TRANSPORT AND FLOW FOR HORIZONTAL SOBOLEV CONTACT VELOCITIES IN CARNOT GROUPS

GIANLUCA SOMMA, SIMONE VERZELLESI, AND DAVIDE VITTONE

Abstract. We establish new well-posedness results for transport and flow equations driven by contact vector fields on Carnot groups. The velocity fields are assumed to have horizontal Sobolev regularity, namely Sobolev regularity only along the horizontal directions determined by the stratified geometry of the group. In the broader sub-Riemannian setting, results of this type were previously known only for Heisenberg groups. Our proof relies on the theory of renormalized solutions as originally introduced by DiPerna and Lions in the Euclidean setting.

1. Introduction

In this paper, we establish new well-posedness results for the *transport equation*

$$\begin{cases} \dfrac{\partial u}{\partial \tau} - \langle \mathbf{b}, \nabla u \rangle + cu = 0 & \text{in } (0,\bar{\tau}) \times \mathbb{G} \\ u(0,\cdot) = u_0 & \text{in } \mathbb{G} \end{cases} \tag{1.1}$$

and for the *flow equation*

$$\begin{cases} \dot{\Phi}(\tau,p) = \mathbf{b}(\tau,\Phi(\tau,p)) & \text{in } (0,\bar{\tau}) \times \mathbb{G} \\ \Phi(0,p) = p & \text{in } \mathbb{G} \end{cases} \tag{1.2}$$

in the non-Euclidean setting of sub-Riemannian *Carnot groups* $\mathbb{G}$. In addition to natural growth conditions and a uniform control on its spatial divergence, the velocity field $\mathbf{b} = \mathbf{b}(\tau,p)$ has the following properties:

(i) *horizontal* Sobolev regularity, i.e., Sobolev regularity along a (possibly small) family of directions;
(ii) a precise *contact structure*, compatible with the above family of directions.

The original motivation for this theory is to address the Lagrangian problem (1.2) beyond the classical Cauchy-Lipschitz framework (see e.g. [4, Chapter 8]). In the Euclidean setting, this problem was solved in the influential paper [15] for Sobolev velocity fields, and was later extended to the BV setting in the breakthrough work [1]. The principle underlying the results of [1, 15] – formalized in great generality in [2, 3] – is to obtain information at the Lagrangian level from the *Eulerian* viewpoint of (1.1). More precisely, well-posedness for distributional solutions to (1.1) yields well-posedness for solutions to (1.2), for instance in the sense of *regular Lagrangian flows* as introduced in [1]. This Eulerian approach has since led to many further results under suitable regularity and structural assumptions on $\mathbf{b}$ (see e.g. [7, 11, 24, 27]). Purely Lagrangian methods have also been developed (see e.g. [12, 20, 29]).

A major step toward extending the theory to non-Euclidean geometries was taken in [6], which established analogous results for Sobolev vector fields on a broad class of metric measure spaces. Since Carnot groups carry a natural metric measure structure, it is natural to ask whether well-posedness results in this setting follow directly from the theory developed in [6]. As explained by L. Ambrosio and the authors in [5, Section 6], this is not the case even for *Heisenberg groups*, the simplest non-abelian examples of Carnot groups. Indeed, although Heisenberg groups satisfy the structural requirements imposed in [6], the corresponding assumptions on the vector field are too restrictive, reducing the applicability of the theory to the trivial case of the null vector field. This limitation was overcome in [5], where well-posedness was established for contact vector fields with horizontal Sobolev regularity on Heisenberg groups. The present paper extends these results to the more general setting of Carnot groups.

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A Carnot group $\mathbb{G}$ is a connected and simply connected Lie group whose Lie algebra $\mathfrak{g}$ of left-invariant vector fields admits the non-trivial *stratification*

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_s, \qquad V_{i+1} = [V_1, V_i] \quad \text{for } i = 1, \dots, s-1, \qquad [V_1, V_s] = \{0\}.$$

The number s of *layers* in the above decomposition is known as *step* of $\mathbb{G}$. Fixing a basis of $\mathfrak{g}$,

$$\left\{X^i_\alpha\right\}_{i,\alpha}, \qquad i = 1, \dots, s, \qquad \alpha = 1, \dots, \dim V_i, \tag{1.3}$$

adapted to the above stratification, uniquely determines a left-invariant Riemannian metric $\langle\cdot,\cdot\rangle$ which makes (1.3) orthonormal. Moreover, exponential coordinates identify $\mathbb{G}$ with $\mathbb{R}^n$ endowed with a typically non-abelian group law. Under this identification, the Haar measure coincides with the Lebesgue measure. Consequently, if read in exponential coordinates, the intrinsic problem (1.1) takes the form of the usual Euclidean transport problem, so that our results are relevant from both the intrinsic and the Euclidean viewpoint.

Both the regularity and the structural assumptions that we impose on $\mathbf{b}$ are naturally expressed in terms of the so-called *horizontal distribution* $\mathcal{H}$, defined by

$$\mathcal{H}_p \coloneqq V_1(p), \qquad p \in \mathbb{G}.$$

On the one hand, our regularity assumptions involve derivatives only along horizontal directions. Since $\mathcal{H}$ does not, in general, exhaust the whole tangent space, these assumptions are weaker than their Euclidean counterparts. For instance, in the relevant case of Sobolev regularity (see e.g. [36]), one typically has

$$W^{\ell,\theta}(\Omega) \subsetneq W^{\ell,\theta}_{\mathcal{H}}(\Omega),$$

where $W^{\ell,\theta}_{\mathcal{H}}(\Omega)$ denotes the corresponding horizontal Sobolev space. In this sense, and as already pointed out in [5], our results improve upon [15] by requiring regularity only along horizontal directions.

On the other hand, the horizontal distribution also underlies the structural condition imposed on the velocity field. In the smooth setting, a vector field is called *contact* if its flow preserves $\mathcal{H}$. This natural class arises in contact geometry (see e.g. [28, 34]), and has been studied thoroughly in the setting of Carnot groups (see e.g. [30, 31]). Particular attention has been devoted to Heisenberg and filiform groups, especially in connection with the theory of quasiconformal mappings (see e.g. [22, 23, 37]). Contact vector fields are characterized by the infinitesimal condition

$$[\mathbf{b}, V_1]_p \subseteq \mathcal{H}_p, \qquad \text{for every } p \in \mathbb{G}. \tag{1.4}$$

As (1.4) is still meaningful under mild regularity assumptions on $\mathbf{b}$, we adopt it as definition. Our main well-posedness result, therefore, applies to (time-dependent) vector fields $\mathbf{b}$ having horizontal Sobolev regularity in their spatial dependence, and satisfying (1.4) in the appropriate weak sense.

The core idea behind the well-posedness of (1.1) exploits the machinery of *renormalized solutions* introduced in [15]: when bounded distributional solutions to (1.1) are renormalized solutions (cf. Section 6), well-posedness follows by natural growth assumptions and uniform bounds on $\operatorname{div}\mathbf{b}$. In turn, the above *renormalization property* follows by adapting the mollification argument introduced in [15]. More precisely, mollifying (1.1) via *group convolution* with appropriate intrinsic kernels yields

$$\frac{\partial u_\varepsilon}{\partial \tau} - \langle \mathbf{b}, \nabla u_\varepsilon\rangle + c u_\varepsilon = \mathscr{C}_\varepsilon \qquad \text{in } (0,\bar\tau) \times \mathbb{G}. \tag{1.5}$$

The error term $\mathscr{C}_\varepsilon$, commonly known as *commutator*, quantifies the lack of commutativity between divergence and convolution. The main technical step consists in proving that $\mathscr{C}_\varepsilon$ converges to 0 in L^1_{loc} as $\varepsilon \searrow 0$. As we explain below, the commutator can be decomposed into terms involving difference quotients of the components of $\mathbf{b}$, of order up to s, and suitable remainder terms. The convergence of the former is ensured by horizontal Sobolev regularity, while the contact condition forces the latter to vanish.

The paper is organized as follows. In Section 2, we introduce the setting of Carnot groups and the tools needed throughout the paper, in particular group mollification and horizontal Sobolev spaces.

In Section 3, we express left-invariant vector fields in terms of right-invariant ones. Namely, if X^i_α is as in (1.3) and $\left(X^i_\alpha\right)^r$ is its right-invariant counterpart, we show (see (3.2) and (3.4) for the notation) that

$$X^{i_0}_{\beta_0} = \left(X^{i_0}_{\beta_0}\right)^r + \sum_{k=1}^{s-i_0} \frac{1}{k!} \sum_{|I_k| \le s-i_0} \sum_{\mathcal{A}_k(I_k)} \sum_{\mathcal{B}_k(i_0,I_k)} c(I_k, i_0 + |I_k|)^{B_k}_{A_k B_{k-1}} x^{I_k}_{A_k} \left(X^{i_0+|I_k|}_{\beta_k}\right)^r. \tag{1.6}$$

As we shall see, this formula will be crucial for relating the structure of $\mathscr{C}_\varepsilon$ to the contact condition (1.4).

In Section 4, we introduce the relevant class of velocity fields. As already mentioned, a *contact vector field* $\mathbf{b}$ with *horizontal Sobolev regularity* is defined by requiring that, for some $\theta \in [1, \infty]$,

$$\mathbf{b} = \sum_{i=1}^{s} \sum_{\alpha=1}^{\dim V_i} b^i_\alpha X^i_\alpha, \qquad b^i_\alpha \in W^{1,\theta}_{\mathcal{H},\mathrm{loc}}(\mathbb{G}), \qquad [\mathbf{b}, V_1]_p \subseteq \mathcal{H}_p \text{ for a.e. } p \in \mathbb{G}. \tag{1.7}$$

By means of (1.7), we show that the first-order Sobolev regularity of the coefficients of $\mathbf{b}$ improves according to the corresponding layer (see Proposition 4.1). The contact condition in (1.7) can be equivalently formulated owing to the family of first-order compatibility conditions

$$X^1_\beta b^k_\gamma = \sum_{\alpha=1}^{\dim V_{k-1}} c(k-1,1)^\gamma_{\alpha\beta}\, b^{k-1}_\alpha \qquad \beta = 1, \dots, \dim V_1,\ k = 2, \dots, s,\ \gamma = 1, \dots, \dim V_k, \tag{1.8}$$

where $c(i,j)^\gamma_{\alpha\beta}$ are the so-called *structural constants* of $\mathfrak{g}$. We show that the improved Sobolev regularity actually upgrades (1.8) to the higher-order system of iterated compatibility conditions

$$X^{I_k}_{A_k} b^i_{\beta_k} = \sum_{\mathcal{B}_{k-1}(i-|I_k|, I_{k-1})} \sum_{\alpha=1}^{\dim V_{i-|I_k|}} c(i, I_k)^{B_k}_{B_{k-1}A_k} b^{i-|I_k|}_\alpha. \tag{1.9}$$

(see Proposition 4.2 and Proposition 4.3). The identities in (1.9) will play an essential role in Section 6, as they allow us to get rid of the aforementioned remainder terms arising in the commutator analysis. The section concludes with a few structural remarks and further insights into the class of contact vector fields.

In Section 5, we investigate the asymptotic behavior of (higher-order) difference quotients for functions with horizontal Sobolev regularity. More precisely, if $f \in W^{\ell,\theta}_{\mathcal{H}}$ and $\{\delta_\varepsilon\}_{\varepsilon>0}$ denotes the appropriate family of *intrinsic dilations* of $\mathbb{G}$, we prove that

$$\frac{f\left(p \cdot \delta_\varepsilon(w)\right) - P^\ell_p f\left(\delta_\varepsilon(w)\right)}{\varepsilon^\ell} \xrightarrow[\varepsilon \searrow 0]{L^\theta} 0, \tag{1.10}$$

where $P^\ell_\cdot f$ is the intrinsic Taylor polynomial of f of *homogeneous order* ℓ. Thus, (1.10) provides an L^θ-Taylor expansion adapted to the homogeneous structure of $\mathbb{G}$, thereby furnishing the second key ingredient in the forthcoming commutator analysis.

In Section 6, we prove the main results of the paper. As stressed above, the key step is the analysis of the commutator $\mathscr{C}_\varepsilon$ in (1.5), which we carry out in Section 6.2. Using (1.6) and performing some algebraic manipulations, one finds that $\mathscr{C}_\varepsilon$ can be written as combination of terms involving the difference quotients of the components of $\mathbf{b}$, namely

$$\int_{\mathbb{G}} \frac{b^i_\alpha(p \cdot \delta_\varepsilon(w)) - P^{i-1}_p b^i_\alpha(\delta_\varepsilon(w))}{\varepsilon^i} u(p \cdot \delta_\varepsilon(w)) X^i_\alpha \varrho(w), dw, \tag{1.11}$$

and remainder terms of the form

$$\left[X^{I_k}_{A_k} b^i_{\beta_k}(p) - \sum_{\mathcal{B}_{k-1}(i-\ell, I_{k-1})} \sum_{\alpha=1}^{\dim V_{i-\ell}} c(i, I_k)^{B_k}_{B_{k-1}A_k} b^{i-\ell}_\alpha(p) \right] \int_{\mathbb{G}} \delta_\varepsilon\left(w^{I_k}_{A_k}\right) u(p \cdot \delta_\varepsilon(w)) \varepsilon^{-i} X^i_{\beta_k} \varrho(w)\, dw. \tag{1.12}$$

On the one hand, (1.10) and the improved Sobolev regularity provided by (1.7) imply that (1.11) converges to 0 in L^1_{loc}. On the other hand, (1.9) shows that (1.12) vanishes identically. Consequently, the renormalization property and the well-posedness of (1.1) follow essentially as in [5], under the above-mentioned growth conditions and controllability assumptions on $\mathbf{b}$. Once the Eulerian problem is settled, well-posedness for the Lagrangian problem (1.2) is deduced by means of the general theory developed in [2, 3] (see Section 6.3).

Finally, in Appendix A we collect some additional results which may be of independent interest, including an inductive representation formula for difference quotients and a quantitative mollification approximation which follows at once by the results of Section 5.

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2. Preliminaries

2.1. **Main notation.** We denote by $\mathbb{N}_+$ the set of nonzero natural numbers. We also write $\infty = +\infty$. If $\bar\tau \in (0,\infty]$, the notation $[0,\bar\tau]$ stays for the usual closed interval if $\bar\tau < \infty$ or $[0,\infty)$ if $\bar\tau = \infty$. Given open sets A, Ω, when $\overline{A}$ is a compact subset of Ω, we write $A \Subset \Omega$. If $\theta \in [1,\infty]$, we denote its Hölder conjugate by $\theta' \in [1,\infty]$, i.e., $\frac{1}{\theta} + \frac{1}{\theta'} = 1$. If f is differentiable at a point q, we denote by df_q its differential at q.

2.2. **Carnot groups.** A Lie algebra $\mathfrak{g}$ is said to be *stratified* if it admits a *stratification*, i.e., there exist nontrivial linear subspaces $V_1, \dots, V_s$ of $\mathfrak{g}$ such that

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_s, \qquad V_{i+1} = [V_1, V_i] \quad \text{for } i = 1, \dots, s-1, \qquad [V_1, V_s] = \{0\}. \tag{2.1}$$

We stress that $[V_i, V_j]$ denotes the linear span of vectors of the form $[X, Y]$ with $X \in V_i$ and $Y \in V_j$. Recall that the Jacobi identity

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \qquad \text{for every } X, Y, Z \in \mathfrak{g} \tag{2.2}$$

holds. Setting $V_k = \{0\}$ if $k > s$, (2.2) implies that $[V_i, V_j] \subseteq V_{i+j}$ for every $i \neq j$. If $(\mathbb{G}, \cdot)$ is a Lie group, we denote by $\mathfrak{g}$ its Lie algebra. Moreover, we define

$$L_p(q) = p \cdot q, \qquad R_p(q) = q \cdot p \qquad \text{for every } p,\, q \in \mathbb{G}$$

to be the left and right translations, respectively. Left-invariant vector fields are complete (see [25, Theorem 9.18]). Accordingly, if $Z \in \mathfrak{g}$ and γ is its integral curve starting from the identity element e, we define the exponential map $\exp : \mathfrak{g} \to \mathbb{G}$ by

$$\exp(Z) \coloneqq \gamma(1).$$

We say that $(\mathbb{G}, \cdot)$ is a *stratified Lie group* if it is connected, simply connected, and its Lie algebra $\mathfrak{g}$ is stratified. Let $\mathfrak{g} = V_1 \oplus \cdots \oplus V_s$ be a stratification and set $n_i \coloneqq \dim V_i$. Set also $n \coloneqq \dim \mathfrak{g} = \dim \mathbb{G}$. Let $\{X^i_\alpha\}_{\alpha=1,\dots,n_i}$ be a basis for V_i for every $i = 1, \dots, s$. Since the exponential map of a stratified Lie group is a diffeomorphism (see [35, Theorem 3.6.2]), we can define a system of *graded coordinates* associated with the basis $\{X^i_\alpha\}_{\substack{i=1,\dots,s\\ \alpha=1,\dots,n_i}}$ for $\mathfrak{g}$:

$$\mathbb{R}^n \ni \left(x^1_1, \dots, x^1_{n_1}, x^2_1, \dots, x^2_{n_2}, \dots, x^s_1, \dots, x^s_{n_s}\right) \mapsto \exp\left(\sum_{i=1}^{s} \sum_{\alpha=1}^{n_i} x^i_\alpha X^i_\alpha\right) \in \mathbb{G}.$$

In these coordinates, the identity element is 0 and the inverse element of $p \in \mathbb{G}$ is $p^{-1} = -p$. We denote by $\log : \mathbb{G} \to \mathfrak{g}$ the inverse of exp. We fix a system of graded coordinates and denote by X^i_α and $x^i_\alpha(p)$ (or simply x^i_α if there is no ambiguity), with $i = 1, \dots, s$ and $\alpha = 1, \dots, n_i$, the α-th vector field of the basis of V_i and the corresponding graded coordinate of $p \in \mathbb{G}$. Sometimes, we will adopt the notation

$$p = \left(x^1, \dots, x^s\right), \qquad x^i = \left(x^i_1, \dots, x^i_{n_i}\right) \in \mathbb{R}^{n_i}.$$

