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On geodesics and the cut locus in the oriented Grassmann manifold

Andrea C. G. Mennucci (*)

Abstract – The Stiefel and Grassmann manifolds have been studied extensively and play a prominent role in many applications. Their geodesic structure is well understood; in particular, minimal geodesics and the cut locus in the Grassmann manifold admit classical descriptions. The oriented Grassmann manifold is less frequently treated explicitly. In this note we review the relevant Stiefel and Grassmann results for Hilbert spaces, including the infinite-dimensional case, give direct proofs adapted to our notation, and derive an explicit minimal-geodesic criterion and a cut-locus characterization for the oriented Grassmann manifold.

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1. Introduction

The interest in Stiefel and Grassmann manifolds is long standing. They provide natural configuration spaces for orthonormal frames and linear subspaces, and they occur in numerical linear algebra, optimization, statistics, and shape analysis. Classical studies go back to [18, 21]. A standard reference for matrix algorithms and the Riemannian geometry of these manifolds is [5]; statistical aspects are treated, for example, in [3]; and Grassmannian descriptions of shape spaces appear in [19, 22, 23].

(*) *(autore corrispondente) Indirizzo dell'A.*: Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa, Italy; andrea.mennucci@sns.it
Author IDs: ORCID 0000-0002-4302-3275

This also motivates a companion paper in preparation on the StiefelCurve software library, where $\mathrm{Gr}^+\big(2, L^2([0,1])\big)$ is used to represent planar curves up to translation, rotation, and scaling in shape analysis.

Interest in these topics continues to this day. Recent work includes surveys and computational treatments of Grassmann manifolds [1], algorithms for logarithms and geodesics on Stiefel manifolds [14, 25], and cut-locus results for the canonical metric [20]. In contrast, this paper uses the metric induced by the ambient Hilbert space.

In this paper we review the definitions of the Stiefel manifold, the Grassmann manifold, and the oriented Grassmann manifold in a possibly infinite-dimensional Hilbert space H. We use the Riemannian metric induced by the ambient Hilbert space on the Stiefel manifold and the corresponding quotient metrics on the Grassmannians. Other natural metrics exist on these manifolds, but they are not considered here.

Main contribution of the paper

The main contributions of the paper concern the oriented Grassmann manifold. The ordinary Grassmann results are recalled and reformulated mainly as preparation and comparison, but in a form that is readily adapted to infinite-dimensional Hilbert spaces through the use of $p \times p$ Gram matrices.

Using the ‘‘angle-eigenvalues’’ Lemma 3.11, Proposition 3.18 shows that a class of Grassmann geodesics[1] admits a geometric decomposition into ‘‘principal spaces’’ that depends only on the endpoints.

In Definition 3.22 we introduce an oriented variant of the singular value decomposition, denoted SVD^+, and the associated oriented principal angles. With these tools, Proposition 3.20 gives a criterion for minimality, Proposition 3.23 gives an explicit construction of a minimal geodesic in $\mathrm{Gr}^+(p, H)$, and Theorem 4.2 gives a geometric characterization of its cut locus. The cut locus is described by three endpoint configurations: a doubly-semi-perpendicular case, a multiplicity condition on the largest ordinary principal angle together with the orientation sign, and the case where the two oriented planes project to the same unoriented plane. Using this characterization, Theorem 4.6 and Corollary 4.7 compute the injectivity radius and the diameter of the oriented Grassmannian.

To the best of our knowledge, the minimality criterion of Proposition 3.20, the construction of minimal geodesics in Proposition 3.23, and the cut-locus characterization in

(1) These geodesics need not be minimal. The geodesic ODE is the same in the standard and oriented Grassmannians.

Theorem 4.2 are not available in the literature in this explicit endpoint/principal-angle form. The injectivity-radius and diameter statements are then recorded as consequences of that characterization.

Structure of the paper

In Sec. 3.1 we recall the classical geodesic formula on the Stiefel manifold and the closed-form description of geodesics. In Sec. 3.2 we do the same for the Grassmann manifold; Sec. 3.2.1 discusses the geometric meaning of the decomposition, and Sec. 3.2.2 relates the angles to the eigenvalues of the associated Gram matrices. All formulas are rewritten in a Hilbert-space form that uses only $p \times p$ Gram matrices; this avoids singular value decompositions that are less transparent when $\dim H = \infty$.

For the Grassmann manifold, in Sec. 3.3 we present the characterization and the known formulas for minimal geodesics and in Sec. 4.2 the classical characterization of the cut locus. Then in Sec. 3.5 and Sec. 4.3 we derive the corresponding statements for the oriented Grassmann manifold, where the endpoint representatives must be oriented representatives and the ordinary singular value decomposition must be replaced by the oriented variant SVD^+ introduced in Definition 3.22. We then use the corresponding *oriented principal angles*.

For the Stiefel manifold itself, the corresponding global minimality and cut-locus problems are more subtle. Explicit systematic criteria are not available in the same form. [2]

For the reader's convenience, this paper is mostly self-contained: we include short proofs of most classical results used below, adapted to the definitions and conventions of this paper.

2. Definitions

We present a short account of the main definitions and the corresponding notation.

Definition 2.1. *Suppose that M is a smooth complete Riemannian manifold without boundary, modeled on a Hilbert space, possibly infinite dimensional.*

(2) For comparison, see the introduction of [14], which discusses the logarithm problem and provides theoretical guarantees for a β-metric family including the embedded metric at $\beta = 1$.

- *Let $P \in M$ and $V \in T_P M$ be fixed. Let $\gamma : \mathbb{R} \to M$ be the geodesic solving*

$$\nabla_{\dot\gamma}\dot\gamma(t) = 0 \,, \quad \gamma(0) = P \,, \ \ \dot\gamma(0) = V \,.$$

  Then

$$\exp_P(V) = \gamma(1)$$

  defines ***the exponential map****.*

- *Consider a geodesic written as $\gamma(t) = \exp_P(tV)$. A* ***conjugate point*** *Q along the geodesic* [3] *is a point $Q = \gamma(r) = \exp_P(rV)$ such that $d\exp_P$ at rV is not invertible. The* ***multiplicity*** *of the point is the dimension of the kernel of $d\exp_P$.* [4]

- *The* ***cut locus*** *of P is the set of all points $Q = \gamma(1)$, for some $V \neq 0$, such that γ restricted to $[0,1]$ is a minimum-length geodesic connecting P to Q, while, for every $t > 1$, γ restricted to $[0,t]$ is no longer a minimum-length geodesic connecting P to $\gamma(t)$.* [5]

These definitions are classical and can be found in standard texts such as [4], [10], or [11].

