THE TRUNCATED OCTAHEDRON MINIMIZES SURFACE AREA AMONG PARALLELOHEDRA OF EQUAL VOLUME

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Abstract. We prove that the regular truncated octahedron uniquely minimizes surface area among all parallelohedra of fixed volume. Equivalently, every three-dimensional parallelohedron P satisfies

$$\frac{\mathcal{H}^2(\partial P)}{|P|^{2/3}} \geq \frac{3(1+2\sqrt{3})}{4^{2/3}},$$

with equality if and only if P is similar to the regular truncated octahedron. Among the non-truncated Fedorov types we prove a stronger sharp bound, attained uniquely by the regular rhombic dodecahedron.

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1. Introduction

A three-dimensional parallelohedron is a convex polytope whose translates tile $\mathbb{R}^3$ face-to-face. In this paper we study the isoperimetric problem within this class: given a parallelohedron P, we consider the scale-invariant quantity

$$\mathrm{Iso}(P) = \frac{\mathcal{H}^2(\partial P)}{|P|^{2/3}}, \tag{1}$$

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and ask which shape minimizes it.

The natural candidate is the regular truncated octahedron. The corresponding statement is usually referred to as the *Truncated Octahedron Conjecture* and is attributed to Bezdek; see [16, Conjecture 1]. The conjecture asserts that, among all three-dimensional parallelohedra of fixed volume, the regular truncated octahedron has the smallest surface area.

This problem may be viewed as a lattice analogue of the classical isoperimetric problem, and is also closely related to the Kelvin problem for equal-volume partitions of space. Other optimization problems on lattices, such as sphere packing and quantization, likewise single out highly symmetric lattices. There is, however, no direct implication between these different variational problems; see [1, 3, 4] for related examples.

One of the main difficulties is the interaction between affine and Euclidean geometry. Three-dimensional parallelohedra admit a well-understood affine classification, whereas the isoperimetric quotient (1) is invariant under similarities but not under general affine transformations. Thus an affine description of the possible combinatorial types does not by itself identify the Euclidean minimizer.

Two particularly important examples are the Voronoi cells of the body-centered cubic lattice (BCC) and face-centered cubic lattice (FCC). The Voronoi cell of BCC is the regular truncated octahedron, while that of FCC is the regular rhombic dodecahedron. Their isoperimetric quotients are

$$\mathrm{Iso}_{\mathrm{BCC}} = \frac{3(1+2\sqrt{3})}{4^{2/3}} \approx 5.314739700, \qquad \mathrm{Iso}_{\mathrm{FCC}} = 3\,2^{5/6} \approx 5.345392309.$$

The small gap between these two values shows how sharp is the problem.

Our main result proves the Truncated Octahedron Conjecture and also identifies the sharp minimizer among all the remaining Fedorov types.

Theorem 1.1 (Truncated Octahedron Conjecture)**.** *For every three-dimensional parallelohedron P, one has*

$$\mathrm{Iso}(P) \geq \mathrm{Iso}_{\mathrm{BCC}} = \frac{3(1+2\sqrt{3})}{4^{2/3}}, \tag{2}$$

with equality if and only if P is similar to the regular truncated octahedron. In particular, for every three-dimensional lattice Λ,

$$\mathrm{Iso}(\mathrm{Vor}(\Lambda)) \geq \mathrm{Iso}_{\mathrm{BCC}},$$

with equality precisely for BCC, up to similarity.

If P is not of truncated-octahedral Fedorov type, then the stronger estimate

$$\mathrm{Iso}(P) \geq \mathrm{Iso}_{\mathrm{FCC}} = 3\,2^{5/6} \tag{3}$$

holds, with equality if and only if P is similar to the regular rhombic dodecahedron. In particular, among lattice Voronoi cells with at most twelve facets, equality occurs precisely at FCC.

Theorem 1.1 does not address arbitrary periodic equal-volume partitions, and hence does not solve the Kelvin problem. It does, however, settle the corresponding problem for convex lattice-translative tiles. Indeed, if a convex body $P \subset \mathbb{R}^3$ tiles space by translations along a lattice, then P is necessarily a parallelohedron; see, for instance, [20].

Let us briefly describe the main ideas of the proof. We first consider lattice Voronoi cells. A classical result of Selling implies that every three-dimensional lattice admits an obtuse superbase. This gives a convenient six-parameter description of its Voronoi cell and, through the geometry of zonotopes, an explicit formula for its surface area. The same description can be encoded by the complete graph K_4, a representation that makes both the volume and the facet structure particularly transparent.

The proof then separates naturally into the interior and the boundary of the parameter space. On the boundary we prove the sharp FCC inequality. The strata with few active generators are covered by the isoperimetric inequality of Joós and Lángi [14], while the remaining case is reduced to a four-point determinant inequality proved here.

The interior case contains the BCC lattice and is the main part of the argument. The determinant expansion associated with the K_4 representation defines a finite determinantal probability measure. Its one-point marginals, or leverage scores, allow us to reduce the original geometric inequality to a small number of scalar constraints. An entropy inequality, ultimately based on the rank-one Ball–Barthe inequality [2], provides the key estimate. The resulting scalar inequality is sharp, and equality forces all six Selling parameters to coincide, which gives the BCC lattice.

The lattice result alone is not sufficient for the full conjecture, since an arbitrary parallelohedron need not be a Voronoi cell in its given Euclidean metric. In the final step we therefore keep the same zonotopal description but allow the ambient Euclidean metric to vary. A determinant inequality controls these additional affine degrees of freedom and shows that no affine deformation can improve the BCC constant. Combining this with the boundary analysis yields Theorem 1.1 for all three-dimensional parallelohedra.

The approach builds on classical reduction theory. Selling's reduction of ternary quadratic forms [21], later incorporated into Voronoi's general reduction theory [23, 24], implies that every three-dimensional lattice is of Voronoi's first kind. Conway and Sloane gave a modern formulation in terms of conorms and vonorms and recovered in this way the five Fedorov types of three-dimensional lattice Voronoi cells [6, 7]. We also use the classical determinant formula for zonotope volumes [22]. Lángi proved that the regular truncated octahedron minimizes mean width among three-dimensional parallelohedra [16].

In our earlier work [5], we derived a closed formula for the isoperimetric quotient in Selling coordinates, proved the strict local minimality of BCC, and showed that FCC and the cubic lattice SC are not local minimizers. The present paper establishes the corresponding global result and extends it from lattice Voronoi cells to arbitrary three-dimensional parallelohedra.

The paper is organized as follows. Section 3 introduces Selling coordinates, the weighted K_4 zonotope, and the surface-area formula. Section 4 proves the sharp FCC boundary inequality, including the four-point determinant inequality. Section 5 treats the positive six-generator case and proves the global BCC inequality for lattice Voronoi cells. Section 6 extends the argument to arbitrary three-dimensional parallelohedra and completes the proof of the Truncated Octahedron Conjecture. The final section discusses some open problems.

We point out that ChatGPT (GPT-5.6 Sol) was used during the preparation of this work to check algebraic identities and to assist in exploring and verifying proof arguments.

After this work was completed, we became aware of an independent proof of the Truncated Octahedron Conjecture by Thomas Hales and Lark Song [13].

2. Notation

For $y = (y_1, \dots, y_d) \in \mathbb{R}^d$, the symbol $e_k(y)$, with $1 \leq k \leq d$ denotes its kth elementary symmetric function, that is the sum of all products of k distinct coordinates y_i of y. Let Ω be a finite probability space, given p and q probability measures on Ω, with the support of p contained in the support of q, we denote the relative entropy of p with respect to q (or Kullback-Leibler divergence) as follows:

$$D_{\mathrm{KL}}(p\|q) = \sum_{\omega\in\Omega} p(\omega) \log \frac{p(\omega)}{q(\omega)}, \tag{4}$$

with the convention $0 \log(0) = 0$.

The Gram matrix G associated with vectors $v_1, \dots, v_m$ is the symmetric matrix with entries $g_{ij} = v_i \cdot v_j$. The notation $d^{\circ 2}$ means entrywise squaring of a matrix d, that is, squaring each entry of the matrix, and $\mathrm{covol}(\Lambda)$ is the Euclidean covolume of a lattice, that is, the volume of its fundamental parallelotope. We also let K_4 be the complete graph on four vertices. A spanning tree of a graph is a connected, acyclic subgraph containing all its vertices. The group S_4 acts on Selling coordinates by relabelling the four vertices of the superbase.

3. The surface-area formula

In this section we introduce and recall the main results about lattice-Voronoi cells, and we derive the exact surface-area formula. The Voronoi cell associated to a lattice Λ in $\mathbb{R}^n$, denoted $\mathrm{Vor}(\Lambda)$, is an n-dimensional convex polytope symmetric about the origin and defined as

$$\mathrm{Vor}(\Lambda) := \{x \in \mathbb{R}^n : |x| \leq |x - \lambda|, \quad \forall \lambda \in \Lambda, \lambda \neq 0\}.$$

Equivalently $\mathrm{Vor}(\Lambda)$ can be defined as the intersection of the half spaces

$$H_\lambda = \{x \in \mathbb{R}^n : 2x \cdot \lambda \leq |\lambda|^2\}$$

for all $\lambda \in \Lambda \setminus \{0\}$.

3.1. **Selling-Voronoi reduction.** Lattices of Voronoi's first kind in $\mathbb{R}^n$ are full-rank lattices which admit an obtuse superbase, that is $n+1$ vectors $v_0, \ldots, v_n$ such that $v_1, \ldots v_n$ is a lattice basis, $v_0 + v_1 + \cdots + v_n = 0$ and $v_i \cdot v_j \leq 0$ for $i \neq j$. In dimension three every lattice is of Voronoi's first kind, see the following result.

Proposition 3.1 (Selling–Voronoi reduction). *Every full-rank lattice $\Lambda \subset \mathbb{R}^3$ admits an obtuse superbase v_0, v_1, v_2, v_3. The Selling parameters are defined as*

$$\rho_{ij} := -v_i \cdot v_j \geq 0 \qquad (i \neq j).$$

Conversely, any six nonnegative numbers ρ_{ij} whose associated Gram matrix is positive definite determine a lattice in $\mathbb{R}^3$, up to orthogonal isometry.

Proof. The existence statement is Selling's reduction theorem, also known as Voronoi's theorem (see [21, 23, 8]). For the definition of the superbase and of Selling parameters ρ_{ij} see [6, Sections 2 and 7]. If $v_0 = -v_1 - v_2 - v_3$, then

$$v_i \cdot v_j = -\rho_{ij}, \qquad v_i \cdot v_i = \rho_{0i} + \sum_{j \in \{1,2,3\} \setminus \{i\}} \rho_{ij}$$

which gives the Gram matrix $A = (v_i \cdot v_j)_{1 \leq i,j \leq 3}$ defined below. From positive definiteness of the matrix, one reconstructs the lattice. □

Let K_4 be the complete graph with vertex set $\{0, 1, 2, 3\}$. We may identify the vertices with the four vectors v_0, v_1, v_2, v_3 of the superbase and the Selling parameter ρ_{ij} with the weight of the edge ij of K_4. Thus every full-rank lattice is associated with a nonnegative edge-weighted realization of K_4. We use throughout the fixed edge order $(01, 02, 03, 12, 13, 23)$ and write

$$\rho = (a, b, c, d, e, f) = (\rho_{01}, \rho_{02}, \rho_{03}, \rho_{12}, \rho_{13}, \rho_{23}) \in \mathbb{R}^6_{\geq 0}.$$

The associated Gram matrix can be written as

$$A(\rho) = \begin{pmatrix} a+d+e & -d & -e \\ -d & b+d+f & -f \\ -e & -f & c+e+f \end{pmatrix}, \qquad D(\rho) = \det A(\rho).$$

A point ρ of the Selling cone may be regarded as a nonnegative weighting of the six edges of K_4, as shown in Figure 1. Passing to the boundary amounts to setting at least one of these edge weights equal to zero; the support graph of the remaining positive edges will therefore provide a convenient combinatorial description of the boundary strata considered below.

A reference realization of this abstract graph K_4 in $\mathbb{R}^3$ is the tetrahedron with vertices

$$p_0 = 0, \qquad p_1 = e_1, \qquad p_2 = e_2, \qquad p_3 = e_3.$$

The corresponding edge vectors are

$$r_a = e_1, \quad r_b = e_2, \quad r_c = e_3, \quad r_d = e_1 - e_2, \quad r_e = e_1 - e_3, \quad r_f = e_2 - e_3.$$

Equivalently, the Gram matrix can be written as

$$A(\rho) = \sum_\xi \xi r_\xi r_\xi^{\mathsf{T}}.$$

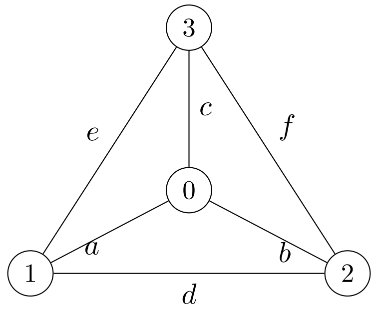

FIGURE 1. The weighted graph K_4 associated with the Selling vector $\rho = (a, b, c, d, e, f) = (\rho_{01}, \rho_{02}, \rho_{03}, \rho_{12}, \rho_{13}, \rho_{23})$.

For a nontrivial subset $S \subset \{0, 1, 2, 3\}$, let

$$c(S) = \sum_{ij \in \delta(S)} \rho_{ij}$$

be the total weight of the cut $S \mid S^c$, where $\delta(S)$ denotes the set of edges with one endpoint in S and the other in S^c. Modulo complementation there are seven nontrivial cuts. Their weights are:

$$\begin{aligned} &c_0 = a + b + c, && c_1 = a + d + e, && c_2 = b + d + f, && c_3 = c + e + f, \\ &c_4 = b + c + d + e, && c_5 = a + c + d + f, && c_6 = a + b + e + f. && \end{aligned} \tag{5}$$

Here $c_i = c(\{i\})$ for $i = 0, 1, 2, 3$, while c_4, c_5, c_6 correspond respectively to the cuts $\{0, 1\} \mid \{2, 3\}$, $\{0, 2\} \mid \{1, 3\}$, and $\{0, 3\} \mid \{1, 2\}$. For every cut $S \mid S^c$, we consider the spanning trees of the induced subgraphs $K_4[S]$ and $K_4[S^c]$, and we compute the weighted spanning-tree polynomial of $S \mid S^c$, i.e. the sum of the products of edge weights over all spanning trees. So the following seven quadratic polynomials are exactly the weighted spanning-tree polynomials associated to the previous seven cuts:

$$\begin{aligned} &q_0 = de + df + ef, && q_1 = bc + bf + cf, && q_2 = ac + ae + ce, && q_3 = ab + ad + bd, \\ &q_4 = af, && q_5 = be, && q_6 = cd. && \end{aligned} \tag{6}$$

The Gram determinant is

$$\begin{aligned} D(\rho) = \det A(\rho) = abc + abe + abf + acd + acf + ade + adf + aef + \\ + bcd + bce + bde + bdf + bef + cde + cdf + cef. \end{aligned} \tag{7}$$

The polynomial $D(\rho)$ is the weighted spanning-tree polynomial of K_4, that is the sum of the products of edge weights over all spanning trees of K_4.

