishing parameters assuming that both are ruled by ε: $\delta=\delta_\varepsilon$ denotes the period of the centers of the perforations, and $r=r_\varepsilon$ denotes the side-length of the cubic perforations.

For the sake of simplicity we suppose that

$$\frac{\delta_\varepsilon}{\varepsilon}\in\mathbb{N}\quad\text{ for every }\varepsilon>0.$$

This hypothesis is convenient for the analysis of the discrete case as it implies that the lattice $\delta_\varepsilon\mathbb{Z}^d$ is coarser than $\varepsilon\mathbb{Z}^d$, so that the perforations are centered on the support of the atomic measure μ_ε; however, it is not needed for the analysis in the continuous framework. For every $\varepsilon>0$, we define the array of perforations as

$$P_\varepsilon:=\bigcup_{i\in\mathbb{Z}^d}\overline{Q}(i\delta_\varepsilon,r_\varepsilon)$$

and we define the domain of our energy as

$$\mathcal{D}_\varepsilon(\Omega;\mathbb{R}^m):=\{u\in\mathcal{A}_{\mu_\varepsilon}(\Omega;\mathbb{R}^m):u(x)=0\text{ for }\mu_\varepsilon\text{-a.e. }x\in P_\varepsilon\}.$$

Then, we define functionals $F_\varepsilon:L^d(\Omega;\mathbb{R}^m)\to[0,+\infty]$ as

$$F_\varepsilon(u):=\begin{cases}\displaystyle\int_{\mathbb{R}^d}\int_{\Omega_{\mu_\varepsilon}(\xi)} f(\xi,u(x+\varepsilon\xi)-u(x))\,d\mu_\varepsilon(x)\,d\mu_1(\xi) & \text{if } u\in\mathcal{D}_\varepsilon(\Omega;\mathbb{R}^m),\\ +\infty & \text{otherwise.}\end{cases}\tag{3.7}$$

Finally, we state our main result.

Theorem 3.4. *Let*

$$\mu_\varepsilon=\frac{1}{\varepsilon^d}\mathcal{L}^d\qquad\textit{or}\qquad\mu_\varepsilon=\sum_{\alpha\in\varepsilon\mathbb{Z}^d}\delta_\alpha$$

for every $\varepsilon>0$ and let $\{F_\varepsilon\}_\varepsilon$ be defined by (3.7) with f satisfying assumptions (H), (G), *and* (L). *Consider two families of positive parameters $\{r_\varepsilon\}_\varepsilon,\{\delta_\varepsilon\}_\varepsilon$ such that*

$$\lim_{\varepsilon\to0}r_\varepsilon=\lim_{\varepsilon\to0}\delta_\varepsilon=0$$

and assume that

$$\lim_{\varepsilon\to0}\frac{\varepsilon}{r_\varepsilon}=0$$

and that there exists $\gamma>0$ such that

$$\lim_{\varepsilon\to0}\frac{\exp(-(\gamma\delta_\varepsilon^d)^{\frac{1}{1-d}})}{r_\varepsilon}=1.$$

Then, the functionals $\{F_\varepsilon\}_\varepsilon$ Γ-converges with respect to the $L^d(\Omega;\mathbb{R}^m)$-topology as $\varepsilon\to0$ to the functional $F:L^d(\Omega;\mathbb{R}^m)\to[0,+\infty]$ defined as

$$F(u):=\begin{cases}\displaystyle\int_\Omega f_{\mathrm{hom}}(\nabla u)\,dx+\gamma\int_\Omega\varphi(u)\,dx & \textit{if } u\in W^{1,d}(\Omega;\mathbb{R}^m),\\ +\infty & \textit{otherwise,}\end{cases}\tag{3.8}$$

where f_{hom} is given by (3.3) *and φ is a convex function that, for every $z\in\mathbb{R}^m$, is given by*

$$\varphi(z):=\lim_{L\to+\infty}(\log L)^{d-1}\min\Big\{\int_{Q(L)}f_{\mathrm{hom}}(\nabla v)\,dx:v-z\in W^{1,d}_0(Q(L);\mathbb{R}^m),v=0\textit{ on }Q(1)\Big\}.\tag{3.9}$$

Remark 3.5. We choose cubic perforations since they are naturally compatible with the cubic lattices used in the discrete setting. This choice is not restrictive since, by a comparison argument, the same conclusion holds true if we replace the reference cube with any bounded set having nonempty interior.

Outline of the proof and convexity of the strange term. We conclude this section by illustrating in broad terms the idea of our proof and how this led us to observe, somewhat indirectly, the convexity of the strange term. Our argument is based on a new variant of the 'joining lemma on varying domains' (see Lemma 6.1) devised by Ansini and Braides in [8]. It allows one to modify a converging family $\{u_\varepsilon\}_\varepsilon$ (at a small energetic cost) by imposing constant boundary conditions on several dyadic concentric annuli surrounding the perforations. In this way, one can isolate the contributions to the energy arising from the perforations and the remaining region. The latter is essentially the unconstrained energy (3.2) and it contributes to the limit energy yielding the first-order term in (3.8).

To estimate the energy due to a single perforation, that we assume to be centered at 0 for simplicity, we perform a dyadic construction: we fix a large $L \in \mathbb{N}$ and cover the reference cell $Q(\delta_\varepsilon)$ by means of concentric cubes

$$Q(r_\varepsilon), Q(r_\varepsilon L), Q(r_\varepsilon L^2), \ldots, Q(r_\varepsilon L^S) \sim Q(\delta_\varepsilon), \qquad S \sim \tfrac{\log(\delta_\varepsilon / r_\varepsilon)}{\log L}.$$

Thanks to an argument that allows us to consider only finite-range interactions, we decompose the energy close to the perforation on each "frame" as follows:

$$F_\varepsilon(u_\varepsilon, Q(\delta_\varepsilon)) \sim \sum_{h=1}^{S} \mathcal{F}_\varepsilon(u_\varepsilon, Q(r_\varepsilon L^h) \setminus Q(r_\varepsilon L^{h-1})).$$

By the joining lemma, we can assume that there exist constants $\{u_\varepsilon^{0,h} : h \in \{0, \ldots, S\}\}$ such that

$$u_\varepsilon = u_\varepsilon^{0,h} \text{ close to } \partial Q(r_\varepsilon L^h) \qquad \text{for every } h \in \{0, \ldots, S\},$$

with $u_\varepsilon^{0,0} = 0$ and, taking into account such boundary conditions, we can optimize the energy on $Q(\delta_\varepsilon)$ to obtain

$$\begin{aligned} F_\varepsilon(u_\varepsilon, Q(\delta_\varepsilon)) \gtrsim \sum_{h=1}^{S} \min\{&\mathcal{F}_\varepsilon(u, Q(r_\varepsilon L^h) \setminus Q(r_\varepsilon L^{h-1})) : \\ &u = u_\varepsilon^{0,h} \text{ close to } \partial Q(r_\varepsilon L^h), u = u_\varepsilon^{0,h-1} \text{ close to } \partial Q(r_\varepsilon L^{h-1})\}. \end{aligned}$$

On a single frame $Q(r_\varepsilon L^h) \setminus Q(r_\varepsilon L^{h-1})$, we use the scaling invariance of the energy and the convergence of minima in Theorem 3.2 to get that

$$\begin{aligned} &\min\{\mathcal{F}_\varepsilon(u, Q(r_\varepsilon L^h) \setminus Q(r_\varepsilon L^{h-1})) : u = u_\varepsilon^{0,h} \text{ close to } \partial Q(r_\varepsilon L^h), u = u_\varepsilon^{0,h-1} \text{ close to } \partial Q(r_\varepsilon L^{h-1})\} \\ &= \min\{\mathcal{F}_{\varepsilon/(r_\varepsilon L^{h-1})}(u, Q(L) \setminus Q(1)) : u = u_\varepsilon^{0,h} \text{ close to } \partial Q(L), u = u_\varepsilon^{0,h-1} \text{ close to } \partial Q(1)\} \\ &\sim \min\Big\{\int_{Q(L)} f_{\mathrm{hom}}(\nabla u)\, dx : u - (u_\varepsilon^{0,h} - u_\varepsilon^{0,h-1}) \in W_0^{1,d}(Q(L); \mathbb{R}^m), u = 0 \text{ on } Q(1)\Big\} \end{aligned} \tag{3.10}$$

as $\varepsilon \to 0$. Note that here we made use of the assumption $\varepsilon / r_\varepsilon \to 0$ and of a uniform-convergence argument that allows one to keep the boundary conditions unaltered in spite of the fact that these actually depend on ε. Recalling (3.9), we have that (3.10) is approximately

$$(\log L)^{1-d} \varphi(u_\varepsilon^{0,h} - u_\varepsilon^{0,h-1}).$$

Since $S \sim \frac{\log(\delta_\varepsilon / r_\varepsilon)}{\log L}$ and $r_\varepsilon \sim \exp(-\delta_\varepsilon^{d/(1-d)})$, we have that $(\log L)^{1-d} \sim \delta_\varepsilon^d S^{d-1}$; hence, summing up over the frames, we have

$$F_\varepsilon(u_\varepsilon, Q(\delta_\varepsilon)) \gtrsim \delta_\varepsilon^d S^{d-1} \sum_{h=1}^{S} \varphi(u_\varepsilon^{0,h} - u_\varepsilon^{0,h-1}).$$

Upon observing that

$$\sum_{h=1}^{S}(u_\varepsilon^{0,h} - u_\varepsilon^{0,h-1}) = u_\varepsilon^{0,S} - u_\varepsilon^{0,0} = u_\varepsilon^{0,S},$$

we can further optimize the previous sum and use the d-homogeneity of φ to get

$$F_\varepsilon(u_\varepsilon, Q(\delta_\varepsilon)) \gtrsim \delta_\varepsilon^d \min\Big\{S^{d-1}\sum_{h=1}^{S}\varphi(y_h) : \sum_{h=1}^{S} y_h = u_\varepsilon^{0,S}\Big\} = \delta_\varepsilon^d \min\Big\{\frac{1}{S}\sum_{h=1}^{S}\varphi(y_h) : \frac{1}{S}\sum_{h=1}^{S} y_h = u_\varepsilon^{0,S}\Big\}.$$

The latter term is seen to be approximately $\delta_\varepsilon^d\varphi^{**}(u_\varepsilon^{0,S})$ as $S \to +\infty$, where φ^{**} denotes the convex envelope of φ. Then, summing up over all the perforations and using that $u_\varepsilon \to u$, we infer that

$$F_\varepsilon\Big(u_\varepsilon, \bigcup_{i\in\mathbb{Z}^d\cap\delta_\varepsilon^{-1}\Omega} Q(i\delta_\varepsilon, \delta_\varepsilon)\Big) \gtrsim \sum_{i\in\mathbb{Z}^d\cap\delta_\varepsilon^{-1}\Omega} \delta_\varepsilon^d\varphi^{**}(u_\varepsilon^{i,S}) \sim \int_\Omega \varphi^{**}(u)\,dx$$

as $\varepsilon \to 0$ and $S \to +\infty$.

The heuristic that yields this lower bound (that is seen to be optimal) seems to be in contrast with the separation of scales argument leading to (3.8) and, to some extent, hints that φ is convex. Conscious of this fact, we directly prove that φ coincides with its convex envelope starting from its very definition (see Proposition 4.11).

4 Preliminary results

In this section we collect some preliminary results that are instrumental for our analysis.

4.1 Auxiliary reference energies

Given r_0 as in (G0), for every $\sigma > 0$, $A \subseteq \mathbb{R}^d$ a μ_σ-measurable set, and $u \in \mathcal{A}_{\mu_\sigma}(A;\mathbb{R}^m)$, we define the reference energies

$$G_\sigma(u, A) := \int_{B(r_0)}\int_{A_{\mu_\sigma}(\xi)} |u(x+\sigma\xi) - u(x)|^d\,d\mu_\sigma(x)\,d\mu_1(\xi), \tag{4.1}$$

with $A_{\mu_\sigma}(\xi)$ as in (3.1). In the continuous case, such functionals can be rewritten as

$$\int_{B(r_0)}\int_{A\cap(A-\sigma\xi)} \Big|\frac{u(x+\sigma\xi) - u(x)}{\sigma}\Big|^d\,dx\,d\xi;$$

in the discrete case, under the additional (and not restrictive) assumption $r_0 < 2$, these can be rewritten as

$$\sum_{k=1}^{d}\sum_{\substack{\alpha\in A\cap\sigma\mathbb{Z}^d\\ [\alpha,\alpha+\sigma e_k]\subset A}} |u(\alpha+\sigma e_k) - u(\alpha)|^d.$$

From now on, we assume without loss of generality that $r_0 \in (1, 2)$, so that the above energies account for nearest neighborhood interactions in the discrete framework.

Concerning these auxiliary functionals, we first recall a lemma that allows us to control interactions of any range with short-range interactions only. This result is obtained recasting the corresponding results in the continuous case (see [2, Lemma 4.1]) and in the discrete one (see [4, Lemma 3.6]).

Lemma 4.1. *There exists a positive constant C such that for every Ω open set, $A \subseteq \Omega$ bounded open set with Lipschitz boundary, $T > 0$, $\xi \in B(T)$, and $\sigma > 0$ such that*

$$\mathrm{dist}_\infty(A, \Omega^c) > (T + 3\sqrt{d})\sigma, \tag{4.2}$$

it holds

$$\int_A |u(x+\sigma\xi) - u(x)|^d\,d\mu_\sigma(x) \le C(1 + |\xi|^d)G_\sigma(u, A + Q((T+3\sqrt{d})\sigma))$$

for every $u \in \mathcal{A}_{\mu_\sigma}(\Omega;\mathbb{R}^m)$.

Proof. Assume that $\mu_\sigma = \sigma^{-d}\mathcal{L}^d$ and let $A + (0,\sigma)\xi := \{x + t\xi : x \in A, t \in (0,\sigma)\}$. By (4.2) and the fact that $r_0 < 2$ we have

$$\mathrm{dist}(A + (0,\sigma)\xi, \Omega^c) > \mathrm{dist}(A + B(\sigma T), \Omega^c) > \mathrm{dist}_\infty(A + Q(\sigma T), \Omega^c) > 3\sqrt{d}\sigma > r_0\sigma,$$

so that, applying [2, Lemma 4.1], we get

$$\int_A \Big| \frac{u(x+\sigma\xi) - u(x)}{\sigma} \Big|^d dx \leq C(1 + |\xi|^d) G_\sigma(u, A + (0,\sigma)\xi + B(r_0\sigma)),$$

and the conclusion follows since $A + (0,\sigma)\xi + B(r_0\sigma) \subset A + Q((T + 3\sqrt{d})\sigma)$.

Assume now that $\mu_\sigma = \sum_{\alpha\in\sigma\mathbb{Z}^d} \delta_\alpha$ and, given $E \subset \Omega$, consider the sets

$$E_\sigma := \{x \in E : \ \mathrm{dist}(x, E^c) > 2\sqrt{d}\sigma\} \qquad \text{and} \qquad R^\xi_\sigma(E) := \{\alpha \in E \cap \sigma\mathbb{Z}^d : [\alpha, \alpha + \sigma\xi] \subset E\}.$$

By [4, Lemma 3.6], we have

$$\sum_{\alpha\in R^\xi_\sigma(E_\sigma)} |u(\alpha + \sigma\xi) - u(\alpha)|^d \leq C|\xi|^d G_\sigma(u, E).$$

The conclusion follows letting $E = A + Q((T + 3\sqrt{d})\sigma)$. Indeed, it holds that

$$[x, x + \sigma\xi] \subset A + B((T + \sqrt{d})\sigma) \qquad \text{for every } x \in A \text{ and } \xi \in B(T)$$

and then, by (4.2), it follows

$$A \cap \sigma\mathbb{Z}^d \subseteq R^\xi_\sigma(A + B((T + \sqrt{d})\sigma)) \subseteq R^\xi_\sigma((A + B((T + 3\sqrt{d})\sigma))_\sigma) \subseteq R^\xi_\sigma((A + Q((T + 3\sqrt{d})\sigma))_\sigma),$$

which yields the thesis. □

Remark 4.2. The above lemma also applies with $\Omega = \mathbb{R}^d$. In particular, there exists a positive constant C such that for every $\xi \in \mathbb{R}^d$ and $\sigma > 0$, it holds

$$\int_{\mathbb{R}^d} |u(x + \sigma\xi) - u(x)|^d \, d\mu_\sigma(x) \leq C(1 + |\xi|^d) G_\sigma(u, \mathbb{R}^d) \tag{4.3}$$

for every $u \in \mathcal{A}_{\mu_\sigma}(\mathbb{R}^d; \mathbb{R}^m)$.

The following proposition is the rescaled version of Poincaré-Wirtinger inequality. We prove it in a slightly more general form in the Appendix (see Proposition A.3). In the continuous setting this can be inferred from [2, Proposition 4.2]. In the discrete setting, we include a proof that also addresses a possible gap in the argument of [17, Lemma 2].

Proposition 4.3. *Let $0 < \ell < L$, consider the set $A := Q(L) \setminus \overline{Q}(\ell)$ or $A := Q(L)$, and let $A' \subseteq A$ be a μ_σ-measurable set and $x_0 \in \mathbb{R}^d$. There exist positive constants σ_0 and C depending on A such that, having set*

$$(u)^{\mu_\sigma}_E := \frac{1}{\mu_\sigma(E)} \int_E u \, d\mu_\sigma$$

for every E μ_σ-measurable set with $\mu_\sigma(E) > 0$, it holds

$$\int_{x_0+\tau A} |u(x) - (u)^{\mu_\sigma}_{x_0+\tau A'}|^d \, d\mu_\sigma(x) \leq C \frac{\mu_\sigma(x_0 + \tau A)}{\mu_\sigma(x_0 + \tau A')} \Big(\frac{\tau}{\sigma}\Big)^d G_\sigma(u, x_0 + \tau A)$$

for every $\tau > 0$, $\sigma \in (0, \sigma_0\tau)$, and $u \in \mathcal{A}_{\mu_\sigma}(x_0 + \tau A; \mathbb{R}^m)$.

We conclude this subsection with the rescaled version of Poincaré inequality. In the continuous case this is proved in [2, Proposition 4.1]; as for the discrete case, a proof can be found in the Appendix (see Proposition A.5).

Proposition 4.4. *Let $T > 1, x_0 \in \mathbb{R}^d$, and let $A \subset \mathbb{R}^d$ be a bounded open set. There exists a positive constant C depending on A such that*

$$\int_{x_0+\tau A} |u(x)|^d \, d\mu_\sigma(x) \leq C\Big(\frac{\tau}{\sigma}\Big)^d G_\sigma(u, x_0 + \tau A)$$

for every $\tau, \sigma > 0$ and for every $u \in \mathcal{A}_{\mu_\sigma}(x_0 + \tau A; \mathbb{R}^m)$ such that $u(x) = 0$ for μ_σ-a.e. $x \in x_0 + \tau A$ such that $\mathrm{dist}_\infty(x, (x_0 + \tau A)^c) \leq \sigma T$.

4.2 Functionals with finite-range interactions

Now we introduce "truncated" versions of the functionals in (3.2). For every $T > 2$ and for every $z \in \mathbb{R}^m$ we let

$$f^T(\xi, z) := \begin{cases} f(\xi, z) & \text{if } |\xi| \leq T, \\ 0 & \text{otherwise,} \end{cases}$$

and, given $\sigma > 0$ and $A \subseteq \mathbb{R}^d$ μ_σ-measurable, we let $\mathcal{F}^T_\sigma(\cdot, A) : L^d(A; \mathbb{R}^m) \to [0, +\infty]$ be defined as

$$\mathcal{F}^T_\sigma(u, A) := \begin{cases} \displaystyle\int_{\mathbb{R}^d} \int_{A_{\mu_\sigma}(\xi)} f^T(\xi, u(x + \sigma\xi) - u(x))\, d\mu_\sigma(x)\, d\mu_1(\xi) & \text{if } u \in \mathcal{A}_{\mu_\sigma}(A; \mathbb{R}^m), \\ +\infty & \text{otherwise.} \end{cases} \tag{4.4}$$

Also in this case, we write $\mathcal{F}^T_\sigma(u)$ in place of $\mathcal{F}^T_\sigma(u, \Omega)$ when $A = \Omega$.

Remark 4.5. Since $T > 2 > r_0$, by (G0) and Remark 3.1(i) it follows

$$CG_\sigma(u, A) \leq \mathcal{F}^T_\sigma(u, A). \tag{4.5}$$

As for the constrained functionals (3.7), for $\varepsilon > 0$ we define $F^T_\varepsilon : L^d(\Omega; \mathbb{R}^m) \to [0, +\infty]$ as

$$F^T_\varepsilon(u) := \begin{cases} \displaystyle\int_{\mathbb{R}^d} \int_{\Omega_{\mu_\varepsilon}(\xi)} f^T(\xi, u(x + \varepsilon\xi) - u(x))\, d\mu_\varepsilon(x)\, d\mu_1(\xi) & \text{if } u \in \mathcal{D}_\varepsilon(\Omega; \mathbb{R}^m), \\ +\infty & \text{otherwise.} \end{cases} \tag{4.6}$$

The Γ-limit of the functionals $\{\mathcal{F}^T_\sigma\}_\sigma$ in (4.4) can be computed making use of Theorem 3.2. According to the following proposition, the corresponding energy densities, denoted by $f^T_{\rm hom}$, converge as $T \to +\infty$ to $f_{\rm hom}$, the energy density of the Γ-limit of the functionals $\{\mathcal{F}_\sigma\}_\sigma$ defined by (3.2).