The following result is fundamental.

THEOREM 2.2. [6] *If a geodesic has a conjugate point of positive multiplicity in its interior, then it is not a minimum-length geodesic between its endpoints.*

Consequently, if a geodesic is minimizing up to a conjugate point $Q = \gamma(r)$ of positive multiplicity, then it cannot be extended past Q as a minimizing geodesic; hence Q is in the cut locus of P.

PROPOSITION 2.3. *Suppose that M is finite-dimensional. Then every conjugate point has positive multiplicity. Moreover, for every $P \in M$, the cut locus of P has the following properties.*

- *The cut locus is closed.*
- *It is rectifiable.*
- *It is the union of a singular part Σ and a conjugate part:*

(3) Definition 1.12.9 in [10].
(4) Note that, if $\dim H = \infty$, then a conjugate point may have multiplicity zero.
(5) Note that P is not in the cut locus of P.
(6) Theorem 1.12.13 in [10], where this assertion holds also if H is infinite dimensional.

  - *Σ is the set where the distance function*

$$Q \mapsto d_M(P, Q)$$

  is not differentiable; equivalently, it is the set of points Q joined to P by at least two distinct minimizing geodesics.

  - *The conjugate part consists of those points Q such that $\exp_P(V) = Q$, $\|V\| = d_M(P, Q)$*[7]*, and $\exp_P$ is not locally invertible at V.*

  (In all of the above we consider only $Q \neq P$ and $V \neq 0$.)

- *If Q is in the cut locus of P and they are connected by infinitely many distinct minimizing geodesics, then Q is both a singular point and a conjugate point to P.*[8]
- *The closure of Σ is the cut locus.*

The classical alternative between ordinary cut points and conjugate cut points is discussed, for instance, in [10]. The relation with the singular set of the distance function, and the rectifiability properties of that singular set, are treated in [13].

2.1 – *Stiefel and Grassmann manifolds*

Let $p \geq 1$ be an integer, and let H be a vector space endowed with a scalar product. Then

$$\mathrm{St}(p, H) = \left\{(e_1, e_2, \dots e_p) \in H^p : \|e_i\| = 1, \left\langle e_i, e_j\right\rangle_H = 0, \quad \forall i, j, i \neq j\right\} \tag{2.1}$$

is the *Stiefel manifold* of orthonormal p-frames. We assume $\dim H \geq p$, so that the Stiefel manifold is nonempty. When H is a Hilbert space, $\mathrm{St}(p, H)$ is a smooth embedded closed submanifold of H^p. With the Riemannian metric induced by the ambient space, it has the following Hopf–Rinow-type properties: it is metrically and geodesically complete, its exponential map is globally defined, and any two points in the same connected component can be joined by a minimizing geodesic. [9]

(7) Equivalently, the geodesic $s \mapsto \exp_P(sV)$, $0 \leq s \leq 1$, is minimizing from P to Q.
(8) Lemma 4.8 in [13].
(9) If $p = \dim H$, then $\mathrm{St}(p, H) = \mathrm{O}(p)$ has two connected components, so the end points must be in the same component. In infinite dimension the existence of minimal geodesics was proven in [7]; see the recap in Appendix A. In infinite dimension, closed bounded subsets need not be compact, contrary to one of the conclusions of the finite-dimensional Hopf–Rinow theorem.

When $H = \mathbb{R}^N$ it is customary to represent the p-frame as an $N \times p$ matrix (with orthonormal columns); in this case the Riemannian scalar product of $A, B \in \mathbb{R}^{N\times p}$ is

$$\langle A, B\rangle_{H^p} = \mathbf{tr}(A^\top B) \ . \tag{2.2}$$

If H is a generic Hilbert space, we agree that

$$\langle A, B\rangle_{H^p} = \mathbf{tr}(A^\top B) = \sum_{i=1}^{p} \langle A_i, B_i\rangle_H \ ,$$

where $A^\top B$ denotes the $p \times p$ matrix with entries $\left\langle A_i, B_j\right\rangle_H$.

The Stiefel manifold also carries other natural Riemannian metrics; see [5, 8, 15]. Here we use only the metric induced by the ambient Hilbert space.

The space $\mathrm{Gr}(p, H)$ of p-dimensional subspaces of H is called the *Grassmann manifold*; it can be identified with the quotient

$$\mathrm{Gr}(p, H) = \mathrm{St}(p, H)/\mathrm{O}(p)$$

where the group $\mathrm{O}(p)$ of $p \times p$ orthogonal matrices acts by right multiplication, changing the columns of a frame in $\mathrm{St}(p, H)$ by an orthogonal linear combination. This action can be defined for any space H (not just $H = \mathbb{R}^N$): if $v, w \in H^p$ and $A \in \mathbb{R}^{p\times p}$ then

$$v = wA \iff v_i = \sum_{j=1}^{p} w_j A_{j,i} \ .$$

When H is a Hilbert space, the Grassmann manifold is again a smooth complete Riemannian manifold.

The space $\mathrm{Gr}^+(p, H)$ of p-dimensional oriented subspaces of H is called the *oriented Grassmann manifold*; we identify it with the quotient

$$\mathrm{Gr}^+(p, H) = \mathrm{St}(p, H)/\mathrm{SO}(p) \ .$$

The manifolds $\mathrm{Gr}(p, H)$ and $\mathrm{Gr}^+(p, H)$ have the same geodesic properties: they are smooth Riemannian manifolds, they are metrically and geodesically complete, their exponential maps are globally defined, and under the standing assumption $p < \dim H$, any two points can be joined by a minimizing geodesic.

Now denote the natural projection maps by

$$\begin{array}{rcll} \mathrm{St}(p, H) & \longrightarrow & \mathrm{Gr}(p, H), & Y \mapsto [Y]_{\mathrm{Gr}}, \\ \mathrm{St}(p, H) & \longrightarrow & \mathrm{Gr}^+(p, H), & Y \mapsto [Y]_{\mathrm{Gr}+}, \\ \mathrm{Gr}^+(p, H) & \longrightarrow & \mathrm{Gr}(p, H), & P \mapsto [P]_{\mathrm{Gr}}. \end{array}$$

When $P \in \mathrm{Gr}(p, H)$ and $P = [Y]_{\mathrm{Gr}}$, we will say that Y is a *representative*. When $P \in \mathrm{Gr}^+(p, H)$ and $P = [Y]_{\mathrm{Gr}+}$, we will say that Y is an *oriented representative*.

