UNIVERSAL VOLUME GROWTH BOUNDS FROM POSITIVE INTERMEDIATE CURVATURE

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Abstract. Let n, m be integers such that $n \geq 2$ and $0 \leq m \leq n-2$. Let (M^n, g) be a complete Riemannian manifold, and let $\mathcal{C}_{m+1}$ be the $(m+1)$-intermediate curvature introduced by Brendle–Hirsch–Johne. We prove

$$\mathrm{Ric} \geq 0, \qquad \mathcal{C}_{m+1} \geq 1 \quad \Longrightarrow \quad \mathrm{Vol}\, B_R(p) \leq C(n,m)R^m,$$

for every $p \in M$ and every $R > 0$. In particular, when $m = n-2$, we deduce that

$$\mathrm{Ric} \geq 0, \qquad \mathrm{Scal} \geq 1 \quad \Longrightarrow \quad \mathrm{Vol}\, B_R(p) \leq C(n)R^{n-2},$$

for every $p \in M$ and every $R > 0$.

1. Introduction

In this note we prove the following result.

Theorem 1.1. *Let $n \geq 2$, and let (M^n, g) be a complete connected Riemannian manifold satisfying*

$$\mathrm{Ric}_g \geq 0, \qquad \mathrm{Scal}_g \geq 1.$$

There is a constant $C = C(n) < \infty$ such that

$$\mathrm{Vol}_g\, B_R(p) \leq CR^{n-2} \qquad \text{for every } R > 0 \text{ and every } p \in M.$$

Theorem 1.1 is the case $m = n-2$ of the more general Theorem 5.1 below, and it answers in the affirmative a question posed by Gromov in [11, §2.A(b)]. Theorem 5.1 can be used to bound the m-dimensional Urysohn width of (M^n, g). Let $k \geq 0$ be an integer. The *k-dimensional Urysohn width* of a metric space X is

$$\mathrm{UW}_k(X) := \inf_f \sup_{y \in P} \mathrm{diam}(f^{-1}(y)),$$

where the infimum is taken over all continuous maps $f\colon X \to P$ to simplicial complexes of dimension at most k. Gromov asked whether every complete n-dimensional Riemannian manifold with $\mathrm{Scal} \geq \sigma^2 > 0$ satisfies $\mathrm{UW}_{n-2}(M) \leq \frac{C(n)}{\sigma}$; see, e.g., [11, § 2.A(c)]. The three-dimensional case of this conjecture is known after [18, Theorem 1.1], and [19, Theorem 1.1].

Combining Theorem 5.1 with [21, Theorem 3.3] (see also [17, Theorem 1.6]) gives the following noncollapsed form of the conjectured estimate for all the intermediate curvatures considered here.

Corollary 1.2. *Let $n \geq 2$, $0 \leq m \leq n-2$, and $v_0 > 0$. There exists a constant $C = C(n, m, v_0) < \infty$ with the following property. If (M^n, g) is a complete connected Riemannian manifold satisfying*

$$\mathrm{Ric}_g \geq 0, \qquad \mathfrak{K}_{m,g} \geq 1, \qquad \inf_{p \in M} \mathrm{Vol}_g\, B_1(p) \geq v_0,$$

(see Section 2.1 for the definition of $\mathfrak{K}_{m,g}$), then $\mathrm{UW}_m(M, g) \leq C$. In particular, since $\mathfrak{K}_{n-2} = \mathrm{Scal}$, we obtain

$$\mathrm{Ric}_g \geq 0, \qquad \mathrm{Scal}_g \geq 1, \qquad \inf_{p \in M} \mathrm{Vol}_g\, B_1(p) \geq v_0 \quad \Longrightarrow \quad \mathrm{UW}_{n-2}(M, g) \leq C(n, v_0).$$

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It would be interesting to remove the assumption $\inf_{p\in M} \mathrm{Vol}_g B_1(p) \geq v_0$ in Corollary 1.2.

History of the problem and previous results. Recently, there have been many contributions by different authors related to Theorem 1.1. We list some of them:

(1) Under the stronger assumption $\mathrm{Sec} \geq 0$, Theorem 1.1 follows from Petrunin's local upper bound for the scalar-curvature integral [22];
(2) B. Zhu stated the linear volume growth conclusion of Theorem 1.1 in dimension three under a uniform lower bound for the volumes of unit balls in [29, Theorem 1.8]. In n dimensions he stated an $O(R^{n-1})$ estimate under the same noncollapsing assumption and an $O(R^{n-2})$ estimate under a uniform injectivity-radius lower bound [29, Corollary 1.9];
(3) From this item on, in this list, assume $n = 3$ and the assumptions of Theorem 1.1. Munteanu–Wang established the linear-growth estimate with a universal constant and also considered scalar-curvature lower bounds decaying at infinity [20];
(4) Chodosh–Li–Stryker subsequently gave a different proof of the result in [20] and extended the power-law decay range, with a constant C depending on the manifold and basepoint [7];
(5) Wei–Xu–Zhang obtained the sharp upper bound for $\limsup_{R\to\infty} \mathrm{Vol}\, B_R(p)/R$ [26];
(6) Y. Wang proved an effective local volume upper bound [25];
(7) Huang–Liu obtained sharp asymptotic estimates in the asymptotically nonnegative-Ricci setting, under some decay assumptions [13].

In the Ricci-limit setting, Wang–Xie–B. Zhu–X. Zhu [24] excluded an $\mathbb{R}^{n-1}$-splitting for noncollapsed limits of complete n-dimensional Riemannian manifolds with $\mathrm{Ric} \geq 0$ and $\mathrm{Scal} \geq 1$; see their Theorem 1.1(1) and the earlier result [30]. In [24] they also proved

$$\inf_{p\in M} \mathrm{Vol}\, B_r(p) \leq c(n) r^{n-2},$$

under $\mathrm{Ric} \geq 0$ and $\mathrm{Scal} \geq 1$; see [24, Theorem 1.9].

X. Zhu obtained related results in [31]. For example, he proved that asymptotic cones of manifolds with $\mathrm{Ric} \geq 0$, $\mathrm{Scal} \geq 1$, and $\inf_{p\in M} \mathrm{Vol}\, B_1(p) > 0$ have essential dimension at most $n-2$; see [31, Theorem 1.6]. Note that Corollary 3.2 below allows one to remove the noncollapsed assumption from both [31, Theorem 1.6] and [24, Theorem 1.1(1)].

We stress that, as discussed in [24, Remark 1.5], for a noncompact M satisfying the assumptions of Theorem 1.1, we can use Theorem 1.1 to show $b_1(M) \leq n-3$ (thus removing the noncollapsing hypothesis from the inequality on b_1 in [31, Theorem 1.3]). The estimate on the first Betti number also has a rigidity statement. Let (M^n, g) be a noncompact Riemannian manifold as in the assumptions of Theorem 1.1. By [1] and Theorem 1.1 applied to the universal cover of M we get that if $b_1(M) = n-3$, then M has at most linear volume growth (and thus exactly linear volume growth by Calabi–Yau lower volume bound). It then follows from [12, Theorem A and Remark 1.2(2)] that $\widetilde{M}$ splits off isometrically an $\mathbb{R}^{n-3}$-factor. This answers the question raised in [31, Remark 5.4] without a uniform noncollapsing assumption.