Proposition 4.6. *Assume $T > 2$ and let*

$$f^T_{\rm hom}(M) := \lim_{h \to +\infty} \frac{1}{h^d} \inf \Big\{ \int_{Q(h)} \int_{Q(h)} f^T(y - x, v(y) - v(x))\, d\mu_1(x)\, d\mu_1(y) : v \in \mathcal{A}^M_{\mu_1}(Q(h)) \Big\} \tag{4.7}$$

for every $M \in \mathbb{R}^{m \times d}$, where $\mathcal{A}^M_{\mu_1}(Q(h))$ is as in (3.4)*. Given $f_{\rm hom}$ as in* (3.3)*, it holds that*

$$\lim_{T \to +\infty} f^T_{\rm hom} = f_{\rm hom} \tag{4.8}$$

monotonically and uniformly on compact subsets of $\mathbb{R}^{m \times d}$.

Proof. The proof of the pointwise convergence in (4.8) is contained in the proof of [2, Theorem 6.1] for the continuous case and in the proof of [4, Proposition 4.2] for the discrete case (in particular, see equation (4.9) therein). As for the uniform convergence on compact subsets of $\mathbb{R}^{m \times d}$, we note that $f^T_{\rm hom} \leq f_{\rm hom}$ for every $T > 2$. By (3.5), this implies that there exists a positive constant C such that

$$\frac{1}{C}|M|^d \leq f^T_{\rm hom}(M) \leq f_{\rm hom}(M) \leq C|M|^d \tag{4.9}$$

for every $T > 2$ and $M \in \mathbb{R}^{m \times d}$. Since $f^T_{\rm hom}$ is a d-homogeneous quasiconvex function, as a consequence of [14, Remark 4.13] and (4.9), there exists a positive constant C such that

$$|f^T_{\rm hom}(M) - f^T_{\rm hom}(M')| \leq C(|M|^{d-1} + |M'|^{d-1})|M - M'| \tag{4.10}$$

for every $T > 2$ and $M, M' \in \mathbb{R}^{m \times d}$. The claim follows by the pointwise convergence, (4.10), and Ascoli-Arzelà Theorem. □

The main advantage in dealing with functionals that account for finite-range interactions only is that, in broad terms, they behave quite similarly to local functionals. For the constrained energies $\{F_\varepsilon\}_\varepsilon$ defined in (3.7) as well, it is possible to perform the asymptotic analysis considering first functionals with finite-range interactions (4.6), and then recovering the general case letting $T \to +\infty$, as stated in the proposition below. The proof is provided in [31, Section 5, Step 1] for the discrete case and, for the continuous case, it is obtained by arguing as in the proof of [2, Lemma 5.1], where the corresponding result for the unconstrained energies $\{\mathcal{F}_\sigma\}_\sigma$ is established.

Proposition 4.7. *Let $\{T_h\}_h$ be an increasing sequence such that $T_h \to +\infty$ as $h \to +\infty$ and let $\{F^{T_h}_\varepsilon\}_\varepsilon$ be defined as in (4.6). For every $h \in \mathbb{N}$, assume that there exists a functional F^{T_h} : $L^d(\Omega; \mathbb{R}^m) \to [0, +\infty]$ such that*

$$\Gamma\text{-}\lim_{\varepsilon \to 0} F^{T_h}_\varepsilon = F^{T_h}.$$

Then, it holds

$$\Gamma\text{-}\lim_{\varepsilon \to 0} F_\varepsilon = \lim_{h \to +\infty} F^{T_h}.$$

The next result accounts for the pre-compactness in the strong L^d-topology of sequences of functions with uniformly bounded energies. We provide a more general statement for non-truncated energies (3.2). The proof of this fact follows combining [2, Theorem 4.2] with (4.5) in the continuous case and can be inferred from the proof of [4, Corollary 3.11] in the discrete case.

Proposition 4.8. *Let $\{\varepsilon_j\}_j$ be a positive sequence such that $\varepsilon_j \to 0$ as $j \to +\infty$, let $\{\mathcal{F}_{\varepsilon_j}\}_j$ be as in (3.2) with $A = \Omega$, and let $u_j \in \mathcal{A}_{\mu_{\varepsilon_j}}(\Omega; \mathbb{R}^m), j \in \mathbb{N}$, be a sequence such that*

$$\sup_j \left(\mathcal{F}_{\varepsilon_j}(u_j) + \|u_j\|_{L^d(\Omega;\mathbb{R}^m)}\right) < +\infty.$$

Then, there exist a (not relabeled) subsequence $u_j \in \mathcal{A}_{\mu_{\varepsilon_j}}(\Omega; \mathbb{R}^m), j \in \mathbb{N}$, and a function $u \in W^{1,d}(\Omega; \mathbb{R}^m)$ such that $u_j \to u$ in $L^d(\Omega; \mathbb{R}^m)$ as $j \to +\infty$.

4.3 Homogenization-type capacitary formula

We illustrate some relevant properties of the density φ appearing in the *strange term* in our main result Theorem 3.4. The following proposition, which coincides with [43, Proposition 5.1] up to replacing spherical perforations with cubic ones, determines the asymptotics of non-linear capacities modelled on d-homogeneous densities.

Proposition 4.9. *Let $g : \mathbb{R}^{m\times d} \to [0, +\infty)$ be a continuous d-homogeneous function and assume that there exists a positive constant C such that, for every $M \in \mathbb{R}^{m\times d}$, it holds*

$$\frac{1}{C}|M|^d \le g(M) \le C|M|^d.$$

Then, there exist

$$\psi(z) := \lim_{L\to+\infty} (\log L)^{d-1} \min\Big\{\int_{Q(L)} g(\nabla v)\,dx : v - z \in W^{1,d}_0(Q(L); \mathbb{R}^m) : v = 0 \text{ on } Q(1)\Big\},$$

and a positive constant C such that, for every $z \in \mathbb{R}^m$, it holds

$$\frac{1}{C}|z|^d \le \psi(z) \le C|z|^d.$$

Our main observation is that the function ψ in Proposition 4.9 is convex. To see this, we first prove the following lemma.

Lemma 4.10. *Let $\psi : \mathbb{R}^m \to [0, +\infty)$ be a coercive continuous function and let ψ^{**} denote its convex envelope. For every $z \in \mathbb{R}^m$ it holds*

$$\psi^{**}(z) = \lim_{S\to+\infty} \min\Big\{\frac{1}{S}\sum_{h=1}^{S} \psi(z_h) : \frac{1}{S}\sum_{h=1}^{S} z_h = z\Big\}. \tag{4.11}$$

Proof. Let $\widetilde{\psi}(z)$ denote the right-hand side of (4.11). We only prove that $\widetilde{\psi}(z) \le \psi^{**}(z)$ as the converse inequality is immediately satisfied.

By Carathéodory's theorem (see [40, Theorem 17.1]), there exist $\{\lambda_1, \dots, \lambda_{m+1}\} \subset [0,1]$ and $\{z_1, \dots, z_{m+1}\} \subset \mathbb{R}^m$ such that

$$\sum_{i=1}^{m+1} \lambda_i \psi(z_i) = \psi^{**}(z), \qquad \sum_{i=1}^{m+1} \lambda_i z_i = z, \qquad \text{and} \qquad \sum_{i=1}^{m+1} \lambda_i = 1.$$

Consider S a positive integer and let

$$(y_1, \dots, y_S) := \Big(\underbrace{\frac{\lambda_1 S}{\lfloor \lambda_1 S \rfloor} z_1, \dots, \frac{\lambda_1 S}{\lfloor \lambda_1 S \rfloor} z_1}_{\lfloor \lambda_1 S \rfloor \text{ times}}, \dots, \underbrace{\frac{\lambda_{m+1} S}{\lfloor \lambda_{m+1} S \rfloor} z_{m+1}, \dots, \frac{\lambda_{m+1} S}{\lfloor \lambda_{m+1} S \rfloor} z_{m+1}}_{\lfloor \lambda_{m+1} S \rfloor \text{ times}}, \underbrace{0, \dots, 0}_{S - \sum_{i=1}^{m+1} \lfloor \lambda_i S \rfloor \text{ times}} \Big).$$

We have that

$$\frac{1}{S} \sum_{h=1}^{S} y_h = \frac{1}{S} \sum_{i=1}^{m+1} \lfloor \lambda_i S \rfloor \frac{\lambda_i S}{\lfloor \lambda_i S \rfloor} z_i = z$$

and

$$\frac{1}{S} \sum_{h=1}^{S} \psi(y_h) = \frac{1}{S} \sum_{i=1}^{m+1} \lfloor \lambda_i S \rfloor \psi\Big(\frac{\lambda_i S}{\lfloor \lambda_i S \rfloor} z_i \Big) + \Big(1 - \frac{1}{S} \sum_{i=1}^{m+1} \lfloor \lambda_i S \rfloor \Big) \psi(0);$$

therefore, passing to the limit as $S \to +\infty$, we obtain

$$\widetilde{\psi}(z) \le \sum_{i=1}^{m+1} \lambda_i \psi(z_i) = \psi^{**}(z),$$

which is the thesis. □

Proposition 4.11. *Let g and ψ be as in Proposition 4.9. Then, ψ is convex.*

Proof. We prove that $\psi \le \psi^{**}$, the other inequality being trivially satisfied. Let $z \in \mathbb{R}^m$. By the d-homogeneity of g, we have that ψ is d-homogeneous as well; hence, in light of Lemma 4.10, we have

$$\psi^{**}(z) = \lim_{S \to +\infty} \min \Big\{ \frac{1}{S} \sum_{h=1}^{S} \psi(z_h) : \frac{1}{S} \sum_{h=1}^{S} z_h = z \Big\} = \lim_{S \to +\infty} S^{d-1} \min \Big\{ \sum_{h=1}^{S} \psi(z_h) : \sum_{h=1}^{S} z_h = z \Big\}. \tag{4.12}$$

Let L, S be fixed positive integers. Consider $(z_1^*, \dots, z_S^*)$ such that $\sum_{h=1}^{S} z_h^* = z$ and

$$\sum_{h=1}^{S} \psi(z_h^*) = \min \Big\{ \sum_{h=1}^{S} \psi(z_h) : \sum_{h=1}^{S} z_h = z \Big\}, \tag{4.13}$$

and, for every $h \in \{1, \dots, S\}$, consider a function v_h such that $v_h - z_h^* \in W_0^{1,d}(Q(L); \mathbb{R}^m)$, $v_h = 0$ on $Q(1)$ and

$$\min \Big\{ \int_{Q(L)} g(\nabla v) \, dx : v - z_h^* \in W_0^{1,d}(Q(L); \mathbb{R}^m) : v = 0 \text{ on } Q(1) \Big\} = \int_{Q(L)} g(\nabla v_h) \, dx. \tag{4.14}$$

Now we define the function $w : Q(L^S) \to \mathbb{R}^m$ as

$$w(x) := \begin{cases} v_1(x) & \text{if } x \in \overline{Q}(L), \\ v_h\Big(\dfrac{x}{L^{h-1}} \Big) + \displaystyle\sum_{l=1}^{h-1} z_l^* & \text{if } x \in Q(L^h) \setminus \overline{Q}(L^{h-1}), h \in \{2, \dots, S\}, \end{cases}$$

and we note that $w - z \in W_0^{1,d}(Q(L^S);\mathbb{R}^m)$ and $w = 0$ on $Q(1)$. Using the d-homogeneity of g and (4.14), we have

$$\begin{aligned}\int_{Q(L^S)} g(\nabla w)\,dx &= \sum_{h=1}^{S} \int_{Q(L^h)\setminus\overline{Q}(L^{h-1})} \frac{1}{(L^{h-1})^d} g\Big(\nabla v_h\Big(\frac{x}{L^{h-1}}\Big)\Big)\,dx \\ &= \sum_{h=1}^{S} \int_{Q(L)} g(\nabla v_h)\,dx \\ &= \sum_{h=1}^{S} \min\Big\{\int_{Q(L)} g(\nabla v)\,dx : v - z_h^* \in W_0^{1,d}(Q(L);\mathbb{R}^m) : v = 0 \text{ on } Q(1)\Big\}.\end{aligned}$$

Therefore, multiplying by $(\log L^S)^{d-1}$ we obtain

$$\begin{aligned}&(\log L^S)^{d-1} \int_{Q(L^S)} g(\nabla w)\,dx \\ &\quad= S^{d-1} \sum_{h=1}^{S} (\log L)^{d-1} \min\Big\{\int_{Q(L)} g(\nabla v)\,dx : v - z_h^* \in W_0^{1,d}(Q(L);\mathbb{R}^m) : v = 0 \text{ on } Q(1)\Big\},\end{aligned}$$

and passing to the limit as $L \to +\infty$, we get

$$\psi(z) \le S^{d-1} \sum_{h=1}^{S} \psi(z_h^*) = S^{d-1} \min\Big\{\sum_{h=1}^{S} \psi(z_h) : \sum_{h=1}^{S} z_h = z\Big\}$$

in virtue of (4.13). Recalling (4.12) and letting $S \to +\infty$ we have $\psi(z) \le \psi^{**}(z)$, which concludes the proof. ☐

5 Approximating capacitary-type energy densities

In this section we introduce some auxiliary functions that, asymptotically, shall constitute our energy density of capacitary-type.

We let T, ℓ, L, σ be positive parameters satisfying

$$T \in \mathbb{N},\ T > 2, \qquad \ell \le 1, \qquad L > 4, \qquad \sigma < 1/T,$$

and recalling the definitions of $\mathcal{F}_\sigma^T$, f_{hom}^T, and f_{hom} in (4.4), (4.7), and (3.3), respectively, we let

$$\begin{aligned}\varphi_{\ell,L,\sigma}^T(z) &:= \inf\{\mathcal{F}_\sigma^T(v, Q(L)\setminus\overline{Q}(\ell)) : v \in \mathcal{A}_{\mu_\sigma}(Q(L);\mathbb{R}^m), \\ &\qquad v(x) = 0 \text{ for } \mu_\sigma\text{-a.e. } x \in \overline{Q}(\ell+\sigma T), v(x) = z \text{ for } \mu_\sigma\text{-a.e. } x \in Q(L-\sigma T)^c\}, \qquad (5.1)\\ \varphi_L^T(z) &:= \min\Big\{\int_{Q(L)} f_{\mathrm{hom}}^T(\nabla v)\,dx : v - z \in W_0^{1,d}(Q(L);\mathbb{R}^m), v = 0 \text{ on } Q(1)\Big\},\\ \varphi^T(z) &:= \lim_{L\to+\infty} (\log L)^{d-1} \varphi_L^T(z), \qquad (5.2)\\ \varphi(z) &:= \lim_{L\to+\infty} (\log L)^{d-1} \min\Big\{\int_{Q(L)} f_{\mathrm{hom}}(\nabla v)\,dx : v - z \in W_0^{1,d}(Q(L);\mathbb{R}^m), v = 0 \text{ on } Q(1)\Big\}, \qquad (5.3)\end{aligned}$$

for every $z \in \mathbb{R}^m$.

Remark 5.1. The following properties hold true:

(i) recalling that the energy $\mathcal{F}_\sigma^T$ accounts for pairwise interactions with range at most σT only, for any function v admissible for $\varphi_{\ell,L,\sigma}^T$ we have

$$\mathcal{F}_\sigma^T(v, Q(L)\setminus\overline{Q}(\ell)) = \mathcal{F}_\sigma^T(v, Q(L)) = \mathcal{F}_\sigma^T(v - z, \mathbb{R}^d);$$

(ii) functions φ_L^T and φ are well-defined as minimum problems since $f_{\rm hom}^T$ and $f_{\rm hom}$ are quasiconvex functions and coerciveness is ensured by the boundary conditions and Poincaré inequality;

(iii) taking into account (4.9) and the fact that $f_{\rm hom}^T$ and $f_{\rm hom}$ are d-homogeneous, we have that φ^T and φ are well defined in light of Proposition 4.9 and are convex by Proposition 4.11;

(iv) by (H) and the d-homogeneity of $f_{\rm hom}^T$ and $f_{\rm hom}$, we have that the functions $\varphi_{\ell,L,\sigma}^T, \varphi_L^T, \varphi^T$, and φ are d-homogeneous;

(v) it holds that

$$\varphi_{\ell,L,\sigma}^T = \varphi_{c\ell,cL,c\sigma}^T$$

for every $c > 0$. Note indeed that, given any function $v \in \mathcal{A}_{\mu_\sigma}(Q(L);\mathbb{R}^m)$ satisfying $v = 0$ μ_σ-a.e. on $\overline{Q}(\ell + \sigma T)$ and $v = z$ μ_σ-a.e. on $Q(L - \sigma T)^c$, we have that

$$w(x) := v\Big(\frac{x}{c}\Big), \qquad x \in \mathbb{R}^d,$$

is such that $w \in \mathcal{A}_{\mu_\sigma}(Q(cL);\mathbb{R}^m)$ with $w = 0$ μ_σ-a.e. on $\overline{Q}(c\ell + c\sigma T)$ and $w = z$ μ_σ-a.e. on $Q(cL - c\sigma T)^c$ and, by (H),

$$\begin{aligned}
&\mathcal{F}_{c\sigma}^T(w, Q(cL) \setminus \overline{Q}(c\ell)) \\
&\quad = \int_{\mathbb{R}^d} \int_{(Q(cL)\setminus\overline{Q}(c\ell))_{\mu_{c\sigma}}(\xi)} f^T(\xi, w(x + c\sigma\xi) - w(x))\, d\mu_{c\sigma}(x)\, d\mu_1(\xi) \\
&\quad = \int_{\mathbb{R}^d} \int_{(Q(L)\setminus\overline{Q}(\ell))_{\mu_\sigma}(\xi)} f^T(\xi, v(x + \sigma\xi) - v(x))\, d\mu_\sigma(x)\, d\mu_1(\xi) = \mathcal{F}_\sigma^T(v, Q(L) \setminus \overline{Q}(\ell));
\end{aligned}$$

(vi) for $\ell \le \ell'$ and $L \le L'$ we have

$$\varphi_{\ell,L,\sigma}^T \le \varphi_{\ell',L,\sigma}^T, \qquad \varphi_{\ell,L',\sigma}^T \le \varphi_{\ell,L,\sigma}^T, \qquad \text{and} \qquad \varphi_{L'}^T \le \varphi_L^T.$$

We establish uniform bounds for the functions in (5.1).

Lemma 5.2. *Let $\varphi_{\ell,L,\sigma}^T$ be as in* (5.1), *then there exists a positive constant C such that*

$$(\log L)^{d-1}\varphi_{\ell,L,\sigma}^T(z) \le C|z|^d \tag{5.4}$$

and

$$|(\log L)^{d-1}\varphi_{\ell,L,\sigma}^T(z) - (\log L)^{d-1}\varphi_{\ell,L,\sigma}^T(z')| \le C(|z|^{d-1} + |z'|^{d-1})|z - z'| \tag{5.5}$$

for every $T \in \mathbb{N}, T > 2, \ell \le 1, L > 4$ and $\sigma < 1/T$, and for every $z, z' \in \mathbb{R}^m$.

Proof. Let $z \in \mathbb{R}^m$ and $u \in C^1(\mathbb{R}^d;\mathbb{R}^m)$ such that $u - z \in C_c^1(Q(L - \sigma(T+1));\mathbb{R}^m)$ and $u = 0$ in $\overline{Q}(\ell + \sigma(T+1))$. We show that there exist a function $\hat{u}$ admissible for the minimum problem (5.1) and a constant C depending on r_0 and d such that

$$G_\sigma(\hat{u} - z, \mathbb{R}^d) \le C \int_{Q(L-\sigma(T+1))} |\nabla u|^d\, dx, \tag{5.6}$$

where $G_\sigma(\hat{u} - z, \mathbb{R}^d)$ is defined as in (4.1) with $A = \mathbb{R}^d$. To this end, we treat separately the continuous and the discrete case.

