UNIQUENESS OF ENTROPY SOLUTIONS TO THE BURGERS–HILBERT EQUATION

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Abstract. We prove uniqueness of L^2-continuous entropy solutions of the Burgers–Hilbert equation on the real line and on the torus. The proofs combine a Kato inequality with logarithmic endpoint estimates for the Hilbert transform and Osgood's lemma.

1. Introduction and main result

We study the Cauchy problem for the *Burgers–Hilbert equation*

$$\partial_t u + \partial_x \left(\frac{u^2}{2}\right) = \mathbf{H}u, \qquad (t,x) \in (0,T) \times \mathbb{R}, \tag{1.1}$$

where $T > 0$ and $\mathbf{H}$ denotes the *Hilbert transform*. In [BH10], Biello and Hunter studied this equation and showed that its weakly nonlinear amplitude equation also describes waves on a planar vorticity discontinuity in a two-dimensional inviscid fluid.

For a Schwartz function g, the Hilbert transform is given by

$$\mathbf{H}g(x) \coloneqq \frac{1}{\pi}\ \mathrm{p.v.} \int_{\mathbb{R}} \frac{g(y)}{x-y}\,\mathrm{d}y. \tag{1.2}$$

With the Fourier-transform convention

$$\widehat{g}(\xi) \coloneqq \int_{\mathbb{R}} e^{-ix\xi} g(x)\,\mathrm{d}x,$$

we have

$$\widehat{\mathbf{H}g}(\xi) = -i\ \mathrm{sign}(\xi)\widehat{g}(\xi), \tag{1.3}$$

and this multiplier defines the unique L^2-extension of $\mathbf{H}$. In particular, we have

$$\|\mathbf{H}g\|_{L^2(\mathbb{R})} = \|g\|_{L^2(\mathbb{R})}, \qquad \langle \mathbf{H}g, g\rangle_{L^2(\mathbb{R})} = 0. \tag{1.4}$$

The natural stability theory for scalar conservation laws is formulated in L^1 (see, e.g., [Daf26; Bre00; Ser99; GR21; HR15; Coc24]). This framework is not directly compatible with the right-hand side of (1.1). Indeed, although $\mathbf{H}$ is bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$, it is not bounded from $L^1(\mathbb{R})$ to itself (see [Ste70, Chapter II, §§ 2 and 4]). By contrast, (1.4) shows that the nonlocal term is particularly well behaved in L^2, whereas the Burgers entropy semigroup is not contractive in L^2 (nevertheless, for L^2-stability theories based on *relative entropy and shifts*, see [Vas23] and references therein). This mismatch is the main source of difficulty in the analysis of (1.1).

We work within the class of entropy-admissible solutions introduced in [BN14, Definition 1.1].

Definition 1.1 (Entropy solution). Let $\bar{u} \in L^2(\mathbb{R})$. An *entropy solution* of (1.1) on $[0,T]$ with initial datum $\bar{u}$ is a function $u \in L^1_{\mathrm{loc}}([0,T]\times\mathbb{R})$ such that

$$u \in C([0,T]; L^2(\mathbb{R})), \qquad u(0,\cdot) = \bar{u},$$

and, for every $k \in \mathbb{R}$ and every non-negative $\phi \in C^1_c((0,T)\times\mathbb{R})$,

$$\iint_{(0,T)\times\mathbb{R}} \left\{ |u-k|\partial_t\phi + \mathrm{sign}(u-k)\frac{u^2-k^2}{2}\partial_x\phi + \mathbf{H}u\ \mathrm{sign}(u-k)\phi \right\} \mathrm{d}x\,\mathrm{d}t \geq 0. \tag{1.5}$$

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Bressan and Nguyen proved in [BN14, Theorem 1.2] that every initial datum $\bar{u} \in L^2(\mathbb{R})$ admits at least one global entropy solution in the sense of Definition 1.1; moreover, this solution has non-increasing L^2 norm. If u is any entropy solution in the sense of Definition 1.1, then the map $t \mapsto \mathbf{H}u(t,\cdot)$ is continuous from $[0,T]$ into $L^2(\mathbb{R})$ by (1.4), and $\sup_{0\le t\le T}\|\mathbf{H}u(t,\cdot)\|_{L^2} = \sup_{0\le t\le T}\|u(t,\cdot)\|_{L^2} < \infty$. Applying [BN14, Lemma 2.2 and Remark 2.3] then yields

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R})} \le C_{T,u}\big(1+t^{-1/3}\big), \qquad 0<t\le T, \tag{1.6}$$

where $C_{T,u}$ depends on T and $\sup_{0\le t\le T}\|u(t,\cdot)\|_{L^2(\mathbb{R})}$. Consequently,

$$t \longmapsto \|u(t,\cdot)\|_{L^\infty(\mathbb{R})} \quad \text{belongs to } L^1(0,T). \tag{1.7}$$