The weighted matrix–tree theorem (see, for instance, [15, Eq. (1)]) gives the following identity:

$$\sum_{i=0}^{6} q_i c_i = 3D(\rho). \tag{8}$$

Indeed, for each nontrivial cut $S \mid S^c$ of K_4, modulo complementation, q_i is the weighted spanning-tree polynomial of the two induced subgraphs $K_4[S]$ and $K_4[S^c]$, while c_i is the total weight of the edges crossing the cut. Hence $q_i c_i$ is the total weight of spanning trees T for which $S \mid S^c$ is obtained by deleting one edge of T. Conversely, deleting any edge of a spanning tree produces exactly one such nontrivial cut. Since every spanning tree of K_4 has three edges, summing over the seven cuts counts every weighted spanning tree exactly three times, which proves (8).

3.2. **Zonotopal representation of the Voronoi cells.** Using the geometric representation r_ξ of the edges of the realization of K_4, we define the centered graphical zonotope

$$Z_\rho = \sum_{\xi\in\{a,b,c,d,e,f\}} \left[-\frac{\xi r_\xi}{2}, \frac{\xi r_\xi}{2}\right]. \tag{9}$$

Lattices of Voronoi's first kind are zonotopal lattices, that is, their Voronoi cells are zonotopes; see, for instance, [18]. In the present Selling coordinates this structure takes the following explicit graphical form.

Proposition 3.2 (Voronoi cell as a graphical zonotope)**.** *Let Λ be a lattice in $\mathbb{R}^3$ with associated Selling parameters ρ_{ij}. If $D(\rho) > 0$, then, up to an orthogonal change of coordinates, the Voronoi cell associated to Λ is given by*

$$\mathrm{Vor}(\Lambda) = A(\rho)^{-1/2} Z_\rho. \tag{10}$$

Moreover $|Z_\rho| = D(\rho)$.

Proof. Choose the lattice basis with Gram matrix $A(\rho)$, so that $\Lambda = A^{1/2}(\rho)\mathbb{Z}^3$. For $n \in \mathbb{Z}^3$, the support function of Z_ρ is

$$h_{Z_\rho}(n) = \frac{1}{2}\sum_\xi \xi |n\cdot r_\xi|.$$

Since $n\cdot r_\xi \in \mathbb{Z}$, one has $|n\cdot r_\xi| \le (n\cdot r_\xi)^2$. Recalling that $A(\rho) = \sum_\xi \xi r_\xi r_\xi^{\mathsf{T}}$,

$$h_{Z_\rho}(n) \le \frac{1}{2} nA(\rho)n^{\mathsf{T}}.$$

These are precisely the Voronoi half-space inequalities for the lattice $A^{1/2}(\rho)\mathbb{Z}^3$ applied to vectors scaled by $A^{-1/2}(\rho)$. Thus $A^{-1/2}(\rho)Z_\rho \subseteq \mathrm{Vor}(\Lambda)$.

By Shephard's zonotope decomposition formula [22, Eq. (57)],

$$|Z_\rho| = \sum_{\substack{I\subset E(K_4)\\ |I|=3}} \left(\prod_{\xi\in I}\rho_\xi\right) |\det(r_\xi)_{\xi\in I}|\,,$$

where ρ_ξ denotes the weight of the edge ξ. For the incidence roots of K_4, the determinant has absolute value one precisely when I is a spanning tree, and vanishes otherwise. Hence

$$|Z_\rho| = \sum_{T\in\mathcal{T}(K_4)} \prod_{\xi\in T}\rho_\xi = D(\rho),$$

and the image of Z_ρ under $A^{-1/2}(\rho)$ has volume $D^{1/2}(\rho) = |\,\mathrm{Vor}(\Lambda)|$. Recalling that $A^{-1/2}Z_\rho \subseteq \mathrm{Vor}(\Lambda)$, this implies the conclusion. $\square$

Finally we recall that the strict inequalities $\rho_{ij} > 0$ for all $i \neq j$ correspond to a lattice whose Voronoi cell has fourteen facets, whereas the class of full-rank lattices with at least one vanishing Selling parameter corresponds to the nondegenerate boundary of the Selling cone and the Voronoi cells associated to such lattices have at most twelve facets.

Given the nonnegative edge-weighted realization of K_4 associated to a lattice with Selling parameters ρ_{ij}, the active graph G_ρ consists of the subgraph of K_4 with the same number of vertices and only with the edges which have positive weight.

We have the following result (for the proof we refer to [6]).

Proposition 3.3. *If $D(\rho) > 0$, the active graph $G_\rho \subset K_4$ is connected. The opposite facet pairs of* $\mathrm{Vor}(\Lambda)$ *are in bijection with unordered nontrivial bipartitions $S \mid S^c$ for which both induced graphs $G_\rho[S]$ and $G_\rho[S^c]$ are connected. Consequently,* $\mathrm{Vor}(\Lambda)$ *has fourteen facets if and only if all six Selling parameters are positive; otherwise* $\mathrm{Vor}(\Lambda)$ *has at most twelve facets.*

Corollary 3.4. *Let $\Lambda \subset \mathbb{R}^3$ be a full-rank lattice and $P = \mathrm{Vor}(\Lambda)$. For any non-negative Selling parameter vector obtained from an obtuse superbase, the following are equivalent:*

(1) *P has at most twelve facets;*

(2) *at least one Selling parameter vanishes;*
(3) *the parameter vector lies on the nondegenerate boundary*

$$\left\{\rho \in \mathbb{R}^6_{\geq 0} : D(\rho) > 0,\ \min_{ij} \rho_{ij} = 0\right\}.$$

Moreover, P has fourteen facets if and only if all six parameters are positive; in this case P is combinatorially a truncated octahedron.

3.3. **The explicit formula for the isoperimetric quotient.** We associate to the seven nontrivial vertex cuts of K_4, modulo complementation, the following vectors

$$\begin{aligned} z_0 &= (1,1,1)^\mathsf{T}, & z_1 &= (1,0,0)^\mathsf{T}, & z_2 &= (0,1,0)^\mathsf{T}, & z_3 &= (0,0,1)^\mathsf{T}, \\ z_4 &= (0,1,1)^\mathsf{T}, & z_5 &= (1,0,1)^\mathsf{T}, & z_6 &= (1,1,0)^\mathsf{T}. \end{aligned} \tag{11}$$

Indeed the facet of Z_ρ exposed by z_i is, up to translation, the planar zonotope generated by the active edges internal to the two sides of the cut.

Proposition 3.5. *For every lattice Λ with $D(\rho) > 0$, the scale-invariant isoperimetric quotient of the lattice Voronoi cell is*

$$\mathrm{Iso}(\mathrm{Vor}(\Lambda)) = F(\rho) = \frac{2N(\rho)}{D(\rho)^{5/6}}, \qquad N(\rho) = \sum_{i=0}^{6} q_i \sqrt{c_i}. \tag{12}$$

The total surface area is $\mathcal{H}^2(\partial\,\mathrm{Vor}(\Lambda)) = 2N(\rho)/\sqrt{D(\rho)}$.

This formula has been derived in [5, Section 2], for the sake of completeness we provide the proof also here.

We recall that

$$\begin{aligned} \mathrm{Iso}_{\mathrm{BCC}} &= F(1,1,1,1,1,1) = \frac{3(1+2\sqrt{3})}{4^{2/3}} := F_{\mathrm{BCC}} \\ \mathrm{Iso}_{\mathrm{FCC}} &= F(0,1,1,1,1,0) = 3\ 2^{\frac{5}{6}} := F_{\mathrm{FCC}}. \end{aligned}$$

Proof. For planar generators $g_1, \dots, g_m$ lying in a plane with unit normal ν, the two-dimensional zonotope formula is

$$\mathcal{H}^2\left(\sum_{k=1}^{m} [0, g_k]\right) = \sum_{1 \leq r < s \leq m} |(g_r \times g_s) \cdot \nu|.$$

We are going to apply this formula to $g_\xi = \xi r_\xi$ with $\xi \in \{a,b,c,d,e,f\}$ and $\nu = z_i/|z_i|$, where z_i are defined in (11).

More precisely, for a singleton cut, say $\{0\} \mid \{1,2,3\}$, the free edges are $12, 13, 23$. Their three cross products are parallel to z_0 and have respective magnitudes $de|z_0|$, $df|z_0|$ and $ef|z_0|$; their sum is $q_0|z_0|$. Relabelling gives the other singleton cuts.

For a $2+2$ cut there is one internal edge on each side; for example $\{0,1\} \mid \{2,3\}$ gives the parallelogram generated by ar_a and fr_f, whose area is $af|z_4| = q_4|z_4|$.

Thus in every case

$$\mathcal{H}^2(F_i(Z_\rho)) = q_i|z_i|. \tag{13}$$

Now we recall by Proposition 3.2 that $\mathrm{Vor}(\Lambda) = A(\rho)^{-1/2} Z_\rho$.

If an invertible linear map L acts on a planar set with normal n, its area is multiplied by $|\det L|\,|L^{-\mathsf{T}} n|/|n|$. Apply this with $L = A^{-1/2}(\rho)$ and $n = z_i$, using the fact that $z_i^\mathsf{T} A(\rho) z_i = c_i$, we obtain from (13) that

$$\mathcal{H}^2(F_i(\mathrm{Vor}(\Lambda))) = D^{-1/2}(\rho) \frac{\sqrt{c_i}}{|z_i|}\, q_i |z_i| = \frac{q_i \sqrt{c_i}}{\sqrt{D(\rho)}}.$$

The opposite facet has the same area. Summing the seven opposite pairs we obtain the total surface area of the Voronoi cell

$$\mathcal{H}^2(\partial \operatorname{Vor}(\Lambda)) = 2\sum_{i=0}^{6} \frac{q_i\sqrt{c_i}}{\sqrt{D(\rho)}}.$$

To conclude we divide it by $D^{1/3}(\rho)$. □

We conclude with an observation that will be useful when proving the interior isoperimetric inequality (that is, among lattices with Selling parameters $\rho_{ij} > 0$ for all i, j). Let

$$M(\rho) = \sum_{i=0}^{6} \frac{q_i}{\sqrt{c_i}} z_i z_i^{\mathsf{T}}, \qquad \mathcal{J}(\rho) = \frac{\det M(\rho)}{D^{3/2}(\rho)}. \tag{14}$$

In the positive cone of $\rho_{ij} > 0$, both $A(\rho)$ and $M(\rho)$ are positive definite. Since $z_i^{\mathsf{T}} A(\rho) z_i = c_i$,

$$\operatorname{tr}(A^{1/2}(\rho)M(\rho)A^{1/2}(\rho)) = \sum_{i=0}^{6} \frac{q_i}{\sqrt{c_i}}\, z_i^{\mathsf{T}} A(\rho) z_i = \sum_{i=0}^{6} q_i\sqrt{c_i} = N(\rho).$$

Let $\lambda_1, \lambda_2, \lambda_3 > 0$ be the eigenvalues of $A^{1/2}(\rho)M(\rho)A^{1/2}(\rho)$. Then

$$N(\rho) = \sum_{j=1}^{3} \lambda_j \geq 3(\lambda_1\lambda_2\lambda_3)^{1/3} = 3(D(\rho)\det M(\rho))^{1/3},$$

by the arithmetic–geometric mean inequality. Therefore, using (12) and (14), we obtain the following inequality relating the isoperimetric quotient with the determinant $\mathcal{J}(\rho)$

$$F(\rho) = \frac{2N(\rho)}{D(\rho)^{5/6}} \geq 6\left(\frac{\det M(\rho)}{D(\rho)^{3/2}}\right)^{1/3} = 6\mathcal{J}^{1/3}(\rho). \tag{15}$$

The BCC lattice has Selling parameters $\rho_{ij} \equiv 1$, that is $a = \cdots = f = 1$. Therefore, a direct substitution gives

$$A^{1/2}(1)M(1)A^{1/2}(1) = (2 + 4\sqrt{3})I_3$$

where I_3 is the identity matrix. For every BCC rescaling, $A^{1/2}(\rho)M(\rho)A^{1/2}(\rho)$ remains a positive scalar multiple of the identity. Thus the three eigenvalues λ_i coincide, and equality is attained in (15). Moreover,

$$F_{\mathrm{BCC}} = F(1) = \frac{3(1+2\sqrt{3})}{4^{2/3}}, \qquad \mathcal{J}_{\mathrm{BCC}} = \mathcal{J}(1) = \left(\frac{F_{\mathrm{BCC}}}{6}\right)^3 = \frac{37+30\sqrt{3}}{128}. \tag{16}$$

Therefore the inequality

$$\mathcal{J}(\rho) \geq \mathcal{J}_{\mathrm{BCC}} \tag{17}$$

implies the isoperimetric inequality $F(\rho) \geq F_{\mathrm{BCC}}$.

4. The FCC theorem

This section is devoted to the analysis of the isoperimetric inequality for full-rank lattices with at least one vanishing Selling parameter. In particular, we prove the sharp boundary theorem for lattice Voronoi cells, showing that the unique minimizer in this case is the Voronoi cell of the FCC lattice.

First of all, we recall a known result by Joós and Lángi [14] showing that among lattices with at least two vanishing Selling parameters, the FCC lattice has the Voronoi cell with minimal isoperimetric quotient.

Proposition 4.1. *If $D(\rho) > 0$ and at most four Selling parameters are positive, then*

$$F(\rho) \geq F_{\mathrm{FCC}} = 3\,2^{5/6}.$$

Equality holds precisely on the S_4-orbit of the FCC *lattice, which has Selling parameters $(0, m, m, m, m, 0)$, for $m > 0$.*

Proof. By Proposition 3.2, $\mathrm{Vor}(\Lambda)$ is a full-dimensional zonotope generated by three or four nonzero segments. Joós and Lángi [14, Theorem 3, with $d = 3$ and $k = 2$] prove that, among three-dimensional zonotopes generated by three or four segments and of fixed volume, the second intrinsic volume, that in dimension 3 is given by $\mathcal{H}^2(\partial\,\mathrm{Vor}(\Lambda))/2$ is uniquely minimized, respectively, by the cube SC which is associated to Selling parameters $(1, 1, 1, 0, 0, 0)$ and by the regular rhombic dodecahedron FCC. Thus their result gives the sharp surface-area bounds

$$F(\rho) \geq \min\{F_{\mathrm{SC}}, F_{\mathrm{FCC}}\} = F_{\mathrm{FCC}} = 3\,2^{5/6},$$

where $F_{\mathrm{SC}} = F(1, 1, 1, 0, 0, 0) = 6$. It remains to identify equality in Selling coordinates. Since $F_{\mathrm{SC}} > F_{\mathrm{FCC}}$, equality requires exactly four positive coordinates and the corresponding zonotope must be a regular rhombic dodecahedron. A connected four-edge graph on four vertices is either a 4-cycle or a triangle with a pendant edge. The latter is impossible, because the three roots belonging to the triangle are linearly dependent, whereas every three generators of a regular rhombic dodecahedron are independent.

On a 4-cycle, orient the four roots cyclically. Their unique linear relation has coefficients of equal absolute value. The physical generators are

$$g_\xi = \rho_\xi A(\rho)^{-1/2} r_\xi.$$

Consequently, the coefficients in their unique dependence are proportional to ρ_ξ^{-1}. For a regular rhombic dodecahedron, after a suitable orientation, the four generators are the vertices of a regular tetrahedron centred at the origin, so the coefficients in their unique dependence have equal absolute value. Hence the four active Selling parameters are equal. The three 4-cycles of K_4 are the complements of the three pairs of opposite edges, which is exactly the S_4-orbit of $(0, m, m, m, m, 0)$, $m > 0$. $\square$

We are therefore left to prove the same result for lattices with exactly one vanishing Selling parameter. In this case the result of Joós and Lángi in [14] does not apply. Although a Voronoi cell with five positive Selling parameters is a five-generator zonotope, the class of all five-generator zonotopes is strictly larger: some members of that larger class have a smaller isoperimetric quotient than the regular rhombic dodecahedron, but they do not satisfy the lattice–Voronoi metric relation.

