A PLANAR SET THAT CAN BE CONNECTED BY ARBITRARILY SHORT SETS BUT BY NO ZERO-LENGTH SET

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Abstract. We provide an example of a planar bounded open set A with countably many connected components that can be connected by adding a set of arbitrarily small positive length (i.e. one-dimensional Hausdorff measure), but cannot be connected by adding any set of zero length. In particular this shows that the existence of a solution to the Steiner problem of finding a set of minimum length connecting a given set cannot be guaranteed when the latter is not compact.

1. The problem and the claim

For an $A \subset \mathbb{R}^2$, define

$$m(A) := \inf\big\{\mathcal{H}^1(S) : S \subset \mathbb{R}^2,\ S \cup A \text{ is connected}\big\}.$$

The question, originally stated by Emanuele Paolini in survey [1], asks whether there exists a bounded, non-closed set A for which this infimum is not attained.

Theorem 1. *There exists a bounded open set $A \subset (0,1) \times (1/2,1)$, with countably many connected components, such that*

$$m(A) = 0,$$

but there is no set $S \subset \mathbb{R}^2$ with $\mathcal{H}^1(S) = 0$ such that $A \cup S$ is connected. In other words, the minimization problem defining $m(A)$ has no solution.

2. Construction

Fix an irrational number α and put

$$\xi_k := \{k\alpha\} \in (0,1), \qquad k \geq 1,$$

where $\{\cdot\}$ denotes fractional part. By Weyl's equidistribution theorem, the sequence (ξ_k) is uniformly distributed in $[0,1]$.In particular, the set

$$D := \{\xi_k : k \geq 1\}$$

is countable and dense in $[0,1]$.

For $t \in (1/2,1)$ and $k \geq 1$, define

$$a_k(t) := (1-t)\left(\xi_k + \sum_{\xi_j < \xi_k} t^j\right), \tag{1}$$

$$b_k(t) := a_k(t) + (1-t)t^k = (1-t)\left(\xi_k + \sum_{\xi_j \leq \xi_k} t^j\right), \tag{2}$$

$$I_k(t) := (a_k(t), b_k(t)). \tag{3}$$

Finally set

$$A_k := \big\{(x,t) : 1/2 < t < 1,\ x \in I_k(t)\big\}, \qquad A := \bigcup_{k=1}^{\infty} A_k.$$

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Lemma 2. *For every $t \in (1/2, 1)$ the intervals $I_k(t)$ are pairwise disjoint subsets of $(0,1)$, and*

$$|I_k(t)| = (1-t)t^k, \qquad \sum_{k=1}^{\infty} |I_k(t)| = t.$$

Moreover, each A_k is open and path connected, and the A_k are precisely the connected components of A.

Proof. If $\xi_k < \xi_\ell$, then

$$a_\ell(t) - b_k(t) = (1-t)\left(\xi_\ell - \xi_k + \sum_{\xi_k < \xi_j < \xi_\ell} t^j\right) > 0.$$

Hence the intervals are disjoint and ordered according to the positions of the ξ_k. Also

$$|I_k(t)| = b_k(t) - a_k(t) = (1-t)t^k,$$

so

$$\sum_{k \geq 1} |I_k(t)| = (1-t)\sum_{k \geq 1} t^k = t.$$

The bounds $0 < a_k(t) < b_k(t) < 1$ follow directly from (1)–(2) and

$$(1-t)\left(1 + \sum_{j \geq 1} t^j\right) = 1.$$

For fixed k, the functions a_k, b_k are continuous on $(1/2, 1)$ and $a_k < b_k$. Thus the region between their graphs is path connected. Since the A_k are pairwise disjoint open subsets and $A \setminus A_k$ is open in A, each A_k is a connected component. $\square$

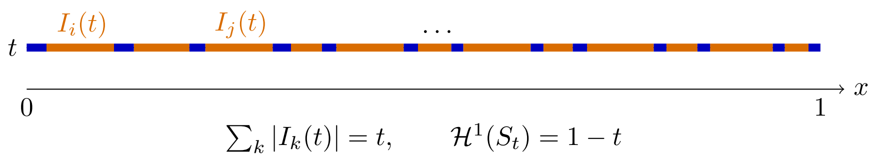

FIGURE 1. A schematic horizontal section. The orange intervals are the sections of the components A_k; the blue complement is the short connector S_t. The picture is not to scale and shows only finitely many intervals.