In [BN14, Theorem 4.3], Bressan and Nguyen established uniqueness in the periodic setting under an additional uniform-in-time bounded-variation hypothesis. In [BZ17, Theorem 2.1], Bressan and Zhang subsequently constructed local-in-time, piecewise-smooth solutions with a single shock for a class of well-prepared initial data and proved uniqueness within that class. In [KV20, Theorem 1.2], Krupa and Vasseur proved stability and uniqueness of these single-shock solutions within a larger class of weak solutions in $L^2((0,T)\times\mathbb{R})\cap L^\infty((0,T)\times\mathbb{R})$ having strong traces and satisfying the (quadratic) entropy inequality. More recently, Ftaka and Nguyen [FN26, Theorem 3.3] proved uniqueness for spatially periodic entropy solutions to a broader class of singular nonlocal balance laws, provided that the total variation of each solution over one period is uniformly bounded on $[0,T]$.

To the best of our knowledge, however, the general uniqueness problem in the class of Definition 1.1 remained open. Our main result, Theorem 1.2, establishes uniqueness on the real line in the L^2-continuous entropy class of Definition 1.1.

Theorem 1.2 (Uniqueness on the real line)**.** *Let u and v be entropy solutions of (1.1) in the sense of Definition 1.1, with the same initial datum $\bar{u} \in L^2(\mathbb{R})$. Then $u = v$ in $C([0,T];L^2(\mathbb{R}))$.*

We also give a direct proof of the periodic analogue of Theorem 1.2 (see Theorem 2.1 below).

1.1. **Outline of the proofs.** The common starting point of the proofs of Theorem 1.2 and Theorem 2.1 is Kružkov's *doubling-of-variables argument* (see [Kru70] and, for our setting, the formulation of [CG11, Theorem 3.2, Eq. (15), and Remark 1.1]), which yields the following *Kato inequality*:

$$\partial_t|u-v| + \partial_x\left(\frac{u+v}{2}|u-v|\right) \le \operatorname{sign}(u-v)\mathbf{H}(u-v) \le |\mathbf{H}(u-v)| \qquad \text{in } \mathcal{D}'((0,T)\times\mathbb{R}). \tag{1.8}$$

The two domains, the torus and the real line, require different ways of turning this inequality into a comparison principle.

We begin with the periodic setting in Section 2. On $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$, let $\mathbf{H}_{\mathbb{T}}$ denote the periodic Hilbert transform. Since $L^2(\mathbb{T}) \subset L^1(\mathbb{T})$, the L^1-distance is finite, and a spatially constant test is admissible in the periodic analogue of (1.8), so the flux term vanishes. The key ingredient is the following logarithmic estimate: for a constant $C_0 > 0$ and every non-zero $g \in L^2(\mathbb{T})$,

$$\|\mathbf{H}_{\mathbb{T}} g\|_{L^1(\mathbb{T})} \le C_0\|g\|_{L^1(\mathbb{T})}\left[1+\log\left(\frac{\|g\|_{L^2(\mathbb{T})}}{\|g\|_{L^1(\mathbb{T})}}\right)\right],$$

which is proved in Lemma 2.2 by combining Kolmogorov's weak-type $(1,1)$ inequality for the periodic Hilbert transform (see [Kol25, Théorème I]) with the L^2-bound. Applied to $g := u-v$, this estimate and the spatially-integrated Kato inequality give an inequality for $Z(t) := \|u(t,\cdot)-v(t,\cdot)\|_{L^1(\mathbb{T})}$ with a modulus of the form $z\big(1+\log(M/z)\big)$, where M is a uniform L^2 bound for $u-v$ on $[0,T]$, which allows us to apply Osgood's lemma and deduce $Z \equiv 0$.

We next turn to the real line in Section 3. Here $L^2(\mathbb{R})$ is not contained in $L^1(\mathbb{R})$, so the L^1-distance need not be finite in the class of Definition 1.1; moreover, $\mathbf{H}$ does not map $L^1(\mathbb{R})$ into itself. We therefore fix $\frac{1}{2} < \alpha < 1$ and let

$$\omega(x) := (1+|x|)^{-\alpha}, \qquad Z_\alpha(g) = \int_{\mathbb{R}} \omega(x)|g(x)|\,\mathrm{d}x.$$

The weight has two roles. Since $\omega \in L^2(\mathbb{R})$, the quantity $Z_\alpha(g)$ is finite for every $g \in L^2(\mathbb{R})$. Moreover,

$$|\omega'(x)| \le \alpha\omega(x) \qquad \text{for almost every } x \in \mathbb{R},$$

so testing (1.8) against compactly supported approximations of ω controls the Burgers flux by

$$\frac{\alpha}{2}\left(\|u(t,\cdot)\|_{L^\infty(\mathbb{R})}+\|v(t,\cdot)\|_{L^\infty(\mathbb{R})}\right)Z_\alpha(u(t,\cdot)-v(t,\cdot)).$$