In order to prove the result, we need to use a different representation of the isoperimetric quotient F.

4.1. **Lattices with at least one null Selling parameter.** By tetrahedral symmetry it suffices to take $a = 0$ and let $f = t$, be a varying parameter (that could also be 0). In this case the weighted spanning-tree polynomials q_i defined in (6) and the weights of the cuts defined in (5) are

$$\begin{aligned}
q_0 &= de + t(d+e), & c_0 &= b + c,\\
q_1 &= bc + t(b+c), & c_1 &= d + e,\\
q_2 &= ce, & c_2 &= b + d + t,\\
q_3 &= bd, & c_3 &= c + e + t,\\
q_4 &= 0 & c_4 &= b + c + d + e,\\
q_5 &= be & c_5 &= c + d + t\\
q_6 &= cd. & c_6 &= b + e + t
\end{aligned}$$

We recall from (12) and (7) that $N(\rho) = \sum_{i=0}^{6} q_i \sqrt{c_i}$

$$D(\rho) = bcd + bce + bde + bdt + bet + cde + cdt + cet.$$

The term containing $\sqrt{c_4}$ vanishes because $q_4 = 0$, so it will be omitted below.

We define

$$d_{01} = \sqrt{c_0},\ d_{23} = \sqrt{c_1},\ d_{02} = \sqrt{c_2},\ d_{13} = \sqrt{c_3},\ d_{12} = \sqrt{c_5},\ d_{03} = \sqrt{c_6} \tag{18}$$

and we observe that the array d is an element of the metric cone on four points according to this definition.

Definition 4.2 (Metric cone on four points)**.** Let us fix four points $0,1,2,3$ and define MET_4 as the set of all possible semimetrics on $\{0,1,2,3\}$. More precisely $d=(d_{ij})_{0\le i,j\le 3}\in\mathrm{MET}_4$ if and only if $d_{ii}=0$, $d_{ij}=d_{ji}\ge 0$, and $d_{ik}\le d_{ij}+d_{jk}$. MET_4 is a cone, that is, it is closed by addition and multiplication by positive scalar.

Lemma 4.3. *Let d be defined as in* (18) *and let us extend it to an array d_{ij}, for all $0\le i,j\le 3$ by imposing $d_{ij}=d_{ji}$ for $i>j$ and $d_{ii}=0$. Then $d\in\mathrm{MET}_4$.*

Proof. It is sufficient to check the triangle inequality, by direct computation. Indeed, for the four sets of indices $ijk=012,013,023,123$, the unordered triples of quantities $d_{ij}^2+d_{ik}^2-d_{jk}^2$ are respectively

$$2(b,c,d+t),\qquad 2(b,c,e+t),\qquad 2(d,e,b+t),\qquad 2(d,e,c+t),$$

and are nonnegative. Hence, $d_{ij}^2\le d_{ik}^2+d_{jk}^2$. Taking square roots gives $d_{ij}\le d_{ik}+d_{jk}$; thus the positive square roots satisfy all triangle inequalities. $\square$

For $d\in\mathrm{MET}_4$, we let

$$\Gamma(d)=\begin{pmatrix} d_{01} & (d_{01}+d_{02}-d_{12})/2 & (d_{01}+d_{03}-d_{13})/2\\ (d_{01}+d_{02}-d_{12})/2 & d_{02} & (d_{02}+d_{03}-d_{23})/2\\ (d_{01}+d_{03}-d_{13})/2 & (d_{02}+d_{03}-d_{23})/2 & d_{03}\end{pmatrix}$$

and

$$\Gamma(d^{\circ 2})=\begin{pmatrix} d_{01}^2 & (d_{01}^2+d_{02}^2-d_{12}^2)/2 & (d_{01}^2+d_{03}^2-d_{13}^2)/2\\ (d_{01}^2+d_{02}^2-d_{12}^2)/2 & d_{02}^2 & (d_{02}^2+d_{03}^2-d_{23}^2)/2\\ (d_{01}^2+d_{03}^2-d_{13}^2)/2 & (d_{02}^2+d_{03}^2-d_{23}^2)/2 & d_{03}^2\end{pmatrix}.$$

We will denote $\mathsf{G}=\Gamma(d^{\circ 2})$, $\mathsf{H}=\Gamma(d)$, where d is as in (18).

Substituting the c_i in the definition of $\Gamma(d)$, $\Gamma(d^{\circ 2})$ gives the concrete matrices

$$\mathsf{H}=\begin{pmatrix} \sqrt{c_0} & \frac{\sqrt{c_0}+\sqrt{c_2}-\sqrt{c_5}}{2} & \frac{\sqrt{c_0}+\sqrt{c_6}-\sqrt{c_3}}{2}\\ \frac{\sqrt{c_0}+\sqrt{c_2}-\sqrt{c_5}}{2} & \sqrt{c_2} & \frac{\sqrt{c_2}+\sqrt{c_6}-\sqrt{c_1}}{2}\\ \frac{\sqrt{c_0}+\sqrt{c_6}-\sqrt{c_3}}{2} & \frac{\sqrt{c_2}+\sqrt{c_6}-\sqrt{c_1}}{2} & \sqrt{c_6}\end{pmatrix}$$

and

$$\mathsf{G}=\begin{pmatrix} c_0 & \frac{c_0+c_2-c_5}{2} & \frac{c_0+c_6-c_3}{2}\\ \frac{c_0+c_2-c_5}{2} & c_2 & \frac{c_2+c_6-c_1}{2}\\ \frac{c_0+c_6-c_3}{2} & \frac{c_2+c_6-c_1}{2} & c_6\end{pmatrix}=\begin{pmatrix} b+c & b & b\\ b & b+d+t & b+t\\ b & b+t & b+e+t\end{pmatrix}.$$

Notice that G has the positive-semidefinite rank-one decomposition

$$\mathsf{G}=b(1,1,1)^{\mathsf{T}}(1,1,1)+ce_1e_1^{\mathsf{T}}+de_2e_2^{\mathsf{T}}+ee_3e_3^{\mathsf{T}}+t(0,1,1)^{\mathsf{T}}(0,1,1).$$

Computing the determinant of G we have

$$\begin{aligned}\det\mathsf{G}=&\frac{1}{4}\Big(-c_0^2c_1-c_0c_1^2+c_0c_1c_2+c_0c_1c_3+c_0c_1c_5+c_0c_1c_6+c_0c_2c_3\\ &-c_0c_2c_5-c_0c_3c_6+c_0c_5c_6+c_1c_2c_3-c_1c_2c_6-c_1c_3c_5+c_1c_5c_6\\ &-c_2^2c_3-c_2c_3^2+c_2c_3c_5+c_2c_3c_6+c_2c_5c_6+c_3c_5c_6-c_5^2c_6-c_5c_6^2\Big)\\ =&\,bcd+bce+bde+bdt+bet+cde+cdt+cet\\ =&\,D(\rho).\end{aligned}$$

Direct differentiation gives

$$\frac{\partial\det\mathsf{G}}{\partial c_i}=q_i$$

for all $0\le i\le 6$, with $i\neq 4$. Note also that there holds

$$\mathsf{H}=\Gamma(d)=\sum_{i\in\{0,1,2,3,5,6\}}\sqrt{c_i}\,\frac{\partial\mathsf{G}}{\partial c_i}.$$

Jacobi's formula, in its polynomial form gives for every $i \neq 4$

$$\sqrt{c_i}\,\frac{\partial \det \mathsf{G}}{\partial c_i} = \sqrt{c_i}\,\mathrm{tr}\left(\mathrm{adj}\,\mathsf{G}\,\frac{\partial \mathsf{G}}{\partial c_i}\right) = \mathrm{tr}\left(\mathrm{adj}\,\mathsf{G}\,\sqrt{c_i}\frac{\partial \mathsf{G}}{\partial c_i}\right)$$

where $\mathrm{adj}\,\mathsf{G}$ is the transpose of the cofactor matrix of G, that is $(\det \mathsf{G})\mathsf{G}^{-1}$. Therefore we get

$$\det \mathsf{G} = D(\rho), \qquad N(\rho) = \sum_{i=0}^{6} q_i\sqrt{c_i} = \mathrm{tr}(\mathrm{adj}\,\mathsf{G}\,\mathsf{H}) = \det \mathsf{G}\,\mathrm{tr}(\mathsf{G}^{-1/2}\,\mathsf{H}\mathsf{G}^{-1/2}). \tag{19}$$

Recalling (12), the isoperimetric quotient can be expressed as

$$F(\rho) = 2\det \mathsf{G}^{1/6}\,\mathrm{tr}(\mathsf{G}^{-1/2}\,\mathsf{H}\mathsf{G}^{-1/2}). \tag{20}$$

4.2. **A determinant inequality.** We provide now a general result relating the determinant of matrices $\Gamma(d)$ and $\Gamma(d^{\circ 2})$ for $d \in \mathrm{MET}_4$.

First of all we introduce the notion of cut cone. We call a nonempty proper subset of $\{0,1,2,3\}$ a cut and identify a cut with its complement. Thus there are four singleton cuts and three $2+2$ cuts.

Definition 4.4 (cut cone)**.** The cut metric associated to a cut S is the semimetric defined as

$$\delta_S(i,j) := \begin{cases} 0 & i,j \in S \\ 0 & i,j \in S^c = \{0,1,2,3\}\setminus S \\ 1 & \text{elsewhere.} \end{cases}$$

The cut cone is the conic hull

$$\mathrm{CUT}_4 = \mathrm{cone}\{\delta_S : S \subsetneq \{0,1,2,3\},\ S \neq \emptyset\} \subset \mathrm{MET}_4.$$

We recall the following result (see [9, Chapter 4]).

Proposition 4.5. *There holds*

$$\mathrm{MET}_4 = \mathrm{CUT}_4.$$

In particular, any semimetric in the metric cone is a nonnegative combination of the seven cut metrics.

Notice that the cut metrics satisfy the identity

$$\delta_{\{0\}} + \delta_{\{1\}} + \delta_{\{2\}} + \delta_{\{3\}} \equiv \delta_{\{0,1\}} + \delta_{\{0,2\}} + \delta_{\{0,3\}}.$$

Starting from any nonnegative cut decomposition, subtract the smallest of the three $2+2$ coefficients from all three of them and add the same quantity to each singleton coefficient. The displayed identity shows that the metric is unchanged, while one $2+2$ coefficient becomes zero.

We shall also use that the two determinants appearing below, $\det\Gamma(d)$ and $\det\Gamma(d^{\circ 2})$, are invariant under relabelling of the four points.

After a permutation of the four points, we may write for every $d \in \mathrm{MET}_4$

$$d = \sum_{i=0}^{3} x_i\delta_{\{i\}} + u\delta_{\{0,1\}} + v\delta_{\{0,2\}}, \qquad x_i, u, v \geq 0. \tag{21}$$

In these coordinates the six distances are

$$\begin{aligned} d_{01} &= x_0 + x_1 + v, & d_{02} &= x_0 + x_2 + u, \\ d_{03} &= x_0 + x_3 + u + v, & d_{12} &= x_1 + x_2 + u + v, \\ d_{13} &= x_1 + x_3 + u, & d_{23} &= x_2 + x_3 + v. \end{aligned}$$

We have the following theorem.

Theorem 4.6 (Determinant inequality)**.** *For every semimetric $d \in \mathrm{MET}_4$,*

$$2\big(\det\Gamma(d)\big)^2 \geq \det\Gamma(d^{\circ 2}), \tag{22}$$

where $(d^{\circ 2})_{ij} = d_{ij}^2$. If $\det\Gamma(d) > 0$, equality holds if and only if the six distances are equal.

Proof. Let $x_0, \ldots, x_3, u, v$ be as in (21). Set

$$s := x_0 + x_1 + x_2 + x_3, \qquad \pi := x_0x_1x_2x_3, \qquad M_i := \prod_{j \neq i} x_j, \quad 0 \le i \le 3.$$

For a cut $S \subset \{0,1,2,3\}$, set

$$r_S := \big(\delta_S(0,1), \delta_S(0,2), \delta_S(0,3)\big)^{\mathsf{T}}.$$

Since

$$\delta_S(i,j) = \delta_S(0,i) + \delta_S(0,j) - 2\delta_S(0,i)\delta_S(0,j),$$

the definition of $\Gamma(\delta_S)$ gives

$$\Gamma(\delta_S) = r_S r_S^{\mathsf{T}}.$$

For the six cuts appearing in (21), the corresponding vectors are

$$\begin{aligned} r_{\{0\}} &= (1,1,1)^{\mathsf{T}}, & r_{\{1\}} &= (1,0,0)^{\mathsf{T}}, & r_{\{2\}} &= (0,1,0)^{\mathsf{T}}, \\ r_{\{3\}} &= (0,0,1)^{\mathsf{T}}, & r_{\{0,1\}} &= (0,1,1)^{\mathsf{T}}, & r_{\{0,2\}} &= (1,0,1)^{\mathsf{T}}. \end{aligned}$$

Hence, by linearity of Γ,

$$\Gamma(d) = \sum_{i=0}^{3} x_i r_{\{i\}} r_{\{i\}}^{\mathsf{T}} + u\, r_{\{0,1\}} r_{\{0,1\}}^{\mathsf{T}} + v\, r_{\{0,2\}} r_{\{0,2\}}^{\mathsf{T}}.$$

Thus $\Gamma(d) = BB^{\mathsf{T}}$, where

$$B := \begin{pmatrix} \sqrt{x_0}\, r_{\{0\}} & \sqrt{x_1}\, r_{\{1\}} & \sqrt{x_2}\, r_{\{2\}} & \sqrt{x_3}\, r_{\{3\}} & \sqrt{u}\, r_{\{0,1\}} & \sqrt{v}\, r_{\{0,2\}} \end{pmatrix}.$$

and

$$\det \Gamma(d) = \sum_{\substack{I \subset \{1,\ldots,6\} \\ |I|=3}} (\det B_I)^2.$$

The nonzero 3×3 minors of the unweighted column configuration have determinant of absolute value one. Among the minors containing $r_{\{0,1\}}$, but not $r_{\{0,2\}}$, the nonzero ones correspond to the products

$$x_0x_2, \quad x_0x_3, \quad x_1x_2, \quad x_1x_3,$$

while those containing $r_{\{0,2\}}$, but not $r_{\{0,1\}}$, correspond to

$$x_0x_1, \quad x_0x_3, \quad x_1x_2, \quad x_2x_3.$$

All four minors containing both of the last two columns are nonzero. Thus Cauchy–Binet formula gives

$$\begin{aligned} P := \det \Gamma(d) &= e_3(x) + u(x_0 + x_1)(x_2 + x_3) \\ &\quad + v(x_0 + x_2)(x_1 + x_3) + uv(x_0 + x_1 + x_2 + x_3) \\ &= e_3(x) + u(x_0 + x_1)(x_2 + x_3) + v(x_0 + x_2)(x_1 + x_3) + uvs. \end{aligned} \tag{23}$$

