HARMONIC MORPHISMS
OF SUB-RIEMANNIAN LIE GROUPS

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Abstract. We prove that harmonic morphisms between sub-Riemannian Lie groups are smooth, are symmetries of the sub-Riemannian Laplacian and are conformal submersions that satisfy a particular PDE. Moreover, we give some partial results for harmonic morphisms beyond Lie groups. We show the existence of harmonic coordinates for harmonically homogeneous spaces, and that smooth harmonic morphisms are symmetries of the Laplacian in all sub-Riemannian manifolds. Finally, we compute the first variation of the horizontal energy along a harmonic morphism of sub-Riemannian Lie groups: it is given by a linear form on the Lie algebra of the target, the modular mismatch, which vanishes for Riemannian and Carnot targets, but surprisingly not in general. Therefore, unlike in the Riemannian case, harmonic morphisms of sub-Riemannian Lie groups need not be harmonic maps.

Contents

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1. Introduction

A *harmonic morphism* is a continuous map that pulls back harmonic functions to harmonic functions. Harmonic morphisms between Riemannian manifolds have been studied for a long time: we invite the reader to consult Baird and Wood's monograph [2] and Gudmundsson's "The Bibliography of Harmonic Morphisms".[1]

Harmonic morphisms between Riemannian manifolds are smooth maps that solve a certain PDE. Fuglede [16] and Ishihara [21] proved that *a smooth map between Riemannian manifolds is a harmonic morphism if and only if it is both a harmonic map and a conformal submersion map.*

A *harmonic map* is a smooth map that is a critical point of the Dirichlet energy (see [2, Definition 3.3.1]). Harmonic functions are real-valued harmonic maps. Harmonic maps between Riemannian manifolds are characterized by the vanishing of the *tension field* (see [2, Definition 3.2.4] and [2, Theorem 3.3.3]).

A map $\phi : M \to N$ between Riemannian manifolds is a *conformal submersion* (a.k.a., *semiconformal map*) if, for every $x \in M$, the differential $\mathrm{d}\phi(x) : T_xM \to T_{\phi(x)}N$ is either zero, or surjective with conformal restriction $\mathrm{d}\phi(x)|_{\ker(\mathrm{d}\phi(x))^\perp}$ from the subspace of T_xM orthogonal to the kernel of $\mathrm{d}\phi(x)$ onto $T_{\phi(x)}N$ (see [2, Definition 2.4.2]). Conformal maps are conformal submersions, but also orthogonal projections and constant maps are conformal submersions. See Section 4.7 for the definition for maps between sub-Riemannian Lie groups. We stress that we include the degenerate case $\mathrm{d}\phi(x) = 0$, even though the map is not a submersion at those points.

Our goal is to extend the Fuglede–Ishihara characterization to sub-Riemannian Lie groups; see Theorem E. Subelliptic and pseudoharmonic maps have been studied in several settings, including Hörmander systems, pseudohermitian and contact manifolds, and sub-Riemannian Lie groups; see, among others, [3, 4, 11, 13, 22,

[1]Available at `https://www.matematik.lu.se/matematiklu/personal/sigma/harmonic/bibliography.html`

17]. Harmonic morphisms associated with subelliptic operators were investigated in [3, 13], and, more recently, in the pseudohermitian and Fefferman settings in [12].

A *(measured) sub-Riemannian manifold* is determined by $(M, \tilde{V}, \rho, \mu)$ where M is a smooth manifold, $\tilde{V} \subset TM$ is a bracket-generating subbundle, ρ is a scalar product on $\tilde{V}$ and μ is a smooth measure on M. We call $\tilde{V}$ the *horizontal distribution*, and a curve γ in M is a *horizontal curve* if it is absolutely continuous and $\dot{\gamma}(t) \in \tilde{V}$ for almost every t. The measured sub-Riemannian structure determines a second-order hypoelliptic differential operator $\triangle_M$, the *sub-Laplacian* of M; see Section 3.5. When $\tilde{V} = TM$, ρ is a Riemannian metric, and μ is its Riemannian volume measure, $\triangle_M$ is the Laplace–Beltrami operator. Sub-Riemannian *harmonic functions* are (smooth) functions $u : M \to \mathbb{R}$ with $\triangle_M u = 0$.

Apart from continuity, the definition of harmonic morphism does not assume any a priori regularity of the map; our first two results concern the regularity of harmonic morphisms.

An application of [8] allows us to prove that harmonic morphisms are smooth whenever they preserve horizontal curves.

Theorem A. *Let M and N be measured sub-Riemannian manifolds and let $F : M \to N$ be a continuous harmonic morphism. Suppose that F maps horizontal curves to horizontal curves. Then F is C^∞-smooth.*

Our second regularity result relies on some particular assumptions concerning the target space. A sub-Riemannian manifold $(M, \tilde{V}, \rho, \mu)$ is said to be *harmonically homogeneous* if there is a transitive Lie group action on M by harmonic morphisms; see Section 3.8. For instance, isometries that preserve the measure μ are harmonic morphisms. So, sub-Riemannian Lie groups or, more generally, isometrically homogeneous sub-Riemannian manifolds, are harmonically homogeneous.

On harmonically homogeneous sub-Riemannian manifolds we can prove that harmonic coordinates exist, and therefore that harmonic morphisms are smooth. We stress that the existence of harmonic coordinates on sub-Riemannian manifolds is an open problem (see [8, 9]) and that the definition of harmonic morphism does not assume any a-priori regularity.

Theorems B and C are proved in Section 5.2.

Theorem B. *Let M be a measured sub-Riemannian manifold that is harmonically homogeneous. Then every point of M has harmonic coordinates, that is, a system of local coordinates whose coordinate functions are harmonic.*

Theorem C. *Let M and N be measured sub-Riemannian manifolds and let $F : M \to N$ be a harmonic morphism. Suppose that N is harmonically homogeneous. Then F is C^∞-smooth.*

Our second set of results describes the structure of harmonic morphisms. In sub-Riemannian manifolds, we show that harmonic morphisms are symmetries of the sub-Riemannian Laplacian. Focusing on sub-Riemannian Lie groups, we can give a geometric description of harmonic morphisms.

Theorem D. *Let M and N be measured sub-Riemannian manifolds and let $F : M \to N$ be a C^k-map, with $k \geq 2$. The following statements are equivalent:*

(i) F is a harmonic morphism.

(ii) There exists $f : M \to \mathbb{R}$ such that, for every $v \in C^\infty(N)$,

$$\triangle_M(v \circ F) = f \cdot (\triangle_N v) \circ F. \tag{1}$$

Moreover, the function f in (1) *is non-negative and of class C^{k-2}, and* (1) *holds for $v \in C^2(N)$.*

The proof of Theorem D is in Section 6. Our approach is based on the theory of Brelot harmonic spaces; see [5, 10, 15, 19, 18].

Our main result is a geometric characterization of harmonic morphisms between sub-Riemannian Lie groups. A *sub-Riemannian Lie group* is a Lie group endowed with a left-invariant sub-Riemannian structure. Sub-Riemannian Lie groups are harmonically homogeneous, via the action of left-translations.

Theorem E. *Let G and H be sub-Riemannian Lie groups, $\Omega \subset G$ open and $F : \Omega \to H$. The following statements are equivalent:*

(i) F is a harmonic morphism.

(ii) F is C^∞-smooth and there exists $\lambda : \Omega \to [0, +\infty)$ such that, for every $v \in C^\infty(H)$,

$$\triangle_G(v \circ F) = \lambda^2 \cdot (\triangle_H v) \circ F. \tag{2}$$

(iii) F is C^∞-smooth and there exists $\lambda : \Omega \to [0, +\infty)$ such that F is a conformal submersion of factor λ, and, for every $p \in \Omega$,

$$\operatorname{tr}_G\big(\mathrm{D}^2 F(p)\big) - \mathrm{D}F(p)[\mathfrak{m}_G] + \lambda(p)^2\, \mathfrak{m}_H = 0, \tag{3}$$

where $\mathfrak{m}_G \in V(G)$ and $\mathfrak{m}_H \in V(H)$ are the modular vectors of G and H, respectively.

Here $V(G) \subset \mathfrak{g} = T_{1_G}G$ is the polarization at the identity, whereas $\tilde{V}(G) \subset TG$ is the horizontal subbundle obtained by left translation of $V(G)$. Thus the modular vectors $\mathfrak{m}_G$ and $\mathfrak{m}_H$ in (3) belong to $V(G)$ and $V(H)$, respectively. For the definition of conformal submersion, see Section 4.7; the modular vector is the horizontal gradient of the logarithm of the modular function, see Section 4.4. The left-hand side of (3) is the *harmonic morphism operator* $\mathscr{T}(F)$ of Section 7.1.Theorem E follows from Theorem D and [23, Theorem A]; we give the proof in Section 7. Our last set of results, in Section 8, compares Theorem E with Fuglede and Ishihara's theorem. In the Riemannian setting, a harmonic morphism is a conformal submersion and a *harmonic map*, that is, a critical point of the Dirichlet energy. Theorem E says that a harmonic morphism of sub-Riemannian Lie groups is a conformal submersion that satisfies the equation $\mathscr{T}(F) = 0$; the question is whether this equation is the Euler–Lagrange equation of the horizontal energy

$$\mathscr{E}(F) = \frac{1}{2} \int_\Omega \sum_{i=1}^m \big|\mathrm{D}F[X_i]\big|_H^2 \,\mathrm{dvol}_G,$$

where $X_1, \dots, X_m$ is an orthonormal basis of the polarization $V(G)$. The critical points of $\mathscr{E}$ among contact maps have been studied by Grong and Markina in [17, Theorem 4.2]. The answer is negative, and the obstruction is a linear form on the Lie algebra of the target alone. Namely, for every complement $\mathfrak{q}$ of the polarization $V(H)$ in $\mathfrak{h}$ we define in (31) the *modular mismatch* $\chi \in \mathfrak{h}^*$ of the splitting $\mathfrak{h} = V(H) \oplus \mathfrak{q}$, and we prove:

Theorem F. *Let G and H be sub-Riemannian Lie groups, $\Omega \subset G$ open, and $F : \Omega \to H$ a harmonic morphism, with conformal factor λ as in Theorem E. Let $\mathfrak{q}$ be a complement of $V(H)$ in $\mathfrak{h}$ and $\chi \in \mathfrak{h}^*$ the modular mismatch of the splitting $\mathfrak{h} = V(H) \oplus \mathfrak{q}$. Let $\mathcal{F} : \Omega \times (-\epsilon, \epsilon) \to H$ be a smooth contact variation of F: writing $F_s(p) := \mathcal{F}(p, s)$, assume that each F_s is a contact map, that $F_0 = F$, and that all the maps F_s agree with F outside a fixed compact subset of Ω. If*

$$\varphi(p) := \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} F(p)^{-1} F_s(p)$$

is its variation field, then

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \mathscr{E}(F_s) = \int_\Omega \lambda^2\, \chi(\varphi)\, \mathrm{dvol}_G.$$

In particular, if $\chi = 0$, then F is a harmonic map.

The quantity χ involves neither F, nor G, nor the scalar products: it only depends on the polarized Lie algebra $(\mathfrak{h}, V(H))$ and on the choice of $\mathfrak{q}$. It vanishes for a suitable $\mathfrak{q}$ when H is Riemannian and when H is a Carnot group, among others, and in these cases harmonic morphisms are harmonic maps, as in the Riemannian theory: see Theorem 8.8 and Corollary 8.9. It does not vanish in general. Indeed, let AN be the simply transitive solvable Iwasawa group model of the complex hyperbolic plane $\mathbb{C}H^2 \simeq PU(2,1)/U(2)$, and polarize AN by a bracket-generating hyperplane of its Lie algebra. In Theorem 9.3 we show that the identity map of this polarized Lie group is a harmonic morphism whose horizontal energy is strictly decreased by an explicit compactly supported variation; see Section 9.1. Thus harmonic morphisms of sub-Riemannian Lie groups need not be harmonic maps. We close the introduction with an open question: are harmonic morphisms smooth in general? One possible approach would be to establish the existence of harmonic coordinates on every measured sub-Riemannian manifold.

Two further questions concerning the results of Section 8 are stated in Section 10.2.

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2. Brelot Harmonic spaces

We recall the standard foundamentals of the theory of Brelot harmonic spaces. Our main references are [5, Chapter 6], [10] and [15]. See also [19]

2.1. **Brelot harmonic space.**

A *harmonic space* is a pair $(X, \mathscr{H})$ where X is a locally compact, Hausdorff topological space and $\mathscr{H}$ is a linear sheaf of real-valued continuous functions.