Let us also discuss the intermediate-curvature conclusion in Theorem 5.1. BiRicci curvature (corresponding to the case $m = 1$ in Theorem 5.1) was introduced by Shen–Ye [23]. In [2, Theorem 3(1)] we proved linear volume growth in dimensions $3 \leq n \leq 5$ under $\mathrm{Ric} \geq 0$ and a uniformly positive spectral biRicci condition outside a compact set; when the spectral parameter is zero, this includes a pointwise biRicci lower bound. Zhou–Zhu [28] later extended the result to $3 \leq n \leq 7$. Note that Theorem 5.1 shows that $\mathrm{Ric} \geq 0$ and $\mathrm{biRic} \geq 1$ imply that M has linear volume growth (and then bounded isoperimetric profile), thus answering the global pointwise version of the first bullet of [2, Question 1] in the negative in every dimension $n \geq 3$. In [3, Definition 1.1] Brendle–Hirsch–Johne introduced and studied the curvature family interpolating between Ric and Scal used in this note.

Theorem 1.1 is closely related to a more general problem posed by Yau [27], which asks whether every complete noncompact n-dimensional Riemannian manifold with $\mathrm{Ric} \geq 0$ satisfies

$$\limsup_{r\to\infty} r^{2-n} \int_{B_r(p)} \mathrm{Scal}\; dV < \infty.$$

Under the additional assumption $\mathrm{Scal} \geq 1$, a solution of Yau's conjecture would immediately imply the volume growth estimate in Theorem 1.1. Up to the author's knowledge, Yau's conjecture remains open in every dimension $n \geq 3$.

Ideas of the proof. The proof has two main ingredients. First, in the setting of Theorem 5.1 (of which Theorem 1.1 is a particular case with $m = n-2$) a ball which is quantitatively close to splitting $m+1$ Euclidean directions has radius at most the curvature scale; this is Corollary 3.2. Moreover, if a ball is close to split only $k \leq m$ directions, Proposition 4.5 finds another ball which is close to split $k+1$ directions while losing only a harmless multiplicative factor in the m-volume ratio. Hence, starting at rank $k = 0$ and iterating finitely many times proves the theorem.

The proof of Proposition 4.5 uses tools contained in the proof of Kapovitch–Wilking rescaling theorem [15, Theorem 5.1]. The result in [14, Lemma 2.12] provides a closely related finite formulation of [15, Theorem 5.1]; compare with our Proposition 4.4. We stress that in order to prove Proposition 4.4 we do not need to use the full strength of [15, Theorem 5.1]; see the paragraph before the statement of Proposition 4.4. We refer the reader to the beginning of the proof of Proposition 4.5 for an explanation of its proof.

The Hodge obstruction and the rank-improvement mechanism described above are inspired by the arguments in [8, 24, 31]; in particular, the Kapovitch–Wilking rescaling theorem [15, Theorem 5.1] is also a central input in the proof of [31, Theorem 1.6].

Moreover, at least two closely related uses of harmonic splitting maps and the Bochner formula precede the argument of Proposition 3.1 and Corollary 3.2. In [24, Remark 3.1], the authors give an alternative proof of [24, Theorem 1.1(1)&(2)] using harmonic splitting maps with computations similar to the ones in Proposition 3.1. On the other hand, in [8], the authors use k-component splitting maps to obtain integral control of the sum R_k of the lowest k Ricci eigenvalues on k-symmetric balls; see [8, Lemma 3.2 and Theorem 3.4]. Moreover, in [8, Theorem 3.8(2)] the authors prove a quantitative obstruction to an $\mathbb{R}^{n-1}$-splitting under almost nonnegative Ricci curvature, integral almost-nonnegativity of Ric_{n-2}, and integral positivity of the truncated scalar curvature. The new feature in the proof of Proposition 3.1 is the addition of a contribution coming from the wedge of the one-forms β^a to make the intermediate curvature appear: compare with (8).

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Disclosure of AI tools. This work made substantial use of OpenAI's GPT-5.6 Sol with Ultra reasoning effort. GPT proposed the central inductive procedure based on the Hodge obstruction and rank improvement behind the proof of Theorem 1.1. I formulated and guided the problem, suggested relevant strategies and literature, and, following the suggested proof strategy of Theorem 1.1, developed this note.

I emphasize that the proof below is quite different from the original argument suggested by GPT. On the one hand, the proof below is effective, while the original suggested strategy proceeded by contradiction: this emerged through discussions with Daniele Semola, to whom I am very grateful. On the other hand, contrary to the original strategy proposed by the LLM, the proof below is less reliant on the full proof of [15, Theorem 5.1]; see the discussion before Proposition 4.4: this emerged through discussions with Elia Bruè and Kai Xu.

After examining the original output, I observed that a modification of the original argument proposed by the LLM gives the extension to the intermediate curvatures of Brendle–Hirsch–Johne, as stated in Theorem 5.1. This extension, motivated by a question I previously had in a joint work with Kai Xu, is my own contribution.

Addendum. After this manuscript was completed and circulated privately, I learned of independent work by Jian Ge [9]. Ge proves Theorem 1.1 by a different method, based on heat-kernel Fisher metric and Nash entropy.

2. Splitting maps and curvature algebra

2.1. **Intermediate curvature.** We use the convention

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \qquad K_{ab} = \langle R(e_a,e_b)e_b, e_a\rangle,$$

so that $\mathrm{Scal} = 2\sum_{a<b} K_{ab}$. Let r be an integer such that $1 \le r \le n-1$. For an r-plane $W \subset T_xM^n$, choose an orthonormal basis $e_1,\dots,e_r$ of W and extend it to an orthonormal basis $e_1,\dots,e_r,e_{r+1},\dots,e_n$ of T_xM. The r-intermediate curvature of Brendle–Hirsch–Johne is

$$\mathcal{C}_r(W) := \sum_{a=1}^{r}\sum_{b=a+1}^{n} K_{ab}. \tag{1}$$

For $0 \le m \le n-2$ set, for every $(m+1)$-dimensional plane $W \subset T_xM$

$$\mathfrak{K}_m(W) := 2\mathcal{C}_{m+1}(W). \tag{2}$$

Thus $\mathcal{C}_1 = \mathrm{Ric}$, $\mathcal{C}_2 = \mathrm{biRic}$, $\mathcal{C}_{n-1} = \mathrm{Scal}/2$, and

$$\mathfrak{K}_0 = 2\,\mathrm{Ric}, \qquad \mathfrak{K}_1 = 2\,\mathrm{biRic}, \qquad \mathfrak{K}_{n-2} = \mathrm{Scal}\,.$$

For a fixed integer $m \ge 0$ define the *m-volume ratio* for $x \in M$ and $r > 0$

$$\mathcal{E}_m(x,r) := \frac{\mathrm{Vol}\,B_r(x)}{r^m}. \tag{3}$$

When $\mathrm{Ric} \ge 0$, Bishop–Gromov comparison gives

$$\frac{\mathrm{Vol}\,B_s(x)}{s^n} \ge \frac{\mathrm{Vol}\,B_r(x)}{r^n} \quad \forall 0 < s \le r, \qquad \mathrm{Vol}\,B_r(x) \le \omega_n r^n \quad \forall r > 0. \tag{4}$$

2.2. **Metric and harmonic splitting.** Whenever a ball occurs as an argument of d_{GH}, or as the domain or target of a Gromov–Hausdorff approximation, it is understood to be the corresponding closed metric ball. Moreover, we always intend that such comparisons are pointed with respect to the displayed centers.