PROPOSITION 2.4. *If $p = \dim H$ then $St(p,H) = \mathrm{O}(p)$, which has two connected components (one being $\mathrm{SO}(p)$), $Gr(p,H)$ is one point, and $Gr^+(p,H)$ is two points.*

If $p+1 \le \dim H$, then $St(p,H)$, $Gr(p,H)$, and $Gr^+(p,H)$ are connected. *The proof is in Appendix C, p. 46.*

For the above reasons we impose this hypothesis.

HYPOTHESIS 2.5. *We will assume that H is a Hilbert space and that $\dim H \ge p+1$ in the following.*

Note that all following results hold also when H is infinite dimensional.

REMARK 2.6. *The Stiefel and Grassmann manifolds can also be defined in different, but equivalent, ways: see the Introduction in* [1]. *Note also that all the different Riemannian metrics on $St(p,H)$ considered in* [5, 15] *project to the usual Riemannian metric on $Gr(p,H)$ and $Gr^+(p,H)$, since they coincide on the horizontal part of the tangent space in $St(p,H)$.*

The results in [7], recalled in Appendix A, show that the cut locus in the possibly infinite-dimensional manifolds studied here is completely determined by the corresponding finite-dimensional cut locus.

For the conjugate points, we add this result.

THEOREM 2.7. *Consider a geodesic $\gamma(t)$ in $St(p,H)$ starting from $P = \gamma(0)$, and let $Q = \gamma(r)$. Then Q is a conjugate point along the geodesic with positive multiplicity if and only if there exists a subspace $E \subseteq H$, with $\dim E \le 3p$, such that $\gamma(t) \in St(p,E)$ and Q is a conjugate point along the geodesic in $St(p,E)$.*

The same holds in the Grassmannian and the oriented Grassmannian.

The proof is in Appendix C, p. 47.

2.2 – *Jordan angles*

The *principal angles* [10] between two p-dimensional subspaces $P, Q \subseteq \mathbb{R}^n$ were introduced by Camille Jordan in his study of the relative position of subspaces under the action of the orthogonal group.

(10) Also called *Jordan angles* in honor of the fundamental paper [9]

Definition 2.8. *Given orthonormal bases $U, V \in \mathbb{R}^{n\times p}$ of P and Q, the Jordan angles*

$$0 \le \theta_1 \le \cdots \le \theta_p \le \frac{\pi}{2}$$

are defined by performing a SVD decomposition of $U^\top V$. If

$$1 \ge \sigma_1 \ge \cdots \ge \sigma_p \ge 0$$

are the singular values, then $\theta_i = \arccos \sigma_i$.

Equivalently, there exist orthonormal bases

$$u_1, \dots, u_p \in P, \qquad v_1, \dots, v_p \in Q,$$

such that

$$\langle u_i, v_j \rangle = 0 \quad (i \ne j), \qquad \langle u_i, v_i \rangle = \cos \theta_i .$$

There is also a purely geometric definition [2], which is useful when the spaces do not have the same dimension.

Definition 2.9. *Let $P, Q \subset \mathbb{R}^n$ be two linear subspaces, with*

$$\dim P = p, \qquad \dim Q = q,$$

and assume, for instance, that $p \le q$. The principal angles

$$0 \le \theta_1 \le \cdots \le \theta_p \le \frac{\pi}{2}$$

between P and Q may be defined recursively by the procedure $P_0 = Q_0 = \{0\}$ and for $i = 1, \dots, p$

$$\begin{aligned}
\sigma_i &= \max_{\substack{u\in P,\ \|u\|=1 \\ u\perp P_{i-1}}} \ \max_{\substack{v\in Q,\ \|v\|=1 \\ v\perp Q_{i-1}}} \langle u, v \rangle, \\
\langle u_i, v_i \rangle &= \sigma_i, \\
\theta_i &= \arccos \sigma_i, \\
P_i &= \operatorname{span}\{u_1, \dots, u_i\}, \qquad Q_i = \operatorname{span}\{v_1, \dots, v_i\}.
\end{aligned}$$

Here $u_i \in P$ and $v_i \in Q$ are unit vectors realizing the maxima.

If $\dim P = \dim Q = p$, this recursive construction gives orthonormal bases satisfying the conditions in the previous definition.

The definitions above extend naturally to the case when $\mathbb{R}^n$ is replaced by a Hilbert space H, possibly infinite dimensional.

The Grassmann manifold is a Riemannian symmetric space, and the Jordan angles form a complete set of invariants for a pair of points $P, Q \in \mathrm{Gr}(p, H)$ under the action of $\mathrm{O}(H)$. If $\dim H \geq 2p$, choosing suitable orthonormal coordinates, one may write

$$P = \mathrm{span}\{e_1, \ldots, e_p\}$$

and

$$Q = \mathrm{span}\left\{\cos\theta_1\, e_1 + \sin\theta_1\, e_{p+1}, \ldots, \cos\theta_p\, e_p + \sin\theta_p\, e_{2p}\right\} .$$

2.3 – *p-plane configurations*

With a slight abuse of terminology, we will call a point $P \in \mathrm{Gr}(p, H)$ a *p-plane*, and a point $P \in \mathrm{Gr}^+(p, H)$ an *oriented p-plane*.

We introduce these convenient definitions.

Definition 2.10. *Let $P_0, P_1 \in Gr(p, H)$, $P_0 \neq P_1$. We say that they are*

(1) perpendicular *if every vector in P_0 is orthogonal to every vector in P_1;*

(2) semi-perpendicular *if there exists a non-zero vector in P_0 that is orthogonal to P_1, or equivalently with the roles of P_0 and P_1 exchanged;*

(3) oblique *if no non-zero vector in either p-plane is orthogonal to the other p-plane.*

Thus perpendicular planes are also semi-perpendicular, while oblique planes are precisely the planes that are not semi-perpendicular. Equivalently, for Stiefel representatives Y_0, Y_1,

(1) *P_0, P_1 are perpendicular iff $Y_0^\top Y_1 = 0$, iff all principal angles are $\pi/2$;*

(2) *P_0, P_1 are semi-perpendicular iff $\det(Y_0^\top Y_1) = 0$, iff at least one principal angle is $\pi/2$;*

(3) *P_0, P_1 are oblique iff $\det(Y_0^\top Y_1) \neq 0$, iff no principal angle is $\pi/2$.*

These conditions do not depend on the choice of representatives. *The proof is in Appendix C, p. 49.*

2.4 – *Tangent and normal spaces*

What follows comes from Section 2.2.1 in [5] (and is valid for the generic case when H is a Hilbert space, using the extended notations presented in the previous section).

