UNIVERSAL VOLUME GROWTH BOUNDS FROM POSITIVE INTERMEDIATE CURVATURE

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Abstract. Let n, m be integers such that $n \geq 2$ and $0 \leq m \leq n-2$. Let $\mathcal{C}_{m+1}$ denote the $(m+1)$-intermediate curvature introduced by Brendle–Hirsch–Johne.

We prove that there are constants $\nu(n,m), C(n,m) > 0$ such that the following holds. If (M^n, g) is complete and connected and, for $\delta \geq 0$,

$$\mathrm{Ric} \geq -\delta^2, \qquad \mathcal{C}_{m+1} \geq 1,$$

then

$$\delta R \leq \nu(n,m) \quad \Longrightarrow \quad \mathrm{Vol}\, B_R(p) \leq C(n,m) R^m \quad \text{for every } p \in M \text{ and } R > 0.$$

In particular, taking $m = n-2$ and $\delta = 0$ gives Gromov's conjectured codimension-two volume growth estimate under $\mathrm{Ric} \geq 0$ and $\mathrm{Scal} \geq 1$.

1. Introduction

In [4, Definition 1.1], Brendle–Hirsch–Johne defined and studied notions of intermediate curvatures interpolating between Ric and Scal on a Riemannian manifold.

Definition 1.1 ([4, Definition 1.1]). Let (M^n, g) be a Riemannian manifold, and let r be an integer such that $1 \leq r \leq n-1$. For an r-plane $W \subset T_xM$ choose an orthonormal basis $\{e_1, \ldots, e_r\}$ of W and extend it to an orthonormal basis $\{e_1, \ldots, e_r, e_{r+1}, \ldots, e_n\}$ of T_xM. The *r-intermediate curvature* of W is

$$\mathcal{C}_r(W) := \sum_{a=1}^{r} \sum_{b=a+1}^{n} \mathrm{Sect}(e_a \wedge e_b). \tag{1}$$

Let $k \in \mathbb{R}$. When we write $\mathcal{C}_r \geq k$ we mean that $\mathcal{C}_r(W) \geq k$ for every r-plane $W \subset T_xM$ and every $x \in M$.

Notice that $\mathcal{C}_1 = \mathrm{Ric}$, $\mathcal{C}_2 = \mathrm{biRic}$ (the biRicci curvature was introduced and studied in [38]), and $\mathcal{C}_{n-1} = \mathrm{Scal}/2$. The aim of this note is to prove the following volume growth result assuming a Ricci curvature lower bound and positive intermediate curvature.

Theorem 1.2. *Let n, m be integers such that $n \geq 2$ and $0 \leq m \leq n-2$. There are constants $\nu(n,m) > 0$ and $C(n,m) > 0$ such that the following holds.*

Let (M^n, g) be a complete connected Riemannian manifold, and let $\delta \geq 0$, $\sigma > 0$. Assume

$$\mathrm{Ric} \geq -\delta^2, \qquad \mathcal{C}_{m+1} \geq \sigma^2.$$

Then, for every $p \in M$ and every $R > 0$,

$$\delta R \leq \nu(n,m) \quad \Longrightarrow \quad \mathrm{Vol}\, B_R(p) \leq C(n,m)\sigma^{-(n-m)} R^m. \tag{2}$$

Taking $m = n-2$, $\delta = 0$, and $\sigma = \frac{\sqrt{2}}{2}$ in Theorem 1.2 we deduce the following.

Corollary 1.3. *Let $n \geq 2$, and let (M^n, g) be a complete connected Riemannian manifold satisfying*

$$\mathrm{Ric} \geq 0, \qquad \mathrm{Scal} \geq 1.$$

There is a constant $C = C(n) > 0$ such that

$$\mathrm{Vol}\, B_R(p) \leq C(n) R^{n-2} \qquad \text{for every } p \in M \text{ and every } R > 0.$$

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Corollary 1.3 answers in the affirmative a question posed by Gromov in [15, §2.A(b)].

Theorem 1.2 can be used to bound the m-dimensional Urysohn width of (M^n, g). Let $m \geq 0$ be an integer. The *m-dimensional Urysohn width* of a metric space X is the infimum, over all continuous maps $f\colon X \to P$ to simplicial complexes of dimension at most m, of $\sup_{y\in P} \operatorname{diam}(f^{-1}(y))$. Gromov asked whether every complete n-dimensional Riemannian manifold with $\mathrm{Scal} \geq \sigma^2 > 0$ satisfies $\mathrm{UW}_{n-2}(M) \leq C(n)\sigma^{-1}$; see, e.g., [15, § 2.A(c)]. The three-dimensional case of this conjecture is known after [30, Theorem 1.1], and [31, Theorem 1.1]. The analogous macroscopic statement, with positive macroscopic scalar curvature (see [18]) in place of a pointwise scalar-curvature lower bound, is false in every dimension $n \geq 4$, as recently shown by Kumar and Sen [27, Theorem A].

Combining Theorem 1.2 with [36, Theorem 3.3] (see also [29, Theorem 1.6] and the related papers [19, 20, 35]) gives the following noncollapsed form of the conjectured width-estimate for all the intermediate curvatures considered here, under an almost non-negative Ricci lower bound.

Theorem 1.4. *Let $n \geq 2$, $0 \leq m \leq n-2$, and $v_0 > 0$. There exist constants $C = C(n, m, v_0) > 0$ and $0 < \delta = \delta(n, m, v_0) < 1$ with the following property. If (M^n, g) is a complete connected Riemannian manifold satisfying*

$$\mathrm{Ric} \geq -\delta^2, \qquad \mathcal{C}_{m+1} \geq 1, \qquad \inf_{p\in M} \operatorname{Vol} B_1(p) \geq v_0,$$

then $\mathrm{UW}_m(M, g) \leq C$. In particular, since $\mathcal{C}_{n-1} = \mathrm{Scal}/2$, we obtain

$$\mathrm{Ric} \geq -\delta(n, v_0)^2, \qquad \mathrm{Scal} \geq 1, \qquad \inf_{p\in M} \operatorname{Vol} B_1(p) \geq v_0 \quad \Longrightarrow \quad \mathrm{UW}_{n-2}(M, g) \leq C(n, v_0).$$

Even when $\mathrm{Ric} \geq 0$, up to the author's knowledge, it is unknown whether the assumption $\inf_{p\in M} \operatorname{Vol} B_1(p) \geq v_0$ in Theorem 1.4 can be dropped in dimension $n \geq 4$.

Finally, we mention a consequence of our results for rationally essential manifolds. The aspherical positive-scalar-curvature conjecture predicts that a closed aspherical manifold admits no metric of positive scalar curvature (it is solved for dimension $n \leq 5$, see [9, 16]), while its stronger rational-essential version replaces asphericity with the condition (3), see, e.g., [17, Section 3.2] (Note that every closed oriented aspherical manifold is rationally essential).

Definition 1.5. Let M^n be a closed connected oriented manifold, let $\Gamma = \pi_1(M)$, and let $c_M\colon M \to B\Gamma$ be a classifying map. We say that M is *rationally essential* if

$$(c_M)_*[M]_{\mathbb{Q}} \neq 0 \quad \text{in } H_n(B\Gamma; \mathbb{Q}). \tag{3}$$

If M is closed with $\mathrm{Ric} \geq 0$, Cheeger–Gromoll [7] gives $\widetilde{M} = N^{n-k} \times \mathbb{R}^k$, where N is simply connected and compact, $0 \leq k \leq n$, and $\pi_1(M)$ contains a finite-index subgroup $\cong \mathbb{Z}^k$. Thus if M is rationally essential, we must have $k = n$, hence $\widetilde{M} = \mathbb{R}^n$, and thus M is flat. The point of the following theorem is that, under a dimensionally controlled negative Ricci lower bound, being rationally essential rules out having a positive lower bound on any intermediate curvature; in particular it rules out having a positive lower bound on Scal. The proof of Theorem 1.6 is a direct consequence of Theorem 1.2 and [3, Theorem 1.3]. For the conclusion on the simplicial volume, compare with [15, Section 3.A.]. For a related result, see [32, Corollary 2.7].

Theorem 1.6. *Let $n \geq 2$ and $0 \leq m \leq n-2$. There exists a constant $\varepsilon = \varepsilon(n, m) > 0$ with the following property. If (M^n, g) is closed, connected, and oriented, and*

$$\mathrm{Ric} \geq -\varepsilon^2, \qquad \mathcal{C}_{m+1} \geq 1,$$

then M is not rationally essential. Hence, M is not aspherical and its simplicial volume vanishes.

History of Corollary 1.3 and previous results. Recently, there have been many contributions by different authors related to Corollary 1.3. We list some of them:

(1) Under the stronger assumption $\mathrm{Sec} \geq 0$, Corollary 1.3 follows from Petrunin's local upper bound for the scalar-curvature integral [37];

(2) B. Zhu stated the linear volume growth conclusion of Corollary 1.3 in dimension three under a uniform lower bound for the volumes of unit balls in [44, Theorem 1.8]. In n dimensions he stated an $O(R^{n-1})$ estimate under the same noncollapsing assumption and an $O(R^{n-2})$ estimate under a uniform injectivity-radius lower bound [44, Corollary 1.9];
(3) From this item on, in this list, assume $n = 3$ and the assumptions of Corollary 1.3. Munteanu–Wang established the linear-growth estimate with a universal constant and also considered scalar-curvature lower bounds decaying at infinity [33];
(4) Chodosh–Li–Stryker subsequently gave a different proof of the result in [33] and extended the power-law decay range, with a constant C depending on the manifold and basepoint [10];
(5) Wei–Xu–Zhang obtained the sharp upper bound for $\limsup_{R\to\infty} \operatorname{Vol} B_R(p)/R$ [41];
(6) Y. Wang proved an effective local volume upper bound [40];
(7) Huang–Liu obtained sharp asymptotic estimates in the asymptotically nonnegative-Ricci setting, under some decay assumptions [22].