Definition 2.1. Let $0 < \varepsilon < 1$ and let k, n be integers such that $0 \le k \le n$. A ball $B_r(x)$ is (k,ε)*-split* if there are a pointed proper length space (Z,z) and a pointed εr-Gromov–Hausdorff approximation

$$B_{\varepsilon^{-1}r}(x) \longrightarrow B_{\varepsilon^{-1}r}((0,z)) \subset \mathbb{R}^k \times Z.$$

Note that every ball is $(0,\varepsilon)$-split.

The following is the quantitative harmonic replacement associated with Definition 2.1. It is due to Cheeger–Colding; see [4, 5] and [6, Lemma 1.21(2)].

Proposition 2.2. *For all integers n, k such that $1 \leq k \leq n$ and $\eta > 0$ there is $\varepsilon = \varepsilon(n, k, \eta) > 0$ with the following property. Let (M^n, g) be a complete Riemannian manifold. If $\mathrm{Ric} \geq 0$ and $B_r(x)$ is (k, ε)-split, then there are harmonic functions*

$$u = (u^1, \dots, u^k) : B_{4r}(x) \longrightarrow \mathbb{R}^k$$

such that for every $a = 1, \dots, k$ on $B_{2r}(x)$ we have

$$|\nabla u^a| \leq C(n), \qquad \frac{1}{\mathrm{Vol}\, B_{2r}(x)} \int_{B_{2r}(x)} \left(|G - I|^2 + r^2 \sum_{a=1}^{k} |\operatorname{Hess} u^a|^2 \right) dV \leq \eta, \tag{5}$$

where $G^{ab} = \langle \nabla u^a, \nabla u^b \rangle$, and I is the identity matrix. Here $|A|$ denotes the Hilbert–Schmidt norm of the matrix A.

Remark 2.3. We shall also use the approximation part of the harmonic-replacement construction. If (M_i^n, g_i) are complete Riemannian manifolds with $\mathrm{Ric}_{g_i} \geq 0$ and

$$(M_i, g_i, p_i) \longrightarrow (\mathbb{R}^k, g_{\mathrm{eu}}, 0)$$

in the pointed Gromov–Hausdorff sense, then the harmonic replacements u_i in Proposition 2.2, normalized by $u_i(p_i) = 0$, may be chosen so that, on every fixed ball, they are uniformly Lipschitz and converge uniformly to the Euclidean coordinate map.

2.3. **The Weitzenböck formula.** Let (M^n, g) be a complete n-dimensional Riemannian manifold, and let $1 \leq p \leq n - 1$ be an integer. We shall use the Weitzenböck formula

$$\Delta_H = d\delta + \delta d = \nabla^* \nabla + q_p(\mathrm{Rm}) \qquad \text{on } \Lambda^p T^* M. \tag{6}$$

With our curvature convention, $q_1(\mathrm{Rm}) = \mathrm{Ric}$. The only algebraic fact needed below is the following standard formula.

Lemma 2.4. *Let $\beta^1, \dots, \beta^n$ be an orthonormal coframe in (M^n, g) and $I \subset \{1, \dots, n\}$ with $|I| = p$. If $\beta^I = \bigwedge_{a \in I} \beta^a$, then*

$$\langle q_p(\mathrm{Rm}) \beta^I, \beta^I \rangle = \sum_{a \in I} \sum_{b \notin I} K_{ab}. \tag{7}$$

Consequently, the following holds. Let $0 \leq m \leq n - 2$ be an integer and $t = m + 1$. If $\beta^1, \dots, \beta^t$ is an orthonormal coframe for a t-plane W, and $\Theta = \beta^1 \wedge \cdots \wedge \beta^t$, then

$$\langle q_t(\mathrm{Rm}) \Theta, \Theta \rangle + \sum_{a=1}^{t} \langle q_1(\mathrm{Rm}) \beta^a, \beta^a \rangle = 2\mathcal{C}_t(W) = \mathfrak{K}_m(W). \tag{8}$$

Proof. Formula (7) is in [16, equation (11)]. For completeness, the first term in (8) is $\sum_{a \leq t < b} K_{ab}$, while the sum of the one-form terms is

$$\sum_{a=1}^{t} \mathrm{Ric}((\beta^a)^\sharp, (\beta^a)^\sharp) = 2 \sum_{a < b \leq t} K_{ab} + \sum_{a \leq t < b} K_{ab}.$$

Their sum is twice (1), thus proving (8), as desired. ∎

3. The quantitative Hodge obstruction

The next proposition is the first core ingredient of our proof.

Proposition 3.1. *Fix an integer $n \geq 2$, an integer $0 \leq m \leq n-2$, and put $t = m+1$. There are constants $\eta(n,m) > 0$ and $C(n,m) < \infty$ with the following property. Let (M^n, g) be a complete Riemannian manifold, and let $\sigma > 0$. Suppose*

$$\mathrm{Ric} \geq 0, \qquad \mathfrak{K}_m \geq \sigma^2 > 0$$

on $B_{2r}(x) \subset M$, and suppose there are harmonic functions $u^1, \dots, u^t$ on $B_{4r}(x)$ satisfying (5) for some $\eta \leq \eta(n,m)$. Then

$$\sigma^2 r^2 \leq C(n,m). \tag{9}$$

Proof. Write $\theta^a = du^a$ and $G^{ab} = \langle \theta^a, \theta^b \rangle$. Let I be the identity matrix. $|A|$ denotes the Hilbert–Schmidt norm of the matrix A. Fix a sufficiently small $\eta_0 = \eta_0(n,m) > 0$. On

$$\Omega := \{|G - I| < 3\eta_0\}$$

the matrix G is positive definite, and we define

$$\begin{pmatrix} \beta^1 \\ \vdots \\ \beta^t \end{pmatrix} := G^{-1/2} \begin{pmatrix} \theta^1 \\ \vdots \\ \theta^t \end{pmatrix}, \qquad \Theta := \beta^1 \wedge \cdots \wedge \beta^t.$$

Thus $\beta^1, \dots, \beta^t$ is orthonormal on Ω.