Given $Y \in \mathrm{St}(p, H)$, the tangent space

$$T_Y \mathrm{St}(p, H) = \{V \in H^p : V^\top Y + Y^\top V = 0\} \tag{2.3}$$

and the normal space is

$$N_Y\mathrm{St}(p,H) = \{YS : S \in \mathbb{R}^{p\times p}, S^\top = S\} \tag{2.4}$$

.

For $A \in \mathbb{R}^{p\times p}$, set

$$\mathrm{sym}(A) = \frac{A + A^\top}{2}, \qquad \mathrm{skew}(A) = \frac{A - A^\top}{2}.$$

Let $Z \in H^p$ and $Y \in \mathrm{St}(p,H)$. The normal projection of Z is

$$\pi_N(Z) = Y\,\mathrm{sym}(Y^\top Z), \tag{2.5}$$

and the tangent projection is

$$\pi_T(Z) = Y\,\mathrm{skew}(Y^\top Z) + (\mathbb{I} - YY^\top)Z. \tag{2.6}$$

Here $YY^\top$ denotes the orthogonal projection of H onto the span of the columns of Y, applied columnwise to Z; $\mathbb{I}$ is the identity operator on H.

For $\mathrm{Gr}(p,H)$, we note that the vertical space associated to the right action of $\mathrm{O}(p)$ on $\mathrm{St}(p,H)$ is

$$\mathrm{Vert}_Y = \{YA : A \in \mathbb{R}^{p\times p}, A^\top = -A\} \tag{2.7}$$

.

Using the horizontal subspace at Y, we may identify the tangent space of $\mathrm{Gr}(p,H)$ with

$$T_{[Y]}\mathrm{Gr}(p,H) \simeq \{V \in H^p : Y^\top V = 0\}. \tag{2.8}$$

These are the directions horizontal to the right action of $\mathrm{O}(p)$.

3. Geodesics

3.1 – *Geodesics in the Stiefel manifold*

Classically, the Stiefel manifold $\mathrm{St}(p,\mathbb{R}^n)$ is defined as the set of all frames composed of p orthonormal vectors in $\mathbb{R}^n$ (with $1 \le p \le n$); those frames are represented as $n \times p$ matrices A with orthonormal columns, that is, $A^\top A = \mathbb{I}$ where $\mathbb{I} \in \mathbb{R}^{p\times p}$ is the identity matrix.

Geodesics in Stiefel manifolds $\mathrm{St}(p,\mathbb{R}^n)$ are known to have closed-form solutions, as demonstrated by [5].[11]

Proposition 3.1 (Geodesics in $\mathrm{St}(p,\mathbb{R}^n)$). *Let $I \subseteq \mathbb{R}$ be an interval. Suppose that $St(p,\mathbb{R}^n)$ is endowed with the Riemannian metric (2.2), namely*

$$\langle A, B\rangle = \mathbf{tr}(A^\top B)\ .$$

Then a smooth path $Y : I \to St(p,\mathbb{R}^n)$ is a geodesic if and only if it satisfies

$$\ddot{Y} + Y(\dot{Y}^\top \dot{Y}) = 0\ . \tag{3.1}$$

The solution is

$$(Y(t)e^{A_0 t}, \dot{Y}(t)e^{A_0 t}) = (Y(0), \dot{Y}(0)) \exp t \begin{pmatrix} A_0 & -S_0 \\ \mathbb{I} & A_0 \end{pmatrix} \tag{3.2}$$

where $A_0 = Y^\top(0)\dot{Y}(0)$, $S_0 = \dot{Y}^\top(0)\dot{Y}(0)$, and $\mathbb{I}$ is the $p \times p$ identity matrix.

Moreover

$$A(t) = Y^\top(t)\dot{Y}(t) \tag{3.3}$$

is constant and equal to A_0 along the geodesic;

$$S(t) = \dot{Y}^\top(t)\dot{Y}(t) = e^{A_0 t} S_0 e^{-A_0 t} \quad ; \tag{3.4}$$

A is skew-symmetric and S is symmetric; hence $e^{A_0 t}$ is an orthogonal matrix. The proof is in Appendix C, p. 49.

A different expression for Stiefel geodesics, covering other choices of Riemannian metric as well, can be found in [24]; see also Theorem 2.1 in [14].

The solution in (3.2), while written for $\mathrm{St}(p,\mathbb{R}^n)$, extends to $\mathrm{St}(p,H)$, where H is a Hilbert space. In this case, A, S are the matrices of all scalar products

$$A_{i,j} = \left\langle Y_i, \dot{Y}_j \right\rangle_H \quad , \quad S_{i,j} = \left\langle \dot{Y}_i, \dot{Y}_j \right\rangle_H \tag{3.5}$$

of vectors in the frames (or, respectively, their derivatives).

Consequently, $\|\dot{Y}(t)\|_{\mathrm{St}}$ is constant along the geodesic and

$$\|\dot{Y}(t)\|_{\mathrm{St}} = \sqrt{\mathbf{tr} S(t)} = \|\dot{Y}(0)\|_{\mathrm{St}}\ .$$

The length of the geodesic, from time 0 to time t, can be written in many different ways

$$\int_0^t \|\dot{Y}(s)\|_{\mathrm{St}}\, \mathrm{d}s = t\|\dot{Y}(0)\|_{\mathrm{St}} = t\sqrt{\mathbf{tr} S_0}\ . \tag{3.6}$$

(11) [5] credits a personal communication by R. A. Lippert for the final closed-form formula (3.2).

3.2 – *Geodesics in the Grassmann manifold*

We first define this notation.