In the Ricci-limit setting, Wang–Xie–B. Zhu–X. Zhu [39] excluded an $\mathbb{R}^{n-1}$-splitting for noncollapsed limits of complete n-dimensional Riemannian manifolds with $\operatorname{Ric} \geq 0$ and $\operatorname{Scal} \geq 1$; see their Theorem 1.1(1) and the earlier result [45]. In [39] they also proved $\inf_{p\in M} \operatorname{Vol} B_r(p) \leq c(n)r^{n-2}$ under $\operatorname{Ric} \geq 0$ and $\operatorname{Scal} \geq 1$; see [39, Theorem 1.9].

X. Zhu obtained related results in [46]. For example, he proved that asymptotic cones of manifolds with $\operatorname{Ric} \geq 0$, $\operatorname{Scal} \geq 1$, and $\inf_{p\in M} \operatorname{Vol} B_1(p) > 0$ have essential dimension at most $n-2$; see [46, Theorem 1.6]. Note that Corollary 3.2 below allows one to remove the noncollapsed assumption from both [46, Theorem 1.6] and [39, Theorem 1.1(1)].

As discussed in [39, Remark 1.5], for a noncompact M satisfying the assumptions of Corollary 1.3, we can use Corollary 1.3 to show $b_1(M) \leq n-3$ (thus removing the noncollapsing hypothesis from the inequality on b_1 in [46, Theorem 1.3]). The estimate on the first Betti number also has a rigidity statement. Let (M^n, g) be a noncompact Riemannian manifold as in the assumptions of Corollary 1.3. By [1] and Corollary 1.3 applied to the universal cover of M we get that if $b_1(M) = n-3$, then M has at most linear volume growth (and thus exactly linear volume growth by Calabi–Yau lower volume bound). It then follows from [21, Theorem A and Remark 1.2(2)] that $\widetilde{M}$ splits off isometrically an $\mathbb{R}^{n-3}$-factor. This answers the question raised in [46, Remark 5.4] without a uniform noncollapsing assumption. Similar results hold for positive lower bounds on arbitrary intermediate curvatures.

Let us comment further on Theorem 1.2. BiRicci curvature (corresponding to the case $m = 1$ in Theorem 1.2) was introduced by Shen–Ye [38]. In [2, Theorem 3(1)] we proved linear volume growth in dimensions $3 \leq n \leq 5$ under $\operatorname{Ric} \geq 0$ and a uniformly positive spectral biRicci condition outside a compact set; when the spectral parameter is zero, this includes a pointwise biRicci lower bound. Zhou–Zhu [43] later extended the result to $3 \leq n \leq 7$. Note that Theorem 1.2 shows that $\operatorname{Ric} \geq 0$ and $\operatorname{biRic} \geq 1$ imply that M has linear volume growth (and then bounded isoperimetric profile), thus answering the global pointwise version of the first bullet of [2, Question 1] in the negative in every dimension $n \geq 3$.

Corollary 1.3 is closely related to a more general problem posed by Yau [42], which asks whether every complete noncompact n-dimensional Riemannian manifold with $\operatorname{Ric} \geq 0$ satisfies

$$\limsup_{r\to\infty} r^{2-n} \int_{B_r(p)} \operatorname{Scal}\, dV < \infty.$$

Under the additional assumption $\operatorname{Scal} \geq 1$, a solution of Yau's conjecture would immediately imply the volume growth estimate in Corollary 1.3. Up to the author's knowledge, Yau's conjecture remains open in every dimension $n \geq 3$. For recent progress towards this problem see, e.g., [11, 34].

Ideas of the proof. The proof of Theorem 1.2 has two main ingredients. First, a ball which is quantitatively close to splitting $m+1$ Euclidean directions has radius at most the curvature scale; this is Corollary 3.2. Moreover, if a ball is close to split only $k \leq m$ directions, Proposition 4.5 finds another ball which is close to split $k+1$ directions while losing only a harmless multiplicative factor in the m-volume ratio. Hence, starting at rank $k=0$ and iterating finitely many times proves the theorem.

The proof of Proposition 4.5 uses tools contained in the proof of Kapovitch–Wilking rescaling theorem [24, Theorem 5.1]. We are also inspired by the result in [23, Lemma 2.12], which contains a closely related finite formulation of [24, Theorem 5.1]; compare with our Proposition 4.4. In order to prove Proposition 4.4 we do not need to use the full strength of [24, Theorem 5.1]; see the paragraph before the statement of Proposition 4.4. We refer the reader to the beginning of the proof of Proposition 4.5 for an explanation of its proof.

The Hodge obstruction and the rank-improvement mechanism described above are inspired by the arguments in [11, 39, 46]; in particular, the Kapovitch–Wilking rescaling theorem [24, Theorem 5.1] is also a central input in the proof of [46, Theorem 1.6].

Moreover, at least two closely related uses of harmonic splitting maps and the Bochner formula precede the argument of Proposition 3.1 and Corollary 3.2. In [39, Remark 3.1], the authors give an alternative proof of [39, Theorem 1.1(1)&(2)] using harmonic splitting maps with computations similar to the ones in Proposition 3.1. On the other hand, in [11], the authors use k-component splitting maps to obtain integral control of the sum R_k of the lowest k Ricci eigenvalues on k-symmetric balls; see [11, Lemma 3.2 and Theorem 3.4]. Moreover, in [11, Theorem 3.8(2)] the authors prove a quantitative obstruction to an $\mathbb{R}^{n-1}$-splitting under almost nonnegative Ricci curvature, integral almost-nonnegativity of Ric_{n-2}, and integral positivity of the truncated scalar curvature. The key new feature in the proof of Proposition 3.1 is the addition of a contribution coming from the wedge of the one-forms β^a to make the intermediate curvature appear: compare with (8). We further notice that a large part of the proof in this note is likely to work in the setting of metric-measure spaces as well; see Remark 4.6.

For Theorem 1.4, noncollapsing and Bishop–Gromov comparison convert the $O(r^m)$ volume bound into an $O(r^m)$ packing bound for a large annulus. Averaging over $O(r)$ distance spheres gives an $O(r^{m-1})$ covering bound for one boundary sphere, and [36, Theorem 3.3] then gives bounded m-dimensional Urysohn width.

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Disclosure of AI tools. This work made substantial use of OpenAI's GPT-5.6 Sol with Ultra reasoning effort and Codex. GPT proposed the central inductive procedure based on the Hodge obstruction and rank improvement behind the proof of Corollary 1.3. I formulated and guided the problem, suggested relevant strategies and literature, and, following the suggested proof strategy of Corollary 1.3, developed this note.

I emphasize that the proof below is quite different from the original argument suggested by GPT. On the one hand, the proof below is effective, while the original suggested strategy proceeded by contradiction: this emerged through discussions with Daniele Semola, to whom I am very grateful. On the other hand, contrary to the original strategy proposed by the LLM, the proof below is less reliant on the full proof of [24, Theorem 5.1]; see the discussion before Proposition 4.4: this emerged through discussions with Elia Bruè and Kai Xu.

After examining the original output, I observed that a modification of the original argument proposed by the LLM gives the extension to the intermediate curvatures of Brendle–Hirsch–Johne, as stated in Theorem 1.2. This extension, motivated by a question I previously had in a joint work with Kai Xu, is my own contribution. Theorems 1.4 and 1.6 emerged through discussions with colleagues, ad described in the Acknowledgments.

Addendum. After this manuscript was completed and circulated privately, I learned of independent work by Jian Ge [12]. Ge proves Corollary 1.3 by a completely different method, based on heat-kernel Fisher metric and Nash entropy. At the same time, I learned of independent work of Bochao Kong and Xingyu Zhu [26]. They also establish Corollary 1.3 using heat-kernel techniques. Both approaches are substantially different from the rank-improvement argument used here. However, I highlight one connection: [26, Lemma 4.2] bears strong similarities with the computations in the proof of Proposition 3.1.

After completing the revision of this manuscript, in which minor modifications of the original proofs led to the present form of Theorem 1.2, I learned of independent work of Robert Koirala [25, Theorem 1.1]. Koirala's result independently implies Theorem 1.2 and sharpens it in the following way: it also gives an all-scale estimate with an exponential factor. His proof, based on heat-kernel techniques, is substantially different from the argument used here.

2. Splitting maps and curvature algebra

2.1. **Intermediate curvature and volume ratio.** Let (M^n, g) be a complete Riemannian manifold. We use the convention

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \qquad K_{ab} = \mathrm{Sect}(e_a\wedge e_b) = \langle R(e_a,e_b)e_b, e_a\rangle,$$

where e_a, e_b are orthonormal. In this way we can define Brendle–Hirsch–Johne's intermediate curvatures as in (1). Recall that $\mathcal{C}_1 = \mathrm{Ric}$, $\mathcal{C}_2 = \mathrm{biRic}$, $\mathcal{C}_{n-1} = \mathrm{Scal}/2$.