If $\mu_\sigma = \sigma^{-d}\mathcal{L}^d$, we let $\hat{u} = u$. Recalling that in this case $G_\sigma(u - z, \mathbb{R}^d)$ corresponds to

$$\int_{B(r_0)} \int_{\mathbb{R}^d} \Big|\frac{u(x + \sigma\xi) - u(x)}{\sigma}\Big|^d\, dx\, d\xi,$$

and using that

$$\frac{u(x + \sigma\xi) - u(x)}{\sigma} = \int_0^1 \partial_\xi u(x + t\sigma\xi)\, dt, \tag{5.7}$$

by Jensen's inequality and Fubini's Theorem we obtain

$$
\begin{aligned}
G_\sigma(u-z,\mathbb{R}^d) &= \int_{B(r_0)}\int_{\mathbb{R}^d}\Big|\int_0^1 \partial_\xi u(x+t\sigma\xi)\,dt\Big|^d\,dx\,d\xi \\
&\le \int_{B(r_0)}\int_{\mathbb{R}^d}\int_0^1 |\partial_\xi u(x+t\sigma\xi)|^d\,dt\,dx\,d\xi \\
&\le \int_{B(r_0)}|\xi|^d\int_0^1\int_{\mathbb{R}^d}|\nabla u(x)|^d\,dx\,dt\,d\xi \le C\int_{Q(L-\sigma(T+1))}|\nabla u|^d\,dx.
\end{aligned}
$$

If $\mu_\sigma=\sum_{\alpha\in\sigma\mathbb{Z}^d}\delta_\alpha$, we define the discrete function

$$
\hat u(\alpha) := \frac{1}{\sigma^d}\int_{\alpha+[0,\sigma)^d} u(x)\,dx, \qquad \alpha\in\sigma\mathbb{Z}^d.
$$

Recalling that now

$$
G_\sigma(\hat u - z,\mathbb{R}^d) = \sum_{k=1}^d\sum_{\alpha\in\sigma\mathbb{Z}^d}|\hat u(\alpha+\sigma e_k)-\hat u(\alpha)|^d,
$$

we apply (5.7) with $\xi\in\{e_1,\dots,e_d\}$ and combine Jensen's inequality with Fubini's Theorem to obtain

$$
\begin{aligned}
G_\sigma(\hat u - z,\mathbb{R}^d) &= \sum_{k=1}^d\sum_{\alpha\in\sigma\mathbb{Z}^d}\Big|\frac{1}{\sigma^d}\int_{\alpha+[0,\sigma)^d}u(x+\sigma e_k)-u(x)\,dx\Big|^d \\
&= \sum_{k=1}^d\sum_{\alpha\in\sigma\mathbb{Z}^d}\Big|\frac{1}{\sigma^d}\int_{\alpha+[0,\sigma)^d}\sigma\int_0^1\partial_{e_k}u(x+t\sigma e_k)\,dt\,dx\Big|^d \\
&\le \sum_{k=1}^d\sum_{\alpha\in\sigma\mathbb{Z}^d}\int_{\alpha+[0,\sigma)^d}\int_0^1\big|\partial_{e_k}u(x+t\sigma e_k)\big|^d\,dt\,dx \\
&\le C\sum_{\alpha\in\sigma\mathbb{Z}^d}\int_0^1\int_{\alpha+[0,2\sigma)^d}|\nabla u(x)|^d\,dx\,dt \le C\int_{Q(L-\sigma(T+1))}|\nabla u|^d\,dx,
\end{aligned}
$$

which proves (5.6).

Taking into account Remark 3.1(i) and Remark 5.1(i), we apply (G1), (4.3), and (5.6) to get

$$
\begin{aligned}
\mathcal{F}^T_\sigma(\hat u, Q(L)\setminus\overline{Q}(\ell)) &= \mathcal{F}^T_\sigma(\hat u - z,\mathbb{R}^d) \\
&\le \int_{\mathbb{R}^d}M(\xi)\int_{\mathbb{R}^d}|\hat u(x+\sigma\xi)-\hat u(x)|^d\,d\mu_\sigma(x)\,d\mu_1(\xi) \\
&\le C\Big(\int_{\mathbb{R}^d}M(\xi)(1+|\xi|^d)\,d\mu_1(\xi)\Big)G_\sigma(\hat u - z,\mathbb{R}^d) \le C\int_{Q(L-\sigma(T+1))}|\nabla u|^d\,dx,
\end{aligned}
$$

and then, by the arbitrariness of u, we conclude

$$
\begin{aligned}
&(\log L)^{d-1}\varphi^T_{\ell,L,\sigma}(z) \\
&\le C(\log L)^{d-1}\inf\Big\{\int_{Q(L-\sigma(T+1))}|\nabla v|^d\,dx : v-z\in C^1_c(Q(L-\sigma(T+1);\mathbb{R}^m), v=0 \text{ on } Q(\ell+\sigma(T+1))\Big\} \\
&= C(\log L)^{d-1}\mathrm{Cap}_d\big(Q(\ell+\sigma(T+1)),Q(L-\sigma(T+1))\big)|z|^d \\
&\le C(\log L)^{d-1}\Big(\log\Big(\frac{L-\sigma(T+1)}{\ell+\sigma(T+1)}\Big)\Big)^{1-d}|z|^d \\
&\le C(\log L)^{d-1}\Big(\log\Big(\frac{L-3/2}{5/2}\Big)\Big)^{1-d}|z|^d \le C|z|^d,
\end{aligned}
$$

which proves (5.4).

Now, we prove (5.5) directly addressing both cases. We first observe that, if $z = 0$ or $z' = 0$, the inequality follows by (5.4). Then, we can suppose z and z' both not null and consider the map $\Theta : \mathbb{R}^m \to \mathbb{R}^m$ defined by

$$\Theta(\zeta) = \frac{|z'|}{|z|}\mathcal{R}_z^{z'}(\zeta), \qquad \zeta \in \mathbb{R}^m,$$

where $\mathcal{R}_z^{z'}$ is a rotation that maps z into $\frac{|z|}{|z'|}z'$. Note that $\Theta(0) = 0$, $\Theta(z) = z'$ and

$$|\Theta(\zeta)| = \frac{|z'|}{|z|}|\zeta|, \qquad |(\Theta - I)(\zeta)| \le C\frac{|z - z'|}{|z|}|\zeta| \tag{5.8}$$

for every $\zeta \in \mathbb{R}^m$, where we let I denote the identity matrix of $\mathbb{R}^{m\times m}$.

Fix $\eta > 0$ and let v_z be (almost) optimal for (5.1); i.e., $\mathcal{F}_\sigma^T(v_z, Q(L) \setminus \overline{Q}(\ell)) \le (1+\eta)\varphi_{\ell,L,\sigma}^T(z)$. We set $v_{z'} := \Theta \circ v_z$ and we note that $v_{z'}$ is admissible for the minimum problem $\varphi_{\ell,L,\sigma}^T(z')$. Then,

$$\begin{aligned}\varphi_{\ell,L,\sigma}^T(z') &\le \mathcal{F}_\sigma^T(v_{z'}, Q(L) \setminus \overline{Q}(\ell))\\ &\le (1+\eta)\varphi_{\ell,L,\sigma}^T(z) + \mathcal{F}_\sigma^T(v_{z'}, Q(L) \setminus \overline{Q}(\ell)) - \mathcal{F}_\sigma^T(v_z, Q(L) \setminus \overline{Q}(\ell)).\end{aligned} \tag{5.9}$$

We set $\Delta_\sigma^\xi w(x) := w(x + \sigma\xi) - w(x)$. By assumption (L) and (5.8) we get

$$\begin{aligned}&|f^T(\xi, \Delta_\sigma^\xi v_{z'}(x)) - f^T(\xi, \Delta_\sigma^\xi v_z(x))|\\ &\le CM(\xi)(|\Delta_\sigma^\xi v_{z'}(x)|^{d-1} + |\Delta_\sigma^\xi v_z(x)|^{d-1})|\Delta_\sigma^\xi v_{z'}(x) - \Delta_\sigma^\xi v_z(x)|\\ &= CM(\xi)(|\Theta(\Delta_\sigma^\xi v_z(x))|^{d-1} + |\Delta_\sigma^\xi v_z(x)|^{d-1})|(\Theta - I)(\Delta_\sigma^\xi v_z(x))|\\ &\le CM(\xi)\Big[\Big(\frac{|z'|}{|z|}\Big)^{d-1}|\Delta_\sigma^\xi v_z(x)|^{d-1} + |\Delta_\sigma^\xi v_z(x)|^{d-1}\Big]\frac{|z - z'|}{|z|}|\Delta_\sigma^\xi v_z(x)|\\ &\le CM(\xi)\frac{|z|^{d-1} + |z'|^{d-1}}{|z|^d}|z - z'||\Delta_\sigma^\xi v_z(x)|^d\end{aligned}$$

for every $x \in \mathbb{R}^d$. Then, we apply (4.3) to get

$$\begin{aligned}&|\mathcal{F}_\sigma^T(v_{z'}, Q(L) \setminus \overline{Q}(\ell)) - \mathcal{F}_\sigma^T(v_z, Q(L) \setminus \overline{Q}(\ell))|\\ &\le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|f^T(\xi, \Delta_\sigma^\xi v_{z'}(x)) - f^T(\xi, \Delta_\sigma^\xi v_z(x))|\,d\mu_\sigma(x)\,d\mu_1(\xi)\\ &\le C\Big(\int_{\mathbb{R}^d} M(\xi)\int_{\mathbb{R}^d}|v_z(x + \sigma\xi) - v_z(x)|^d\,d\mu_\sigma(x)\,d\mu_1(\xi)\Big)\frac{|z|^{d-1} + |z'|^{d-1}}{|z|^d}|z - z'|\\ &\le C\Big(\int_{\mathbb{R}^d} M(\xi)\,(1 + |\xi|^d)\,d\mu_1(\xi)\Big)G_\sigma(v_z - z, \mathbb{R}^d)\frac{|z|^{d-1} + |z'|^{d-1}}{|z|^d}|z - z'|.\end{aligned}$$

Since $T > 2 > r_0$, inequality (4.5) holds, and by (G1) and Remark 5.1(i) we obtain

$$\begin{aligned}|\mathcal{F}_\sigma^T(v_{z'}, Q(L) \setminus \overline{Q}(\ell)) - \mathcal{F}_\sigma^T(v_z, Q(L) \setminus \overline{Q}(\ell))| &\le C\mathcal{F}_\sigma^T(v_z - z, \mathbb{R}^d)\frac{|z|^{d-1} + |z'|^{d-1}}{|z|^d}|z - z'|\\ &\le C(1+\eta)\varphi_{\ell,L,\sigma}^T(z)\frac{|z|^{d-1} + |z'|^{d-1}}{|z|^d}|z - z'|.\end{aligned}$$

Multiplying by $(\log L)^{d-1}$ on both sides of (5.9) and using (5.4) we infer

$$\begin{aligned}(\log L)^{d-1}\varphi_{\ell,L,\sigma}^T(z') &\le (1+\eta)(\log L)^{d-1}\varphi_{\ell,L,\sigma}^T(z)\\ &\quad + (\log L)^{d-1}|\mathcal{F}_\sigma^T(v_{z'}, Q(L) \setminus \overline{Q}(\ell)) - \mathcal{F}_\sigma^T(v_z, Q(L) \setminus \overline{Q}(\ell))|\\ &\le (1+\eta)\Big[(\log L)^{d-1}\varphi_{\ell,L,\sigma}^T(z) + C(\log L)^{d-1}\varphi_{\ell,L,\sigma}^T(z)\frac{|z|^{d-1} + |z'|^{d-1}}{|z|^d}|z - z'|\Big]\\ &\le (1+\eta)\Big[(\log L)^{d-1}\varphi_{\ell,L,\sigma}^T(z) + C(|z|^{d-1} + |z'|^{d-1})|z - z'|\Big].\end{aligned}$$

Letting $\eta \to 0$ and reversing the roles of z and z' we obtain (5.5), which concludes the proof. $\square$

The following result illustrates the asymptotics of the energy densities introduced at the beginning of this section.

Proposition 5.3. *The following hold:*

(i) let $T \in \mathbb{N}, T > 2, \ell \le 1$, and $L > 4$, and assume that

$$\mu_\sigma = \frac{1}{\sigma^d}\mathcal{L}^d \qquad \text{or} \qquad \mu_\sigma = \sum_{\alpha\in\sigma\mathbb{Z}^d} \delta_\alpha$$

for every $\sigma \in (0, 1/T)$. We have

$$\lim_{\sigma\to 0} \frac{\varphi^T_{\ell,L,\sigma}}{\varphi^T_{L/\ell}} = 1$$

uniformly on $\mathbb{R}^m \setminus \{0\}$;

(ii) for every $T \in \mathbb{N}, T > 2$ we have

$$\lim_{L\to+\infty} \frac{(\log L)^{d-1}\varphi^T_L}{\varphi^T} = 1$$

uniformly on $\mathbb{R}^m \setminus \{0\}$;

(iii) we have that $\{\varphi^T(z)\}_T$ is an increasing sequence for every $z \in \mathbb{R}^m$ and

$$\lim_{T\to+\infty} \varphi^T = \varphi$$

uniformly on compact subsets of $\mathbb{R}^m$.

Proof. To prove (i), we first show that

$$\lim_{\sigma\to 0} \varphi^T_{\ell,L,\sigma} = \varphi^T_{L/\ell} \tag{5.10}$$

uniformly on compact subsets of $\mathbb{R}^m$. By (5.5) it is enough to prove the pointwise convergence. Recalling (3.6), we have that

$$\begin{aligned}
&\{v \in \mathcal{A}_{\mu_\sigma}(Q(L);\mathbb{R}^m) : v(x) = 0 \text{ for } \mu_\sigma\text{-a.e. } x \in \overline{Q}(\ell + \sigma T), v(x) = z \text{ for } \mu_\sigma\text{-a.e. } x \in Q(L - \sigma T)^c\}\\
&\quad = \{v \in \mathcal{A}_{\mu_\sigma}(Q(L);\mathbb{R}^m) : v(x) = g(x) \text{ for } \mu_\sigma\text{-a.e. } x \in Q(L) \setminus \overline{Q}(\ell), \mathrm{dist}_\infty(x, [Q(L) \setminus \overline{Q}(\ell)]^c) \le \sigma T\}\\
&\quad = \mathcal{A}^{\sigma T,g}_{\mu_\sigma}(Q(L) \setminus \overline{Q}(\ell)),
\end{aligned}$$

where $g : \mathbb{R}^d \to \mathbb{R}^m$ is a Lipschitz function such that

$$g(x) = \begin{cases} 0 & \text{if } x \in \overline{Q}(\ell + 1),\\ z & \text{if } x \in Q(L - 1)^c.\end{cases}$$

Then, we can rewrite $\varphi^T_{\ell,L,\sigma}(z)$ as

$$\inf\{\mathcal{F}^T_\sigma(v, Q(L) \setminus \overline{Q}(\ell)) : v \in \mathcal{A}^{\sigma T,g}_{\mu_\sigma}(Q(L) \setminus \overline{Q}(\ell))\}$$

and, applying the second part of Theorem 3.2, we obtain

$$\lim_{\sigma\to 0} \varphi^T_{\ell,L,\sigma}(z) = \min\Big\{\int_{Q(L)\setminus\overline{Q}(\ell)} f^T_{\mathrm{hom}}(\nabla v)\, dx : v - g \in W^{1,d}_0(Q(L) \setminus \overline{Q}(\ell);\mathbb{R}^m)\Big\}.$$

Upon setting $v = 0$ in $Q(\ell)$, since $f^T_{\mathrm{hom}}(0) = 0$, we get

$$\begin{aligned}
\lim_{\sigma\to 0} \varphi^T_{\ell,L,\sigma}(z) &= \min\Big\{\int_{Q(L)} f^T_{\mathrm{hom}}(\nabla v)\, dx : v - z \in W^{1,d}_0(Q(L);\mathbb{R}^m), v = 0 \text{ on } Q(\ell)\Big\}\\
&= \min\Big\{\int_{Q(L/\ell)} f^T_{\mathrm{hom}}(\nabla v)\, dx : v - z \in W^{1,d}_0(Q(L/\ell);\mathbb{R}^m), v = 0 \text{ on } Q(1)\Big\} = \varphi^T_{L/\ell}(z),
\end{aligned}$$

where the second equality follows by a change of variables and the d-homogeneity of $f^T_{\hom}$, and this proves (5.10).

Now, we observe that there exists a positive constant C such that

$$\frac{1}{C}|z|^d \le (\log(L/\ell))^{d-1}\varphi^T_{L/\ell}(z) \le C|z|^d \tag{5.11}$$

for every T, ℓ, L and for every $z \in \mathbb{R}^m$. The upper bound readily follows passing to the limit in (5.4) by (5.10). For the lower bound, it suffices to note that $f^T_{\hom}(M) \ge |M|^d/C$ by (4.9), and the estimate follows by (2.2).

We conclude the proof of (i). By Remark 5.1(iv), the function $\varphi^T_{\ell,L,\sigma}/\varphi^T_{L/\ell}$ is 0-homogeneous for every $\sigma \in (0, 1/T)$. Letting then $\zeta := z/|z|$ for every $z \in \mathbb{R}^m \setminus \{0\}$ and using the lower bound in (5.11), we have

$$\left|\frac{\varphi^T_{\ell,L,\sigma}(z)}{\varphi^T_{L/\ell}(z)} - 1\right| = \left|\frac{\varphi^T_{\ell,L,\sigma}(\zeta)}{\varphi^T_{L/\ell}(\zeta)} - 1\right| = \left|\frac{\varphi^T_{\ell,L,\sigma}(\zeta) - \varphi^T_{L/\ell}(\zeta)}{\varphi^T_{L/\ell}(\zeta)}\right| \le C(\log(L/\ell))^{d-1}|\varphi^T_{\ell,L,\sigma}(\zeta) - \varphi^T_{L/\ell}(\zeta)|,$$

and the conclusion follows since the convergence in (5.10) is uniform on $\mathbb{S}^{m-1}$.

The proof of (ii) follows the same line as the proof of (i) upon observing that we have

$$\lim_{L\to+\infty}(\log L)^{d-1}\varphi^T_L = \varphi^T$$

uniformly on compact subsets of $\mathbb{R}^m$ for every $T > 2$. To see this, we apply (5.10) with $\ell = 1$ in order to pass to the limit in (5.5) as $\sigma \to 0$ and get

$$|(\log L)^{d-1}\varphi^T_L(z) - (\log L)^{d-1}\varphi^T_L(z')| \le C(|z|^{d-1} + |z'|^{d-1})|z - z'| \tag{5.12}$$

for every $T > 2, L > 4$, and for every $z, z' \in \mathbb{R}^m$. As a consequence, $\{(\log L)^{d-1}\varphi^T_L\}_L$ is a family of locally equi-lipschitz continuous functions on $\mathbb{R}^m$, and since the pointwise convergence follows by Proposition 4.9, the uniform convergence on compact subsets of $\mathbb{R}^m$ is proved.

Finally, we prove (iii). Let $z \in \mathbb{R}^m$. Recalling the definitions of $\varphi^T(z)$ and $\varphi(z)$ in (5.2) and (5.3), respectively, our aim is to prove that

$$\begin{aligned}\lim_{T\to+\infty}&\left[\lim_{L\to+\infty}(\log L)^{d-1}\min\Big\{\int_{Q(L)} f^T_{\hom}(\nabla v)\,dx : v - z \in W^{1,d}_0(Q(L);\mathbb{R}^m), v = 0 \text{ on } Q(1)\Big\}\right]\\ &= \lim_{L\to+\infty}(\log L)^{d-1}\min\Big\{\int_{Q(L)} f_{\hom}(\nabla v)\,dx : v - z \in W^{1,d}_0(Q(L);\mathbb{R}^m), v = 0 \text{ on } Q(1)\Big\}\end{aligned}$$

uniformly on compact subsets of $\mathbb{R}^m$.

We claim that there exists a positive constant C with the property that for every $\eta > 0$ there exists $T_\eta > 0$ such that

$$\begin{aligned}(\log L)^{d-1}\Big|&\min\Big\{\int_{Q(L)} f_{\hom}(\nabla v)\,dx : v - z \in W^{1,d}_0(Q(L);\mathbb{R}^m), v = 0 \text{ on } Q(1)\Big\}\\ &- \min\Big\{\int_{Q(L)} f^T_{\hom}(\nabla v)\,dx : v - z \in W^{1,d}_0(Q(L);\mathbb{R}^m), v = 0 \text{ on } Q(1)\Big\}\Big| \le C\eta|z|^d\end{aligned} \tag{5.13}$$

for every $L > 4, T > T_\eta$, and $z \in \mathbb{R}^m$.