Choose a smooth function $\chi = \chi(|G-I|^2)$ which equals one when $|G - I| \leq \eta_0$, vanishes when $|G - I| \geq 2\eta_0$, and takes values in $[0,1]$. Since

$$\nabla G^{ab} = \operatorname{Hess} u^a(\,\cdot\,, \nabla u^b) + \operatorname{Hess} u^b(\,\cdot\,, \nabla u^a),$$

the gradient bound in (5) and differentiation of $A \mapsto A^{-1/2}$ give

$$|\nabla \chi|^2 + |\nabla \Theta|^2 + \sum_{a=1}^t |\nabla \beta^a|^2 \leq C(n,m,\eta_0) \sum_{a=1}^t |\operatorname{Hess} u^a|^2, \tag{10}$$

on Ω. Note that the forms $\chi\Theta$ and $\chi\beta^a$ extend smoothly by zero across the boundary of Ω.

Let f be a smooth cutoff function supported in $B_{2r}(x)$, equal to one on $B_r(x)$, and satisfying $|\nabla f| \leq C(n)/r$. Set

$$\widetilde{\omega} = f\chi\Theta, \qquad \omega^a = f\chi\beta^a \quad \forall 1 \leq a \leq t.$$

Integrating the identity (6) gives, for any compactly supported p-form ξ,

$$\int \langle q_p(\mathrm{Rm})\xi, \xi \rangle = \int \left(|d\xi|^2 + |\delta\xi|^2 - |\nabla \xi|^2 \right) \leq C(n) \int |\nabla \xi|^2. \tag{11}$$

Equation (8) and the curvature hypothesis give the pointwise lower bound on $B_{2r}(x)$:

$$\langle q_t(\mathrm{Rm})\widetilde{\omega}, \widetilde{\omega} \rangle + \sum_{a=1}^t \langle q_1(\mathrm{Rm})\omega^a, \omega^a \rangle \geq \sigma^2 f^2 \chi^2.$$

Summing (11) for ξ in the list $(\widetilde{\omega}, \omega^1, \dots, \omega^t)$ and using (10) and (5) therefore yields

$$\begin{aligned} \sigma^2 \int_{B_{2r}(x)} f^2 \chi^2 &\leq C(n,m,\eta_0) \left(r^{-2} \operatorname{Vol} B_{2r}(x) + \int_{B_{2r}(x)} \sum_{a=1}^t |\operatorname{Hess} u^a|^2 \right) \\ &\leq C(n,m,\eta_0)(1+\eta) r^{-2} \operatorname{Vol} B_{2r}(x). \end{aligned} \tag{12}$$

It remains to bound the left-hand side of the above inequality from below. Markov's inequality and (5) give

$$\operatorname{Vol}\big(B_{2r}(x) \cap \{|G - I| > \eta_0\}\big) \leq \eta_0^{-2} \eta \operatorname{Vol} B_{2r}(x),$$

whereas Bishop–Gromov gives $\operatorname{Vol} B_r(x) \geq 2^{-n} \operatorname{Vol} B_{2r}(x)$. Choose $\eta(n,m)$ additionally so small that $\eta(n,m) \leq 2^{-n-1}\eta_0^2$. Since $f = \chi = 1$ on $B_r(x) \cap \{|G - I| \leq \eta_0\}$, we thus deduce, using the previous inequalities,

$$\int f^2\chi^2 \geq 2^{-n-1} \operatorname{Vol} B_{2r}(x).$$

Combining this with (12) proves (9), as desired. □

Combining Propositions 2.2 and 3.1 gives the following.

Corollary 3.2. *For every integer $n \geq 2$ and integer $0 \leq m \leq n-2$ there are $\varepsilon_H(n,m) > 0$ and $C(n,m) < \infty$ such that, if a complete manifold (M^n, g) satisfies*

$$\operatorname{Ric} \geq 0, \qquad \mathfrak{K}_m \geq \sigma^2 > 0,$$

and $B_r(x) \subset M$ is $(m+1, \varepsilon_H)$-split, then

$$r \leq C(n,m)\sigma^{-1}.$$

4. Critical scales and effective rank improvement

We record a consequence of Cheeger–Colding's almost splitting theorem [4, 5]. One can give a simple proof by contradiction of Lemma 4.1, based on Gigli's splitting theorem [10].

Lemma 4.1. *For every integer $n \geq 1$, every integer $0 \leq k < n$, and $\varepsilon > 0$, there are $L = L(n,k,\varepsilon) > \varepsilon^{-1}$ and $\delta_s = \delta_s(n,k,\varepsilon) > 0$ with the following property. Let (M^n, g) be a complete Riemannian manifold with* Ric ≥ 0*, let $s > 0$, and let (Z, d_Z, z_0) be a pointed proper geodesic space. Assume the closed ball $\overline{B}_{Ls}(q)$ is $\delta_s s$-close to the closed Ls-ball centered at $(0, z_0)$ in $\mathbb{R}^k \times Z$. If z_0 is the midpoint of a minimizing segment $\gamma : [-Ls, Ls] \to Z$, then $B_s(q)$ is $(k+1, \varepsilon)$-split.*

Remark 4.2. Corollary 3.2 immediately gives a diameter bound for maximal residual factors. Namely, if $0 \leq m \leq n-2$ and

$$(M_i, g_i, p_i) \longrightarrow (\mathbb{R}^m \times K, (0, z)), \qquad \operatorname{Ric}_{g_i} \geq 0, \quad \mathfrak{K}_{m,g_i} \geq \sigma^2,$$

then

$$\operatorname{diam} K \leq C(n,m)\sigma^{-1}. \tag{13}$$

Indeed, recentering at the midpoint of a long segment in K and applying Lemma 4.1 produces an $(m+1, \varepsilon_H)$-split ball of comparable radius, which is bounded by Corollary 3.2. In the scalar-curvature case this extends the corresponding diameter results of [30] and [24, Theorem 1.1]. Note that the present formulation requires no noncollapsing assumption and applies to all the intermediate curvatures $\mathfrak{K}_m$. The optimal constant $C(n,m)$ in (13) is presently unknown. The optimal constant C in Theorem 1.1 and Theorem 5.1, except in a few cases, is also unknown.

We state an elementary lemma. For a Lipschitz $\mathbb{R}^k$-valued map b defined on a ball $B_2(q) \subset (M^n, g)$, put, for $s \in (0,1)$, and d being the Riemannian distance on M

$$D_{q,b}(s) := \sup_{x,y \in \overline{B}_s(q)} \big| d(x,y) - |b(x) - b(y)| \big|. \tag{14}$$

Lemma 4.3. *The function $s \mapsto D_{q,b}(s)$ is locally Lipschitz on $(0,1)$.*

Proof. The function $(x,y) \mapsto |d(x,y) - |b(x) - b(y)||$ is Lipschitz on $\overline{B}_s(q) \times \overline{B}_s(q)$, with Lipschitz constant bounded from above by $1+\operatorname{Lip}(b)$. Since $d_H(\overline{B}_s(q), \overline{B}_{s'}(q)) \leq |s - s'|$ for every $s, s' \in (0,1)$, the conclusion follows. □

The next proposition relies on ideas in the proof of the Kapovitch—Wilking rescaling theorem [15, Theorem 5.1], and it is inspired by Jansen's formulation [14, Lemma 2.12].