We equip $\mathbb{G}$ with the Riemannian metric $\langle \cdot, \cdot \rangle$ that makes the above basis orthonormal. Accordingly, the Riemannian gradient reads as

$$\nabla f = \sum_{i=1}^{s} \sum_{\alpha=1}^{n_i} \left(X^i_\alpha f\right) X^i_\alpha.$$

We denote by $\left(X^i_\alpha\right)^r$ the right-invariant vector field corresponding to X^i_α, that is,

$$\left(X^i_\alpha\right)^r(p) = \left(dR_p\right)_e \left(X^i_\alpha(e)\right) \qquad \text{for every } p \in \mathbb{G}.$$

Moreover, we define the *structural constants* $c(i,j)^\gamma_{\alpha\beta} \in \mathbb{R}$ by

$$\left[X^i_\alpha, X^j_\beta\right] \coloneqq \sum_{\gamma=1}^{n_{i+j}} c(i,j)^\gamma_{\alpha\beta} X^{i+j}_\gamma \tag{2.3}$$

for every $i, j = 1, \dots, s$ such that $i + j \le s$, $\alpha = 1, \dots, n_i$, $\beta = 1, \dots, n_j$. Observe that

$$c(i,j)^\gamma_{\alpha\beta} = -c(j,i)^\gamma_{\beta\alpha}. \tag{2.4}$$

We endow $\mathbb{G}$ with a *homogeneous structure* (see [8]). Precisely, for any $\lambda > 0$, we define the *intrinsic dilations* by setting $\delta_\lambda Z \coloneqq \lambda^i Z$ if $Z \in V_i$ and extending it by linearity on the whole algebra $\mathfrak{g}$. Then, considering their image through the exponential map, we get a one-parameter group of dilations on $\mathbb{G}$, which in graded coordinates are expressed as follows:

$$\delta_\lambda(p) := \left(\lambda x_1^1, \dots, \lambda x_{n_1}^1, \lambda^2 x_1^2, \dots, \lambda^2 x_{n_2}^2, \dots, \lambda^s x_1^s, \dots, \lambda^s x_{n_s}^s\right) \qquad \text{for every } \lambda > 0,\ p = \left(x^1, \dots, x^s\right) \in \mathbb{G}.$$

The *horizontal distribution* $\mathcal{H}$ is defined by

$$\mathcal{H}_p := V_1(p) \qquad \text{for every } p \in \mathbb{G}.$$

We endow $\mathbb{G}$ with the so-called *Carnot-Carathéodory distance* d (see [33]): given $p, q \in \mathbb{G}$, set

$$d(p,q) := \inf\left\{\int_0^1 \sqrt{\langle\dot\gamma,\dot\gamma\rangle}\, d\tau \ :\ \gamma : [0,1] \to \mathbb{G} \text{ is absolutely continuous, horizontal, } \gamma(0) = p,\ \gamma(1) = q\right\},$$

where an absolutely continuous curve is *horizontal* if $\dot\gamma(\tau) \in \mathcal{H}_{\gamma(\tau)}$ for a.e. τ. By Chow's theorem, d is finite. The metric space $(\mathbb{G}, d)$ is called a *Carnot group*. We indicate by $B(p,r) := \{q \in \mathbb{G} : d(p,q) < r\}$ the open ball with center $p \in \mathbb{G}$ and radius $r > 0$. The Carnot-Carathéodory distance is compatible with the group and homogeneous structures:

$$\begin{aligned} d(w\cdot p, w \cdot q) &= d(p,q),\\ d(\delta_\lambda(p), \delta_\lambda(q)) &= \lambda d(p,q) \end{aligned} \qquad \text{for every } p, q, w \in \mathbb{G},\ \lambda > 0.$$

The Lebesgue measure $|\cdot|$ is a Haar measure:

$$\begin{aligned} |L_p(E)| &= |E|,\\ |R_p(E)| &= |E| \end{aligned} \qquad \text{for every } E \subseteq \mathbb{G} \text{ measurable},\ p \in \mathbb{G}. \tag{2.5}$$

Furthermore,

$$|\delta_\lambda(E)| = \lambda^Q |E| \qquad \text{for every } E \subseteq \mathbb{G} \text{ measurable},\ \lambda > 0, \tag{2.6}$$

where $Q := \sum_{i=1}^s i n_i$ is the *homogeneous dimension* of $(\mathbb{G}, \cdot)$ (see [33]). Using arguments identical to those in [5, Lemma 2.1] and [5, Lemma 2.2], the following simple properties hold.

Lemma 2.1. *Define* $\iota : \mathbb{G} \to \mathbb{G}$ *by* $\iota(q) = q^{-1}$. *Then*

$$X_\alpha^i(\varphi \circ \iota) = -\left(\left(X_\alpha^i\right)^r \varphi\right) \circ \iota \qquad \textit{for every } \varphi \in C^1(\mathbb{G}),\ i = 1, \dots, s,\ \alpha = 1, \dots, n_i.$$

In particular, if $\varphi = \varphi \circ \iota$, *then* $X_\alpha^i \varphi = -\left(\left(X_\alpha^i\right)^r \varphi\right) \circ \iota$.

Lemma 2.2. *Let* $\theta \in [1, \infty)$ *and* $f \in L^\theta(\mathbb{G})$. *Then* $\lim_{q \to 0} \|f \circ R_q - f\|_{L^\theta(\mathbb{G})} = 0$.

We denote by $\mathbb{X}(\mathbb{G})$ the class of locally integrable vector fields in $\mathbb{G}$, and we adopt the notation

$$\mathbf{b} = \sum_{i=1}^s \sum_{\alpha=1}^{n_i} b_\alpha^i X_\alpha^i \qquad \text{for every } \mathbf{b} \in \mathbb{X}(\mathbb{G}). \tag{2.7}$$

Since the frame $\left\{X_\alpha^i\right\}_{\substack{i=1,\dots,s\\ \alpha=1,\dots,n_i}}$ is orthonormal and its coefficients matrix with respect to the canonical basis of $\mathbb{R}^n$ has determinant equal to 1 (cf. [8, Corollary 1.3.19]), the Riemannian volume coincides with the Lebesgue measure. Hence, we define the divergence of a vector field $\mathbf{b} \in \mathbb{X}(\mathbb{G})$ as the distribution $\operatorname{div} \mathbf{b}$ such that

$$\langle \operatorname{div} \mathbf{b}, \varphi\rangle := -\int_{\mathbb{G}} d\varphi(\mathbf{b})\, dp \qquad \text{for every } \varphi \in C_c^\infty(\mathbb{G}).$$

Thanks to (2.5), one can prove that left and right-invariant vector fields are divergence-free.

2.3. **Group convolution.** We refer to [17] for more details. Given $f, g \in L^1_{\mathrm{loc}}(\mathbb{G})$, their *group convolution* $f * g$ is defined by

$$(f * g)(p) := \int_{\mathbb{G}} f(q) g\left(q^{-1}\cdot p\right)\, dq \overset{(2.5)}{=} \int_{\mathbb{G}} f\left(p \cdot q^{-1}\right) g(q)\, dq \qquad \text{for every } p \in \mathbb{G},$$

provided that the integrals converge. If the domain of f and g is an arbitrary open set $\Omega \subseteq \mathbb{G}$, we can still define $f * g$ by extending them to be 0 outside Ω. Fix a mollifier $\varrho \in C_c^\infty(\mathbb{G})$ such that

$$\varrho \geq 0, \quad \varrho(p) = \varrho\left(p^{-1}\right), \quad \int_{\mathbb{G}} \varrho\, dp = 1 \quad \text{and} \quad \varrho \equiv 0 \text{ in } B(0,1)^c \tag{2.8}$$

and set

$$\varrho_\varepsilon(p) = \frac{1}{\varepsilon^Q}\varrho\left(\delta_{\frac{1}{\varepsilon}}(p)\right) \qquad \text{for every } \varepsilon > 0,\ p \in \mathbb{G}. \tag{2.9}$$

Observe that

$$\left(X^i_\alpha\right)^r \varrho_\varepsilon(p) = \frac{1}{\varepsilon^{Q+i}}\left(\left(X^i_\alpha\right)^r \varrho\right)\left(\delta_{\frac{1}{\varepsilon}}(p)\right) \qquad \text{for every } i = 1,\dots,s,\ \alpha = 1,\dots,n_i,\ p \in \mathbb{G}. \tag{2.10}$$

The following standard convergence properties of group mollification hold.

Proposition 2.3. *Let $\theta \in [1,\infty)$. Let $u \in L^\theta(\mathbb{G})$ and set $u_\varepsilon = u * \varrho_\varepsilon$ for every $\varepsilon > 0$. Then*

(i) $u_\varepsilon \xrightarrow{L^\theta} u$ *as* $\varepsilon \searrow 0$,
(ii) $\|u_\varepsilon\|_{L^\theta(\mathbb{G})} \le \|u\|_{L^\theta(\mathbb{G})}$ *for every* $\varepsilon > 0$.

2.4. **Horizontal Sobolev spaces.** We refer to [16, 9] for further information. Fix an open subset $\Omega \subseteq \mathbb{G}$ and $u \in L^1_{\mathrm{loc}}(\Omega)$. If $i = 1,\dots,s$ and $\alpha = 1,\dots,n_i$, the distribution $X^i_\alpha u$ is defined by

$$\langle X^i_\alpha u, \varphi\rangle := -\int_\Omega u X^i_\alpha \varphi \, dp \qquad \text{for every } \varphi \in C^\infty_c(\Omega).$$

If $\theta \in [1,\infty]$, we define the *horizontal Sobolev space* $W^{1,\theta}_{\mathcal{H}}(\Omega)$ by

$$W^{1,\theta}_{\mathcal{H}}(\Omega) := \left\{u \in L^\theta(\Omega) \,:\, X^1_\alpha u \in L^\theta(\Omega) \text{ for every } \alpha = 1,\dots,n_1\right\}.$$

The spaces $W^{1,\theta}_{\mathcal{H},\mathrm{loc}}(\Omega)$ and $W^{\ell,\theta}_{\mathcal{H}}(\Omega)$, for $\ell = 2,3,\dots$, are defined accordingly. It is easy to see that the vector space $W^{\ell,\theta}_{\mathcal{H}}(\Omega)$, endowed with the norm

$$\|u\|_{W^{\ell,\theta}_{\mathcal{H}}(\Omega)} := \|u\|_{L^\theta(\Omega)} + \sum_{\alpha=1}^{n_1}\left\|X^1_\alpha u\right\|_{L^\theta(\Omega)} + \dots + \sum_{\alpha_1,\dots,\alpha_\ell=1}^{n_1}\left\|X^1_{\alpha_1}\dots X^1_{\alpha_\ell} u\right\|_{L^\theta(\Omega)},$$

is a Banach space for every $1 \le \theta \le \infty$ and $\ell \in \mathbb{N}_+$. Moreover, the following Meyers-Serrin-type approximation result holds (see [18, Theorem A.2]).

Theorem 2.4. *Let $\Omega \subseteq \mathbb{G}$ be an open set. Let $\ell \in \mathbb{N}_+$ and $\theta \in [1,\infty)$. Let $u \in W^{\ell,\theta}_{\mathcal{H}}(\Omega)$. There exists a sequence $\{u_h\}_{h\in\mathbb{N}} \subseteq C^\infty(\Omega) \cap W^{\ell,\theta}_{\mathcal{H}}(\Omega)$ such that*

$$\lim_{h\to\infty}\|u_h - u\|_{W^{\ell,\theta}_{\mathcal{H}}(\Omega)} = 0.$$

Observe that, if $\mathbf{b}$ is as in (2.7) and $b^i_\alpha \in W^{1,\theta}_{\mathcal{H},\mathrm{loc}}(\Omega)$ for every $i = 1,\dots,s$ and $\alpha = 1,\dots,n_i$, then

$$\operatorname{div}\mathbf{b} = \sum_{i=1}^{s}\sum_{\alpha=1}^{n_i} X^i_\alpha b^i_\alpha. \tag{2.11}$$

3. Left and right-invariant vector fields

In this section, we express left-invariant vector fields with respect to the right-invariant frame, providing some relevant examples. The key ingredient is the differential at the identity element e of the conjugation map: for any $p \in \mathbb{G}$, set $C_p := L_p \circ R_{p^{-1}}$ and $\mathrm{Ad}_p := (dC_p)_e$. Denote by $\mathrm{End}(\mathfrak{g})$ the set of all endomorphisms of $\mathfrak{g}$ and define the operator $\mathrm{ad}_Z : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ by $\mathrm{ad}_Z X := [Z,X]$ for every $X, Z \in \mathfrak{g}$. Then, it is known that (see [21, Proposition 1.91])

$$\mathrm{Ad}_{\exp(Z)} = e^{\mathrm{ad}_Z}, \tag{3.1}$$

where $e^A := \sum_{k=0}^\infty A^k/k! \in \mathrm{End}(\mathfrak{g})$ for every $A \in \mathrm{End}(\mathfrak{g})$. We introduce a specific notation to simplify the next computations. For all $k \in \mathbb{N}_+$, we denote a generic k-multi-index by $I_k = (i_1,\dots,i_k) \in \{1,\dots,s\}^k$, and we set $|I_k| := i_1 + \dots + i_k$. Moreover, given $i = 1,\dots,s-1$, we define the sets

$$\begin{aligned}
\mathcal{A}_k(I_k) &= \left\{(\alpha_1,\dots,\alpha_k) \in \mathbb{N}^k : 1 \le \alpha_j \le n_{i_j}\right\},\\
\mathcal{B}_k(i,I_k) &= \left\{(\beta_1,\dots,\beta_k) \in \mathbb{N}^k : 1 \le \beta_j \le n_{i+i_1+\dots+i_j}\right\}, \qquad \text{provided that } i + |I_k| \le s.
\end{aligned} \tag{3.2}$$

We adopt the notation

$$\sum_{\mathcal{A}_k(I_k)} = \sum_{(\alpha_1,\dots,\alpha_k)\in\mathcal{A}_k}, \qquad \sum_{\mathcal{B}_k(i,I_k)} = \sum_{(\beta_1,\dots,\beta_k)\in\mathcal{B}_k(i,I_k)}.$$

Lemma 3.1. *Let $i_0 = 1, \dots, s-1$ and $\beta_0 = 1, \dots, n_{i_0}$. If $Z = \sum_{i=1}^{s} \sum_{\alpha=1}^{n_i} x_\alpha^i X_\alpha^i \in \mathfrak{g}$ and $k = 1, \dots, s - i_0$, then*

$$(\mathrm{ad}_Z)^k X_{\beta_0}^{i_0} = \sum_{|I_k| \le s-i_0} \sum_{\mathcal{A}_k(I_k)} \sum_{\mathcal{B}_k(i_0,I_k)} c(i_1, i_0)_{\alpha_1\beta_0}^{\beta_1} \dots c(i_k, i_0 + i_1 + \dots + i_{k-1})_{\alpha_k\beta_{k-1}}^{\beta_k} x_{\alpha_1}^{i_1} \dots x_{\alpha_k}^{i_k} X_{\beta_k}^{i_0+|I_k|}. \tag{3.3}$$

Proof. We prove the formula by induction on k. A direct computation shows that

$$\mathrm{ad}_Z X_{\beta_0}^{i_0} = \left[Z, X_{\beta_0}^{i_0}\right] = \sum_{i_1=1}^{s-i_0} \sum_{\alpha_1=1}^{n_{i_1}} x_{\alpha_1}^{i_1} \left[X_{\alpha_1}^{i_1}, X_{\beta_0}^{i_0}\right] \overset{(2.3)}{=} \sum_{i_1=1}^{s-i_0} \sum_{\alpha_1=1}^{n_{i_1}} \sum_{\beta_1=1}^{n_{i_0+i_1}} c(i_1, i_0)_{\alpha_1\beta_0}^{\beta_1} x_{\alpha_1}^{i_1} X_{\beta_1}^{i_0+i_1},$$

which is (3.3) with $k = 1$. Then, assume $k > 1$ and that the thesis holds for $k - 1$. Hence, the inductive hypothesis and the base case ensure

$$\begin{aligned}
(\mathrm{ad}_Z)^k X_{\beta_0}^{i_0} &= \underbrace{\left[Z, \left[Z, \dots, \left[Z, X_{\beta_0}^{i_0}\right]\dots\right]\right]}_{k} \\
&= \sum_{|I_{k-1}| \le s-i_0} \sum_{\mathcal{A}_{k-1}(I_{k-1})} \sum_{\mathcal{B}_{k-1}(i_0,I_{k-1})} c(i_1, i_0)_{\alpha_1\beta_0}^{\beta_1} \dots c(i_{k-1}, i_0 + \dots + i_{k-2})_{\alpha_{k-1}\beta_{k-2}}^{\beta_{k-1}} x_{\alpha_1}^{i_1} \dots x_{\alpha_{k-1}}^{i_{k-1}} \mathrm{ad}_Z X_{\beta_{k-1}}^{i_0+|I_{k-1}|} \\
&\overset{(2.3)}{=} \sum_{|I_k| \le s-i_0} \sum_{\mathcal{A}_k(I_k)} \sum_{\mathcal{B}_k(i_0,I_k)} c(i_1, i_0)_{\alpha_1\beta_0}^{\beta_1} \dots c(i_k, i_0 + \dots + i_{k-1})_{\alpha_k\beta_{k-1}}^{\beta_k} x_{\alpha_1}^{i_1} \dots x_{\alpha_k}^{i_k} X_{\beta_k}^{i_0+|I_k|},
\end{aligned}$$

which is the thesis. □

From now on, whenever $k \in \mathbb{N}_+$, $A_k = (\alpha_1, \dots, \alpha_k) \in \mathcal{A}_k(I_k)$ and $B_k = (\beta_1, \dots, \beta_k) = (B_{k-1}, \beta_k) \in \mathcal{B}_k(i, I_k)$, we may adopt the compact notation

$$\begin{aligned}
x_{A_k}^{I_k} &:= x_{\alpha_1}^{i_1} \dots x_{\alpha_k}^{i_k} && \text{for every } \left(x^1, \dots, x^s\right) \in \mathbb{G}, \\
X_{A_k}^{I_k} &:= X_{\alpha_1}^{i_1} \dots X_{\alpha_k}^{i_k}, \\
c(I_k, i)_{A_k B_{k-1}}^{B_k} &:= c(i_1, i - i_k - \dots - i_1)_{\alpha_1\alpha}^{\beta_1} \dots c(i_k, i - i_k)_{\alpha_k\beta_{k-1}}^{\beta_k} && \text{for every } |I_k| < i,\ \alpha = 1, \dots n_i, \\
c(i, I_k)_{B_{k-1} A_k}^{B_k} &:= c(i - i_k - \dots - i_1, i_1)_{\alpha\alpha_1}^{\beta_1} \dots c(i - i_k, i_k)_{\beta_{k-1}\alpha_k}^{\beta_k} && \text{for every } |I_k| < i,\ \alpha = 1, \dots n_i.
\end{aligned} \tag{3.4}$$

Theorem 3.2. *Let $i_0 = 1, \dots, s-1$ and $\beta_0 = 1, \dots, n_{i_0}$. Then*

$$X_{\beta_0}^{i_0} = \left(X_{\beta_0}^{i_0}\right)^r + \sum_{k=1}^{s-i_0} \frac{1}{k!} \sum_{|I_k| \le s-i_0} \sum_{\mathcal{A}_k(I_k)} \sum_{\mathcal{B}_k(i_0,I_k)} c(I_k, i_0 + |I_k|)_{A_k B_{k-1}}^{B_k} x_{A_k}^{I_k} \left(X_{\beta_k}^{i_0+|I_k|}\right)^r. \tag{3.5}$$

Instead, if $i_0 = s$, then $X_{\beta_0}^s = \left(X_{\beta_0}^s\right)^r$.