LEMMA 3.2. *Let $t \in \mathbb{R}$ and $A \in \mathbb{R}^{p\times p}$. We use the notation $B = \sqrt{A}$ only formally: the following expressions are defined by their power series in A, and therefore do not require that an actual square root B exists. We define the following symbols*

$$\cos(tB) = \sum_{k=0}^{\infty} \frac{(-1)^k A^k t^{2k}}{(2k)!} ,$$

$$B\sin(tB) = \sum_{k=0}^{\infty} \frac{(-1)^k A^{k+1} t^{2k+1}}{(2k+1)!} , \qquad B^{-1}\sin(tB) = \sum_{k=0}^{\infty} \frac{(-1)^k A^k t^{2k+1}}{(2k+1)!} .$$

The third series is well defined even when A is not invertible. If such a matrix $B = \sqrt{A}$ exists and is invertible, these definitions agree with the usual matrix functions.

With these definitions, we have

$$\frac{\partial}{\partial t}\cos(tB) = -\,B\sin(tB) ,$$

$$\frac{\partial}{\partial t}B\sin(tB) = B^2\cos(tB) , \qquad \frac{\partial}{\partial t}B^{-1}\sin(tB) = \cos(tB) .$$

where again these formulas do not depend on the existence of a matrix $B = \sqrt{A}$. The proof is in Appendix C, p. 50.

The group $\mathrm{O}(p)$ of $p \times p$ orthogonal matrices acts by right multiplication. This action is isometric; hence, by Noether's theorem, there is an associated conserved momentum along geodesics: this quantity is the matrix $A(t) = Y^\top(t)\dot{Y}(t)$ defined in equation (3.3), and it is called *angular momentum*.

As discussed in the introduction, the *Grassmann manifold* can be identified with the quotient

$$\mathrm{St}(p, H)/\mathrm{O}(p) .$$

A geodesic in the Grassmann manifold can be represented by lifting it to a geodesic in the Stiefel manifold that is *horizontal* with respect to the action, that is, such that $A = 0$. This means that all velocities $\dot{Y}_i$ are orthogonal to all positions Y_j. In this case $S(t) = S_0$ is constant, as shown by the relation (3.4), hence the geodesic equation is simpler, so the geodesic formula (3.2) can be expressed more explicitly:

Proposition 3.3. *Let $Y(t)$ be a horizontal Stiefel geodesic representing a geodesic in $Gr(p, H)$. Set $S = \dot{Y}(0)^\top \dot{Y}(0)$ and let $R = \sqrt{S}$.*[12] *Then*

$$(Y(t), \dot{Y}(t)) = (Y(0), \dot{Y}(0)) \begin{pmatrix} \cos(tR) & -R\sin(tR) \\ R^{-1}\sin(tR) & \cos(tR) \end{pmatrix} \tag{3.7}$$

which can be summarized as

$$Y(t) = Y(0)\cos(tR) + \dot{Y}(0)R^{-1}\sin(tR) \quad ; \tag{3.8}$$

It can be directly verified that this is the solution to (3.1). *The proof is in Appendix C, p. 50.*

An alternative finite-dimensional description of the above results can be found in Section 2.5.1 of [5], or in Proposition 3.3 of [1]. That description uses an SVD of $\dot{Y}(0)$, whereas the formulation above uses only the finite $p \times p$ Gram matrix $S = \dot{Y}(0)^\top \dot{Y}(0)$. This makes the formula more directly adapted to Hilbert spaces.

A further simplification is then possible: we may right-multiply the frames by an orthogonal matrix, and they will still represent the same point in the Grassmann manifold. By the Spectral Theorem, [13] let $O \in \mathrm{SO}(p)$ be an orthogonal matrix such that $D = O^\top R O$ is diagonal (and then $D^2 = O^\top S O$); define $Z(t) = Y(t)O$, then

$$Z(t) = Z(0)\cos(tD) + \dot{Z}(0)D^{-1}\sin(tD) \quad ;$$

let $v_1, v_2, \ldots, v_p \in \mathbb{R}_{\geq 0}$ be the diagonal entries of D, after reordering so that $0 \leq v_1 \leq \cdots \leq v_p$.

We obtain a total decoupling of the above formula: each column evolves according to its own velocity v_i. When $v_i = 0$, the column $Z_i(t)$ is constant. For each i for which $v_i \neq 0$, the corresponding column evolves as

$$Z_i(t) = Z_i(0)\cos(tv_i) + \frac{\dot{Z}_i(0)}{v_i}\sin(tv_i) \, . \tag{3.9}$$

Note that $v_i = \|\dot{Z}_i\|_H$, so $\dot{Z}_i(0)/v_i$ is a unit vector: each $Z_i(t)$ moves on a *great circle* in H, with speed v_i; all circles are orthogonal.

The length of the geodesic from time 0 to time t is

$$t\sqrt{\sum_{i=1}^{p} v_i^2} = t\sqrt{\mathbf{tr}(S)} = t\|\dot{Z}\| = t\|\dot{Y}\| \quad , \tag{3.10}$$

(12) Here R denotes the unique symmetric positive semidefinite square root of S.

(13) Usually the Spectral Theorem is stated as ''there exists an $O \in \mathrm{O}(p)$'' but, up to changing the sign of one column, it is equivalent to this formulation.

in accordance with the previous relation (3.6).

Moreover, for any two times t, τ, the matrix $Z(t)^\top Z(\tau)$ is diagonal, with diagonal entries $\cos((t-\tau)v_i)$. In particular $Z(0)^\top Z(t)$ has the values $\cos(tv_i)$ on the diagonal.

REMARK 3.4. *In general, $v_i \geq 0$. If the above is the lift of the minimal Grassmann geodesic connecting $[Z(0)]_{Gr}$ to $[Z(1)]_{Gr}$, then the values $v_i \in [0, \pi/2]$ are the principal angles. Indeed $\langle Z_i(0), Z_i(1)\rangle = \cos(v_i)$.*

REMARK 3.5. *More generally, for $t, \tau \in \mathbb{R}$, the principal angles between the p-planes $[Z(t)]$ and $[Z(\tau)]$ are, up to ordering,*

$$\arccos|\cos((t-\tau)v_i)|\ .$$

This does not depend on the representation Z of the geodesic in the Grassmann manifold.

In particular, if $P(t)$ is a geodesic in $Gr(p, H)$, there is $\varepsilon > 0$ such that the principal angles between $P(t), P(\tau)$ are, up to ordering, $|(t-\tau)v_i|$ when $|t-\tau| < \varepsilon$. This characterizes the ''velocities'' of the geodesic $P(t)$.