Set now

$$R := 2P^2 - \det \Gamma(d^{\circ 2})$$

and define

$$\begin{aligned} U &:= ((x_0 + x_1)(x_2 + x_3))^2 - 8\pi, \\ V &:= ((x_0 + x_2)(x_1 + x_3))^2 - 8\pi, \\ W &:= (x_0x_3 + x_1x_2)^2 - 8\pi, \end{aligned}$$

together with

$$\begin{aligned} \Lambda_u &:= 4[x_0x_1(x_0 + x_1)(x_2 - x_3)^2 + x_2x_3(x_2 + x_3)(x_0 - x_1)^2], \\ \Lambda_v &:= 4[x_0x_2(x_0 + x_2)(x_1 - x_3)^2 + x_1x_3(x_1 + x_3)(x_0 - x_2)^2], \\ B_{21} &:= 4[x_0x_2(x_0 + x_2) + x_0x_3(x_0 + x_3) + x_1x_2(x_1 + x_2) + x_1x_3(x_1 + x_3)], \\ B_{12} &:= 4[x_0x_1(x_0 + x_1) + x_0x_3(x_0 + x_3) + x_1x_2(x_1 + x_2) + x_2x_3(x_2 + x_3)]. \end{aligned}$$

Using

$$\det \begin{pmatrix} a & p & q \\ p & b & r \\ q & r & c \end{pmatrix} = abc + 2pqr - ar^2 - bq^2 - cp^2$$

and collecting the terms with the same powers of u and v, we obtain

$$\begin{aligned} R = {} & 2\sum_{i<j}(M_i - M_j)^2 + \Lambda_u u + \Lambda_v v + 2(Uu^2 + 2Wuv + Vv^2) \\ & + B_{21}u^2v + B_{12}uv^2 + 2u^2v^2\big(2(u+v)^2 + 4s(u+v) + 3s^2\big). \end{aligned} \tag{24}$$

All the terms on the right-hand side are manifestly nonnegative, except a priori the quadratic form $Uu^2 + 2Wuv + Vv^2$. Indeed,

$$(x_0 + x_1)(x_2 + x_3) \geq 4(x_0x_1x_2x_3)^{1/2} = 4\sqrt{\pi},$$

and similarly

$$(x_0 + x_2)(x_1 + x_3) \geq 4\sqrt{\pi}.$$

Hence

$$U \geq 8\pi, \qquad V \geq 8\pi.$$

Moreover,

$$x_0x_3 + x_1x_2 \geq 2\sqrt{x_0x_1x_2x_3} = 2\sqrt{\pi},$$

and therefore

$$W \geq -4\pi.$$

Since $u, v \geq 0$, it follows that

$$\begin{aligned} Uu^2 + 2Wuv + Vv^2 &\geq 8\pi u^2 - 8\pi uv + 8\pi v^2 \\ &= 8\pi(u^2 - uv + v^2) \geq 0. \end{aligned}$$

Thus every term in (24) is nonnegative, and hence

$$2\big(\det\Gamma(d)\big)^2 \geq \det\Gamma(d^{\circ 2}).$$

We now discuss the equality case. Assume $P = \det\Gamma(d) > 0$ and $R = 0$. The last term in (24) forces

$$uv = 0.$$

Suppose, for instance, that $u > 0$ and $v = 0$. Since all terms in (24) are nonnegative, equality implies $U = 0$. But

$$U = (x_0^2 + x_1^2)(x_2 + x_3)^2 + 2x_0x_1(x_2 - x_3)^2.$$

Thus either $x_0 = x_1 = 0$ or $x_2 = x_3 = 0$. In either case (23) gives $P = 0$, a contradiction. The case $v > 0$ and $u = 0$ is identical, using

$$V = (x_0^2 + x_2^2)(x_1 + x_3)^2 + 2x_0x_2(x_1 - x_3)^2.$$

Therefore

$$u = v = 0.$$

Equality in (24) then yields

$$M_0 = M_1 = M_2 = M_3.$$

Since $u = v = 0$,

$$P = e_3(x) = M_0 + M_1 + M_2 + M_3 > 0.$$

Hence their common value is $P/4 > 0$, so in particular every M_i is positive. It follows that

$$x_0, x_1, x_2, x_3 > 0.$$

Writing

$$M_i = \frac{\pi}{x_i},$$

the equality of the four M_i implies

$$x_0 = x_1 = x_2 = x_3.$$

Since $u = v = 0$, the six distances in (21) are therefore equal.

Conversely, if all six distances are equal to $m > 0$, then, writing $\mathbf{e}$ for the equilateral metric on four points,

$$\Gamma(d) = m\Gamma(\mathbf{e}), \qquad \Gamma(d^{\circ 2}) = m^2\Gamma(\mathbf{e}), \qquad \det\Gamma(\mathbf{e}) = \frac{1}{2}.$$

Hence equality in (22) follows directly. $\square$

4.3. **Main result.** We are now ready to prove the FCC theorem.

Theorem 4.7. *For every nondegenerate point* $(0, b, c, d, e, t)$*,*

$$F(0, b, c, d, e, t) \geq 3\,2^{5/6}.$$

Equality holds if and only if $(0, b, c, d, e, t) = (0, m, m, m, m, 0)$ *for some* $m > 0$*.*

Proof. Let $\mu_1, \mu_2, \mu_3 \geq 0$ be the eigenvalues of $\mathsf{G}^{-1/2}\mathsf{H}\mathsf{G}^{-1/2}$. Then

$$\operatorname{tr}\mathsf{G}^{-1/2}\mathsf{H}\mathsf{G}^{-1/2} = \mu_1 + \mu_2 + \mu_3 \quad \text{and} \quad \det\mathsf{H} = (\det\mathsf{G})\mu_1\mu_2\mu_3.$$

Therefore from (20) and from the inequality $\mu_1 + \mu_2 + \mu_3 \geq 3\sqrt[3]{\mu_1\mu_2\mu_3}$ we get

$$F(\rho) = 2\det\mathsf{G}^{1/6}(\mu_1 + \mu_2 + \mu_3) \geq 6\det\mathsf{G}^{1/6}\sqrt[3]{\mu_1\mu_2\mu_3} = 6\left(\frac{(\det\mathsf{H})^2}{\det\mathsf{G}}\right)^{1/6}.$$

Theorem 4.6 yields $2(\det\mathsf{H})^2 \geq \det\mathsf{G}$, so, substituting in the previous inequality we get

$$F(\rho) \geq 6\left(\frac{(\det\mathsf{H})^2}{\det\mathsf{G}}\right)^{1/6} \geq 3\ 2^{5/6} = F_{\mathrm{FCC}}.$$

This gives the minimality of FCC.

If equality holds in the final estimate, then equality holds both in the spectral arithmetic–geometric mean and in Theorem 4.6. Since $D(\rho) = \det\mathsf{G} > 0$, the latter theorem forces all six metric lengths d_{ij}, and hence all six c_i, $i \neq 4$ to agree. The relations $c_3 = c_5$, $c_3 = c_6$, $c_0 = c_1$, and $c_1 = c_2$ imply $b = c = d = e$ and $t = 0$. Conversely, on the FCC ray all six c_i, for $i \neq 4$, are equal and H is a positive scalar multiple of G, so there is equality in both constituent inequalities. $\square$

Theorem 4.8 (FCC minimality)**.** *Let* $\operatorname{Vor}(\Lambda)$ *be the Voronoi cell of a full-rank lattice in* $\mathbb{R}^3$*. If* $\operatorname{Vor}(\Lambda)$ *has at most twelve facets, then*

$$\operatorname{Iso}(\operatorname{Vor}(\Lambda)) = F(\rho) \geq 3\,2^{5/6}. \tag{25}$$

Equality holds if and only if $\operatorname{Vor}(\Lambda)$ *is similar to the regular rhombic dodecahedron, equivalently if and only if the lattice is similar to FCC.*

Proof. Corollary 3.4 identifies the stated class with the nondegenerate boundary of the Selling cone. Proposition 4.1 handles strata with at most four active parameters, and Theorem 4.7 handles the six open five-generator facets. Their equality classifications agree exactly on the FCC orbit. $\square$

5. The BCC theorem

The aim of this section is to prove the sharp determinant inequality (17) that, as we noted at the end of Subsection 3.3, controls the interior of the Selling cone and singles out the BCC ray.

We do not minimize the determinant $\mathcal{J}(\rho)$ directly in the six Selling coordinates. Instead, the seven cut directions introduced above in (11) allow us to separate the argument into three steps.

First, by Cauchy–Binet we rewrite the determinant $\mathcal{J}$ as a sum over the 29 nonzero bases of these directions; grouping the bases according to the number of double cuts produces a four-state lower bound. Second, the geometry of the positive Selling cone is encoded by two inequalities for the three double-cut leverage scores: the product estimate in Theorem 5.3 and the correlation estimate in Theorem 5.6. Finally, these constraints reduce the remaining optimization to the scalar estimate of Theorem 5.8.

In this way, we separate the algebraic steps and the final optimization result.

5.1. **Determinantal probability measures and entropy contraction.** Recall the definition of the seven cut directions z_i in (11).

For every $u \in \mathbb{R}^7$, let

$$A_u = \sum_i u_i z_i z_i^\mathsf{T} = \begin{pmatrix} u_0+u_1+u_5+u_6 & u_0+u_6 & u_0+u_5 \\ u_0+u_6 & u_0+u_2+u_4+u_6 & u_0+u_4 \\ u_0+u_5 & u_0+u_4 & u_0+u_3+u_4+u_5 \end{pmatrix}$$

and

$$\mathcal{P}(u) := \det A_u. \tag{26}$$

For every $I \subseteq \{0,\dots,6\}$, with $|I| = 3$, Z_I is the 3×3 matrix whose columns are the vectors z_i, $i \in I$, and define

$$\kappa_I = (\det Z_I)^2.$$

Of the $\binom{7}{3} = 35$ triples I, exactly 29 matrices Z_I have nonzero determinant. For 28 of them the square of the determinant is 1, while $(\det(z_4, z_5, z_6))^2 = 4$.

The Cauchy–Binet theorem gives

$$\mathcal{P}(u) = \sum_{\substack{I\subset\{0,\dots,6\}\\|I|=3}} \det(Z_I)^2 \prod_{i\in I} u_i = \sum_{\substack{I\subset\{0,\dots,6\}\\|I|=3}} \kappa_I \prod_{i\in I} u_i. \tag{27}$$

From the Cauchy-Binet formula above one derives the explicit expression

$$\begin{aligned} \mathcal{P}(u) = {} & e_3(u_0,u_1,u_2,u_3) + u_4(u_0+u_1)(u_2+u_3) \\ & + u_5(u_0+u_2)(u_1+u_3) + u_6(u_0+u_3)(u_1+u_2) \\ & + (u_0+u_1+u_2+u_3)e_2(u_4,u_5,u_6) + 4u_4u_5u_6. \end{aligned} \tag{28}$$

To every fixed $u \in \mathbb{R}^7$ with $u_i \ge 0$ and $\mathcal{P}(u) > 0$ we associate a probability measure on the space $\Omega := \{I \subset \{0,\dots,6\}, |I| = 3, \kappa_I > 0\}$ defined as follows:

$$\tau_u(I) = \frac{\kappa_I \prod_{i\in I} u_i}{\mathcal{P}(u)}, \qquad |I| = 3, \qquad \kappa_I := \det(Z_I)^2. \tag{29}$$

Note that τ_u is a probability measure due to (27). τ_u is the probability law on the 29 triples for which $\kappa_I > 0$ measuring the relative contribution of each basis to the determinant $\mathcal{P}(u)$ and it is called the finite projection determinantal measure associated with the rank-three subspace spanned by the weighted columns; see [17, Sections 2–3]. The one-point marginals associated to τ_u are defined for every $i \in \{0,\dots,6\}$ as

$$\ell_i(u) := \sum_{I \text{ s.t. } i\in I} \tau_u(I). \tag{30}$$

It is straightforward to check using the definition that

$$\ell_i(u) = u_i \frac{\partial}{\partial u_i} \log \mathcal{P}(u). \tag{31}$$

By Jacobi's formula, recalling that $\operatorname{adj} A_u = \mathcal{P}(u) A_u^{-1}$ and $\frac{\partial A_u}{\partial u_i} = z_i z_i^\mathsf{T}$, we obtain

$$\begin{aligned} u_i \frac{\partial}{\partial u_i} \mathcal{P}(u) &= u_i \operatorname{tr}\left(\operatorname{adj} A_u \frac{\partial A_u}{\partial u_i}\right) \\ &= u_i \mathcal{P}(u)\, z_i^\mathsf{T} A_u^{-1} z_i, \end{aligned} \tag{32}$$

$$\ell_i(u) = u_i \frac{\partial}{\partial u_i} \log \mathcal{P}(u) = u_i z_i^\mathsf{T} A_u^{-1} z_i. \tag{33}$$

The next lemma estimates the relative entropy of τ_u with respect to τ_v by comparing only their one-point marginals; this is the entropy contraction that will be used later.

Lemma 5.1 (Rank-one determinantal entropy contraction)**.** *Let $u, v \in \mathbb{R}^7_{\geq 0}$ be such that $\mathcal{P}(u) > 0$ and $\mathcal{P}(v) > 0$, and assume that*

$$\operatorname{supp} \tau_u \subseteq \operatorname{supp} \tau_v.$$

Then

$$D_{\mathrm{KL}}(\tau_u \| \tau_v) \geq \sum_{i=0}^{6} \ell_i(u) \log \frac{\ell_i(u)}{\ell_i(v)}, \tag{34}$$

where terms with $\ell_i(u) = 0$ are understood to be zero.

Proof. Since $\mathcal{P}(u) > 0$ and $\mathcal{P}(v) > 0$, the matrices

$$A_u = \sum_{i=0}^{6} u_i z_i z_i^{\mathsf{T}}, \qquad A_v = \sum_{i=0}^{6} v_i z_i z_i^{\mathsf{T}}$$

are positive definite.

Recall that, by Jacobi's formula,

$$\ell_i(u) = u_i \frac{\partial}{\partial u_i} \log \mathcal{P}(u) = u_i z_i^{\mathsf{T}} A_u^{-1} z_i,$$

and similarly

$$\ell_i(v) = v_i z_i^{\mathsf{T}} A_v^{-1} z_i.$$

Let

$$S := \{i \in \{0, \ldots, 6\} : \ell_i(u) > 0\}.$$

For every $i \in S$, there exists a basis I containing i such that $\tau_u(I) > 0$. Since $\operatorname{supp} \tau_u \subseteq \operatorname{supp} \tau_v$, we also have $\tau_v(I) > 0$. Hence $u_i > 0$, $v_i > 0$, and $\ell_i(v) > 0$ for every $i \in S$.