To control the nonlocal source, we use the weighted logarithmic estimate proved in Lemma 3.1 (using [LOP09, Theorem 1.1]): for a constant $C_\alpha>0$ and every non-zero $g\in L^2(\mathbb{T})$,

$$\int_{\mathbb{R}}\omega(x)|\mathbf{H}g(x)|\,\mathrm{d}x\le C_\alpha Z_\alpha(g)\left[1+\log\left(1+\frac{C_\alpha\|g\|_{L^2(\mathbb{R})}}{Z_\alpha(g)}\right)\right].$$

Applied to $g:=u-v$, this estimate gives the same logarithmic Osgood modulus for the weighted distance $Z_\alpha(u-v)$.

In summary, as in [BN14], both proofs reduce uniqueness to Osgood's lemma: the periodic argument uses Lemma 2.2 (in place of the BV estimate previously employed in [BN14, Lemma 4.2]), while the real-line argument uses its weighted counterpart, Lemma 3.1.

2. Proof on the torus

In this section, we prove uniqueness on the torus. We work on

$$\mathbb{T}:=\mathbb{R}/(2\pi\mathbb{Z}),\qquad \mathrm{d}\mu(x):=\frac{\mathrm{d}x}{2\pi},$$

so that $\mu(\mathbb{T})=1$. Throughout this section, all $L^p(\mathbb{T})$ norms are taken with respect to μ. For $g\in C^\infty(\mathbb{T})$, the periodic Hilbert transform is defined by

$$\mathbf{H}_{\mathbb{T}}g(x):=\mathrm{p.\,v.}\int_{\mathbb{T}}g(y)\cot\left(\frac{x-y}{2}\right)\mathrm{d}\mu(y).\tag{2.1}$$

With the Fourier convention

$$\widehat{g}(k):=\int_{\mathbb{T}}g(x)e^{-ikx}\,\mathrm{d}\mu(x),\qquad k\in\mathbb{Z},$$

(2.1) is equivalently given by

$$\widehat{\mathbf{H}_{\mathbb{T}}g}(k)=-i\,\mathrm{sign}(k)\widehat{g}(k).\tag{2.2}$$

The multiplier formulation defines $\mathbf{H}_{\mathbb{T}}$ on $L^2(\mathbb{T})$ and yields

$$\|\mathbf{H}_{\mathbb{T}}g\|_{L^2(\mathbb{T})}^2=\|g\|_{L^2(\mathbb{T})}^2-|\widehat{g}(0)|^2\le\|g\|_{L^2(\mathbb{T})}^2.\tag{2.3}$$

We consider the periodic Burgers–Hilbert equation

$$\partial_t u+\partial_x\left(\frac{u^2}{2}\right)=\mathbf{H}_{\mathbb{T}}u\qquad (t,x)\in(0,T)\times\mathbb{T}.\tag{2.4}$$

By a *periodic entropy solution* with initial datum $\bar u$ we mean the periodic analogue of Definition 1.1 (as in [BN14, Definition 4.1]): namely, $u\in C([0,T];L^2(\mathbb{T}))$, $u(0,\cdot)=\bar u$, and (1.5), with $\mathbb{R}$ and $\mathbf{H}$ replaced by $\mathbb{T}$ and $\mathbf{H}_{\mathbb{T}}$, and with $\mathrm{d}x$ replaced by $\mathrm{d}\mu$, holds for every $k\in\mathbb{R}$ and every non-negative $\phi\in C_c^1((0,T)\times\mathbb{T})$.

Theorem 2.1 (Uniqueness on the torus). *Let u and v be periodic entropy solutions of (2.4) with the same initial datum $\bar u\in L^2(\mathbb{T})$. Then*

$$u=v\qquad \textit{in } C([0,T];L^2(\mathbb{T})).$$

Because $\mu(\mathbb{T})=1$, every $L^2(\mathbb{T})$ function belongs to $L^1(\mathbb{T})$, so the L^1-distance between two solutions is finite. The difficulty is that $\mathbf{H}_{\mathbb{T}}$ is not bounded on $L^1(\mathbb{T})$. The following logarithmic estimate replaces the missing L^1-bound.