For a fixed integer $m \geq 0$ define the *m-volume ratio* for $x \in M$ and $r > 0$

$$\mathcal{E}_m(x,r) := \frac{\mathrm{Vol}\, B_r(x)}{r^m}. \tag{4}$$

2.2. **Metric and harmonic splitting.** Whenever a ball occurs as an argument of d_{GH}, or as the domain or target of a Gromov–Hausdorff approximation, it is understood to be the corresponding closed metric ball. Moreover, we always intend that such comparisons are pointed with respect to the displayed centers.

Definition 2.1. Let $0 < \varepsilon < 1$ and let k, n be integers such that $0 \leq k \leq n$. A ball $B_r(x)$ is *(k,ε)-split* if there are a pointed proper length space (Z, z) and a pointed εr-Gromov–Hausdorff approximation

$$B_{\varepsilon^{-1}r}(x) \longrightarrow B_{\varepsilon^{-1}r}((0,z)) \subset \mathbb{R}^k \times Z.$$

Note that every ball is $(0,\varepsilon)$-split.

The following is the quantitative harmonic replacement associated with Definition 2.1. It is due to Cheeger–Colding; see [5, 6] and [8, Lemma 1.21(2)].

Proposition 2.2. *For all integers n, k such that $1 \leq k \leq n$ and $\eta > 0$ there is $0 < \varepsilon = \varepsilon(n,k,\eta) < 1$ with the following property. Let (M^n, g) be a complete Riemannian manifold, let $x \in M$ and $r > 0$. If $\mathrm{Ric} \geq -\varepsilon r^{-2}$ and $B_r(x)$ is (k,ε)-split, then there are harmonic functions*

$$u = (u^1, \ldots, u^k) : B_{4r}(x) \longrightarrow \mathbb{R}^k$$

such that for every $a = 1, \ldots, k$ on $B_{2r}(x)$ we have $|\nabla u^a| \leq C(n)$ on $B_{2r}(x)$, and moreover

$$\frac{1}{\mathrm{Vol}\, B_{2r}(x)} \int_{B_{2r}(x)} \left(|G - I|^2 + r^2 \sum_{a=1}^k |\mathrm{Hess}\, u^a|^2 \right) dV \leq \eta, \tag{5}$$

where $G^{ab} := \langle \nabla u^a, \nabla u^b\rangle$, and I is the identity matrix. Here $|A|$ denotes the Hilbert–Schmidt norm of the matrix A.

Remark 2.3. Let $\{\varepsilon_i\}_{i\in\mathbb{N}}$ be a sequence of nonnegative numbers such that $\varepsilon_i \to 0$. If (M_i^n, g_i) are complete Riemannian manifolds with $\mathrm{Ric}_{g_i} \geq -\varepsilon_i$ and

$$(M_i, g_i, p_i) \longrightarrow (\mathbb{R}^k, g_{\mathrm{eu}}, 0)$$

in the pointed Gromov–Hausdorff sense, then, up to subsequences, the harmonic functions u_i in Proposition 2.2 can be chosen in the following way:

(1) $u_i : B_i(p_i) \to B_{i+1}^{\mathbb{R}^k}(0)$ and $u_i(p_i) = 0$;
(2) u_i is a 2^{-i}-GH approximation from $B_i(p_i)$ to $B_i^{\mathbb{R}^k}(0)$.

2.3. **The Weitzenböck formula.** Let (M^n, g) be a complete n-dimensional Riemannian manifold, and let $1 \leq p \leq n-1$ be an integer. We shall use the Weitzenböck formula

$$\Delta_H = d\delta + \delta d = \nabla^*\nabla + q_p(\mathrm{Rm}) \qquad \text{on } \Lambda^p T^*M. \tag{6}$$

With our curvature convention, $q_1(\mathrm{Rm}) = \mathrm{Ric}$. The only algebraic fact needed below is the following standard formula.

Lemma 2.4. *Let $\beta^1, \ldots, \beta^n$ be an orthonormal coframe in (M^n, g) and $I \subset \{1, \ldots, n\}$ with $|I| = p$. If $\beta^I = \bigwedge_{a\in I} \beta^a$, then*

$$\langle q_p(\mathrm{Rm})\beta^I, \beta^I\rangle = \sum_{a\in I}\sum_{b\notin I} K_{ab}. \tag{7}$$

Consequently, the following holds. Let $0 \leq m \leq n-2$ be an integer. If $\beta^1, \ldots, \beta^{m+1}$ is an orthonormal coframe for a $(m+1)$-plane W, and $\Theta = \beta^1 \wedge \cdots \wedge \beta^{m+1}$, then

$$\langle q_{m+1}(\mathrm{Rm})\Theta, \Theta\rangle + \sum_{a=1}^{m+1} \langle q_1(\mathrm{Rm})\beta^a, \beta^a\rangle = 2\mathcal{C}_{m+1}(W). \tag{8}$$

Proof. Formula (7) is in [28, equation (11)]. For completeness, the first term in (8) is $\sum_{a\leq m+1<b} K_{ab}$, while the sum of the one-form terms is

$$\sum_{a=1}^{m+1} \mathrm{Ric}((\beta^a)^\sharp, (\beta^a)^\sharp) = 2\sum_{a<b\leq m+1} K_{ab} + \sum_{a\leq m+1<b} K_{ab}.$$

Their sum is twice (1), thus proving (8), as desired. $\square$

3. The quantitative Hodge obstruction

The next proposition is the first core ingredient of our proof.

Proposition 3.1. *Fix an integer $n \geq 2$, and an integer $0 \leq m \leq n-2$. There are constants $\eta(n,m) > 0$ and $C(n,m) < \infty$ with the following property. Let (M^n, g) be a complete Riemannian manifold, let $\sigma > 0$, $0 \leq \kappa \leq 1$, $x \in M$, and $r > 0$. Assume*

$$\mathrm{Ric} \geq -\kappa^2 r^{-2}, \qquad \mathcal{C}_{m+1} \geq \sigma^2,$$

on $B_{2r}(x) \subset M$, and suppose there are harmonic functions $u^1, \ldots, u^{m+1}$ on $B_{4r}(x)$ satisfying (5) (and the gradient bound stated before it) for some $\eta \leq \eta(n,m)$. Then

$$\sigma^2 r^2 \leq C(n,m). \tag{9}$$

Proof. For $1 \leq a, b \leq m+1$, write $\theta^a = du^a$ and $G^{ab} = \langle\theta^a, \theta^b\rangle$. Let I be the $(m+1)\times(m+1)$-identity matrix. $|A|$ denotes the Hilbert–Schmidt norm of the matrix A. Fix a sufficiently small $\eta_0 = \eta_0(n,m) > 0$ so that on

$$\Omega := \{|G - I| < 3\eta_0\}$$

the matrix G is invertible. On Ω define

$$\begin{pmatrix} \beta^1 \\ \vdots \\ \beta^{m+1} \end{pmatrix} := G^{-1/2} \begin{pmatrix} \theta^1 \\ \vdots \\ \theta^{m+1} \end{pmatrix}, \qquad \Theta := \beta^1 \wedge \cdots \wedge \beta^{m+1}.$$

Thus $\beta^1, \ldots, \beta^{m+1}$ are orthonormal on Ω.

Choose a smooth function $\chi = \chi(|G - I|^2)$ which equals one when $|G - I| \leq \eta_0$, vanishes when $|G - I| \geq 2\eta_0$, and takes values in $[0, 1]$. Since

$$\nabla G^{ab} = \operatorname{Hess} u^a(\,\cdot\,, \nabla u^b) + \operatorname{Hess} u^b(\,\cdot\,, \nabla u^a),$$

the gradient bound stated before (5) and differentiation of $A \mapsto A^{-1/2}$ give

$$|\nabla\chi|^2 + |\nabla\Theta|^2 + \sum_{a=1}^{m+1} |\nabla\beta^a|^2 \leq C(n, m, \eta_0) \sum_{a=1}^{m+1} |\operatorname{Hess} u^a|^2, \tag{10}$$

on Ω. Note that the forms $\chi\Theta$ and $\chi\beta^a$ extend smoothly by zero across the boundary of Ω.

Let f be a smooth cutoff function supported in $B_{2r}(x)$, equal to one on $B_r(x)$, and satisfying $|\nabla f| \leq C(n)/r$. Set

$$\widetilde{\omega} := f\chi\Theta, \qquad \omega^a := f\chi\beta^a \quad \forall 1 \leq a \leq m+1.$$

Integrating the identity (6) gives, for any compactly supported p-form ξ,

$$\int \langle q_p(\mathrm{Rm})\xi, \xi\rangle = \int (|d\xi|^2 + |\delta\xi|^2 - |\nabla\xi|^2) \leq C(n) \int |\nabla\xi|^2. \tag{11}$$

Equation (8) and the curvature hypothesis give the pointwise lower bound on $B_{2r}(x)$:

$$\langle q_{m+1}(\mathrm{Rm})\widetilde{\omega}, \widetilde{\omega}\rangle + \sum_{a=1}^{m+1} \langle q_1(\mathrm{Rm})\omega^a, \omega^a\rangle \geq 2\sigma^2 f^2\chi^2.$$

Summing (11) for ξ in the list $(\widetilde{\omega}, \omega^1, \ldots, \omega^{m+1})$ and using (10) and (5) therefore yields

$$\begin{aligned} 2\sigma^2 \int_{B_{2r}(x)} f^2\chi^2 &\leq C(n, m, \eta_0) \left(r^{-2} \operatorname{Vol} B_{2r}(x) + \int_{B_{2r}(x)} \sum_{a=1}^{m+1} |\operatorname{Hess} u^a|^2 \right) \\ &\leq C(n, m, \eta_0)(1 + \eta) r^{-2} \operatorname{Vol} B_{2r}(x). \end{aligned} \tag{12}$$

It remains to bound the left-hand side of the above inequality from below. Markov's inequality and (5) give

$$\operatorname{Vol}\big(B_{2r}(x) \cap \{|G - I| > \eta_0\}\big) \leq \eta_0^{-2}\eta \operatorname{Vol} B_{2r}(x),$$

whereas Bishop–Gromov gives $\operatorname{Vol} B_r(x) \geq c(n) \operatorname{Vol} B_{2r}(x)$. Choose $\eta(n, m) < 1$ so small that $\eta(n, m) \leq c(n)\eta_0^2/2$. Since $f = \chi = 1$ on $B_r(x) \cap \{|G - I| \leq \eta_0\}$, we thus deduce, using the previous inequalities,

$$\int f^2\chi^2 \geq c(n) \operatorname{Vol} B_{2r}(x)/2.$$

Combining this with (12) proves (9), as desired. □

Combining Propositions 2.2 and 3.1 gives the following.