For $i \in S$, define

$$w_i(u) := \sqrt{u_i}\, A_u^{-1/2} z_i, \qquad e_i(u) := \frac{w_i(u)}{\sqrt{\ell_i(u)}}.$$

Then

$$\|w_i(u)\|^2 = u_i z_i^{\mathsf{T}} A_u^{-1} z_i = \ell_i(u),$$

so that each $e_i(u)$ is a unit vector. Moreover,

$$\begin{aligned}
\sum_{i \in S} \ell_i(u) e_i(u) e_i(u)^{\mathsf{T}} &= \sum_{i=0}^{6} w_i(u) w_i(u)^{\mathsf{T}} \\
&= A_u^{-1/2} \left(\sum_{i=0}^{6} u_i z_i z_i^{\mathsf{T}} \right) A_u^{-1/2} \\
&= I_3.
\end{aligned}$$

Set

$$B := A_u^{-1/2} A_v A_u^{-1/2}.$$

Then B is positive definite and

$$\det B = \frac{\det A_v}{\det A_u} = \frac{\mathcal{P}(v)}{\mathcal{P}(u)}.$$

By the definitions of τ_u and τ_v,

$$\begin{aligned}
D_{\mathrm{KL}}(\tau_u \| \tau_v) &= \sum_{I \in \operatorname{supp} \tau_u} \tau_u(I) \log \frac{\tau_u(I)}{\tau_v(I)} \\
&= \sum_{I \in \operatorname{supp} \tau_u} \tau_u(I) \left(\log \frac{\mathcal{P}(v)}{\mathcal{P}(u)} + \sum_{i \in I} \log \frac{u_i}{v_i} \right) \\
&= \log \frac{\mathcal{P}(v)}{\mathcal{P}(u)} + \sum_{i \in S} \ell_i(u) \log \frac{u_i}{v_i}
\end{aligned}$$

$$= \log\det B + \sum_{i\in S} \ell_i(u) \log \frac{u_i}{v_i}.$$

Since

$$A_v^{-1} = A_u^{-1/2} B^{-1} A_u^{-1/2},$$

for every $i \in S$ we have

$$\begin{aligned}\ell_i(v) &= v_i z_i^\top A_v^{-1} z_i \\ &= v_i z_i^\top A_u^{-1/2} B^{-1} A_u^{-1/2} z_i \\ &= \frac{v_i}{u_i}\, \ell_i(u)\, e_i(u)^\top B^{-1} e_i(u).\end{aligned}$$

Therefore

$$\frac{u_i \ell_i(v)}{v_i \ell_i(u)} = e_i(u)^\top B^{-1} e_i(u).$$

It follows that

$$\begin{aligned}&D_{\mathrm{KL}}(\tau_u \| \tau_v) - \sum_{i=0}^{6} \ell_i(u) \log \frac{\ell_i(u)}{\ell_i(v)} \\ &\quad = \log\det B + \sum_{i\in S} \ell_i(u) \log\Big(e_i(u)^\top B^{-1} e_i(u)\Big).\end{aligned}$$

By the spectral Jensen inequality, for every unit vector e,

$$\log(e^\top B^{-1} e) \ge e^\top (\log B^{-1}) e.$$

Thus

$$\begin{aligned}\sum_{i\in S} \ell_i(u) \log\Big(e_i(u)^\top B^{-1} e_i(u)\Big) &\ge \sum_{i\in S} \ell_i(u) e_i(u)^\top (\log B^{-1}) e_i(u) \\ &= \operatorname{tr}\left((\log B^{-1}) \sum_{i\in S} \ell_i(u) e_i(u) e_i(u)^\top\right) \\ &= \operatorname{tr}(\log B^{-1}) \\ &= \log\det B^{-1} \\ &= -\log\det B.\end{aligned}$$

Consequently,

$$\log\det B + \sum_{i\in S} \ell_i(u) \log\Big(e_i(u)^\top B^{-1} e_i(u)\Big) \ge 0,$$

which proves (34). □

Let $(q_i)_{i=0,\dots,6}$ be the weighted spanning-tree polynomials defined in (6). Direct substitution gives the matrix identity

$$A_q = \sum_{i=0}^{6} q_i z_i z_i^\top = \operatorname{adj} A(\rho) = (\det A(\rho)) A(\rho)^{-1}. \tag{35}$$

We define $\mathcal{P}(q)$ as in (26), τ_q as in (29), and denote its one-point marginals by $\ell_i(q)$.

Lemma 5.2. *There hold*

$$\mathcal{P}(q) = \det\operatorname{adj} A(\rho) = D(\rho)^2, \qquad \ell_i(q) = \frac{q_i c_i}{D(\rho)}, \qquad q_i \frac{\partial}{\partial q_i} \mathcal{P}(q) = D(\rho)^2 \ell_i(q). \tag{36}$$

Proof. For a 3×3 matrix A,

$$\det(\operatorname{adj} A) = (\det A)^2, \qquad \operatorname{adj}(\operatorname{adj} A) = (\det A) A.$$

Since $\det A(\rho) = D(\rho)$, the first identity follows from (35). Moreover, $A_q^{-1} = D(\rho)^{-1}A(\rho)$, and hence, using $z_i^{\mathsf{T}} A(\rho) z_i = c_i$ and (33),

$$\ell_i(q) = q_i z_i^{\mathsf{T}} A_q^{-1} z_i = \frac{q_i c_i}{D(\rho)}.$$

Finally, (32) gives

$$q_i \frac{\partial}{\partial q_i} \mathcal{P}(q) = \mathcal{P}(q)\ell_i(q) = D(\rho)^2 \ell_i(q).$$

□

5.2. **A sharp constraint on the three double leverages.** From now on we denote the marginals in (36) as the leverage scores

$$\ell_i = \frac{q_i c_i}{D(\rho)}, \qquad \sum_{i=0}^{6} \ell_i = 3. \tag{37}$$

Set

$$L = \ell_0 + \ell_1 + \ell_2 + \ell_3, \qquad T = abc + ade + bdf + cef = D(\rho)\left[1 - \ell_4 - \ell_5 - \ell_6\right]. \tag{38}$$

The spanning-tree identity (8) gives

$$L - 2 = 1 - (\ell_4 + \ell_5 + \ell_6) = \frac{T}{D(\rho)}. \tag{39}$$

We observe the following property.

Lemma 5.3. *For every positive Selling vector ρ,*

$$\ell_4 \ell_5 \ell_6 \leq \frac{(L-2)^2}{4}. \tag{40}$$

Equality holds precisely on the opposite-edge family

$$a = f, \qquad b = e, \qquad c = d. \tag{41}$$

Proof. Since $q_4 q_5 q_6 = abcdef$, due to (39), inequality (40) is equivalent to

$$4abcdef\, c_4 c_5 c_6 \leq T^2 D(\rho). \tag{42}$$

Let us introduce the three opposite-pair sums and differences

$$S_a = a + f, \quad S_b = b + e, \quad S_c = c + d, \qquad x = a - f, \quad y = b - e, \quad z = c - d,$$

and normalize

$$\alpha = \frac{x}{S_a}, \qquad \beta = \frac{y}{S_b}, \qquad \gamma = \frac{z}{S_c}.$$

Then $|\alpha|, |\beta|, |\gamma| < 1$. Put

$$K = (S_a + S_b)(S_a + S_c)(S_b + S_c),$$

$$\lambda_a = \frac{S_a^2(S_b + S_c)}{K}, \quad \lambda_b = \frac{S_b^2(S_a + S_c)}{K}, \quad \lambda_c = \frac{S_c^2(S_a + S_b)}{K}, \quad \eta = \frac{S_a S_b S_c}{K}.$$

Note that

$$\lambda_a + \lambda_b + \lambda_c + 2\eta = 1. \tag{43}$$

Then by direct computations we obtain

$$T = \frac{S_a S_b S_c + xyz}{2} = \frac{S_a S_b S_c}{2}(1 + \alpha\beta\gamma),$$

$$D(\rho) = \frac{K}{4}\Theta,$$

where

$$\Theta = \lambda_a(1 - \alpha^2) + \lambda_b(1 - \beta^2) + \lambda_c(1 - \gamma^2) + 2\eta(1 + \alpha\beta\gamma). \tag{44}$$

Furthermore

$$abcdef = \frac{S_a^2 S_b^2 S_c^2}{64}(1 - \alpha^2)(1 - \beta^2)(1 - \gamma^2), \qquad c_4 c_5 c_6 = K.$$

Therefore proving (42) (and so (40)) is equivalent to prove

$$(1+\alpha\beta\gamma)^2\Theta \geq (1-\alpha^2)(1-\beta^2)(1-\gamma^2). \tag{45}$$

First of all by weighted AM–GM and (43), recalling (44)

$$(1-\alpha^2)^{\lambda_a}(1-\beta^2)^{\lambda_b}(1-\gamma^2)^{\lambda_c}(1+\alpha\beta\gamma)^{2\eta} \leq \lambda_a(1-\alpha^2)+\lambda_b(1-\beta^2)+\lambda_c(1-\gamma^2)+2\eta(1+\alpha\beta\gamma) = \Theta. \tag{46}$$

Now we observe that for every $|h|, |k|, |g| < 1$,

$$(1-k^2)(1-g^2) \leq (1+hkg)^2. \tag{47}$$

Indeed

$$1+hkg \geq 1-|kg|, \qquad (1-|kg|)^2-(1-k^2)(1-g^2) = (|k|-|g|)^2 \geq 0.$$

Therefore, using (47) for α, β, γ we get

$$\begin{aligned}
(1-\beta^2)^{\lambda_a+\eta}(1-\gamma^2)^{\lambda_a+\eta} &\leq (1+\alpha\beta\gamma)^{2\lambda_a+2\eta}\\
(1-\gamma^2)^{\lambda_b+\eta}(1-\alpha^2)^{\lambda_b+\eta} &\leq (1+\alpha\beta\gamma)^{2\lambda_b+2\eta}\\
(1-\beta^2)^{\lambda_c+\eta}(1-\alpha^2)^{\lambda_c+\eta} &\leq (1+\alpha\beta\gamma)^{2\lambda_c+2\eta}
\end{aligned}$$

Multiplying the three lines and recalling (43), we get

$$(1-\alpha^2)^{1-\lambda_a}(1-\beta^2)^{1-\lambda_b}(1-\gamma^2)^{1-\lambda_c} \leq (1+\alpha\beta\gamma)^{2+2\eta}$$

Multiplying this inequality by (46)

$$(1+\alpha\beta\gamma)^2\Theta \geq (1-\alpha^2)(1-\beta^2)(1-\gamma^2),$$

which is (45). This gives the conclusion.

All four AM–GM weights $\lambda_a, \lambda_b, \lambda_c, \eta$ are positive in the interior. Equality in (46) occurs when

$$1-\alpha^2 = 1-\beta^2 = 1-\gamma^2 = 1+\alpha\beta\gamma,$$

which implies $\alpha = \beta = \gamma = 0$. This gives $a = f$, $b = e$, and $c = d$, that is (41). □

5.3. **A weak correlation estimate.** We provide now a parametrization of q_i in (6) by six positive variables and will be used in the correlation estimate below.

Lemma 5.4. *Set*

$$A = af, \quad B = be, \quad C = cd, \qquad X = ef, \quad Y = df, \quad Z = de \tag{48}$$

so that

$$q_0 = X+Y+Z, \tag{49}$$

$$q_1 = \frac{BC+CX+BY}{Z}, \tag{50}$$

$$q_2 = \frac{AC+CX+AZ}{Y}, \tag{51}$$

$$q_3 = \frac{AB+BY+AZ}{X}, \tag{52}$$

$$q_4 = A, \qquad q_5 = B, \qquad q_6 = C. \tag{53}$$

Conversely, given $(A,B,C,X,Y,Z) \in \mathbb{R}^6$ with $A,B,C,X,Y,Z > 0$ it is possible to define a positive Selling vector (a,b,c,d,e,f) by

$$d = \sqrt{\frac{YZ}{X}}, \quad e = \sqrt{\frac{XZ}{Y}}, \quad f = \sqrt{\frac{XY}{Z}}, \qquad a = \frac{A}{f}, \quad b = \frac{B}{e}, \quad c = \frac{C}{d}. \tag{54}$$

Therefore (49)–(53) parameterize exactly the image of the positive Selling cone in the space of weighted spanning-tree polynomials q_i.

Proof. It is a straightforward computation, by using definition (6). □

Define

$$\Delta = ABC + ABY + ABZ + ACX + ACZ + AXZ + AYZ + AZ^2 \\ + BCX + BCY + BXY + BY^2 + BYZ + CX^2 + CXY + CXZ, \tag{55}$$

and

$$h_4 = A(BY + CX + XZ + YZ),$$
$$h_5 = B(AZ + CX + XY + YZ),$$
$$h_6 = C(AZ + BY + XY + XZ).$$

We have

$$\mathcal{P}(q) = \frac{\Delta^2}{XYZ}, \qquad \ell_i = \frac{h_i}{\Delta} \quad (i = 4, 5, 6). \tag{56}$$

Indeed, substituting (49)–(53) into (28) and collecting terms gives

$$\mathcal{P}(q) = \frac{\Delta^2}{XYZ}.$$

Since $\mathcal{P}(q) = D(\rho)^2$ by (36), and all quantities are positive on the positive Selling cone, it follows that

$$D(\rho) = \frac{\Delta}{\sqrt{XYZ}}.$$

Moreover, using (37), (5), (53), and (54), we obtain, for instance,

$$\ell_4 = \frac{q_4 c_4}{D(\rho)} = \frac{A(b + c + d + e)\sqrt{XYZ}}{\Delta} \\ = \frac{A(BY + CX + XZ + YZ)}{\Delta} = \frac{h_4}{\Delta}.$$

The identities

$$\ell_5 = \frac{h_5}{\Delta}, \qquad \ell_6 = \frac{h_6}{\Delta}$$

follow in the same way.

Let now

$$H = h_4 + h_5 + h_6, \qquad \mathscr{L} = \Delta(\Delta - H)^2 - 4h_4h_5h_6, \tag{57}$$

and

$$\mathscr{C} = A^2BCZ^2 + AB^2CY^2 + ABC^2X^2 + ABY^2Z^2 \\ + ACX^2Z^2 + BCX^2Y^2 - 6ABCXYZ. \tag{58}$$

Lemma 5.5 (Projection-kernel identities)**.** *Let I be distributed according to the probability measure τ_q associated to q_i. Write $\delta_j = \ell_{3+j}$, $j = 1, 2, 3$, and*

$$J_2 = \sum_{1 \le i < j \le 3} \Pr(i + 3, j + 3 \in I), \qquad J_3 = \Pr(4, 5, 6 \in I).$$

Then

$$J_2 = e_2(\delta_1, \delta_2, \delta_3) - \frac{\mathscr{C}}{\Delta^2}, \qquad J_3 \le \delta_1\delta_2\delta_3. \tag{59}$$

In particular $\mathscr{C} \ge 0$.

Proof. The inclusion probabilities of a projection determinantal process are the principal minors of its kernel. The one-point marginals are δ_j.

By (29) and recalling the definition of $\mathcal{P}$, for $i \ne j$ the two-point inclusion probabilities can be written as

$$\Pr(i, j \in I) = \frac{q_i q_j \, \partial_{ij}\mathcal{P}(q)}{\mathcal{P}(q)}, \qquad \partial_{ij} := \frac{\partial^2}{\partial q_i \partial q_j}.$$

Hence

$$J_2 = \frac{q_4q_5 \, \partial_{45}\mathcal{P}(q) + q_4q_6 \, \partial_{46}\mathcal{P}(q) + q_5q_6 \, \partial_{56}\mathcal{P}(q)}{\mathcal{P}(q)}.$$

Differentiating (28) in the three double fields, substituting (49)–(53), and collecting terms gives

$$\begin{aligned} &q_4q_5\,\partial_{45}\mathcal{P}(q)+q_4q_6\,\partial_{46}\mathcal{P}(q)+q_5q_6\,\partial_{56}\mathcal{P}(q)\\ &\quad=\frac{h_4h_5+h_4h_6+h_5h_6-\mathscr{C}}{XYZ}. \end{aligned}$$

Using (56), we therefore obtain

$$\begin{aligned} J_2&=\frac{h_4h_5+h_4h_6+h_5h_6}{\Delta^2}-\frac{\mathscr{C}}{\Delta^2}\\ &=e_2(\delta_1,\delta_2,\delta_3)-\frac{\mathscr{C}}{\Delta^2}, \end{aligned}$$

which proves the first identity in (59).

Moreover,

$$\frac{\mathscr{C}}{\Delta^2}=\sum_{1\le i<j\le 3}(\delta_i\delta_j-\Pr(i+3,j+3\in I))\ge 0,$$

since each two-point principal minor is at most the product of the corresponding diagonal entries. Thus $\mathscr{C}\ge 0$.