Let $\eta > 0$. By Proposition 4.6, there exists T_η such that

$$|f_{\hom}(M) - f^T_{\hom}(M)| \le \eta \tag{5.14}$$

for every $M \in \mathbb{R}^{m\times d}$ with $|M| = 1$ and for every $T > T_\eta$. Let now $\overline{v}$ be a minimizer for $\varphi^T_L(z)$; i.e., for

$$\min\Big\{\int_{Q(L)} f^T_{\hom}(\nabla v)\,dx : v - z \in W^{1,d}_0(Q(L);\mathbb{R}^m), v = 0 \text{ on } Q(1)\Big\}.$$

Recalling that $f^T_{\rm hom}$ and $f_{\rm hom}$ are d-homogeneous and that $f^T_{\rm hom}(0)=f_{\rm hom}(0)=0$, we use (4.9), the upper bound in (5.11) with $\ell=1$, and (5.14) to get

$$\begin{aligned}
&(\log L)^{d-1}\Big[\min\Big\{\int_{Q(L)} f_{\rm hom}(\nabla v)\,dx : v-z\in W^{1,d}_0(Q(L);\mathbb{R}^m), v=0 \text{ on } Q(1)\Big\}-\varphi^T_L(z)\Big]\\
&\quad\le(\log L)^{d-1}\Big[\int_{Q(L)} f_{\rm hom}(\nabla\overline{v})\,dx-\int_{Q(L)} f^T_{\rm hom}(\nabla\overline{v})\,dx\Big]\\
&\quad=(\log L)^{d-1}\int_{Q(L)\setminus\{\nabla\overline{v}=0\}}\Big[f_{\rm hom}\Big(\frac{\nabla\overline{v}}{|\nabla\overline{v}|}\Big)-f^T_{\rm hom}\Big(\frac{\nabla\overline{v}}{|\nabla\overline{v}|}\Big)\Big]|\nabla\overline{v}|^d\,dx\\
&\quad\le(\log L)^{d-1}\eta\int_{Q(L)}|\nabla\overline{v}|^d\,dx\\
&\quad\le C(\log L)^{d-1}\eta\int_{Q(L)} f^T_{\rm hom}(\nabla\overline{v})\,dx\\
&\quad=C\eta(\log L)^{d-1}\varphi^T_L(z)\le C\eta|z|^d.
\end{aligned}$$

At this point we can repeat the same argument choosing $\overline{v}$ minimizer for

$$\min\Big\{\int_{Q(L)} f_{\rm hom}(\nabla v)\,dx : v-z\in W^{1,d}_0(Q(L);\mathbb{R}^m), v=0 \text{ on } Q(1)\Big\},$$

which proves the other inequality in order to get our claim. Finally, passing to the limit as $L\to+\infty$ in (5.13) we get

$$|\varphi(z)-\varphi^T(z)|\le C\eta|z|^d$$

for every $T>T_\eta$ and $z\in\mathbb{R}^m$; then the conclusion follows by the arbitrariness of η. Since, by definition, $\varphi^T\le\varphi^{T+1}$ for every $T>2$, the convergence is monotone and the proof is concluded. $\square$

Remark 5.4. By Proposition 5.3(*ii*), letting $L\to+\infty$ in (5.12) we get

$$|\varphi^T(z)-\varphi^T(z')|\le C(|z|^{d-1}+|z'|^{d-1})|z-z'| \tag{5.15}$$

for every $T>2$, and for every $z,z'\in\mathbb{R}^m$, and then $\{\varphi^T\}_T$ are locally equi-lipschitz continuous functions on $\mathbb{R}^m$.

6 Asymptotic analysis

This section is devoted to the proof of Theorem 3.4. We fix $\{\varepsilon_j\}_j,\{\delta_j\}_j,\{r_j\}_j$ positive sequences tending to 0 as $j\to+\infty$ and such that

$$\lim_{j\to+\infty}\frac{\varepsilon_j}{r_j}=0,\qquad \lim_{j\to+\infty}\frac{\exp(-(\gamma\delta_j^d)^{\frac{1}{1-d}})}{r_j}=1 \tag{6.1}$$

for some $\gamma>0$. We immediately observe that (6.1) implies

$$\lim_{j\to+\infty}\frac{|\log r_j|^{1-d}}{\delta_j^d}=\gamma\qquad\text{and}\qquad\lim_{j\to+\infty}\frac{\big(\log\big(\frac{\delta_j}{2r_j}\big)\big)}{|\log r_j|}=1,$$

which, in turn, yields

$$\lim_{j\to+\infty}\frac{\big(\log\big(\frac{\delta_j}{2r_j}\big)\big)^{1-d}}{\delta_j^d}=\gamma. \tag{6.2}$$

6.1 Auxiliary results

The main ingredient of our proof is the following 'joining lemma on varying domains', which is a variant of that originally devised in [8]; see also, e.g., [44]. This is used to modify a sequence with equi-bounded energy so that constant Dirichlet boundary conditions are imposed on several concentric dyadic 'frames' surrounding the perforations. To make the construction more transparent and consistent with the discrete case, we consider frames with side-lengths that are multiples of ε_j, that corresponds to the spacing of the cubic lattice in the discrete setting.

Lemma 6.1. *Let T, k, N be fixed integers such that $T > 2$, $2 \le k \le N-3$, and let $\{S_j\}_j$ be a sequence of positive integers such that*

$$r_j 2^{NS_j} < \frac{\delta_j}{2} \tag{6.3}$$

for every $j \in \mathbb{N}$. Let $u_j \in \mathcal{A}_{\varepsilon_j}(\Omega;\mathbb{R}^m), j \in \mathbb{N}$, be a sequence such that $u_j \to u$ in $L^d(\Omega;\mathbb{R}^m)$ for some $u \in W^{1,d}(\Omega;\mathbb{R}^m)$ as $j \to +\infty$ and let

$$Z_j(\Omega) := \Big\{ i \in \mathbb{Z}^d \cap \frac{1}{\delta_j}\Omega : \mathrm{dist}_\infty(i\delta_j, \Omega^c) > \delta_j \Big\} \tag{6.4}$$

for every $j \in \mathbb{N}$. Assume that

$$\lim_{j\to+\infty} \frac{\varepsilon_j}{r_j} = 0 \tag{6.5}$$

and

$$\sup_j \mathcal{F}^T_{\varepsilon_j}(u_j) < +\infty. \tag{6.6}$$

Then, there exists a sequence $w_j \in \mathcal{A}_{\varepsilon_j}(\Omega;\mathbb{R}^m), j \in \mathbb{N}$, converging to u in $L^d(\Omega;\mathbb{R}^m)$ as $j \to +\infty$ with the property that for every j large enough, for every $i \in Z_j(\Omega)$, and for every $h \in \{1,\dots,S_j\}$ there exists $k_{i,h} \in \{0,\dots,k-1\}$ such that, having set

$$C_j^{i,h} := Q\Big(i\delta_j, \Big\lfloor \frac{r_j}{\varepsilon_j} 2^{Nh-k_{i,h}} \Big\rfloor \varepsilon_j\Big) \setminus \overline{Q}\Big(i\delta_j, \Big\lfloor \frac{r_j}{\varepsilon_j} 2^{Nh-k_{i,h}-1} \Big\rfloor \varepsilon_j\Big), \tag{6.7}$$

$$\rho_j^{i,h} := \Big\lfloor \frac{3}{4}\frac{r_j}{\varepsilon_j} 2^{Nh-k_{i,h}} \Big\rfloor \varepsilon_j,$$

$$u_j^{i,h} := \frac{1}{\mu_{\varepsilon_j}(C_j^{i,h})} \int_{C_j^{i,h}} u_j \, d\mu_{\varepsilon_j},$$

the following hold:

$$w_j = u_j \qquad \mu_{\varepsilon_j}\text{-a.e. on } \Omega \setminus \Big[\bigcup_{\substack{i\in Z_j(\Omega)\\ h\in\{1,\dots,S_j\}}} C_j^{i,h} \Big], \tag{6.8}$$

$$w_j = u_j^{i,h} \qquad \mu_{\varepsilon_j}\text{-a.e. on } \overline{Q}(i\delta_j, \rho_j^{i,h} + \varepsilon_j T) \setminus Q(i\delta_j, \rho_j^{i,h} - \varepsilon_j T), \tag{6.9}$$

$$|\mathcal{F}^T_{\varepsilon_j}(u_j) - \mathcal{F}^T_{\varepsilon_j}(w_j)| \le \frac{C}{k} \tag{6.10}$$

for some $C > 0$ independent of j.

Proof. For every $j \in \mathbb{N}, i \in Z_j(\Omega)$, $h \in \{1,\dots,S_j\}$, and $l \in \{0,\dots,k-1\}$, consider

$$C_{j,l}^{i,h} := Q\Big(i\delta_j, \Big\lfloor \frac{r_j}{\varepsilon_j} 2^{Nh-l} \Big\rfloor \varepsilon_j\Big) \setminus \overline{Q}\Big(i\delta_j, \Big\lfloor \frac{r_j}{\varepsilon_j} 2^{Nh-l-1} \Big\rfloor \varepsilon_j\Big),$$

$$\rho_{j,l}^{h} := \Big\lfloor \frac{3}{4}\frac{r_j}{\varepsilon_j} 2^{Nh-l} \Big\rfloor \varepsilon_j,$$

$$u_{j,l}^{i,h} := \frac{1}{\mu_{\varepsilon_j}(C_{j,l}^{i,h})} \int_{C_{j,l}^{i,h}} u_j \, d\mu_{\varepsilon_j}.$$

We define the open sets $C^{i,h}_{j,l,T}$ as

$$C^{i,h}_{j,l,T} := Q\Big(i\delta_j, \Big\lfloor \frac{r_j}{\varepsilon_j} 2^{Nh-l} \Big\rfloor \varepsilon_j - \varepsilon_j T\Big) \setminus \overline{Q}\Big(i\delta_j, \Big\lfloor \frac{r_j}{\varepsilon_j} 2^{Nh-l-1} \Big\rfloor \varepsilon_j + \varepsilon_j T\Big),$$

and cut-off functions $\chi^{i,h}_{j,l} \in C^\infty_c(C^{i,h}_{j,l,T})$ such that

$$\begin{cases} 0 \leq \chi^{i,h}_{j,l} \leq 1 & \text{on } \mathbb{R}^d, \\ \chi^{i,h}_{j,l} = 1 & \text{on } \overline{Q}(i\delta_j, \rho^h_{j,l} + \varepsilon_j T) \setminus Q(i\delta_j, \rho^h_{j,l} - \varepsilon_j T), \\ |\nabla \chi^{i,h}_{j,l}| \leq \frac{C}{\rho^h_{j,l}} & \text{on } \mathbb{R}^d. \end{cases}$$

We remark that, if we assume that j is large enough, then $C^{i,h}_{j,l,T}$ and $\chi^{i,h}_{j,l}$ are well defined for every $i \in Z_j(\Omega), h \in \{1, \ldots, S_j\}$, and $l \in \{0, \ldots, k-1\}$. It suffices to note that, by a straightforward computation, the inequalities

$$\Big\lfloor \frac{r_j}{\varepsilon_j} 2^{Nh-l} \Big\rfloor \varepsilon_j - \varepsilon_j T > \rho^h_{j,l} + \varepsilon_j T \qquad \text{and} \qquad \rho^h_{j,l} - \varepsilon_j T > \Big\lfloor \frac{r_j}{\varepsilon_j} 2^{Nh-l-1} \Big\rfloor \varepsilon_j + \varepsilon_j T$$

are satisfied for every h, l provided that $r_j/\varepsilon_j > 4(2T+1)$, which holds for j sufficiently large in light of the assumption (6.5). Similarly, the estimate on $|\nabla \chi^{i,h}_{j,l}|$ follows by

$$|\nabla \chi^{i,h}_{j,l}| \leq \frac{C}{\Big(\Big\lfloor \frac{r_j}{\varepsilon_j} 2^{Nh-l} \Big\rfloor \varepsilon_j - \varepsilon_j T\Big) - (\rho^h_{j,l} + \varepsilon_j T)} \leq \frac{C}{\rho^h_{j,l}}.$$

Let then

$$w^{i,h}_{j,l}(x) := \chi^{i,h}_{j,l}(x) u^{i,h}_{j,l} + (1 - \chi^{i,h}_{j,l}(x)) u_j(x) \qquad \text{for } \mu_{\varepsilon_j}\text{-a.e. } x \in \Omega.$$

We aim at estimating $\mathcal{F}^T_{\varepsilon_j}(w^{i,h}_{j,l}, C^{i,h}_{j,l})$. Using Remark 3.1(i), we get

$$\begin{aligned} \mathcal{F}^T_{\varepsilon_j}(w^{i,h}_{j,l}, C^{i,h}_{j,l}) &= \int_{\mathbb{R}^d} \int_{(C^{i,h}_{j,l})_{\mu_{\varepsilon_j}}(\xi)} f^T(\xi, w^{i,h}_{j,l}(x + \varepsilon_j \xi) - w^{i,h}_{j,l}(x))\, d\mu_{\varepsilon_j}(x)\, d\mu_1(\xi) \\ &\leq \int_{B(T)} M(\xi) \int_{(C^{i,h}_{j,l})_{\mu_{\varepsilon_j}}(\xi)} |w^{i,h}_{j,l}(x + \varepsilon_j \xi) - w^{i,h}_{j,l}(x)|^d\, d\mu_{\varepsilon_j}(x)\, d\mu_1(\xi). \end{aligned} \tag{6.11}$$

By definition, we have

$$w^{i,h}_{j,l}(x+\varepsilon_j\xi) - w^{i,h}_{j,l}(x) = (u^{i,h}_{j,l} - u_j(x+\varepsilon_j\xi))(\chi^{i,h}_{j,l}(x+\varepsilon_j\xi) - \chi^{i,h}_{j,l}(x)) + (1 - \chi^{i,h}_{j,l}(x))(u_j(x+\varepsilon_j\xi) - u_j(x)),$$

and then, by the estimate on $\nabla \chi^{i,h}_{j,l}$, it holds that

$$|w^{i,h}_{j,l}(x + \varepsilon_j \xi) - w^{i,h}_{j,l}(x)|^d \leq C |u^{i,h}_{j,l} - u_j(x + \varepsilon_j \xi)|^d \Big| \frac{\varepsilon_j}{\rho^h_{j,l}} \xi \Big|^d + C |u_j(x + \varepsilon_j \xi) - u_j(x)|^d.$$

Substituting in (6.11) and recalling (G1), we obtain

$$\begin{aligned} \mathcal{F}^T_{\varepsilon_j}(w^{i,h}_{j,l}, C^{i,h}_{j,l}) &\leq C \Big(\frac{\varepsilon_j}{\rho^h_{j,l}}\Big)^d \int_{B(T)} M(\xi) |\xi|^d \int_{(C^{i,h}_{j,l})_{\mu_{\varepsilon_j}}(\xi)} |u^{i,h}_{j,l} - u_j(x + \varepsilon_j \xi)|^d\, d\mu_{\varepsilon_j}(x)\, d\mu_1(\xi) \\ &\quad + C \int_{B(T)} M(\xi) \int_{(C^{i,h}_{j,l})_{\mu_{\varepsilon_j}}(\xi)} |u_j(x + \varepsilon_j \xi) - u_j(x)|^d\, d\mu_{\varepsilon_j}(x)\, d\mu_1(\xi) \\ &\leq C \Big(\frac{\varepsilon_j}{\rho^h_{j,l}}\Big)^d \Big(\int_{B(T)} M(\xi) |\xi|^d d\mu_1(\xi)\Big) \int_{C^{i,h}_{j,l}} |u^{i,h}_{j,l} - u_j(x)|^d\, d\mu_{\varepsilon_j}(x) \\ &\quad + C \int_{B(T)} M(\xi) \int_{C^{i,h}_{j,l}} |u_j(x + \varepsilon_j \xi) - u_j(x)|^d\, d\mu_{\varepsilon_j}(x)\, d\mu_1(\xi), \end{aligned} \tag{6.12}$$

where in the last inequality we used that, upon assuming j large enough,

$$\operatorname{dist}_\infty(C_{j,l}^{i,h}, \Omega^c) > \frac{\delta_j}{2} > (T + 3\sqrt{d})\varepsilon_j \tag{6.13}$$

in light of (6.3), (6.4), and the fact that $\varepsilon_j/\delta_j \to 0$; and therefore $C_{j,l}^{i,h} + \varepsilon_j \xi \subset \Omega$ for every $\xi \in B(T), i \in Z_j(\Omega), h \in \{1, \dots, S_j\}$, and $l \in \{0, \dots, k-1\}$.

For the sake of convenience, we introduce the sets

$$E_{j,l}^{i,h} := Q\Big(i\delta_j, 4\Big\lfloor \frac{r_j}{\varepsilon_j} 2^{Nh-l-1} \Big\rfloor \varepsilon_j\Big) \setminus \overline{Q}\Big(i\delta_j, \frac{1}{2}\Big\lfloor \frac{r_j}{\varepsilon_j} 2^{Nh-l-1} \Big\rfloor \varepsilon_j\Big), \tag{6.14}$$

and we note that, in light of (6.5), it holds that

$$\frac{\mu_{\varepsilon_j}(E_{j,l}^{i,h})}{\mu_{\varepsilon_j}(C_{j,l}^{i,h})} \le 2^{d+1} \tag{6.15}$$

and that

$$\begin{aligned}
&C_{j,l}^{i,h} + Q((T + 3\sqrt{d})\varepsilon_j) \\
&= Q\Big(i\delta_j, \Big\lfloor \frac{r_j}{\varepsilon_j} 2^{Nh-l} \Big\rfloor \varepsilon_j + (T + 3\sqrt{d})\varepsilon_j\Big) \setminus \overline{Q}\Big(i\delta_j, \Big\lfloor \frac{r_j}{\varepsilon_j} 2^{Nh-l-1} \Big\rfloor \varepsilon_j - (T + 3\sqrt{d})\varepsilon_j\Big) \\
&\subseteq E_{j,l}^{i,h} \subset \Omega
\end{aligned} \tag{6.16}$$

for every j large enough and for every i, h, l.

To estimate the first term in (6.12), we apply the rescaled version of Poincaré-Wirtinger inequality in Proposition 4.3, with

$$\sigma = \varepsilon_j, \qquad \tau = \Big\lfloor \frac{r_j}{\varepsilon_j} 2^{Nh-l-1} \Big\rfloor \varepsilon_j, \qquad x_0 = i\delta_j,$$

$$A = \frac{1}{\Big\lfloor \frac{r_j}{\varepsilon_j} 2^{Nh-l-1} \Big\rfloor \varepsilon_j}(E_{j,l}^{i,h} - i\delta_j) = Q(4) \setminus \overline{Q}(1/2), \qquad A' = \frac{1}{\Big\lfloor \frac{r_j}{\varepsilon_j} 2^{Nh-l-1} \Big\rfloor \varepsilon_j}(C_{j,l}^{i,h} - i\delta_j).$$

Note indeed that, given σ_0 as in Proposition 4.3, for j large enough we have $\varepsilon_j < \sigma_0 \big\lfloor \frac{r_j}{\varepsilon_j} 2^{Nh-l-1} \big\rfloor \varepsilon_j$ for every h, l by (6.5). Upon choosing j large enough, it also holds that $r_j 2^{Nh-l-1} \le 2\rho_{j,l}^h$ for every h, l; then, combining this observation with (4.5) and (6.15), we infer that

$$\begin{aligned}
\int_{C_{j,l}^{i,h}} |u_{j,l}^{i,h} - u_j(x)|^d \, d\mu_{\varepsilon_j}(x) &\le \int_{E_{j,l}^{i,h}} |u_{j,l}^{i,h} - u_j(x)|^d \, d\mu_{\varepsilon_j}(x) \\
&\le C \frac{\mu_{\varepsilon_j}(E_{j,l}^{i,h})}{\mu_{\varepsilon_j}(C_{j,l}^{i,h})} \Big\lfloor \frac{r_j}{\varepsilon_j} 2^{Nh-l-1} \Big\rfloor^d G_{\varepsilon_j}(u_j, E_{j,l}^{i,h}) \\
&\le C \Big(\frac{\rho_{j,l}^h}{\varepsilon_j}\Big)^d G_{\varepsilon_j}(u_j, E_{j,l}^{i,h}) \le C \Big(\frac{\rho_{j,l}^h}{\varepsilon_j}\Big)^d \mathcal{F}_{\varepsilon_j}^T(u_j, E_{j,l}^{i,h}),
\end{aligned} \tag{6.17}$$

where C depends on the Poincaré-Wirtinger constant of $Q(4) \setminus \overline{Q}(1/2)$ and is independent of all the involved indexes.

Now, we estimate the second term in (6.12). By (6.13), we are in position to apply Lemma 4.1 with

$$\sigma = \varepsilon_j, \qquad A = C_{j,l}^{i,h},$$

and this, combined with (G1), (4.5), and (6.16), yields

$$\begin{aligned}
&\int_{B_T} M(\xi) \int_{C_{j,l}^{i,h}} |u_j(x + \varepsilon_j \xi) - u_j(x)|^d \, d\mu_{\varepsilon_j}(x) \, d\mu_1(\xi) \\
&\le C\Big(\int_{\mathbb{R}^d} M(\xi)(1 + |\xi|^d) \, d\mu_1(\xi)\Big) G_{\varepsilon_j}(u_j, E_{j,l}^{i,h}) \\
&\le C \mathcal{F}_{\varepsilon_j}^T(u_j, E_{j,l}^{i,h}),
\end{aligned} \tag{6.18}$$

where, also in this case, the constant C does not depend on the involved indexes.