3. Arbitrarily short connectors

For $t \in (1/2, 1)$ let

$$K_t := [0,1] \setminus \bigcup_{k \geq 1} I_k(t), \qquad S_t := K_t \times \{t\}.$$

Because the $I_k(t)$ are disjoint open intervals, K_t is compact and

$$\mathcal{H}^1(S_t) = \mathcal{L}^1(K_t) = 1 - \sum_{k \geq 1} |I_k(t)| = 1 - t.$$

Furthermore,

$$\big(A \cap ([0,1] \times \{t\})\big) \cup S_t = [0,1] \times \{t\}.$$

Thus $A \cup S_t$ contains the entire horizontal segment $[0,1] \times \{t\}$. Every component A_k meets this segment, hence $A \cup S_t$ is connected. Therefore

$$m(A) \leq 1 - t \qquad \text{for every } t < 1,$$

and consequently

$$m(A) = 0. \tag{4}$$

The remaining point is to prove that the value 0 cannot be attained.

4. Separating curves and non-attainment

It remains to show that no connector of zero length exists. The argument is geometric.

For $c \in (0,1) \setminus D$ define, for $t < 1$,

$$g_c(t) := (1-t)\left(c + \sum_{\xi_j < c} t^j\right). \tag{5}$$

We claim that

$$(1-t)\sum_{\xi_j<c} t^j \longrightarrow c \qquad (t \uparrow 1). \tag{6}$$

Indeed, let

$$u_j := \mathbf{1}_{\{\xi_j < c\}}, \qquad A_N(c) := \sum_{j=1}^{N} u_j = \#\{1 \le j \le N : \xi_j < c\}.$$

By uniform distribution,

$$\frac{A_N(c)}{N} \longrightarrow c,$$

that is,

$$A_N(c) = cN + o(N).$$

Since

$$u_N = A_N(c) - A_{N-1}(c),$$

summation by parts gives

$$\begin{aligned}\sum_{\xi_j<c} t^j &= \sum_{N\ge1} u_N t^N \\ &= \sum_{N\ge1} \big(A_N(c) - A_{N-1}(c)\big)t^N \\ &= (1-t)\sum_{N\ge1} A_N(c)t^N.\end{aligned}$$

Therefore

$$\begin{aligned}(1-t)\sum_{\xi_j<c} t^j &= (1-t)^2\sum_{N\ge1} A_N(c)t^N \\ &= c(1-t)^2\sum_{N\ge1} Nt^N + (1-t)^2\sum_{N\ge1} o(N)t^N.\end{aligned}$$

Now if $t \to 1$ then

$$(1-t)^2\sum_{N\ge1} Nt^N = t \longrightarrow 1,$$

while

$$(1-t)^2\sum_{N\ge1} o(N)t^N \longrightarrow 0.$$

Hence (6) follows.

Finally,

$$g_c(t) = (1-t)c + (1-t)\sum_{\xi_j<c} t^j \longrightarrow 0 + c = c.$$

For $t < 1$ we also have

$$0 < g_c(t) \le (1-t)c + t < 1.$$

We therefore put $g_c(1) := c$ and define

$$\Gamma_c := \{(g_c(t), t) : 1/2 \le t \le 1\} \subset Q, \qquad Q := [0,1]\times[1/2,1].$$

Lemma 3. *For every $c \in (0,1)\setminus D$, the curve Γ_c is disjoint from A. Every component A_k with $\xi_k < c$ lies strictly to the left of Γ_c, while every component with $\xi_k > c$ lies strictly to its right.*

Proof. If $\xi_k < c$, then

$$g_c(t) - b_k(t) = (1-t)\left(c - \xi_k + \sum_{\xi_k < \xi_j < c} t^j\right) > 0.$$

If $\xi_k > c$, then

$$a_k(t) - g_c(t) = (1-t)\left(\xi_k - c + \sum_{c < \xi_j < \xi_k} t^j\right) > 0.$$

This proves the claim. □

Since D is dense, for every $c \in (0,1) \setminus D$ there are components of A on both sides of Γ_c. The graph Γ_c joins the bottom and top sides of Q, and

$$Q \setminus \Gamma_c = \{(x,t) : x < g_c(t)\} \dot\cup \{(x,t) : x > g_c(t)\}.$$

Thus any connected subset of Q meeting connected components of A on both sides must meet Γ_c.