Proof. For any $p \in \mathbb{G}$, the identity $L_p = R_p \circ C_p$ holds. Therefore, the left-invariance of $X_{\beta_0}^{i_0}$ implies

$$X_{\beta_0}^{i_0}(p) = (dL_p)_e \left(X_{\beta_0}^{i_0}(e)\right) = (dR_p)_e \left(\mathrm{Ad}_p X_{\beta_0}^{i_0}(e)\right).$$

Set $Z = \log(p)$. Then,

$$X_{\beta_0}^{i_0}(p) \overset{(3.1)}{=} (dR_p)_e \left(e^{\mathrm{ad}_Z} X_{\beta_0}^{i_0}(e)\right) = \sum_{k=0}^{s-i_0} \frac{1}{k!} (dR_p)_e \left((\mathrm{ad}_Z)^k X_{\beta_0}^{i_0}(e)\right),$$

which allows us to conclude if $i_0 = s$. If, instead, $i_0 < s$, then

$$\begin{aligned}
X_{\beta_0}^{i_0}(p) &\overset{(3.3)}{=} (dR_p)_e \left(X_{\beta_0}^{i_0}(e)\right) + \sum_{k=1}^{s-i_0} \frac{1}{k!} \sum_{|I_k| \le s-i_0} \sum_{\mathcal{A}_k(I_k)} \sum_{\mathcal{B}_k(i_0,I_k)} c(I_k, i_0 + |I_k|)_{A_k B_{k-1}}^{B_k} x_{A_k}^{I_k}(p) \, (dR_p)_e \left(X_{\beta_k}^{i_0+|I_k|}(e)\right) \\
&= \left(X_{\beta_0}^{i_0}\right)^r(p) + \sum_{k=1}^{s-i_0} \frac{1}{k!} \sum_{|I_k| \le s-i_0} \sum_{\mathcal{A}_k(I_k)} \sum_{\mathcal{B}_k(i_0,I_k)} c(I_k, i_0 + |I_k|)_{A_k B_{k-1}}^{B_k} x_{A_k}^{I_k}(p) \left(X_{\beta_k}^{i_0+|I_k|}\right)^r(p),
\end{aligned}$$

which is the thesis. □

Example 3.3. The Lie algebra of the n-th *Heisenberg group* (see e.g. [8]) is generated by the vector fields

$$X^1_\alpha = \partial_{x^1_\alpha} + 2x^1_{\alpha+n}\partial_{x^2_1}, \qquad X^1_{\alpha+n} = \partial_{x^1_{\alpha+n}} - 2x^1_\alpha\partial_{x^2_1}, \qquad X^2_1 = \partial_{x^2_1} \qquad \text{for every } \alpha = 1,\dots,n.$$

In this case, $s = 2$. We have $c(1,1)^1_{\alpha\beta} = 0$ if $\alpha < \beta$ and $\beta \neq \alpha + n$, and $c(1,1)^1_{\alpha,\alpha+n} = -4$ for every $\alpha = 1,\dots,n$. Recall that $\left(X^2_1\right)^r = X^2_1$. Regarding X^1_α and $X^1_{\alpha+n}$, we obtain

$$X^1_\alpha \overset{(3.5)}{=} \left(X^1_\alpha\right)^r + \sum_{\alpha_1=1}^{2n} c(1,1)^1_{\alpha_1,\alpha}x^1_{\alpha_1}X^2_1 = \left(X^1_\alpha\right)^r + c(1,1)^1_{\alpha+n,\alpha}x^1_{\alpha+n}X^2_1 = \left(X^1_\alpha\right)^r + 4x^1_{\alpha+n}X^2_1,$$

$$X^1_{\alpha+n} \overset{(3.5)}{=} \left(X^1_{\alpha+n}\right)^r + \sum_{\alpha_1=1}^{2n} c(1,1)^1_{\alpha_1,\alpha+n}x^1_{\alpha_1}X^2_1 = \left(X^1_{\alpha+n}\right)^r + c(1,1)^1_{\alpha,\alpha+n}x^1_\alpha X^2_1 = \left(X^1_{\alpha+n}\right)^r - 4x^1_\alpha X^2_1.$$

Example 3.4. The Lie algebra of the *Engel group* (see e.g. [8]) is generated by the vector fields

$$X^1_1 = \partial_{x^1_1} + \frac{x^1_2}{2}\partial_{x^2_1} + \left(\frac{x^2_1}{2} - \frac{x^1_1x^1_2}{12}\right)\partial_{x^3_1}, \quad X^1_2 = \partial_{x^1_2} - \frac{x^1_1}{2}\partial_{x^2_1} + \frac{(x^1_1)^2}{12}\partial_{x^3_1}, \quad X^2_1 = \partial_{x^2_1} - \frac{x^1_1}{2}\partial_{x^3_1}, \quad X^3_1 = \partial_{x^3_1}.$$

In this case, $s = 3$. The structural constants are zero except for $c(1,1)^1_{12} = -1$ and $c(1,2)^1_{11} = -1$. Recall that $\left(X^3_1\right)^r = X^3_1$. First, we compute X^1_1 and X^1_2 in terms of the right-invariant frame:

$$\begin{aligned}
X^1_1 &\overset{(3.5)}{=} \left(X^1_1\right)^r + \sum_{i_1=1}^{2}\sum_{\alpha_1=1}^{2} c(i_1,1)^1_{\alpha_1,1}x^{i_1}_{\alpha_1}\left(X^{1+i_1}_1\right)^r + \frac{1}{2}\sum_{\alpha_1=1}^{2}\sum_{\alpha_2=1}^{2} c(1,1)^1_{\alpha_1,1}c(1,2)^1_{\alpha_2,1}x^1_{\alpha_1}x^1_{\alpha_2}X^3_1 \\
&= \left(X^1_1\right)^r + c(1,1)^1_{21}x^1_2\left(X^2_1\right)^r + c(2,1)^1_{11}x^2_1X^3_1 + \frac{1}{2}c(1,1)^1_{21}c(1,2)^1_{11}x^1_2x^1_1X^3_1 \\
&= \left(X^1_1\right)^r + x^1_2\left(X^2_1\right)^r + x^2_1X^3_1 - \frac{1}{2}x^1_1x^1_2X^3_1 \\
&= \left(X^1_1\right)^r + x^1_2\left(X^2_1\right)^r + \left(x^2_1 - \frac{x^1_1x^1_2}{2}\right)X^3_1, \\
X^1_2 &\overset{(3.5)}{=} \left(X^1_2\right)^r + \sum_{i_1=1}^{2}\sum_{\alpha_1=1}^{2} c(i_1,1)^1_{\alpha_1,2}x^{i_1}_{\alpha_1}\left(X^{1+i_1}_1\right)^r + \frac{1}{2}\sum_{\alpha_1=1}^{2}\sum_{\alpha_2=1}^{2} c(1,1)^1_{\alpha_1,2}c(1,2)^1_{\alpha_2,1}x^1_{\alpha_1}x^1_{\alpha_2}X^3_1 \\
&= \left(X^1_2\right)^r + c(1,1)^1_{12}x^1_1\left(X^2_1\right)^r + \frac{1}{2}c(1,1)^1_{12}c(1,2)^1_{11}\left(x^1_1\right)^2X^3_1 \\
&= \left(X^1_2\right)^r - x^1_1\left(X^2_1\right)^r + \frac{\left(x^1_1\right)^2}{2}X^3_1.
\end{aligned}$$

Finally,

$$X^2_1 \overset{(3.5)}{=} \left(X^2_1\right)^r + \sum_{\alpha_1=1}^{2} c(1,2)^1_{\alpha_1,1}x^1_{\alpha_1}X^3_1 = \left(X^2_1\right)^r + c(1,2)^1_{11}x^1_1X^3_1 = \left(X^2_1\right)^r - x^1_1X^3_1.$$

4. Contact vector fields

As stressed in the introduction, *contact vector fields* constitute a natural class in the setting of stratified Lie groups. A smooth vector field $\mathbf{b} \in \mathbb{X}(\mathbb{G})$ is a contact vector field if

$$[\mathbf{b}, V_1]_p \subseteq \mathcal{H}_p \qquad \text{for every } p \in \mathbb{G}. \tag{4.1}$$

The interest in this class lies in the fact that contact vector fields are precisely those vector fields whose flows both preserve the horizontal distribution and are (locally) Lipschitz continuous in the metric space $(\mathbb{G}, d)$. Since we are interested in non-smooth vector fields, we notice that (4.1) is well-posed (in the weak sense) under mild regularity assumptions on $\mathbf{b}$. For instance, when the components of $\mathbf{b}$ have horizontal Sobolev regularity, then $[\mathbf{b}, Z]$ is a well-defined vector field in $\mathbb{X}(\mathbb{G})$ for every $Z \in V_1$. Accordingly, we say that $\mathbf{b} \in \mathbb{X}(\mathbb{G})$ is a *contact vector field with horizontal Sobolev regularity* if, for some $\theta \in [1, \infty]$,

$$\mathbf{b} = \sum_{i=1}^{s}\sum_{\alpha=1}^{n_i} b^i_\alpha X^i_\alpha, \qquad b^i_\alpha \in W^{1,\theta}_{\mathcal{H},\mathrm{loc}}(\mathbb{G}), \qquad [\mathbf{b}, V_1]_p \subseteq \mathcal{H}_p \ \text{ for a.e. } p \in \mathbb{G}. \tag{4.2}$$

The contact property in (4.2) can be written as a family of compatibility conditions. Indeed, if $\beta = 1, \dots, n_1$,

$$\begin{aligned}
\left[\mathbf{b}, X^1_\beta\right] &= \sum_{k=1}^{s}\sum_{\alpha=1}^{n_k} b^k_\alpha \left[X^k_\alpha, X^1_\beta\right] - \sum_{k=1}^{s}\sum_{\gamma=1}^{n_k}\left(X^1_\beta b^k_\gamma\right) X^k_\gamma \\
&\overset{(2.3)}{=} \sum_{k=1}^{s-1}\sum_{\gamma=1}^{n_{k+1}}\sum_{\alpha=1}^{n_k} c(k,1)^\gamma_{\alpha\beta}\, b^k_\alpha\, X^{k+1}_\gamma - \sum_{k=2}^{s}\sum_{\gamma=1}^{n_k}\left(X^1_\beta b^k_\gamma\right) X^k_\gamma - \sum_{\gamma=1}^{n_1}\left(X^1_\beta b^1_\gamma\right) X^1_\gamma \\
&= \sum_{k=2}^{s}\sum_{\gamma=1}^{n_k}\left(\sum_{\alpha=1}^{n_{k-1}} c(k-1,1)^\gamma_{\alpha\beta}\, b^{k-1}_\alpha - X^1_\beta b^k_\gamma\right) X^k_\gamma - \sum_{\gamma=1}^{n_1}\left(X^1_\beta b^1_\gamma\right) X^1_\gamma.
\end{aligned}$$

In particular, (4.2) is equivalent to

$$X^1_\beta b^k_\gamma = \sum_{\alpha=1}^{n_{k-1}} c(k-1,1)^\gamma_{\alpha\beta}\, b^{k-1}_\alpha \qquad \text{for every } \beta = 1,\dots,n_1,\ k = 2,\dots,s,\ \gamma = 1,\dots,n_k. \tag{4.3}$$

By (4.3), the regularity of $\mathbf{b}$ improves in the following stratified way.

Proposition 4.1. *Let* $\mathbf{b}$ *satisfy* (4.2)*. Then, for every* $i = 1, \dots, s$,

$$b^i_\alpha \in W^{i,\theta}_{\mathcal{H},\mathrm{loc}}(\mathbb{G}) \qquad \text{for every } \alpha = 1,\dots,n_i. \tag{4.4}$$

Proof. We argue by induction on i. The case $i = 1$ is part of the assumption (4.2). Assume $i > 1$ and that (4.4) holds for every $\alpha = 1, \dots, n_{i-1}$. Then, for any $\beta = 1, \dots, n_1$ and $\gamma = 1, \dots, n_i$,

$$X^1_\beta b^i_\gamma \overset{(4.3)}{=} \sum_{\alpha=1}^{n_{i-1}} c(i-1,1)^\gamma_{\alpha\beta} b^{i-1}_\alpha \in W^{i-1,\theta}_{\mathcal{H},\mathrm{loc}}(\mathbb{G}).$$

In particular, $b^i_\gamma \in W^{i,\theta}_{\mathcal{H},\mathrm{loc}}(\mathbb{G})$, and the thesis follows. □

The improved regularity provided by Proposition 4.1 allows to enlarge the class of compatibility conditions.

Proposition 4.2. *Let* $\mathbf{b}$ *satisfy* (4.2)*. Then*

$$X^j_\beta b^k_\gamma = \sum_{\alpha=1}^{n_{k-j}} c(k-j,j)^\gamma_{\alpha\beta} b^{k-j}_\alpha \tag{4.5}$$

for every $j = 1, \dots, s-1$, $\beta = 1, \dots, n_j$, $k = j+1, \dots, s$ *and* $\gamma = 1, \dots, n_k$.

Proof. We argue by induction on j. The case $j = 1$ is true by (4.3). Assume $j > 1$, and assume that (4.5) holds for $j-1$ and for any $\beta = 1, \dots, n_{j-1}$, $k = j, \dots, s$ and $\gamma = 1, \dots, n_k$. Fix $\delta = 1, \dots, n_1$, $\beta = 1, \dots, n_{j-1}$, $k = j+1, \dots, s$, $\gamma = 1, \dots, n_k$. By induction,

$$X^{j-1}_\beta b^k_\gamma = \sum_{\alpha=1}^{n_{k-j+1}} c(k-j+1, j-1)^\gamma_{\alpha\beta}\, b^{k-j+1}_\alpha.$$

Therefore,

$$X^1_\delta\left(X^{j-1}_\beta b^k_\gamma\right) = \sum_{\alpha=1}^{n_{k-j+1}} c(k-j+1,j-1)^\gamma_{\alpha\beta}\, X^1_\delta b^{k-j+1}_\alpha \overset{(4.3)}{=} \sum_{\alpha=1}^{n_{k-j+1}}\sum_{\mu=1}^{n_{k-j}} c(k-j+1,j-1)^\gamma_{\alpha\beta}\, c(k-j,1)^\alpha_{\mu\delta}\, b^{k-j}_\mu.$$

On the other hand, by (4.3),

$$X^1_\delta b^k_\gamma = \sum_{\alpha=1}^{n_{k-1}} c(k-1,1)^\gamma_{\alpha\delta}\, b^{k-1}_\alpha.$$

Hence, by induction,

$$X^{j-1}_\beta\left(X^1_\delta b^k_\gamma\right) = \sum_{\alpha=1}^{n_{k-1}} c(k-1,1)^\gamma_{\alpha\delta}\, X^{j-1}_\beta b^{k-1}_\alpha = \sum_{\alpha=1}^{n_{k-1}}\sum_{\mu=1}^{n_{k-j}} c(k-1,1)^\gamma_{\alpha\delta}\, c(k-j,j-1)^\alpha_{\mu\beta}\, b^{k-j}_\mu.$$

Combining the above equations,

$$\left[X^1_\delta, X^{j-1}_\beta\right] b^k_\gamma = \sum_{\mu=1}^{n_{k-j}}\left(\sum_{\alpha=1}^{n_{k-j+1}} c(k-j,1)^\alpha_{\mu\delta}\, c(k-j+1,j-1)^\gamma_{\alpha\beta} - \sum_{\alpha=1}^{n_{k-1}} c(k-j,j-1)^\alpha_{\mu\beta}\, c(k-1,1)^\gamma_{\alpha\delta}\right) b^{k-j}_\mu.$$

By the Jacobi identity (2.2), if $\mu = 1, \dots, n_{k-j}$,

$$\left[\left[X^{k-j}_\mu, X^1_\delta\right], X^{j-1}_\beta\right] - \left[\left[X^{k-j}_\mu, X^{j-1}_\beta\right], X^1_\delta\right] = \left[X^{k-j}_\mu, \left[X^1_\delta, X^{j-1}_\beta\right]\right].$$

Applying (2.3) to the above three terms, we obtain

$$\left[\left[X^{k-j}_\mu, X^1_\delta\right], X^{j-1}_\beta\right] = \sum_{\alpha=1}^{n_{k-j+1}} \sum_{\eta=1}^{n_k} c(k-j,1)^\alpha_{\mu\delta}\, c(k-j+1, j-1)^\eta_{\alpha\beta}\, X^k_\eta,$$

$$\left[\left[X^{k-j}_\mu, X^{j-1}_\beta\right], X^1_\delta\right] = \sum_{\alpha=1}^{n_{k-1}} \sum_{\eta=1}^{n_k} c(k-j,j-1)^\alpha_{\mu\beta}\, c(k-1,1)^\eta_{\alpha\delta}\, X^k_\eta,$$

$$\left[X^{k-j}_\mu, \left[X^1_\delta, X^{j-1}_\beta\right]\right] = \sum_{\lambda=1}^{n_j} \sum_{\eta=1}^{n_k} c(1,j-1)^\lambda_{\delta\beta}\, c(k-j,j)^\eta_{\mu\lambda}\, X^k_\eta.$$

Since $\{X^k_1, \dots, X^k_{n_k}\}$ is a basis for V_k, we conclude that

$$\sum_{\alpha=1}^{n_{k-j+1}} c(k-j,1)^\alpha_{\mu\delta}\, c(k-j+1,j-1)^\gamma_{\alpha\beta} - \sum_{\alpha=1}^{n_{k-1}} c(k-j,j-1)^\alpha_{\mu\beta}\, c(k-1,1)^\gamma_{\alpha\delta} = \sum_{\lambda=1}^{n_j} c(1,j-1)^\lambda_{\delta\beta}\, c(k-j,j)^\gamma_{\mu\lambda}.$$

In particular,

$$\left[X^1_\delta, X^{j-1}_\beta\right] b^k_\gamma = \sum_{\lambda=1}^{n_j} \sum_{\mu=1}^{n_{k-j}} c(1,j-1)^\lambda_{\delta\beta}\, c(k-j,j)^\gamma_{\mu\lambda}\, b^{k-j}_\mu. \tag{4.6}$$

On the other hand,

$$\left[X^1_\delta, X^{j-1}_\beta\right] b^k_\gamma \overset{(2.3)}{=} \sum_{\lambda=1}^{n_j} c(1,j-1)^\lambda_{\delta\beta}\, X^j_\lambda b^k_\gamma. \tag{4.7}$$

Therefore, comparing (4.6) and (4.7) and setting

$$X = \sum_{\lambda=1}^{n_j} \left(X^j_\lambda b^k_\gamma - \sum_{\mu=1}^{n_{k-j}} c(k-j,j)^\gamma_{\mu\lambda}\, b^{k-j}_\mu \right) X^j_\lambda,$$

we have

$$\left\langle \left[X^1_\delta, X^{j-1}_\beta\right], X \right\rangle = \sum_{\lambda=1}^{n_j} c(1,j-1)^\lambda_{\delta\beta} \left(X^j_\lambda b^k_\gamma - \sum_{\mu=1}^{n_{k-j}} c(k-j,j)^\gamma_{\mu\lambda}\, b^{k-j}_\mu \right) = 0$$

for every $\delta = 1, \dots, n_1$ and $\beta = 1, \dots, n_{j-1}$. This fact and (2.1) grant that $X = 0$, whence (4.5) follows. □

The forthcoming commutator estimates will require the following careful iterations of (4.5).