We substitute (6.17) and (6.18) in (6.12) to obtain

$$\mathcal{F}^T_{\varepsilon_j}(w^{i,h}_{j,l}, C^{i,h}_{j,l}) \le C\mathcal{F}^T_{\varepsilon_j}(u_j, E^{i,h}_{j,l}),$$

and we deduce

$$|\mathcal{F}^T_{\varepsilon_j}(u_j, C^{i,h}_{j,l}) - \mathcal{F}^T_{\varepsilon_j}(w^{i,h}_{j,l}, C^{i,h}_{j,l})| \le C\mathcal{F}^T_{\varepsilon_j}(u_j, E^{i,h}_{j,l}). \tag{6.19}$$

Since the sets $C^{i,h}_{j,l}$ are pairwise disjoint, by construction it follows that the sets $E^{i,h}_{j,l}$ mutually intersect at most four at a time when i, h are fixed and l ranges in $\{0, \dots, k-1\}$, and

$$\bigcup_{l=0}^{k-1} E^{i,h}_{j,l} \subseteq Q(i\delta_j, r_j 2^{Nh+1}) \setminus \overline{Q}(i\delta_j, r_j 2^{Nh-k-1} - \varepsilon_j/2) =: E^{i,h}_j.$$

Hence, summing over $l \in \{0, \dots, k-1\}$ in (6.19), we obtain

$$\sum_{l=0}^{k-1} |\mathcal{F}^T_{\varepsilon_j}(u_j, C^{i,h}_{j,l}) - \mathcal{F}^T_{\varepsilon_j}(w^{i,h}_{j,l}, C^{i,h}_{j,l})| \le C\mathcal{F}^T_{\varepsilon_j}(u_j, E^{i,h}_j),$$

and therefore, there exists $k_{i,h} \in \{0, \dots, k-1\}$ such that

$$|\mathcal{F}^T_{\varepsilon_j}(u_j, C^{i,h}_{j,k_{i,h}}) - \mathcal{F}^T_{\varepsilon_j}(w^{i,h}_{j,k_{i,h}}, C^{i,h}_{j,k_{i,h}})| \le \frac{C}{k}\mathcal{F}^T_{\varepsilon_j}(u_j, E^{i,h}_j). \tag{6.20}$$

We obtain that conditions (6.8), (6.9), and (6.10) are satisfied choosing

$$C^{i,h}_j := C^{i,h}_{j,k_{i,h}}, \qquad u^{i,h}_j := u^{i,h}_{j,k_{i,h}}, \qquad \rho^{i,h}_j := \rho^h_{j,k_{i,h}},$$

and

$$w_j(x) := \begin{cases} w^{i,h}_{j,k_{i,h}}(x) & \text{for } \mu_{\varepsilon_j}\text{-a.e. } x \in C^{i,h}_j,\ i \in Z_j(\Omega), h \in \{1, \dots, S_j\}, \\ u_j(x) & \text{otherwise.} \end{cases}$$

Indeed, (6.8) is satisfied by definition and, since $\chi^{i,h}_{j,k_{i,h}} = 1$ on $\overline{Q}(i\delta_j, \rho^{i,h}_j + \varepsilon_j T) \setminus Q(i\delta_j, \rho^{i,h}_j - \varepsilon_j T)$, (6.9) follows. Since $k \le N-3$ by assumption, we have that

$$r_j 2^{Nh-k-1} - \varepsilon_j/2 > r_j 2^{N(h-1)+1}$$

for every h provided that $\varepsilon_j/r_j < 1$; hence, for j large and for fixed $i \in Z_j(\Omega)$, the sets $E^{i,h}_j, h \in \{1, \dots, S_j\}$, are pairwise disjoint. Moreover, by assumption (6.3), we have that $E^{i,h}_j \subseteq Q(i\delta_j, \delta_j)$ for every j, i, h, and therefore, the sets $E^{i,h}_j$ are pairwise disjoint also when i ranges in $Z_j(\Omega)$. Resorting to (6.8) and (6.20), in virtue of the assumption (6.6), we obtain

$$\begin{aligned} |\mathcal{F}^T_{\varepsilon_j}(u_j) - \mathcal{F}^T_{\varepsilon_j}(w_j)| &\le \sum_{i \in Z_j(\Omega)} \sum_{h=1}^{S_j} |\mathcal{F}^T_{\varepsilon_j}(u_j, C^{i,h}_j) - \mathcal{F}^T_{\varepsilon_j}(w_j, C^{i,h}_j)| \\ &\le \frac{C}{k} \sum_{i \in Z_j(\Omega)} \sum_{h=1}^{S_j} \mathcal{F}^T_{\varepsilon_j}(u_j, E^{i,h}_j), \\ &\le \frac{C}{k}\mathcal{F}^T_{\varepsilon_j}(u_j) \le \frac{C}{k}, \end{aligned}$$

which proves (6.10).

Finally, we prove that $w_j \to u$ in $L^d(\Omega; \mathbb{R}^m)$. Since $u_j \to u$, we prove that $u_j - w_j \to 0$. By (6.8) and the definition of w_j, we have

$$\begin{aligned} \lim_{j \to +\infty} \int_\Omega |u_j - w_j|^d\,dx &\le \lim_{j \to +\infty} \sum_{i \in Z_j(\Omega)} \sum_{h=1}^{S_j} \int_{C^{i,h}_j} |u_j - u^{i,h}_j|^d\,dx \\ &= \lim_{j \to +\infty} \sum_{i \in Z_j(\Omega)} \sum_{h=1}^{S_j} \varepsilon_j^d \int_{C^{i,h}_j} |u_j - u^{i,h}_j|^d\,d\mu_{\varepsilon_j}, \end{aligned} \tag{6.21}$$

where, in the discrete case, the last equality follows identifying the function u_j with a piecewise constant function and using that $r_j/\varepsilon_j \to +\infty$. Using (6.3) and (6.17), we obtain

$$\varepsilon_j^d \int_{C_j^{i,h}} |u_j - u_j^{i,h}|^d \, d\mu_{\varepsilon_j} \leq C(\rho_{j,k_{i,h}}^h)^d \mathcal{F}_{\varepsilon_j}^T(u_j, \widetilde{E}_j^{i,h}) \leq C\delta_j^d \mathcal{F}_{\varepsilon_j}^T(u_j, \widetilde{E}_j^{i,h}),$$

and then, summing over $i \in Z_j(\Omega)$ and $h \in \{1, \dots, S_j\}$ and substituting in (6.21), we infer

$$\lim_{j\to+\infty} \int_\Omega |u_j - w_j|^d \, dx \leq C \lim_{j\to+\infty} \delta_j^d \mathcal{F}_{\varepsilon_j}^T(u_j) = 0$$

by (6.6). This concludes the proof. □

The following lemma allows us to obtain our *strange term* as the sum of the contributions to the total energy occurring near the perforations.

Lemma 6.2. *Let $u_j \in \mathcal{A}_{\varepsilon_j}(\Omega; \mathbb{R}^m)$, $j \in \mathbb{N}$, be a sequence such that $u_j \to u$ in $L^d(\Omega; \mathbb{R}^m)$ as $j \to +\infty$ with $u \in W^{1,d}(\Omega; \mathbb{R}^m)$, and let $T, k, N, \{S_j\}_j$, and $\{u_j^{i,S_j}\}_j$ be as in Lemma 6.1 for every j large enough and for every $i \in Z_j(\Omega)$. Assume that*

$$r_j 2^{NS_j} < \frac{\delta_j}{2} \leq r_j 2^{N(S_j+1)} \tag{6.22}$$

for every $j \in \mathbb{N}$, that

$$\lim_{j\to+\infty} \frac{\varepsilon_j}{r_j} = 0, \tag{6.23}$$

and

$$\sup_j (\mathcal{F}_{\varepsilon_j}^T(u_j) + \|u_j\|_{L^\infty(\Omega;\mathbb{R}^m)}) < +\infty. \tag{6.24}$$

Then, letting φ^T be defined by (5.2), it holds that

$$\lim_{j\to+\infty} \sum_{i\in Z_j(\Omega)} \delta_j^d \varphi^T(u_j^{i,S_j}) = \int_\Omega \varphi^T(u) \, dx.$$

Proof. To simplify the notation we set $Q_j^i := Q(i\delta_j, \delta_j)$. Using (5.15) and (6.24), the following estimate holds:

$$\begin{aligned}
&\Big| \sum_{i\in Z_j(\Omega)} \delta_j^d \varphi^T(u_j^{i,S_j}) - \int_\Omega \varphi^T(u) \, dx \Big| \\
&= \Big| \sum_{i\in Z_j(\Omega)} \int_{Q_j^i} (\varphi^T(u_j^{i,S_j}) - \varphi^T(u)) \, dx - \int_{\Omega\setminus\bigcup_{i\in Z_j(\Omega)} Q_j^i} \varphi^T(u) \, dx \Big| \\
&\leq \sum_{i\in Z_j(\Omega)} \int_{Q_j^i} |\varphi^T(u_j^{i,S_j}) - \varphi^T(u_j)| \, dx + \sum_{i\in Z_j(\Omega)} \int_{Q_j^i} |\varphi^T(u_j) - \varphi^T(u)| \, dx \\
&\quad + \int_{\Omega\setminus\bigcup_{i\in Z_j(\Omega)} Q_j^i} \varphi^T(u) \, dx \\
&\leq C \sum_{i\in Z_j(\Omega)} \int_{Q_j^i} |u_j^{i,S_j} - u_j| \, dx + C \sum_{i\in Z_j(\Omega)} \int_{Q_j^i} |u_j - u| \, dx \\
&\quad + \int_{\Omega\setminus\bigcup_{i\in Z_j(\Omega)} Q_j^i} \varphi^T(u) \, dx =: A_j^1 + A_j^2 + A_j^3.
\end{aligned}$$

We have

$$A_j^2 \leq C\|u_j - u\|_{L^1(\Omega;\mathbb{R}^m)},$$

which tends to 0 as $j \to +\infty$ by assumption. Moreover, using (6.24) and the fact that

$$\Big|\Omega \setminus \bigcup_{i \in Z_j(\Omega)} Q_j^i\Big| \to 0$$

as $j \to +\infty$, we infer that $A_j^3 \to 0$ as well.

To establish the claim, it remains to prove that $A_j^1 \to 0$ as $j \to +\infty$. Arguing as in the last part of the proof of Lemma 6.1, we have

$$\lim_{j \to +\infty} \sum_{i \in Z_j(\Omega)} \int_{Q_j^i} |u_j^{i,S_j} - u_j| \, dx = \lim_{j \to +\infty} \sum_{i \in Z_j(\Omega)} \varepsilon_j^d \int_{Q_j^i} |u_j^{i,S_j} - u_j| \, d\mu_{\varepsilon_j}.$$

By Hölder's inequality we get

$$\begin{aligned}
\sum_{i \in Z_j(\Omega)} \varepsilon_j^d \int_{Q_j^i} |u_j^{i,S_j} - u_j| \, d\mu_{\varepsilon_j} &\leq \varepsilon_j^d \sum_{i \in Z_j(\Omega)} (\mu_{\varepsilon_j}(Q_j^i))^{1-\frac{1}{d}} \Big(\int_{Q_j^i} |u_j^{i,S_j} - u_j|^d \, d\mu_{\varepsilon_j}\Big)^{\frac{1}{d}} \\
&\leq \varepsilon_j^d \Big(\frac{\delta_j}{\varepsilon_j}\Big)^{d-1} \sum_{i \in Z_j(\Omega)} \Big(\int_{Q_j^i} |u_j^{i,S_j} - u_j|^d \, d\mu_{\varepsilon_j}\Big)^{\frac{1}{d}} \\
&= \delta_j^{d-1} \sum_{i \in Z_j(\Omega)} \Big(\varepsilon_j^d \int_{Q_j^i} |u_j^{i,S_j} - u_j|^d \, d\mu_{\varepsilon_j}\Big)^{\frac{1}{d}}. \qquad (6.25)
\end{aligned}$$

By the first inequality in (6.22) we have $C_j^{i,S_j} \subset Q_j^i$, where C_j^{i,S_j} is as in (6.7); then, recalling that

$$u_j^{i,S_j} = \frac{1}{\mu_{\varepsilon_j}(C_j^{i,S_j})} \int_{C_j^{i,S_j}} u_j \, d\mu_{\varepsilon_j},$$

we apply the rescaled version of Poincaré-Wirtinger inequality in Proposition 4.3 with

$$\sigma = \varepsilon_j, \qquad \tau = \delta_j, \qquad x_0 = i\delta_j, \qquad A = \frac{1}{\delta_j}(Q_j^i - i\delta_j) = Q(1), \qquad A' = \frac{1}{\delta_j}(C_j^{i,S_j} - i\delta_j).$$

Note indeed that, given σ_0 as in Proposition 4.3, we have $\varepsilon_j < \sigma_0 \delta_j$ for j large enough since $\varepsilon_j/\delta_j \to 0$ as $j \to +\infty$ in light of (6.23). We obtain that

$$\begin{aligned}
\varepsilon_j^d \int_{Q_j^i} |u_j^{i,S_j} - u_j|^d \, d\mu_{\varepsilon_j} &\leq C\delta_j^d \frac{\mu_{\varepsilon_j}(Q_j^i)}{\mu_{\varepsilon_j}(C_j^{i,S_j})} G_{\varepsilon_j}(u_j, Q_j^i) \\
&\leq C\delta_j^d \frac{\delta_j^d}{(2^d - 1)(r_j 2^{NS_j - k_{i,S_j} - 1})^d} G_{\varepsilon_j}(u_j, Q_j^i) \\
&\leq C 2^{(2N-2)d} \delta_j^d G_{\varepsilon_j}(u_j, Q_j^i),
\end{aligned}$$

where C depends on d and on the Poincaré-Wirtinger constant of $Q(1)$ but it is independent of j and i, and we used that

$$r_j 2^{NS_j - k_{i,S_j} - 1} \geq \delta_j 2^{-2N+2}$$

in light of the second inequality in (6.22) and the fact that $0 \leq k_{i,S_j} \leq N - 4$. Hence, we obtain

$$\varepsilon_j^d \int_{Q_j^i} |u_j^{i,S_j} - u_j|^d \, d\mu_{\varepsilon_j} \leq C\delta_j^d G_{\varepsilon_j}(u_j, Q_j^i),$$

with the constant C independent of j. Substituting this inequality in (6.25) and using the concavity of the function $x \mapsto x^{\frac{1}{d}}$, (4.5) and (6.24), we conclude that

$$\begin{aligned}\sum_{i\in Z_j(\Omega)} \varepsilon_j^d \int_{Q_j^i} |u_j^{i,S_j} - u_j| \, d\mu_{\varepsilon_j} &\le C\delta_j^{d-1} \sum_{i\in Z_j(\Omega)} \delta_j (G_{\varepsilon_j}(u_j, Q_j^i))^{\frac{1}{d}} \\ &\le C\delta_j^d (\#Z_j(\Omega))^{1-1/d} \Big(\sum_{i\in Z_j(\Omega)} G_{\varepsilon_j}(u_j, Q_j^i) \Big)^{\frac{1}{d}} \\ &\le C\delta_j (G_{\varepsilon_j}(u_j, \Omega))^{\frac{1}{d}} \\ &\le C\delta_j (\mathcal{F}^T_{\varepsilon_j}(u_j))^{\frac{1}{d}} \le C\delta_j,\end{aligned}$$

and we conclude that $A_j^1 \to 0$ as $\delta_j \to 0$. □

Finally, we recall a lemma that asserts that the asymptotic analysis can be reduced to sequences of functions that are equi-bounded in L^∞, see [5, Lemma 6.2] in the continuous case and [44, Lemma 7.2] in the discrete case.

Lemma 6.3. *Let $T > 2$ and $u_j \in \mathcal{A}_{\varepsilon_j}(\Omega; \mathbb{R}^m)$, $j \in \mathbb{N}$, be a sequence such that*

$$\sup_j (\mathcal{F}^T_\varepsilon(u_j) + \|u_j\|_{L^d(\Omega;\mathbb{R}^m)}) < +\infty.$$

Then, for every $\eta > 0$ and $L \in \mathbb{N}$, there exist a subsequence (not relabeled), a constant $R_L > L$ and a 1-Lipschitz function $t_L : \mathbb{R}^m \to \mathbb{R}^m$ such that $t_L(z) = z$ if $|z| < R_L$, $t_L(z) = 0$ if $|z| > 2R_L$, and

$$\liminf_{j\to+\infty} \mathcal{F}^T_{\varepsilon_j}(t_L(u_j)) \le \liminf_{j\to+\infty} \mathcal{F}^T_{\varepsilon_j}(u_j) + \eta.$$

6.2 Proof of Theorem 3.4

For the sake of exposition, we divide the proof into three steps.

Step 1. Let $T > 2$ be a fixed natural. Let $u \in W^{1,d}(\Omega; \mathbb{R}^m)$, $u_j \to u$ in $L^d(\Omega; \mathbb{R}^m)$ and, without loss of generality, suppose that $\sup_j F^T_{\varepsilon_j}(u_j) < +\infty$. We claim that

$$\liminf_{j\to+\infty} F^T_{\varepsilon_j}(u_j) \ge \int_\Omega f^T_{\mathrm{hom}}(\nabla u)\, dx + \gamma \int_\Omega \varphi^T(u)\, dx, \tag{6.26}$$

where f^T_{hom} and φ^T are defined by (4.7) and (5.2), respectively.

In a first instance, we further assume that

$$\sup_j \|u_j\|_{L^\infty(\Omega;\mathbb{R}^m)} < +\infty. \tag{6.27}$$

Fix $k \ge 2$ natural and $\eta \in (0, 1)$. Applying Proposition 5.3(ii) with $L = 2^{N+k}$, we have that

$$\frac{(\log 2^{N+k})^{d-1} \varphi^T_{2^{N+k}}}{\varphi^T} \to 1 \qquad \text{as } N \to +\infty$$

uniformly on $\mathbb{R}^m \setminus \{0\}$. Recalling that $\varphi^T_{2^{N+k}}(0) = \varphi^T(0) = 0$, we deduce that there exists N^* such that

$$(\log 2^N)^{d-1} \varphi^T_{2^{N+k}}(z) \ge (1-\eta)\varphi^T(z) \qquad \text{if } N \ge N^* \tag{6.28}$$

for every $z \in \mathbb{R}^m$. Then, we fix $N > \max\{N^*, k+3\}$. Similarly, applying Proposition 5.3(i), with $\ell = 1$ and $L = 2^{N+k}$, we have

$$\frac{\varphi^T_{1,2^{N+k},\sigma}}{\varphi^T_{2^{N+k}}} \to 1 \qquad \text{as } \sigma \to 0$$

uniformly on $\mathbb{R}^m \setminus \{0\}$, and therefore there exists $\sigma^* < 1/T$ such that

$$\varphi^T_{1,2^{N+k},\sigma}(z) \ge (1-\eta)\varphi^T_{2^{N+k}}(z) \qquad \text{if } \sigma \le \sigma^* \tag{6.29}$$

for every $z \in \mathbb{R}^m$. Since k is fixed and $\varepsilon_j/r_j \to 0$, we assume that $2^k\varepsilon_j/r_j < \sigma^*$ for every $j \in \mathbb{N}$.

We apply Lemma 6.1 with T, k, N fixed as above and a sequence of naturals $\{S_j\}_j$ such that

$$r_j 2^{NS_j} < \frac{\delta_j}{2} \leq r_j 2^{N(S_j+1)} \tag{6.30}$$

for every $j \in \mathbb{N}$. Note that, upon considering j large enough, condition (6.30) is compatible with the choice of the parameter N since $\delta_j/r_j \to +\infty$ as $j \to +\infty$. We obtain functions $w_j \in \mathcal{A}_{\varepsilon_j}(\Omega;\mathbb{R}^m)$, $j \in \mathbb{N}$, converging to u in $L^d(\Omega;\mathbb{R}^m)$ such that, for every j large enough, we have

$$w_j = u_j \qquad \mu_{\varepsilon_j}\text{-a.e. on } \Omega \setminus \Big[\bigcup_{\substack{i\in Z_j(\Omega)\\ h\in\{1,\dots,S_j\}}} C_j^{i,h} \Big], \tag{6.31}$$

$$w_j = u_j^{i,h} \qquad \mu_{\varepsilon_j}\text{-a.e. on } \overline{Q}(i\delta_j, \rho_j^{i,h} + \varepsilon_j T) \setminus Q(i\delta_j, \rho_j^{i,h} - \varepsilon_j T), \tag{6.32}$$

$$F^T_{\varepsilon_j}(u_j) \geq \mathcal{F}^T_{\varepsilon_j}(w_j) - \frac{C}{k} \tag{6.33}$$

for every $i \in Z_j(\Omega)$, $h \in \{1,\dots,S_j\}$, and some $C > 0$ independent of j.

We set

$$E_j := \bigcup_{i\in Z_j(\Omega)} Q(i\delta_j, \rho_j^{i,S_j})$$

and we first estimate from below the contribution to the energy of the region $\Omega \setminus E_j$; that is, far from the perforations that are strictly contained in Ω.

Analysis far from the perforations. We define $v_j \in \mathcal{A}_{\varepsilon_j}(\Omega;\mathbb{R}^m)$, $j \in \mathbb{N}$, as

$$v_j(x) := \begin{cases} u_j^{i,S_j} & \text{for } \mu_{\varepsilon_j}\text{-a.e. } x \in Q(i\delta_j, \rho_j^{i,S_j}), i \in Z_j(\Omega), \\ w_j(x) & \text{otherwise,} \end{cases}$$

and we claim that $v_j \to u$ in $L^d(\Omega;\mathbb{R}^m)$. To this end, we note that, by (6.32) and (6.33), it holds

$$\mathcal{F}^T_{\varepsilon_j}(v_j) = \mathcal{F}^T_{\varepsilon_j}(w_j, \Omega \setminus E_j) \leq F^T_{\varepsilon_j}(u_j) + C,$$

so that $\sup_j \mathcal{F}^T_{\varepsilon_j}(v_j) < +\infty$. Moreover, by (6.27) and the construction of Lemma 6.1, we have

$$\sup_j \|v_j\|_{L^\infty(\Omega;\mathbb{R}^m)} < +\infty;$$

and then, by applying Proposition 4.8, we infer that a (not relabeled) subsequence of $\{v_j\}_j$ converges in $L^d(\Omega;\mathbb{R}^m)$ to a certain v. Let χ denote the periodic extension of the characteristic function of the cube $Q(2/3)$. In virtue of (6.30), we have

$$\bigcup_{h=1}^{S_j} C_j^{i,h} \subset Q\Big(i\delta_j, \frac{\delta_j}{2}\Big)$$

for every $i \in Z_j(\Omega)$; hence, recalling (6.31), we get

$$u_j(x)\Big(1 - \chi\Big(\frac{x}{\delta_j}\Big)\Big) = v_j(x)\Big(1 - \chi\Big(\frac{x}{\delta_j}\Big)\Big) \qquad \text{for } \mathcal{L}^d\text{-a.e. } x \in \Omega, \tag{6.34}$$

up to identifying u_j and v_j with piecewise constant functions in the discrete case. Since $\chi(\cdot/\delta_j)$ weakly*-converges to the constant $(2/3)^d$ in $L^\infty(\Omega;\mathbb{R}^m)$ and since $u_j \to u, v_j \to v$ in $L^d(\Omega;\mathbb{R}^m)$, we pass to the limit in (6.34) to infer that $u = v$ $\mathcal{L}^d$-a.e. on Ω.