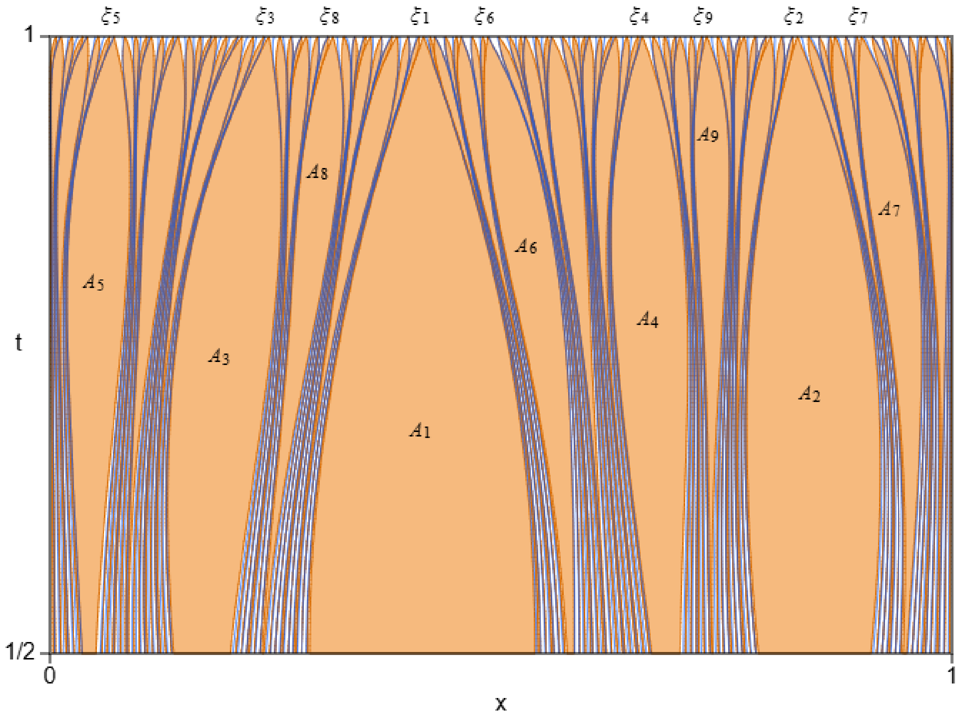

FIGURE 2. The first hundred components $A_1, \ldots, A_{100}$ (orange sets) together with a sample of the separating curves Γ_c (blue curves).

We next show that a set of zero length cannot meet Γ_c for almost every parameter c.

Lemma 4. *Fix $\rho \in (1/2, 1)$ and put*

$$Q_\rho := [0,1] \times [1/2, \rho], \qquad \mathcal{G}_\rho := \bigcup_{c \in (0,1)\setminus D} (\Gamma_c \cap Q_\rho).$$

The curves Γ_c are pairwise disjoint, so the map

$$p_\rho : \mathcal{G}_\rho \to (0,1), \qquad p_\rho(z) = c \quad \text{if } z \in \Gamma_c,$$

is well defined. Moreover p_ρ is Lipschitz.

Proof. For $c < d$ and $t \leq \rho$,

$$g_d(t) - g_c(t) = (1-t)\left(d - c + \sum_{c < \xi_j < d} t^j\right) \geq (1-\rho)(d-c). \tag{7}$$

Also, termwise differentiation in (5) is valid on $t \le \rho$, and uniformly in c,

$$|g_c'(t)| \le 1 + \sum_{j\ge1} t^j + (1-t)\sum_{j\ge1} jt^{j-1} \le \frac{2}{1-\rho}.$$

Take $z = (g_c(t), t) \in \Gamma_c$ and $w = (g_d(s), s) \in \Gamma_d$. Using (7) at height t and the preceding derivative bound,

$$\begin{aligned}(1-\rho)|d-c| &\le |g_d(t) - g_c(t)| \\ &\le |g_d(t) - g_d(s)| + |g_d(s) - g_c(t)| \\ &\le \frac{2}{1-\rho}|t-s| + |z-w| \\ &\le \left(\frac{2}{1-\rho} + 1\right)|z-w|.\end{aligned}$$

Hence

$$|p_\rho(z) - p_\rho(w)| = |d-c| \le \left(\frac{2}{(1-\rho)^2} + \frac{1}{1-\rho}\right)|z-w|.$$

□

Proof that the infimum is not attained. Suppose that $S \subset \mathbb{R}^2$ satisfies

$$\mathcal{H}^1(S) = 0, \qquad A \cup S \text{ connected}.$$

Let $P : \mathbb{R}^2 \to Q$ be the nearest-point projection onto the convex rectangle Q. It is 1-Lipschitz and fixes A, hence

$$A \cup P(S) = P(A \cup S)$$

is connected and $\mathcal{H}^1(P(S)) = 0$. Replacing S by $P(S)$, we may therefore assume $S \subset Q$.