In a harmonic space $(X, \mathscr{H})$, an open set $U \subset X$ is *$\mathscr{H}$-regular* if

(1) (Dirichlet principle) for every $f \in C^0(\partial U)$ there exists $h \in C^0(\bar{U})$ with $h|_U \in \mathscr{H}(U)$ and $h|_{\partial U} = f$; and
(2) (comparison principle) for every $f, g \in C^0(\partial U)$ and every $h_f, h_g \in C^0(\bar{U})$ with $h_f|_U \in \mathscr{H}(U)$, $h_f|_{\partial U} = f$, $h_g|_U \in \mathscr{H}(U)$, and $h_g|_{\partial U} = g$, if $f \leq g$ then $h_f \leq h_g$.

Notice that the second condition implies the uniqueness of the function h in the first condition. The function h is called the *harmonic extension* of f to U and denoted by H_U^f. The operator $C^0(\partial U) \to C^0(U)$, $f \mapsto H_U^f$, is easily shown to be linear and monotone. It follows that there is a family $\{\mu_x^U\}_{x \in U}$ of positive measures supported on ∂U such that, for every $f \in C^0(\partial U)$ and $x \in U$, we have $H_U^f(x) = \int_{\partial U} f\, \mathrm{d}\mu_x^U$. Such measures are called *harmonic measures*.

A harmonic space $(X, \mathscr{H})$ is a *Brelot harmonic space* if

(1) $\mathscr{H}$-regular pre-compact domains form a basis of the topology of X;

(2) (Harnack's principle) the limit of every increasing sequence of harmonic functions on a domain is either harmonic or identically infinite.

2.2. **Hyperharmonic and superharmonic functions.** Let $(X, \mathscr{H})$ be a Brelot harmonic space. A function

$$f : X \to (-\infty, +\infty]$$

is called *hyperharmonic* if it is lower semicontinuous and, for every relatively compact $\mathscr{H}$-regular open set $U \subset X$,

$$f(x) \geq \int_{\partial U} f \, \mathrm{d}\mu_x^U \qquad \text{for every } x \in U.$$

A hyperharmonic function f is called *superharmonic* if it is finite on a dense subset of X.

2.3. **Harmonic morphism.** Let $(X, \mathscr{H})$ and $(Y, \mathscr{K})$ be two harmonic spaces. A continuous function $F : X \to Y$ is a *harmonic morphism* if, for every $V \subset Y$ open and $v \in \mathscr{K}(V)$, we have $v \circ F \in \mathscr{H}(F^{-1}V)$, where $F^{-1}V$ denotes the preimage of V.

Proposition 2.1 ([10, Cor. 3.2])**.** *Let M and N be harmonic spaces, with M connected, and $F : M \to N$ a non-constant harmonic morphism. If $V \subset N$ is open and $f : V \to \mathbb{R}$ is superharmonic, then $f \circ F : F^{-1}(V) \to \mathbb{R}$ is also superharmonic.*

3. sub-Riemannian manifolds

3.1. **General notations.** If M is a smooth manifold, we denote by TM its tangent bundle and by $\Gamma(TM)$ the space of all C^∞-smooth sections of TM. If $F : M \to N$ is a C^1 map between smooth manifolds, we denote by $\mathrm{d}F : TM \to TN$ its standard differential-geometric differential.

3.2. **Measured sub-Riemannian manifold.** A *measured sub-Riemannian manifold* is given by $(M, \tilde{V}, \rho, \mu)$ where M is a smooth manifold, $\tilde{V} \subset TM$ is a bracket-generating subbundle, ρ is a scalar product on $\tilde{V}$ and μ is a smooth measure on M. We note that there are more general definitions of sub-Riemannian manifolds, such as [1, Definition 3.2]. In our definition, the rank of $\tilde{V}$, i.e., the dimension of the fiber $\tilde{V}|_p$ for $p \in M$, is constant.

3.3. **Gradient.** Let $(M, \tilde{V}, \rho, \mu)$ be a measured sub-Riemannian manifold. The *(horizontal) gradient* of a C^1 function $f : M \to \mathbb{R}$ at $x \in M$ is the unique vector field $\tilde{\nabla}_M f \in \Gamma(\tilde{V}) \subset \Gamma(TM)$ such that

$$\langle \tilde{\nabla}_M f(p), v \rangle_\rho = \mathrm{d}f(p)[v], \quad \text{for all } v \in \tilde{V}_p. \tag{4}$$

3.4. **Divergence.** Let M be a smooth manifold and μ a smooth measure on M. The *μ-divergence* of a vector field $\tilde{v} \in \Gamma(TM)$ is the function $\mathrm{div}_\mu \tilde{v} \in C^\infty(M)$ such that, for every $\phi \in C_c^\infty(M)$,

$$\int_M (\mathrm{div}_\mu \tilde{v}) \phi \, \mathrm{d}\mu = - \int_M \mathrm{d}\phi[\tilde{v}] \, \mathrm{d}\mu. \tag{5}$$

One can express the divergence of a smooth vector field $\tilde{v} \in \Gamma(TM)$ using the Lie derivative $\mathcal{L}_{\tilde{v}}$ of the volume form $\mathrm{d}\mu$ as

$$(\mathrm{div}_{\mathrm{vol}_\mu} \tilde{v}) \, \mathrm{d}\mu = \mathcal{L}_{\tilde{v}} \, \mathrm{d}\mu = \left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_{t=0} \Phi_t^* \, \mathrm{d}\mu, \tag{6}$$

where Φ_t is the flow of $\tilde{v}$, see [23].

We recall that, if $\tilde{X} \in \Gamma(TM)$ is a vector field and $f \in C^\infty(M)$, then (see [26, Proposition 12.32] and [23])

$$\operatorname{div}_\mu(f\tilde{X}) = df[\tilde{X}] + f \operatorname{div}_\mu(\tilde{X}). \tag{7}$$

3.5. **Sub-Riemannian Laplacian.** Let $M = (M, \tilde{V}, \rho, \mu)$ be a measured sub-Riemannian manifold. The *sub-Riemannian Laplacian on M* is the differential operator

$$\triangle_M u := \operatorname{div}_\mu(\tilde{\nabla}_M u), \qquad \forall u \in C^2(M).$$

3.6. **Sub-Riemannian Harmonic function.** Let $M = (M, \tilde{V}, \rho, \mu)$ be a measured sub-Riemannian manifold. A function $u : M \to \mathbb{R}$ is *harmonic* or *M-harmonic* if $\triangle_M u = 0$. By the hypoellipticity of $\triangle_M$, see [20], harmonic functions are smooth.

3.7. **Sub-Riemannian manifolds as Brelot harmonic spaces.**

Proposition 3.1 ([7, Theorem 8.2])**.** *Let M be a measured sub-Riemannian manifold and $\mathscr{H}$ the sheaf of $\triangle_M$-harmonic functions on M. Then $(M, \mathscr{H})$ is a Brelot harmonic space.*

Proposition 3.2 ([6, Theorem 4.2])**.** *Let M be a measured sub-Riemannian manifold and $u \in C^2(M)$. The following statements are equivalent:*

(i) u is superharmonic (or sub-harmonic, resp.) in the abstract sense of Brelot spaces;
(ii) $\triangle_M u \leq 0$ (or $\triangle_M u \geq 0$, resp.).

Let M and N be measured sub-Riemannian manifolds. By the definition in Section 2.3, a continuous function $F : M \to N$ is a harmonic morphism if for every $\Omega_N \subset N$ and every N-harmonic function $u : \Omega_N \to \mathbb{R}$, the pull back $F^*u = u \circ F : F^{-1}(\Omega_N) \to \mathbb{R}$ is M-harmonic. Propositions 2.1 and 3.2 imply that, if $F : M \to N$ is a harmonic morphism, then whenever $\triangle_N u \geq 0$ we have $\triangle_M(F^*u) \geq 0$.

3.8. **Harmonically homogeneous sub-Riemannian manifolds.** A measured sub-Riemannian manifold M, is *harmonically homogeneous*, if there is a Lie group K with a transitive action $K \times M \to M$, such that for every $k \in K$ the map $x \mapsto kx$ is a harmonic morphism of M.

Isometries of M that preserve the measure of M commute with $\triangle_M$, see [23], and thus they are harmonic morphisms. Therefore, isometrically homogeneous measured sub-Riemannian manifolds, and in particular sub-Riemannian Lie groups with their left translations, are harmonically homogeneous.

4. sub-Riemannian Lie groups

For sub-Riemannian Lie groups, we use the same notation as in [23]. Here we will only list the main objects. For further details on the theory of sub-Riemannian Lie groups, see [25].

4.1. **Lie groups.** Given a Lie group G, we identify, as vector spaces, its Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ with the tangent space $T_{1_G}G$ of G at the identity element 1_G of G. We denote by L_p the left translation $G \to G$ by p: $L_p(x) = px$. If $v : E \to \mathfrak{g}$ for some $E \subset G$, then we define $\tilde{v} : E \to TG$ by the following formula

$$\tilde{v}(p) := dL_p[v] \in T_pG, \qquad \forall p \in E.$$

If $v \in \mathfrak{g}$, then $\tilde{v}$ is the left-invariant vector field with value v at 1.

If $f:\Omega\to W$ is a C^1-function from an open set $\Omega\subset G$ to a finite-dimensional vector space W, and $v:\Omega\to\mathfrak{g}$ is continuous, then we define $\tilde v f:\Omega\to W$ by

$$\tilde v f(p) := df(p)[\tilde v(p)] = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(p\exp(tv(p))).$$

4.2. **Lie differential.** Let G and H be Lie groups and $\Omega\subset G$ open. The *Lie differential of order one* of a smooth function $F:\Omega\to H$ at $p\in\Omega$ is the map $\mathrm{D}F:\Omega\to\mathfrak{h}\otimes\mathfrak{g}^*$ defined by

$$\mathrm{D}F(p)[v] := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} F(p)^{-1}F(p\exp(tv)), \qquad \forall v\in\mathfrak{g} \tag{8}$$

Let G and H be two polarized Lie groups, $\Omega\subset G$ open and $F:\Omega\to H$ a C^2 function. Notice that the function $\mathrm{D}F:\Omega\to\mathfrak{h}\otimes\mathfrak{g}^*$ is a C^1 function valued in the Abelian Lie group $\mathfrak{h}\otimes\mathfrak{g}^*$. As such, we define the *Lie differential of order two* of F as the Lie differential $\mathrm{D}(\mathrm{D}F)$ of $\mathrm{D}F$, that is, as the function $\mathrm{D}^2F:\Omega\to(\mathfrak{h}\otimes\mathfrak{g}^*)\otimes\mathfrak{g}^*$,

$$\begin{aligned}
\mathrm{D}^2F(p)[v,w] &:= (\mathrm{D}(\mathrm{D}F)(p)[w])[v] \overset{(8)}{=} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\mathrm{D}F(p\exp(tw))[v]-\mathrm{D}F(p)[v])\\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{D}F(p\exp(tw))[v] \qquad (9)\\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} F(p\exp(tw))^{-1}F(p\exp(tw)\exp(sv)).
\end{aligned}$$

Notice that $D^2F(p)[v,w]$ need not be symmetric in $[v,w]$.

4.3. **Sub-Riemannian Lie groups.** A *polarized Lie group* is a pair $(G,V(G))$ where G is a connected Lie group and $V(G)\subset\mathfrak{g}$ is a bracket-generating subspace of the Lie algebra $\mathfrak{g}$ of G. We call $V(G)$ the *polarization* of G. By *bracket-generating subspace* we mean a linear subspace that Lie generates the Lie algebra.

Consequently, the bundle of left translates

$$\tilde V(G) := \bigcup_{p\in G} dL_p(V(G))$$

forms a *horizontal subbundle* of TG, which is bracket generating.

A *measured sub-Riemannian Lie group* is the measured sub-Riemannian manifold determined by $(G,\tilde V(G),\langle\cdot,\cdot\rangle_G,\mathrm{vol}_G)$ where $(G,V(G))$ is a polarized Lie group, $\langle\cdot,\cdot\rangle_G$ is a left-invariant scalar product on $\tilde V(G)$ and vol_G is a left Haar measure on G. Notice that, by left-invariance, the scalar product is determined by its value on $V(G)\subset T_{1_G}G$.

If $\Omega\subset G$ is open and $f\in C^1(\Omega)$, then the horizontal gradient $\tilde\nabla_G f\in\Gamma(\tilde V(G))$ from Section 3.3 defines a map $\nabla_G f:\Omega\to V(G)$ such that

$$\tilde\nabla_G f(p) = \mathrm{d}L_p|_{1_G}[\nabla_G f(p)],$$

that is, $\nabla_G f(p)$ is the unique vector of $V(G)$ such that $\langle\nabla_G f(p),v\rangle_G = \mathrm{D}f(p)[v]$ for every $v\in V(G)$.