Using now that $v_j \to u$ and resorting to Theorem 3.2, we obtain

$$\liminf_{j\to+\infty} \mathcal{F}^T_{\varepsilon_j}(w_j, \Omega \setminus E_j) = \liminf_{j\to+\infty} \mathcal{F}^T_{\varepsilon_j}(v_j) \geq \int_\Omega f^T_{\text{hom}}(\nabla u)\,dx, \tag{6.35}$$

which is the desired lower bound.

Analysis near the perforations. Now we perform the asymptotic analysis on E_j.
Fix $i \in Z_j(\Omega)$ and let

$$w_j^{i,h}(x) := \begin{cases} 0 & \text{if } x \in \overline{Q}(\rho_j^{i,h} + \varepsilon_j T), \\ w_j(x + i\delta_j) - u_j^{i,h} & \text{if } x \in Q(\rho_j^{i,h+1} - \varepsilon_j T) \setminus \overline{Q}(\rho_j^{i,h} + \varepsilon_j T), \\ u_j^{i,h+1} - u_j^{i,h} & \text{if } x \in Q(\rho_j^{i,h+1} - \varepsilon_j T)^c, \end{cases}$$

for every $h \in \{1, \dots, S_j - 1\}$, where the equality is valid for μ_{ε_j}-a.e. $x \in \mathbb{R}^d$. To deal with the Dirichlet boundary condition imposed on the perforation, we let

$$w_j^{i,0}(x) := \begin{cases} 0 & \text{if } x \in \overline{Q}(r_j), \\ w_j(x + i\delta_j) & \text{if } x \in Q(\rho_j^{i,1} - \varepsilon_j T) \setminus \overline{Q}(r_j), \\ u_j^{i,1} & \text{if } x \in Q(\rho_j^{i,1} - \varepsilon_j T)^c, \end{cases}$$

where the equality is valid for μ_{ε_j}-a.e. $x \in \mathbb{R}^d$. We observe that $r_j 2^{-k} + \varepsilon_j T \le r_j$ for j large enough; hence, it also holds

$$w_j^{i,0} = \begin{cases} 0 & \text{for } \mu_{\varepsilon_j}\text{-a.e. } x \in \overline{Q}(r_j 2^{-k} + \varepsilon_j T), \\ u_j^{i,1} & \text{for } \mu_{\varepsilon_j}\text{-a.e. } x \in Q(\rho_j^{i,1} - \varepsilon_j T)^c. \end{cases}$$

We have

$$\begin{aligned} \mathcal{F}_{\varepsilon_j}^T(w_j, Q(i\delta_j, \rho_j^{i,S_j})) &\ge \mathcal{F}_{\varepsilon_j}^T(w_j, Q(i\delta_j, \rho_j^{i,1}) \setminus \overline{Q}(i\delta_j, r_j)) + \sum_{h=1}^{S_j-1} \mathcal{F}_{\varepsilon_j}^T(w_j, Q(i\delta_j, \rho_j^{i,h+1}) \setminus \overline{Q}(i\delta_j, \rho_j^{i,h})) \\ &= \mathcal{F}_{\varepsilon_j}^T(w_j^{i,0}, Q(\rho_j^{i,1}) \setminus \overline{Q}(r_j 2^{-k})) + \sum_{h=1}^{S_j-1} \mathcal{F}_{\varepsilon_j}^T(w_j^{i,h}, Q(\rho_j^{i,h+1}) \setminus \overline{Q}(\rho_j^{i,h})) \\ &\ge \varphi^T_{r_j 2^{-k}, \rho_j^{i,1}, \varepsilon_j}(u_j^{i,1}) + \sum_{h=1}^{S_j-1} \varphi^T_{\rho_j^{i,h}, \rho_j^{i,h+1}, \varepsilon_j}(u_j^{i,h+1} - u_j^{i,h}). \end{aligned}$$

For every j large enough, it holds

$$r_j 2^{Nh-k} \le \rho_j^{i,h} \le r_j 2^{Nh}$$

for every $i \in Z_j(\Omega)$ and $h \in \{1, \dots, S_j\}$; hence, using Remark 5.1(vi) we infer

$$\mathcal{F}_{\varepsilon_j}^T(w_j, Q(i\delta_j, \rho_j^{i,S_j})) \ge \sum_{h=0}^{S_j-1} \varphi^T_{r_j 2^{Nh-k}, r_j 2^{N(h+1)}, \varepsilon_j}(u_j^{i,h+1} - u_j^{i,h}),$$

where we let $u_j^{i,0} := 0$. Upon setting

$$\sigma_j^h := \frac{\varepsilon_j}{r_j} 2^{-Nh+k}, \qquad j \in \mathbb{N},\ h \in \{0, \dots, S_j - 1\},$$

we apply Remark 5.1(v) to get

$$\varphi^T_{r_j 2^{Nh-k}, r_j 2^{N(h+1)}, \varepsilon_j}(u_j^{i,h+1} - u_j^{i,h}) = \varphi^T_{1, 2^{N+k}, \sigma_j^h}(u_j^{i,h+1} - u_j^{i,h})$$

for every $h \in \{0, \dots, S_j - 1\}$. As a consequence, we obtain

$$\mathcal{F}_{\varepsilon_j}^T(w_j, Q(i\delta_j, \rho_j^{i,S_j})) \ge \sum_{h=0}^{S_j-1} \varphi^T_{1, 2^{N+k}, \sigma_j^h}(u_j^{i,h+1} - u_j^{i,h}).$$

We note that $\sigma_j^h \le 2^k\varepsilon_j/r_j < \sigma^*$ for every $j\in\mathbb{N}$ and $h\in\{0,\dots,S_j-1\}$. Then, using (6.28) and (6.29), we get

$$
\begin{aligned}
\mathcal{F}^T_{\varepsilon_j}(w_j, Q(i\delta_j,\rho_j^{i,S_j})) &\ge (1-\eta)\sum_{h=0}^{S_j-1}\varphi^T_{2^{N+k}}(u_j^{i,h+1}-u_j^{i,h})\\
&\ge (1-\eta)^2(\log 2^N)^{1-d}\sum_{h=0}^{S_j-1}\varphi^T(u_j^{i,h+1}-u_j^{i,h}).
\end{aligned}
\tag{6.36}
$$

Now, recalling that $u_j^{i,0}=0$, we observe that

$$
\sum_{h=0}^{S_j-1}(u_j^{i,h+1}-u_j^{i,h}) = u_j^{i,S_j}-u_j^{i,0} = u_j^{i,S_j},
$$

and therefore, using that the function φ^T is convex and d-homogeneous (see Remark 5.1(iii),(iv)), we have

$$
\sum_{h=0}^{S_j-1}\varphi^T(u_j^{i,h+1}-u_j^{i,h}) \ge S_j\varphi^T\Big(\frac{u_j^{i,S_j}}{S_j}\Big) = S_j^{1-d}\varphi^T(u_j^{i,S_j}). \tag{6.37}
$$

We substitute (6.37) in (6.36) and recall that $S_j\log 2^N \le \log(\delta_j/(2r_j))$ by the first inequality in (6.30). We obtain

$$
\mathcal{F}^T_{\varepsilon_j}(w_j, Q(i\delta_j,\rho_j^{i,S_j})) \ge (1-\eta)^2(S_j\log 2^N)^{1-d}\varphi^T(u_j^{i,S_j}) \ge (1-\eta)^2\Big(\log\Big(\frac{\delta_j}{2r_j}\Big)\Big)^{1-d}\varphi^T(u_j^{i,S_j}).
$$

Summing over $i\in Z_j(\Omega)$ we get

$$
\begin{aligned}
\liminf_{j\to+\infty}\mathcal{F}^T_{\varepsilon_j}(w_j,E_j) &= \liminf_{j\to+\infty}\sum_{i\in Z_j(\Omega)}\mathcal{F}^T_{\varepsilon_j}(w_j, Q(i\delta_j,\rho_j^{i,S_j}))\\
&\ge (1-\eta)^2\liminf_{j\to+\infty}\sum_{i\in Z_j(\Omega)}\Big(\log\Big(\frac{\delta_j}{2r_j}\Big)\Big)^{1-d}\varphi^T(u_j^{i,S_j})\\
&= (1-\eta)^2\liminf_{j\to+\infty}\sum_{i\in Z_j(\Omega)}\gamma\delta_j^d\varphi^T(u_j^{i,S_j}),
\end{aligned}
$$

where the last equality follows by (6.2). Since $\{u_j\}_j$ is bounded in $L^\infty(\Omega;\mathbb{R}^m)$ by (6.27), we apply Lemma 6.2 and we infer

$$
\liminf_{j\to+\infty}\mathcal{F}^T_{\varepsilon_j}(w_j,E_j) \ge (1-\eta)^2\gamma\int_\Omega\varphi^T(u)\,dx. \tag{6.38}
$$

Substituting (6.35) and (6.38) in (6.33), we obtain

$$
\liminf_{j\to+\infty}F^T_{\varepsilon_j}(u_j) \ge \int_\Omega f^T_{\mathrm{hom}}(\nabla u)\,dx + (1-\eta)^2\gamma\int_\Omega\varphi^T(u)\,dx - \frac{C}{k}
$$

and we infer (6.26) by letting first $\eta\to 0$ and then $k\to+\infty$.

To conclude, we prove the claim removing the assumption (6.27). Let $u\in W^{1,d}(\Omega;\mathbb{R}^m)$, $u_j\to u$ in $L^d(\Omega;\mathbb{R}^m)$ and suppose that $\sup_j\mathcal{F}^T_{\varepsilon_j}(u_j)<+\infty$. By Lemma 6.3 and the lower bound we just proved for a sequence bounded in $L^\infty(\Omega;\mathbb{R}^m)$, up to extracting a not relabeled subsequence we have

$$
\begin{aligned}
\liminf_{j\to+\infty}\mathcal{F}^T_{\varepsilon_j}(u_j) &\ge \liminf_{j\to+\infty}\mathcal{F}^T_{\varepsilon_j}(t_L(u_j)) - \eta\\
&\ge \int_\Omega f^T_{\mathrm{hom}}(t_L(u))\,dx + \gamma\int_\Omega\varphi^T(t_L(u))\,dx - \eta,
\end{aligned}
$$

and then, letting $L \to +\infty$, we get

$$\liminf_{j\to+\infty} \mathcal{F}^T_{\varepsilon_j}(u_j) \geq \int_\Omega f^T_{\mathrm{hom}}(u)\,dx + \gamma \int_\Omega \varphi^T(u)\,dx - \eta.$$

The conclusion follows by the arbitrariness of η.

Step 2. Let $T > 2$ be a fixed natural and let $u \in W^{1,d}(\Omega;\mathbb{R}^m)$. We prove that there exists a sequence $u_j \to u$ in $L^d(\Omega;\mathbb{R}^m)$ such that

$$\limsup_{j\to+\infty} F^T_{\varepsilon_j}(u_j) \leq \int_\Omega f^T_{\mathrm{hom}}(\nabla u)\,dx + \gamma \int_\Omega \varphi^T(u)\,dx. \tag{6.39}$$

Fix $\widetilde{\Omega}$ a bounded open subset of $\mathbb{R}^d$ with Lipschitz boundary such that $\overline{\Omega} \subset \widetilde{\Omega}$ and, by Theorem 3.2, consider a sequence $\widetilde{u}_j \in \mathcal{A}_{\mu_{\varepsilon_j}}(\widetilde{\Omega};\mathbb{R}^m)$, $j \in \mathbb{N}$, converging to u in $L^d(\widetilde{\Omega};\mathbb{R}^m)$ such that

$$\lim_{j\to+\infty} \mathcal{F}^T_{\varepsilon_j}(\widetilde{u}_j,\widetilde{\Omega}) = \int_{\widetilde{\Omega}} f^T_{\mathrm{hom}}(\nabla u)\,dx. \tag{6.40}$$

It is not restrictive to assume that u is bounded and then, resorting to Lemma 6.3, we may also suppose that

$$\sup_j \|\widetilde{u}_j\|_{L^\infty(\widetilde{\Omega};\mathbb{R}^m)} < +\infty. \tag{6.41}$$

As in the first step, we fix $k \geq 2$ natural and $\eta \in (0,1)$, and we use Proposition 5.3(ii) with $L = 2^{N-k}$ to obtain that there exists N^* such that

$$(\log 2^N)^{d-1}\varphi^T_{2^{N-k}}(z) \leq (1+\eta)\varphi^T(z) \qquad \text{if } N \geq N^* \tag{6.42}$$

for every $z \in \mathbb{R}^m$. Then, we fix $N > \max\{N^*, k+3\}$. We apply Proposition 5.3(i) with $\ell = 1$ and $L = 2^{N-k}$ to obtain that there exists $\sigma^* < 1/T$ such that

$$\varphi^T_{1,2^{N-k},\sigma}(z) \leq (1+\eta)\varphi^T_{2^{N-k}}(z) \qquad \text{if } \sigma \leq \sigma^* \tag{6.43}$$

for every $z \in \mathbb{R}^m$, and we assume that $\varepsilon_j/r_j < \sigma^*$ for every $j \in \mathbb{N}$.

We apply Lemma 6.1 to the sequence $\{\widetilde{u}_j\}_j$ on $\widetilde{\Omega}$, with T, k, N fixed as above and $\{S_j\}_j$ a sequence of naturals such that

$$r_j 2^{NS_j} < \frac{\delta_j}{2} \leq r_j 2^{N(S_j+1)} \tag{6.44}$$

for every $j \in \mathbb{N}$. We obtain a sequence $w_j \in \mathcal{A}_{\varepsilon_j}(\widetilde{\Omega};\mathbb{R}^m)$, $j \in \mathbb{N}$, converging to u in $L^d(\widetilde{\Omega};\mathbb{R}^m)$ such that, for every j large enough, the following hold:

$$w_j = \widetilde{u}_j \qquad \mu_{\varepsilon_j}\text{-a.e. on } \widetilde{\Omega} \setminus \Big[\bigcup_{\substack{i\in Z_j(\widetilde{\Omega})\\ h\in\{1,\dots,S_j\}}} C^{i,h}_j\Big],$$

$$w_j = \widetilde{u}^{i,S_j}_j \qquad \mu_{\varepsilon_j}\text{-a.e. on } \overline{Q}(i\delta_j, \rho^{i,S_j}_j + \varepsilon_j T) \setminus Q(i\delta_j, \rho^{i,S_j}_j - \varepsilon_j T), \tag{6.45}$$

$$\mathcal{F}^T_{\varepsilon_j}(w_j,\widetilde{\Omega}) \leq \mathcal{F}^T_{\varepsilon_j}(\widetilde{u}_j,\widetilde{\Omega}) + \frac{C}{k} \tag{6.46}$$

for every $i \in Z_j(\widetilde{\Omega})$ and some $C > 0$ independent of j.

In analogy with the first step, we let

$$E_j := \bigcup_{i\in Z_j(\Omega)} Q(i\delta_j, \rho^{i,S_j}_j)$$

and

$$\widetilde{E}_j := \bigcup_{i\in Z_j(\widetilde{\Omega})} Q(i\delta_j, \rho^{i,S_j}_j)$$

and we start by defining our recovery sequence on $\Omega \setminus \widetilde{E}_j$.

Analysis far from the perforations. We set

$$u_j(x) := w_j(x) \qquad \text{for } \mu_{\varepsilon_j}\text{-a.e. } x \in \Omega \setminus \widetilde{E}_j,$$

and observe that by (6.46) we have

$$\mathcal{F}^T_{\varepsilon_j}(u_j, \Omega \setminus \widetilde{E}_j) \leq \mathcal{F}^T_{\varepsilon_j}(w_j, \widetilde{\Omega}) \leq \mathcal{F}^T_{\varepsilon_j}(\widetilde{u}_j, \widetilde{\Omega}) + \frac{C}{k}.$$

Hence, by (6.40) we obtain

$$\limsup_{j\to+\infty} \mathcal{F}^T_{\varepsilon_j}(u_j, \Omega \setminus \widetilde{E}_j) \leq \int_{\widetilde{\Omega}} f^T_{\text{hom}}(\nabla u)\, dx + \frac{C}{k}, \tag{6.47}$$

which is the desired upper bound.

Analysis near the perforations. Now we further modify the functions $\{w_j\}_j$ near the perforations that are far from $\partial\widetilde{\Omega}$. Recalling the definition of $Z_j(\widetilde{\Omega})$ in (6.4), upon assuming that j is large enough, we have that

$$\mathbb{Z}^d \cap \frac{1}{\delta_j}\Omega \subset Z_j(\widetilde{\Omega}),$$

hence, this amounts to modifying the functions $\{w_j\}_j$ on all the perforations of Ω, including those that intersect $\partial\Omega$.

For every $j \in \mathbb{N}$, $i \in Z_j(\widetilde{\Omega})$, and $h \in \{0, \ldots, S_j - 2\}$, consider $\overline{v}^{i,h}_j \in \mathcal{A}_{\varepsilon_j}(\mathbb{R}^d; \mathbb{R}^m)$ an admissible function for the minimum problem $\varphi^T_{r_j 2^{Nh}, r_j 2^{N(h+1)}, \varepsilon_j}(\widetilde{u}^{i,S_j}_j / S_j)$ such that

$$\mathcal{F}^T_{\varepsilon_j}(\overline{v}^{i,h}_j, Q(r_j 2^{N(h+1)}) \setminus \overline{Q}(r_j 2^{Nh})) \leq (1+\eta)\varphi^T_{r_j 2^{Nh}, r_j 2^{N(h+1)}, \varepsilon_j}(\widetilde{u}^{i,S_j}_j / S_j). \tag{6.48}$$

The case $h = S_j - 1$ has to be treated separately taking into account that ρ^{i,S_j}_j differs from $r_j 2^{NS_j}$. In such a case, we consider $\overline{v}^{i,S_j-1}_j \in \mathcal{A}_{\varepsilon_j}(\mathbb{R}^d; \mathbb{R}^m)$ an admissible function for the minimum problem $\varphi^T_{r_j 2^{N(S_j-1)}, \rho^{i,S_j}_j, \varepsilon_j}(\widetilde{u}^{i,S_j}_j / S_j)$ such that

$$\mathcal{F}^T_{\varepsilon_j}(\overline{v}^{i,S_j-1}_j, Q(\rho^{i,S_j}_j) \setminus \overline{Q}(r_j 2^{N(S_j-1)})) \leq (1+\eta)\varphi^T_{r_j 2^{N(S_j-1)}, \rho^{i,S_j}_j, \varepsilon_j}(\widetilde{u}^{i,S_j}_j / S_j). \tag{6.49}$$

For every $h \in \{0, \ldots, S_j - 1\}$ we let

$$v^{i,h}_j(x) := \overline{v}^{i,h}_j(x - i\delta_j) + \frac{h}{S_j}\widetilde{u}^{i,S_j}_j \qquad \text{for } \mu_{\varepsilon_j}\text{-a.e. } x \in \mathbb{R}^d,$$

and we note that, by the admissibility of $\overline{v}^{i,h}_j$, we have

$$v^{i,h}_j = \begin{cases} \dfrac{h}{S_j}\widetilde{u}^{i,S_j}_j & \mu_{\varepsilon_j}\text{-a.e. on } \overline{Q}(i\delta_j, r_j 2^{Nh} + \varepsilon_j T), \\ \dfrac{h+1}{S_j}\widetilde{u}^{i,S_j}_j & \mu_{\varepsilon_j}\text{-a.e. on } [Q(i\delta_j, r_j 2^{N(h+1)} - \varepsilon_j T)]^c, \end{cases}$$

for every $h \in \{0, \ldots, S_j - 2\}$ and

$$v^{i,S_j-1}_j = \begin{cases} \dfrac{S_j - 1}{S_j}\widetilde{u}^{i,S_j}_j & \mu_{\varepsilon_j}\text{-a.e. on } \overline{Q}(i\delta_j, r_j 2^{N(S_j-1)} + \varepsilon_j T), \\ \widetilde{u}^{i,S_j}_j & \mu_{\varepsilon_j}\text{-a.e. on } [Q(i\delta_j, \rho^{i,S_j}_j - \varepsilon_j T)]^c. \end{cases}$$