Finally, J_3 is the determinant of the 3×3 principal kernel on the three double channels. Hadamard's inequality gives

$$J_3\le \delta_1\delta_2\delta_3,$$

completing the proof. □

Theorem 5.6 (Weak correlation control)**.** *On the positive chart, we have*

$$\mathscr{L}\ge(\Delta-H)\mathscr{C}. \tag{60}$$

Equivalently, with

$$t=1-(\delta_1+\delta_2+\delta_3)=\frac{\Delta-H}{\Delta},\qquad c=e_2(\delta_1,\delta_2,\delta_3)-J_2,$$

one has

$$c\le\frac{t^2-4\delta_1\delta_2\delta_3}{t}. \tag{61}$$

Equality in (60) *is possible only on the opposite-edge locus*

$$X^2=AB,\qquad Y^2=AC,\qquad Z^2=BC.$$

Proof. Write

$$A=\mathfrak{a}^2,\quad B=\mathfrak{b}^2,\quad C=\mathfrak{c}^2,\qquad X=\mathfrak{ab}x,\quad Y=\mathfrak{ac}y,\quad Z=\mathfrak{bc}z,$$

with $x,y,z>0$. A direct collection of $\mathscr{L}-(\Delta-H)\mathscr{C}$ in the scale variables gives

$$\mathscr{L}-(\Delta-H)\mathscr{C}=\mathfrak{a}^5\mathfrak{b}^5\mathfrak{c}^5\,\Psi, \tag{62}$$

where

$$\begin{aligned} \Psi=\;&\mathfrak{a}^2\mathfrak{b}\,(xz+y)E_z^-E_z^++\mathfrak{a}^2\mathfrak{c}\,(x+yz)E_z^-E_z^+\\ &+\mathfrak{a}\mathfrak{b}^2\,(xy+z)E_y^-E_y^++\mathfrak{b}^2\mathfrak{c}\,(x+yz)E_y^-E_y^+\\ &+\mathfrak{a}\mathfrak{c}^2\,(xy+z)E_x^-E_x^++\mathfrak{b}\mathfrak{c}^2\,(xz+y)E_x^-E_x^+\\ &+\mathfrak{abc}\,Q(x,y,z), \end{aligned}$$

with

$$\begin{aligned} E_z^-&=(x-y)^2+(z-1)^2, & E_z^+&=(x+y)^2+(z+1)^2,\\ E_y^-&=(x-z)^2+(y-1)^2, & E_y^+&=(x+z)^2+(y+1)^2,\\ E_x^-&=(y-z)^2+(x-1)^2, & E_x^+&=(y+z)^2+(x+1)^2. \end{aligned}$$

It remains only to show $Q\ge 0$. Set

$$s_1=x^2+y^2+z^2,\qquad s_2=x^2y^2+y^2z^2+z^2x^2,\qquad r=xyz.$$

The remaining symmetric coefficient is

$$Q = s_1^3 + 2s_1^2 + 2s_1 + 1 - (s_1+9)s_2 - 8r^2 - 2r(s_1+1). \tag{63}$$

For the three positive numbers x^2, y^2, z^2,

$$s_2 \le \frac{s_1^2}{3}, \qquad r^2 \le \left(\frac{s_1}{3}\right)^3, \qquad r \le \left(\frac{s_1}{3}\right)^{3/2}.$$

Since all three quantities occur in (63) with negative coefficients,

$$Q \ge \frac{10}{27}s_1^3 - s_1^2 + 2s_1 + 1 - 2(s_1+1)\left(\frac{s_1}{3}\right)^{3/2}.$$

Putting $\tau = \sqrt{s_1/3}$, the right-hand side factors as

$$(\tau-1)^2\left(10\tau^4 + 14\tau^3 + 9\tau^2 + 2\tau + 1\right) \ge 0.$$

Thus every term in (62) is nonnegative, proving (60). Notice also that

$$t = 1 - (\ell_4 + \ell_5 + \ell_6) = \frac{T}{D} > 0.$$

Dividing by $t\Delta^3$ gives (61), because

$$\frac{\mathscr{C}}{\Delta^2} = c, \qquad \frac{h_4h_5h_6}{\Delta^3} = \delta_1\delta_2\delta_3.$$

If equality holds, the six manifestly nonnegative scale terms force $x = y = z = 1$, which is the stated opposite-edge locus. $\square$

Let now

$$s = 1 - t, \qquad L = 2 + t, \qquad \nu = e_2(\delta_1, \delta_2, \delta_3), \qquad p = \delta_1\delta_2\delta_3.$$

Since $\delta_j > 0$ and $\delta_1 + \delta_2 + \delta_3 = 1 - t$, one has $0 < t < 1$ and $s > 0$. Let $I \sim \tau_q$ and write

$$K = |I \cap \{4,5,6\}|, \qquad P_k = \Pr(K = k), \quad 0 \le k \le 3.$$

With J_2, J_3 as in Lemma 5.5,

$$\begin{aligned} &P_0 = t + J_2 - J_3, \quad P_1 = s - 2J_2 + 3J_3,\\ &P_2 = J_2 - 3J_3, \qquad P_3 = J_3. \end{aligned} \tag{64}$$

Indeed $\mathbb{E}K = s$, $\mathbb{E}\binom{K}{2} = J_2$, and $\Pr(K=3) = J_3$.

Lemma 5.7 (Four-state determinant bound)**.** *For every positive Selling vector ρ,*

$$\mathcal{J}(\rho) \ge \Phi(\rho) := \frac{4P_0^{3/2}}{L^{3/2}} + \frac{2P_1^{3/2}}{L\sqrt{s}} + \frac{P_2^{3/2}}{\sqrt{L\nu}} + \frac{P_3^{3/2}}{2\sqrt{p}}. \tag{65}$$

At BCC equality holds in (65)*.*

Proof. For one of the 29 bases I with $\kappa_I > 0$, put

$$a_I = \kappa_I \prod_{i\in I} \ell_i.$$

By Cauchy–Binet and (14),

$$\begin{aligned} \mathcal{J}(\rho) &= \frac{1}{D^{3/2}(\rho)} \sum_{I:\,\kappa_I>0} \kappa_I \prod_{i\in I} \frac{q_i}{\sqrt{c_i}}\\ &= \sum_{I:\,\kappa_I>0} \frac{\kappa_I \prod_{i\in I} q_i}{D^{3/2}(\rho)\sqrt{\prod_{i\in I} c_i}}. \end{aligned}$$

On the other hand, by (29), (36), and (37),

$$\tau_q(I) = \frac{\kappa_I \prod_{i\in I} q_i}{D^2(\rho)}, \qquad a_I = \frac{\kappa_I \prod_{i\in I} q_i c_i}{D^3(\rho)},$$

where we used $|I| = 3$. Therefore

$$\frac{\tau_q(I)^{3/2}}{\sqrt{a_I}} = \frac{\kappa_I \prod_{i\in I} q_i}{D^{3/2}(\rho)\sqrt{\prod_{i\in I} c_i}}.$$

As a consequence,

$$\mathcal{J}(\rho) = \sum_{I:\,\kappa_I>0} \frac{\tau_q(I)^{3/2}}{\sqrt{a_I}}. \tag{66}$$

For $\mathcal{B}_k = \{I : \kappa_I > 0,\ |I \cap \{4,5,6\}| = k\}$, Hölder's inequality gives

$$\sum_{I\in\mathcal{B}_k} \frac{\tau_q(I)^{3/2}}{\sqrt{a_I}} \geq \frac{P_k^{3/2}}{\sqrt{M_k}}, \qquad M_k := \sum_{I\in\mathcal{B}_k} a_I. \tag{67}$$

Write $x_i = \ell_i$, $0 \leq i \leq 3$. Equation (28) applied to the field $(x_0, \dots, x_3; \delta_1, \delta_2, \delta_3)$ gives

$$\begin{aligned} M_0 &= e_3(x_0, x_1, x_2, x_3), \\ M_1 &= \delta_1(x_0 + x_1)(x_2 + x_3) + \delta_2(x_0 + x_2)(x_1 + x_3) \\ &\quad + \delta_3(x_0 + x_3)(x_1 + x_2), \\ M_2 &= L\nu, \qquad M_3 = 4p. \end{aligned}$$

Recalling that $x_0 + \cdots + x_3 = L$, from Maclaurin's inequality

$$e_3(x_0, x_1, x_2, x_3) \leq \binom{4}{3}\left(\frac{x_0 + x_1 + x_2 + x_3}{4}\right)^3 = \frac{L^3}{16}$$

and from the inequality $(x_i + x_j)(L - x_i - x_j) \leq L^2/4$, we get

$$M_0 \leq \frac{L^3}{16}, \qquad M_1 \leq \frac{L^2 s}{4}, \qquad M_2 = L\nu, \qquad M_3 = 4p. \tag{68}$$

Substitution in (67), followed by summation in k, proves (65). At BCC

$$(t, s, \nu, p) = \left(\frac{1}{4}, \frac{3}{4}, \frac{3}{16}, \frac{1}{64}\right), \qquad (P_0, P_1, P_2, P_3) = \frac{1}{64}(27, 27, 9, 1),$$

and direct substitution gives $\Phi = \mathcal{J}_{\mathrm{BCC}} = \mathcal{J}(1,1,1,1,1,1)$. □

We shall use repeatedly the elementary support inequality

$$\frac{x^{3/2}}{\sqrt{m}} \geq hx - \frac{4m}{27}h^3, \qquad x \geq 0, \quad m > 0, \quad h \geq 0. \tag{69}$$

Indeed the minimum of the left-hand side minus hx occurs at $x = 4mh^2/9$.

Theorem 5.8 (Scalar closure from the physical constraints)**.** *For every positive Selling vector ρ,*

$$\Phi(\rho) \geq \mathcal{J}_{\mathrm{BCC}} = \frac{37 + 30\sqrt{3}}{128} \tag{70}$$

where Φ is defined in (65)*. Equality is possible only when*

$$t = \delta_1 = \delta_2 = \delta_3 = \frac{1}{4}. \tag{71}$$

Proof. The physical constraints used below are

$$p \leq \frac{t^2}{4}, \qquad J_2 \geq \nu - \frac{t^2 - 4p}{t}, \qquad J_3 \leq p, \tag{72}$$

by Theorem 5.3, Theorem 5.6, and Lemma 5.5. We also use Schur's inequality

$$\nu \leq \frac{s^3 + 9p}{4s}, \tag{73}$$

and the AM–GM bound $p \leq s^3/27$.

Range I: $0 < t \le 1/8$. Apply (69) to the first three terms of (65) with the common slope

$$h = \frac{3}{2\sqrt{2}},$$

and discard the last nonnegative term. Since $4h^3/27 = 1/(4\sqrt{2})$,

$$\Phi(\rho) \ge h(1-P_3) - \frac{1}{4\sqrt{2}}\left(\frac{L^3}{16} + \frac{L^2 s}{4} + L\nu\right).$$

Using $P_3 = J_3 \le p \le t^2/4$ and, from (73),

$$\nu \le \frac{s^3 + 9t^2/4}{4s},$$

we obtain

$$\Phi(\rho) \ge B(t) := -\frac{\sqrt{2}\,(t-4)(t+2)(t^2+10t-8)}{128(t-1)}. \tag{74}$$

Moreover

$$B'(t) = -\frac{3\sqrt{2}\,t(t^3+4t^2-20t+24)}{128(t-1)^2} < 0 \qquad (0 < t \le 1/8),$$

since $t^3 + 4t^2 - 20t + 24 > 24 - 20/8 > 0$. Hence $B(t) \ge B(1/8)$, where

$$B(1/8) = \frac{227137\sqrt{2}}{458752}.$$

The elementary rational estimates

$$\sqrt{2} > \frac{140}{99}, \qquad \sqrt{3} < \frac{26}{15}$$

follow by squaring, and give

$$B(1/8) - \mathcal{J}_{\mathrm{BCC}} > \frac{7877}{1622016} > 0.$$

Thus the inequality is strict in Range I.

Ranges II–III: $1/8 \le t < 1$. Apply (69) to all four terms in (65) with

$$(\sigma_0, \sigma_1, \sigma_2, \sigma_3) = \left(\frac{2}{\sqrt{3}}, 1, \frac{\sqrt{3}}{2}, \frac{3}{4}\right). \tag{75}$$

These are precisely the four support slopes at BCC. Thus

$$\Phi(\rho) \ge \sum_{k=0}^{3} \sigma_k P_k - \frac{4}{27}\left(\frac{L^3}{16}\sigma_0^3 + \frac{L^2 s}{4}\sigma_1^3 + L\nu\sigma_2^3 + 4p\sigma_3^3\right). \tag{76}$$

The coefficients of J_2 and J_3 in the right-hand side are

$$\frac{-12+7\sqrt{3}}{6} > 0, \qquad \frac{45-26\sqrt{3}}{12} < 0,$$

respectively (the signs follow from $7^2 \cdot 3 > 12^2$ and $26^2 \cdot 3 > 45^2$). Hence (72) permits the substitutions

$$J_2 = \nu - \frac{t^2-4p}{t}, \qquad J_3 = p.$$

After these substitutions the coefficient of ν is

$$-\frac{\sqrt{3}\,t - 19\sqrt{3} + 36}{18} < 0,$$

so Schur's upper bound (73) may also be inserted; here $36 > 19\sqrt{3}$. Denote by $G(t,p)$ the resulting right-hand side of (76) minus $\mathcal{J}_{\mathrm{BCC}}$. A direct collection gives

$$\frac{\partial G}{\partial p} = -\frac{N(t)}{24t(1-t)}, \tag{77}$$

where

$$N(t) = (84 - 49\sqrt{3})t^2 + (-168 + 107\sqrt{3})t + 192 - 112\sqrt{3}.$$

Since $84 - 49\sqrt{3} < 0$, $N(t)$ is concave on $[0, 1]$. Observing that $N'(1) = 9\sqrt{3} > 0$, it follows that $N(t)$ is also increasing on $[0, 1]$. Moreover

$$N(1/8) = \frac{11028 - 6361\sqrt{3}}{64} > 0,$$

because $11028^2 - 3 \cdot 6361^2 = 229821 > 0$. As a consequence we get

$$\frac{\partial G}{\partial p} < 0 \qquad (1/8 \le t < 1). \tag{78}$$

If $1/8 \le t \le 1/4$, then $p \le t^2/4$, and (78) gives $G(t, p) \ge G(t, t^2/4)$. The endpoint factors as

$$G\left(t, \frac{t^2}{4}\right) = \frac{(72 - 43\sqrt{3})(4t - 1)Q_-(t)}{3763584(t - 1)}, \tag{79}$$

where

$$Q_-(t) = 484t^3 - 12(345\sqrt{3} + 119)t^2 - 9(2467\sqrt{3} + 3326)t + 1718 - 873\sqrt{3}.$$

Since $72 - 43\sqrt{3} < 0$ (because $72^2 < 3 \cdot 43^2$), it remains only to note that $Q_-(t) < 0$ on this interval. Indeed

$$Q_-'(t) = 3\{484t^2 - (2760\sqrt{3} + 952)t - (7401\sqrt{3} + 9978)\} < 0,$$

because $t \le 1/4$ gives $484t^2 \le 121/4 < 7401\sqrt{3} + 9978$, and

$$Q_-(1/8) = -\frac{59409\sqrt{3}}{16} - \frac{261775}{128} < 0.$$

All signs in (79) therefore give $G(t, t^2/4) \ge 0$, with equality only at $t = 1/4$.