We stress that, using the full strength of [15, Theorem 5.1] (namely, the fact that the residual factor K in there is unique), one can prove a more general version of Proposition 4.4 below. We decided to include the weaker statement below for two reasons: first, it has the advantage that its proof (which is inspired by [15, 14]) is essentially self-contained and only leverages basic facts of Cheeger–Colding theory, [15, Lemma 2.1], and a classical maximal-function argument; second, Proposition 4.4 is enough to prove our sought Proposition 4.5.

Proposition 4.4. *Fix an integer $n \geq 1$, an integer $0 \leq k < n$, and $0 < A < \infty$, $0 < \eta < 1$. There are $\delta_c = \delta_c(n,k,A,\eta) > 0$, $c = c(n) > 0$, $C = C(k,n) < \infty$, and $c_0 = c_0(n)$ with the following property. Suppose (M^n, g, p) is complete, has $\mathrm{Ric}_g \geq 0$, and*

$$d_{GH}(B_{\delta_c^{-1}}(p), B^{\mathbb{R}^k}_{\delta_c^{-1}}(0)) \leq \delta_c.$$

Then there are a set $G \subset B_{1/4}(p)$ and an integer $N \geq 1$ for which the following holds. For every $1 \leq j \leq N$ there are $q_j \in B_{1/4}(p)$, $0 < t_j \leq 1$, pairwise disjoint $E_j \subset G$, pointed proper geodesic spaces (Y_j, d_{Y_j}, y_j), and $a_j \in Y_j$ such that:

(i) *$G = \sqcup_{j=1}^N E_j$ and $\mathrm{Vol}(G) \geq c \,\mathrm{Vol}\, B_1(p)$;*
(ii) *$\sum_{j=1}^N t_j^k \leq C$, and, for every $1 \leq j \leq N$, $E_j \subset B_{Ct_j}(q_j)$;*
(iii) *For every $1 \leq j \leq N$, $d_{Y_j}(y_j, a_j) \geq c_0 t_j$, and*

$$d_{GH}(B_{At_j}(q_j), B^{\mathbb{R}^k \times Y_j}_{At_j}((0, y_j))) \leq \eta t_j.$$

Proof. Set $c(n) := 4^{-n-1}$, $c_*(n) := 10^{-n^2}$, $c_0(n) := c_*(n)/4$ and fix $C(k,n) > 1$ large enough so that

$$1 - C(k,n)^{-1}/2 > 5c_*/4, \qquad C(k,n) \geq \alpha(k,n)(2\sqrt{k}\mathcal{K}(n) + 1)^k, \tag{15}$$

where $\mathcal{K}(n)$ is the constant in the first inequality of (5), and $\alpha(k,n)$ is chosen large enough, only depending on k, n (in order for (20) below to hold).

The case $k = 0$ is elementary. In that case, the assertion is true choosing $\delta_c \ll 1$ and: $N = 1$, $G = E_1 = B_{1/4}(p)$, $q_1 = p$, $t_1 = \mathrm{diam}\, M < \infty$, $(Y_1, d_{Y_1}, y_1) = (M, d_g, p)$ and $a_1 \in M$ such that $d(a_1, p) \geq \mathrm{diam}\, M/2$. Hence, from now on, assume $k \geq 1$.

Suppose that no $\delta_c > 0$ works the choices of constants above. Taking $\delta_i \to 0$ we thus have a contradicting sequence

$$(M_i, g_i, p_i) \longrightarrow (\mathbb{R}^k, g_{\mathrm{eu}}, 0)$$

for which the conclusion is false. Proposition 2.2 (see Remark 2.3) and a diagonal argument give radii $R_i \to \infty$ and harmonic maps

$$b_i = (b_i^1, \ldots, b_i^k) : B_{R_i}(p_i) \longrightarrow \mathbb{R}^k, \qquad b_i(p_i) = 0,$$

such that the following hold:

(a) For every $R > 0$, the maps b_i have a uniform Lipschitz bound on $B_R(p_i)$ for i large enough, and their distance distortion on $B_R(p_i)$ tends to zero as $i \to \infty$;
(b) Let $h_i := \sum_{a,b=1}^k \left| \langle \nabla b_i^a, \nabla b_i^b \rangle - \delta_{ab} \right|^2 + \sum_{a=1}^k |\mathrm{Hess}\, b_i^a|^2$. For every fixed $R > 2$,

$$\frac{1}{\mathrm{Vol}\, B_R(p_i)} \int_{B_R(p_i)} h_i \, dV_{g_i} \longrightarrow 0.$$

Set $\alpha_i := i^{-1} + \left(\frac{1}{\operatorname{Vol} B_2(p_i)} \int_{B_2(p_i)} h_i \, dV_{g_i}\right)^{1/2} > 0$, and define

$$H_i := \left\{ q \in B_{1/4}(p_i) : \sup_{0<r\leq 3/4} \frac{1}{\operatorname{Vol} B_r(q)} \int_{B_r(q)} h_i \, dV_{g_i} \leq \alpha_i \right\}.$$

A standard maximal-function estimate and Bishop–Gromov comparison imply

$$\frac{\operatorname{Vol}(B_{1/4}(p_i) \setminus H_i)}{\operatorname{Vol} B_1(p_i)} \longrightarrow 0, \qquad \operatorname{Vol}(H_i) \geq \frac{1}{2} 4^{-n} \operatorname{Vol} B_1(p_i), \tag{16}$$

for all sufficiently large i.

For $q \in H_i$ define the distortion, for $r \in (0, 3/4]$,

$$D_{q,b_i}(r) := \sup_{x,y \in \overline{B}_r(q)} \left| d_i(x,y) - |b_i(x) - b_i(y)| \right|.$$

Let $\rho_i(q)$ be the largest real number r in $(0, 3/4]$ such that $D_{q,b_i}(r) = c_* r$. It is readily seen that such an r exists for i large enough. Indeed, since $k < n$, one has $\liminf_{r\to 0} \frac{D_{q,b_i}(r)}{r} \geq 1$. On the other hand, property (a) above gives, uniformly for $q \in H_i$, $D_{q,b_i}(3/4) < 3c_*/4$, for all sufficiently large i. Lemma 4.3 therefore gives the existence of r. Moreover, notice $\sup_{q\in H_i} \rho_i(q) \leq c_*^{-1} \operatorname{dis}(b_i|_{\overline{B}_1(p_i)}) \to 0$ as $i \to \infty$.