Corollary 3.2. *For every integer $n \geq 2$ and integer $0 \leq m \leq n-2$ there are $0 < \varepsilon_H(n, m) < 1$ and $C(n, m) > 0$ such that the following holds. Let (M^n, g) be a complete Riemannian manifold, and fix $x \in M$, $r, \sigma > 0$. Let us assume that*

$$\mathrm{Ric} \geq -\varepsilon_H r^{-2}, \qquad \mathcal{C}_{m+1} \geq \sigma^2,$$

and $B_r(x)$ is $(m+1, \varepsilon_H)$-split. Then

$$r \leq C(n, m)\sigma^{-1}.$$

4. Critical scales and effective rank improvement

We record a consequence of Cheeger–Colding's almost splitting theorem [5, 6]. One can give a simple proof by contradiction of Lemma 4.1, based on Gigli's splitting theorem [13].

Lemma 4.1. *For every integer $n \geq 1$, every integer $0 \leq k < n$, and $0 < \varepsilon < 1$, there are $L = L(n,k,\varepsilon) > \varepsilon^{-1}$ and $0 < \delta_s = \delta_s(n,k,\varepsilon) < 1$ with the following property. Let (M^n, g) be a complete Riemannian manifold, $s > 0$, assume $\mathrm{Ric} \geq -\delta_s s^{-2}$, and let (Z, d_Z, z_0) be a pointed proper geodesic space. Assume the closed ball $\overline{B}_{Ls}(q)$ is $\delta_s s$-close to the closed Ls-ball centered at $(0, z_0)$ in $\mathbb{R}^k \times Z$. If z_0 is the midpoint of a minimizing segment $\gamma : [-Ls, Ls] \to Z$, then $B_s(q)$ is $(k+1, \varepsilon)$-split.*

Remark 4.2. Corollary 3.2 immediately gives the following diameter bound. If $0 \leq m \leq n-2$, $\{\varepsilon_i\}_{i \in \mathbb{N}}$ is an infinitesimal sequence, and

$$(M_i, g_i, p_i) \longrightarrow (\mathbb{R}^m \times K, (0, z)), \qquad \mathrm{Ric}_{g_i} \geq -\varepsilon_i, \quad \mathcal{C}_{m+1, g_i} \geq \sigma^2 > 0,$$

then

$$\operatorname{diam} K \leq C(n,m)\sigma^{-1}. \tag{13}$$

Indeed, recentering at the midpoint of a long segment in K and applying Lemma 4.1 produces an $(m+1, \varepsilon_H)$-split ball of comparable radius, which is bounded by Corollary 3.2. In the scalar-curvature case this extends the corresponding diameter results of [45] and [39, Theorem 1.1]. Note that the present formulation requires no noncollapsing assumption and applies to all the intermediate curvatures $\mathcal{C}_{m+1}$. The optimal constant $C(n,m)$ in (13) is presently unknown. The optimal constant C in Corollary 1.3 and Theorem 1.2 is also unknown in general.

We state an elementary lemma. For a Lipschitz $\mathbb{R}^k$-valued map b defined on a ball $B_2(q) \subset (M^n, g)$, put, for $s \in (0,1)$, and d being the Riemannian distance on M,

$$D_{q,b}(s) := \sup_{x,y \in \overline{B}_s(q)} \big| d(x,y) - |b(x) - b(y)| \big|. \tag{14}$$

Lemma 4.3. *The function $s \mapsto D_{q,b}(s)$ is locally Lipschitz on $(0,1)$.*

Proof. The function $(x,y) \mapsto |d(x,y) - |b(x) - b(y)||$ is Lipschitz on $\overline{B}_s(q) \times \overline{B}_s(q)$, with Lipschitz constant controlled by $1 + \mathrm{Lip}(b)$. Since $d_H(\overline{B}_s(q), \overline{B}_{s'}(q)) \leq |s - s'|$ for every $s, s' \in (0,1)$, the conclusion follows. □

The next proposition relies on ideas in the proof of the Kapovitch—Wilking rescaling theorem [24, Theorem 5.1], and it is inspired by Jansen's formulation [23, Lemma 2.12].

We stress that, using the full strength of [24, Theorem 5.1] (namely, the fact that the residual factor K in there is unique), one can prove a more general version of Proposition 4.4 below. We decided to include the weaker statement below for two reasons: first, it has the advantage that its proof (which is inspired by [24, 23]) is essentially self-contained as it only leverages basic facts of Cheeger–Colding theory, [24, Lemma 2.1], and a classical maximal-function argument; second, Proposition 4.4 is enough to prove our sought Proposition 4.5.

Proposition 4.4. *Fix an integer $n \geq 1$, an integer $0 \leq k < n$, and $0 < A < \infty$, $0 < \eta < 1$. There are $0 < \delta_c = \delta_c(n,k,A,\eta) < 1$, $c = c(n) > 0$, $C = C(k,n) < \infty$, and $c_0 = c_0(n) > 0$ with the following property. Suppose (M^n, g, p) is complete, has $\mathrm{Ric}_g \geq -\delta_c$, and*

$$d_{GH}(B_{\delta_c^{-1}}(p), B^{\mathbb{R}^k}_{\delta_c^{-1}}(0)) \leq \delta_c.$$

Then there are a measurable set $G \subset B_{1/4}(p)$ and an integer $N \geq 1$ for which the following holds. For every $1 \leq j \leq N$ there are $q_j \in B_{1/4}(p)$, $0 < t_j \leq 1$, pairwise disjoint measurable $E_j \subset G$, pointed proper geodesic spaces (Y_j, d_{Y_j}, y_j), and $a_j \in Y_j$ such that:

(i) *$G = \sqcup_{j=1}^N E_j$ and $\mathrm{Vol}(G) \geq c \operatorname{Vol} B_1(p)$;*
(ii) *$\sum_{j=1}^N t_j^k \leq C$, and, for every $1 \leq j \leq N$, $E_j \subset B_{Ct_j}(q_j)$;*

(iii) *For every* $1 \le j \le N$, $d_{Y_j}(y_j, a_j) \ge c_0 t_j$, *and*

$$d_{GH}(B_{At_j}(q_j), B^{\mathbb{R}^k \times Y_j}_{At_j}((0, y_j))) \le \eta t_j.$$

Proof. Set $c(n) := 4^{-n-1}$, $c_*(n) := 10^{-n^2}$, $c_0(n) := c_*(n)/4$ and fix $C(k,n) > 1$ large enough so that

$$1 - C(k,n)^{-1}/2 > 5c_*/4, \qquad C(k,n) \ge \alpha(k,n)(2\sqrt{k}\mathcal{K}(n) + 1)^k, \tag{15}$$

where $\mathcal{K}(n)$ is the constant in the gradient bound stated before (5), and $\alpha(k,n)$ is chosen large enough, only depending on k, n (in order for (20) below to hold).

The case $k = 0$ is elementary. In that case, the assertion is true choosing $\delta_c \ll 1$ and: $N = 1$, $G = E_1 = B_{1/4}(p)$, $q_1 = p$, $t_1 = \operatorname{diam} M < \infty$, $(Y_1, d_{Y_1}, y_1) = (M, d_g, p)$ and $a_1 \in M$ such that $d(a_1, p) \ge \operatorname{diam} M/2$. Hence, from now on, assume $k \ge 1$.