We will use the following elementary reparametrization fact.

LEMMA 3.6. *Let γ be a geodesic in $Gr(p, H)$ or $Gr^+(p, H)$, with principal velocities (v_i). If $r > 0$ and $\theta(t) = \gamma(rt)$, then θ is a geodesic with principal velocities $\tilde{v}_i = rv_i$.*

Thus, when discussing whether a geodesic is minimizing, we may assume without loss of generality that $t \in [0, 1]$.

3.2.1. Geometric decomposition. The complete geometric picture can also be described as follows. We note this lemma.

LEMMA 3.7. *Let $P_t = [Y(t)]_{Gr}$, and let $I_Y = \bigcap_t P_t$ be the intersection of these vector spaces. We call it the* invariant space *for the geodesic. The following are equivalent.*

- *$\dim I_Y = k$;*
- *$v_j = 0$ for $1 \leq j \leq k$ and $v_j > 0$ for $k+1 \leq j \leq p$.*

PROPOSITION 3.8. *Define $Y(t), Z(t), v_i$ as in Section 3.2. The vector space K_Y spanned by the columns $Y_i(t), \dot{Y}_i(t)$ (for $i = 1, \ldots, p$ and $t \in \mathbb{R}$) is the same as the vector space spanned by the columns $Z_i(t), \dot{Z}_i(t)$. This vector space decomposes as the orthogonal direct sum of*

- *I_Y, where the columns $Z_i(t)$ (for $i = 1, \ldots k$) are constant in t, and form an orthonormal basis for I_Y;*

- *$K_Y \cap I_Y^\perp$, where each column $Z_i(t)$ (for $i = k+1, \dots p$) moves in the two-dimensional plane* [14]
$$\pi_i = \mathrm{span}\{Z_i(t), \dot Z_i(t)\},$$
along a circle, with speed v_i, as in (3.9). For any fixed t, $Z_i(t), \dot Z_i(t)/v_i$ (for $i = k+1, \dots p$) are an orthonormal basis for $K_Y \cap I_Y^\perp$.

The vectore space K_Y is the direct sum of the orthogonal spaces

$$K_Y = I_Y \oplus \pi_{k+1} \oplus \cdots \oplus \pi_p \ . \tag{3.11}$$

We have $\dim K_Y = 2p - k$.

REMARK 3.9. *If all $v_i \neq 0$, we have reduced to the simplified situation where*

$$Z_1(t), \dots, Z_p(t), \frac{\dot Z_1(t)}{v_1}, \dots, \frac{\dot Z_p(t)}{v_p}$$

is a $2p$-orthonormal frame in H, at all times. (This can happen only if $\dim H \geq 2p$, *obviously.)*

3.2.2. The angle-eigenvalues Lemma. Using the Hilbert-space formulation (3.8) for geodesics, we can prove this useful lemma.

REMARK 3.10. *If $Y(t)$ is a horizontal geodesic lift of a geodesic $\gamma(t)$ in $Gr(p, H)$, then every other horizontal geodesic lift of the same geodesic is of the form $Y(t)Q$, with $Q \in \mathrm{O}(p)$. For $Gr^+(p, H)$, the same statement holds with $Q \in \mathrm{SO}(p)$.*

LEMMA 3.11 (The angle-eigenvalues Lemma). *Fix two p-planes $P_0, P_1 \in Gr(p, H)$. Suppose that $Y_0, Y_1 \in St(p, H)$ are representatives of P_0, P_1.*

Consider a geodesic connecting P_0 to P_1. Up to lifting and possibly rotating Y_1 by $Q \in \mathrm{O}(p)$, this geodesic is represented by a horizontal geodesic $Y : [0, 1] \to St(p, H)$ with $Y_0 = Y(0)$ and $Y_1 Q = Y(1)$. Let $W = \dot Y(0)$ be its initial velocity.

Then the matrices

$$M = Y(0)^\top Y(1) = Y_0^\top Y_1 Q \quad \text{and} \quad N = \dot Y(0)^\top Y(1) = W^\top Y_1 Q$$

are symmetric and, together with $S = \dot Y(0)^\top \dot Y(0)$, admit a common orthonormal eigenbasis. If $z \in \mathbb{R}^p$ is a common eigenvector and σ, θ, λ are the eigenvalues as in

$$Mz = \sigma z, \qquad Nz = \theta z, \qquad Sz = \lambda z,$$

[14] π_i does not depend on t.

the following holds:

$$\sigma = \cos\sqrt{\lambda}\ ,\quad \theta = \sqrt{\lambda}\sin\sqrt{\lambda}\ . \tag{3.12}$$

The above results also hold if we fix two oriented p-planes $P_0, P_1 \in Gr^+(p, H)$. In this case the endpoint rotation satisfies $Q \in \mathrm{SO}(p)$.

The proof is in Appendix C, p. 51.

So the relation (3.12) gives a strong constraint on the eigenvalues of S, for any horizontal geodesic representing a geodesic connecting P_0, P_1.

In particular we can use this to characterize the minimum-length geodesic and the cut locus, as we will show.

A simplified statement is this.

Corollary 3.12. *Under the same hypotheses,*

$$\sigma_i = |\cos\sqrt{\lambda_i}|\ ,\quad \theta_i = \sqrt{\lambda_i}\ |\sin\sqrt{\lambda_i}| \tag{3.13}$$

where σ_i are the singular values of $Y_0^\top Y_1$, θ_i are the singular values of $\dot{Y}(0)^\top Y_1$, and λ_i are the eigenvalues of $S = \dot{Y}(0)^\top \dot{Y}(0)$; the identity holds up to reordering. *The proof is in Appendix C, p. 51.*

3.2.3. Geodesics in the oriented Grassmann manifold. The same infinitesimal formulas apply to the oriented Grassmannian $\mathrm{Gr}^+(p, H) = \mathrm{St}(p, H)/\mathrm{SO}(p)$, since $\mathrm{SO}(p)$ has the same Lie algebra of skew-symmetric matrices as $\mathrm{O}(p)$. [15] The difference appears for minimal geodesics, as we will see in the next sections.

3.3 – *Minimal geodesics in the Grassmann manifold*

We now present the simple (and well-known) characterization of minimal geodesics in the Grassmann manifold.