Proposition 4.3. *Let* $\mathbf{b}$ *satisfy* (4.2). *Let* $k \in \mathbb{N}_+$, $i = 2, \dots, s$, $|I_k| < i$, $A_k \in \mathcal{A}_k(I_k)$, $\beta_k = 1, \dots, n_i$. *Then*

$$X^{I_k}_{A_k} b^i_{\beta_k} = \sum_{\mathcal{B}_{k-1}(i-|I_k|, I_{k-1})} \sum_{\alpha=1}^{n_{i-|I_k|}} c(i, I_k)^{B_k}_{B_{k-1}A_k}\, b^{i-|I_k|}_\alpha \tag{4.8}$$

Proof. We argue by induction on k. We know that

$$X^{i_1}_{\alpha_1} b^i_{\beta_1} \overset{(4.5)}{=} \sum_{\alpha=1}^{n_{i-i_1}} c(i-i_1, i_1)^{\beta_1}_{\alpha\alpha_1}\, b^{i-i_1}_\alpha,$$

which is the thesis for $k = 1$. Then, assume $k > 1$ and that the thesis holds for $k-1$, so that

$$\begin{aligned} X^{i_2}_{\alpha_2} \dots X^{i_k}_{\alpha_k} b^i_{\beta_k} &= \sum_{\substack{(\beta_2,\dots,\beta_{k-1}) \in \mathbb{N}^{k-2} \\ 1 \le \beta_j \le n_{i-i_{j+1}-\dots-i_k}}} \sum_{\beta_1=1}^{n_{i-i_2-\dots-i_k}} c(i-i_k-\dots-i_2, i_2)^{\beta_2}_{\beta_1\alpha_2} \dots c(i-i_k, i_k)^{\beta_k}_{\beta_{k-1}\alpha_k}\, b^{i-i_2-\dots-i_k}_{\beta_1} \\ &= \sum_{\mathcal{B}_{k-1}(i-|I_k|, I_{k-1})} c(i-i_k-\dots-i_2, i_2)^{\beta_2}_{\beta_1\alpha_2} \dots c(i-i_k, i_k)^{\beta_k}_{\beta_{k-1}\alpha_k}\, b^{i-i_2-\dots-i_k}_{\beta_1}. \end{aligned}$$

Differentiating along $X^{i_1}_{\alpha_1}$, we finally obtain

$$\begin{aligned} X^{I_k}_{A_k} b^i_{\beta_k} &= \sum_{\mathcal{B}_{k-1}(i-|I_k|,I_{k-1})} c(i-i_k-\cdots-i_2,i_2)^{\beta_1}_{\beta_1\alpha_2}\dots c(i-i_k,i_k)^{\beta_k}_{\beta_{k-1}\alpha_k} X^{i_1}_{\alpha_1} b^{i-i_2-\cdots-i_k}_{\beta_1} \\ &\overset{(4.5)}{=} \sum_{\mathcal{B}_{k-1}(i-|I_k|,I_{k-1})} \sum_{\alpha=1}^{n_{i-|I_k|}} c(i-|I_k|,i_1)^{\beta_1}_{\alpha\alpha_1} c(i-i_k-\cdots-i_2,i_2)^{\beta_1}_{\beta_1\alpha_2}\dots c(i-i_k,i_k)^{\beta_k}_{\beta_{k-1}\alpha_k} b^{i-|I_k|}_{\alpha} \\ &= \sum_{\mathcal{B}_{k-1}(i-|I_k|,I_{k-1})} \sum_{\alpha=1}^{n_{i-|I_k|}} c(i,I_k)^{B_k}_{B_{k-1}A_k} b^{i-|I_k|}_{\alpha}, \end{aligned}$$

which is the desired formula. $\square$

Remark 4.4. We proved that a vector field $\mathbf{b}$ as in (4.2) satisfies (4.8). On the other hand, if

$$\mathbf{b} = \sum_{i=1}^{s}\sum_{\alpha=1}^{n_i} b^i_\alpha X^i_\alpha, \qquad b^i_\alpha \in L^\theta_{\mathrm{loc}}(\mathbb{G})$$

and (4.8) holds, clearly $\mathbf{b}$ satisfies (4.2). Therefore, (4.2), (4.3), (4.5) and (4.8) are actually equivalent.

4.1. **Further remarks on contact vector fields.** Proposition 4.2 may suggest that the components along the last layer uniquely determine a contact vector field. Actually, this is not true, as the next example shows.

Example 4.5. Denote by $\mathbb{H}^n$ the n-th Heisenberg group and consider the direct product of groups $\mathbb{H}^n \times \mathbb{R}$. Using the notation of Example 3.3 and denoting by ∂_x the canonical vector field on $\mathbb{R}$, we have that $\mathbb{H}^n\times\mathbb{R}$ is a stratified Lie group of step 2, and its stratification is

$$V_1 = \operatorname{span}\left\{X^1_1,\dots,X^1_{2n},\partial_x\right\}, \qquad V_2 = \operatorname{span}\left\{X^2_1\right\}.$$

In this setting, a contact vector field $\mathbf{b}$ decomposes as the sum of a contact vector field on $\mathbb{H}^n$ and a vector field on $\mathbb{R}$. In particular, the component of $\mathbf{b}$ along ∂_x is arbitrary.

Anyway, the orthogonal projection onto the *center* of $\mathfrak{g}$ uniquely determines a contact vector field. Recall that the center $\mathcal{Z}(\mathfrak{g})$ of $\mathfrak{g}$ is the subspace of vector fields $Z\in\mathfrak{g}$ such that $[Z,X]=0$ for every $X\in\mathfrak{g}$.

Proposition 4.6. *Let* $\mathbf{b},\bar{\mathbf{b}}\in\mathbb{X}(\mathbb{G})$ *satisfy* (4.2)*. Assume that* $\langle\mathbf{b},\mathcal{Z}(\mathfrak{g})\rangle = \langle\bar{\mathbf{b}},\mathcal{Z}(\mathfrak{g})\rangle$ *a.e. on* $\mathbb{G}$*. Then* $\mathbf{b}=\bar{\mathbf{b}}$*.*

Proof. First, notice that

$$\mathcal{Z}(\mathfrak{g}) = \mathcal{Z}(\mathfrak{g})\cap V_1\oplus\cdots\oplus\mathcal{Z}(\mathfrak{g})\cap V_s.$$

It is clear that $\mathcal{Z}(\mathfrak{g})\cap V_1\oplus\cdots\oplus\mathcal{Z}(\mathfrak{g})\cap V_s\subseteq\mathcal{Z}(\mathfrak{g})$. Conversely, let $X\in\mathcal{Z}(\mathfrak{g})$, and decompose it uniquely along $V_1,\dots,V_s$ as $X=X_1+\cdots+X_s$. Fix $i=1,\dots,s$ and $Y\in V_i$. Then

$$0=[X,Y]=\sum_{j=1}^{s}[X_j,Y].$$

Noticing that $\langle[X_j,Y],[X_k,Y]\rangle = 0$ for every $j,k=1,\dots,s$ and $j\neq k$, we get $[X_j,Y]=0$ for every $j=1,\dots,s$, whence $X\in\mathcal{Z}(\mathfrak{g})\cap V_1\oplus\cdots\oplus\mathcal{Z}(\mathfrak{g})\cap V_s$. Therefore, up to changing and rearranging the basis, we can assume that $\left\{X^i_1,\dots,X^i_{m_i}\right\}$ is a basis for $\mathcal{Z}(\mathfrak{g})\cap V_i$ for every $i=1,\dots,s$ and a suitable $0\le m_i\le n_i$. By linearity, we can suppose $\bar{\mathbf{b}}=0$. Hence, if $\mathbf{b}=\sum_{i=1}^{s}\sum_{\alpha=1}^{n_i} b^i_\alpha X^i_\alpha$, we have $b^i_\alpha=0$ for every $i=1,\dots,s$ and $\alpha=1,\dots,m_i$. It suffices to prove that $b^k := \left(b^k_1,\dots,b^k_{n_k}\right)=0$ for every $k=1,\dots,s$. We prove this fact by reverse induction on k. Since $V_s\subseteq\mathcal{Z}(\mathfrak{g})$ by definition of stratification, we have $b^s=0$ by assumption. Hence, assume $k<s$ and $b^{k+1}=0$. Fix $p\in\mathbb{G}$ and define the vector field $B=\sum_{\alpha=1}^{n_k} b^k_\alpha(p)X^k_\alpha\in\mathfrak{g}$. Then, we deduce

$$\left[B,X^1_\beta\right] \overset{(2.3)}{=} \sum_{\gamma=1}^{n_{k+1}}\sum_{\alpha=1}^{n_k} c(k,1)^\gamma_{\alpha\beta}\, b^k_\alpha(p) X^{k+1}_\gamma \overset{(4.5)}{=} \sum_{\gamma=1}^{n_{k+1}}\left(X^1_\beta b^{k+1}_\gamma\right)X^{k+1}_\gamma = 0$$

for every $\beta=1,\dots,n_1$, so that $B\in\mathcal{Z}(\mathfrak{g})$. On the other hand, we know that $B=\sum_{\alpha=m_i+1}^{n_i} b^k_\alpha(p)X^k_\alpha$ with $X^k_\alpha\notin\mathcal{Z}(\mathfrak{g})$. Hence, the only possibility is $B=0$. This is the thesis, since $p\in\mathbb{G}$ was arbitrary. $\square$

On the other hand, it is not true that all the central components are free. Although this is well-known from the theory of Tanaka prolongation (see [34]), we include the following example for the sake of completeness.

Example 4.7. Consider $\mathbb{F}_{32}$, i.e., the 6-dimensional stratified Lie group of step 2 such that

$$V_1 = \operatorname{span}\{X_1, X_2, X_3\}, \qquad V_2 = \operatorname{span}\{Y_{12}, Y_{13}, Y_{23}\}, \qquad [X_i, X_j] = Y_{ij} \text{ for every } i, j = 1, 2, 3,\ i \neq j.$$

Let $\mathbf{b} = f_1 X_1 + f_2 X_2 + f_3 X_3 + f_{12} Y_{12} + f_{13} Y_{13} + f_{23} Y_{23} \in \mathbb{X}(\mathbb{F}_{32})$ be a smooth contact vector field. A direct computation shows that, for every $i = 1, 2, 3$,

$$[X_i, \mathbf{b}] = (X_i f_1)X_1 + (X_i f_2)X_2 + (X_i f_3)X_3 + (X_i f_{12})Y_{12} + (X_i f_{13})Y_{13} + (X_i f_{23})Y_{23} + f_1 Y_{i1} + f_2 Y_{i2} + f_3 Y_{i3},$$

where we use the convention $Y_{ij} = 0$ whenever $i = j$. By (4.1), the components of $[X_i, \mathbf{b}]$ along Y_{12}, Y_{13}, Y_{23} must be zero. In particular,

$$X_1 f_{23} = X_2 f_{13} = X_3 f_{12} = 0,$$

which means that f_{12}, f_{13}, f_{23} are not arbitrary.

5. Difference quotients of horizontal Sobolev functions

It is well known that Euclidean Sobolev functions are characterized by the convergence (in the suitable weak sense) of their difference quotients. In this section, we prove the analogous result for horizontal Sobolev functions in Carnot groups. This will be crucial for the regularization property of the next section. We stress that we deal with difference quotients computed along intrinsic dilations, involving certain polynomials which will turn out to be Taylor polynomials in the sense of [17, 8]. Throughout this section, $\Omega \subseteq \mathbb{G}$ is a fixed open set. Moreover, whenever $k \in \mathbb{N}$, $f \in C^k(\Omega)$ and $Z \in \mathfrak{g}$, we adopt the notation

$$Z^k f = \underbrace{Z \dots Z}_{k} f.$$

First, we recall the following two results. See [35, Lemma 2.12.1] for the proof of the first one.

Lemma 5.1. *Let $k \in \mathbb{N}$. Let $f \in C^k(\Omega)$. Fix $p \in \Omega$, $t > 0$ and $Z \in \mathfrak{g}$. Assume that $p \cdot \exp(tZ) \in \Omega$. Then*

$$\frac{d^k}{dt^k} f(p \cdot \exp(tZ)) = Z^k f(p \cdot \exp(tZ)). \tag{5.1}$$

In particular,

$$\left.\frac{d^k}{dt^k}\right|_{t=0} f(p \cdot \exp(tZ)) = Z^k f(p). \tag{5.2}$$

As an immediate consequence of (5.2), observe that, by the Baker-Campbell-Hausdorff formula (cf. [35]), for every $i, j = 1, \dots, s$, $\alpha = 1, \dots, n_i$ and $\beta = 1, \dots, n_j$,

$$X^i_\alpha x^i_\alpha = 1, \qquad X^i_\alpha x^i_\beta = 0 \quad \text{if } \alpha \neq \beta, \qquad X^i_\alpha x^j_\beta = 0 \quad \text{if } j < i. \tag{5.3}$$

Proposition 5.2. *Let $k \in \mathbb{N}_+$ and $f \in C^{k+1}(\Omega)$. Fix $p \in \Omega$ and $W = \sum_{i=1}^{s} \sum_{\alpha=1}^{n_i} w^i_\alpha X^i_\alpha \in \mathfrak{g}$. Then*

$$\left.\frac{d^k}{d\tau^k}\right|_{\tau=0} f(p \cdot \exp(\delta_\tau W)) = k! \sum_{j=1}^{k} \frac{1}{j!} \sum_{\substack{i_1, \dots, i_j = 1 \\ i_1 + \dots + i_j = k}}^{s} \sum_{\alpha_1 = 1}^{n_{i_1}} \cdots \sum_{\alpha_j = 1}^{n_{i_j}} w^{i_1}_{\alpha_1} \dots w^{i_j}_{\alpha_j} X^{i_1}_{\alpha_1} \dots X^{i_j}_{\alpha_j} f(p). \tag{5.4}$$

Proof. Since Ω is open, there exists $\tau > 0$ such that $p \cdot \exp(t\delta_\tau W) \in \Omega$ for every $t \in [0, 1]$. Set $Z = \delta_\tau W$. Then, thanks to (5.1), (5.2) and Taylor's formula with Lagrange remainder,

$$f(p \cdot \exp(tZ)) = \sum_{j=0}^{k} \frac{Z^j f(p)}{j!} t^j + \frac{Z^{k+1} f(p \cdot \exp(\sigma Z))}{(k+1)!} t^{k+1}$$

for a suitable $\sigma \in [0, t]$. In particular, evaluating at $t = 1$,

$$f(p \cdot \exp(Z)) = \sum_{j=0}^{k} \frac{Z^j f(p)}{j!} + \frac{Z^{k+1} f(p \cdot \exp(\sigma Z))}{(k+1)!}.$$

Hence, noticing that, for every $j \geq 1$,

$$Z^j = (\delta_\tau W)^j = \left(\sum_{i=1}^{s} \sum_{\alpha=1}^{n_i} \tau^i w^i_\alpha X^i_\alpha \right)^j = \sum_{i_1, \dots, i_j = 1}^{s} \sum_{\alpha_1 = 1}^{n_{i_1}} \cdots \sum_{\alpha_j = 1}^{n_{i_j}} \tau^{i_1 + \dots + i_j} w^{i_1}_{\alpha_1} \dots w^{i_j}_{\alpha_j} X^{i_1}_{\alpha_1} \dots X^{i_j}_{\alpha_j},$$

we obtain

$$f(p\cdot\exp(\delta_\tau W)) = f(p) + \sum_{j=1}^{k}\frac{1}{j!}\sum_{i_1,\dots,i_j=1}^{s}\sum_{\alpha_1=1}^{n_{i_1}}\cdots\sum_{\alpha_j=1}^{n_{i_j}}\tau^{i_1+\dots+i_j}w^{i_1}_{\alpha_1}\dots w^{i_j}_{\alpha_j}X^{i_1}_{\alpha_1}\dots X^{i_j}_{\alpha_j}f(p)$$
$$+\underbrace{\frac{1}{(k+1)!}\sum_{i_1,\dots,i_{k+1}=1}^{s}\sum_{\alpha_1=1}^{n_{i_1}}\cdots\sum_{\alpha_{k+1}=1}^{n_{i_{k+1}}}\tau^{i_1+\dots+i_{k+1}}w^{i_1}_{\alpha_1}\dots w^{i_{k+1}}_{\alpha_{k+1}}X^{i_1}_{\alpha_1}\dots X^{i_{k+1}}_{\alpha_{k+1}}f(p\cdot\exp(\sigma\delta_\tau W))}_{=:g(\tau)}.$$