By Lemma 3, connectedness forces

$$S \cap \Gamma_c \neq \varnothing \qquad \text{for every } c \in (0,1) \setminus D. \tag{8}$$

On the other hand choose $\rho_m \uparrow 1$, for example $\rho_m = 1 - 1/m$, $m \ge 3$. For each m, Lemma 4 and the Lipschitz invariance of zero $\mathcal{H}^1$-measure give

$$\mathcal{H}^1\big(p_{\rho_m}(S \cap \mathcal{G}_{\rho_m})\big) = 0.$$

Thus the set of parameters c for which S meets Γ_c at some point with $t < 1$ has one-dimensional measure zero, being a countable union of such images. On the top edge, Γ_c meets $\{t = 1\}$ only at $(c, 1)$, so

$$\{c : (c,1) \in S\}$$

also has one-dimensional measure zero because $\mathcal{H}^1(S) = 0$. Therefore

$$\mathcal{H}^1\big(\{c \in (0,1) \setminus D : S \cap \Gamma_c \neq \varnothing\}\big) = 0.$$

This contradicts (8), since D is countable and hence $(0,1) \setminus D$ has measure 1.

Thus no zero-length connector exists. Together with (4), this proves Theorem 1. □

Remark 5. *For each fixed t, the family of intervals $I_k(t)$ can be viewed as an order-preserving blow-up of the dense set $D = \{\xi_k : k \ge 1\}$: each point ξ_k is replaced by an interval of length $(1-t)t^k$, while the order of the points is preserved. This is analogous to the blow-up step in the classical construction of Denjoy counterexamples; see* [5, 6]. *No dynamical system is involved here.*

5. Additional: Why finitely many components cannot work

The failure of existence cannot occur for a bounded set with only finitely many connected components. This explains why a genuinely infinite-component construction is needed.

Proposition 6. *Let $A \subset \mathbb{R}^d$ be bounded and assume that A has only finitely many connected components. Then the infimum*

$$m(A) := \inf\big\{\mathcal{H}^1(S) : S \subset \mathbb{R}^d,\ S \cup A \text{ is connected}\big\}$$

is attained.

Proof. Write the connected components of A as $A_1, \dots, A_N$ and set

$$C := \overline{A} = \bigcup_{i=1}^{N} \overline{A_i}.$$

Then C is compact and has only finitely many connected components. If $S \cup A$ is connected, then

$$S \cup A \subset S \cup C \subset \overline{S \cup A},$$

and every set lying between a connected set and its closure is connected. Hence the same S is admissible for the compact datum C, and therefore

$$m(C) \leq m(A).$$

By the existence theorem [3], there is a minimizer S_0 for C. Replacing S_0 by $S_0 \setminus C$ does not change the connected set $C \cup S_0$ and cannot increase its length, so we may assume $S_0 \cap C = \varnothing$. Since C has finitely many connected components, the structural part of the same theorem implies that S_0 has only finitely many connected components and that the closure of each is a finite geodesic graph with finitely many endpoints on C. Let $P \subset C$ be the finite set of all these attachment points.

It remains only to replace each connected component of C by the corresponding pieces of A without losing connectivity. A connected component of C is a union of finitely many compact connected sets $\overline{A_i}$. Their intersection graph is connected; choose a spanning tree in this finite graph and, for every edge of the spanning tree, one point of the corresponding nonempty intersection. Let $F \subset C$ be the finite set of all points chosen in this way. Adding also the finite attachment set P, the set

$$A \cup F \cup P$$

is connected inside each connected component of C and contains every point where S_0 attaches to C.

Consequently

$$S_* := S_0 \cup F \cup P$$

is admissible for A, while finite sets have zero $\mathcal{H}^1$-measure. Thus

$$\mathcal{H}^1(S_*) = \mathcal{H}^1(S_0) = m(C).$$

Hence $m(A) \leq m(C)$. Together with the reverse inequality above, $m(A) = m(C)$ and S_* is a minimizer for A. $\square$

Thus any bounded counterexample in $\mathbb{R}^d$ must have infinitely many connected components. Our construction has countably many.

The statement of Proposition 6 admits a natural generalization. For a subset A of a metric space X, put

$$m_X(A) := \inf\{\mathcal{H}^1(S) : S \subset X,\ S \cup A \text{ is connected}\}.$$

Proposition 7. *Let X be a proper connected metric space, and let $A \subset X$ be precompact. If A has only finitely many connected components, then the infimum $m_X(A)$ is attained.*

The proof is the same as in the Euclidean case. Since X is proper and A is precompact, $\overline{A}$ is compact. The existence and structural results for the Steiner problem in proper metric spaces (see [3]) then apply to $\overline{A}$, and the finite-component argument above carries over verbatim.

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