Suppose that $Y_0, Y_1 \in \mathrm{St}(p, H)$ are representatives of two p-planes $P_0, P_1 \in \mathrm{Gr}(p, H)$. We define

$$B = Y_0^\top Y_1$$

and compute the singular value decomposition of B, so

$$B = U\Sigma V^\top\ . \tag{3.14}$$

([15]) Indeed, $\mathrm{SO}(p)$ is the connected component of $\mathrm{O}(p)$ containing the identity.

Let $\sigma_i = \Sigma_{i,i}$ be the singular values. [16]

REMARK 3.13. *Since Y_0 and Y_1 are Stiefel frames, for every $v \in \mathbb{R}^p$ we have*

$$\|Bv\| = \|Y_0^\top Y_1 v\| \leq \|Y_1 v\| = \|v\|.$$

Thus $\|B\|_{\mathrm{op}} \leq 1$, and all singular values satisfy $\sigma_i \in [0, 1]$.

We now define $Z_0 = Y_0 U$ and $Z_1 = Y_1 V$: these represent the same p-planes in the Grassmann manifold. Let $z_{0,i}$ and $z_{1,i}$ denote their i-th columns.

We define

$$v_i = \arccos(\sigma_i), \qquad 0 \leq v_i \leq \pi/2. \tag{3.15}$$

As aforementioned, the values v_i are known as *principal angles*. Let $D = \mathrm{diag}(v_1, \dots, v_p)$.

We then define an initial velocity frame $W = (w_1, \dots, w_p)$: if $v_i > 0$, then

$$w_i = v_i \hat{z}_i\ ,\ \hat{z}_i = \frac{1}{\sin v_i}\left(z_{1,i} - \sigma_i z_{0,i}\right)\ ;$$

(note that $\hat{z}_i$ is a unit vector, parallel to the projection of $z_{1,i}$ onto the orthogonal complement of $z_{0,i}$); if $v_i = 0$, then $z_{1,i} = z_{0,i}$ and we set $w_i = 0$.

PROPOSITION 3.14. *The equation*

$$Z(t) = Z_0 \cos(tD) + W D^{-1} \sin(tD) \tag{3.16}$$

(where $D^{-1}\sin(tD)$ is understood in the sense defined above in Lemma 3.2) then represents a minimum-length geodesic such that $Z(0) = Z_0$, $Z(1) = Z_1$, $\dot{Z}(0) = W$.

The same geodesic can also be written columnwise as

$$Z_i(t) = z_{0,i} \cos(t v_i) + \hat{z}_i \sin(t v_i). \tag{3.17}$$

We will show two proofs in the following sections.

The length of the geodesic is given by (3.10).

This SVD/principal-angle construction gives a minimizing Grassmann geodesic; see Section 4.3 in [5]. The presentation in [5] extends, with minor adaptations, to p-planes in Hilbert spaces, including the infinite-dimensional case. In the shape-analysis

(16) We will assume that they are in decreasing order, although this is not relevant in this section.

literature it has already been used by [23] and then in [19]; for a modern matrix-manifold reference see [1].

The above are called *Neretin geodesics* in [23] in reference to the paper [17] that describes some of the above ideas.

The above is also related to this criterion for minimality.

PROPOSITION 3.15. *As in Lemma 3.11, fix two p-planes $P_0, P_1 \in Gr(p, H)$, suppose that $Y_0, Y_1 \in St(p, H)$ are representatives of P_0, P_1. Consider a geodesic γ connecting P_0 to P_1. Up to lifting and possibly rotating Y_1 by $Q \in \mathrm{O}(p)$, this geodesic is represented by a horizontal geodesic $Y : [0, 1] \to St(p, H)$ with $Y_0 = Y(0)$ and $Y_1 Q = Y(1)$.*

Then γ is a minimizing geodesic connecting P_0 to P_1 if and only if all the eigenvalues of $S = \dot{Y}(0)^\top \dot{Y}(0)$ are less than or equal to $\pi^2/4$.

3.3.1. Proof of minimality. For convenience of the reader, we prove that the above geodesic built in Prop. 3.14 is indeed the minimum-length geodesic. We will need the following lemma.

LEMMA 3.16 (Minkowski's integral inequality). *Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces. Let $f : X \times Y \to [0, \infty]$ be $\mathcal{M} \otimes \mathcal{N}$-measurable. If $1 \le q < \infty$, then*

$$\left[\int_X \left(\int_Y f(x, y)\,\mathrm{d}\nu(y)\right)^q \mathrm{d}\mu(x)\right]^{1/q} \le \int_Y \left[\int_X f(x, y)^q\,\mathrm{d}\mu(x)\right]^{1/q} \mathrm{d}\nu(y).$$

This is Theorem 6.19(a) in [6].

Let $Y_0 = (e_1, \ldots, e_p)$ and $Y_1 = (f_1, \ldots, f_p)$ be two fixed Stiefel frames, and let $Y(t) = (y_1(t), \ldots, y_p(t))$ be any Stiefel path joining them. Each column $y_i(t)$ is a unit-sphere path from e_i to f_i, hence

$$\int_0^1 \|\dot{y}_i(t)\|\, dt \ge \arccos\langle e_i, f_i\rangle.$$

We apply Lemma 3.16, in the form

$$\left(\sum_{i=1}^p \left(\int_0^1 a_i(t)\,\mathrm{d}t\right)^2\right)^{1/2} \le \int_0^1 \left(\sum_{i=1}^p a_i(t)^2\right)^{1/2} \mathrm{d}t$$

for non-negative functions a_i, which is obtained by taking μ to be the counting measure on $\{1,\dots p\}$. We obtain

$$\begin{aligned}\operatorname{len}(Y) = \int_0^1 \|\dot{Y}\|_{\mathrm{St}}\,\mathrm{d}t = \int_0^1 \sqrt{\left(\sum_{i=1}^{p} \|\dot{y}_i(t)\|_H^2\right)}\,\mathrm{d}t \geq \\ \geq \sqrt{\sum_{i=1}^{p}\left(\int_0^1 \|\dot{y}_i(t)\|_H\,\mathrm{d}t\right)^2} \geq \sqrt{\left(\sum_{i=1}^{p} \arccos^2\langle e_i, f_i\rangle\right)}\,.\end{aligned}$$

For a Grassmann path, the terminal frame may be replaced by any equivalent representative $Y_1 Q$, with $Q \in \mathrm{O}(p)$. Thus the global lower bound is obtained by minimizing the right-hand side over all such Q. The principal-angle statement is precisely that this minimum is attained by the SVD alignment and is equal to

$$\left(\sum_{i=1}^{p} \arccos^2(\sigma_i)\right)^{1/2}.$$

The geodesic constructed above has exactly this length by (3.10), hence it is minimizing.