We next claim that for every sufficiently large i and every $q \in H_i$ there are a pointed proper geodesic space (Y, d_Y, y) and $a \in Y$ such that

$$d_Y(y, a) \geq c_0, \qquad d_{GH}\left(B_A^{(M_i, \rho_i(q)^{-2} g_i)}(q), B_A^{\mathbb{R}^k \times Y}((0, y))\right) \leq \eta. \tag{17}$$

Indeed, if this failed, choose violating points $q_i \in H_i$ and put

$$\widetilde{g}_i := \rho_i(q_i)^{-2} g_i, \qquad u_i := \rho_i(q_i)^{-1}(b_i - b_i(q_i)).$$

By what was proved above $\rho_i(q_i) \to 0$. Fix $R > 0$. For i large enough, the definition of H_i gives

$$\frac{1}{\operatorname{Vol}^{\widetilde{g}_i}(B_R^{\widetilde{g}_i}(q_i))} \left(\int_{B_R^{\widetilde{g}_i}(q_i)} \left(\sum_{a,b=1}^k \left| \langle \nabla u_i^a, \nabla u_i^b \rangle - \delta_{ab} \right|^2 + \sum_{a=1}^k |\operatorname{Hess}_{\widetilde{g}_i} u_i^a|^2 \right) dV_{\widetilde{g}_i} \right) \leq (1 + \rho_i(q_i)^2) \alpha_i. \tag{18}$$

Hence, using (18), we can apply the result in [15, Lemma 2.1] to get, after passing to subsequences,

$$(M_i, \widetilde{g}_i, q_i, u_i) \longrightarrow (\mathbb{R}^k \times Y, (0, y), \operatorname{pr}_{\mathbb{R}^k}), \tag{19}$$

for a pointed proper geodesic space (Y, y), where $\operatorname{pr}_{\mathbb{R}^k}$ denotes the projection onto the $\mathbb{R}^k$ factor.

Choose $x_i^-, x_i^+ \in \overline{B}_1^{\widetilde{g}_i}(q_i)$ realizing the supremum in $D_{q_i,b_i}(\rho_i(q_i))$. After passing to a subsequence, their limits are $x^\pm = (v^\pm, y^\pm)$ in $\mathbb{R}^k \times Y$. Convergence in (19), and the fact that $D_{q_i,b_i}(\rho_i(q_i)) = c_* \rho_i(q_i)$ give

$$\left| d_{\mathbb{R}^k \times Y}(x^-, x^+) - |v^- - v^+| \right| = d_{\mathbb{R}^k \times Y}(x^-, x^+) - |v^- - v^+| = c_*.$$

Using the product formula for $d_{\mathbb{R}^k \times Y}$, we thus deduce $d_Y(y^-, y^+) \geq c_*$. Hence, at least one of y^-, y^+ lies at distance at least $c_*/2$ from y. Since Y is geodesic, there is therefore a point $a \in Y$ with $d_Y(y, a) = c_0 = c_*/4$. This and (19) contradict the assumed failure of (17). Thus (17) holds uniformly over H_i.

We now construct the E_j's. Apply Vitali's covering Lemma to $\left\{ B_{\rho_i(q)/10}^{\mathbb{R}^k}(b_i(q)) : q \in H_i \right\}$. We thus have a countable pairwise disjoint subfamily, indexed by points $q_{i,j} \in H_i$, whose five-times-enlargements cover $b_i(H_i)$. Since the functions b_i are $\sqrt{k}\mathcal{K}(n)$-Lipschitz on $B_1(p_i)$, all the pairwise disjoint balls $B_{\rho_i(q_{i,j})/10}(b_i(q_{i,j}))$ lie in $B^{\mathbb{R}^k}_{2\sqrt{k}\mathcal{K}(n)+1}(0)$. Euclidean volume comparison gives (recall (15)), for a constant $\beta(k,n)$,

$$\sum_j \rho_i(q_{i,j})^k \leq \beta(k,n)\mathcal{L}^k(B^{\mathbb{R}^k}_{2\sqrt{k}\mathcal{K}(n)+1}(0)) \leq C(n,k). \tag{20}$$

The measurable sets $F_{i,j} := H_i \cap b_i^{-1}\left(B^{\mathbb{R}^k}_{\rho_i(q_{i,j})/2}(b_i(q_{i,j}))\right)$ cover H_i. Define $E_{i,1} := F_{i,1}$ and $E_{i,j} := F_{i,j} \setminus \cup_{\ell<j} F_{i,\ell}$. Notice that $\{E_{i,j}\}_j$ are pairwise disjoint. We claim that

$$E_{i,j} \subset B_{C\rho_i(q_{i,j})}(q_{i,j}). \tag{21}$$

If $x \in E_{i,j} \subset F_{i,j}$ and $D = d(x, q_{i,j}) \geq C\rho_i(q_{i,j})$, then $D \leq 1/2$ (recall $H_i \subset B_{1/4}(p_i)$) and (15) gives

$$D_{q_{i,j},b_i}(5D/4) \geq D - |b_i(x) - b_i(q_{i,j})| \geq D - \rho_i(q_{i,j})/2 > c_*\, 5D/4.$$

Since $5D/4 < 3/4$, the latter inequality and $D_{q_{i,j},b_i}(3/4) < 3c_*/4$ (for i large enough) together with Lemma 4.3, gives a contradiction to the definition of $\rho_i(q_{i,j})$ and proves (21). Indeed, by the previous inequalities and Lemma 4.3, there is $r \in (5D/4, 3/4)$ such that $D_{q_{i,j},b_i}(r) = c_* r$. Consequently, by definition of $\rho_i(q_{i,j})$, we have $\rho_i(q_{i,j}) \geq r > 5D/4 > D \geq C\rho_i(q_{i,j}) > \rho_i(q_{i,j})$, a contradiction.

Now, let us retain finitely many $\{E_{i,j}\}_{j=1}^{N_i}$ and set $G_i := \sqcup_{j=1}^{N_i} E_{i,j}$ so that $\mathrm{Vol}(G_i) \geq \mathrm{Vol}(H_i)/2$. For each $q_{i,j}$, denote $\widehat{Y}_{i,j} := (Y_{i,j}, \rho_i(q_{i,j})d_{Y_{i,j}}, y_{i,j})$ where $Y_{i,j}$ is the pointed metric space in (17) associated with $q_{i,j}$; retain also the associated points $a_{i,j} \in \widehat{Y}_{i,j}$. Now, for one fixed sufficiently large i, equations (16), (17), (20), and (21) give all three conclusions of the proposition with $G = G_i$, $N = N_i$, $q_j = q_{i,j}$, $t_j = \rho_i(q_{i,j})$, $E_j = E_{i,j}$, $a_{i,j} \in \widehat{Y}_{i,j}$ defined above, thus resulting in a contradiction. □

We now prove the second core ingredient of the proof.