Lemma 2.2 (Periodic endpoint estimate). *There is a universal constant $C_0>0$ such that every non-zero $g\in L^2(\mathbb{T})$ satisfies*

$$\|\mathbf{H}_{\mathbb{T}}g\|_{L^1(\mathbb{T})}\le C_0\|g\|_{L^1(\mathbb{T})}\left[1+\log\left(\frac{\|g\|_{L^2(\mathbb{T})}}{\|g\|_{L^1(\mathbb{T})}}\right)\right].\tag{2.5}$$

Proof. Since $\mu(\mathbb{T}) = 1$, we have $\|g\|_{L^1(\mathbb{T})} \le \|g\|_{L^2(\mathbb{T})}$, so the logarithm in (2.5) is non-negative. Kolmogorov's weak-type $(1,1)$ inequality (see [Kol25, Théorème I]) gives, for every $\lambda > 0$,

$$\mu\big(\{x \in \mathbb{T} : |\mathbf{H}_{\mathbb{T}} g(x)| > \lambda\}\big) \le \frac{C\|g\|_{L^1(\mathbb{T})}}{\lambda},$$

while Chebyshev's inequality and (2.3) give

$$\mu\big(\{x \in \mathbb{T} : |\mathbf{H}_{\mathbb{T}} g(x)| > \lambda\}\big) \le \frac{\|g\|^2_{L^2(\mathbb{T})}}{\lambda^2}.$$

Combining these bounds via the *layer-cake formula* (see, e.g., [LL01, Theorem 1.13]), with the integration split at $\|g\|_{L^1(\mathbb{T})}$ and $\|g\|^2_{L^2(\mathbb{T})}/\|g\|_{L^1(\mathbb{T})}$, we obtain

$$\begin{aligned}
\|\mathbf{H}_{\mathbb{T}} g\|_{L^1(\mathbb{T})} &= \int_0^\infty \mu\big(\{x \in \mathbb{T} : |\mathbf{H}_{\mathbb{T}} g(x)| > \lambda\}\big)\, \mathrm{d}\lambda \\
&\le \|g\|_{L^1(\mathbb{T})} + C\|g\|_{L^1(\mathbb{T})} \int_{\|g\|_{L^1(\mathbb{T})}}^{\|g\|^2_{L^2(\mathbb{T})}/\|g\|_{L^1(\mathbb{T})}} \frac{\mathrm{d}\lambda}{\lambda} + \|g\|^2_{L^2(\mathbb{T})} \int_{\|g\|^2_{L^2(\mathbb{T})}/\|g\|_{L^1(\mathbb{T})}}^\infty \frac{\mathrm{d}\lambda}{\lambda^2} \\
&= 2\|g\|_{L^1(\mathbb{T})} \left[1 + C \log\left(\frac{\|g\|_{L^2(\mathbb{T})}}{\|g\|_{L^1(\mathbb{T})}}\right)\right] \le C_0 \|g\|_{L^1(\mathbb{T})} \left[1 + \log\left(\frac{\|g\|_{L^2(\mathbb{T})}}{\|g\|_{L^1(\mathbb{T})}}\right)\right].
\end{aligned}$$

□

Remark 2.3. Alternatively, Lemma 2.2 follows from [Pic72, Theorem 3.7], which gives

$$\|\mathbf{H}_{\mathbb{T}}\|_{L^p(\mathbb{T};\mathbb{R}) \to L^p(\mathbb{T};\mathbb{R})} = \tan\left(\frac{\pi}{2p}\right) \le \frac{3}{\pi(p-1)}, \qquad (1 < p \le 3/2). \tag{2.6}$$

Choosing

$$p = 1 + \frac{1}{2 + \log\left(\frac{\|g\|_{L^2(\mathbb{T})}}{\|g\|_{L^1(\mathbb{T})}}\right)},$$

Hölder's inequality, (2.6), and interpolation yield

$$\begin{aligned}
\|\mathbf{H}_{\mathbb{T}} g\|_{L^1(\mathbb{T})} &\le \|\mathbf{H}_{\mathbb{T}} g\|_{L^p(\mathbb{T})} \le \frac{3}{\pi(p-1)} \|g\|_{L^p(\mathbb{T})} \le \frac{3}{\pi(p-1)} \|g\|_{L^1(\mathbb{T})} \left(\frac{\|g\|_{L^2(\mathbb{T})}}{\|g\|_{L^1(\mathbb{T})}}\right)^{2(p-1)/p} \\
&= \frac{3}{\pi} \|g\|_{L^1(\mathbb{T})} \left[2 + \log\left(\frac{\|g\|_{L^2(\mathbb{T})}}{\|g\|_{L^1(\mathbb{T})}}\right)\right] \exp\left(\frac{2\log\left(\frac{\|g\|_{L^2(\mathbb{T})}}{\|g\|_{L^1(\mathbb{T})}}\right)}{3 + \log\left(\frac{\|g\|_{L^2(\mathbb{T})}}{\|g\|_{L^1(\mathbb{T})}}\right)}\right) \\
&\le \frac{6e^2}{\pi} \|g\|_{L^1(\mathbb{T})} \left[1 + \log\left(\frac{\|g\|_{L^2(\mathbb{T})}}{\|g\|_{L^1(\mathbb{T})}}\right)\right].
\end{aligned} \tag{2.7}$$

Using Lemma 2.2, we can now prove Theorem 2.1.