For every $i \in Z_j(\widetilde{\Omega})$, we set

$$u_j(x) := \begin{cases} v_j^{i,0}(x) & \text{if } x \in \overline{Q}(i\delta_j, r_j 2^N), \\ v_j^{i,h}(x) & \text{if } x \in Q(i\delta_j, r_j 2^{N(h+1)}) \setminus \overline{Q}(i\delta_j, r_j 2^{Nh}), h \in \{1, \dots, S_j - 2\}, \\ v_j^{i,S_j-1}(x) & \text{if } x \in Q(i\delta_j, \rho_j^{i,S_j}) \setminus Q(i\delta_j, r_j 2^{N(S_j-1)}), \end{cases}$$

where the equality is valid for μ_{ε_j}-a.e. $x \in Q(i\delta_j, \rho_j^{i,S_j})$, and we observe that

$$u_j(x) = v_j^{i,0}(x) = 0, \qquad \text{for } \mu_{\varepsilon_j}\text{-a.e. } x \in \overline{Q}(i\delta_j, r_j). \tag{6.50}$$

We now estimate the energy near the perforations that are far from $\partial\widetilde{\Omega}$. Recalling (6.48) and (6.49), we get

$$\begin{aligned} \mathcal{F}_{\varepsilon_j}^T(u_j, Q(i\delta_j, \rho_j^{i,S_j})) &= \sum_{h=0}^{S_j-2} \mathcal{F}_{\varepsilon_j}^T(v_j^{i,h}, Q(i\delta_j, r_j 2^{N(h+1)}) \setminus Q(i\delta_j, r_j 2^{Nh})) \\ &\quad + \mathcal{F}_{\varepsilon_j}^T(v_j^{i,S_j-1}, Q(i\delta_j, \rho_j^{i,S_j}) \setminus Q(i\delta_j, r_j 2^{N(S_j-1)})) \\ &= \sum_{h=0}^{S_j-2} \mathcal{F}_{\varepsilon_j}^T(\overline{v}_j^{i,h}, Q(r_j 2^{N(h+1)}) \setminus Q(r_j 2^{Nh})) \\ &\quad + \mathcal{F}_{\varepsilon_j}^T(\overline{v}_j^{i,S_j-1}, Q(\rho_j^{i,S_j}) \setminus Q(r_j 2^{N(S_j-1)})) \\ &\le \sum_{h=0}^{S_j-2} (1+\eta)\varphi^T_{r_j 2^{Nh}, r_j 2^{N(h+1)}, \varepsilon_j}(\widetilde{u}_j^{i,S_j}/S_j) + (1+\eta)\varphi^T_{r_j 2^{N(S_j-1)}, \rho_j^{i,S_j}, \varepsilon_j}(\widetilde{u}_j^{i,S_j}/S_j) \\ &\le (1+\eta) \sum_{h=0}^{S_j-1} \varphi^T_{r_j 2^{Nh}, r_j 2^{N(h+1)-k}, \varepsilon_j}(\widetilde{u}_j^{i,S_j}/S_j), \end{aligned} \tag{6.51}$$

where the last inequality follows by Remark 5.1(vi) and by the fact that, for j large enough,

$$\rho_j^{i,S_j} = \Big\lfloor \frac{3}{4}\frac{r_j}{\varepsilon_j} 2^{NS_j - k_{i,S_j}} \Big\rfloor \varepsilon_j \ge r_j 2^{NS_j - k}$$

for every i since $k_{i,S_j} \le k-1$. Upon setting

$$\sigma_j^h := \frac{\varepsilon_j}{r_j} 2^{-Nh}, \qquad j \in \mathbb{N}, h \in \{0, \dots S_j - 1\},$$

Remark 5.1(v) yields

$$\varphi^T_{r_j 2^{Nh}, r_j 2^{N(h+1)-k}, \varepsilon_j}(\widetilde{u}_j^{i,S_j}/S_j) = \varphi^T_{1, 2^{N-k}, \sigma_j^h}(\widetilde{u}_j^{i,S_j}/S_j),$$

and then (6.51) can be rewritten as

$$\mathcal{F}_{\varepsilon_j}^T(u_j, Q(i\delta_j, \rho_j^{i,S_j})) \le (1+\eta) \sum_{h=0}^{S_j-1} \varphi^T_{1, 2^{N-k}, \sigma_j^h}(\widetilde{u}_j^{i,S_j}/S_j).$$

As $\sigma_j^h \le \varepsilon_j / r_j < \sigma^*$ for every $j \in \mathbb{N}$ and $h \in \{0, \dots, S_j - 1\}$, we use (6.42) and (6.43) and we employ the d-homogeneity of φ^T to get

$$\begin{aligned} \mathcal{F}_{\varepsilon_j}^T(u_j, Q(i\delta_j, \rho_j^{i,S_j})) &\le (1+\eta)^2 S_j \varphi^T_{2^{N-k}}(\widetilde{u}_j^{i,S_j}/S_j) \\ &\le (1+\eta)^3 S_j (\log 2^N)^{1-d} \varphi^T(\widetilde{u}_j^{i,S_j}/S_j) = (1+\eta)^3 (S_j \log 2^N)^{1-d} \varphi^T(\widetilde{u}_j^{i,S_j}). \end{aligned}$$

By the second inequality in (6.44), we have

$$S_j \log 2^N \geq \log\Big(\frac{\delta_j}{2r_j}\Big) - \log 2^N$$

for every $j \in \mathbb{N}$; hence, taking into account that $\delta_j/r_j \to +\infty$ as $j \to +\infty$ and that N and η are fixed, it is not restrictive to suppose that

$$S_j \log 2^N \geq (1+\eta)^{\frac{1}{1-d}} \log\Big(\frac{\delta_j}{2r_j}\Big)$$

for every $j \in \mathbb{N}$. We obtain that

$$\mathcal{F}^T_{\varepsilon_j}(u_j, Q(i\delta_j, \rho_j^{i,S_j})) \leq (1+\eta)^4 \Big(\log\Big(\frac{\delta_j}{2r_j}\Big)\Big)^{1-d} \varphi^T(\widetilde{u}_j^{i,S_j}).$$

Finally, we sum over $i \in Z_j(\widetilde{\Omega})$. By (6.41), we are in position to apply Lemma 6.2 and then, also using (6.2), we get

$$\begin{aligned}
\limsup_{j\to+\infty} \mathcal{F}^T_{\varepsilon_j}(u_j, \widetilde{E}_j) &= \limsup_{j\to+\infty} \sum_{i\in Z_j(\widetilde{\Omega})} \mathcal{F}^T_{\varepsilon_j}(u_j, Q(i\delta_j, \rho_j^{i,S_j})) \\
&\leq (1+\eta)^4 \limsup_{j\to+\infty} \sum_{i\in Z_j(\widetilde{\Omega})} \Big(\log\Big(\frac{\delta_j}{2r_j}\Big)\Big)^{1-d} \varphi^T(\widetilde{u}_j^{i,S_j}) \\
&= (1+\eta)^4 \limsup_{j\to+\infty} \sum_{i\in Z_j(\widetilde{\Omega})} \gamma\delta_j^d \varphi^T(\widetilde{u}_j^{i,S_j}) \\
&= (1+\eta)^4 \gamma \int_{\widetilde{\Omega}} \varphi^T(u)\,dx. \qquad (6.52)
\end{aligned}$$

Using (6.45) and combining (6.47) and (6.52), we conclude that

$$\begin{aligned}
\limsup_{j\to+\infty} \mathcal{F}^T_{\varepsilon_j}(u_j) &= \limsup_{j\to+\infty} (\mathcal{F}^T_{\varepsilon_j}(u_j, \Omega\setminus\widetilde{E}_j) + \mathcal{F}^T_{\varepsilon_j}(u_j, \widetilde{E}_j)) \\
&\leq \int_{\widetilde{\Omega}} f^T_{\mathrm{hom}}(\nabla u)\,dx + (1+\eta)^4\gamma \int_{\widetilde{\Omega}} \varphi^T(u)\,dx + \frac{C}{k},
\end{aligned}$$

and (6.39) follows by the arbitrariness of η, k and $\widetilde{\Omega}$ and the fact that, in virtue of (6.50), $\{u_j\}_j$ is an admissible sequence for $\{F^T_{\varepsilon_j}\}_j$.

It remains to verify that $u_j \to u$ in $L^d(\Omega;\mathbb{R}^m)$. By construction, we have

$$\begin{aligned}
\lim_{j\to+\infty} \int_{\widetilde{\Omega}} |u_j - w_j|^d\,dx &= \lim_{j\to+\infty} \int_{\widetilde{E}_j} |u_j - w_j|^d\,dx \\
&= \lim_{j\to+\infty} \sum_{i\in Z_j(\widetilde{\Omega})} \int_{Q(i\delta_j, \rho_j^{i,S_j})} |u_j - w_j|^d\,dx \\
&= \lim_{j\to+\infty} \sum_{i\in Z_j(\widetilde{\Omega})} \varepsilon_j^d \int_{Q(i\delta_j, \rho_j^{i,S_j})} |u_j - w_j|^d\,d\mu_{\varepsilon_j}.
\end{aligned}$$

By (6.45) we have that $u_j - w_j = 0$ μ_{ε_j}-a.e. on $Q(i\delta_j, \rho_j^{i,S_j}) \setminus Q(i\delta_j, \rho_j^{i,S_j} - \varepsilon_j T)$ for every $i \in Z_j(\widetilde{\Omega})$. Therefore, we apply a rescaled version of Poincaré inequality in Proposition 4.4 with

$$\sigma = \varepsilon_j, \qquad \tau = \rho_j^{i,S_j}, \qquad x_0 = i\delta_j, \qquad A = \frac{1}{\rho_j^{i,S_j}}(Q(i\delta_j, \rho_j^{i,S_j}) - i\delta_j) = Q(1),$$

and we obtain

$$
\begin{aligned}
&\lim_{j\to+\infty} \sum_{i\in Z_j(\widetilde{\Omega})} \varepsilon_j^d \int_{Q(i\delta_j,\rho_j^{i,S_j})} |u_j - w_j|^d \, d\mu_{\varepsilon_j} \\
&\quad \le C \lim_{j\to+\infty} \sum_{i\in Z_j(\widetilde{\Omega})} (\rho_j^{i,S_j})^d G_{\varepsilon_j}(u_j - w_j, Q(i\delta_j,\rho_j^{i,S_j})) \\
&\quad \le C \lim_{j\to+\infty} \delta_j^d \sum_{i\in Z_j(\widetilde{\Omega})} \big(G_{\varepsilon_j}(u_j, Q(i\delta_j,\rho_j^{i,S_j})) + G_{\varepsilon_j}(w_j, Q(i\delta_j,\rho_j^{i,S_j}))\big) \\
&\quad \le C \lim_{j\to+\infty} \delta_j^d (\mathcal{F}^T_{\varepsilon_j}(u_j, \widetilde{E}_j) + \mathcal{F}^T_{\varepsilon_j}(w_j, \widetilde{\Omega})) = 0
\end{aligned}
$$

in light of (6.40), (6.46), and (6.52). Taking into account that $w_j \to u$ by Lemma 6.1, we infer that $u_j \to u$.

Step 3. In the first two steps we proved that, for every $T > 2$ natural, it holds

$$\Gamma\text{-}\lim_{j\to+\infty} F^T_{\varepsilon_j}(u) = \int_\Omega f^T_{\mathrm{hom}}(\nabla u)\, dx + \gamma \int_\Omega \varphi^T(u)\, dx, \qquad u \in W^{1,d}(\Omega;\mathbb{R}^m).$$

Hence, resorting to Proposition 4.6, Proposition 4.7, and Proposition 5.3(*iii*), we obtain

$$\Gamma\text{-}\lim_{j\to+\infty} F_{\varepsilon_j}(u) = \int_\Omega f_{\mathrm{hom}}(\nabla u)\, dx + \gamma \int_\Omega \varphi(u)\, dx,$$

which concludes the proof of Theorem 3.4.

A Appendix

In this appendix we prove the Poincaré-Wirtinger and the Poincaré inequality corresponding to Propositions 4.3 and 4.4, with a particular focus on the discrete case. For the sake of completeness, we consider a slightly more general setting. For every $\sigma > 0$, $A \subseteq \mathbb{R}^d$ a μ_σ-measurable set, and $p \in [1, +\infty)$, we let

$$\mathcal{A}_{\mu_\sigma}(A;\mathbb{R}^m) := \{u : \mathbb{R}^d \to \mathbb{R}^m : u(x) = u(\alpha) \text{ if } x \in \alpha + [0,\sigma)^d, \alpha \in A \cap \sigma\mathbb{Z}^d\} \cap L^p(A;\mathbb{R}^m)$$

in the discrete case and

$$\mathcal{A}_{\mu_\sigma}(A;\mathbb{R}^m) := \{u : \mathbb{R}^d \to \mathbb{R}^m : u \text{ is } \mathcal{L}^d\text{-measurable}\} \cap L^p(A;\mathbb{R}^m)$$

in the continuous case. We consider reference energies

$$G^p_\sigma(u, A) := \int_{B(r_0)} \int_{A_{\mu_\sigma}(\xi)} |u(x + \sigma\xi) - u(x)|^p \, d\mu_\sigma(x)\, d\mu_1(\xi),$$

with $r_0 \in (1, 2)$ as in (G0) and $u \in \mathcal{A}_{\mu_\sigma}(A;\mathbb{R}^m)$.

In the discrete case, the following lemma allows one to control interactions of pairs of points in a possibly non-convex domain by means of interactions of pairs of points that are connected by a segment that lies in the domain.

Lemma A.1. *Let $A \subset \mathbb{R}^d$ be a connected set which is finite union of open rectangles $A = \cup_{h=1}^N A_h$ with $N \ge 2$ and such that*

$$A_h \cap A_{h+1} \cap \sigma\mathbb{Z}^d \neq \emptyset \qquad \text{for every } h \in \{1, \dots, N-1\}.$$

There exist positive constants σ_0 and C depending on $A_1, \dots, A_N$ such that it holds

$$\sum_{1\le j<\ell\le N} \sum_{\alpha\in(A_j\setminus A_\ell)\cap\sigma\mathbb{Z}^d} \sum_{\beta\in(A_\ell\setminus A_j)\cap\sigma\mathbb{Z}^d} |u(\alpha) - u(\beta)|^p \le CN^{p+1} \sum_{h=1}^N \sum_{\alpha\in A_h\cap\sigma\mathbb{Z}^d} \sum_{\beta\in A_h\cap\sigma\mathbb{Z}^d} |u(\alpha) - u(\beta)|^p \tag{A.1}$$

for every $\sigma < \sigma_0$ and $u : A \cap \sigma\mathbb{Z}^d \to \mathbb{R}^m$.

Proof. We consider $Q_1, \ldots, Q_{N-1}$ open cubes with sides parallel to the coordinate axes with the property that $Q_h \subseteq A_h \cap A_{h+1}$ for every $h \in \{1, \ldots, N-1\}$,

$$|Q_1| = \cdots = |Q_{N-1}| \geq \frac{1}{2} \min\{|A_h \cap A_{h+1}| : h \in \{1, \ldots, N-1\}\},$$

and

$$C(\sigma) := \#(Q_1 \cap \sigma\mathbb{Z}^d) = \cdots = \#(Q_{N-1} \cap \sigma\mathbb{Z}^d)$$

for every σ small enough. We label the elements of each set $Q_h \cap \sigma\mathbb{Z}^d$ as $\{q_h^1, \ldots, q_h^{C(\sigma)}\}$.

We define surjective functions

$$I_h : A_h \cap \sigma\mathbb{Z}^d \to \{1, \ldots, C(\sigma)\}, \qquad h \in \{1, \ldots, N\},$$

such that

$$\#(I_h^{-1}(\{k\})) \leq \left\lceil \frac{\#(A_h \cap \sigma\mathbb{Z}^d)}{C(\sigma)} \right\rceil$$

for every $h \in \{1, \ldots, N\}$ and $k \in \{1, \ldots, C(\sigma)\}$, where we let $\lceil x \rceil$ denote the upper integer part of x. Upon assuming that σ is sufficiently small, we can then suppose that

$$\#(I_h^{-1}(\{k\})) \leq 8|A_h|/\min\{|A_h \cap A_{h+1}| : h \in \{1, \ldots, N-1\}\} \tag{A.2}$$

for every $h \in \{1, \ldots, N\}$ and $k \in \{1, \ldots, C(\sigma)\}$.

Let $1 \leq j < \ell \leq N$ and consider points $\alpha \in (A_j \setminus A_\ell) \cap \sigma\mathbb{Z}^d$ and $\beta \in (A_\ell \setminus A_j) \cap \sigma\mathbb{Z}^d$. We define the intermediate nodes that shall allow us to connect these points. We set

$$p_{j+k} := \begin{cases} q_{j+k}^{I_\ell(\beta)} & \text{if } k \text{ is even,} \\ q_{j+k}^{I_j(\alpha)} & \text{if } k \text{ is odd,} \end{cases} \qquad k \in \{0, \ldots, \ell - j - 1\}.$$

For sake of notation, it is convenient to set

$$n(j, \ell) := \begin{cases} \ell & \text{if } \ell - j \text{ is even,} \\ \ell + 1 & \text{if } \ell - j \text{ is odd,} \end{cases}$$

and

$$p_{j-1} := \alpha, \qquad p_{n(j,\ell)} := \beta.$$

Now we distinguish two cases:

- if $\ell - j$ is even, we have that $p_{\ell-1} = q_{\ell-1}^{I_j(\alpha)}$ and we define
$$\mathcal{P}(\alpha, \beta) := \{\alpha = p_{j-1}, p_j, \ldots, p_{\ell-1}, p_\ell = p_{n(j,\ell)} = \beta\},$$
- if $\ell - j$ is odd, we have that $p_{\ell-1} = q_{\ell-1}^{I_\ell(\beta)}$, then we set $p_\ell := q_{\ell-1}^{I_j(\alpha)}$ and we define
$$\mathcal{P}(\alpha, \beta) := \{\alpha = p_{j-1}, p_j, \ldots, p_\ell, p_{\ell+1} = p_{n(j,\ell)} = \beta\}.$$

Finally, we connect α and β using the path obtained connecting through segments the points of $\mathcal{P}(\alpha, \beta)$ in the established order.

If $\ell - j$ is even, the path associated with $\mathcal{P}(\alpha, \beta)$ is made up of at most $\ell - j + 1$ segments and

$$[p_h, p_{h+1}] \subset A_{h+1} \quad \text{for every } h \in \{j-1, \ldots, \ell-1\}.$$

If $\ell - j$ is odd, the path associated with $\mathcal{P}(\alpha, \beta)$ is made up of at most $\ell - j + 2$ segments and

$$[p_h, p_{h+1}] \subset A_{h+1} \quad \text{for every } h \in \{j-1, \ldots, \ell-1\}, \qquad [p_\ell, p_{\ell+1}] \subset A_\ell.$$

Let us consider a segment $[\lambda, \eta] \subset A_h$ for some $h \in \{1, \ldots, N\}$. We let $S(\lambda, \eta)$ denote the set of pairs $(\alpha, \beta) \in (A_j \setminus A_\ell) \cap \sigma\mathbb{Z}^d \times (A_\ell \setminus A_j) \cap \sigma\mathbb{Z}^d, 1 \leq j < \ell \leq N$, such that λ and η are two consecutive points belonging to the path associated with $\mathcal{P}(\alpha, \beta)$.

To estimate the cardinality of $S(\lambda, \eta)$, let us assume that $\ell - j$ is even and consider $(\alpha, \beta) \in S(\lambda, \eta)$. Upon exchanging the roles of λ and η, we have the following possibilities:

(i) $\lambda = \alpha = p_{j-1}, \eta = p_j$,

(ii) $\lambda = p_{\ell-1}, \eta = \beta = p_\ell$,

(iii) $\lambda = p_m, \eta = p_{m+1}$ for some $m \in \{j, \ldots, \ell - 2\}$.