Proposition 4.5. *Fix an integer $n \geq 2$, integers m, k such that $0 \leq m \leq n-2$, $0 \leq k \leq m$, and fix $0 < \varepsilon < 1$. There are $\delta = \delta(n,m,k,\varepsilon) > 0$ and $c = c(n,m,k,\varepsilon) > 0$ such that the following holds. Let (M^n, g) be a complete Riemannian manifold. If $\mathrm{Ric} \geq 0$ and $B_r(p)$ is (k,δ)-split, then there are $q \in M$ and $s > 0$ such that $B_s(q)$ is $(k+1,\varepsilon)$-split and*

$$\mathcal{E}_m(q,s) \geq c\mathcal{E}_m(p,r). \tag{22}$$

Proof. All constants denoted by $c, C, C' > 0$ in this proof depend only on n, m, k, ε and may change from line to line. It is enough to prove the rescaled version with $r = 1$. So, from now on, let us assume $r = 1$. Notice

$$\mathcal{E}_m(p,1) = \mathrm{Vol}\, B_1(p).$$

Assume $B_1(p)$ is (k,δ)-split. Let

$$\Phi : B_{\delta^{-1}}(p) \longrightarrow B_{\delta^{-1}}((0,z)) \subset \mathbb{R}^k \times Z \tag{23}$$

be the δ-approximation given by Definition 2.1.

The proof is divided into two cases. If the residual factor Z has a long enough segment through z, Lemma 4.1 directly produces an additional Euclidean direction. Otherwise Proposition 4.4 gives a collection of good cells E_j in M at different scales, with bounded diameter. Since their scales are summable, we can select one cell carrying a definite fraction of the m-volume ratio. A segment in the space Y_j associated to such a cell supplies the additional Euclidean direction. Let us carefully perform this strategy.

Let $L(n,k,\varepsilon), \delta_s(n,k,\varepsilon)$ be the constants in Lemma 4.1, and let c_0 be the constant in Proposition 4.4. Choose

$$\vartheta \leq c_0/(32L), \quad \eta \leq \min\{1/2, \delta_s\vartheta/C_1\}, \quad A > c_0/4 + L\vartheta + 2, \quad d_0 = \delta_c/20, \quad R_0 = 4\delta_c^{-1},$$

where C_1 is a large universal constant that controls restriction and recentering of a Gromov–Hausdorff approximation, and $\delta_c(n,k,A,\eta)$ is the corresponding constant in Proposition 4.4. Finally choose δ sufficiently small so that:

$$1+\frac{3d_0}{4}\le\delta^{-1},\qquad C_1\delta\le\frac{\delta_s d_0}{4L},\qquad \delta_c^{-1}<\frac{\delta^{-1}+1}{10},\quad C_1\delta\le\frac{\delta_c}{2}. \tag{24}$$

Case 1: There is a long enough segment through z in Z. Suppose that some $z'\in B_{R_0}(z)$ satisfies $d_Z(z,z')\ge d_0$. Let $\gamma:[0,d_0]\to Z$ be the unit-speed initial segment of a minimizing geodesic from z to z'. Set

$$\bar z:=\gamma(d_0/2),\qquad s_0:=\frac{d_0}{4L}.$$

Since $Ls_0=d_0/4$, the restriction of γ to $[d_0/4,3d_0/4]$, reparametrized on $[-Ls_0,Ls_0]$, is a minimizing segment centered at $\bar z$. Choose $q\in M$ corresponding to $(0,\bar z)$ under the approximation (23). First $\overline{B}_{Ls_0}((0,\bar z))\subset\overline{B}_{3d_0/4}((0,z))$, so this ball lies inside the original comparison region since $3d_0/4\le\delta^{-1}$ (see (24)). Restriction and recentering give

$$d_{GH}\Big(\overline{B}_{Ls_0}(q),\overline{B}_{Ls_0}((0,\bar z))\Big)\le C_1\delta,$$

for a universal constant $C_1>0$. Since $C_1\delta\le\delta_s s_0$ (see (24)), all the hypotheses of Lemma 4.1 are satisfied, and consequently $B_{s_0}(q)$ is $(k+1,\varepsilon)$-split.

Since p and q correspond respectively to $(0,z)$ and $(0,\bar z)$,

$$d(p,q)\le d_Z(z,\bar z)+C_1\delta=\frac{d_0}{2}+C_1\delta\le C(n,m,k,\varepsilon),$$

and hence $B_1(p)\subset B_C(q)$ for a possibly larger constant $C>\max\{1,s_0\}$. Bishop–Gromov monotonicity then gives, with $c:=c(n,m,k,\varepsilon)$

$$\mathcal{E}_m(q,s_0)=s_0^{n-m}\frac{\operatorname{Vol}B_{s_0}(q)}{s_0^n}\ge\frac{s_0^{n-m}}{C^n}\operatorname{Vol}B_C(q)\ge c\operatorname{Vol}B_1(p)=c\mathcal{E}_m(p,1).$$

We thus proved that $B_{s_0}(q)$ is $(k+1,\varepsilon)$-split and $\mathcal{E}_m(q,s_0)\ge c\mathcal{E}_m(p,1)$. Hence, the proof is concluded in the first case.

Case 2: The residual factor Z is macroscopically small. Suppose now that every point of $B_{R_0}(z)$ lies within d_0 of z. Since $\delta_c^{-1}<R_0$, projection onto the Euclidean factor gives

$$d_{GH}\Big(B^{\mathbb{R}^k\times Z}_{\delta_c^{-1}}((0,z)),B^{\mathbb{R}^k}_{\delta_c^{-1}}(0)\Big)\le 2d_0.$$

Hence, using (23) and the third inequality in (24)

$$d_{GH}\Big(B_{\delta_c^{-1}}(p),B^{\mathbb{R}^k}_{\delta_c^{-1}}(0)\Big)\le\delta_c,$$

where we also used $C_1\delta\le\delta_c/2$, see (24). Apply Proposition 4.4, and denote its output by G and $\{q_j,t_j,E_j,Y_j,y_j,a_j\}_{j=1}^N$. Since $k\le m$ and $0<t_j\le 1$, we have $\sum_{j=1}^N t_j^m\le\sum_{j=1}^N t_j^k\le C$. Consequently, for some j,

$$\frac{\operatorname{Vol}(E_j)}{t_j^m}\ge\frac{\operatorname{Vol}(G)}{\sum_{\ell=1}^N t_\ell^m}\ge c\operatorname{Vol}B_1(p).$$

Since $E_j\subset B_{Ct_j}(q_j)$,

$$\mathcal{E}_m(q_j,Ct_j)=\frac{\operatorname{Vol}B_{Ct_j}(q_j)}{(Ct_j)^m}\ge c\frac{\operatorname{Vol}(E_j)}{t_j^m}\ge c\operatorname{Vol}B_1(p)=c\mathcal{E}_m(p,1). \tag{25}$$

Let $\gamma:[0,c_0t_j/2]\to Y_j$ be the initial part of a minimizing geodesic from y_j to a_j, and put $\bar y:=\gamma(c_0t_j/4)$, and $s:=\vartheta t_j$. Thus $\bar y$ is the midpoint of a segment of half-length $c_0t_j/4$. Since

$Ls = L\vartheta t_j \le c_0 t_j/32$, the central subsegment of half-length Ls satisfies the segment hypothesis of Lemma 4.1. Moreover, by the choice of A

$$\overline{B}_{Ls}((0,\bar{y})) \subset \overline{B}_{(c_0/4+L\vartheta)t_j}((0,y_j)) \subset \overline{B}_{At_j}((0,y_j)).$$

Choose $q \in M$ corresponding to $(0,\bar{y})$ in the product approximation at q_j. Restriction and recentering give

$$d_{GH}\Big(\overline{B}_{Ls}(q), \overline{B}_{Ls}((0,\bar{y}))\Big) \le C_1 \eta t_j \le \delta_s \vartheta t_j = \delta_s s.$$

Lemma 4.1 therefore shows that $B_s(q) = B_{\vartheta t_j}(q)$ is $(k+1,\varepsilon)$-split.