4.4. **Haar measure and the modular function.** Let G be a sub-Riemannian Lie group with polarization $V(G)\subset\mathfrak{g}$ and left-invariant Haar measure vol_G. The volume form dvol_G is left invariant, in the sense that

$$\forall a,x\in G \qquad dL_a|_x^*\,\mathrm{dvol}_G(ax) = \mathrm{dvol}_G(x).$$

The *modular function* of a Lie group G is the function $\mu_G:G\to(0,\infty)$ given by

$$\mu_G(g) := \det(\mathrm{Ad}_g),$$

where $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ is the adjoint representation of G. More explicitly, $\mathrm{Ad}_g = DC_g$ where $C_g(x) = gxg^{-1}$ is the conjugation by g. The defining property of the modular function is that, for all $p \in G$,

$$C_p^* \,\mathrm{dvol}_G = \mu_G(p)\,\mathrm{dvol}_G, \qquad \text{and} \qquad R_{p^{-1}}^* \,\mathrm{dvol}_G = \mu_G(p)\,\mathrm{dvol}_G. \tag{10}$$

See [14, Chapter 11] for further details.

We write

$$\kappa_G := \mathrm{tr} \circ \mathrm{ad} \in \mathfrak{g}^*,$$

that is, $\kappa_G(v) = \mathrm{tr}(\mathrm{ad}_v)$ for all $v \in \mathfrak{g}$. Since κ_G is linear, there exists a unique vector $\mathfrak{m}_G \in V(G)$, which we call the *modular vector* of G, such that

$$\langle \mathfrak{m}_G, v \rangle_G = \kappa_G(v) \quad \forall v \in V(G). \tag{11}$$

If $X_1, \dots, X_m$ is an orthonormal basis of $V(G)$, then

$$\mathfrak{m}_G = \sum_{i=1}^m \kappa_G(X_i) X_i. \tag{12}$$

Lemma 4.1. *Let G be a sub-Riemannian Lie group with polarization $V(G) \subset \mathfrak{g}$, left Haar measure vol_G, modular function μ_G and modular vector $\mathfrak{m}_G$. Let $X_1, \dots, X_m$ be an orthonormal basis of $V(G)$.*

(a) $\log \mu_G : G \to (\mathbb{R}, +)$ is a Lie group morphism and

$$\mathrm{D}(\log \mu_G) = \kappa_G, \tag{13}$$

i.e., $\tilde{v}(\log \mu_G) \equiv \kappa_G(v)$ for every $v \in \mathfrak{g}$. In particular,

$$\nabla_G \log \mu_G = \mathfrak{m}_G \in V(G). \tag{14}$$

(b) For every $v \in \mathfrak{g}$ we have $\mathrm{div}_{\mathrm{vol}_G} \tilde{v} = -\kappa_G(v)$. Consequently, for every $\phi \in C_c^\infty(G)$ and every $v \in \mathfrak{g}$,

$$\int_G \tilde{v}\phi \,\mathrm{dvol}_G = \kappa_G(v) \int_G \phi \,\mathrm{dvol}_G. \tag{15}$$

(c) For every open set $\Omega \subset G$ and every $u \in C^2(\Omega)$,

$$\triangle_G u = \sum_{i=1}^m \tilde{X}_i^2 u - \widetilde{\mathfrak{m}_G} u. \tag{16}$$

Proof. (a) From $\mathrm{Ad}_{gh} = \mathrm{Ad}_g \mathrm{Ad}_h$ we get

$$\mu_G(gh) = \mu_G(g)\mu_G(h) \qquad \forall g, h \in G.$$

Using the Jacobi formula, we get

$$\begin{aligned} \log \mu_G(g \exp(tv)) &= \log \mu_G(g) + \log \det \left(e^{t\,\mathrm{ad}_v} \right) \\ &= \log \mu_G(g) + \log \left(e^{t\,\mathrm{tr}(\mathrm{ad}_v)} \right) = \log \mu_G(g) + t\,\kappa_G(v). \end{aligned}$$

Differentiating at $t = 0$ gives $\mathrm{D}(\log \mu_G)[v] = \tilde{v}(\log \mu_G) = \kappa_G(v)$, which is (13). Then (11) is the definition (4) of the horizontal gradient, and so we get (14).

(b) From (10) and (13) we get $R_a^* \,\mathrm{dvol}_G = \mu_G(a)^{-1} \,\mathrm{dvol}_G$ for every $a \in G$. The flow of $\tilde{v}$ is $\Phi_t = R_{\exp(tv)}$, hence

$$\begin{aligned} (\mathrm{div}_{\mathrm{vol}_G} \tilde{v})\,\mathrm{dvol}_G &\overset{(6)}{=} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi_t^* \,\mathrm{dvol}_G \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mu_G(\exp(tv))^{-1} \,\mathrm{dvol}_G = -\kappa_G(v)\,\mathrm{dvol}_G. \end{aligned}$$

Identity (15) follows from (5).

(c) Since $\tilde\nabla_G u = \sum_i (\tilde X_i u)\tilde X_i$ and (7), part (b) gives $\triangle_G u = \sum_i \tilde X_i^2 u - \sum_i \kappa_G(X_i)\tilde X_i u$, which is (16) by (12). □

Remark 4.2. As an example, consider the affine group $G = \{(a,b) : a > 0\}$ with $(a,b)(a',b') = (aa', b + ab')$ and $V(G) = \mathfrak{g}$: here $\tilde X = a\partial_a$ and $\tilde Y = a\partial_b$ are orthonormal left-invariant fields, $[X,Y] = Y$, hence $\mathfrak{m}_G = X$ and

$$\triangle_G = (a\partial_a)^2 + (a\partial_b)^2 - a\partial_a = a^2(\partial_a^2 + \partial_b^2)$$

is the Laplace–Beltrami operator of the hyperbolic plane, as it must be.

4.5. **Contact maps.** Let G and H be polarized Lie groups and $\Omega \subset G$ open. A C^1-map $F : \Omega \to H$ is a *contact map*, or a *horizontal map*, if

$$\mathrm{D}F(p)[V(G)] \subseteq V(H), \qquad \forall p \in \Omega.$$

Let G and H be sub-Riemannian Lie groups, $\Omega \subset G$ open, $X_1, \dots, X_m$ an orthonormal basis of $V(G)$, and $F : \Omega \to H$ a C^1 contact map. We shall use the functions

$$A_i : \Omega \to V(H) \subset \mathfrak{h}, \qquad A_i(p) := \mathrm{D}F(p)[X_i], \qquad i = 1, \dots, m. \tag{17}$$

If F is a C^2 contact map, then $\mathrm{D}^2F(p)|_{V(G)}$ is valued in $V(H)$ and the *trace* of the bilinear map $\mathrm{D}^2F(p)|_{V(G)} : V(G) \times V(G) \to V(H)$ is

$$\mathrm{tr}_G(\mathrm{D}^2F(p)) := \sum_{i=1}^m \mathrm{D}^2F(p)[X_i, X_i],$$

for one (thus every) orthonormal basis $X_1, \dots, X_m$ of $V(G)$. By (9), we have $\tilde X_i A_i = \mathrm{D}^2F[X_i, X_i]$, and thus

$$\mathrm{tr}_G(\mathrm{D}^2F) = \sum_{i=1}^m \tilde X_i A_i. \tag{18}$$

4.6. **Homothetic projections.** Let V, W be Hilbert spaces. The *transpose* of a linear map $L : V \to W$ is the linear map $L^T : W \to V$ such that

$$\langle L^T w, v\rangle_V = \langle w, Lv\rangle_W$$

for all $v \in V$ and $w \in W$.

A linear map $L : V \to W$ is a *homothetic projection* of factor $\lambda \geq 0$ if

$$\langle L^T w_1, L^T w_2\rangle_V = \lambda^2 \langle w_1, w_2\rangle_W, \tag{19}$$

for all $w_1, w_2 \in W$, i.e., the transpose $L^T : W \to V$ is a *homothetic embedding* of factor λ. It follows that, if $\lambda = 0$ then $L = 0$, while if $\lambda > 0$ then L is surjective.

Lemma 4.3. *Let V and W be Hilbert spaces with $X_1, \dots, X_m$ an orthonormal basis of V, $L : V \to W$ linear, and $\lambda \in [0, +\infty)$. Then L is a homothetic projection of factor λ if and only if*

$$\sum_{i=1}^m \langle w, LX_i\rangle_W \cdot LX_i = \lambda^2\, w \quad \forall w \in W. \tag{20}$$

In this case, for every linear map $P : W \to W$,

$$\sum_{i=1}^m \langle PLX_i, LX_i\rangle_W = \lambda^2\, \mathrm{tr}(P). \tag{21}$$

Proof. For $w \in W$ we have $L^T w = \sum_i \langle L^T w, X_i\rangle_V X_i = \sum_i \langle w, LX_i\rangle_W X_i$, hence

$$\langle L^T w_1, L^T w_2\rangle_V = \sum_{i=1}^m \langle w_1, LX_i\rangle_W \langle w_2, LX_i\rangle_W,$$

and by (19) this equals $\lambda^2\langle w_1, w_2\rangle_W$ for all $w_1, w_2 \in W$ if and only if (20) holds. For (21), expand the trace of $P \circ \sum_i LX_i \otimes LX_i = \lambda^2 P$ in an orthonormal basis $Y_1, \dots, Y_n$ of W:

$$\lambda^2 \operatorname{tr}(P) = \sum_{a=1}^n \left\langle P \sum_{i=1}^m \langle Y_a, LX_i\rangle_W LX_i, Y_a \right\rangle_H = \sum_{i=1}^m \langle PLX_i, LX_i\rangle_H.$$

□

4.7. **Conformal submersion of sub-Riemannian Lie groups.** Let G and H be sub-Riemannian Lie groups, and $\Omega \subset G$ open. A *C^1-conformal submersion* of factor $\lambda : \Omega \to [0, +\infty)$ is a C^1 contact map $F : \Omega \to H$, such that, for every $p \in \Omega$, the restriction $\mathrm{D}F(p)|_{V(G)} : V(G) \to V(H)$ is a linear homothetic projection of factor $\lambda(p)$. Conformal submersions are also known as *semiconformal maps.*

We stress that the term "submersion" is slightly abused since $\lambda(p)$ may be zero, in which case $\mathrm{D}F(p)|_{V(G)} = 0$. However, *if F is a C^∞-smooth conformal submersion of factor λ, then F is a submersion on $\{\lambda \neq 0\}$*. Indeed, if $\lambda(p) \neq 0$, then $\mathrm{D}F(p)[V(G)] = V(H)$. Since $V(H)$ is bracket-generating, F is a smooth submersion on $\{\lambda \neq 0\}$, that is, the Lie differential $\mathrm{D}F(p) : \mathfrak{g} \to \mathfrak{h}$ is surjective for all $p \in \Omega \cap \{\lambda \neq 0\}$, see Lemma A.1.

Let $X_1, \dots, X_m$ be an orthonormal basis of $V(G)$, $A_1, \dots, A_m$ as in (17), and $Y_1, \dots, Y_n$ an orthonormal basis of $V(H)$. Taking $w = Y_a$ in (20) and summing over a, we see that the factor λ of a conformal submersion F is determined by F, because

$$\sum_{i=1}^m |A_i|_H^2 = n\,\lambda^2. \tag{22}$$

4.8. **Horizontal energy and harmonic maps.** Let G and H be sub-Riemannian Lie groups, $\Omega \subset G$ open, and $X_1, \dots, X_m$ an orthonormal basis of $V(G)$. The *horizontal energy* of a C^1 contact map $F : \Omega \to H$ is

$$\mathscr{E}(F) := \frac{1}{2} \int_\Omega \sum_{i=1}^m \left|\mathrm{D}F[X_i]\right|_H^2 \,\mathrm{dvol}_G \;\in [0, +\infty], \tag{23}$$

which does not depend on the choice of the orthonormal basis.

A *contact variation* of a smooth contact map $F : \Omega \to H$ is a smooth family $(F_s)_{s\in(-\epsilon,\epsilon)}$ of contact maps $\Omega \to H$ with $F_0 = F$ and such that $F_s = F$ on $\Omega \setminus K$, for every s, for some compact set $K \subset \Omega$; the *variation field* of $(F_s)_s$ is

$$\varphi \in C_c^\infty(\Omega; \mathfrak{h}), \qquad \varphi(p) := \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} F(p)^{-1} F_s(p), \qquad \forall p \in \Omega.$$

Since $F_s = F$ outside K, the derivative at $s = 0$ of $s \mapsto \frac{1}{2}\int_K \sum_i |\mathrm{D}F_s[X_i]|_H^2 \,\mathrm{dvol}_G$ does not depend on the choice of K: we denote it by $\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \mathscr{E}(F_s)$, also when $\mathscr{E}(F) = +\infty$. The map F is a *harmonic map* if

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \mathscr{E}(F_s) = 0 \qquad \text{for every variation } (F_s)_s \text{ of } F.$$

We stress that every F_s is required to be a contact map.

5. Regularity of harmonic morphisms

5.1. **Smoothness of horizontal harmonic morphisms.** In the following theorem, we show that harmonic morphisms are smooth if we assume continuity and a weak condition on the preservation of the horizontal structures. We stress that the definition of harmonic morphism does not assume the map to be differentiable or to preserve the horizontal distributions.