Finally let $1/4 \le t < 1$. Here $p \le s^3/27$, so $G(t, p) \ge G(t, s^3/27)$. This endpoint factors as

$$G\left(t, \frac{s^3}{27}\right) = \left(-\frac{5}{54} + \frac{13\sqrt{3}}{243}\right) \frac{t - 1/4}{t} Q_+(t), \tag{80}$$

where

$$Q_+(t) = t^3 + \left(\frac{37}{4} + 6\sqrt{3}\right) t^2 + \left(\frac{9589}{16} + \frac{705\sqrt{3}}{2}\right) t - 96 - 48\sqrt{3}.$$

The prefactor is positive because $(26\sqrt{3})^2 > 45^2$. Furthermore $Q_+'(t) > 0$ for $t > 0$ and

$$Q_+(1/4) = \frac{3483}{64} + \frac{81\sqrt{3}}{2} > 0.$$

Thus (80) is nonnegative, again with equality only at $t = 1/4$.

We have proved (70). Equality forces $t = 1/4$ and, from the decreasing p-step above, $p = t^2/4 = 1/64$. Since also $s = 3/4$, equality in AM–GM gives $\delta_1 = \delta_2 = \delta_3 = 1/4$, proving (71). □

5.4. **Main result.**

Theorem 5.9 (Sharp seven-direction determinant inequality). *For every positive Selling vector ρ,*

$$\mathcal{J}(\rho) \ge \mathcal{J}_{\mathrm{BCC}} = \frac{37 + 30\sqrt{3}}{128}.$$

Equality holds only on the BCC ray.

Proof. Lemma 5.7 and Theorem 5.8 give

$$\mathcal{J}(\rho) \geq \Phi(\rho) \geq \mathcal{J}_{\mathrm{BCC}}.$$

If equality holds, Theorem 5.8 gives $t = \delta_1 = \delta_2 = \delta_3 = 1/4$. In particular

$$\delta_1\delta_2\delta_3 = \frac{1}{64} = \frac{t^2}{4},$$

so equality holds in Theorem 5.3. Its equality classification gives the opposite-edge family $a = f = A$, $b = e = B$, $c = d = C$. Equality of the three double leverage scores then reads

$$A^2(B + C) = B^2(A + C) = C^2(A + B).$$

Since

$$A^2(B + C) - B^2(A + C) = (A - B)\{AB + C(A + B)\},$$

and the bracket is positive, $A = B$; cyclically $A = B = C$. Thus the Selling vector lies on the BCC ray. Conversely direct substitution gives equality at BCC. □

Theorem 5.10 (Global minimality and uniqueness of BCC)**.** *For every nonnegative Selling vector ρ defining a nondegenerate full-rank lattice Voronoi cell,*

$$F(\rho) \geq F_{\mathrm{BCC}}.$$

Equality holds only on the BCC ray. Therefore the regular truncated octahedron is the unique global minimizer among all three-dimensional lattice Voronoi cells, up to similarity.

Proof. In the positive interior, (15) and the preceding theorem give

$$F(\rho) \geq 6\mathcal{J}^{1/3}(\rho) \geq 6\mathcal{J}_{\mathrm{BCC}}^{1/3} = F_{\mathrm{BCC}}.$$

Equality is attained only at BCC. On the nondegenerate boundary, Theorem 4.8 gives $F(\rho) \geq 3\,2^{5/6} > F_{\mathrm{BCC}}$. The strict comparison is exact: after cubing and cancelling the common positive factor it is equivalent to

$$64\sqrt{2} > 37 + 30\sqrt{3}.$$

Indeed $\sqrt{2} > 7/5$ and $\sqrt{3} < 7/4$, so that $64\sqrt{2} > 448/5 > 179/2 > 37 + 30\sqrt{3}$. □

6. Extension to all parallelohedra

This section develops the part of the Truncated Octahedron Conjecture which is not contained in the lattice–Voronoi theorem. The key point is that the weighted K_4 combinatorics used throughout the paper survives unchanged: what is lost outside the Voronoi subclass is only the relation tying the Euclidean metric to the six graphical weights.

6.1. A graphical parametrization.

We retain from Section 3 the identification

$$E(K_4) = \{01, 02, 03, 12, 13, 23\}$$

with the fixed edge order

$$(01, 02, 03, 12, 13, 23),$$

and the corresponding incidence roots

$$r_a = e_1, \quad r_b = e_2, \quad r_c = e_3, \quad r_d = e_1 - e_2, \quad r_e = e_1 - e_3, \quad r_f = e_2 - e_3.$$

Thus $r_a, \ldots, r_f$ are, up to sign, the six edge vectors of the reference realization of K_4 on

$$p_0 = 0, \qquad p_1 = e_1, \qquad p_2 = e_2, \qquad p_3 = e_3.$$

We now regard

$$\omega = (a, b, c, d, e, f) \in \mathbb{R}_{\geq 0}^{E(K_4)}$$

as an arbitrary nonnegative weighting of the six edges and put

$$Z_\omega = \sum_{\xi \in \{a,b,c,d,e,f\}} \left[-\frac{\omega_\xi r_\xi}{2}, \frac{\omega_\xi r_\xi}{2}\right]. \tag{81}$$

The polynomials $D(\omega)$ and $q_i(\omega)$ are the same spanning-tree and facet polynomials as in Section 3.

In the lattice–Voronoi subclass these graphical weights are precisely the Selling conorms,

$$\omega_{ij} = \rho_{ij},$$

and they determine the Selling Gram matrix

$$A(\omega) = \sum_{\xi \in \{a,b,c,d,e,f\}} \omega_\xi r_\xi r_\xi^{\mathsf{T}}.$$

Equivalently, $A(\omega)$ is the reduced weighted Laplacian of K_4. In the general affine problem the same weighted graphical zonotope Z_ω is retained, while the Euclidean metric is allowed to vary independently.

Theorem 6.1 (Graphical affine parametrization)**.** *Every three-dimensional parallelohedron is, up to translation, of the form*

$$P = LZ_\omega \tag{82}$$

for some $L \in GL(3,\mathbb{R})$ and some $\omega \in \mathbb{R}_{\geq 0}^{E(K_4)}$ with $D(\omega) > 0$.

For an edge $ij \in E(K_4)$, let $\overline{ij} = kl$ denote the complementary edge, namely

$$\{i,j,k,l\} = \{0,1,2,3\}.$$

After relabelling the four vertices, the five Fedorov types correspond to the images, under the edge-complement involution

$$ij \longmapsto \overline{ij},$$

of the support patterns described in [16, Section 2]*. In particular, they have respectively* $6, 5, 4, 4, 3$ *active graphical generators.*

Proof. Lángi's representation of three-dimensional parallelohedra [16] shows that a translate of every such polytope can be written as

$$\sum_{0 \leq i < j \leq 3} [0, \beta_{ij}(v_i \times v_j)], \qquad \beta_{ij} \geq 0,$$

where

$$v_0 + v_1 + v_2 + v_3 = 0.$$

The vectors $v_0, \ldots, v_3$ form a nondegenerate centered tetrahedral configuration: every three are independent and their unique linear relation is $v_0 + v_1 + v_2 + v_3 = 0$. The lower Fedorov types are obtained from the same nondegenerate quadruple by setting the appropriate coefficients β_{ij} equal to zero.

Choose the centered reference tetrahedron

$$u_0 = -(e_1 + e_2 + e_3), \qquad u_1 = e_1, \qquad u_2 = e_2, \qquad u_3 = e_3.$$

Since both $(u_i)_{i=0}^3$ and $(v_i)_{i=0}^3$ satisfy the same unique linear relation with all coefficients equal to one, there exists $S \in GL(3,\mathbb{R})$ such that

$$v_i = Su_i, \qquad 0 \leq i \leq 3.$$

The covariance formula for the cross product gives

$$(Su_i) \times (Su_j) = \det(S) S^{-\mathsf{T}} (u_i \times u_j).$$

Thus all tetrahedral shape variables are absorbed in the single invertible linear map

$$L := \det(S) S^{-\mathsf{T}}.$$

We now compare the six cross products of the reference tetrahedron with the incidence roots

$$r_{01} = e_1, \qquad r_{02} = e_2, \qquad r_{03} = e_3, \qquad r_{12} = e_1 - e_2, \qquad r_{13} = e_1 - e_3, \qquad r_{23} = e_2 - e_3.$$

A direct computation gives

$$\begin{aligned} &u_2 \times u_3 = r_{01}, \quad &u_1 \times u_3 = -r_{02}, \quad &u_1 \times u_2 = r_{03}, \\ &u_0 \times u_3 = -r_{12}, \quad &u_0 \times u_2 = r_{13}, \quad &u_0 \times u_1 = -r_{23}. \end{aligned}$$

Equivalently,

$$u_i \times u_j = \varepsilon_{ij} r_{\overline{ij}}, \qquad \varepsilon_{ij} \in \{-1, 1\}. \tag{83}$$

Hence the coefficient attached in Lángi's representation to the pair ij becomes the graphical weight on the complementary edge:

$$\omega_{\overline{ij}} := \beta_{ij}. \tag{84}$$

Using (83), we therefore obtain, up to translation,

$$\begin{aligned} P &= L \sum_{0 \le i < j \le 3} [0, \beta_{ij}(u_i \times u_j)] \\ &= L \sum_{0 \le i < j \le 3} [0, \varepsilon_{ij} \omega_{\overline{ij}} r_{\overline{ij}}]. \end{aligned}$$

For any vector g, the segment $[0, g]$ is a translate of $[-g/2, g/2]$, and replacing g by $-g$ does not change the latter centered segment. Consequently the last Minkowski sum is, up to translation, precisely

$$Z_\omega = \sum_{\xi \in E(K_4)} \left[-\frac{\omega_\xi r_\xi}{2}, \frac{\omega_\xi r_\xi}{2} \right].$$

This proves

$$P = L Z_\omega$$

up to translation.

Since P is three-dimensional and L is invertible, Z_ω is full-dimensional. The graphical zonotope Z_ω is full-dimensional if and only if the support graph of ω is connected. By the weighted matrix–tree theorem, this is equivalent to

$$D(\omega) > 0.$$

Finally, (84) shows that the support of the graphical weight ω is obtained from the support of the coefficients β_{ij} by the edge-complement involution $ij \mapsto \overline{ij}$. Thus the Fedorov support patterns are the complementary-edge images of those listed in [16, Section 2]. Since edge complementation is a bijection of the six edges of K_4, the five types have respectively $6, 5, 4, 4, 3$ active graphical generators. □

For $P = L Z_\omega$, define the determinant-one metric

$$Q = |\det L|^{2/3} L^{-1} L^{-\mathsf{T}}, \qquad Q > 0, \qquad \det Q = 1. \tag{85}$$

Thus Q records the five Euclidean shape degrees of freedom left after removing scale and rotations.

Proposition 6.2 (Exact affine surface formula)**.** *For every $D(\omega) > 0$ and every $Q > 0$ with $\det Q = 1$,*

$$\operatorname{Iso}(\omega, Q) = \frac{2}{D(\omega)^{2/3}} \sum_{i=0}^{6} q_i(\omega) \sqrt{z_i^{\mathsf{T}} Q z_i}. \tag{86}$$

The Voronoi realization is the codimension-five slice

$$Q_{\mathrm{Vor}}(\omega) = D(\omega)^{-1/3} A(\omega). \tag{87}$$

On this slice (86) *reduces exactly to* (12)*.*

Proof. By (13), the facet of Z_ω with normal z_i has area $q_i |z_i|$. Under L, a planar area with normal z_i is multiplied by $|\det L|\, |L^{-\mathsf{T}} z_i| / |z_i|$. From (85),

$$|L^{-\mathsf{T}} z_i| = |\det L|^{-1/3} \sqrt{z_i^{\mathsf{T}} Q z_i}.$$

Summing opposite facets gives

$$\mathcal{H}^2(\partial P) = 2 |\det L|^{2/3} \sum_i q_i \sqrt{z_i^{\mathsf{T}} Q z_i}.$$

Since $|P| = |\det L| D(\omega)$, division by $|P|^{2/3}$ proves (86). For a lattice Voronoi cell, $L = A^{-1/2}$ and $\det A = D$, which gives (87). $\square$

6.2. **Affine duality.** For the global inequality we keep the affine metric arbitrary. The two steps needed in the original dual formulation—the Ball–Barthe certificate and its entropy comparison with $\mathcal{J}$—can be combined into one canonical bridge. This is the main simplification in the passage from lattices to arbitrary parallelohedra.

We first record a useful physical matrix identity. For a graphical edge weight ω let $A(\omega)$ be the matrix associated with the weighted K_4 as above, and let $q = q(\omega)$ be the seven facet field. Then

$$H(q) = \sum_{i=0}^{6} q_i z_i z_i^{\mathsf{T}} = \operatorname{adj} A(\omega) = D(\omega) A(\omega)^{-1}. \tag{88}$$

In particular $\mathcal{P}(q) = D^2$. This is the same adjugate identity as (35), now written for an arbitrary positive edge weighting ω.

For a positive field u, put

$$H(u) = \sum_{i=0}^{6} u_i z_i z_i^{\mathsf{T}}, \qquad \ell_i(u) = u_i z_i^{\mathsf{T}} H(u)^{-1} z_i = u_i \partial_i \log \mathcal{P}(u).$$

By homogeneity, $\sum_i \ell_i(u) = 3$.

For $q = q(\omega)$, let c_i be the corresponding cut weights and introduce the canonical surface field

$$p_i := \frac{q_i}{\sqrt{c_i}}, \qquad 0 \le i \le 6. \tag{89}$$

Then $H(p) = M$ from (14), so

$$\mathcal{P}(p) = \det M, \qquad \mathcal{J}(\omega) = \frac{\mathcal{P}(p)}{D^{3/2}(\omega)}. \tag{90}$$

Theorem 6.3 (Canonical affine determinant bridge)**.** *For every positive graphical weight ω, every $Q > 0$ with $\det Q = 1$, and $q = q(\omega)$, one has*

$$\frac{N_Q(q)^3}{27\mathcal{P}(q)} \ge \mathcal{J}, \qquad N_Q(q) := \sum_{i=0}^{6} q_i \sqrt{z_i^{\mathsf{T}} Q z_i}. \tag{91}$$

If equality holds, then, with p as in (89) and $\widehat{\ell}_i = \ell_i(p)$, the seven quantities

$$\frac{q_i \sqrt{z_i^{\mathsf{T}} Q z_i}}{\widehat{\ell}_i} \qquad (0 \le i \le 6) \tag{92}$$

are equal.