Suppose that no $\delta_c > 0$ works with the choices of constants above. Taking $\delta_i \to 0$ we thus have a contradicting sequence

$$(M_i, g_i, p_i) \longrightarrow (\mathbb{R}^k, g_{\mathrm{eu}}, 0)$$

with $\mathrm{Ric}_{g_i} \ge -\delta_i$ for which the conclusion is false. Proposition 2.2 (see Remark 2.3) and a diagonal argument give radii $R_i \to \infty$ and harmonic maps

$$b_i = (b_i^1, \dots, b_i^k) : B_{R_i}(p_i) \longrightarrow \mathbb{R}^k, \qquad b_i(p_i) = 0,$$

such that the following hold:

(a) For every $R > 0$, the maps b_i have a uniform Lipschitz bound on $B_R(p_i)$ for i large enough, and their distance distortion on $B_R(p_i)$ tends to zero as $i \to \infty$;

(b) Let $h_i := \sum_{a,b=1}^k \left|\langle \nabla b_i^a, \nabla b_i^b \rangle - \delta_{ab}\right|^2 + \sum_{a=1}^k |\operatorname{Hess} b_i^a|^2$. For every fixed $R \ge 2$,

$$\frac{1}{\operatorname{Vol} B_R(p_i)} \int_{B_R(p_i)} h_i \, dV_{g_i} \longrightarrow 0.$$

Set $\alpha_i := i^{-1} + \left(\frac{1}{\operatorname{Vol} B_2(p_i)} \int_{B_2(p_i)} h_i \, dV_{g_i}\right)^{1/2} > 0$, and define

$$H_i := \left\{ q \in B_{1/4}(p_i) : \sup_{0<r\le 3/4} \frac{1}{\operatorname{Vol} B_r(q)} \int_{B_r(q)} h_i \, dV_{g_i} \le \alpha_i \right\}.$$

Notice $\alpha_i \to 0$. A standard maximal-function estimate and Bishop–Gromov comparison imply

$$\frac{\operatorname{Vol}(B_{1/4}(p_i) \setminus H_i)}{\operatorname{Vol} B_1(p_i)} \longrightarrow 0, \qquad \operatorname{Vol}(H_i) \ge \frac{1}{2} 4^{-n} \operatorname{Vol} B_1(p_i), \tag{16}$$

for all sufficiently large i.

For $q \in H_i$ define the distortion, for $r \in (0, 3/4]$,

$$D_{q,b_i}(r) := \sup_{x,y \in \overline{B}_r(q)} \left| d_i(x,y) - |b_i(x) - b_i(y)| \right|.$$

Let $\rho_i(q)$ be the largest real number r in $(0, 3/4]$ such that $D_{q,b_i}(r) = c_* r$. It is readily seen that such an r exists for i large enough. Indeed, since $k < n$, one has $\liminf_{r\to 0} \frac{D_{q,b_i}(r)}{r} \ge 1$. On the other hand, property (a) above gives, uniformly for $q \in H_i$, $D_{q,b_i}(3/4) < 3c_*/4$, for all sufficiently large i. Lemma 4.3 therefore gives the existence of such an r. Moreover, notice $\sup_{q \in H_i} \rho_i(q) \le c_*^{-1} \operatorname{dis}(b_i|_{\overline{B}_1(p_i)}) \to 0$ as $i \to \infty$.

We next claim that for every sufficiently large i and every $q \in H_i$ there are a pointed proper geodesic space (Y, d_Y, y) and $a \in Y$ such that

$$d_Y(y, a) \ge c_0, \qquad d_{GH}\left(B_A^{(M_i, \rho_i(q)^{-2} g_i)}(q), B_A^{\mathbb{R}^k \times Y}((0, y))\right) \le \eta. \tag{17}$$

Indeed, if this failed, choose violating points $q_i \in H_i$ and put

$$\widetilde{g}_i := \rho_i(q_i)^{-2} g_i, \qquad u_i := \rho_i(q_i)^{-1}(b_i - b_i(q_i)).$$

By what was proved above $\rho_i(q_i) \to 0$. Fix $R > 0$. For i large enough, the definition of H_i gives

$$\frac{1}{\mathrm{Vol}^{\widetilde{g}_i}(B_R^{\widetilde{g}_i}(q_i))}\left(\int_{B_R^{\widetilde{g}_i}(q_i)}\left(\sum_{a,b=1}^{k}\left|\langle\nabla u_i^a,\nabla u_i^b\rangle-\delta_{ab}\right|^2+\sum_{a=1}^{k}|\operatorname{Hess}_{\widetilde{g}_i} u_i^a|^2\right)dV_{\widetilde{g}_i}\right)\leq(1+\rho_i(q_i)^2)\alpha_i. \tag{18}$$

Hence, using (18), we can apply the result in [24, Lemma 2.1] to get, after passing to subsequences,

$$(M_i,\widetilde{g}_i,q_i,u_i)\longrightarrow(\mathbb{R}^k\times Y,(0,y),\mathrm{pr}_{\mathbb{R}^k}), \tag{19}$$

for a pointed proper geodesic space (Y,y), where $\mathrm{pr}_{\mathbb{R}^k}$ denotes the projection onto the $\mathbb{R}^k$ factor.

Choose $x_i^-, x_i^+ \in \overline{B}_1^{\widetilde{g}_i}(q_i)$ realizing the supremum in $D_{q_i,b_i}(\rho_i(q_i))$. After passing to a subsequence, their limits are $x^\pm = (v^\pm, y^\pm)$ in $\mathbb{R}^k \times Y$. Convergence in (19), and the fact that $D_{q_i,b_i}(\rho_i(q_i)) = c_*\rho_i(q_i)$ give

$$\left|d_{\mathbb{R}^k\times Y}(x^-,x^+)-|v^--v^+|\right|=d_{\mathbb{R}^k\times Y}(x^-,x^+)-|v^--v^+|=c_*.$$

Using the product formula for $d_{\mathbb{R}^k\times Y}$, we thus deduce $d_Y(y^-,y^+)\geq c_*$. Hence, at least one of y^-, y^+ lies at distance at least $c_*/2$ from y. Since Y is geodesic, there is therefore a point $a \in Y$ with $d_Y(y,a) = c_0 = c_*/4$. This and (19) contradict the assumed failure of (17). Thus (17) holds uniformly over H_i, as desired.

We now construct the E_j's. Apply Vitali's covering Lemma to $\left\{B^{\mathbb{R}^k}_{\rho_i(q)/10}(b_i(q)) : q \in H_i\right\}$. We thus have a countable pairwise disjoint subfamily, indexed by points $q_{i,j} \in H_i$, whose five-times-enlargements cover $b_i(H_i)$. Since the functions b_i are $\sqrt{k}\mathcal{K}(n)$-Lipschitz on $B_1(p_i)$, all the pairwise disjoint balls $B_{\rho_i(q_{i,j})/10}(b_i(q_{i,j}))$ lie in $B^{\mathbb{R}^k}_{2\sqrt{k}\mathcal{K}(n)+1}(0)$. Euclidean volume comparison gives (recall (15)), for a constant $\beta(k,n)$,

$$\sum_j \rho_i(q_{i,j})^k\leq\beta(k,n)\mathcal{L}^k(B^{\mathbb{R}^k}_{2\sqrt{k}\mathcal{K}(n)+1}(0))\leq C(n,k). \tag{20}$$

The measurable sets $F_{i,j} := H_i \cap b_i^{-1}\left(B^{\mathbb{R}^k}_{\rho_i(q_{i,j})/2}(b_i(q_{i,j}))\right)$ cover H_i. Define $E_{i,1} := F_{i,1}$ and $E_{i,j} := F_{i,j} \setminus \cup_{\ell<j} F_{i,\ell}$ for $j > 1$. Notice that $\{E_{i,j}\}_j$ are pairwise disjoint. We claim that

$$E_{i,j}\subset B_{C\rho_i(q_{i,j})}(q_{i,j}). \tag{21}$$

If $x \in E_{i,j} \subset F_{i,j}$ and $D = d(x, q_{i,j}) \geq C\rho_i(q_{i,j})$, then $D \leq 1/2$ (recall $H_i \subset B_{1/4}(p_i)$) and (15) gives

$$D_{q_{i,j},b_i}(5D/4)\geq D-|b_i(x)-b_i(q_{i,j})|\geq D-\rho_i(q_{i,j})/2>c_*\,5D/4.$$

Since $5D/4 < 3/4$, the latter inequality and $D_{q_{i,j},b_i}(3/4) < 3c_*/4$ (for i large enough) together with Lemma 4.3, gives a contradiction to the definition of $\rho_i(q_{i,j})$ and proves (21). Indeed, by the previous inequalities and Lemma 4.3, there is $r \in (5D/4, 3/4)$ such that $D_{q_{i,j},b_i}(r) = c_*r$. Consequently, by definition of $\rho_i(q_{i,j})$, we have $\rho_i(q_{i,j}) \geq r > 5D/4 > D \geq C\rho_i(q_{i,j}) > \rho_i(q_{i,j})$, a contradiction.

Now, let us retain finitely many $\{E_{i,j}\}_{j=1}^{N_i}$ and set $G_i := \sqcup_{j=1}^{N_i} E_{i,j}$ so that $\mathrm{Vol}(G_i) \geq \mathrm{Vol}(H_i)/2$. For each $q_{i,j}$, denote $\widehat{Y}_{i,j} := (Y_{i,j}, \rho_i(q_{i,j})d_{Y_{i,j}}, y_{i,j})$ where $Y_{i,j}$ is the pointed metric space in (17) associated with $q_{i,j}$; retain also the associated points $a_{i,j} \in \widehat{Y}_{i,j}$. Now, for one fixed sufficiently large i, equations (16), (17), (20), and (21) give all three conclusions of the proposition with $G = G_i$, $N = N_i$, $q_j = q_{i,j}$, $t_j = \rho_i(q_{i,j})$, $E_j = E_{i,j}$, $a_{i,j} \in \widehat{Y}_{i,j}$ defined above, thus resulting in a contradiction. □

We now prove the second core ingredient of the proof.