A sub-Riemannian manifold M with horizontal bundle $\tilde{V} \subset TM$ is *equiregular* if, for every $k \in \mathbb{N}$, the set $\tilde{V}^{(k)} \subset TM$ is a subbundle of TM, where $\tilde{V}^{(k)}$ is the set generated k-iterated brackets of sections of $\tilde{V}$. We use equiregularity only in the following Theorem 5.1, because it is required in [8, Theorem 4.4].

Theorem 5.1 (Theorem A)**.** *Let M and N be measured equiregular sub-Riemannian manifolds and let $F : M \to N$ be a continuous harmonic morphism. Suppose that F maps horizontal curves to horizontal curves, that is, if γ is an AC horizontal curve in M, then $F \circ \gamma$ is an AC horizontal curve in N. Then F is C^∞-smooth.*

Proof. The proof is a direct application of results from [8].

Fix $p \in M$ and $q = F(p) \in N$. First, by [8, Theorem 4.4], there is a system of coordinates $y_1, \dots, y_n$ for N at q so that $y_1, \dots, y_r$ are $\triangle_N$-harmonic functions, where r is the rank of N. Since F is harmonic, the functions $F_k = y_k \circ F$ are $\triangle_M$-harmonic for $k \in \{1, \dots, r\}$, and thus smooth. Since F maps horizontal curves to horizontal curves, we can apply [8, Proposition 4.14] to obtain that F is C^∞-smooth. □

5.2. **Harmonic coordinates.** Recall from Section 3.8 that a measured sub-Riemannian manifold is harmonically homogeneous if it carries a transitive Lie group action by harmonic morphisms, and that sub-Riemannian Lie groups are harmonically homogeneous.

Theorem 5.2 (Theorem B)**.** *Let M be a measured sub-Riemannian manifold that is harmonically homogeneous. Then every point of M has harmonic coordinates, that is, the coordinate functions are harmonic functions.*

Proof. Let K be a Lie group and let $K \times M \to M$ be a transitive action by harmonic morphisms of M. For every $p \in M$, define

$$\tilde{W}_p := \{v \in T_pM : \mathrm{d}u(p)[v] = 0 \text{ for every harmonic function } u \text{ defined near } p\}.$$

If $k \in K$ and u is harmonic near kp, then $u \circ k$ is harmonic near p, because k acts as a harmonic morphism; hence $\mathrm{d}u(kp)[\mathrm{d}k(p)[v]] = \mathrm{d}(u \circ k)(p)[v] = 0$ for every $v \in \tilde{W}_p$, that is, $\mathrm{d}k(p)[\tilde{W}_p] \subseteq \tilde{W}_{kp}$. Applying the same argument to k^{-1}, we get $\mathrm{d}k(p)[\tilde{W}_p] = \tilde{W}_{kp}$. Since the action is transitive, the set $\tilde{W} := \bigsqcup_{p \in M} \tilde{W}_p \subset TM$ is a K-invariant smooth vector subbundle of TM.

By construction, for every $X \in \Gamma(\tilde{W})$, we have that $Xu = 0$ for every harmonic function u on M.

Arguing by contradiction, assume that $\tilde{W} \neq 0$, and thus there is a non-zero smooth vector field $X \in \Gamma(\tilde{W})$. Fix $p \in M$ with $X(p) \neq 0$ and $\gamma : (-\epsilon, \epsilon) \to M$ with $\gamma(0) = p$ and $\dot{\gamma}(t) = X(\gamma(t))$ for every $t \in (-\epsilon, \epsilon)$, for some $\epsilon > 0$.

Let f be a smooth map defined on some open subset Ω of M such that $Xf \equiv 1$ and $p \in \Omega$.

Since measured sub-Riemannian manifolds are Brelot harmonic spaces, see Proposition 3.1, we can find a $\triangle_M$-regular domain $U \Subset \Omega$ with $p \in U$ and $\gamma((-\epsilon, \epsilon)) \cap M \setminus U \neq \emptyset$. In particular, since $\gamma^{-1}(U)$ is an open subset of $\mathbb{R}$ containing 0, there are $t_- < 0$ and $t_+ > 0$ such that $\gamma(t_-) \in \partial U$, $\gamma(t_+) \in \partial U$, and $\gamma(t) \in U$ for every $t \in (t_-, t_+)$.

Let $u \in C(\bar{U})$ be a harmonic function in U and so that $u = f$ on ∂U. Since u is harmonic, then $Xu = 0$ and thus $u(\gamma(t_-)) = u(\gamma(t_+))$. Since $Xf = 1$, then $f(\gamma(t_+)) - f(\gamma(t_-)) = t_+ - t_- > 0$: contradiction.

We conclude that $\tilde{W} = 0$ and thus, for every $p \in M$, there are harmonic functions $u_1, \dots, u_n$ so that $\mathrm{d}u_1(p), \dots, \mathrm{d}u_n(p)$ is a basis of T_p^*M, and thus these functions form a system of coordinates at p. □

Theorem 5.3 (Theorem C)**.** *Let M and N be measured sub-Riemannian manifolds and $F : M \to N$ a harmonic morphism. Suppose that N is harmonically homogeneous. Then F is C^∞-smooth.*

Proof. Fix $p \in M$. By Theorem 5.2, there are an open neighborhood Ω_N of $F(p)$ in N and harmonic functions $u_1, \dots, u_n$ on Ω_N that form a system of coordinates on Ω_N. Since F is a harmonic morphism, the functions $u_k \circ F$ are harmonic on the open set $F^{-1}(\Omega_N)$, and thus they are smooth, by the hypoellipticity of $\triangle_M$. Since $(u_1, \dots, u_n)$ is a chart of N, we conclude that F is smooth on $F^{-1}(\Omega_N)$. □

Since sub-Riemannian Lie groups are harmonically homogeneous, we obtain the following.

Corollary 5.4. *Let G and H be sub-Riemannian Lie groups, $\Omega \subset G$ open, and $F : \Omega \to H$ a harmonic morphism. Then F is C^∞-smooth.*

6. Proof of Theorem D

6.1. **Pull back of differential operators.** A part of classical results for harmonic spaces, Proposition 6.2, which follows from Lemma 6.1, covers the main step of the proof of Theorem D.

Lemma 6.1. *Let V be a vector space over a field $\mathbb{K}$ and $\alpha, \beta \in \mathtt{Lin}_{\mathbb{K}}(V; \mathbb{K})$. If $\ker(\alpha) \subset \ker(\beta)$, then there exists $f \in \mathbb{K}$ such that*

$$\beta v = f \cdot \alpha v, \qquad \forall v \in V. \tag{24}$$

Proof. If $\alpha = 0$, then $\beta = 0$ as well and we can take $f = 0$. Suppose $\alpha \neq 0$: then there is $\hat{v} \in V$ such that $\alpha\hat{v} \neq 0$. If $v \in V$, then $\alpha\left((\alpha\hat{v})v - (\alpha v)\hat{v}\right) = 0$ and the assumption $\ker(\alpha) \subset \ker(\beta)$ implies that $\beta\left((\alpha\hat{v})v - (\alpha v)\hat{v}\right) = 0$. It follows by linearity that $(\alpha\hat{v})(\beta v) = (\alpha v)(\beta\hat{v})$ which proves (24) with $f = \beta\hat{v}/\alpha\hat{v}$. □

Proposition 6.2. *Let M, N be real manifolds and $F : M \to N$ a map of class C^k for some $k \in \mathbb{N}$. Let P_M and P_N be differential operators of class C^p and order d on M and N, respectively, with $p, d \leq k$. Assume that, for every $y \in N$, $P_N|_y \neq 0$. Assume also that for every open set $U \subset N$ and $u \in C^d(U; \mathbb{R})$, if $P_N u \geq 0$ on U, then $P_M(u \circ F) \geq 0$ on $F^{-1}(U)$.*

Then there exists a function $f : M \to [0, +\infty)$ of class C^s with $s = \min\{k - d, p\}$ such that

$$P_M \circ F^* = f \cdot (F^* \circ P_N). \tag{25}$$

Equivalently, for every $U \subset N$ open and $u \in C^k(U)$,

$$P_M(u \circ F) = f \cdot (P_N u) \circ F \qquad \text{on } F^{-1}(U).$$

Proof. Fix $x \in M$, put $y = Fx$, and let C_y^k denote the set of germs of C^k functions at y. Consider the two linear functionals $\alpha, \beta : C_y^k \to \mathbb{R}$ defined by: $\beta u = (P_M(u \circ F))x$ and $\alpha u = (P_N u)y$.

We claim that

$$\ker(\alpha) \subset \ker(\beta). \tag{26}$$

Indeed, since $P_N|_y \neq 0$, then there exists $w \in C^k_y$ such that $P_N w(y) > 0$. If $u \in C^k_y$ satisfies $\alpha u = 0$, then for every $\epsilon > 0$, there is a neighborhood of y where $P_N(u + \epsilon w) = P_N u + \epsilon P_N w > 0$. It follows that for every $\epsilon > 0$ there is a neighborhood of x where $P_M((u + \epsilon w) \circ F) \geq 0$ and we conclude that $P_M(u \circ F)x \geq 0$. Similarly, for every $\epsilon > 0$ there is a neighborhood of y where $P_N(u - \epsilon w) = P_N u - \epsilon P_N w < 0$. Hence for every $\epsilon > 0$ there is a neighborhood of x where $P_M((u - \epsilon w) \circ F) \leq 0$ and we conclude that $P_M(u \circ F)x \leq 0$. It follows that $\beta u = P_M(u \circ F)x = 0$ which proves the claim at (26).

By Lemma 6.1, there exists $f \in \mathbb{R}$ such that $\beta = f\alpha$. So, there exists a function $f : M \to \mathbb{R}$ such that, for every $U \subset N$ open and every $u \in C^k(U; \mathbb{R})$, $P_M(u \circ F) = f \cdot ((P_N u) \circ F)$, which proves (25).

To check the regularity of f we begin by observing that if $x \in M$ then $P_N|_{F(x)} \neq 0$, and so there is a neighborhood V of $F(x)$ in N and $u \in C^k(V; \mathbb{R})$ such that $P_N u(y) > 0$ for all $y \in V$. Therefore, $((P_N u) \circ F)(z) > 0$ for all $z \in F^{-1}(V)$, where $F^{-1}(V)$ is an open neighborhood of x since F is continuous. Identity (25) implies that,

$$\forall z \in F^{-1}(V), \qquad f(z) = \frac{P_M(u \circ F)(z)}{(P_N u) \circ F(z)} \geq 0.$$

Set $s = \min\{k - d, p\}$. Notice that $P_N u \in C^s(V; \mathbb{R})$ and thus $(P_N u) \circ F \in C^s(F^{-1}(V); \mathbb{R})$. Moreover, $u \circ F \in C^k(F^{-1}(V); \mathbb{R})$ and thus

$$P_M(u \circ F) \in C^s(F^{-1}(V); \mathbb{R}).$$

We conclude that $f \in C^s(F^{-1}(V); \mathbb{R})$. □

6.2. **Proof of Theorem D.** It is clear that $(ii) \Rightarrow (i)$. We need to show the reverse implication $(i) \Rightarrow (ii)$.

First of all, we note that M and N are Brelot harmonic spaces by Proposition 3.1. If F is a harmonic morphism in the sense of Section 2.3, then Proposition 2.1 implies that F pulls back superharmonic functions from N to M. Proposition 3.2 ensures that the hypothesis of Proposition 6.2 are satisfied and thus we get (1) from (25).

Proposition 6.2 implies also that $f \geq 0$ and that f is of class C^{k-2}, which concludes the proof of Theorem D. □

7. Harmonic morphisms of sub-Riemannian Lie groups

In this section we prove Theorem E. The main ingredients of the proof are Theorem D and [23, Theorem A]. We will restate the latter in Proposition 7.1 using the harmonic morphism operator defined in Section 7.1.

7.1. **The harmonic morphism operator.** Let G and H be sub-Riemannian Lie groups, $\Omega \subset G$ open, and $F : \Omega \to H$ a C^2 conformal submersion of factor λ. The *harmonic morphism operator* of F is the function

$$\mathscr{T}(F) := \mathrm{tr}_G(\mathrm{D}^2 F) - \mathrm{D}F[\mathfrak{m}_G] + \lambda^2\, \mathfrak{m}_H \ :\ \Omega \to V(H), \tag{27}$$

where $\mathfrak{m}_G \in V(G)$ and $\mathfrak{m}_H \in V(H)$ are the modular vectors of G and H. By (22), the operator $\mathscr{T}(F)$ depends only on F. Equation (3) of Theorem E reads as $\mathscr{T}(F) = 0$.

7.2. **Symmetries of the sub-Riemannian Laplacian.** We recall a result from [23] and show a consequence we need for harmonic morphisms in Corollary 7.2.