Therefore, since $g(\tau)=o\left(\tau^k\right)$ as $\tau\searrow 0$, the thesis follows by the uniqueness of Taylor's polynomial. $\square$

Formula (5.4) motivates the following definition. Given $p\in\Omega$, $q\in\mathbb{G}$ and $f\in C^\infty(\Omega)$, set

$$T^k_p f(q) := \sum_{j=1}^{k}\frac{1}{j!}\sum_{\substack{i_1,\dots,i_j=1\\ i_1+\dots+i_j=k}}^{s}\sum_{\alpha_1=1}^{n_{i_1}}\cdots\sum_{\alpha_j=1}^{n_{i_j}} q^{i_1}_{\alpha_1}\dots q^{i_j}_{\alpha_j}X^{i_1}_{\alpha_1}\dots X^{i_j}_{\alpha_j}f(p) \tag{5.5}$$

and

$$P^\ell_p f(q) := f(p) + \sum_{k=1}^{\ell} T^k_p f(q). \tag{5.6}$$

We say that $P^\ell_p f$ is the *Taylor polynomial of f of order ℓ centered at p*. Combining [8, Corollary 20.2.10] and [8, Proposition 20.3.15],

$$X^{I_k}_{A_k}P^\ell_p f(0) = X^{I_k}_{A_k}f(p) \qquad \text{for every } k\le\ell,\ |I_k|\le\ell \text{ and } A_k\in\mathcal{A}_k(I_k). \tag{5.7}$$

Equivalently, $P^\ell_p f$ is the only polynomial with *homogeneous degree* (cf. [17, 8]) at most ℓ such that (5.7) holds. We can now proceed with the asymptotic analysis of the difference quotients and prove the main result of the section. For any $\varepsilon>0$ and $w\in\mathbb{G}$, set

$$R^w_{\ell,\varepsilon}f(p) := \frac{f\left(p\cdot\delta_\varepsilon(w)\right) - P^\ell_p f\left(\delta_\varepsilon(w)\right)}{\varepsilon^\ell}. \tag{5.8}$$

Theorem 5.3. *Let $\ell\ge 1$ and $\theta\in[1,\infty)$. Let $f\in W^{\ell,\theta}_{\mathcal{H}}(\Omega)$. Let $A\subseteq\Omega$ be open with $\operatorname{dist}(A,\partial\Omega)>0$. Then*

$$\overset{L^\theta(A)}{\lim_{\varepsilon\searrow 0}} R^w_{\ell,\varepsilon}f(p) = 0 \qquad \textit{for every } w\in\mathbb{G}. \tag{5.9}$$

In addition, there exist $C_1=C_1(\mathbb{G})>0$ and $C_2=C_2(\mathbb{G},\ell)>0$ such that, if $\varepsilon>0$ and $w\in\mathbb{G}$ satisfy

$$C_1\,\varepsilon\, d(w,0) < \operatorname{dist}(A,\partial\Omega), \tag{5.10}$$

then

$$\left\|R^w_{\ell,\varepsilon}f\right\|_{L^\theta(A)} \le C_2\, d(w,0)^\ell \sum_{\alpha_1,\dots,\alpha_\ell=1}^{n_1}\left\|X^1_{\alpha_1}\dots X^1_{\alpha_\ell}f\right\|_{L^\theta(\Omega)}. \tag{5.11}$$

Finally, if $f\in W^{\ell,\infty}_{\mathcal{H}}(\Omega)$, then

$$\left\|R^w_{\ell,\varepsilon}f\right\|_{L^\infty(A)} \le C_2\, d(w,0)^\ell \sum_{\alpha_1,\dots,\alpha_\ell=1}^{n_1}\left\|X^1_{\alpha_1}\dots X^1_{\alpha_\ell}f\right\|_{L^\infty(\Omega)}. \tag{5.12}$$

Proof. Assume first that $\theta\in[1,\infty)$. By the density of $C^\infty(\Omega)\cap W^{\ell,\theta}_{\mathcal{H}}(\Omega)$ in $W^{\ell,\theta}_{\mathcal{H}}(\Omega)$ (recall Theorem 2.4), it suffices to prove the thesis when $f\in C^\infty(\Omega)\cap W^{\ell,\theta}_{\mathcal{H}}(\Omega)$. Fix $p\in A$. Since $\operatorname{dist}(A,\partial\Omega)>0$, there exists an open neighborhood of 0, say U, such that the function $F_p: U\to\mathbb{R}$ given by

$$F_p(q) := f(p\cdot q) - P^\ell_p f(q)$$

is well-defined. Notice that

$$X^1_{\alpha_1}\dots X^1_{\alpha_j}F_p(0) \overset{(5.7)}{=} 0 \qquad \text{for every } j=0,\dots,\ell \text{ and } \alpha_1,\dots,\alpha_j=1,\dots,n_1. \tag{5.13}$$

Moreover, by the definition of $P^\ell_p f$,

$$X^1_{\alpha_1}\dots X^1_{\alpha_\ell}F_p(q) = \left(X^1_{\alpha_1}\dots X^1_{\alpha_\ell}f\right)(p\cdot q) - X^1_{\alpha_1}\dots X^1_{\alpha_\ell}f(p) \qquad \text{for every } \alpha_1,\dots\alpha_\ell=1,\dots,n_1. \tag{5.14}$$

By [17, Lemma 1.40], there exist constants $N \in \mathbb{N}_+$ and $C > 0$, both depending only on $\mathbb{G}$, such that, for any $w \in \mathbb{G}$, the following properties hold:

$$\begin{aligned} &w = w_1 \cdot w_2 \cdot \ldots \cdot w_N \qquad \text{for some } w_1, \ldots, w_N \in \exp(V_1), \\ &d(w_j, 0) \le C d(w, 0) \qquad \text{for every } j = 1, \ldots, N. \end{aligned} \tag{5.15}$$

Set $C_1 = NC$. Fix $w \in \mathbb{G}$ and $\varepsilon > 0$ such that (5.10) holds. Define $\gamma : [0,1] \to \mathbb{G}$ by means of the following construction. For every $j = 1, \ldots, N$, set

$$t_0 = 0, \qquad t_j := \frac{j}{N}, \qquad J_j := [t_{j-1}, t_j], \qquad W_j := \log w_j.$$

Define γ by setting

$$\gamma|_{J_j}(\tau) := \delta_\varepsilon(w_1) \cdot \ldots \cdot \delta_\varepsilon(w_{j-1}) \cdot \delta_{\varepsilon N(\tau - t_{j-1})}(w_j).$$

In particular,

$$\dot\gamma|_{J_j}(\tau) = \varepsilon N \, W_j|_{\gamma(\tau)}.$$

By construction, $p \cdot \gamma$ is a horizontal, piecewise smooth curve joining $\gamma(0) = p$ and $\gamma(1) = p \cdot \delta_\varepsilon(w)$. Moreover,

$$d(p \cdot \gamma(\tau), p) = d(\gamma(\tau), 0) \le \varepsilon \sum_{i=1}^N d(w_i, 0) \overset{(5.15)}{\le} NC\varepsilon d(w,0) = C_1 \varepsilon d(w, 0). \tag{5.16}$$

Therefore, by (5.10), $p \cdot \gamma(\tau) \in \Omega$ for every $\tau \in [0,1]$. By the fundamental theorem of calculus, writing $\tau_0 := 1$,

$$F_p(\delta_\varepsilon(w)) \overset{(5.13)}{=} F_p(\gamma(\tau_0)) - F_p(\gamma(0)) = \int_0^{\tau_0} \frac{d}{d\tau_1}(F_p(\gamma(\tau_1))\, d\tau_1 = \varepsilon N \sum_{i_1=1}^N \int_{J_{i_1}} (W_{i_1} F_p)(\gamma(\tau_1))\, d\tau_1.$$

Fix $i_1 = 1, \ldots, N$ and $\tau_1 \in J_{i_1}$. Since $W_{i_1} \in V_1$, (5.13) yields $(W_{i_1} F_p)(0) = 0$. Therefore, arguing as above,

$$(W_{i_1} F_p)(\gamma(\tau_1)) \overset{(5.13)}{=} (W_{i_1} F_p)(\gamma(\tau_1)) - (W_{i_1} F_p)(\gamma(0)) = \varepsilon N \sum_{i_2=1}^N \int_{J_{i_2} \cap [0, \tau_1]} (W_{i_2} W_{i_1} F_p)(\gamma(\tau_2))\, d\tau_2.$$

Combining the above identities,

$$F_p(\delta_\varepsilon(w)) = (\varepsilon N)^2 \sum_{i_1, i_2 = 1}^N \int_{J_{i_1}} \int_{J_{i_2} \cap [0, \tau_1]} (W_{i_2} W_{i_1} F_p)(\gamma(\tau_2))\, d\tau_2\, d\tau_1.$$

Iterating the above procedure ℓ times,

$$F_p(\delta_\varepsilon(w)) = (\varepsilon N)^\ell \sum_{i_1, \ldots, i_\ell = 1}^N \int_{J_{i_1}} \cdots \int_{J_{i_\ell} \cap [0, \tau_{\ell-1}]} (W_{i_\ell} \ldots W_{i_1} F_p)(\gamma(\tau_\ell))\, d\tau_\ell \ldots d\tau_1.$$

Hence, by (5.14) and noticing that $F_p(\delta_\varepsilon(w)) = \varepsilon^\ell R^w_{\ell,\varepsilon} f(p)$,

$$R^w_{\ell,\varepsilon} f(p) = N^\ell \sum_{i_1, \ldots, i_\ell = 1}^N \int_{J_{i_1}} \cdots \int_{J_{i_\ell} \cap [0, \tau_{\ell-1}]} \Big[(W_{i_\ell} \ldots W_{i_1} f)(p \cdot \gamma(\tau_\ell)) - (W_{i_\ell} \ldots W_{i_1} f)(p) \Big]\, d\tau_\ell \ldots d\tau_1.$$

In particular, Minkowski's integral inequality (cf. [26, Corollary B.83]) implies

$$\left\| R^w_{\ell,\varepsilon} f \right\|_{L^\theta(A)} \le N^\ell \sum_{i_1, \ldots, i_\ell = 1}^N \int_0^1 \underbrace{\left(\int_A \left| (W_{i_\ell} \ldots W_{i_1} f)(p \cdot \gamma(\tau)) - (W_{i_\ell} \ldots W_{i_1} f)(p) \right|^\theta dp \right)^{\frac{1}{\theta}}}_{=: g_{i_1, \ldots, i_\ell}(\tau)} d\tau. \tag{5.17}$$

First, (5.16) and Lemma 2.2 imply that

$$g_{i_1, \ldots, i_\ell}(\tau) \to 0 \text{ as } \varepsilon \searrow 0 \text{ for a.e. } \tau \in (0,1). \tag{5.18}$$

For any $i = 1, \ldots, N$, set $W_i = \sum_{j=1}^{n_1} w_i^j X_j^1$ and denote by $|w_i|_{\mathrm{eu}}$ the Euclidean norm of $(w_1^1, \ldots, w_i^{n_1})$. By [8, Example 5.1.2] and [8, Proposition 5.1.4], there exists a constant $\tilde C = \tilde C(\mathbb{G}) > 0$ such that

$$|w_i|_{\mathrm{eu}} \le \tilde C d(w_i, 0). \tag{5.19}$$

Then

$$
\begin{aligned}
|W_{i_\ell}\dots W_{i_1} f(q)| &\le (n_1)^\ell |w_{i_\ell}|_{\mathrm{eu}}\dots|w_{i_1}|_{\mathrm{eu}} \sum_{j_1,\dots,j_\ell=1}^{n_1} \left|X^1_{j_\ell}\dots X^1_{j_1} f(q)\right| \\
&\overset{(5.19)}{\le} \left(\tilde{C} n_1\right)^\ell d(w_{i_\ell},0)\dots d(w_{i_1},0) \sum_{j_1,\dots,j_\ell=1}^{n_1} \left|X^1_{j_\ell}\dots X^1_{j_1} f(q)\right| \\
&\overset{(5.15)}{\le} \left(C\tilde{C} n_1\right)^\ell d(w,0)^\ell \sum_{j_1,\dots,j_\ell=1}^{n_1} \left|X^1_{j_\ell}\dots X^1_{j_1} f(q)\right|.
\end{aligned}
$$

Therefore, (2.5) implies that

$$
g_{i_1,\dots,i_\ell}(\tau) \le 2\left(C\tilde{C} n_1\right)^\ell d(w,0)^\ell \sum_{j_1,\dots,j_\ell=1}^{n_1} \left\|X^1_{j_\ell}\dots X^1_{j_1} f\right\|_{L^\theta(\Omega)}. \tag{5.20}
$$

By (5.17), (5.18), (5.20) and the dominated convergence theorem, (5.9) follows. Moreover, (5.11) follows as well from (5.17) and (5.20). Eventually, assume that $f \in W^{\ell,\infty}_{\mathcal{H}}(\Omega)$. By [19],

$$
X^1_{i_1}\dots X^1_{i_k} f \in C(\Omega) \qquad \text{for every } k = 1,\dots,\ell-1 \text{ and } i_1,\dots,i_k = 1,\dots,n_1. \tag{5.21}
$$

Combining (5.21) with [32, Proposition 2.4] and [32, Proposition 2.5], one argues as above to deduce that

$$
R^w_{\ell,\varepsilon}(p) = \frac{N^{\ell-1}}{\varepsilon} \sum_{i_1,\dots,i_{\ell-1}=1}^{N} \int_{J_{i_1}} \cdots \int_{J_{i_{\ell-1}}\cap[0,\tau_{\ell-2}]} \left(W_{i_{\ell-1}}\dots W_{i_1} F_p\right)(\gamma(\tau_{\ell-1}))\, d\tau_{\ell-1}\dots d\tau_1. \tag{5.22}
$$

Combining (5.13) and (5.16) with [10, Proposition 3.4], [32, Proposition 2.4] and [32, Proposition 2.5],

$$
\left|\left(W_{i_{\ell-1}}\dots W_{i_1} F_p\right)(\gamma(\tau_{\ell-1}))\right| \le 2\varepsilon N \left\|W_{i_\ell}\dots W_{i_1} f\right\|_{L^\infty(\Omega)}. \tag{5.23}
$$

By (5.22) and (5.23), (5.12) follows as in the previous case. □

6. Well-posedness for transport and flow equations driven by contact velocities

In this section, we prove that contact vector fields with horizontal Sobolev regularity satisfy the renormalization property. As a consequence, we obtain well-posedness for the transport and flow equations. We provide only the proofs specific to our setting, as the remaining arguments follows as in Euclidean spaces and Heisenberg groups. We refer to [15] and [5], respectively, for the corresponding proofs and further details.

6.1. **Transport equation, distributional solutions, and renormalized solutions.** Fix $\bar{\tau} \in (0,\infty]$. Given $\alpha,\beta \in [1,\infty]$, we consider (see [26]) the space $L^\alpha\left(0,\bar{\tau}; L^\beta(\mathbb{G})\right)$ of Borel functions $u : (0,\bar{\tau}) \times \mathbb{G} \to \mathbb{R}$ such that

$$
\|u\|_{L^\alpha(0,\bar{\tau};L^\beta(\mathbb{G}))} := \begin{cases} \left(\displaystyle\int_0^{\bar{\tau}} \|u(\cdot,\tau)\|^\alpha_{L^\beta(\mathbb{G})}\, d\tau\right)^{\frac{1}{\alpha}} & \text{if } \alpha \in [1,\infty) \\ \operatorname*{ess\,sup}_{\tau\in(0,\bar{\tau})} \|u(\cdot,\tau)\|_{L^\beta(\mathbb{G})} & \text{if } \alpha = \infty \end{cases}
$$

is finite. Its local version $L^\alpha\left(0,\bar{\tau}; L^\beta_{\mathrm{loc}}(\mathbb{G})\right)$ is defined similarly. Moreover, when $\alpha = \beta$, $L^\alpha\left(0,\bar{\tau}; L^\alpha(\mathbb{G})\right)$ naturally identifies with $L^\alpha\left((0,\bar{\tau}) \times \mathbb{G}\right)$. The Cauchy problem for the *transport equation* reads

$$
\begin{cases} \dfrac{\partial u}{\partial \tau} - \langle \mathbf{b}, \nabla u\rangle + cu = 0 & \text{in } (0,\bar{\tau}) \times \mathbb{G} \\ u(0,\cdot) = u_0 & \text{in } \mathbb{G}, \end{cases} \tag{$\mathscr{T}$}
$$

where a velocity field $\mathbf{b} \in \mathbb{X}(\mathbb{G})$, a reaction term c and an initial condition u_0 are given. Fix $\theta \in [1,\infty]$. From now on, we suppose that

$$
\mathbf{b} \in L^1\left(0,\bar{\tau}; \left[L^{\theta'}_{\mathrm{loc}}(\mathbb{G})\right]^n\right), \qquad c, \operatorname{div}\mathbf{b} \in L^1\left(0,\bar{\tau}; L^{\theta'}_{\mathrm{loc}}(\mathbb{G})\right), \qquad u_0 \in L^\theta_{\mathrm{loc}}(\mathbb{G}). \tag{6.1}
$$

We emphasize that, in the first assumption above (and similarly in the analogous ones below), the integrability of $\mathbf{b}$ is understood component-wise with respect to the fixed left-invariant basis. We say that $u \in L^\infty\left(0,\bar{\tau}; L^\theta_{\mathrm{loc}}(\mathbb{G})\right)$ ($u \in L^\infty((0,\bar{\tau}) \times \mathbb{G})$ if $\theta = \infty$) is a *distributional solution to* ($\mathscr{T}$) if

$$
\langle \mathscr{T}_{u_0,\mathbf{b},c}(u), \varphi\rangle := -\int_{\mathbb{G}} u_0(p)\varphi(0,p)\, dp + \int_0^{\bar{\tau}} \int_{\mathbb{G}} u\left[-\partial_\tau \varphi + \langle \mathbf{b}, \nabla\varphi\rangle + (c + \operatorname{div}\mathbf{b})\varphi\right] dp\, d\tau = 0
$$

for every $\varphi \in C_c^\infty([0,\bar\tau)\times\mathbb{G})$. The existence of solutions to ($\mathscr{T}$) is ensured by the next result, similar to [15, Proposition II.1].