Proof of Theorem 2.1. Step 1. *The comparison functional.* We define

$$Z(t) := \|u(t,\cdot) - v(t,\cdot)\|_{L^1(\mathbb{T})}.$$

Since $u - v \in C([0,T]; L^2(\mathbb{T}))$, (2.3) and $\mu(\mathbb{T}) = 1$ show that

$$t \longmapsto Z(t) \quad \text{and} \quad t \longmapsto \|\mathbf{H}_{\mathbb{T}}(u(t,\cdot) - v(t,\cdot))\|_{L^1(\mathbb{T})}$$

are continuous on $[0,T]$. If $Z(t) = 0$, then $u(t,\cdot) = v(t,\cdot)$ almost everywhere.

Step 2. *The Kato inequality.* Applying the same doubling-of-variables argument leading to (1.8) to the periodic equation gives

$$\partial_t |u - v| + \partial_x \left(\frac{u+v}{2} |u - v|\right) \le |\mathbf{H}_{\mathbb{T}}(u - v)| \qquad \text{in } \mathcal{D}'((0,T) \times \mathbb{T}).$$

Testing with $\phi(t,x) = \eta(t)$, where $\eta \in C_c^\infty(0,T)$ is non-negative, gives

$$Z'(t) \le \|\mathbf{H}_{\mathbb{T}}(u(t,\cdot) - v(t,\cdot))\|_{L^1(\mathbb{T})} \qquad \text{in } \mathcal{D}'(0,T).$$

Integrating in time yields

$$Z(t) \le Z(s) + \int_s^t \|\mathbf{H}_{\mathbb{T}}(u(\tau,\cdot) - v(\tau,\cdot))\|_{L^1(\mathbb{T})} \, \mathrm{d}\tau, \qquad 0 \le s \le t \le T. \tag{2.8}$$

<u>Step 3.</u> *The logarithmic Osgood modulus.* Let $M_T := \max_{0\le t\le T} \|u(t,\cdot)-v(t,\cdot)\|_{L^2(\mathbb{T})}$. If $M_T = 0$, the conclusion is immediate, so we assume that $M_T > 0$. Since

$$Z(t) \le \|u(t,\cdot) - v(t,\cdot)\|_{L^2(\mathbb{T})} \le M_T,$$

Lemma 2.2 gives, whenever $Z(t) > 0$,

$$\|\mathbf{H}_{\mathbb{T}}(u(t,\cdot) - v(t,\cdot))\|_{L^1(\mathbb{T})} \le C_0 Z(t) \left[1 + \log\left(\frac{M_T}{Z(t)}\right)\right]. \tag{2.9}$$

for every $t \in [0,T]$ (with the right-hand side defined by continuity at zero).

We define

$$\Phi_T(z) := z\left[1 + \log\left(\frac{M_T}{z}\right)\right], \qquad 0 < z \le M_T, \qquad \Phi_T(0) := 0. \tag{2.10}$$

Then Φ_T is continuous and non-decreasing on $[0, M_T]$, positive on $(0, M_T]$, and

$$\int_{0+}^{M_T} \frac{\mathrm{d}z}{\Phi_T(z)} = \int_1^\infty \frac{\mathrm{d}q}{q} = \infty, \qquad q = 1 + \log\left(\frac{M_T}{z}\right). \tag{2.11}$$

<u>Step 4.</u> *Conclusion by Osgood's lemma.* Since u and v have the same initial datum, $Z(0) = 0$. Combining (2.8) and (2.9) with $s = 0$ yields

$$Z(t) \le C_0 \int_0^t \Phi_T(Z(\tau)) \, \mathrm{d}\tau, \qquad 0 \le t \le T.$$

Osgood's lemma (see [BCD11, Lemma 3.4 and Corollary 3.5]), together with (2.11), implies that $Z(t) = 0$ for every $t \in [0,T]$. Therefore

$$\begin{aligned} u(t,\cdot) &= v(t,\cdot) && \text{almost everywhere on } \mathbb{T} \text{ for every } t \in [0,T], \\ u &= v && \text{in } C([0,T]; L^2(\mathbb{T})). \end{aligned}$$

□

3. Proof on the real line

In this section, we prove the uniqueness result on the real line. Although the Hilbert transform is not bounded on $L^1(\mathbb{R})$, we can recover the following weighted L^1-estimate from its quantitative weighted L^p-bounds. To formulate this estimate, let us fix

$$\frac{1}{2} < \alpha < 1, \qquad \omega(x) := (1 + |x|)^{-\alpha}. \tag{3.1}$$

Then $\omega \in L^2(\mathbb{R})$, $\omega > 0$, and

$$|\omega'(x)| \le \alpha\omega(x) \quad \text{for almost every } x \in \mathbb{R}. \tag{3.2}$$