Proposition 7.1 ([23, Theorem A]). *Let G and H be sub-Riemannian Lie groups, $\Omega_G \subset G$ and $\Omega_H \subset H$ open, $F : \Omega_G \to \Omega_H$ a C^2-smooth map, and $\lambda : \Omega_G \to [0, +\infty)$, $b : \Omega_G \to V(H)$ and $c : \Omega_G \to \mathbb{R}$ continuous functions. The following statements are equivalent:*

(i) for every $v \in C^2(\Omega_H)$,

$$\triangle_G(v \circ F) = \lambda^2 \cdot (\triangle_H v) \circ F + \langle b, (\nabla_H v) \circ F \rangle_H + c \cdot (v \circ F); \tag{28}$$

(ii) F *is a conformal submersion of factor* λ, $c \equiv 0$ *and* $b = \mathscr{T}(F)$.

Corollary 7.2 (Composition formula)**.** *Let* G *and* H *be sub-Riemannian Lie groups,* $\Omega \subset G$ *open, and* $F : \Omega \to H$ *a* C^2 *conformal submersion of factor* $\lambda : \Omega \to [0, +\infty)$. *Then, for every every* $v \in C^2(H)$ *and every* $p \in \Omega$,

$$\triangle_G(v \circ F)(p) = \lambda(p)^2\,(\triangle_H v)(F(p)) + \mathrm{D}v(F(p))\big[\mathscr{T}(F)(p)\big]. \tag{29}$$

Consequently, F *is a harmonic morphism if and only if* $\mathscr{T}(F) \equiv 0$.

Proof. Apply Proposition 7.1 to F with $b := \mathscr{T}(F)$, which is a continuous $V(H)$-valued function, and with $c \equiv 0$: statement (ii) holds by assumption, and thus (28) holds. Since $\mathscr{T}(F)(p) \in V(H)$, the definition of the horizontal gradient gives $\langle \mathscr{T}(F)(p), \nabla_H v(F(p)) \rangle_H = \mathrm{D}v(F(p))[\mathscr{T}(F)(p)]$, and (29) follows.

If $\mathscr{T}(F) \equiv 0$, then (29) shows that $v \circ F$ is harmonic whenever v is harmonic, that is, F is a harmonic morphism. Conversely, suppose that F is a harmonic morphism. By Theorem D, there is $f : \Omega \to [0, +\infty)$ such that $\triangle_G(v \circ F) = f \cdot (\triangle_H v) \circ F$ for every $v \in C^2(H)$; the function f is continuous, and so is $\sqrt{f}$. Hence statement (i) of Proposition 7.1 holds with $\sqrt{f}$, $b = 0$ and $c = 0$ in place of λ, b and c. Statement (ii) then implies that F is a conformal submersion of factor $\sqrt{f}$ and that $\mathscr{T}(F) = 0$. By (22), the factor of a conformal submersion is unique, so that $\sqrt{f} = \lambda$. □

7.3. **Proof of Theorem E.** (i)⇒(ii). By Corollary 5.4, the map F is C^∞-smooth. Theorem D then gives a function $f : \Omega \to [0, +\infty)$ such that $\triangle_G(v \circ F) = f \cdot (\triangle_H v) \circ F$ for every $v \in C^\infty(H)$. Set $\lambda := \sqrt{f}$.

(ii)⇒(iii). Both sides of (2) at a point $p \in \Omega$ depend only on the 2-jet of v at $F(p)$, and every 2-jet at a point of H is the 2-jet of a function in $C^\infty(H)$. Therefore, statement (ii) implies statement (i) of Proposition 7.1 with $b = 0$ and $c = 0$, for every $\Omega_H \subset H$ open. The function λ is continuous because $\lambda^2 = \triangle_G(v \circ F)/(\triangle_H v) \circ F$ for any $v \in C^\infty(H)$ with $\triangle_H v > 0$ near $F(p)$. Hence statement (ii) of Proposition 7.1 holds, that is, F is a conformal submersion of factor λ and $\mathscr{T}(F) = 0$, which is (3).

(iii)⇒(i). This is the last statement of Corollary 7.2. □

8. Harmonicity of harmonic morphisms: the first variation for harmonic morphisms

The theorem of Fuglede [16] and Ishihara [21] states that a map between Riemannian manifolds is a harmonic morphism if and only if it is a conformal submersion (a.k.a. semiconformal maps, see Section 4.7) and a harmonic map. In the following three sections we discuss the corresponding question in the setting of sub-Riemannian Lie groups: *is a harmonic morphism a harmonic map, i.e., a critical point of the horizontal energy* (23)*?* Conjecturally, is the equation $\mathscr{T}(F) = 0$ in Theorem E the Euler–Lagrange equation of the horizontal energy among contact maps? Such Euler–Lagrange equations have been obtained by Grong and Markina in [17, Theorem 4.2].

The answer is negative, and the obstruction is a linear form on the Lie algebra of the target alone, the modular mismatch χ of Section 8.1. The crucial hinge to this section is the pointwise identity of Proposition 8.5, which we prove in Section 8.2: it computes the first variation of the horizontal energy along a conformal submersion in terms of $\mathscr{T}(F)$ and of χ. From it we deduce that harmonic morphisms are harmonic maps whenever $\chi = 0$ for some complement of the polarization of the target (Section 8.3), which is the case for Riemannian targets and for Carnot targets, and that this fails in general (Section 9.1). In Section 10.1 we compare our identity with the Euler–Lagrange equations of [17].

Remark 8.1. The reason for the discrepancy is already visible in the definitions. The horizontal energy (23) does not see the measure of the target: only the scalar product on $V(H)$ enters it. The harmonic morphism equation (3), on the contrary, sees vol_H through the modular vector $\mathfrak{m}_H$, because $\triangle_H$ does. The two agree in particular when the modular data of H are compatible with a splitting $\mathfrak{h} = V(H)\oplus\mathfrak{q}$, which is exactly the condition $\chi = 0$. In the Riemannian case there is no splitting to be compatible with; in the Carnot case the group is unimodular *and* the polarization is the bottom layer of a grading. Both times the mismatch is invisible.

8.1. **The modular mismatch.**

Definition 8.2. Let H be a polarized Lie group with polarization $V(H) \subset \mathfrak{h}$ and let $\mathfrak{q} \subset \mathfrak{h}$ be a complement of $V(H)$ in $\mathfrak{h}$, that is,

$$\mathfrak{h} = V(H) \oplus \mathfrak{q}, \tag{30}$$

with projections $\pi_{V(H)} : \mathfrak{h} \to V(H)$ and $\pi_{\mathfrak{q}} : \mathfrak{h} \to \mathfrak{q}$. The *horizontal* and the *vertical modular characters* of the splitting (30) are the linear forms $\kappa_{V(H)}, \kappa_{\mathfrak{q}} \in \mathfrak{h}^*$,

$$\kappa_{V(H)}(B) := \mathrm{tr}\big(\pi_{V(H)} \circ \mathrm{ad}_B|_{V(H)}\big), \qquad \kappa_{\mathfrak{q}}(B) := \mathrm{tr}\big(\pi_{\mathfrak{q}} \circ \mathrm{ad}_B|_{\mathfrak{q}}\big), \qquad \forall B \in \mathfrak{h},$$

so that $\kappa_{V(H)} + \kappa_{\mathfrak{q}} = \kappa_H$. The *modular mismatch* of the splitting (30) is

$$\chi := \kappa_H \circ \pi_{V(H)} - \kappa_{V(H)} = \kappa_{\mathfrak{q}} \circ \pi_{V(H)} - \kappa_{V(H)} \circ \pi_{\mathfrak{q}} \ \in \mathfrak{h}^*. \tag{31}$$

We say that the complement $\mathfrak{q}$ is *modularly balanced* if $\chi = 0$, that is, if

$$\mathrm{tr}\big(\pi_{V(H)}\,\mathrm{ad}_Z|_{V(H)}\big) = 0 \quad \forall Z \in \mathfrak{q}, \quad \text{and} \quad \mathrm{tr}\big(\pi_{\mathfrak{q}}\,\mathrm{ad}_B|_{\mathfrak{q}}\big) = 0 \quad \forall B \in V(H). \tag{32}$$

Notice that χ depends only on the polarized Lie algebra $(\mathfrak{h}, V(H))$ and on the choice of $\mathfrak{q}$: it involves neither a map F, nor a source group G, nor the scalar products.

Proposition 8.3. *The following polarized Lie algebras* $(\mathfrak{h}, V(H))$ *admit a modularly balanced complement:*

(a) Riemannian Lie groups*: if* $V(H) = \mathfrak{h}$*;*
(b) if $V(H)$ *admits a complement* $\mathfrak{q}$ *that is an ideal of* $\mathfrak{h}$ *with* $\mathrm{tr}(\mathrm{ad}_B|_{\mathfrak{q}}) = 0$ *for every* $B \in V(H)$*;*
(c) Carnot groups*: if* $\mathfrak{h}$ *is stratified by* $\mathfrak{h} = V_1 \oplus \cdots \oplus V_s$ *and* $V(H) = V_1$*;*
(d) Unimodular Lie groups with trace-free horizontal action*: if* $\kappa_H = 0$ *and* $\kappa_{V(H)} = 0$ *for some complement* $\mathfrak{q}$.

Proof. (a) Take $\mathfrak{q} = \{0\}$: then $\pi_{\mathfrak{q}} = 0$ and $\kappa_{\mathfrak{q}} = 0$, so $\chi = 0$.

(b) If $\mathfrak{q}$ is an ideal and $Z \in \mathfrak{q}$, then $[Z, V(H)] \subseteq \mathfrak{q}$, hence $\pi_{V(H)}\,\mathrm{ad}_Z|_{V(H)} = 0$ and $\kappa_{V(H)}|_{\mathfrak{q}} = 0$. Moreover $\mathrm{ad}_B(\mathfrak{q}) \subseteq \mathfrak{q}$ for every $B \in \mathfrak{h}$, so $\kappa_{\mathfrak{q}}(B) = \mathrm{tr}(\mathrm{ad}_B|_{\mathfrak{q}})$, which vanishes on $V(H)$ by assumption.

(c) Take $\mathfrak{q} = V_2 \oplus \cdots \oplus V_s$, which is an ideal; for $B \in V_1$ the map ad_B sends V_j into V_{j+1}, so $\mathrm{ad}_B|_{\mathfrak{q}}$ is nilpotent and traceless. Apply (b).

(d) $\kappa_{\mathfrak{q}} = \kappa_H - \kappa_{V(H)} = 0$, so $\chi = 0$ by (31). $\square$

8.2. **The first variation of the horizontal energy.** We first compute the derivative of the Lie differential along a variation. Recall from Section 4.8 that all the maps of a variation are contact maps.

Lemma 8.4. *Let G and H be sub-Riemannian Lie groups, $\Omega \subset G$ open, $X_1, \dots, X_m$ an orthonormal basis of $V(G)$, and let $F : \Omega \to H$ be a smooth contact map with $A_1, \dots, A_m$ as in* (17)*. Let $(F_s)_{s\in(-\epsilon,\epsilon)}$ be a variation of F with variation field φ. Then*

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \mathrm{D}F_s[v] = \tilde{v}\varphi + [\mathrm{D}F_s[v], \varphi], \qquad \forall v \in \mathfrak{g}. \tag{33}$$

This implies that

$$\tilde{v}\varphi + [\mathrm{D}F_s[v], \varphi] \in V(H), \qquad \forall v \in V(G), \tag{34}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \mathscr{E}(F_s) = \int_\Omega \sum_{i=1}^{m} \big\langle A_i, \tilde{X}_i\varphi + [A_i, \varphi]\big\rangle_H \,\mathrm{dvol}_G. \tag{35}$$

Proof. Set $c_s(p) := F(p)^{-1}F_s(p)$, so that $c_0 \equiv 1$ and $\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0} c_s = \varphi$. For $v \in \mathfrak{g}$ we have

$$F_s(p)^{-1}F_s(p\exp(tv)) = \Big(c_s(p)^{-1}\big(F(p)^{-1}F(p\exp(tv))\big)c_s(p)\Big) \cdot \Big(c_s(p)^{-1}c_s(p\exp(tv))\Big),$$

which is a product of two curves through the identity of H whose derivatives at $t = 0$ therefore add:

$$\mathrm{D}F_s(p)[v] = \mathrm{Ad}\big(c_s(p)^{-1}\big)\mathrm{D}F(p)[v] + \mathrm{D}c_s(p)[v]. \tag{36}$$

We differentiate (36) at $s = 0$. On the one hand, $\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0} \mathrm{Ad}(c_s^{-1}) = -\operatorname{ad}_\varphi$. On the other hand, since $c_0 \equiv 1$, for every fixed t the derivative at $s = 0$ of $s \mapsto c_s(p)^{-1}c_s(p\exp(tv))$ is $\varphi(p\exp(tv)) - \varphi(p)$, hence $\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0} \mathrm{D}c_s(p)[v] = \tilde{v}\varphi(p)$. We have obtained (33).