Proposition 6.1. *Let $\theta\in[1,\infty]$ and assume (6.1). Assume, in addition, that $u_0\in L^\theta(\mathbb{G})$ and*

$$\begin{cases} c+\frac{1}{\theta}\operatorname{div}\mathbf{b}\in L^1(0,\bar\tau;L^\infty(\mathbb{G})) & \text{if } \theta\in(1,\infty]\\ c,\operatorname{div}\mathbf{b}\in L^1(0,\bar\tau;L^\infty(\mathbb{G})) & \text{if } \theta=1.\end{cases}$$

Then there exists a distributional solution $u\in L^\infty\left(0,\bar\tau;L^\theta(\mathbb{G})\right)$ to ($\mathscr{T}$).

The following property (cf. [13]) allows us to handle the initial condition. See [5, Proposition 4.3] for the proof.

Proposition 6.2. *Let $u\in L^\infty\left(0,\bar\tau;L^\theta_{\mathrm{loc}}(\mathbb{G})\right)$ be a distributional solution to ($\mathscr{T}$). Extend $\mathbf{b},c,u$ to $(-\infty,\bar\tau)\times\mathbb{G}$ by setting*

$$\mathbf{b}(\tau,p)=\begin{cases}0 & \text{if } \tau<0\\ \mathbf{b}(\tau,p) & \text{otherwise,}\end{cases}\qquad c(\tau,p)=\begin{cases}0 & \text{if } \tau<0\\ c(\tau,p) & \text{otherwise,}\end{cases}\qquad u(\tau,p)=\begin{cases}u_0(p) & \text{if } \tau<0\\ u(\tau,p) & \text{otherwise.}\end{cases}\tag{6.2}$$

Then

$$\partial_\tau u-\langle\mathbf{b},\nabla u\rangle+cu=0\qquad \text{in } (-\infty,\bar\tau)\times\mathbb{G}$$

in the sense of distributions, namely $\langle\mathscr{T}_{\mathbf{b},c}(u),\varphi\rangle=0$ for every $\varphi\in C_c^\infty((-\infty,\bar\tau)\times\mathbb{G})$, where

$$\langle\mathscr{T}_{\mathbf{b},c}(u),\varphi\rangle:=\int_{-\infty}^{\bar\tau}\int_{\mathbb{G}}u\left[-\partial_\tau\varphi+\langle\mathbf{b},\nabla\varphi\rangle+(c+\operatorname{div}\mathbf{b})\varphi\right]dp\,d\tau.\tag{6.3}$$

We recall the notion of renormalized solution, introduced by DiPerna and Lions in [15]. A function $u\in L^\infty\left(0,\bar\tau;L^\theta(\mathbb{G})\right)$ is a *renormalized solution* to ($\mathscr{T}$) if, for every $\beta\in C^1(\mathbb{R})$ with β' bounded, the function $\beta(u)$ is a distributional solution to

$$\begin{cases}\dfrac{\partial\beta(u)}{\partial\tau}-\langle\mathbf{b},\nabla\beta(u)\rangle+cu\beta'(u)=0 & \text{in } (0,\bar\tau)\times\mathbb{G}\\ \beta(u)(0,\cdot)=\beta(u_0) & \text{in } \mathbb{G},\end{cases}$$

namely if, for every $\varphi\in C_c^\infty\left([0,\bar\tau)\times\mathbb{G}\right)$,

$$-\int_{\mathbb{G}}\beta(u_0)(p)\varphi(0,p)\,dp+\int_0^{\bar\tau}\int_{\mathbb{G}}\beta(u)\left[-\partial_\tau\varphi+\langle\mathbf{b},\nabla\varphi\rangle+\operatorname{div}\mathbf{b}\,\varphi\right]+cu\beta'(u)\varphi\,dp\,d\tau=0.$$

If $\theta=\infty$, the condition that β' is bounded is not required. Choosing β to be the identity map, it is clear that renormalized solutions are distributional solutions. The converse implication does not hold in general (see [14]). However, if it holds for all u_0 and c as in (6.1), we say that $\mathbf{b}$ enjoys the *renormalization property*. We show that contact vector fields with horizontal Sobolev regularity have the renormalization property.

Theorem 6.3. *Let $\theta\in[1,\infty]$ and $u_0\in L^\theta(\mathbb{G})$. Assume that $\mathbf{b}\in L^1\left(0,\bar\tau;\left[W^{1,\theta'}_{\mathcal{H},\mathrm{loc}}(\mathbb{G})\right]^n\right)$ is a time-dependent contact vector field and $c\in L^1\left(0,\bar\tau;L^{\theta'}_{\mathrm{loc}}(\mathbb{G})\right)$. Then any distributional solution $u\in L^\infty\left(0,\bar\tau;L^\theta(\mathbb{G})\right)$ to ($\mathscr{T}$) is a renormalized solution. In particular, $\mathbf{b}$ has the renormalization property.*

6.2. **Regularization.** The proof of Theorem 6.3 follows the mollification scheme of [15]: mollifying a distributional solution to the transport equation yields a smooth (in space) solution up to an error term, known as commutator. Here, mollification is meant with respect to the group convolution as in Section 2.3. The next result provides an integral representation of the commutator. Its proof is analogous to [5, Proposition 4.9].

Proposition 6.4. *For $\theta\in[1,\infty]$, assume (6.1). Fix a distributional solution $u\in L^\infty\left(0,\bar\tau;L^\theta(\mathbb{G})\right)$ to ($\mathscr{T}$) and extend $\mathbf{b},c,u$ to $(-\infty,\bar\tau)\times\mathbb{G}$ as in (6.2). For every $\varepsilon>0$, let $u_\varepsilon=u*\varrho_\varepsilon$ be the spatial group mollification of u by ϱ_ε, with ϱ_ε as in (2.9). Then, the distribution $\mathscr{T}_{\mathbf{b},c}(u_\varepsilon)$ in (6.3) is representable by integration of the* commutator *$\mathscr{C}_\varepsilon$, i.e.,*

$$\langle\mathscr{T}_{\mathbf{b},c}(u_\varepsilon),\varphi\rangle=\int_{-\infty}^{\bar\tau}\int_{\mathbb{G}}\varphi(\tau,p)\mathscr{C}_\varepsilon(\tau,p)\,dp\,d\tau\qquad \text{for every } \varphi\in C_c^\infty((-\infty,\bar\tau)\times\mathbb{G}),$$

where $\mathscr{C}_\varepsilon = \mathscr{C}_\varepsilon^1 + \mathscr{C}_\varepsilon^2 \in L^1\left(-\infty, \bar{\tau}, L^1_{\mathrm{loc}}(\mathbb{G})\right)$ *is defined by*

$$\mathscr{C}_\varepsilon^1(\tau,p) \coloneqq -\left((u\operatorname{div}\mathbf{b}) * \varrho_\varepsilon\right)(\tau,p) - \int_{\mathbb{G}} u\left(\tau, p\cdot q^{-1}\right)\sum_{i=1}^{s}\sum_{\alpha=1}^{n_i}\left[b^i_\alpha(\tau,p)X^i_\alpha\varrho_\varepsilon(q) - b^i_\alpha\left(\tau,p\cdot q^{-1}\right)\left(X^i_\alpha\right)^r\varrho_\varepsilon(q)\right]dq,$$

$$\mathscr{C}_\varepsilon^2(\tau,p) \coloneqq \int_{\mathbb{G}} u\left(\tau,p\cdot q^{-1}\right)\varrho_\varepsilon(q)\left[c(\tau,p) - c\left(\tau,p\cdot q^{-1}\right)\right]dq$$

for a.e. $(\tau,p)\in(-\infty,\bar{\tau})\times\mathbb{G}$.

The following lemma constitutes the crucial step in the proof of Theorem 6.3: the commutator vanishes in the limit, provided that the velocity field is a contact vector field. Recall that, if $\mathbf{b}\in\left[W^{1,\theta}_{\mathcal{H},\mathrm{loc}}(\mathbb{G})\right]^n$ is a contact vector field, then (4.8) holds. Recall also the notation (3.2) and (3.4).

Lemma 6.5. *Let* $\theta\in[1,\infty]$. *Fix* $u_0\in L^\theta_{\mathrm{loc}}(\mathbb{G})$. *Let* $\mathbf{b}\in L^1\left(0,\bar{\tau};\left[W^{1,\theta'}_{\mathcal{H},\mathrm{loc}}(\mathbb{G})\right]^n\right)$ *be a time-dependent contact vector field and* $c\in L^1\left(0,\bar{\tau};L^{\theta'}_{\mathrm{loc}}(\mathbb{G})\right)$. *For* $u\in L^\infty\left(0,\bar{\tau};L^\theta_{\mathrm{loc}}(\mathbb{G})\right)$, *extend* $\mathbf{b},c,u$ *to* $(-\infty,\bar{\tau})\times\mathbb{G}$ *as in* (6.2). *Then* $\mathscr{C}_\varepsilon\to 0$ *in* $L^1\left(-\infty,\bar{\tau};L^1_{\mathrm{loc}}(\mathbb{G})\right)$ *as* $\varepsilon\searrow 0$.

Proof. Every argument will be carried out for τ fixed, so we assume without loss of generality that $u,\mathbf{b}$ and c do not depend on τ. We only prove the local L^1-convergence of $\mathscr{C}^1_\varepsilon$, since the proof for $\mathscr{C}^2_\varepsilon$ is identical to the corresponding case in [5, Lemma 4.10]. For $p\in\mathbb{G}$, one has

$$\begin{aligned}
-\mathscr{C}^1_\varepsilon(p) &= \left((u\operatorname{div}\mathbf{b})*\varrho_\varepsilon\right)(p) + \int_{\mathbb{G}} u\left(p\cdot q^{-1}\right)\sum_{i=1}^{s}\sum_{\alpha=1}^{n_i}\left[b^i_\alpha(p)X^i_\alpha\varrho_\varepsilon(q) - b^i_\alpha\left(p\cdot q^{-1}\right)\left(X^i_\alpha\right)^r\varrho_\varepsilon(q)\right]dq\\
&\overset{(3.5)}{=} \left((u\operatorname{div}\mathbf{b})*\varrho_\varepsilon\right)(p) + \sum_{i=1}^{s}\sum_{\alpha=1}^{n_i}\Bigg\{\int_{\mathbb{G}}\left[b^i_\alpha(p) - b^i_\alpha\left(p\cdot q^{-1}\right)\right]u\left(p\cdot q^{-1}\right)\left(X^i_\alpha\right)^r\varrho_\varepsilon(q)\,dq\\
&\quad+\sum_{k=1}^{s-i}\frac{1}{k!}\sum_{|I_k|\le s-i}\sum_{\mathcal{A}_k(I_k)}\sum_{\mathcal{B}_k(i,I_k)} c(I_k,i+|I_k|)^{B_k}_{A_kB_{k-1}}\int_{\mathbb{G}} b^i_\alpha(p)u\left(p\cdot q^{-1}\right)q^{I_k}_{A_k}\left(X^{i+|I_k|}_{\beta_k}\right)^r\varrho_\varepsilon(q)\,dq\Bigg\}\\
&\overset{(2.10)}{=} \left((u\operatorname{div}\mathbf{b})*\varrho_\varepsilon\right)(p) + \sum_{i=1}^{s}\sum_{\alpha=1}^{n_i}\Bigg\{\int_{\mathbb{G}}\frac{b^i_\alpha(p) - b^i_\alpha\left(p\cdot q^{-1}\right)}{\varepsilon^{Q+i}}u\left(p\cdot q^{-1}\right)\left(\left(X^i_\alpha\right)^r\varrho\right)\left(\delta_{\frac{1}{\varepsilon}}(q)\right)dq\\
&\quad+\sum_{k=1}^{s-i}\frac{1}{k!}\sum_{|I_k|\le s-i}\sum_{\mathcal{A}_k(I_k)}\sum_{\mathcal{B}_k(i,I_k)} c(I_k,i+|I_k|)^{B_k}_{A_kB_{k-1}}\int_{\mathbb{G}}\frac{b^i_\alpha(p)}{\varepsilon^{Q+i+|I_k|}}u\left(p\cdot q^{-1}\right)q^{I_k}_{A_k}\left(\left(X^{i+|I_k|}_{\beta_k}\right)^r\varrho\right)\left(\delta_{\frac{1}{\varepsilon}}(q)\right)dq\Bigg\}.
\end{aligned}$$

We perform the change of variables $w=\delta_{\frac{1}{\varepsilon}}(q)$. Observe that

$$q^{I_k}_{A_k} = q^{i_1}_{\alpha_1}\dots q^{i_k}_{\alpha_k} = \varepsilon^{i_1+\dots+i_k}w^{i_1}_{\alpha_1}\dots w^{i_k}_{\alpha_k} = \varepsilon^{|I_k|}w^{I_k}_{A_k}.$$

Since $\left(\left(X^i_\alpha\right)^r\varrho\right)\left(w^{-1}\right) = -X^i_\alpha\varrho(w)$ for every $i=1,\dots,s$ and $\alpha=1,\dots,n_i$ by Lemma 2.1 and (2.8), then

$$\begin{aligned}
-\mathscr{C}^1_\varepsilon(p) &\overset{(2.6)}{=} \left((u\operatorname{div}\mathbf{b})*\varrho_\varepsilon\right)(p) + \sum_{i=1}^{s}\sum_{\alpha=1}^{n_i}\Bigg\{\int_{\mathbb{G}}\frac{b^i_\alpha(p) - b^i_\alpha\left(p\cdot\delta_\varepsilon\left(w^{-1}\right)\right)}{\varepsilon^i}u\left(p\cdot\delta_\varepsilon\left(w^{-1}\right)\right)\left(X^i_\alpha\right)^r\varrho(w)\,dw\\
&\quad+\sum_{k=1}^{s-i}\frac{1}{k!}\sum_{|I_k|\le s-i}\sum_{\mathcal{A}_k(I_k)}\sum_{\mathcal{B}_k(i,I_k)} c(I_k,i+|I_k|)^{B_k}_{A_kB_{k-1}}\int_{\mathbb{G}}\frac{b^i_\alpha(p)}{\varepsilon^i}u\left(p\cdot\delta_\varepsilon\left(w^{-1}\right)\right)w^{I_k}_{A_k}\left(X^{i+|I_k|}_{\beta_k}\right)^r\varrho(w)\,dw\Bigg\}\\
&= \left((u\operatorname{div}\mathbf{b})*\varrho_\varepsilon\right)(p) + \sum_{i=1}^{s}\sum_{\alpha=1}^{n_i}\Bigg\{\underbrace{\int_{\mathbb{G}}\frac{b^i_\alpha(p\cdot\delta_\varepsilon(w)) - b^i_\alpha(p)}{\varepsilon^i}u(p\cdot\delta_\varepsilon(w))X^i_\alpha\varrho(w)\,dw}_{=:C^i_\alpha(p)}\\
&\quad\underbrace{-\sum_{k=1}^{s-i}\frac{1}{k!}\sum_{|I_k|\le s-i}\sum_{\mathcal{A}_k(I_k)}\sum_{\mathcal{B}_k(i,I_k)}(-1)^k c(I_k,i+|I_k|)^{B_k}_{A_kB_{k-1}}\int_{\mathbb{G}}\frac{b^i_\alpha(p)}{\varepsilon^i}u(p\cdot\delta_\varepsilon(w))w^{I_k}_{A_k}X^{i+|I_k|}_{\beta_k}\varrho(w)\,dw\Bigg\}}_{=:D^i_\alpha(p)}.
\end{aligned}$$