Lemma 3.1 (Weighted logarithmic estimate)**.** *Let ω be given by (3.1). For every non-zero $g \in L^2(\mathbb{R})$,*

$$\int_{\mathbb{R}} \omega(x)|g(x)| \, \mathrm{d}x < \infty.$$

and the following estimate holds:

$$\int_{\mathbb{R}} \omega(x)|\mathbf{H}g(x)| \, \mathrm{d}x \le C_\alpha \left(\int_{\mathbb{R}} \omega(x)|g(x)| \, \mathrm{d}x\right) \left[1 + \log\left(1 + C_\alpha \mathcal{R}_\omega(g)\right)\right], \tag{3.3}$$

where

$$\mathcal{R}_\omega(g) := \frac{\|g\|_{L^2(\mathbb{R})}}{\displaystyle\int_{\mathbb{R}} \omega(x)|g(x)| \, \mathrm{d}x}.$$

and $C_\alpha > 0$ depends only on α.

Proof. By Cauchy–Schwarz, $\int_{\mathbb{R}} \omega|g|\,\mathrm{d}x \le \|\omega\|_{L^2(\mathbb{R})}\|g\|_{L^2(\mathbb{R})} < \infty$; in particular $\|\omega\|_{L^2(\mathbb{R})}\mathcal{R}_\omega(g) \ge 1$. If $\int_{\mathbb{R}} \omega|g|\,\mathrm{d}x = 0$, then $g = 0$ a.e. (as $\omega > 0$) and there is nothing to prove, so we assume it is positive; by density we may take $g \in C_c^\infty(\mathbb{R})$.

Step 1. *A quantitative weighted L^p estimate.* Fix $1 < p \le 3/2$. For a bounded interval $I \subset \mathbb{R}$ with $R := \sup_{x\in I}|x|$, we have $\operatorname{ess\,inf}_I (1+|x|)^{-\alpha(2-p)} = (1+R)^{-\alpha(2-p)}$, while

$$\frac{1}{|I|}\int_I (1+|x|)^{-\alpha(2-p)}\,\mathrm{d}x \le \begin{cases} 2^{\alpha(2-p)}(1+R)^{-\alpha(2-p)}, & |I| < (1+R)/2, \\ \dfrac{4(1+R)^{-\alpha(2-p)}}{1-\alpha(2-p)}, & |I| \ge (1+R)/2, \end{cases}$$

the second bound following from $\frac{1}{|I|}\int_I \le \frac{2}{|I|}\int_0^R$. Hence, in the sense of [Gra14, Definition 7.1.1],

$$\omega^{2-p} \in A_1, \qquad [\omega^{2-p}]_{A_1} \le \frac{4}{1-\alpha(2-p)} \le \frac{4}{1-\alpha}.$$

Since $pp' = p^2/(p-1) \le 9/[4(p-1)]$, the sharp A_1 bound [LOP09, Theorem 1.1], applied to $\mathbf{H}$ with weight ω^{2-p}, gives

$$\left(\int_{\mathbb{R}} |\mathbf{H}g|^p\omega^{2-p}\,\mathrm{d}x\right)^{1/p} \le \frac{C_\alpha}{p-1}\left(\int_{\mathbb{R}} |g|^p\omega^{2-p}\,\mathrm{d}x\right)^{1/p}, \tag{3.4}$$

with C_α depending only on α.

Step 2. *Interpolation and optimization in p.* We now choose

$$p = 1 + \left[2 + \log\left(\|\omega\|_{L^2(\mathbb{R})}\mathcal{R}_\omega(g)\right)\right]^{-1},$$

which satisfies $1 < p \le 3/2$ since $\|\omega\|_{L^2(\mathbb{R})}\mathcal{R}_\omega(g) \ge 1$, and for which

$$\begin{aligned} &\frac{1}{p-1}\left(\|\omega\|_{L^2(\mathbb{R})}\mathcal{R}_\omega(g)\right)^{2(p-1)/p} \\ &= \left[2 + \log\left(\|\omega\|_{L^2(\mathbb{R})}\mathcal{R}_\omega(g)\right)\right]\exp\left(\frac{2\log\left(\|\omega\|_{L^2(\mathbb{R})}\mathcal{R}_\omega(g)\right)}{3+\log\left(\|\omega\|_{L^2(\mathbb{R})}\mathcal{R}_\omega(g)\right)}\right) \\ &\le 2e^2\left[1 + \log\left(\|\omega\|_{L^2(\mathbb{R})}\mathcal{R}_\omega(g)\right)\right]. \end{aligned} \tag{3.5}$$