Each F_s is a contact map, so $\mathrm{D}F_s[v] \in V(H)$ for every $v \in V(G)$ and every s, and (34) follows by differentiating at $s = 0$. Finally, all the F_s agree outside a fixed compact subset of Ω, so we may differentiate (23) under the integral sign, and (35) follows from (33). □

The following identity contains all the results of this section.

Proposition 8.5 (Key identity)**.** *Let G and H be sub-Riemannian Lie groups, $\Omega \subset G$ open, $X_1, \dots, X_m$ an orthonormal basis of $V(G)$, and $F : \Omega \to H$ a C^2-smooth conformal submersion of factor λ, with $A_1, \dots, A_m$ as in* (17)*. Let $\mathfrak{q}$ be a complement of $V(H)$ in $\mathfrak{h}$ and let χ be the modular mismatch of the splitting $\mathfrak{h} = V(H) \oplus \mathfrak{q}$. Then, for every $\varphi \in C^\infty(\Omega; \mathfrak{h})$,*

$$\sum_{i=1}^{m} \big\langle A_i, \pi_{V(H)}\big(\tilde{X}_i\varphi + [A_i, \varphi]\big)\big\rangle_H = \operatorname{div}_{\mathrm{vol}_G}\Big(\sum_{i=1}^{m} \langle A_i, \pi_{V(H)}\varphi\rangle_H \,\tilde{X}_i\Big) - \big\langle \mathscr{T}(F), \pi_{V(H)}\varphi\big\rangle_H + \lambda^2\,\chi(\varphi), \tag{37}$$

where $\mathscr{T}(F)$ is the harmonic morphism operator (27)*.*

Proof. We compute the two summands of the left-hand side of (37).

For the second one, observe that $P_\varphi := \pi_{V(H)} \circ \operatorname{ad}_\varphi|_{V(H)}$ is an endomorphism of $V(H)$, that $\pi_{V(H)}[A_i, \varphi] = -P_\varphi A_i$, and that $A_i \in V(H)$; hence, by (21),

$$\sum_{i=1}^{m} \big\langle A_i, \pi_{V(H)}[A_i, \varphi]\big\rangle_H = -\sum_{i=1}^{m} \big\langle A_i, P_\varphi A_i\big\rangle_H = -\lambda^2\,\mathrm{tr}(P_\varphi) = -\lambda^2\,\kappa_{V(H)}(\varphi).$$

For the first summand, since $\pi_{V(H)}$ is a fixed linear map, the Leibniz rule and (18) give

$$\sum_{i=1}^{m} \big\langle A_i, \pi_{V(H)}(\tilde{X}_i\varphi)\big\rangle_H = \sum_{i=1}^{m} \tilde{X}_i\langle A_i, \pi_{V(H)}\varphi\rangle_H - \big\langle \mathrm{tr}_G(\mathrm{D}^2F), \pi_{V(H)}\varphi\big\rangle_H.$$

By Lemma 4.1(b), by (12) and by (7), writing $f_i := \langle A_i, \pi_{V(H)}\varphi\rangle_H$,

$$\begin{aligned}\sum_{i=1}^m \tilde{X}_i f_i &= \operatorname{div}_{\mathrm{vol}_G}\Big(\sum_{i=1}^m f_i\tilde{X}_i\Big) + \sum_{i=1}^m \kappa_G(X_i) f_i \\ &= \operatorname{div}_{\mathrm{vol}_G}\Big(\sum_{i=1}^m f_i\tilde{X}_i\Big) + \big\langle \mathrm{D}F[\mathfrak{m}_G], \pi_{V(H)}\varphi\big\rangle_H,\end{aligned}$$

where in the last equality we used the linearity of $\mathrm{D}F$.

Adding the two summands, and using $\langle \mathfrak{m}_H, \pi_{V(H)}\varphi\rangle_H = \kappa_H(\pi_{V(H)}\varphi)$, which holds by (11) on H because $\pi_{V(H)}\varphi \in V(H)$, we get that the left-hand side of (37) equals

$$\begin{aligned}&\operatorname{div}_{\mathrm{vol}_G}\Big(\sum_{i=1}^m f_i\tilde{X}_i\Big) - \big\langle \operatorname{tr}_G(\mathrm{D}^2F) - \mathrm{D}F[\mathfrak{m}_G], \pi_{V(H)}\varphi\big\rangle_H - \lambda^2\kappa_{V(H)}(\varphi) \\ &\quad= \operatorname{div}_{\mathrm{vol}_G}\Big(\sum_{i=1}^m f_i\tilde{X}_i\Big) - \big\langle \mathscr{T}(F), \pi_{V(H)}\varphi\big\rangle_H + \lambda^2\kappa_H(\pi_{V(H)}\varphi) - \lambda^2\kappa_{V(H)}(\varphi),\end{aligned}$$

which is (37) by (31). □

Theorem 8.6. *Let G and H be sub-Riemannian Lie groups, $\Omega \subset G$ open, and $F : \Omega \to H$ a C^2-smooth conformal submersion of factor λ. Let $\mathfrak{q}$ be a complement of $V(H)$ in $\mathfrak{h}$ and let χ be the modular mismatch of the splitting $\mathfrak{h} = V(H) \oplus \mathfrak{q}$. Then, for every variation $(F_s)_{s\in(-\epsilon,\epsilon)}$ of F with variation field φ,*

$$\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \mathscr{E}(F_s) = \int_\Omega \Big(\lambda^2\,\chi(\varphi) - \big\langle \mathscr{T}(F), \pi_{V(H)}\varphi\big\rangle_H\Big)\,\mathrm{dvol}_G. \tag{38}$$

In particular, the right-hand side of (38) *does not depend on the choice of $\mathfrak{q}$, although both its summands do.*

Proof. By (34) we have $\tilde{X}_i\varphi + [A_i, \varphi] \in V(H)$, and $A_i \in V(H)$, so that $\langle A_i, \tilde{X}_i\varphi + [A_i,\varphi]\rangle_H = \langle A_i, \pi_{V(H)}(\tilde{X}_i\varphi + [A_i,\varphi])\rangle_H$. Hence (38) follows by integrating (37) over Ω and comparing with (35): indeed, the vector field $\sum_i f_i\tilde{X}_i$ has compact support, because so does φ, and thus the integral of its divergence vanishes, by (5). Finally, the left-hand side of (38) does not involve $\mathfrak{q}$. □

Since harmonic morphisms are the conformal submersions with $\mathscr{T}(F) = 0$, we obtain Theorem F:

Corollary 8.7 (Theorem F)**.** *Let G and H be sub-Riemannian Lie groups, $\Omega \subset G$ open, and $F : \Omega \to H$ a harmonic morphism, with conformal factor λ as in Theorem E. Let $\mathfrak{q}$ be a complement of $V(H)$ in $\mathfrak{h}$ and let χ be the modular mismatch of the splitting $\mathfrak{h} = V(H) \oplus \mathfrak{q}$. Then, for every variation $(F_s)_{s\in(\epsilon,\epsilon)}$ of F with variation field φ,*

$$\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \mathscr{E}(F_s) = \int_\Omega \lambda^2\,\chi(\varphi)\,\mathrm{dvol}_G. \tag{39}$$

Consequently, F is a harmonic map if and only if $\int_\Omega \lambda^2\chi(\varphi)\,\mathrm{dvol}_G = 0$ for every variation field φ of F.

Proof. By Theorem E, F is a smooth conformal submersion of factor λ, and $\mathscr{T}(F) = 0$. Apply Theorem 8.6. □

8.3. **When harmonic morphisms are harmonic maps.**

Theorem 8.8. *Let G and H be sub-Riemannian Lie groups, $\Omega \subset G$ open, and $F : \Omega \to H$ a harmonic morphism. If the polarized Lie algebra $(\mathfrak{h}, V(H))$ admits a modularly balanced complement, then F is a harmonic map.*

Proof. Immediate from Corollary 8.7 with $\chi = 0$. □

Corollary 8.9. *Let G and H be sub-Riemannian Lie groups, $\Omega \subset G$ open, and $F : \Omega \to H$ a harmonic morphism. If H is Riemannian, or if H is a Carnot group, or, more generally, if the polarized Lie algebra $(\mathfrak{h}, V(H))$ is as in Proposition 8.3, then F is a harmonic map.*

9. A harmonic morphism that is not a harmonic map

9.1. **A harmonic morphism that is not a harmonic map.**

Definition 9.1. Let $\mathfrak{h}$ be the 4-dimensional real Lie algebra with basis Y_1, Y_2, Y_3, Z and brackets

$$[Y_1, Y_2] = Z, \qquad [Y_3, Y_1] = Y_1, \qquad [Y_3, Y_2] = Y_2, \qquad [Y_3, Z] = 2Z, \tag{40}$$

all other brackets of basis vectors being zero. Let H be the corresponding connected and simply connected Lie group, polarized by

$$V(H) := \langle Y_1, Y_2, Y_3 \rangle,$$

with Y_1, Y_2, Y_3 declared orthonormal, and endowed with a left Haar measure vol_H.

Lemma 9.2. *The algebra $\mathfrak{h}$ of Definition 9.1 is solvable and isomorphic to $\mathbb{R}Y_3 \ltimes \mathfrak{h}_1$, where $\mathfrak{h}_1 = \langle Y_1, Y_2, Z \rangle$ is the 3-dimensional Heisenberg algebra and $\mathrm{ad}_{Y_3}|_{\mathfrak{h}_1}$ is the derivation $\mathrm{diag}(1, 1, 2)$; equivalently, $\mathfrak{h}$ is the Iwasawa algebra $\mathfrak{a} \oplus \mathfrak{n}$ of $\mathfrak{su}(2, 1)$, that is, the Lie algebra of the group of the complex hyperbolic plane $\mathbb{C}H^2$. The subspace $V(H)$ is a bracket-generating hyperplane, H is not unimodular, and, for the complement $\mathfrak{q} := \mathbb{R}Z$,*

$$\kappa_H = 4Y_3^*, \qquad \mathfrak{m}_H = 4Y_3, \qquad \kappa_{V(H)} = \kappa_{\mathfrak{q}} = 2Y_3^*, \qquad \chi = 2Y_3^* \neq 0,$$

where Y_3^* is the dual basis vector of Y_3.

Proof. The endomorphism $\delta := \mathrm{diag}(1, 1, 2)$ of $\mathfrak{h}_1$ is a derivation, because $\delta[Y_1, Y_2] = 2Z = [\delta Y_1, Y_2] + [Y_1, \delta Y_2]$; hence the semidirect product $\mathbb{R}Y_3 \ltimes_\delta \mathfrak{h}_1$ is a Lie algebra with the brackets (40). The subspace $V(H)$ is bracket generating because $[Y_1, Y_2] = Z$. In the basis (Y_1, Y_2, Y_3, Z) the matrices of $\mathrm{ad}_{Y_1}, \mathrm{ad}_{Y_2}, \mathrm{ad}_Z$ have zero diagonal, while $\mathrm{ad}_{Y_3} = \mathrm{diag}(1, 1, 0, 2)$; therefore $\kappa_H = 4Y_3^*$ and, by (11) on H, $\mathfrak{m}_H = 4Y_3$. With $\mathfrak{q} = \mathbb{R}Z$ we have $\pi_{V(H)} \mathrm{ad}_{Y_3}|_{V(H)} = \mathrm{diag}(1, 1, 0)$, and $\pi_{V(H)} \mathrm{ad}_B|_{V(H)}$ has zero diagonal for $B \in \{Y_1, Y_2\}$, while $\mathrm{ad}_Z Y_1 = \mathrm{ad}_Z Y_2 = 0$ and $\pi_{V(H)} \mathrm{ad}_Z Y_3 = \pi_{V(H)}(-2Z) = 0$; hence $\kappa_{V(H)} = 2Y_3^*$. Similarly $\kappa_{\mathfrak{q}}(B)$ is the coefficient of Z in $[B, Z]$, so $\kappa_{\mathfrak{q}} = 2Y_3^*$. Finally $\chi = \kappa_{\mathfrak{q}} \circ \pi_{V(H)} - \kappa_{V(H)} \circ \pi_{\mathfrak{q}} = 2Y_3^*$, because $\kappa_{V(H)}(Z) = 0$. □

Theorem 9.3. *Let H be the sub-Riemannian Lie group of Definition 9.1, let $\Omega \subset H$ be a bounded open set and let $F := \mathrm{Id}|_\Omega : \Omega \to H$. Then:*

- *(i) F is a harmonic morphism, it is a conformal submersion of factor $\lambda \equiv 1$, and $\mathscr{T}(F) = 0$;*
- *(ii) F is* not *a harmonic map: there are variations of F along which the horizontal energy strictly decreases.*

Proof. (i) The identity is trivially a harmonic morphism. Moreover $\mathrm{D}F(p) = \mathrm{Id}_{\mathfrak{h}}$ for every p, so $\mathrm{D}F(p)|_{V(H)}$ is a homothetic projection of factor 1, and $\mathrm{D}^2 F = 0$; since the source and the target coincide, $\mathscr{T}(F) = 0 - \mathfrak{m}_H + \mathfrak{m}_H = 0$, in accordance with Corollary 7.2.