First, we focus on C^i_α. Recalling (5.6), we have

$$C^i_\alpha(p) = \underbrace{\int_{\mathbb{G}} \frac{b^i_\alpha(p\cdot\delta_\varepsilon(w)) - P^{i-1}_p b^i_\alpha(\delta_\varepsilon(w))}{\varepsilon^i} u(p\cdot\delta_\varepsilon(w)) X^i_\alpha \varrho(w)\, dw}_{=:F^i_\alpha(p)}$$
$$+ \underbrace{\int_{\mathbb{G}} \sum_{\ell=1}^{i-1} T^\ell_p b^i_\alpha(\delta_\varepsilon(w))\varepsilon^{-i} u(p\cdot\delta_\varepsilon(w)) X^i_\alpha \varrho(w)\, dw}_{=:G^i_\alpha(p)},$$

where we agree that, when $i=1$, $P^0_p b^1_\alpha(\delta_\varepsilon(w)) = b^1_\alpha(p)$ and $G^1_\alpha(p) = 0$ for every $\alpha = 1,\dots,n_1$. We compute the L^1_{loc}-limit of F^i_α. Assume first $\theta' < \infty$. Since $\mathbf{b}$ is a contact vector field, $b^i_\alpha \in W^{i,\theta'}_{\mathcal{H},\mathrm{loc}}(\mathbb{G})$ by Proposition 4.1. Recalling (5.5) and (5.8), the dominated convergence theorem and Theorem 5.3 imply that, for every $K \Subset \mathbb{G}$,

$$\int_K \left| \int_{\mathbb{G}} \left[\frac{b^i_\alpha(p\cdot\delta_\varepsilon(w)) - P^{i-1}_p b^i_\alpha(\delta_\varepsilon(w))}{\varepsilon^i} u(p\cdot\delta_\varepsilon(w)) - T^i_p b^i_\alpha(w) u(p) \right] X^i_\alpha\varrho(w)\, dw \right| dp$$
$$\le \int_{\mathbb{G}} \left| X^i_\alpha \varrho(w) \right| \left(\int_K \left| \frac{b^i_\alpha(p\cdot\delta_\varepsilon(w)) - P^{i-1}_p b^i_\alpha(\delta_\varepsilon(w))}{\varepsilon^i} u(p\cdot\delta_\varepsilon(w)) - T^i_p b^i_\alpha(w) u(p) \right| dp \right) dw \xrightarrow[\varepsilon\searrow 0]{} 0,$$

where we also used the L^θ-continuity of right translations (Lemma 2.2) to replace $u(p\cdot\delta_\varepsilon(w))$ with $u(p)$ before applying Theorem 5.3. If instead $\theta' = \infty$, the argument is similar to the corresponding case in the proof of [5, Lemma 4.10]. Hence,

$$\begin{aligned}
\overset{L^1_{\mathrm{loc}}}{\lim_{\varepsilon\searrow 0}} \sum_{i=1}^s \sum_{\alpha=1}^{n_i} F^i_\alpha(p) &= \sum_{i=1}^s \sum_{\alpha=1}^{n_i} \int_{\mathbb{G}} T^i_p b^i_\alpha(w) u(p) X^i_\alpha \varrho(w)\, dw \\
&\overset{(5.5)}{=} \sum_{i=1}^s \sum_{\alpha=1}^{n_i} \sum_{j=1}^i \frac{1}{j!} \sum_{\substack{i_1,\dots,i_j=1 \\ i_1+\dots+i_j=i}}^s \sum_{\alpha_1=1}^{n_{i_1}} \cdots \sum_{\alpha_j=1}^{n_{i_j}} X^{i_1}_{\alpha_1}\dots X^{i_j}_{\alpha_j} b^i_\alpha(p) u(p) \int_{\mathbb{G}} w^{i_1}_{\alpha_1}\dots w^{i_j}_{\alpha_j} X^i_\alpha \varrho(w)\, dw \\
&\overset{(5.3)}{=} \sum_{i=1}^s \sum_{\alpha=1}^{n_i} \Bigg\{ \sum_{\alpha_1=1}^{n_i} X^i_{\alpha_1} b^i_\alpha(p) u(p) \int_{\mathbb{G}} w^i_{\alpha_1} X^i_\alpha \varrho(w)\, dw \\
&\quad + \sum_{j=2}^i \frac{1}{j!} \sum_{\substack{i_1,\dots,i_j=1 \\ i_1+\dots+i_j=i}}^s \sum_{\alpha_1=1}^{n_{i_1}} \cdots \sum_{\alpha_j=1}^{n_{i_j}} X^{i_1}_{\alpha_1}\dots X^{i_j}_{\alpha_j} b^i_\alpha(p) u(p) \underbrace{\int_{\mathbb{G}} X^i_\alpha \left(w^{i_1}_{\alpha_1}\dots w^{i_j}_{\alpha_j} \varrho(w) \right) dw}_{=0} \Bigg\} \\
&\overset{(5.3)}{=} \sum_{i=1}^s \sum_{\alpha=1}^{n_i} \sum_{\substack{\alpha_1=1 \\ \alpha_1\neq\alpha}}^{n_i} X^i_{\alpha_1} b^i_\alpha(p) u(p) \underbrace{\int_{\mathbb{G}} X^i_\alpha \left(w^i_{\alpha_1} \varrho(w) \right) dw}_{=0} + \sum_{i=1}^s \sum_{\alpha=1}^{n_i} X^i_\alpha b^i_\alpha(p) u(p) \int_{\mathbb{G}} w^i_\alpha X^i_\alpha \varrho(w)\, dw \\
&= \sum_{i=1}^s \sum_{\alpha=1}^{n_i} X^i_\alpha b^i_\alpha(p) u(p) \Bigg(\underbrace{\int_{\mathbb{G}} X^i_\alpha \left(w^i_\alpha \varrho(w) \right) dw}_{=0} - \int_{\mathbb{G}} \varrho(w) \underbrace{X^i_\alpha w^i_\alpha}_{\overset{(5.3)}{=} 1} dw \Bigg) \\
&= -u(p) \sum_{i=1}^s \sum_{\alpha=1}^{n_i} X^i_\alpha b^i_\alpha(p) \\
&\overset{(2.11)}{=} -u(p) \operatorname{div} \mathbf{b}(p).
\end{aligned}$$

It remains to analyze $D^i_\alpha(p)+G^i_\alpha(p)$. Observe that

$$\begin{aligned}\sum_{i=1}^{s}\sum_{\alpha=1}^{n_i}&\left[D^i_\alpha(p)+G^i_\alpha(p)\right]\\ &=-\sum_{i=1}^{s-1}\sum_{\alpha=1}^{n_i}\sum_{k=1}^{s-i}\frac{1}{k!}\sum_{|I_k|\le s-i}\sum_{\mathcal{A}_k(I_k)}\sum_{\mathcal{B}_k(i,I_k)}(-1)^k c(I_k,i+|I_k|)^{B_k}_{A_kB_{k-1}}\int_{\mathbb{G}}\frac{b^i_\alpha(p)}{\varepsilon^i}u(p\cdot\delta_\varepsilon(w))w^{I_k}_{A_k}X^{i+|I_k|}_{\beta_k}\varrho(w)\,dw\\ &\quad+\sum_{i=2}^{s}\sum_{\alpha=1}^{n_i}\sum_{\ell=1}^{i-1}\int_{\mathbb{G}}T^\ell_p b^i_\alpha(\delta_\varepsilon(w))\varepsilon^{-i}u(p\cdot\delta_\varepsilon(w))X^i_\alpha\varrho(w)\,dw.\end{aligned}$$

For the first term, we get

$$\begin{aligned}&\sum_{i=1}^{s-1}\sum_{\alpha=1}^{n_i}\sum_{k=1}^{s-i}\frac{1}{k!}\sum_{|I_k|\le s-i}\sum_{\mathcal{A}_k(I_k)}\sum_{\mathcal{B}_k(i,I_k)}(-1)^k c(I_k,i+|I_k|)^{B_k}_{A_kB_{k-1}}\int_{\mathbb{G}}\frac{b^i_\alpha(p)}{\varepsilon^i}u(p\cdot\delta_\varepsilon(w))w^{I_k}_{A_k}X^{i+|I_k|}_{\beta_k}\varrho(w)\,dw\\ &\overset{(2.4)}{=}\sum_{i=1}^{s-1}\sum_{\alpha=1}^{n_i}\sum_{k=1}^{s-i}\frac{1}{k!}\sum_{|I_k|\le s-i}\sum_{\mathcal{A}_k(I_k)}\sum_{\mathcal{B}_k(i,I_k)}c(i+|I_k|,I_k)^{B_k}_{B_{k-1}A_k}\frac{b^i_\alpha(p)}{\varepsilon^{i+|I_k|}}\int_{\mathbb{G}}u(p\cdot\delta_\varepsilon(w))\delta_\varepsilon\left(w^{I_k}_{A_k}\right)X^{i+|I_k|}_{\beta_k}\varrho(w)\,dw\\ &=\sum_{i=1}^{s-1}\sum_{\alpha=1}^{n_i}\sum_{\ell=1}^{s-i}\sum_{k=1}^{\ell}\frac{1}{k!}\sum_{|I_k|=\ell}\sum_{\mathcal{A}_k(I_k)}\sum_{\mathcal{B}_k(i,I_k)}c(i+\ell,I_k)^{B_k}_{B_{k-1}A_k}\frac{b^i_\alpha(p)}{\varepsilon^{i+\ell}}\int_{\mathbb{G}}u(p\cdot\delta_\varepsilon(w))\delta_\varepsilon\left(w^{I_k}_{A_k}\right)X^{i+\ell}_{\beta_k}\varrho(w)\,dw\\ &=\sum_{i=2}^{s}\sum_{\ell=1}^{i-1}\sum_{k=1}^{\ell}\frac{1}{k!}\sum_{|I_k|=\ell}\sum_{\mathcal{A}_k(I_k)}\sum_{\mathcal{B}_k(i-\ell,I_k)}\sum_{\alpha=1}^{n_{i-\ell}}c(i,I_k)^{B_k}_{B_{k-1}A_k}\frac{b^{i-\ell}_\alpha(p)}{\varepsilon^i}\int_{\mathbb{G}}\delta_\varepsilon\left(w^{I_k}_{A_k}\right)u(p\cdot\delta_\varepsilon(w))X^i_{\beta_k}\varrho(w)\,dw\\ &=\sum_{i=2}^{s}\sum_{\ell=1}^{i-1}\sum_{k=1}^{\ell}\frac{1}{k!}\sum_{|I_k|=\ell}\sum_{\mathcal{A}_k(I_k)}\sum_{\beta_k=1}^{n_i}\sum_{\mathcal{B}_{k-1}(i-\ell,I_{k-1})}\sum_{\alpha=1}^{n_{i-\ell}}c(i,I_k)^{B_k}_{B_{k-1}A_k}\frac{b^{i-\ell}_\alpha(p)}{\varepsilon^i}\int_{\mathbb{G}}\delta_\varepsilon\left(w^{I_k}_{A_k}\right)u(p\cdot\delta_\varepsilon(w))X^i_{\beta_k}\varrho(w)\,dw.\end{aligned}$$

Instead, the second one is

$$\begin{aligned}\sum_{i=2}^{s}\sum_{\alpha=1}^{n_i}\sum_{\ell=1}^{i-1}&\int_{\mathbb{G}}T^\ell_p b^i_\alpha(\delta_\varepsilon(w))\varepsilon^{-i}u(p\cdot\delta_\varepsilon(w))X^i_\alpha\varrho(w)\,dw\\ &\overset{(5.5)}{=}\sum_{i=2}^{s}\sum_{\alpha=1}^{n_i}\sum_{\ell=1}^{i-1}\sum_{k=1}^{\ell}\frac{1}{k!}\sum_{|I_k|=\ell}\sum_{\mathcal{A}_k(I_k)}\int_{\mathbb{G}}\delta_\varepsilon\left(w^{I_k}_{A_k}\right)X^{I_k}_{A_k}b^i_\alpha(p)\varepsilon^{-i}u(p\cdot\delta_\varepsilon(w))X^i_\alpha\varrho(w)\,dw\\ &=\sum_{i=2}^{s}\sum_{\ell=1}^{i-1}\sum_{k=1}^{\ell}\frac{1}{k!}\sum_{|I_k|=\ell}\sum_{\mathcal{A}_k(I_k)}\sum_{\beta_k=1}^{n_i}X^{I_k}_{A_k}b^i_{\beta_k}(p)\int_{\mathbb{G}}\delta_\varepsilon\left(w^{I_k}_{A_k}\right)u(p\cdot\delta_\varepsilon(w))\varepsilon^{-i}X^i_{\beta_k}\varrho(w)\,dw.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{i=1}^{s}\sum_{\alpha=1}^{n_i}&\left[D^i_\alpha(p)+G^i_\alpha(p)\right]\\ &=\sum_{i=2}^{s}\sum_{\ell=1}^{i-1}\sum_{k=1}^{\ell}\frac{1}{k!}\sum_{|I_k|=\ell}\sum_{\mathcal{A}_k(I_k)}\sum_{\beta_k=1}^{n_i}\\ &\quad\left[X^{I_k}_{A_k}b^i_{\beta_k}(p)-\sum_{\mathcal{B}_{k-1}(i-\ell,I_{k-1})}\sum_{\alpha=1}^{n_{i-\ell}}c(i,I_k)^{B_k}_{B_{k-1}A_k}b^{i-\ell}_\alpha(p)\right]\int_{\mathbb{G}}\delta_\varepsilon\left(w^{I_k}_{A_k}\right)u(p\cdot\delta_\varepsilon(w))\varepsilon^{-i}X^i_{\beta_k}\varrho(w)\,dw\\ &\overset{(4.8)}{=}0.\end{aligned}$$

In conclusion,

$$\overset{L^1_{\rm loc}}{\lim_{\varepsilon\searrow0}}-\mathscr{C}^1_\varepsilon=\overset{L^1_{\rm loc}}{\lim_{\varepsilon\searrow0}}(u\operatorname{div}\mathbf{b})*\varrho_\varepsilon+\overset{L^1_{\rm loc}}{\lim_{\varepsilon\searrow0}}\left(\sum_{i=1}^{s}\sum_{\alpha=1}^{n_i}F^i_\alpha\right)+\overset{L^1_{\rm loc}}{\lim_{\varepsilon\searrow0}}\left(\underbrace{\sum_{i=1}^{s}\sum_{\alpha=1}^{n_i}\left[D^i_\alpha+G^i_\alpha\right]}_{=0}\right)=u\operatorname{div}\mathbf{b}-u\operatorname{div}\mathbf{b}=0,$$

where the second equality holds due to Proposition 2.3, since $u \operatorname{div} \mathbf{b} \in L^1_{\mathrm{loc}}(\mathbb{G})$. □

Proof of Theorem 6.3. Owing to Lemma 6.5, the proof is just an adaptation of that of [5, Theorem 4.6]. □

6.3. **Well-posedness results.** As a consequence of Theorem 6.3 and Proposition 6.1, we get existence and uniqueness of distributional solutions to ($\mathscr{T}$) under natural growth conditions. We refer to [5, Theorem 4.7] for its proof in the Heisenberg groups case and [15, Theorem II.2] in the Euclidean one.

Theorem 6.6. *Let $\theta \in [1, \infty]$ and $u_0 \in L^\theta(\mathbb{G})$. Let $\mathbf{b} \in L^1\left(0, \bar{\tau}; \left[W^{1,\theta'}_{\mathcal{H},\mathrm{loc}}(\mathbb{G})\right]^n\right)$ be a time-dependent contact vector field. Assume, in addition, that $c, \operatorname{div} \mathbf{b} \in L^1(0, \bar{\tau}; L^\infty(\mathbb{G}))$ and that*

$$\frac{|\mathbf{b}|}{1 + d(p, 0)} \in L^1\left(0, \bar{\tau}; L^1(\mathbb{G})\right) + L^1(0, \bar{\tau}; L^\infty(\mathbb{G})), \tag{6.4}$$

where $|\mathbf{b}|$ is the norm of $\mathbf{b}$ with respect to the Riemannian metric $\langle \cdot, \cdot \rangle$. Then there exists a unique distributional solution u to ($\mathscr{T}$) in $L^\infty\left(0, \bar{\tau}; L^\theta(\mathbb{G})\right)$ corresponding to the initial condition u_0.

A slight adaptation of the abstract framework introduced in [1] – and subsequently formalized in [2, 3] – allows us to leverage Eulerian well-posedness to well-posedness at Lagrangian level. Accordingly, we recall that a map $\Phi(\tau, p)$ is a *regular Lagrangian flow* associated with $\mathbf{b} \in L^1(0, \bar{\tau}; \mathbb{X}(\mathbb{G}))$ if

(a) for a.e. $p \in \mathbb{G}$, $\tau \mapsto \Phi(\tau, p)$ is an absolutely continuous integral curve of $\mathbf{b}$, i.e.,

$$\begin{cases} \dfrac{d}{d\tau}\Phi(\tau, p) = \mathbf{b}(\tau, \Phi(\tau, p)) & \text{for a.e. } \tau \in (0, \bar{\tau}) \\ \Phi(0, p) = p, \end{cases}$$

(b) there exists a constant $C \in (0, \infty)$ such that $\Phi(\tau, \cdot)_\# \mathcal{L}^n \le C \mathcal{L}^n$ for all $\tau \in [0, \bar{\tau})$.

In the above definition, $\mathcal{L}^n$ is the Lebesgue measure and $\Phi(\tau, \cdot)_\# \mathcal{L}^n$ the push-forward measure.

Theorem 6.7. *Assume that $\mathbf{b} \in L^1\left(0, \bar{\tau}; \left[W^{1,1}_{\mathcal{H},\mathrm{loc}}(\mathbb{G})\right]^n\right)$ is a time-dependent contact vector field. Assume that $\operatorname{div} \mathbf{b} \in L^1(0, \bar{\tau}; L^\infty(\mathbb{G}))$ and that (6.4) holds. Then there exists a regular Lagrangian flow associated with $\mathbf{b}$. Moreover, if Φ_1 and Φ_2 are regular Lagrangian flows associated with $\mathbf{b}$, then*

$$\Phi_1(\cdot, p) = \Phi_2(\cdot, p) \qquad \textit{for a.e. } p \in \mathbb{G}.$$

Remark 6.8. Stability properties, both at Eulerian and at Lagrangian level, are a direct consequence of uniqueness as in Theorem 6.6, and follows with minor modifications as in [15, Section II] and [2, Section 5].

We refer to [5, Section 5] for a summary of further consequences in the case of Heisenberg groups, and for several additional references. With small adjustments, these results carry over to the Carnot group setting.

Appendix A. Further properties of difference quotients

In this final appendix, we collect some additional properties of difference quotients. Although they are not needed for the purposes of this paper, we believe they may be of independent interest. The first result provides an inductive characterization of difference quotients.

Proposition A.1. *Let $\Omega \subseteq \mathbb{G}$ be open. Fix $p \in \Omega$ and $w \in \mathbb{G}$. Let $\varepsilon > 0$ be such that $p \cdot \delta_{\tau\varepsilon}(w) \in \Omega$ for every $\tau \in [0, 1]$. If $w = \exp(W) = \exp(W_1 + \cdots + W_s)$ with $W_i \in V_i$ for every $i = 1, \ldots, s$, set*

$$D^w_m = \sum_{k=0}^{s-1} \frac{(-1)^k}{(k+1)!} \sum_{|I_{k+1}|=m} i_1 [W_{i_{k+1}}, \ldots, [W_{i_2}, W_{i_1}] \ldots].$$

Let $\ell \ge 1$ and $f \in C^\ell(\Omega)$. Then

$$R^w_{\ell,\varepsilon} f(p) = \sum_{m=1}^{\ell} \int_0^1 \tau^{\ell-1} R^w_{\ell-m,\tau\varepsilon}(D^w_m f)(p)\, d\tau + \sum_{m=\ell+1}^{s} \varepsilon^{m-\ell} \int_0^1 \tau^{m-1} D^w_m f(p \cdot \delta_{\tau\varepsilon}(w))\, d\tau,$$

where we agree that the second term does not appear when $\ell \ge s$.