Proposition 4.5. *Fix an integer $n \geq 2$, integers m, k such that $0 \leq m \leq n-2$, $0 \leq k \leq m$, and fix $0 < \varepsilon < 1$. There are $0 < \delta = \delta(n,m,k,\varepsilon) < 1$ and $c = c(n,m,k,\varepsilon) > 0$ such that the following holds. Let (M^n, g) be a complete Riemannian manifold, $p \in M$ and $r > 0$. If $\mathrm{Ric} \geq -\delta r^{-2}$ on M and $B_r(p)$ is (k,δ)-split, then there are $q \in M$ and $0 < s \leq r$ such that $B_s(q)$ is $(k+1,\varepsilon)$-split and*

$$\mathcal{E}_m(q,s)\geq c\,\mathcal{E}_m(p,r). \tag{22}$$

Proof. All constants denoted by $c, C, C' > 0$ in this proof depend only on n, m, k, ε and may change from line to line. It is enough to prove the rescaled version with $r = 1$. So, from now on, let us assume $r = 1$. Notice

$$\mathcal{E}_m(p, 1) = \operatorname{Vol} B_1(p).$$

Assume $B_1(p)$ is (k, δ)-split. Let

$$\Phi : B_{\delta^{-1}}(p) \longrightarrow B_{\delta^{-1}}((0, z)) \subset \mathbb{R}^k \times Z \tag{23}$$

be the δ-approximation given by Definition 2.1.

The proof is divided into two cases. If the residual factor Z has a long enough segment through z, Lemma 4.1 directly produces an additional Euclidean direction. Otherwise Proposition 4.4 gives a collection of good cells E_j in M at different scales, with bounded diameter. Since their scales are summable, we can select one cell carrying a definite fraction of the m-volume ratio. A segment in the space Y_j associated to such a cell supplies the additional Euclidean direction. Let us carefully perform this strategy.

Let $L(n, k, \varepsilon) > \varepsilon^{-1} > 1, 0 < \delta_s(n, k, \varepsilon) < 1$ be the constants in Lemma 4.1, and let c_0 be the constant in Proposition 4.4. Choose

$$0 < \vartheta \le c_0/(32L), \quad 0 < \eta \le \min\{1/2, \delta_s\vartheta/C_1\}, \quad A > c_0/4 + L\vartheta + 2, \quad d_0 = \delta_c/20, \quad R_0 = 4\delta_c^{-1},$$

where C_1 is a large universal constant that controls restriction and recentering of a Gromov–Hausdorff approximation, and $\delta_c(n, k, A, \eta) < 1$ is the corresponding constant in Proposition 4.4. Finally choose δ sufficiently small so that $0 < \delta \le \min\{1, \delta_s, \delta_c\}$ and:

$$1 + \frac{3d_0}{4} \le \delta^{-1}, \qquad C_1\delta \le \frac{\delta_s d_0}{4L}, \qquad \delta_c^{-1} < \frac{\delta^{-1} + 1}{10}, \quad C_1\delta \le \frac{\delta_c}{2}. \tag{24}$$

Case 1: There is a long enough segment through z in Z. Suppose that some $z' \in B_{R_0}(z)$ satisfies $d_Z(z, z') \ge d_0$. Let $\gamma : [0, d_0] \to Z$ be the unit-speed initial segment of a minimizing geodesic from z to z'. Set

$$\bar{z} := \gamma(d_0/2), \qquad s_0 := \frac{d_0}{4L}.$$

Since $Ls_0 = d_0/4$, the restriction of γ to $[d_0/4, 3d_0/4]$, reparametrized on $[-Ls_0, Ls_0]$, is a minimizing segment centered at $\bar{z}$. Choose $q \in M$ corresponding to $(0, \bar{z})$ under the approximation (23). First $\overline{B}_{Ls_0}((0, \bar{z})) \subset \overline{B}_{3d_0/4}((0, z))$, so this ball lies inside the original comparison region since $1 + 3d_0/4 \le \delta^{-1}$ (see (24)). Restriction and recentering give

$$d_{GH}\Big(\overline{B}_{Ls_0}(q), \overline{B}_{Ls_0}((0, \bar{z}))\Big) \le C_1\delta,$$

for a universal constant $C_1 > 0$. Since $C_1\delta \le \delta_s s_0$ (see (24)), and $\mathrm{Ric} \ge -\delta \ge -\delta_s s_0^{-2}$ (using $s_0 \le 1$) all the hypotheses of Lemma 4.1 are satisfied, and consequently $B_{s_0}(q)$ is $(k + 1, \varepsilon)$-split.

Since p and q correspond respectively to $(0, z)$ and $(0, \bar{z})$ in the approximation given by (23),

$$d(p, q) \le d_Z(z, \bar{z}) + C_1\delta = \frac{d_0}{2} + C_1\delta \le C(n, m, k, \varepsilon),$$

and hence $B_1(p) \subset B_C(q)$ for a possibly larger constant $C > \max\{1, s_0\}$. Bishop–Gromov monotonicity then gives, with $c := c(n, m, k, \varepsilon)$ possibly changing from line to line

$$\mathcal{E}_m(q, s_0) = \frac{\operatorname{Vol} B_{s_0}(q)}{s_0^m} \ge \frac{c}{s_0^m} \operatorname{Vol} B_C(q) \ge c \operatorname{Vol} B_1(p) = c\mathcal{E}_m(p, 1).$$

We thus proved that $B_{s_0}(q)$ is $(k + 1, \varepsilon)$-split and $\mathcal{E}_m(q, s_0) \ge c\mathcal{E}_m(p, 1)$. Hence, the proof is concluded in the first case. Notice that we have $s_0 = d_0/4L \le 1 = r$, as desired.

Case 2: The residual factor Z is macroscopically small. Suppose now that every point of $B_{R_0}(z)$ lies within d_0 of z. Since $\delta_c^{-1} < R_0$, projection onto the Euclidean factor gives

$$d_{GH}\Big(B^{\mathbb{R}^k \times Z}_{\delta_c^{-1}}((0, z)), B^{\mathbb{R}^k}_{\delta_c^{-1}}(0)\Big) \le 2d_0.$$

Hence, using (23) and the third inequality in (24)

$$d_{GH}\Big(B_{\delta_c^{-1}}(p), B^{\mathbb{R}^k}_{\delta_c^{-1}}(0)\Big) \leq \delta_c,$$

where we also used $C_1\delta \leq \delta_c/2$, see (24). Moreover $\mathrm{Ric} \geq -\delta \geq -\delta_c$. Apply Proposition 4.4, and denote its output by G and $\{q_j, t_j, E_j, Y_j, y_j, a_j\}_{j=1}^N$. Since $k \leq m$ and $0 < t_j \leq 1$, we have $\sum_{j=1}^N t_j^m \leq \sum_{j=1}^N t_j^k \leq C$. Consequently, for some j

$$\frac{\mathrm{Vol}(E_j)}{t_j^m} \geq \frac{\mathrm{Vol}(G)}{\sum_{\ell=1}^N t_\ell^m} \geq c\,\mathrm{Vol}\,B_1(p).$$

Since $E_j \subset B_{Ct_j}(q_j)$, we have the following inequality where the constant c might change from line to line

$$\mathcal{E}_m(q_j, Ct_j) = \frac{\mathrm{Vol}\,B_{Ct_j}(q_j)}{(Ct_j)^m} \geq c\frac{\mathrm{Vol}(E_j)}{t_j^m} \geq c\,\mathrm{Vol}\,B_1(p) = c\mathcal{E}_m(p,1). \tag{25}$$

Let $\gamma : [0, c_0t_j/2] \to Y_j$ be the initial part of a minimizing geodesic from y_j to a_j, and put $\bar{y} := \gamma(c_0t_j/4)$, and $s := \vartheta t_j$. Thus $\bar{y}$ is the midpoint of a segment of half-length $c_0t_j/4$. Since $Ls = L\vartheta t_j \leq c_0t_j/32$, the central subsegment of half-length Ls satisfies the segment hypothesis of Lemma 4.1. Moreover, by the choice of A

$$\overline{B}_{Ls}((0,\bar{y})) \subset \overline{B}_{(c_0/4+L\vartheta)t_j}((0,y_j)) \subset \overline{B}_{At_j}((0,y_j)).$$

Choose $q \in M$ corresponding to $(0,\bar{y})$ in the product approximation given by Proposition 4.4 at q_j. Restriction and recentering give

$$d_{GH}\Big(\overline{B}_{Ls}(q), \overline{B}_{Ls}((0,\bar{y}))\Big) \leq C_1\eta t_j \leq \delta_s\vartheta t_j = \delta_s s.$$

Since $\mathrm{Ric} \geq -\delta \geq -\delta_s s^{-2}$ (using $s = \vartheta t_j \leq 1$), Lemma 4.1 therefore shows that $B_s(q) = B_{\vartheta t_j}(q)$ is $(k+1,\varepsilon)$-split.