(ii) Fix $\varphi_3 \in C_c^\infty(\Omega;\mathbb{R})$ with $\varphi_3 \geq 0$ and $\varphi_3 \not\equiv 0$, and define

$$F_s(p) := p \exp\big(s\, \varphi_3(p) Y_3\big), \qquad s \in \mathbb{R}. \tag{41}$$

Each F_s is smooth and agrees with F outside the support of φ_3. Write $\psi := s\varphi_3$ and $c(p) := \exp(\psi(p)Y_3)$. For $v \in \mathfrak{h}$ we have

$$F_s(p)^{-1} F_s(p\exp(tv)) = \big(c(p)^{-1}\exp(tv)c(p)\big) \cdot \big(c(p)^{-1}c(p\exp(tv))\big),$$

a product of two curves through the identity, whose derivatives at $t = 0$ therefore add. Since $c(p)^{-1}c(p\exp(tv)) = \exp\big((\psi(p\exp(tv)) - \psi(p))Y_3\big)$, both factors lying in the one-parameter subgroup generated by Y_3, we get

$$\mathrm{D}F_s(p)[v] = \mathrm{Ad}\big(\exp(-\psi(p)Y_3)\big)v + (\tilde{v}\psi)(p)\, Y_3. \tag{42}$$

The map $\mathrm{Ad}(\exp(-\psi Y_3)) = e^{-\psi\, \mathrm{ad}_{Y_3}}$ acts as $e^{-\psi}$ on $\langle Y_1, Y_2\rangle$, as the identity on Y_3 and as $e^{-2\psi}$ on Z; in particular it preserves $V(H)$. As also $Y_3 \in V(H)$, formula (42) shows that F_s *is a contact map for every* s, so that $(F_s)_s$ is a variation of F in the sense of Section 4.8. Explicitly,

$$\mathrm{D}F_s[Y_j] = e^{-\psi}Y_j + (\tilde{Y}_j\psi)Y_3 \quad (j = 1,2), \qquad \mathrm{D}F_s[Y_3] = (1 + \tilde{Y}_3\psi)Y_3,$$

so that the energy density in (23) is

$$\sum_{i=1}^{3} \big|\mathrm{D}F_s[Y_i]\big|_H^2 = 2e^{-2\psi} + (1 + \tilde{Y}_3\psi)^2 + (\tilde{Y}_1\psi)^2 + (\tilde{Y}_2\psi)^2.$$

We differentiate at $s = 0$, where $\psi = 0$ and $\frac{\partial\psi}{\partial s} = \varphi_3$; the two squares of first derivatives do not contribute, and we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \mathscr{E}(F_s) = \frac{1}{2}\int_\Omega \big(-4\varphi_3 + 2\,\tilde{Y}_3\varphi_3\big)\,\mathrm{dvol}_H = \int_\Omega \big(\tilde{Y}_3\varphi_3 - 2\varphi_3\big)\,\mathrm{dvol}_H.$$

By (15) and Lemma 9.2, $\int_\Omega \tilde{Y}_3\varphi_3\,\mathrm{dvol}_H = \kappa_H(Y_3)\int_\Omega \varphi_3\,\mathrm{dvol}_H$, and $\kappa_H(Y_3) = 4$. Therefore

$$\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \mathscr{E}(F_s) = 2\int_\Omega \varphi_3\,\mathrm{dvol}_H > 0,$$

in accordance with Corollary 8.7 and $\chi = 2Y_3^*$. Hence $\mathscr{E}(F_s) < \mathscr{E}(F)$ for $s < 0$ small, and F is not a harmonic map. □

Remark 9.4. Some comments on Theorem 9.3.

(a) The variation (41) is explicit and elementary, and the proof is independent of the theory of [17]: in particular it does not use the compactness and simple connectedness assumptions, nor the dichotomy between regular and singular points. It is a statement about the horizontal energy alone. This is relevant, because Theorem 9.3 rules out the abnormal equation of [17, Theorem 4.2(a)] as well: a map that is not a critical point of the energy is not a harmonic map, whichever equation one writes.

(b) By Corollary 8.9, no such example exists with a Riemannian, or Carnot, or unimodular target with $\kappa_{V(H)} = 0$. It is the interplay between the non-unimodularity of H and the polarization $V(H)$ that produces the phenomenon, and not the non-unimodularity alone.

(c) The map $F = \mathrm{Id}$ is an isometry preserving the measure; the example shows in particular that isometries of sub-Riemannian Lie groups need not be critical points of the horizontal energy.

10. The normal equation for harmonic morphisms

10.1. **Comparison with the Grong–Markina equations.** The critical points of the horizontal energy among contact maps are studied in [17, Theorem 4.2] by means of Lagrange multipliers for the constraint (34). In this section we recast the resulting *normal equation* in our notation and we compare it with the key identity (37).

Let G and H be sub-Riemannian Lie groups, $\Omega \subset G$ open, $X_1, \dots, X_m$ an orthonormal basis of $V(G)$, and $F : \Omega \to H$ a smooth contact map with $A_1, \dots, A_m$ as in (17). For $\varphi \in C^\infty(\Omega; \mathfrak{h})$ we set

$$L_F\varphi : \Omega \to \mathfrak{h} \otimes V(G)^*, \qquad (L_F\varphi)[v] := \tilde{v}\varphi + [\mathrm{D}F[v], \varphi], \quad \forall v \in V(G),$$

which is the derivative (33) of $\mathrm{D}F$ along a variation with field φ, and, for $\eta : \Omega \to \mathfrak{h}^* \otimes V(G)^*$, writing $\eta_i := \eta[X_i]$,

$$(L_F^*\eta)(B) := -\sum_{i=1}^m \tilde{X}_i\big(\eta_i(B)\big) + \sum_{i=1}^m \kappa_G(X_i)\,\eta_i(B) + \sum_{i=1}^m \eta_i\big([A_i, B]\big), \qquad \forall B \in \mathfrak{h}. \tag{43}$$

Lemma 10.1. *The operator L_F^* is the formal adjoint of L_F: for every $\eta : \Omega \to \mathfrak{h}^* \otimes V(G)^*$ of class C^1 and every $\varphi \in C_c^\infty(\Omega; \mathfrak{h})$,*

$$\int_\Omega \sum_{i=1}^m \eta_i\big((L_F\varphi)[X_i]\big)\,\mathrm{dvol}_G = \int_\Omega (L_F^*\eta)(\varphi)\,\mathrm{dvol}_G. \tag{44}$$

Proof. By the Leibniz rule and (15),

$$\begin{aligned}\int_\Omega \eta_i(\tilde{X}_i\varphi)\,\mathrm{dvol}_G &= \int_\Omega \Big(\tilde{X}_i\big(\eta_i(\varphi)\big) - (\tilde{X}_i\eta_i)(\varphi)\Big)\,\mathrm{dvol}_G \\ &= \int_\Omega \Big(\kappa_G(X_i)\,\eta_i - \tilde{X}_i\eta_i\Big)(\varphi)\,\mathrm{dvol}_G.\end{aligned}$$

Summing over i and adding $\sum_i \eta_i([A_i, \varphi])$ gives (44). □

Definition 10.2. A smooth contact map $F : \Omega \to H$ is a *normal harmonic map* if there exists $\eta : \Omega \to \mathfrak{h}^* \otimes V(G)^*$ of class C^1, called a *multiplier*, such that

$$\eta_i(B) = \langle A_i, B\rangle_H \quad \forall B \in V(H),\ i = 1, \dots, m, \qquad \text{and} \qquad L_F^*\eta = 0. \tag{45}$$

Lemma 10.3. *A normal harmonic map is a harmonic map.*

Proof. Let η be a multiplier as in (45) and let φ be the variation field of a variation of F. By (34) we have $(L_F\varphi)[X_i] \in V(H)$, hence $\langle A_i, (L_F\varphi)[X_i]\rangle_H = \eta_i\big((L_F\varphi)[X_i]\big)$. By (35) and (44),

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \mathscr{E}(F_s) = \int_\Omega \sum_{i=1}^m \eta_i\big((L_F\varphi)[X_i]\big)\,\mathrm{dvol}_G = \int_\Omega (L_F^*\eta)(\varphi)\,\mathrm{dvol}_G = 0.$$

□

Given a complement $\mathfrak{q}$ of $V(H)$ in $\mathfrak{h}$, we write

$$A^\flat := \langle A, \pi_{V(H)}(\,\cdot\,)\rangle_H \in \mathfrak{h}^*, \qquad A \in V(H), \tag{46}$$

so that $A^\flat|_{V(H)} = \langle A, \cdot\rangle_H$ and $A^\flat|_{\mathfrak{q}} = 0$. If $\mathrm{Ann}(V(H))$ denotes the space of linear forms on $\mathfrak{h}$ that vanish on $V(H)$, the first condition in (45) holds if and only if

$$\eta_i = A_i^\flat + \nu_i \qquad \text{with} \qquad \nu_i : \Omega \to \mathrm{Ann}(V(H)), \quad i = 1, \dots, m, \tag{47}$$

and the ν_i are uniquely determined by η.

Corollary 10.4. *Let G and H be sub-Riemannian Lie groups, $\Omega \subset G$ open, $F : \Omega \to H$ a smooth conformal submersion of factor λ, let $\mathfrak{q}$ be a complement of $V(H)$ in $\mathfrak{h}$ with modular mismatch χ, and let η be as in* (47)*. Then, for every $B \in \mathfrak{h}$,*

$$(L_F^*\eta)(B) = -\big\langle \mathscr{T}(F), \pi_{V(H)}B \big\rangle_H + \lambda^2\,\chi(B) + (L_F^*\nu)(B). \tag{48}$$

Proof. By the linearity of L_F^*, it is enough to prove (48) for $\nu = 0$, that is, for $\eta_i = A_i^\flat$. Apply the key identity (37) to the constant function $\varphi \equiv B$, and observe that $\langle A_i, \pi_{V(H)}\varphi\rangle_H = A_i^\flat(B)$ and $\tilde{X}_i\varphi = 0$, so that the left-hand side of (37) is $\sum_i A_i^\flat([A_i, B])$, while

$$\begin{aligned}\operatorname{div}_{\mathrm{vol}_G}\Big(\sum_{i=1}^m A_i^\flat(B)\tilde{X}_i\Big) &= \sum_{i=1}^m \Big(\tilde{X}_i\big(A_i^\flat(B)\big) - \kappa_G(X_i)A_i^\flat(B)\Big)\\ &= -(L_F^*\eta)(B) + \sum_{i=1}^m A_i^\flat\big([A_i, B]\big),\end{aligned}$$

by Lemma 4.1(b) and (43). $\square$

Theorem 10.5. *Let G and H be sub-Riemannian Lie groups, $\Omega \subset G$ open, and $F : \Omega \to H$ a smooth conformal submersion. Assume that the polarized Lie algebra $(\mathfrak{h}, V(H))$ admits a modularly balanced complement $\mathfrak{q}$. Then the following statements are equivalent:*

(i) F is a harmonic morphism;
(ii) $\mathscr{T}(F) = 0$, that is, F satisfies (3)*;*
(iii) F is a normal harmonic map, with the multiplier η given by $\eta_i = A_i^\flat$.

Proof. (i) $\iff$ (ii) is Corollary 7.2. For $\eta_i = A_i^\flat$, Corollary 10.4 with $\nu = 0$ and $\chi = 0$ gives $(L_F^*\eta)(B) = -\langle \mathscr{T}(F), \pi_{V(H)}B\rangle_H$ for every $B \in \mathfrak{h}$. If $\mathscr{T}(F) = 0$, this vanishes and η is a multiplier, because $A_i^\flat|_{V(H)} = \langle A_i, \cdot\rangle_H$ by (46); conversely, if it vanishes for every $B \in \mathfrak{h}$, then $\mathscr{T}(F) = 0$, because $\pi_{V(H)}$ is onto $V(H)$ and $\mathscr{T}(F) \in V(H)$. $\square$

The next result shows that the obstruction to the existence of a multiplier is pointwise and purely Lie-algebraic. Given a complement $\mathfrak{q}$ of $V(H)$ in $\mathfrak{h}$, we denote by $\mathscr{B}(A, B) := \pi_{\mathfrak{q}}[A, B]$, for $A, B \in V(H)$, the Levi form of $(\mathfrak{h}, V(H))$.