Proof. We know from [35, Theorem 2.14.3] that, for every $Z \in \mathfrak{g}$,

$$(d\exp)_Z = \sum_{k=0}^{s-1} \frac{(-1)^k}{(k+1)!} (\mathrm{ad}_Z)^k. \tag{A.1}$$

Setting $\gamma(\tau) = p \cdot \delta_\tau(w) = p \cdot \exp(\delta_\tau W)$ for every $\tau \in [0, \varepsilon]$, we have

$$\begin{aligned}
\dot\gamma(\tau) &= (dL_p)_{\exp(\delta_\tau W)} (d\exp)_{\delta_\tau W} \left(\frac{d}{d\tau} \delta_\tau W \right) \\
&= (dL_p)_{\exp(\delta_\tau W)} (d\exp)_{\delta_\tau W} \left(\sum_{i_1=1}^{s} i_1 \tau^{i_1 - 1} W_{i_1}(\delta_\tau W) \right) \\
&\overset{(A.1)}{=} \sum_{k=0}^{s-1} \frac{(-1)^k}{(k+1)!} \sum_{i_1=1}^{s} i_1 \tau^{i_1-1} (dL_p)_{\exp(\delta_\tau W)} (\mathrm{ad}_{\delta_\tau W})^k (W_{i_1}(\delta_\tau W)) \\
&= \sum_{k=0}^{s-1} \frac{(-1)^k}{(k+1)!} \sum_{i_1=1}^{s} i_1 \tau^{i_1-1} \sum_{i_2, \dots, i_{k+1}=1}^{s} \tau^{i_2 + \dots + i_{k+1}} [W_{i_{k+1}}, \dots, [W_{i_2}, W_{i_1}] \dots](\gamma(\tau)) \\
&= \sum_{k=0}^{s-1} \frac{(-1)^k}{(k+1)!} \sum_{|I_{k+1}| \le s} i_1 \tau^{|I_{k+1}|-1} [W_{i_{k+1}}, \dots, [W_{i_2}, W_{i_1}] \dots](\gamma(\tau)) \\
&= \sum_{k=0}^{s-1} \frac{(-1)^k}{(k+1)!} \sum_{m=1}^{s} \sum_{|I_{k+1}|=m} i_1 \tau^{m-1} [W_{i_{k+1}}, \dots, [W_{i_2}, W_{i_1}] \dots](\gamma(\tau)) \\
&= \sum_{m=1}^{s} \tau^{m-1} D_m^w(\gamma(\tau)).
\end{aligned}$$

Therefore,

$$\frac{d}{d\tau} f(p \cdot \delta_\tau(w)) = \sum_{m=1}^{s} \tau^{m-1} D_m^w f(p \cdot \delta_\tau(w)) \tag{A.2}$$

and so

$$\begin{aligned}
R_{\ell,\varepsilon}^w f(p) &\overset{(5.8),(5.6)}{=} \frac{1}{\varepsilon^\ell} \left[f(p \cdot \delta_\varepsilon(w)) - f(p) - \sum_{k=1}^{\ell} T_p^k f(\delta_\varepsilon(w)) \right] \\
&\overset{(5.5)}{=} \frac{1}{\varepsilon^\ell} \left[\int_0^1 \frac{d}{d\tau} f(p \cdot \delta_{\tau\varepsilon}(w))\, d\tau - \sum_{k=1}^{\ell} \varepsilon^k T_p^k f(w) \right] \\
&\overset{(A.2)}{=} \underbrace{\frac{1}{\varepsilon^\ell} \left[\sum_{m=1}^{\ell} \int_0^1 \tau^{m-1} \varepsilon^m D_m^w f(p \cdot \delta_{\tau\varepsilon}(w))\, d\tau - \sum_{k=1}^{\ell} \varepsilon^k T_p^k f(w) \right]}_{=:F} \\
&\quad + \sum_{m=\ell+1}^{s} \varepsilon^{m-\ell} \int_0^1 \tau^{m-1} D_m^w f(p \cdot \delta_{\tau\varepsilon}(w))\, d\tau.
\end{aligned}$$

The second term appears in the desired formula, so we focus on the first one. We have

$$\begin{aligned}
F &= \sum_{m=1}^{\ell} \int_0^1 \frac{1}{(\tau\varepsilon)^{\ell-m}} \tau^{\ell-1} D_m^w f(p \cdot \delta_{\tau\varepsilon}(w))\, d\tau - \frac{1}{\varepsilon^\ell} \sum_{k=1}^{\ell} \varepsilon^k T_p^k f(w) \\
&\overset{(5.8)}{=} \sum_{m=1}^{\ell} \int_0^1 \tau^{\ell-1} R_{\ell-m,\tau\varepsilon}^w (D_m^w f)(p)\, d\tau + \underbrace{\sum_{m=1}^{\ell} \frac{1}{\varepsilon^{\ell-m}} \int_0^1 \tau^{\ell-1} P_{\ell-m}(D_m^w f)(p, \delta_{\tau\varepsilon}(w))\, d\tau - \frac{1}{\varepsilon^\ell} \sum_{k=1}^{\ell} \varepsilon^k T_p^k f(w)}_{=:G}.
\end{aligned}$$

Hence, it suffices to prove that $G = 0$. Using the convention $T_p^0(D_m^w f)(w) = (D_m^w f)(p)$, we get

$$\begin{aligned}
G &\overset{(5.6)}{=} \frac{1}{\varepsilon^\ell}\left[\sum_{m=1}^{\ell}\int_0^1 \tau^{m-1}\varepsilon^m \sum_{h=0}^{\ell-m}(\tau\varepsilon)^h T_p^h(D_m^w f)(w)\,d\tau - \sum_{k=1}^{\ell}\varepsilon^k T_p^k f(w)\right] \\
&= \frac{1}{\varepsilon^\ell}\left[\sum_{m=1}^{\ell}\sum_{h=0}^{\ell-m}\frac{\varepsilon^{m+h}}{m+h}T_p^h(D_m^w f)(w) - \sum_{k=1}^{\ell}\varepsilon^k T_p^k f(w)\right] \\
&= \frac{1}{\varepsilon^\ell}\left[\sum_{m=1}^{\ell}\sum_{k=m}^{\ell}\frac{\varepsilon^{k}}{k}T_p^{k-m}(D_m^w f)(w) - \sum_{k=1}^{\ell}\varepsilon^k T_p^k f(w)\right] \\
&= \frac{1}{\varepsilon^\ell}\sum_{k=1}^{\ell}\varepsilon^k\left[\frac{1}{k}\sum_{m=1}^{k}T_p^{k-m}(D_m^w f)(w) - T_p^k f(w)\right].
\end{aligned}$$

We claim that the term inside the parentheses is 0. Indeed, by the general Leibniz rule,

$$\begin{aligned}
T_p^k f(w) &\overset{(5.5),(5.4)}{=} \frac{1}{k!}\frac{d^k}{d\tau^k}\bigg|_{\tau=0} f(p\cdot\delta_\tau(w)) \\
&\overset{(A.2)}{=} \frac{1}{k!}\sum_{m=1}^{s}\frac{d^{k-1}}{d\tau^{k-1}}\bigg|_{\tau=0}\tau^{m-1}D_m^w f(p\cdot\delta_\tau(w)) \\
&= \frac{1}{k!}\sum_{m=1}^{s}\sum_{h=0}^{\min\{k-1,m-1\}}\binom{k-1}{h}\frac{d^h}{d\tau^h}\bigg|_{\tau=0}\tau^{m-1}\frac{d^{k-h-1}}{d\tau^{k-h-1}}\bigg|_{\tau=0}D_m^w f(p\cdot\delta_\tau(w)) \\
&\overset{(5.4)}{=} \frac{1}{k!}\sum_{m=1}^{s}\sum_{h=0}^{\min\{k-1,m-1\}}\binom{k-1}{h}\frac{(m-1)!}{(m-1-h)!}\tau^{m-1-h}|_{\tau=0}(k-h-1)!T_p^{k-h-1}(D_m^w f)(w) \\
&= \frac{1}{k!}\sum_{m=1}^{k}\binom{k-1}{m-1}(m-1)!(k-m)!T_p^{k-m}(D_m^w f)(w) \\
&= \frac{1}{k}\sum_{m=1}^{k}T_p^{k-m}(D_m^w f)(w).
\end{aligned}$$

This shows that $G = 0$, so the proof is concluded. □

Second, the boundedness of difference quotients yields a quantitative mollification error estimate. For simplicity, we consider global horizontal Sobolev functions, for which Theorem 5.3 holds in the following form.

Theorem A.2. *Let $\ell \geq 1$ and $\theta \in [1,\infty)$. Let $f \in W_{\mathcal{H}}^{\ell,\theta}(\mathbb{G})$. Then*

$$\lim_{\varepsilon\searrow 0}^{L^\theta(\mathbb{G})} R_{\ell,\varepsilon}^w f(p) = 0 \qquad \textit{for every } w \in \mathbb{G}.$$

In addition, there exists $C_2 = C_2(\mathbb{G},\ell) > 0$ such that, for every $\varepsilon > 0$ and $w \in \mathbb{G}$,

$$\left\|R_{\ell,\varepsilon}^w f\right\|_{L^\theta(\mathbb{G})} \leq C_2\, d(w,0)^\ell \sum_{\alpha_1,\dots,\alpha_\ell=1}^{n_1}\left\|X_{\alpha_1}^1\dots X_{\alpha_\ell}^1 f\right\|_{L^\theta(\mathbb{G})}. \tag{A.3}$$

Finally, if $f \in W_{\mathcal{H}}^{\ell,\infty}(\mathbb{G})$, one still has

$$\left\|R_{\ell,\varepsilon}^w f\right\|_{L^\infty(\mathbb{G})} \leq C_2\, d(w,0)^\ell \sum_{\alpha_1,\dots,\alpha_\ell=1}^{n_1}\left\|X_{\alpha_1}^1\dots X_{\alpha_\ell}^1 f\right\|_{L^\infty(\mathbb{G})}. \tag{A.4}$$

A simple consequence of Theorem A.2 is the following quantitative version of Proposition 2.3.

Theorem A.3. *Let $\ell \geq 1$ and $\theta \in [1,\infty]$. Let $f \in W_{\mathcal{H}}^{\ell,\theta}(\mathbb{G})$. Let $\varrho \in C_c^\infty(B(0,1))$ be such that*

$$\int_{\mathbb{G}}\varrho\,dp = 1, \qquad \textit{and} \qquad \int_{\mathbb{G}}\varrho\,P\,dp = 0 \tag{A.5}$$

for every non-constant polynomial P of homogeneous degree less than or equal to $\ell-1$ and such that $P(0)=0$. Define ϱ_ε as in (2.9), and set $u_\varepsilon := u * \varrho_\varepsilon$. Then, there exists $C_3 = C_3(\mathbb{G},\ell) > 0$ such that, for every $\varepsilon > 0$,

$$\|u_\varepsilon - u\|_{L^\theta(\mathbb{G})} \le C_3\, \varepsilon^\ell \sum_{\alpha_1,\dots,\alpha_\ell=1}^{n_1} \left\|X^1_{\alpha_1}\dots X^1_{\alpha_\ell} u\right\|_{L^\theta(\mathbb{G})}.$$

Proof. Fix $p \in \mathbb{G}$. We have

$$u_\varepsilon(p) - u(p) \overset{(2.6),(A.5)}{=} \int_{\mathbb{G}} \left[u\left(p\cdot\delta_\varepsilon\left(q^{-1}\right)\right) - u(p)\right]\varrho(q)\,dq \overset{(5.8),(A.5)}{=} \varepsilon^\ell \int_{\mathbb{G}} \left[R^{q^{-1}}_{\ell,\varepsilon} u(p) + T^\ell_p u\left(q^{-1}\right)\right]\varrho(q)\,dq.$$

Therefore,

$$\|u_\varepsilon - u\|_{L^\theta(\mathbb{G})} \le \varepsilon^\ell \int_{\mathbb{G}} \left(\left\|R^{q^{-1}}_{\ell,\varepsilon} u\right\|_{L^\theta(\mathbb{G})} + \left\|T^\ell_\cdot u\left(q^{-1}\right)\right\|_{L^\theta(\mathbb{G})}\right)|\varrho(q)|\,dq.$$

A simple estimate of $T^\ell_\cdot u$, combined with (A.3) and (A.4), concludes the proof. □

References

[1] L. Ambrosio. Transport equation and Cauchy problem for *BV* vector fields. *Invent. Math.*, 158(2):227–260, 2004.

[2] L. Ambrosio and G. Crippa. Existence, uniqueness, stability and differentiability properties of the flow associated to weakly differentiable vector fields. In *Transport equations and multi-D hyperbolic conservation laws*, volume 5 of *Lect. Notes Unione Mat. Ital.*, pages 3–57. Springer, Berlin, 2008.

[3] L. Ambrosio and G. Crippa. Continuity equations and ODE flows with non-smooth velocity. *Proc. Roy. Soc. Edinburgh Sect. A*, 144(6):1191–1244, 2014.

[4] L. Ambrosio, N. Gigli, and G. Savaré. *Gradient flows in metric spaces and in the space of probability measures*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, second edition, 2008.

[5] L. Ambrosio, G. Somma, S. Verzellesi, and D. Vittone. Renormalization of contact vector fields with horizontal Sobolev regularity in Heisenberg groups. Preprint, https://doi.org/10.48550/arXiv.2602.00804, 2026.

[6] L. Ambrosio and D. Trevisan. Well-posedness of Lagrangian flows and continuity equations in metric measure spaces. *Anal. PDE*, 7(5):1179–1234, 2014.

[7] S. Bianchini and P. Bonicatto. A uniqueness result for the decomposition of vector fields in $\mathbb{R}^d$. *Invent. Math.*, 220(1):255–393, 2020.

[8] A. Bonfiglioli, E. Lanconelli, and F. Uguzzoni. *Stratified Lie groups and potential theory for their sub-Laplacians*. Springer Monographs in Mathematics. Springer, Berlin, 2007.

[9] M. Bramanti and L. Brandolini. *Hörmander operators*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2023.

[10] L. Capogna, G. Giovannardi, A. Pinamonti, and S. Verzellesi. The asymptotic p-Poisson equation as $p \to \infty$ in Carnot-Carathéodory spaces. *Math. Ann.*, 390(2):2113–2153, 2024.

[11] I. Capuzzo Dolcetta and B. Perthame. On some analogy between different approaches to first order PDE's with nonsmooth coefficients. *Adv. Math. Sci. Appl.*, 6(2):689–703, 1996.

[12] G. Crippa and C. De Lellis. Estimates and regularity results for the DiPerna-Lions flow. *J. Reine Angew. Math.*, 616:15–46, 2008.

[13] C. De Lellis. Ordinary differential equations with rough coefficients and the renormalization theorem of Ambrosio [after Ambrosio, DiPerna, Lions]. In *Séminaire Bourbaki - Volume 2006/2007 - Exposés 967-981*, number 317 in Astérisque, pages 175–203. Société mathématique de France, 2008. talk:972.

[14] N. Depauw. Non unicité des solutions bornées pour un champ de vecteurs BV en dehors d'un hyperplan. *C. R. Math. Acad. Sci. Paris*, 337(4):249–252, 2003.

[15] R. J. DiPerna and P.-L. Lions. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.*, 98(3):511–547, 1989.

[16] G. B. Folland. Subelliptic estimates and function spaces on nilpotent Lie groups. *Ark. Mat.*, 13(2):161–207, 1975.

[17] G. B. Folland and E. M. Stein. *Hardy spaces on homogeneous groups*, volume 28 of *Mathematical Notes*. Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1982.

[18] N. Garofalo and D.-M. Nhieu. Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces. *Comm. Pure Appl. Math.*, 49(10):1081–1144, 1996.

[19] N. Garofalo and D.-M. Nhieu. Lipschitz continuity, global smooth approximations and extension theorems for Sobolev functions in Carnot-Carathéodory spaces. *J. Anal. Math.*, 74:67–97, 1998.

[20] P.-E. Jabin. Differential equations with singular fields. *J. Math. Pures Appl. (9)*, 94(6):597–621, 2010.

[21] A. W. Knapp. *Lie groups beyond an introduction*, volume 140 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, second edition, 2002.

[22] A. Korányi and H. M. Reimann. Quasiconformal mappings on the Heisenberg group. *Invent. Math.*, 80(2):309–338, 1985.

[23] A. Korányi and H. M. Reimann. Foundations for the theory of quasiconformal mappings on the Heisenberg group. *Adv. Math.*, 111(1):1–87, 1995.

[24] C. Le Bris and P.-L. Lions. Renormalized solutions of some transport equations with partially $W^{1,1}$ velocities and applications. *Ann. Mat. Pura Appl. (4)*, 183(1):97–130, 2004.

[25] J. M. Lee. *Introduction to smooth manifolds*, volume 218 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2013.

[26] G. Leoni. *A first course in Sobolev spaces*, volume 181 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2017.

[27] N. Lerner. Transport equations with partially BV velocities. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 3(4):681–703, 2004.

[28] P. Libermann. Sur les automorphismes infinitésimaux des structures symplectiques et des structures de contact. In *Colloque Géom. Diff. Globale (Bruxelles, 1958)*, pages 37–59. Librairie Universitaire, Louvain, 1959.

[29] Q.-H. Nguyen. Quantitative estimates for regular Lagrangian flows with BV vector fields. *Comm. Pure Appl. Math.*, 74(6):1129–1192, 2021.

[30] A. Ottazzi. A sufficient condition for nonrigidity of Carnot groups. *Math. Z.*, 259(3):617–629, 2008.

[31] A. Ottazzi and B. Warhurst. Contact and 1-quasiconformal maps on Carnot groups. *J. Lie Theory*, 21(4):787–811, 2011.

[32] A. Pinamonti, S. Verzellesi, and C. Wang. The Aronsson equation for absolute minimizers of supremal functionals in Carnot-Carathéodory spaces. *Bull. Lond. Math. Soc.*, 55(2):998–1018, 2023.

[33] F. Serra Cassano. Some topics of geometric measure theory in Carnot groups. In *Geometry, analysis and dynamics on sub-Riemannian manifolds. Vol. 1*, EMS Ser. Lect. Math., pages 1–121. Eur. Math. Soc., Zürich, 2016.

[34] N. Tanaka. On differential systems, graded Lie algebras and pseudogroups. *J. Math. Kyoto Univ.*, 10:1–82, 1970.

[35] V. S. Varadarajan. *Lie groups, Lie algebras, and their representations*, volume 102 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1984. Reprint of the 1974 edition.

[36] S. Verzellesi. Variational properties of local functionals driven by arbitrary anisotropies. *Calc. Var. Partial Differential Equations*, 65(1):Paper No. 3, 24, 2026.

[37] B. Warhurst. Contact and quasiconformal mappings on real model filiform groups. *Bull. Austral. Math. Soc.*, 68(2):329–343, 2003.

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