In the first case α is uniquely determined and $\eta \in Q_j$, so that $\eta = q_j^k$ for a certain $k \in \{1, \ldots, C(\sigma)\}$. Hence, we have that

$$q_j^k = \eta = p_j = q_j^{I_\ell(\beta)}.$$

Therefore, $\beta \in I_\ell^{-1}(\{k\})$ and, by (A.2), the number of possible pairs (α, β) is less than or equal to

$$\#(I_\ell^{-1}(\{k\})) \leq 8|A_\ell| / \min\{|A_h \cap A_{h+1}| : h \in \{1, \ldots, N-1\}\}.$$

In the second case β is uniquely determined and $\lambda \in Q_{\ell-1}$, so that $\lambda = q_{\ell-1}^k$ for a certain $k \in \{1, \ldots, C(\sigma)\}$. Hence, we have that

$$q_{\ell-1}^k = \lambda = p_{\ell-1} = q_{\ell-1}^{I_j(\alpha)}.$$

Therefore, $\alpha \in I_j^{-1}(\{k\})$ and the number of possible pairs (α, β) is less than or equal to

$$\#(I_j^{-1}(\{k\})) \leq 8|A_j| / \min\{|A_h \cap A_{h+1}| : h \in \{1, \ldots, N-1\}\}.$$

In the third case, since $\lambda \in Q_m$ and $\eta \in Q_{m+1}$, we have that $\lambda = q_m^k$ and $\eta = q_{m+1}^{k'}$ for some $k, k' \in \{1, \ldots, C(\sigma)\}$, and then

$$\begin{cases} q_m^k = q_m^{I_\ell(\beta)}, \\ q_{m+1}^{k'} = q_{m+1}^{I_j(\alpha)}, \end{cases} \qquad \text{or} \qquad \begin{cases} q_m^k = q_m^{I_j(\alpha)}, \\ q_{m+1}^{k'} = q_{m+1}^{I_\ell(\beta)}. \end{cases}$$

Therefore,

$$\begin{cases} \alpha \in I_j^{-1}(\{k'\}), \\ \beta \in I_\ell^{-1}(\{k\}), \end{cases} \qquad \text{or} \qquad \begin{cases} \alpha \in I_j^{-1}(\{k\}), \\ \beta \in I_\ell^{-1}(\{k'\}), \end{cases}$$

and the number of possible pairs (α, β) in this case is less than or equal to

$$\begin{aligned} &\max\{\#(I_j^{-1}(\{k'\})) \times \#(I_\ell^{-1}(\{k\})), \#(I_j^{-1}(\{k\})) \times \#(I_\ell^{-1}(\{k'\}))\} \\ &\leq 64|A_j||A_\ell| / \min\{|A_h \cap A_{h+1}| : h \in \{1, \ldots, N-1\}\}^2. \end{aligned}$$

The case $\ell - j$ odd is analogous, provided that the additional segment $[p_{\ell-1}, p_\ell]$ is also considered. For this segment, upon exchanging the roles of λ and η, one has

$$\lambda = p_{\ell-1} = q_{\ell-1}^{I_\ell(\beta)}, \qquad \eta = p_\ell = q_{\ell-1}^{I_j(\alpha)}.$$

Therefore, the number of admissible pairs (α, β) is bounded by

$$\#I_\ell^{-1}(\{k\}) \times \#I_j^{-1}(\{k'\}) \leq 64|A_j||A_\ell| / \min\{|A_h \cap A_{h+1}| : h \in \{1, \ldots, N-1\}\}^2,$$

which is the same estimate obtained for (iii) in the case $\ell - j$ is even. All other segments are treated as in the even case.

Letting j, ℓ vary in $\{1, \ldots, N\}$, we obtain that

$$\begin{aligned} \#S(\lambda, \eta) &\leq 2(64 + 8) \sum_{1 \leq j < \ell \leq N} |A_j||A_\ell| / \min\{|A_h \cap A_{h+1}| : h \in \{1, \ldots, N-1\}\}^2 \\ &\leq N^2 144 \Big(\frac{\max\{|A_h| : h \in \{1, \ldots, N\}\}}{\min\{|A_h \cap A_{h+1}| : h \in \{1, \ldots, N-1\}\}} \Big)^2 =: N^2 C. \end{aligned} \tag{A.3}$$

By Jensen's inequality we have

$$
\begin{aligned}
|u(\alpha)-u(\beta)|^p &= \Big|\sum_{h=j-1}^{n(j,\ell)-1} u(p_{h+1})-u(p_h)\Big|^p \\
&\le (n(j,\ell)-j+1)^{p-1}\sum_{h=j-1}^{n(j,\ell)-1}|u(p_{h+1})-u(p_h)|^p \le (2N)^{p-1}\sum_{h=j-1}^{n(j,\ell)-1}|u(p_{h+1})-u(p_h)|^p.
\end{aligned}
$$

Recalling that $[p_h, p_{h+1}] \subset A_{h+1}$ for every $h \in \{j-1,\dots,\ell-1\}$ and $[p_{n(j,\ell)-1}, p_{n(j,\ell)}] \subset A_\ell$, and using (A.3), we sum over α and β and $1 \le j < \ell \le N$ to get

$$
\begin{aligned}
&\sum_{1\le j<\ell\le N}\ \sum_{\alpha\in(A_j\setminus A_\ell)\cap\sigma\mathbb{Z}^d}\ \sum_{\beta\in(A_\ell\setminus A_j)\cap\sigma\mathbb{Z}^d} |u(\alpha)-u(\beta)|^p \\
&\qquad\qquad \le 2^{p-1}N^{p+1}C\sum_{h=1}^{N}\ \sum_{\lambda\in A_h\cap\sigma\mathbb{Z}^d}\ \sum_{\eta\in A_h\cap\sigma\mathbb{Z}^d}|u(\lambda)-u(\eta)|^p,
\end{aligned}
$$

which concludes the proof. □

Remark A.2. The constant in (A.1) is far from optimal as the paths connecting pairs of points are not constructed following the geodesics inside A. Nevertheless, we note that such a constant is invariant under rescaling of A as it depends on the number of rectangles $A_1, \dots, A_N$ that constitute A and on

$$\frac{\max\{|A_h| : h \in \{1,\dots,N\}\}}{\min\{|A_h\cap A_{h+1}| : h\in\{1,\dots,N-1\}\}},$$

which are unaffected by homotheties of A.

The following result is a more general version of Proposition 4.3.

Proposition A.3 (Poincaré-Wirtinger inequality on union of rectangles)**.** *Let $A \subset \mathbb{R}^d$ be a connected set which is finite union of open rectangles with sides parallel to the coordinate axes $A = \cup_{h=1}^N A_h$ such that, if $N \ge 2$, it holds*

$$A_h \cap A_{h+1} \cap \sigma\mathbb{Z}^d \ne \emptyset \qquad \textit{for every } h \in \{1,\dots,N-1\};$$

let $A' \subseteq A$ be a μ_σ-measurable set and $x_0 \in \mathbb{R}^d$. There exist positive constants σ_0 and C depending on $A, A_1, \dots, A_N, N$ such that, having set

$$(u)_E^{\mu_\sigma} := \frac{1}{\mu_\sigma(E)}\int_E u\, d\mu_\sigma$$

for every E μ_σ-measurable set with $\mu_\sigma(E) > 0$, it holds

$$\int_{x_0+\tau A}|u(x)-(u)^{\mu_\sigma}_{x_0+\tau A'}|^p\, d\mu_\sigma(x) \le C\frac{\mu_\sigma(x_0+\tau A)}{\mu_\sigma(x_0+\tau A')}\Big(\frac{\tau}{\sigma}\Big)^p G^p_\sigma(u, x_0+\tau A) \tag{A.4}$$

for every $\tau > 0$, $\sigma \in (0, \sigma_0\tau)$, and $u \in \mathcal{A}_{\mu_\sigma}(x_0+\tau A;\mathbb{R}^m)$.

Proof. It suffices to prove the statement in the case $x_0 = 0$ and $\tau = 1$ as the general case follows by a translation and scaling argument. Once the statement is proved in the case $A' = A$, the general case follows observing that

$$|(u)^{\mu_\sigma}_A - (u)^{\mu_\sigma}_{A'}|^p = \Big|\frac{1}{\mu_\sigma(A')}\int_{A'}(u)^{\mu_\sigma}_A - u\, d\mu_\sigma\Big|^p \le \frac{1}{\mu_\sigma(A')}\int_{A'}|(u)^{\mu_\sigma}_A - u|^p\, d\mu_\sigma,$$

so that (A.4) applied with $A = A'$, $x_0 = 0$, and $\tau = 1$ yield

$$
\begin{aligned}
\int_A |u(x) - (u)^{\mu_\sigma}_{A'}|^p \, d\mu_\sigma(x) &\leq 2^{p-1}\Big[\int_A |u(x) - (u)^{\mu_\sigma}_A|^p \, d\mu_\sigma(x) + \int_A |(u)^{\mu_\sigma}_A - (u)^{\mu_\sigma}_{A'}|^p \, d\mu_\sigma(x)\Big] \\
&\leq 2^{p-1}\Big[\int_A |u(x) - (u)^{\mu_\sigma}_A|^p \, d\mu_\sigma(x) + \frac{\mu_\sigma(A)}{\mu_\sigma(A')}\int_{A'} |u(x) - (u)^{\mu_\sigma}_A|^p \, d\mu_\sigma(x)\Big] \\
&\leq 2^p \frac{\mu_\sigma(A)}{\mu_\sigma(A')}\int_A |u(x) - (u)^{\mu_\sigma}_A|^d \, d\mu_\sigma(x) \leq C \frac{\mu_\sigma(A)}{\mu_\sigma(A')}\frac{1}{\sigma^p} G^p_\sigma(u, A).
\end{aligned}
$$

We then prove the statement further assuming that $A' = A$.

In the continuous setting, the proof can be found in [2, Proposition 4.2] and, in fact, the result holds for general bounded open sets with Lipschitz boundary. In the discrete case we have to prove that

$$
\sum_{\alpha \in A \cap \sigma\mathbb{Z}^d} |u(\alpha) - (u)_A|^p \leq C \frac{1}{\sigma^p} \sum_{k=1}^d \sum_{\alpha \in R^{e_k}_\sigma(A)} |u(\alpha + \sigma e_k) - u(\alpha)|^p,
$$

where

$$
(u)_A = \frac{1}{\#(A \cap \sigma\mathbb{Z}^d)} \sum_{\alpha \in A \cap \sigma\mathbb{Z}^d} u(\alpha)
$$

and $R^{e_k}_\sigma(A) := \{\alpha \in A \cap \sigma\mathbb{Z}^d : [\alpha, \alpha + \sigma e_k] \subset A\}$ for every $k \in \{1, \dots, d\}$.

We have

$$
\begin{aligned}
\sum_{\alpha \in A \cap \sigma\mathbb{Z}^d} |u(\alpha) - (u)_A|^p &= \sum_{\alpha \in A \cap \sigma\mathbb{Z}^d} \Big| u(\alpha) - \frac{1}{\#(A \cap \sigma\mathbb{Z}^d)} \sum_{\beta \in A \cap \sigma\mathbb{Z}^d} u(\beta)\Big|^p \\
&= \sum_{\alpha \in A \cap \sigma\mathbb{Z}^d} \Big| \frac{1}{\#(A \cap \sigma\mathbb{Z}^d)} \sum_{\beta \in A \cap \sigma\mathbb{Z}^d} (u(\alpha) - u(\beta))\Big|^p \\
&\leq \frac{1}{\#(A \cap \sigma\mathbb{Z}^d)} \sum_{\alpha \in A \cap \sigma\mathbb{Z}^d} \sum_{\beta \in A \cap \sigma\mathbb{Z}^d} |u(\alpha) - u(\beta)|^p.
\end{aligned}
\tag{A.5}
$$

Suppose $N = 1$; i.e., A is a rectangle with sides parallel to the coordinate axes and, without loss of generality, further assume that it is centered at the origin. Slightly adapting [4, Lemma 3.6], we get

$$
\begin{aligned}
\sum_{\alpha \in A \cap \sigma\mathbb{Z}^d} \sum_{\beta \in A \cap \sigma\mathbb{Z}^d} |u(\alpha) - u(\beta)|^p &\leq \sum_{\xi \in \frac{2}{\sigma} A \cap \mathbb{Z}^d} \sum_{\alpha \in R^\xi_\sigma(A)} |u(\alpha + \sigma\xi) - u(\alpha)|^p \\
&\leq C \sum_{\xi \in \frac{2}{\sigma} A \cap \mathbb{Z}^d} |\xi|^p \sum_{k=1}^d \sum_{\alpha \in R^{e_k}_\sigma(A)} |u(\alpha + \sigma e_k) - u(\alpha)|^p \\
&\leq C \#(2A \cap \sigma\mathbb{Z}^d) \Big(\frac{\text{diam } A}{\sigma}\Big)^p \sum_{k=1}^d \sum_{\alpha \in R^{e_k}_\sigma(A)} |u(\alpha + \sigma e_k) - u(\alpha)|^p.
\end{aligned}
\tag{A.6}
$$

Recalling (A.5) we infer

$$
\begin{aligned}
\sum_{\alpha \in A \cap \sigma\mathbb{Z}^d} |u(\alpha) - (u)_A|^p &\leq C \frac{\#(2A \cap \sigma\mathbb{Z}^d)}{\#(A \cap \sigma\mathbb{Z}^d)} \Big(\frac{\text{diam } A}{\sigma}\Big)^p \sum_{k=1}^d \sum_{\alpha \in R^{e_k}_\sigma(A)} |u(\alpha + \sigma e_k) - u(\alpha)|^p \\
&\leq C 2^{d+1} (\text{diam } A)^p \frac{1}{\sigma^p} \sum_{k=1}^d \sum_{\alpha \in R^{e_k}_\sigma(A)} |u(\alpha + \sigma e_k) - u(\alpha)|^p,
\end{aligned}
$$

where we used that $\#(2A_h \cap \sigma\mathbb{Z}^d) \leq 2^{d+1} \#(A \cap \sigma\mathbb{Z}^d)$ for σ sufficiently small.

If $N \geq 2$, by Lemma A.1 we have

$$\sum_{\alpha\in A\cap\sigma\mathbb{Z}^d}\sum_{\beta\in A\cap\sigma\mathbb{Z}^d}|u(\alpha)-u(\beta)|^p \leq (1+CN^{p+1})\sum_{h=1}^{N}\sum_{\alpha\in A_h\cap\sigma\mathbb{Z}^d}\sum_{\beta\in A_h\cap\sigma\mathbb{Z}^d}|u(\alpha)-u(\beta)|^p$$

and then, arguing as in (A.6) for fixed $h \in \{1,\dots,N\}$, we get

$$\begin{aligned}&\sum_{\alpha\in A\cap\sigma\mathbb{Z}^d}\sum_{\beta\in A\cap\sigma\mathbb{Z}^d}|u(\alpha)-u(\beta)|^p\\ &\leq C(1+CN^{p+1})\sum_{h=1}^{N}\#(2A_h\cap\sigma\mathbb{Z}^d)\Big(\frac{\mathrm{diam}\ A_h}{\sigma}\Big)^p\sum_{k=1}^{d}\sum_{\alpha\in R^{e_k}_\sigma(A_h)}|u(\alpha+\sigma e_k)-u(\alpha)|^p\\ &\leq CN(1+CN^{p+1})\#(2A\cap\sigma\mathbb{Z}^d)\Big(\frac{\mathrm{diam}\ A}{\sigma}\Big)^p\sum_{k=1}^{d}\sum_{\alpha\in R^{e_k}_\sigma(A)}|u(\alpha+\sigma e_k)-u(\alpha)|^p.\end{aligned}$$

Therefore, recalling (A.5), we conclude

$$\sum_{\alpha\in A\cap\sigma\mathbb{Z}^d}|u(\alpha)-(u)_A|^p \leq CN(1+CN^{p+1})2^{d+1}(\mathrm{diam}\ A)^p\frac{1}{\sigma^p}\sum_{k=1}^{d}\sum_{\alpha\in R^{e_k}_\varepsilon(A)}|u(\alpha+\sigma e_k)-u(\alpha)|^p,$$

for σ sufficiently small. This proves the thesis for the case $A' = A$ in the discrete setting and concludes the proof. ☐

Remark A.4. In the discrete case, the constant in inequality (A.4) depends on a universal constant coming from [4, Lemma 3.6], the scaling-invariant constant of inequality (A.1), the number of rectangles that constitute A and $(\mathrm{diam}\ A)^p$. Hence, such a constant exhibits the correct behaviour under homothety of the domain, in contrast with [17, Lemma 2 and Lemma 3]. In particular, if we let $C(A)$ denote the constant associated with A, we have that $C(\tau A) = \tau^p C(A)$.

We conclude this appendix with the proof of Proposition 4.4, the rescaled version of Poincaré inequality.

Proposition A.5 (Discrete Poincaré inequality)**.** *Let $T > 1, x_0 \in \mathbb{R}^d$, and let $A \subset \mathbb{R}^d$ be a bounded open set. There exists a positive constant C depending on A such that, having set*

$$R^{e_k}_\sigma(E) := \{\alpha \in E\cap\sigma\mathbb{Z}^d : [\alpha,\alpha+\sigma e_k] \subset E\}$$

for every set E such that $\#(E\cap\sigma\mathbb{Z}^d) \neq 0$ and $k \in \{1,\dots,d\}$, it holds

$$\sum_{\alpha\in(x_0+\tau A)\cap\sigma\mathbb{Z}^d}|u(\alpha)|^p \leq C\Big(\frac{\tau}{\sigma}\Big)^p\sum_{k=1}^{d}\sum_{\alpha\in R^{e_k}_\sigma(x_0+\tau A)}|u(\alpha+\sigma e_k)-u(\alpha)|^p$$

for every $\tau,\sigma > 0$ and for every $u : (x_0+\tau A)\cap\sigma\mathbb{Z}^d \to \mathbb{R}^m$ such that $u(\alpha) = 0$ if $dist_\infty(\alpha,(x_0+\tau A)^c) \leq \sigma T$.

Proof. We prove the result for $x_0 = 0$ and $\tau = 1$ since the general case follows by a translation and scaling argument. We argue associating to any admissible function u a piecewise-affine function obtained by linearly interpolating the values of u on a triangulation of the lattice $\sigma\mathbb{Z}^d$. For the sake of simplicity, we prove in detail the case $d = 2$ and we remark that the same argument can be generalized to any dimension resorting to the Kuhn's decomposition of a cube [35] (see for example [6]).

We identify u with its extension to the whole $\sigma\mathbb{Z}^2$ that equals zero outside $A\cap\sigma\mathbb{Z}^2$. We set

$$T^- := \{(x_1,x_2) \in [0,1]^2 : x_2 \leq 1-x_1\},$$

$$T^+ := \{(x_1,x_2) \in [0,1]^2 : x_2 \geq 1-x_1\},$$

and, given any admissible function $u : \sigma\mathbb{Z}^2 \to \mathbb{R}^m$, we define $\hat{u} : \mathbb{R}^2 \to \mathbb{R}^m$ as

$$\hat{u}(x) := u(\alpha) + \frac{u(\alpha+\sigma e_1) - u(\alpha)}{\sigma}(x_1 - \alpha_1) + \frac{u(\alpha+\sigma e_2) - u(\alpha)}{\sigma}(x_2 - \alpha_2) \quad \text{if } x \in \alpha + \sigma T^-,$$

$$\begin{aligned}\hat{u}(x) := u(\alpha + \sigma(e_1+e_2)) &+ \frac{u(\alpha+\sigma(e_1+e_2)) - u(\alpha+\sigma e_2)}{\sigma}(x_1 - \alpha_1 - \sigma) \\ &+ \frac{u(\alpha+\sigma(e_1+e_2)) - u(\alpha+\sigma e_1)}{\sigma}(x_2 - \alpha_2 - \sigma) \quad \text{if } x \in \alpha + \sigma T^+\end{aligned}$$

for every $\alpha \in \sigma\mathbb{Z}^2$. By the boundary condition on u we infer that $\hat{u} \in W_0^{1,p}(A;\mathbb{R}^m)$. Hence the standard Poincaré inequality holds and we have

$$\int_A |\hat{u}(x)|^p \, dx \leq C \int_A |\nabla \hat{u}(x)|^p \, dx. \tag{A.7}$$

Let us set $P_\alpha^{\sigma,1} := \alpha + \sigma P_1$ and $P_\alpha^{\sigma,2} := \alpha + \sigma P_2$ where

$$P_1 = \{(x_1,x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1, -x_1 \leq x_2 \leq 1 - x_1\},$$

$$P_2 = \{(x_1,x_2) \in \mathbb{R}^2 : 0 \leq x_2 \leq 1, -x_2 \leq x_1 \leq 1 - x_2\}.$$

One can easily show that

$$\partial_{e_k} \hat{u}(x) = \frac{u(\alpha+\sigma e_k) - u(\alpha)}{\sigma}, \qquad \text{for every } \alpha \in \sigma\mathbb{Z}^2, k \in \{1,2\}, \text{ and } x \in P_\alpha^{\sigma,k},$$

and since $|P_\alpha^{\sigma,1}| = |P_\alpha^{\sigma,2}| = \sigma^2$, we get the following estimate

$$\begin{aligned}\int_A |\nabla \hat{u}|^p &\leq C \int_A |\partial_{e_1}\hat{u}|^p + |\partial_{e_2}\hat{u}|^p \, dx \\ &= C \sum_{\alpha \in A \cap \sigma\mathbb{Z}^2} \Big(\int_{P_\alpha^{\sigma,1} \cap A} |\partial_{e_1}\hat{u}|^p \, dx + \int_{P_\alpha^{\sigma,2} \cap A} |\partial_{e_2}\hat{u}|^p \, dx \Big) \\ &= C \sum_{k=1}^2 \sum_{\alpha \in R_\sigma^{e_k}(A)} \sigma^2 \Big| \frac{u(\alpha+\sigma e_k) - u(\alpha)}{\sigma} \Big|^p. \end{aligned} \tag{A.8}$$

On the other hand, we have that

$$\sum_{\alpha \in A \cap \sigma\mathbb{Z}^2} \sigma^2 |u(\alpha)|^p = \sum_{\alpha \in A \cap \sigma\mathbb{Z}^2} \sigma^2 |\hat{u}(\alpha)|^p \leq C \int_A |\hat{u}(x)|^p \, dx \tag{A.9}$$

for a positive constant C independent of σ and u. Hence, combining (A.7), (A.8) and (A.9) we obtain

$$\sum_{\alpha \in A \cap \sigma\mathbb{Z}^2} \sigma^2 |u(\alpha)|^p \leq C \sum_{k=1}^2 \sum_{\alpha \in R_\sigma^{e_k}(A)} \sigma^2 \Big| \frac{u(\alpha+\sigma e_k) - u(\alpha)}{\sigma} \Big|^p,$$

which is the thesis. □

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