The proof is now concluded as in Case 1: we have $d(q,q_j) \le Ct_j$, so $B_{Ct_j}(q_j) \subset B_{C't_j}(q)$. Bishop–Gromov monotonicity and (25) yield, possibly by changing constants c, C,

$$\begin{aligned}
\mathcal{E}_m(q,\vartheta t_j) &= (\vartheta t_j)^{n-m}\frac{\operatorname{Vol} B_{\vartheta t_j}(q)}{(\vartheta t_j)^n} \\
&\ge \frac{\vartheta^{n-m}}{(C')^n}\frac{\operatorname{Vol} B_{C't_j}(q)}{t_j^m} \ge c\frac{\operatorname{Vol} B_{Ct_j}(q_j)}{t_j^m} = c\mathcal{E}_m(q_j, Ct_j) \ge c\mathcal{E}_m(p,1),
\end{aligned}$$

thus concluding the proof. □

5. Volume bounds from positive intermediate curvature

We can now state and prove the main result.

Theorem 5.1. *Let n, m be integers such that $n \ge 2$ and $0 \le m \le n-2$, and let $\sigma > 0$. Let (M^n, g) be a complete connected Riemannian manifold satisfying*

$$\operatorname{Ric}_g \ge 0, \qquad \mathfrak{K}_{m,g} \ge \sigma^2.$$

Then there is a constant $C = C(n,m) < \infty$ such that

$$\operatorname{Vol}_g B_R(p) \le C(n,m)\sigma^{-(n-m)}R^m, \tag{26}$$

for every $p \in M$ and every $R > 0$.

Proof. Let $\varepsilon_{m+1} := \varepsilon_H(n,m)$ be the constant in Corollary 3.2. For $k = m, m-1, \dots, 0$, choose recursively

$$\varepsilon_k := \delta(n,m,k,\varepsilon_{k+1}), \qquad c_k := c(n,m,k,\varepsilon_{k+1}),$$

where δ and c are the constants in Proposition 4.5.

Fix a ball $B_R(p) \subset M$. Since this ball is $(0,\varepsilon_0)$-split, successive applications of Proposition 4.5 give balls $B_{r_k}(p_k)$, $0 \le k \le m+1$, such that

$$r_0 = R, \qquad p_0 = p,$$

$B_{r_k}(p_k)$ is (k,ε_k)-split, and

$$\mathcal{E}_m(p_{k+1}, r_{k+1}) \ge c_k \mathcal{E}_m(p_k, r_k) \qquad \forall 0 \le k \le m. \tag{27}$$

The final ball is $(m+1,\varepsilon_H)$-split, so Corollary 3.2 and the absolute Bishop bound in (4) give

$$\mathcal{E}_m(p_{m+1}, r_{m+1}) \le \omega_n r_{m+1}^{n-m} \le C(n,m)\sigma^{-(n-m)}.$$

Iterating (27) yields

$$\frac{\operatorname{Vol} B_R(p)}{R^m} = \mathcal{E}_m(p_0, r_0) \le \left(\prod_{k=0}^m c_k^{-1}\right)\mathcal{E}_m(p_{m+1}, r_{m+1}) \le C(n,m)\sigma^{-(n-m)},$$

which is (26), as desired. □

Since $\mathfrak{K}_{n-2} = \operatorname{Scal}$, Theorem 1.1 follows from Theorem 5.1 with $m = n-2$ and $\sigma = 1$.

Remark 5.2 (Sharp exponent). The exponent m in Theorem 5.1 is sharp. The product $\mathbb{R}^m \times \mathbb{S}^{n-m}$ has $\mathrm{Ric} \geq 0$, uniformly positive $\mathcal{C}_{m+1}$, and volume growth of exact order R^m.

Proof of Corollary 1.2. The case $m = 0$ is a consequence of Bonnet–Myers theorem. We may therefore assume $1 \leq m \leq n-2$.

By Theorem 5.1, there is $A = A(n,m)$ such that

$$\mathrm{Vol}_g B_R(p) \leq AR^m, \tag{28}$$

for every $p \in M$ and $R > 0$. Fix $x \in M$ and $r \geq 2$. In the closed annulus $\overline{B}_{2r}(x) \setminus B_r(x)$ choose a maximal family of points $p_1, \ldots, p_N$ for which the open balls $B_{1/10}(p_i)$ are pairwise disjoint. Bishop–Gromov comparison and the noncollapsing assumption give

$$\mathrm{Vol}_g B_{1/10}(p_i) \geq 10^{-n} \, \mathrm{Vol}_g B_1(p_i) \geq 10^{-n} v_0.$$

All these balls lie in $B_{2r+1/10}(x)$; hence (28) implies $N \leq C_0(n,m,v_0) r^m$.

By maximality, the balls $B_{1/5}(p_i)$ cover the annulus. Consider the distance spheres $\partial B_j(x)$ for $j \in \mathbb{N} \cap (r, 2r)$. Each open 1/5-ball meets at most one of these unit-spaced spheres. Since $N \leq C_0 r^m$, there is $j \in (r, 2r)$ for which $\partial B_j(x)$ is covered by at most $C_1(n,m,v_0) r^{m-1}$ balls of radius 1/5. Denoting by HC the Hausdorff content, we thus have

$$\mathrm{HC}_m(\partial B_j(x)) \leq C_2(n,m,v_0) r^{m-1}. \tag{29}$$

Let $\varepsilon_{m+1} > 0$ be the constant in [21, Theorem 3.3], and now fix $r = r(n,m,v_0) \geq 2$ so large that the right-hand side of (29) is $\leq \varepsilon_{m+1} r^m$. For each x, set $U_x = B_j(x)$, choosing $j = j(x)$ as above. Then

$$B_r(x) \subset U_x \subset B_{2r}(x) \subset B_{10r}(x), \qquad \mathrm{HC}_m(\partial U_x) \leq \varepsilon_{m+1} r^m.$$

Hence [21, Theorem 3.3] applied with parameter $m+1$ gives $\mathrm{UW}_m(M,g) \leq r$, as desired. $\square$

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