Writing $|g|^p\omega^{2-p} = (\omega|g|)^{2-p}(|g|^2)^{p-1}$, Hölder's inequality, (3.4), and (3.5) yield

$$\begin{aligned} \int_{\mathbb{R}} \omega|\mathbf{H}g|\,\mathrm{d}x &\le \left(\int_{\mathbb{R}} |\mathbf{H}g|^p\omega^{2-p}\,\mathrm{d}x\right)^{1/p}\left(\int_{\mathbb{R}} \omega^2\,\mathrm{d}x\right)^{1/p'} \\ &\le \frac{C_\alpha}{p-1}\left(\int_{\mathbb{R}} \omega|g|\,\mathrm{d}x\right)\left(\|\omega\|_{L^2(\mathbb{R})}\mathcal{R}_\omega(g)\right)^{2(p-1)/p} \\ &\le 2e^2C_\alpha\left(\int_{\mathbb{R}} \omega|g|\,\mathrm{d}x\right)\left[1+\log\left(\|\omega\|_{L^2(\mathbb{R})}\mathcal{R}_\omega(g)\right)\right] \\ &\le C_\alpha\left(\int_{\mathbb{R}} \omega|g|\,\mathrm{d}x\right)\left[1+\log\left(1+C_\alpha\mathcal{R}_\omega(g)\right)\right], \end{aligned}$$

where, after enlarging and relabeling C_α, the last step uses $\|\omega\|^2_{L^2(\mathbb{R})} = \frac{2}{2\alpha-1}$. □

Remark 3.2. The lower bound $\alpha > 1/2$ in (3.1) ensures that $\omega \in L^2(\mathbb{R})$, whereas the upper bound $\alpha < 1$ keeps the A_1 characteristics used in the proof of Lemma 3.1 uniformly bounded as $p \downarrow 1$.

We now combine the Kato inequality in (1.8) with the weighted Hilbert-transform estimate in Lemma 3.1.

Proof of Theorem 1.2. Step 1. *The weighted distance.* Let ω be given by (3.1), and define

$$Z(t) := Z_\alpha(u(t,\cdot) - v(t,\cdot)) := \int_{\mathbb{R}} \omega(x)|u(t,x) - v(t,x)|\,\mathrm{d}x. \tag{3.6}$$

Since $\omega \in L^2(\mathbb{R})$ and $u - v \in C([0,T]; L^2(\mathbb{R}))$, the function Z is continuous and

$$Z(0) = 0, \qquad |Z(t) - Z(s)| \le \|\omega\|_{L^2(\mathbb{R})}\|(u-v)(t) - (u-v)(s)\|_{L^2(\mathbb{R})}. \tag{3.7}$$

If $Z(t) = 0$, then $\omega > 0$ implies that $u(t,\cdot) = v(t,\cdot)$ almost everywhere. Let $M_T := \max_{0\le\tau\le T} \|u(\tau,\cdot) - v(\tau,\cdot)\|_{L^2(\mathbb{R})}$. If $M_T = 0$, then the conclusion is also immediate. We therefore assume that $M_T > 0$. By (1.7),

$$\int_0^T \left(\|u(\tau,\cdot)\|_{L^\infty(\mathbb{R})} + \|v(\tau,\cdot)\|_{L^\infty(\mathbb{R})}\right) \mathrm{d}\tau < \infty. \tag{3.8}$$

Step 2. *The Kato inequality.* By the Kato inequality (1.8),

$$\partial_t|u-v| + \partial_x\left(\frac{u+v}{2}|u-v|\right) \le |\mathbf{H}(u-v)| \quad \text{in } \mathcal{D}'((0,T)\times\mathbb{R}). \tag{3.9}$$

We choose an even function $\zeta \in C_c^\infty(\mathbb{R})$ such that

$$0 \le \zeta \le 1, \qquad \zeta = 1 \text{ on } [-1,1], \qquad \zeta = 0 \text{ on } \mathbb{R}\setminus[-2,2],$$

and, for $R \ge 1$, define

$$\omega_R(x) := \omega(x)\zeta(x/R).$$

Using (3.2), we obtain

$$|\partial_x\omega_R(x)| \le \alpha\omega(x)\zeta(x/R) + \frac{C}{R}\omega(x)\mathbb{1}_{\{R\le|x|\le 2R\}} \quad \text{for almost every } x\in\mathbb{R}. \tag{3.10}$$