The proof is now concluded as in Case 1: we have $d(q,q_j) \leq Ct_j$, so $B_{Ct_j}(q_j) \subset B_{C't_j}(q)$. Bishop–Gromov monotonicity and (25) yield, possibly by changing constants c, C from line to line,

$$\begin{aligned}\mathcal{E}_m(q,\vartheta t_j) &= \frac{\mathrm{Vol}\,B_{\vartheta t_j}(q)}{(\vartheta t_j)^m}\\ &\geq \frac{c}{(\vartheta t_j)^m}\mathrm{Vol}\,B_{C't_j}(q) \geq c\frac{\mathrm{Vol}\,B_{Ct_j}(q_j)}{t_j^m} \geq c\mathcal{E}_m(q_j,Ct_j) \geq c\mathcal{E}_m(p,1),\end{aligned}$$

thus concluding the proof. Notice that we also have $\vartheta t_j \leq 1 = r$, as desired. □

Remark 4.6. Proposition 4.5 is likely to work, with minimal changes, in the setting of metric-measure spaces $(X, \mathrm{d}, \mathcal{H}^n)$ that are $\mathrm{RCD}(-\delta r^{-2}, n)$. In particular, in the proof of Theorem 1.2, the smoothness of M and the lower bound on the intermediate curvature enters into play only in the computations of Proposition 3.1, especially when one uses the equality (8). It would be interesting to understand how (and to which extent) these computations can be adapted in the metric-measure space setting. We do not pursue this line of research here.

5. Proof of the results

Proof of Theorem 1.2. Let $\varepsilon_{m+1} := \varepsilon_H(n,m)$ be the constant in Corollary 3.2. For $k = m, m-1, \ldots, 0$, choose recursively

$$\varepsilon_k := \delta(n,m,k,\varepsilon_{k+1}), \qquad c_k := c(n,m,k,\varepsilon_{k+1}),$$

where δ and c are the constants in Proposition 4.5. Set

$$\nu(n,m) := \min\{1, \sqrt{\varepsilon_0}, \ldots, \sqrt{\varepsilon_{m+1}}\} > 0.$$

Fix a ball $B_R(p) \subset M$ such that $\delta R \le \nu(n,m)$ (Here, δ is the one in the statement of Theorem 1.2). Recall that the output scale s in Proposition 4.5 is such that $0 < s \le r$. Since $B_R(p)$ is $(0, \varepsilon_0)$-split, successive applications of Proposition 4.5 give balls $B_{r_k}(p_k)$, $0 \le k \le m+1$, such that

$$r_0 = R, \qquad p_0 = p, \qquad R = r_0 \ge r_1 \ge \cdots \ge r_{m+1} > 0,$$

$B_{r_k}(p_k)$ is (k, ε_k)-split, and

$$\mathcal{E}_m(p_{k+1}, r_{k+1}) \ge c_k \mathcal{E}_m(p_k, r_k) \qquad \forall 0 \le k \le m. \tag{26}$$

Indeed, at the kth step, the choice of $\nu(n,m)$ and $r_k \le R$ give $\delta^2 r_k^2 \le \delta^2 R^2 \le \nu(n,m)^2 \le \varepsilon_k$ and hence $\mathrm{Ric}_g \ge -\delta^2 g \ge -\varepsilon_k r_k^{-2} g$ which is the Ricci hypothesis required in Proposition 4.5.

The final ball is $(m+1, \varepsilon_H)$-split and, again by the choice of $\nu(n,m)$, we have $\delta^2 r_{m+1}^2 \le \varepsilon_{m+1} = \varepsilon_H(n,m)$. Therefore Corollary 3.2 gives $r_{m+1} \le C(n,m)\sigma^{-1}$. Moreover, Bishop volume comparison and $\delta r_{m+1} \le \delta R \le \nu(n,m) \le 1$ give $\operatorname{Vol} B_{r_{m+1}}(p_{m+1}) \le C(n,m) r_{m+1}^n$. Consequently,

$$\mathcal{E}_m(p_{m+1}, r_{m+1}) \le C(n,m) r_{m+1}^{n-m} \le C(n,m)\sigma^{-(n-m)}.$$

Iterating (26) yields the sought conclusion

$$\frac{\operatorname{Vol} B_R(p)}{R^m} = \mathcal{E}_m(p_0, r_0) \le \left(\prod_{k=0}^{m} c_k^{-1} \right) \mathcal{E}_m(p_{m+1}, r_{m+1}) \le C(n,m)\sigma^{-(n-m)}. \qquad \square$$

Remark 5.1. It is likely that the techniques developed in this note can be used to show versions of Theorem 1.2 where the positive intermediate curvature condition $\mathcal{C}_{m+1} \ge \sigma^2$ is weakened to a lower bound that goes to 0 at infinity like a (properly chosen) negative power of the distance from a point (cf., the results in [33, 10]). Since this is out of the scope of this note, we do not pursue it here.

Proof of Theorem 1.4. The case $m = 0$ is a consequence of Bonnet–Myers theorem. We may therefore assume $1 \le m \le n-2$.

We will be choosing δ small enough during the proof. By Theorem 1.2, there is $A = A(n,m)$ such that for every $p \in M$, if $\delta R \le \nu(n,m)$, then

$$\operatorname{Vol}_g B_R(p) \le A R^m. \tag{27}$$

Fix $x \in M$ and $r \ge 2$ a large enough integer, only depending on n, m, v_0 (see below). In the closed annulus $\overline{B}_{2r}(x) \setminus B_r(x)$ choose a maximal family of points $p_1, \dots, p_N$ for which the open balls $B_{1/10}(p_i)$ are pairwise disjoint. Bishop–Gromov comparison and the noncollapsing assumption give

$$\operatorname{Vol}_g B_{1/10}(p_i) \ge c(n) \operatorname{Vol}_g B_1(p_i) \ge c(n) v_0.$$

All these balls lie in $B_{2r+1/10}(x)$; hence, if δ is chosen sufficiently small, $\delta(2r + 1/10) \le \nu(n,m)$ and (27), together with the previous inequality, implies $N \le C_0(n,m,v_0) r^m$.

By maximality, the balls $B_{1/5}(p_i)$ cover the annulus. Consider the distance spheres $\partial B_j(x)$ for $j \in \mathbb{N} \cap (r, 2r)$. Each open 1/5-ball meets at most one of these unit-spaced spheres. Since $N \le C_0 r^m$, there is $j \in (r, 2r)$ for which $\partial B_j(x)$ is covered by at most $C_1(n,m,v_0) r^{m-1}$ balls of radius 1/5. Denoting by HC the Hausdorff content, we thus have

$$\mathrm{HC}_m(\partial B_j(x)) \le C_2(n,m,v_0) r^{m-1}. \tag{28}$$

Let $\varepsilon_{m+1} > 0$ be the constant in [36, Theorem 3.3]. The choice of $r = r(n,m,v_0) \ge 2$ above should be made in such a way that the right-hand side of (28) is $\le \varepsilon_{m+1} r^m$ (and then, accordingly, δ must be chosen small with respect to this choice of r). For each x, set $U_x = B_j(x)$, choosing $j = j(x)$ as above. Then

$$B_r(x) \subset U_x \subset B_{2r}(x) \subset B_{10r}(x), \qquad \mathrm{HC}_m(\partial U_x) \le \varepsilon_{m+1} r^m.$$

Hence [36, Theorem 3.3] applied with parameter $m+1$ gives $\mathrm{UW}_m(M,g) \le r$, as desired. $\square$

Proof of Theorem 1.6. Suppose that M is rationally essential. By [3, Theorem 1.3], there is $c(n) > 0$ such that the Riemannian universal cover $(\widetilde{M}, \widetilde{g})$ satisfies

$$\sup_{\widetilde{p} \in \widetilde{M}} \mathrm{Vol}_{\widetilde{g}} B_R(\widetilde{p}) > c(n) R^n, \qquad \text{for every } R > 0.$$

On the other hand, Theorem 1.2 applied to $(\widetilde{M}, \widetilde{g})$, gives constants $C = C(n,m)$ and $\nu = \nu(n,m) > 0$ such that

$$\sup_{\widetilde{p} \in \widetilde{M}} \mathrm{Vol}_{\widetilde{g}} B_R(\widetilde{p}) \leq C R^m \qquad \text{whenever } \varepsilon R \leq \nu.$$

Choose $L = L(n,m)$ sufficiently large that $CL^m < c(n)L^n$, and then choose $\varepsilon = \varepsilon(n,m) > 0$ sufficiently small so that $\varepsilon L \leq \nu$. Hence, taking $R = L$ contradicts the two previous inequalities.

Thus M is not rationally essential. Finally, an aspherical M has c_M as a homotopy equivalence, hence M is not aspherical as well; and Gromov's mapping theorem [14, Section 3.1, Corollary (B)] gives $\|M\| = \|(c_M)_*[M]_{\mathbb{R}}\|_1 = 0$. $\square$