Lemma 10.6. *Let G and H be sub-Riemannian Lie groups, $\Omega \subset G$ open, $F : \Omega \to H$ a smooth conformal submersion of factor λ, and $\mathfrak{q}$ a complement of $V(H)$ in $\mathfrak{h}$. Let $p \in \Omega$ with $\lambda(p) \neq 0$, and let $\nu_1, \dots, \nu_m \in \operatorname{Ann}(V(H))$. The following statements are equivalent:*

(i) $\displaystyle\sum_{i=1}^m \nu_i\big(\mathscr{B}(A_i(p), B)\big) = -\lambda(p)^2\,\kappa_{\mathfrak{q}}(B)$ for every $B \in V(H)$;
(ii) the linear map $\Psi : \mathfrak{q} \to V(H)$, $\Psi(Z) := \lambda(p)^{-2}\sum_i \nu_i(Z)A_i(p)$, satisfies

$$\operatorname{tr}\Big(V(H) \ni A \longmapsto \Psi\big(\mathscr{B}(A, B)\big)\Big) = -\kappa_{\mathfrak{q}}(B) \qquad \forall B \in V(H). \tag{49}$$

Moreover, for $\Psi \in \operatorname{Hom}(\mathfrak{q}, V(H))$ the subspace $\mathfrak{q}_\Psi := \{Z - \Psi Z : Z \in \mathfrak{q}\}$ is a complement of $V(H)$ in $\mathfrak{h}$ with

$$\kappa_{\mathfrak{q}_\Psi}(B) = \kappa_{\mathfrak{q}}(B) + \operatorname{tr}\Big(A \mapsto \Psi\big(\mathscr{B}(A, B)\big)\Big), \qquad B \in V(H), \tag{50}$$

so that (49) *holds if and only if $\kappa_{\mathfrak{q}_\Psi}|_{V(H)} = 0$.*

Proof. (ii)$\Rightarrow$(i): the map $A \mapsto \Psi(\mathscr{B}(A, B)) = \lambda^{-2}\sum_i \nu_i(\mathscr{B}(A, B))A_i$ is a sum of rank-one maps $c_i \otimes A_i$ with $c_i := \nu_i(\mathscr{B}(\,\cdot\,, B)) \in V(H)^*$, whose trace is $c_i(A_i)$; hence its trace is $\lambda^{-2}\sum_i \nu_i(\mathscr{B}(A_i, B))$. The converse implication is the same computation.

For (50), identify $\mathfrak{q}_\Psi$ with $\mathfrak{q}$ by $Z \mapsto Z - \Psi Z$ and observe that, modulo $V(H)$, the projections onto $\mathfrak{q}$ and onto $\mathfrak{q}_\Psi$ agree; therefore, for $B \in V(H)$,

$$\kappa_{\mathfrak{q}_\Psi}(B) = \operatorname{tr}\big(Z \mapsto \pi_{\mathfrak{q}}[B, Z - \Psi Z]\big) = \kappa_{\mathfrak{q}}(B) - \operatorname{tr}\big(\pi_{\mathfrak{q}} \circ \operatorname{ad}_B \circ \Psi\big).$$

On the other hand, writing $S : V(H) \to \mathfrak{q}$, $S(A) := \mathscr{B}(A, B)$, and using $\operatorname{tr}_{V(H)}(\Psi S) = \operatorname{tr}_{\mathfrak{q}}(S\Psi)$,

$$\begin{aligned}-\operatorname{tr}\big(\pi_{\mathfrak{q}} \circ \operatorname{ad}_B \circ \Psi\big) &= \operatorname{tr}\big(Z \mapsto \pi_{\mathfrak{q}}[\Psi Z, B]\big) \\ &= \operatorname{tr}_{\mathfrak{q}}(S\Psi) = \operatorname{tr}_{V(H)}(\Psi S) = \operatorname{tr}\Big(A \mapsto \Psi\big(\mathscr{B}(A, B)\big)\Big),\end{aligned}$$

and (50) follows. □

Theorem 10.7. *Let G and H be sub-Riemannian Lie groups, $\Omega \subset G$ open, and $F : \Omega \to H$ a harmonic morphism with conformal factor λ as in Theorem E, with $\lambda(p) \neq 0$ for some $p \in \Omega$. If F is a normal harmonic map, then $V(H)$ admits a complement $\mathfrak{q}'$ in $\mathfrak{h}$ such that*

$$\kappa_{\mathfrak{q}'}|_{V(H)} = 0, \qquad \textit{that is,} \qquad \operatorname{tr}\big(\pi_{\mathfrak{q}'} \operatorname{ad}_B|_{\mathfrak{q}'}\big) = 0 \quad \forall B \in V(H). \tag{51}$$

Condition (51) *depends only on the polarized Lie algebra* $(\mathfrak{h}, V(H))$*: it involves neither F, nor G, nor the scalar products.*

Proof. Fix a complement $\mathfrak{q}$ of $V(H)$ in $\mathfrak{h}$, let η be a multiplier, written as in (47), and evaluate the identity (48) at p on a vector $B \in V(H)$. The functions $\nu_i(B)$ vanish identically, because $\nu_i \in \operatorname{Ann}(V(H))$; hence the first two summands of $(L_F^*\nu)(B)$ in (43) vanish, and the third one is $\sum_i \nu_i([A_i, B]) = \sum_i \nu_i(\mathscr{B}(A_i, B))$, again because ν_i annihilates $V(H)$. Since $\mathscr{T}(F) = 0$ and $\chi|_{V(H)} = \kappa_{\mathfrak{q}}|_{V(H)}$, the equation $L_F^*\eta = 0$ gives

$$0 = \lambda(p)^2 \kappa_{\mathfrak{q}}(B) + \sum_{i=1}^{m} \nu_i\big(\mathscr{B}(A_i(p), B)\big) \qquad \forall B \in V(H),$$

which is statement (i) of Lemma 10.6. The lemma then produces $\Psi \in \operatorname{Hom}(\mathfrak{q}, V(H))$ with $\kappa_{\mathfrak{q}_\Psi}|_{V(H)} = 0$. □

Corollary 10.8. *For the polarized Lie algebra $(\mathfrak{h}, V(H))$ of Definition 9.1 there is* no *complement $\mathfrak{q}'$ of $V(H)$ in $\mathfrak{h}$ with $\kappa_{\mathfrak{q}'}|_{V(H)} = 0$. Consequently, the map F of Theorem 9.3 is not a normal harmonic map, and no multiplier as in* (45) *exists for it.*

Proof. By Lemma 10.6 it suffices to show that (49) has no solution $\Psi \in \operatorname{Hom}(\mathfrak{q}, V(H))$, where $\mathfrak{q} = \mathbb{R}Z$. Set $P := \Psi(Z) \in V(H)$ and write $\mathscr{B}(A, B) = \beta(A, B)Z$, where β is the skew-symmetric bilinear form on $V(H)$ with $\beta(Y_1, Y_2) = 1$ and $\beta(Y_3, \cdot) = 0$. The map $A \mapsto \Psi(\mathscr{B}(A, B)) = \beta(A, B)P$ has trace $\beta(P, B)$, so (49) reads

$$\beta(P, B) = -\kappa_{\mathfrak{q}}(B) = -2Y_3^*(B) \qquad \forall B \in V(H).$$

Taking $B = Y_3$ gives $\beta(P, Y_3) = -2$, which is impossible because Y_3 belongs to the radical of β, so that $\beta(P, Y_3) = 0$. The last statement follows from Theorem 10.7, or directly from Lemma 10.3 and Theorem 9.3(ii). □

10.2. **Two questions.** The first question concerns the gap between the sufficient condition of Theorem 10.5, that is $\chi = 0$, and the necessary condition (51) of Theorem 10.7, which is the first half of (32) only. Notice that in this intermediate case the residual unknown ν of (47) has to solve the underdetermined first-order linear system $(L_F^*\nu)|_{\mathfrak{q}} = -\lambda^2\chi|_{\mathfrak{q}}$.

Question 1. Are there polarized Lie algebras $(\mathfrak{h}, V(H))$ that admit a complement $\mathfrak{q}$ with $\kappa_{\mathfrak{q}}|_{V(H)} = 0$ but no modularly balanced complement? If so, are harmonic morphisms into the corresponding sub-Riemannian Lie groups normal harmonic maps?

The second question is suggested by Remark 8.1: the horizontal energy (23) ignores the measure of the target, while harmonic morphisms do not.

Question 2. Is there a natural modification of the horizontal energy, involving the Haar measure of the target, whose critical points among contact maps are exactly the maps satisfying $\mathscr{T}(F) = 0$? By (39), the required correction of the first variation is $-\int_\Omega \lambda^2 \chi(\varphi)\, \mathrm{dvol}_G$.

Appendix A. Sub-Riemannian submersions

The following lemma is a simple technical result in differential geometry.

Lemma A.1. *Let M, N be smooth manifolds, and let $X_1, \dots, X_m \in \Gamma(TM)$ and $Y_1, \dots, Y_n \in \Gamma(TN)$ be two families of vector fields (not necessarily linearly independent). Let $\tilde{V} \subset TM$ and $\tilde{W} \subset TN$ be the sets*

$$\tilde{V} = \bigsqcup_{x \in M} \mathrm{Span}_{\mathbb{R}}\{X_1(x), \dots, X_m(x)\} \text{ and } \tilde{W} = \bigsqcup_{y \in M} \mathrm{Span}_{\mathbb{R}}\{Y_1(y), \dots, Y_n(y)\}.$$

Let $F : M \to N$ be a C^∞-smooth map with

$$\mathrm{d}F(x)[\tilde{V}|_x] = \tilde{W}|_{F(x)} \qquad \forall x \in M. \tag{52}$$

If $\tilde{W}$ is a bracket generating subbundle of TM, then F is a submersion, that is

$$\mathrm{d}F(p)[T_pM] = T_{F(p)}N \qquad \forall p \in M. \tag{53}$$

Proof. Fix $p \in M$. Up to taking an open neighborhood of $F(p)$ in N and selecting a subset of $\{Y_1(y), \dots, Y_n(y)\}$, we assume that $Y_1(y), \dots, Y_n(y)$ are linearly independent for every $y \in N$. By (52), up to reshuffling, we assume

$$\tilde{W}|_{F(p)} = \mathrm{Span}_{\mathbb{R}}\{\mathrm{d}F(p)[X_1(p)], \dots, \mathrm{d}F(p)[X_n(p)]\}.$$

This implies that $\{X_1, \dots, X_n\}$ are linearly independent in a neighborhood of p, and, up to shrinking M, we assume this neighborhood is M. By (52), there are $A_i^k \in C^\infty(M)$ so that

$$Y_i(F(x)) = \sum_{k=1}^n A_i^k(x) \mathrm{d}F(x)[X_k(x)] = \mathrm{d}F(x)\left[\sum_{k=1}^n A_i^k(x) X_k(x)\right] \qquad \forall i \in \{1, \dots, n\}. \tag{54}$$

Let $A_i := \sum_{k=1}^m A_i^k X_k \in \Gamma(TM)$ for all $i \in \{1, \dots, n\}$. The identity (54) means that A_i and Y_i are F*-related*, as in [24, p. 119] or [26, p. 182].

Let $\mathfrak{f}_n$ be the free Lie algebra with n generators $f_1, \dots, f_n$. Then we have two morphisms of Lie algebras $\phi_M : \mathfrak{f}_n \to \Gamma(TM)$ and $\phi_N : \mathfrak{f}_n \to \Gamma(TN)$ so that $\phi_M(f_j) = A_j$ and $\phi_N(f_j) = Y_j$, for every $j \in \{1, \dots, n\}$. By [24, p .119] or [26, Prop. 8.30], for every $v \in \mathfrak{f}_n$, $\phi_M v$ and $\phi_N v$ are F-related, i.e.,

$$(\phi_N v)(F(x)) = \mathrm{d}F(x)[(\phi_M v)(x)] \qquad \forall x \in M,\ \forall v \in \mathfrak{f}_n. \tag{55}$$

Since $\tilde{W}$ is bracket-generating, $\phi_N(\mathfrak{f}_n)|_{F(p)} = T_{F(p)}N$. From (55), we conclude $\mathrm{d}F(p)[T_pM] = T_{F(p)}N$.

$\square$

Remark A.2. Lemma A.1 requires $\tilde{W}$ to be a subbundle, that is, $y \mapsto \dim(\tilde{W}|_y)$ is constant. This hypothesis is necessary, as the following example shows. Consider $M = N = \mathbb{R}^2$, with the vector fields

$$X_1 = \partial_x, \quad X_2 = \partial_y,$$
$$Y_1 = \partial_x, \quad Y_2 = x\partial_y.$$

Notice that $\{Y_1, Y_2\}$ is bracket generating. The map $F : M \to N$, $F(x, y) := (x, x^2 y)$, does satisfy (52), but not (53) on $\{x = 0\}$.

Remark A.3. The C^∞ smoothness of F is required in (55). If $\tilde{W}$ has step s, we can use (55) only for $v \in \mathfrak{f}_n$ up to step s, and thus we only need F to be C^s-smooth.

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