Proof. Write $P_p = \mathcal{P}(p)$, $P_q = \mathcal{P}(q) = D^2$ and set

$$\ell_i = \ell_i(q), \qquad \widehat{\ell}_i = \ell_i(p), \qquad \rho_i = z_i^{\mathsf{T}} H(p)^{-1} z_i = \frac{\widehat{\ell}_i}{p_i}.$$

The unit vectors

$$e_i = \frac{H(p)^{-1/2} z_i}{\sqrt{\rho_i}}$$

form the isotropic frame $\sum_i \widehat{\ell}_i e_i e_i^{\mathsf{T}} = I_3$. Applying the rank-one Ball–Barthe inequality to $B = H(p)^{1/2} Q H(p)^{1/2}$, whose determinant is P_p, gives

$$\prod_i (z_i^{\mathsf{T}} Q z_i)^{\widehat{\ell}_i/2} \ge P_p^{1/2} \prod_i \left(\frac{\widehat{\ell}_i}{p_i}\right)^{\widehat{\ell}_i/2}. \tag{93}$$

Weighted AM–GM, with weights $\widehat{\ell}_i/3$, followed by (93), yields

$$\frac{N_Q(q)^3}{27P_q} \geq \mathcal{R} := \frac{P_p^{1/2}}{P_q} \prod_i \left(\frac{q_i^2}{p_i\widehat{\ell}_i}\right)^{\widehat{\ell}_i/2}. \tag{94}$$

It remains to compare $\mathcal{R}$ with $\mathcal{J} = P_p/D^{3/2}$. From the definition of $\mathcal{R}$, using $P_q = D^2$, one obtains

$$2\log\frac{\mathcal{R}}{\mathcal{J}} = -\log P_p - \frac{1}{2}\log P_q + \sum_i \widehat{\ell}_i \log\frac{q_i^2}{p_i\widehat{\ell}_i}.$$

On the other hand,

$$D_{\mathrm{KL}}(\tau_p\|\tau_q) - \sum_i \widehat{\ell}_i \log\frac{\widehat{\ell}_i}{\ell_i} = \log\frac{P_q}{P_p} + \sum_i \widehat{\ell}_i \log\frac{p_i\ell_i}{q_i\widehat{\ell}_i}.$$

Since $p_i/q_i = c_i^{-1/2}$, $c_i = D\ell_i/q_i$, and $\sum_i \widehat{\ell}_i = 3$, the two displayed expressions agree. Therefore

$$2\log\frac{\mathcal{R}}{\mathcal{J}} = D_{\mathrm{KL}}(\tau_p\|\tau_q) - \sum_i \widehat{\ell}_i \log\frac{\widehat{\ell}_i}{\ell_i}. \tag{95}$$

Lemma 5.1, applied with $u = p$ and $v = q$, shows that the right-hand side is nonnegative. Combining (94) and (95) proves (91). Equality in the final bound forces equality in the weighted AM–GM step, which is precisely (92). □

Theorem 6.4 (Positive six-generator parallelohedra)**.** *Let $\omega \in \mathbb{R}^6_{>0}$ and let $Q > 0$, $\det Q = 1$. Then*

$$\mathrm{Iso}(\omega, Q) \geq F_{\mathrm{BCC}}. \tag{96}$$

Equality holds only for the regular truncated octahedron, up to similarity.

Proof. Let $q = q(\omega)$ and $N_Q = \sum_i q_i\sqrt{z_i^\top Q z_i}$. Since $\mathcal{P}(q) = D(\omega)^2$, Proposition 6.2 gives

$$\left(\frac{\mathrm{Iso}(\omega, Q)}{6}\right)^3 = \frac{N_Q^3}{27\mathcal{P}(q)}.$$

Apply the canonical affine bridge Theorem 6.3, and then the sharp lattice determinant inequality Theorem 5.9:

$$\left(\frac{\mathrm{Iso}(\omega, Q)}{6}\right)^3 \geq \mathcal{J}(\omega) \geq \mathcal{J}_{\mathrm{BCC}}. \tag{97}$$

Since $F_{\mathrm{BCC}} = 6\mathcal{J}_{\mathrm{BCC}}^{1/3}$, this proves (96).

If equality holds in the final conclusion, then $\mathcal{J} = \mathcal{J}_{\mathrm{BCC}}$. The equality statement of Theorem 5.9 forces ω onto the BCC ray. Normalize $q = (3, 3, 3, 3; 1, 1, 1)$. Then $c = (3, 3, 3, 3; 4, 4, 4)$ and hence

$$p = (\sqrt{3}, \sqrt{3}, \sqrt{3}, \sqrt{3}; 1/2, 1/2, 1/2) = \frac{1}{\sqrt{3}}(3, 3, 3, 3; \sqrt{3}/2, \sqrt{3}/2, \sqrt{3}/2).$$

Leverage scores are invariant under a common scaling of the field, so direct substitution gives two leverage values

$$\ell_S = \frac{9}{11} - \frac{3\sqrt{3}}{22} \quad (i < 4), \qquad \ell_D = -\frac{1}{11} + \frac{2\sqrt{3}}{11} \quad (i \geq 4),$$

with

$$\frac{3\ell_D}{\ell_S} = \frac{2}{\sqrt{3}}.$$

The equality condition (92) therefore forces the seven metric lengths $w_i = \sqrt{z_i^\top Q z_i}$ to satisfy

$$w_0 = w_1 = w_2 = w_3 =: S, \qquad w_4 = w_5 = w_6 = \frac{2S}{\sqrt{3}}.$$

Since w_1^2, w_2^2, w_3^2 are the diagonal entries of Q, they are all S^2. The three double-cut equalities give all off-diagonal entries equal to $-S^2/3$; equivalently

$$Q = \frac{S^2}{3}\begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}.$$

The condition $\det Q = 1$ fixes the scalar. This is exactly the regular BCC metric, so the resulting parallelohedron is similar to the regular truncated octahedron. The converse is immediate. ☐

6.3. **The truncated octahedron theorem.** The strata with at most four generators are already controlled in the full zonotopal class by Joós–Lángi [14]. The only new boundary case is the five-generator elongated rhombic dodecahedron. The following result extends Theorem 4.7 from the Voronoi slice

$$Q = Q_{\mathrm{Vor}}(\omega) = D(\omega)^{-1/3} A(\omega)$$

to an arbitrary determinant-one affine metric Q. The same four-point determinant inequality from Section 4 remains the key input, but the Voronoi metric relation is no longer imposed.

Theorem 6.5 (Five-generator affine boundary)**.** *Every three-dimensional parallelohedron generated by five segments satisfies*

$$\mathrm{Iso}(P) > 3\,2^{5/6}. \tag{98}$$

On the closure of the five-generator stratum,

$$\mathrm{Iso}(P) \geq 3\,2^{5/6}, \tag{99}$$

with equality exactly at the regular rhombic dodecahedron, up to similarity.

Proof. By tetrahedral symmetry set $a = 0$ and $f = t$. Relabel the six active facet polynomials and cut weights of Section 4 as follows:

$$\begin{aligned}
C_1 &= de + t(d+e), & L_1 &= b + c,\\
C_2 &= bc + t(b+c), & L_2 &= d + e,\\
C_3 &= ce, & L_3 &= b + d + t,\\
C_4 &= bd, & L_4 &= c + e + t,\\
C_5 &= be, & L_5 &= c + d + t,\\
C_6 &= cd, & L_6 &= b + e + t.
\end{aligned}$$

Thus

$$(C_1, \dots, C_6) = (q_0, q_1, q_2, q_3, q_5, q_6), \qquad (L_1, \dots, L_6) = (c_0, c_1, c_2, c_3, c_5, c_6).$$

For a six-tuple $x = (x_1, \dots, x_6)$ in this subsection, write

$$\Gamma_*(x_1, \dots, x_6) := \Gamma(d), \qquad (d_{01}, d_{23}, d_{02}, d_{13}, d_{12}, d_{03}) = (x_1, x_2, x_3, x_4, x_5, x_6).$$

This convention records explicitly the nonstandard edge order used below. Let

$$\mathsf{G}_\omega = \Gamma_*(L_1, \dots, L_6).$$

As in (19),

$$\det \mathsf{G}_\omega = D, \qquad \partial_{L_i} \det \mathsf{G}_\omega = C_i. \tag{100}$$

Now let $Q > 0$, $\det Q = 1$ be an arbitrary affine metric. For the six active cut directions $z_0, z_1, z_2, z_3, z_5, z_6$, set

$$(R_1, \dots, R_6) = (z_0^\top Q z_0, z_1^\top Q z_1, z_2^\top Q z_2, z_3^\top Q z_3, z_5^\top Q z_5, z_6^\top Q z_6)$$

and use the edge order

$$d_{01}^2 = R_1, \quad d_{23}^2 = R_2, \quad d_{02}^2 = R_3, \quad d_{13}^2 = R_4, \quad d_{12}^2 = R_5, \quad d_{03}^2 = R_6.$$

These are squared Euclidean distances: if $p_1 = z_0$, $p_2 = z_2$, $p_3 = z_6$ and $p_0 = 0$, then the remaining differences are $p_1 - p_2 = z_5$, $p_1 - p_3 = z_3$, and $p_2 - p_3 = -z_1$. Hence

$$\Gamma_*(R_1, \dots, R_6) = U^\top Q U, \qquad U = (z_0\ z_2\ z_6), \qquad |\det U| = 1, \tag{101}$$

so

$$\det \Gamma_*(R_1, \dots, R_6) = 1. \tag{102}$$

Put

$$\mathsf{H}_Q = \Gamma_*(\sqrt{R_1}, \dots, \sqrt{R_6}).$$

By linearity of Γ and Jacobi's formula,

$$N_Q := \sum_{i=1}^{6} C_i \sqrt{R_i} = \operatorname{tr}(\operatorname{adj} \mathsf{G}_\omega \, \mathsf{H}_Q). \tag{103}$$

If μ_1, μ_2, μ_3 are the eigenvalues of $\mathsf{G}_\omega^{-1/2} \mathsf{H}_Q \mathsf{G}_\omega^{-1/2}$, then

$$N_Q = D(\mu_1 + \mu_2 + \mu_3), \qquad \det \mathsf{H}_Q = D\mu_1\mu_2\mu_3.$$

Therefore

$$N_Q^3 \geq 27D^2 \det \mathsf{H}_Q. \tag{104}$$

Theorem 4.6, applied to the metric d, and (102) give

$$2(\det \mathsf{H}_Q)^2 \geq 1, \qquad \det \mathsf{H}_Q \geq 2^{-1/2}.$$

Since the affine quotient is $2N_Q/D^{2/3}$, we conclude

$$\operatorname{Iso}(P) \geq 6\, 2^{-1/6} = 3\, 2^{5/6}.$$

If equality holds, Theorem 4.6 forces all six d_{ij} to be equal, equivalently all six $R_i = d_{ij}^2$ are equal. Equality in (104) forces $\mathsf{H}_Q = \lambda \mathsf{G}_\omega$ for some $\lambda > 0$. Since

$$\mathsf{H}_Q = \Gamma_*(\sqrt{R_1}, \dots, \sqrt{R_6}), \qquad \mathsf{G}_\omega = \Gamma_*(L_1, \dots, L_6),$$

and Γ is injective on the six edge variables, it follows that

$$\sqrt{R_i} = \lambda L_i \qquad (1 \leq i \leq 6).$$

As the R_i are all equal, all six L_i are equal. The relations $L_3 = L_5$, $L_3 = L_6$, $L_1 = L_2$, and $L_1 = L_3$ yield $b = c$, $d = e$, $b = d$, and finally $t = 0$. Thus the graphical weights lie on the four-generator FCC ray $(0, m, m, m, m, 0)$. Together with the equality of all six distances d_{ij}, this forces the determinant-one metric to be the tetrahedrally symmetric FCC metric. Hence the limiting polytope is the regular rhombic dodecahedron, up to similarity. Equality is not attained in the open five-generator stratum. □

Corollary 6.6 (Non-truncated parallelohedra)**.** *Every three-dimensional parallelohedron which is not of truncated-octahedral type satisfies*

$$\operatorname{Iso}(P) \geq 3\, 2^{5/6} > F_{\mathrm{BCC}}, \tag{105}$$

with equality in the first inequality only for the regular rhombic dodecahedron.

Proof. The three- and four-generator strata follow from [14, Theorem 3]; Theorem 6.5 handles the remaining five-generator type. □

Proof of Theorem 1.1. The sharp estimate for the non-truncated Fedorov types, together with its equality case, is Corollary 6.6. It remains to prove the global BCC bound.

By Theorem 6.1, every three-dimensional parallelohedron is represented by a connected nonnegative edge weighting ω of K_4 and a free affine metric. If all six graphical generators are active, Theorem 6.4 gives the BCC bound and its equality classification. If at least one generator is absent, the polytope belongs to a non-truncated Fedorov stratum and Corollary 6.6 gives

$$\operatorname{Iso}(P) \geq 3\, 2^{5/6} > F_{\mathrm{BCC}}.$$

Thus the regular truncated octahedron is the unique global minimizer up to similarity, and Theorem 1.1 follows. □

7. Open problems

For a full-rank lattice $\Lambda \subset \mathbb{R}^n$, define

$$\mathrm{Iso}_n(\mathrm{Vor}(\Lambda)) = \frac{\mathcal{H}^{n-1}(\partial\,\mathrm{Vor}(\Lambda))}{\mathcal{H}^n(\mathrm{Vor}(\Lambda))^{(n-1)/n}}, \qquad \gamma_n = \inf_\Lambda \mathrm{Iso}_n(\mathrm{Vor}(\Lambda)).$$

The planar problem is minimized by the hexagonal lattice [12], and Theorem 5.10 determines γ_3. Already in dimension four, two special features of the present proof disappear. Not every lattice is of Voronoi's first kind, so there is no global nonnegative-conorm chart of the type used here [19]; and the combinatorial stratification grows rapidly. There are 52 four-dimensional types of parallelohedra [11], while the classification of lattice Voronoi polytopes in dimension five contains 110244 affine types [10].

A simple comparison also shows that the sequence A_n^* cannot be adopted uncritically as the higher-dimensional answer. The Voronoi cell of D_4 is a regular 24-cell [7, Chap. 4]. In the normalization with edge length 1 it has volume 2 and 24 regular-octahedral facets, each of volume $\sqrt{2}/3$. Hence

$$\mathrm{Iso}_4(\mathrm{Vor}(D_4)) = \frac{24(\sqrt{2}/3)}{2^{3/4}} = 2^{11/4} \approx 6.727171322.$$

The Voronoi cell of A_4^* is, up to rescaling, the graphical zonotope of K_5 [7, Chap. 21]. In the root-lattice realization, the matrix-tree formula gives volume $5^3\sqrt{5} = 125\sqrt{5}$. There are ten facets of cut type $1|4$, each a K_4 graphical zonotope of volume 32, and twenty facets of type $2|3$, each the orthogonal product of a segment of length $\sqrt{2}$ and a K_3 graphical zonotope of area $3\sqrt{3}$, hence of volume $3\sqrt{6}$. Thus $|\,\mathrm{Vor}(A_4^*)| = 125\sqrt{5}$, $\mathcal{H}^3(\partial\,\mathrm{Vor}(A_4^*)) = 320 + 60\sqrt{6}$, and therefore

$$\mathrm{Iso}_4(\mathrm{Vor}(A_4^*)) = \frac{320 + 60\sqrt{6}}{(125\sqrt{5})^{3/4}} = \frac{4\,5^{3/8}}{25}(16 + 3\sqrt{6}) \approx 6.831123656,$$

hence D_4 is strictly better than A_4^*.

Open problem 7.1 (The four-dimensional case)**.** Determine γ_4 and classify all minimizers. Is the regular 24-cell, that is the Voronoi cell of D_4, the unique minimizer up to similarity? A complete proof would have to control all four-dimensional combinatorial types together with their metric cones, or replace the stratum-by-stratum analysis by a genuinely higher-dimensional invariant inequality.

Open problem 7.2 (Lattices of Voronoi's first kind)**.** For general n, restrict to lattices admitting an obtuse superbase. Their Voronoi-relevant vectors are indexed by the $2^n - 1$ nontrivial cuts of K_{n+1}, so a higher-rank analogue of the determinant and entropy framework is available. Determine the minimizer of Iso_n in this subclass. In particular, is A_n^* optimal?

Open problem 7.3 (Quantitative stability)**.** Find a distance on the scale-free moduli of three-dimensional parallelohedra and the sharp constant in a global estimate of the form

$$\mathrm{Iso}(P) - \mathrm{Iso}(\mathrm{Vor}(\mathrm{BCC})) \geq c \ \mathrm{dist}(P, \mathrm{BCC})^2.$$

Open problem 7.4 (Beyond parallelohedra)**.** Theorem 1.1 settles the isoperimetric problem inside the class of three-dimensional parallelohedra. Determine the corresponding infimum among all equal-volume translative fundamental domains, or among periodic equal-cell partitions (compare the periodic existence theory in [3, 4]).

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