References

1. Michael T. Anderson, *On the topology of complete manifolds of non-negative Ricci curvature*, Topology **29** (1990), no. 1, 41–55, doi:10.1016/0040-9383(90)90024-E.
2. Gioacchino Antonelli and Kai Xu, *New spectral Bishop–Gromov and Bonnet–Myers theorems and applications to isoperimetry*, Accepted for publication in *J. Eur. Math. Soc. (JEMS)*; arXiv:2405.08918, 2024.
3. Sabine Braun and Roman Sauer, *Volume and macroscopic scalar curvature*, Geom. Funct. Anal. **31** (2021), no. 6, 1321–1376, doi:10.1007/s00039-021-00588-y; arXiv:2012.08999.
4. Simon Brendle, Sven Hirsch, and Florian Johne, *A generalization of Geroch's conjecture*, Comm. Pure Appl. Math. **77** (2024), no. 1, 441–456, doi:10.1002/cpa.22137.
5. Jeff Cheeger and Tobias H. Colding, *Lower bounds on Ricci curvature and the almost rigidity of warped products*, Ann. of Math. (2) **144** (1996), no. 1, 189–237, doi:10.2307/2118589.
6. ———, *On the structure of spaces with Ricci curvature bounded below. I*, J. Differential Geom. **46** (1997), no. 3, 406–480, doi:10.4310/jdg/1214459974.
7. Jeff Cheeger and Detlef Gromoll, *The splitting theorem for manifolds of nonnegative Ricci curvature*, J. Differential Geom. **6** (1971), no. 1, 119–128, doi:10.4310/jdg/1214430220.
8. Jeff Cheeger and Aaron Naber, *Regularity of Einstein manifolds and the codimension 4 conjecture*, Ann. of Math. (2) **182** (2015), no. 3, 1093–1165, doi:10.4007/annals.2015.182.3.5.
9. Otis Chodosh and Chao Li, *Generalized soap bubbles and the topology of manifolds with positive scalar curvature*, Ann. of Math. (2) **199** (2024), no. 2, 707–740, doi:10.4007/annals.2024.199.2.3; arXiv:2008.11888. MR 4713021
10. Otis Chodosh, Chao Li, and Douglas Stryker, *Volume growth of 3-manifolds with scalar curvature lower bounds*, Proc. Amer. Math. Soc. **151** (2023), no. 10, 4501–4511, doi:10.1090/proc/16521.
11. Alessandro Cucinotta and Andrea Mondino, *On manifolds with almost non-negative Ricci curvature and integrally-positive k^{th}-scalar curvature*, Math. Ann. **394** (2026), no. 2, Paper No. 49, doi:10.1007/s00208-026-03406-8.
12. Jian Ge, *Heat kernel geometry and Gromov's volume growth conjecture*, arXiv:2608.13553, 2026.
13. Nicola Gigli, *The splitting theorem in non-smooth context*, Mem. Amer. Math. Soc. **317** (2026), no. 1609, 117 pp., doi:10.1090/memo/1609; arXiv:1302.5555.
14. Mikhail Gromov, *Volume and bounded cohomology*, Publ. Math. Inst. Hautes Études Sci. **56** (1982), 5–99, Numdam.
15. ———, *Large Riemannian manifolds*, Curvature and Topology of Riemannian Manifolds (Katata, 1985) (Berlin) (Katsuhiro Shiohama, Takashi Sakai, and Toshikazu Sunada, eds.), Lecture Notes in Mathematics, vol. 1201, Springer-Verlag, 1986, doi:10.1007/BFb0075649, pp. 108–121.
16. ———, *No metrics with positive scalar curvatures on aspherical 5-manifolds*, arXiv:2009.05332, 2020.
17. ———, *Four lectures on scalar curvature*, Perspectives in Scalar Curvature (Mikhail L. Gromov and H. Blaine Lawson, Jr., eds.), vol. 1, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2023, doi:10.1142/9789811273223_0001; arXiv:1908.10612, pp. 1–514. MR 4577903
18. Larry Guth, *Metaphors in systolic geometry*, Proceedings of the International Congress of Mathematicians. Volume II (New Delhi), Hindustan Book Agency, 2010, doi:10.1142/9789814324359_0072; arXiv:1003.4247, pp. 745–768. MR 2827817
19. ———, *Volumes of balls in large Riemannian manifolds*, Ann. of Math. (2) **173** (2011), no. 1, 51–76, doi:10.4007/annals.2011.173.1.2; arXiv:math/0610212. MR 2753599
20. ———, *Volumes of balls in Riemannian manifolds and Uryson width*, J. Topol. Anal. **9** (2017), no. 2, 195–219, doi:10.1142/S1793525317500029; arXiv:1504.07886. MR 3622232

21. Hongzhi Huang and Xian-Tao Huang, *Splitting and slow volume growth for open manifolds with nonnegative Ricci curvature*, Revised preprint; arXiv:2510.22708v2, 2025.
22. Xian-Tao Huang and Shuai Liu, *Optimal asymptotic volume ratio for noncompact* 3*-manifolds with asymptotically nonnegative Ricci curvature and a uniformly positive scalar curvature lower bound*, J. Geom. Anal. **35** (2025), no. 3, Paper No. 78, 21 pp., doi:10.1007/s12220-025-01919-3.
23. Dorothea Jansen, *Existence of typical scales for manifolds with lower Ricci curvature bound*, Ph.D. thesis, Westfälische Wilhelms-Universität Münster, 2016, arXiv:1703.09590.
24. Vitali Kapovitch and Burkhard Wilking, *Structure of fundamental groups of manifolds with Ricci curvature bounded below*, Revised preprint; arXiv:1105.5955v2, 2011.
25. Robert Koirala, *Volume growth under positive intermediate curvature and a Ricci lower bound*, 2026, doi:10.48550/arXiv.2608.17977; arXiv:2608.17977.
26. Bochao Kong and Xingyu Zhu, *Positive scalar curvature and volume growth*, 2026, arXiv:2608.14438.
27. Aditya Kumar and Balarka Sen, *Urysohn width and macroscopic scalar curvature*, arXiv:2601.14669, 2026.
28. Mohammed-Larbi Labbi, *On Weitzenböck curvature operators*, Math. Nachr. **288** (2015), no. 4, 402–411, doi:10.1002/mana.201300352; arXiv:math/0607521.
29. Yevgeny Liokumovich, Boris Lishak, Alexander Nabutovsky, and Regina Rotman, *Filling metric spaces*, Duke Math. J. **171** (2022), no. 3, 595–632, doi:10.1215/00127094-2021-0039; arXiv:1905.06522.
30. Yevgeny Liokumovich and Davi Maximo, *Waist inequality for* 3*-manifolds with positive scalar curvature*, Perspectives in Scalar Curvature (Mikhail L. Gromov and H. Blaine Lawson, Jr., eds.), vol. 2, World Scientific Publishing, Hackensack, NJ, 2023, MR 4577931; doi:10.1142/9789811273230_0022; arXiv:2012.12478, pp. 799–831.
31. Yevgeny Liokumovich and Zhichao Wang, *On the waist and width inequality in complete* 3*-manifolds with positive scalar curvature*, Cambridge J. Math. **14** (2026), no. 2, 349–375, doi:10.4310/CJM.260514230515; arXiv:2308.04044.
32. John Lott, *Some obstructions to positive scalar curvature on a noncompact manifold*, J. Reine Angew. Math. **829** (2025), 247–283, doi:10.1515/crelle-2025-0072; arXiv:2402.13239.
33. Ovidiu Munteanu and Jiaping Wang, *Geometry of three-dimensional manifolds with positive scalar curvature*, Amer. J. Math. **148** (2026), no. 1, 131–160, doi:10.1353/ajm.2026.a980770.
34. ______, *Sharp integral bound of scalar curvature on* 3*-manifolds*, Trans. Amer. Math. Soc. (2026), doi:10.1090/tran/9744; arXiv:2505.10520.
35. Alexander Nabutovsky, *Linear bounds for constants in Gromov's systolic inequality and related results*, Geom. Topol. **26** (2022), no. 7, 3123–3142, doi:10.2140/gt.2022.26.3123; arXiv:1909.12225. MR 4540902
36. Panos Papasoglu, *Uryson width and volume*, Geom. Funct. Anal. **30** (2020), no. 2, 574–587, doi:10.1007/s00039-020-00533-5.
37. Anton M. Petrunin, *An upper bound for the curvature integral*, St. Petersburg Math. J. **20** (2009), no. 2, 255–265, doi:10.1090/S1061-0022-09-01046-2.
38. Ying Shen and Rugang Ye, *On stable minimal surfaces in manifolds of positive bi-Ricci curvatures*, Duke Math. J. **85** (1996), no. 1, 109–116, doi:10.1215/S0012-7094-96-08505-1.
39. Jinmin Wang, Zhizhang Xie, Bo Zhu, and Xingyu Zhu, *Positive scalar curvature meets Ricci limit spaces*, Manuscripta Math. **175** (2024), no. 3–4, 943–969, doi:10.1007/s00229-024-01596-6.
40. Yipeng Wang, *Effective volume growth of three-manifolds with positive scalar curvature*, Proc. Amer. Math. Soc. **154** (2026), no. 1, 329–337, doi:10.1090/proc/17082.
41. Guodong Wei, Guoyi Xu, and Shuai Zhang, *Volume growth and positive scalar curvature*, Trans. Amer. Math. Soc. **378** (2025), no. 9, 6109–6136, doi:10.1090/tran/9280.
42. Shing-Tung Yau, *Open problems in geometry*, S. S. Chern: A Great Geometer of the Twentieth Century (Shing-Tung Yau, ed.), Monographs in Geometry and Topology, International Press, Hong Kong, 1992, See Section I, p. 278, Problem 9, pp. 275–319.
43. Jie Zhou and Jintian Zhu, *Optimal volume bound and volume growth for Ricci-nonnegative manifolds with positive bi-Ricci curvature*, J. Reine Angew. Math. **821** (2025), 1–21, doi:10.1515/crelle-2024-0100.
44. Bo Zhu, *Geometry of positive scalar curvature on complete manifold*, J. Reine Angew. Math. **791** (2022), 225–246, doi:10.1515/crelle-2022-0049.
45. Bo Zhu and Xingyu Zhu, *Optimal diameter estimate of three-dimensional Ricci limit spaces*, Proc. Amer. Math. Soc. **152** (2024), no. 2, 815–821, doi:10.1090/proc/16529.
46. Xingyu Zhu, *Two-dimension vanishing, splitting and positive scalar curvature*, Math. Ann. **392** (2025), no. 2, 2635–2656, doi:10.1007/s00208-025-03139-0.

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