Testing (3.9) with spatial mollifications of ω_R and smooth time cut-offs converging to $\mathbb{1}_{(s,t)}$, and using L^2-continuity, gives, for every $0 \le s < t \le T$,

$$\begin{aligned}
&\int_{\mathbb{R}} \omega_R(x)|u(t,x)-v(t,x)|\,\mathrm{d}x \\
&\quad \le \int_{\mathbb{R}} \omega_R(x)|u(s,x)-v(s,x)|\,\mathrm{d}x \\
&\quad\quad + \frac{\alpha}{2}\int_s^t \left(\|u(\tau,\cdot)\|_{L^\infty(\mathbb{R})} + \|v(\tau,\cdot)\|_{L^\infty(\mathbb{R})}\right) Z(\tau)\,\mathrm{d}\tau \\
&\quad\quad + \frac{C}{R}\int_s^t \left(\|u(\tau,\cdot)\|_{L^\infty(\mathbb{R})} + \|v(\tau,\cdot)\|_{L^\infty(\mathbb{R})}\right) Z(\tau)\,\mathrm{d}\tau \\
&\quad\quad + \int_s^t\int_{\mathbb{R}} \omega_R(x)|\mathbf{H}(u(\tau,\cdot)-v(\tau,\cdot))(x)|\,\mathrm{d}x\,\mathrm{d}\tau.
\end{aligned} \tag{3.11}$$

Step 3. *Removal of the cut-off.* After increasing C_α if necessary, assume that the estimate in Lemma 3.1 holds with $C_\alpha \ge \alpha/2$. Define the function

$$\Phi(z) := z\left[1 + \log\left(1 + \frac{C_\alpha M_T}{z}\right)\right], \qquad z > 0, \qquad \Phi(0) := 0, \tag{3.12}$$

which is continuous on $[0,\infty)$. When $Z(\tau) > 0$, Lemma 3.1 applied to $u(\tau,\cdot) - v(\tau,\cdot)$ gives

$$\begin{aligned}
&\int_{\mathbb{R}} \omega(x)|\mathbf{H}(u(\tau,\cdot)-v(\tau,\cdot))(x)|\,\mathrm{d}x \\
&\quad \le C_\alpha Z(\tau)\left[1 + \log\left(1 + \frac{C_\alpha\|u(\tau,\cdot)-v(\tau,\cdot)\|_{L^2(\mathbb{R})}}{Z(\tau)}\right)\right] \\
&\quad \le C_\alpha \Phi(Z(\tau)).
\end{aligned} \tag{3.13}$$

Integrating in time and letting $R\uparrow\infty$ in (3.11) and using (3.11) gives

$$Z(t) \le Z(s) + \int_s^t \left[\frac{\alpha}{2}\left(\|u(\tau,\cdot)\|_{L^\infty(\mathbb{R})} + \|v(\tau,\cdot)\|_{L^\infty(\mathbb{R})}\right) Z(\tau) + C_\alpha\Phi(Z(\tau))\right] \mathrm{d}\tau \tag{3.14}$$

for $0 \le s < t \le T$.

Step 3. *The logarithmic Osgood modulus.* We have, for $z > 0$,

$$\Phi'(z) = 1 + \log\left(1 + \frac{C_\alpha M_T}{z}\right) - \frac{C_\alpha M_T}{C_\alpha M_T + z} > 0.$$

Moreover,

$$\Phi(z) \sim z\log\left(\frac{C_\alpha M_T}{z}\right) \qquad \text{as } z\downarrow 0.$$

Thus Φ is continuous and strictly increasing on $[0,\infty)$, is positive on $(0,\infty)$, and satisfies

$$\int_{0+} \frac{\mathrm{d}z}{\Phi(z)} = \infty. \tag{3.15}$$

In addition, the definition of Φ gives

$$z \le \Phi(z), \qquad z \ge 0. \tag{3.16}$$

Step 5. *Conclusion by Osgood's lemma.* We define

$$a(\tau) := C_\alpha \left(1 + \|u(\tau,\cdot)\|_{L^\infty(\mathbb{R})} + \|v(\tau,\cdot)\|_{L^\infty(\mathbb{R})}\right) \tag{3.17}$$

and note that, by (3.8), $a \in L^1(0,T)$. Since $C_\alpha \ge \alpha/2$, (3.16) shows that taking $s = 0$ in (3.14) and using (3.7) gives

$$Z(t) \le \int_0^t a(\tau)\Phi(Z(\tau))\,\mathrm{d}\tau, \qquad 0 \le t \le T. \tag{3.18}$$

Osgood's lemma (see [BCD11, Lemma 3.4 and Corollary 3.5]) implies that $Z(t) = 0$ for every $t \in [0,T]$. Since ω is strictly positive, we conclude that

$$\begin{aligned} u(t,\cdot) &= v(t,\cdot) && \text{almost everywhere on } \mathbb{R} \text{ for every } t \in [0,T], \\ u &= v && \text{in } C([0,T]; L^2(\mathbb{R})). \end{aligned}$$

□

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