The eigenvalue criterion in Prop. 3.15 is proved below.

In the oriented case the same argument applies, but the admissible terminal representatives are restricted to $Q \in \mathrm{SO}(p)$; this is the reason for the SVD$^+$ construction below.

3.3.2. Another minimality proof. The next proof gives a self-contained derivation of the principal-angle lower bound. We will use Corollary 3.12. After reordering, set $d_i = \sqrt{\lambda_i}$, and rewrite the relation (3.13) as

$$|\cos d_i| = \sigma_i\ . \tag{3.18}$$

Since the length of such a geodesic is

$$\|\dot{Y}(0)\|_{\mathrm{St}} = \sqrt{\mathbf{tr} S} = \left(\sum_i d_i^2\right)^{1/2},$$

among all $d_i \geq 0$ satisfying the equation (3.18), the smallest possible value is

$$v_i = \arccos \sigma_i \in [0, \pi/2].$$

Therefore any geodesic connecting P_0 and P_1 has length at least $(\sum_i v_i^2)^{1/2}$. This lower bound is attained by the geodesic constructed in Proposition 3.14 in the previous section.

The above reasoning provides a proof of Prop. 3.15.

3.4 – *Invariant geometric decomposition*

For certain geodesics, the decomposition in Proposition 3.8 may be transformed into a decomposition that depends only on the endpoints.

Proposition 3.17 (Principal spaces). *Let $P, Q \subseteq H$ be two p-dimensional subspaces. Let the principal angles be*

$$0 \le \theta_1 \le \cdots \le \theta_p \le \frac{\pi}{2}$$

and let $0 < \tau_1 < \cdots < \tau_m$ be the distinct positive values among them, so that

$$\{\tau_j : j \le m\} = \{\theta_i : i \le p\} \setminus \{0\}.$$

Let $I = P \cap Q$ and $K = P + Q$. Then there exists a unique decomposition in mutually orthogonal spaces

$$K = I \oplus V_1 \oplus \cdots \oplus V_m \tag{3.19}$$

that is characterized by the facts that, for each j,

- $$V_j = (P \cap V_j) + (Q \cap V_j)\ , \quad (P \cap Q \cap V_j) = \{0\}\ ,$$
- *the subspaces $P \cap V_j$ and $Q \cap V_j$ have the same dimension,*
- *and their only principal angle is τ_j.*

Moreover $\dim I = k$ *iff* 0 *is a principal angle with multiplicity k. Note that the spaces V_j have even dimension.* *The proof is in Appendix C, p. 51.*

Proposition 3.18. *Consider a geodesic $\gamma : [0, 1] \to Gr(p, H)$, let $P_0 = \gamma(0), P_1 = \gamma(1)$. Let v_i be the velocities of γ, and suppose that $0 \le v_i < \pi$.*

Let

$$w_i = \arccos(|\cos v_i|) = \min\{v_i, \pi - v_i\}$$

be the reduced velocities. Then the w_i, up to ordering, are also the principal angles of P_0, P_1.

Let $0 < \tau_1 < \cdots < \tau_m$ be the distinct positive principal angles of P_0, P_1, so that

$$\{\tau_j : j \le m\} = \{w_i : i \le p\} \setminus \{0\}\ .$$

Let $I = P_0 \cap P_1, K = P_0 + P_1$, and decompose

$$K = I \oplus V_1 \oplus \cdots \oplus V_m$$

as in Proposition 3.17.

Let $Y(t) : [0,1] \to St(p,H)$ be a horizontal geodesic that is a lift of $\gamma(t)$. Apply Proposition 3.8 to Y to define K_Y, I_Y, π_i, v_i so that

$$K_Y = I_Y \oplus \pi_{k+1} \oplus \cdots \oplus \pi_p \ .$$

Then

- *$K = K_Y$, so that $P(t) \subseteq K$ for all t;*
- *$I = I_Y$;*
- *the principal spaces are built as*

$$V_j = \bigoplus_{i \colon w_i = \tau_j} \pi_i \ ; \tag{3.20}$$

- *in each space V_j, the geodesic has the form*

$$\gamma(t) \cap V_j = \operatorname{span}\big\{\cos(ta_1)e_1 + \sin(ta_1)e_{h+1}, \ldots, \cos(ta_h)e_h + \sin(ta_h)e_{2h}\big\}$$

  where, for each i, a_i is either τ_j or $\pi - \tau_j$. Here

  - *$\dim V_j = 2h$, and $e_1, \ldots, e_{2h}$ is an orthonormal basis of V_j;*
  - *$e_1, \ldots, e_h$ is an orthonormal basis of $P_0 \cap V_j$;*
  - $$\cos(a_1)e_1 + \sin(a_1)e_{h+1}, \ldots, \cos(a_h)e_h + \sin(a_h)e_{2h}$$

    is an orthonormal basis of $P_1 \cap V_j$.
- *On the invariant part,*

$$\gamma(t) \cap I = I$$

  for all t; that is, the planes do not move there. Equivalently, the indices with $v_i = 0$ give constant columns in the lifted geodesic, and these columns span the invariant term $I_Y = I$.

Note that the above statements are valid for any lift $Y(t)$ of $\gamma(t)$.

The proof is in Appendix C, p. 52.

REMARK 3.19. *The above proposition can be generalized from the hypothesis $0 \leq v_i < \pi$ to the assumption that either $v_i = 0$, or $v_i \notin \pi\mathbb{Z}$. The zero speeds still give the invariant term I. For the non-zero speeds, the reduced velocities are still $\arccos|\cos v_i|$, but the branch speeds should be recorded modulo π: in a principal block with angle τ_j, one has $v_i \equiv \tau_j \pmod{\pi}$ or $v_i \equiv -\tau_j \pmod{\pi}$. We do not need this extra generality here.*

3.5 – *Minimal geodesics in the oriented Grassmann manifold*

For minimal geodesics between oriented subspaces one must choose oriented representatives, equivalently use rotations in $\mathrm{SO}(p)$ rather than arbitrary matrices in $\mathrm{O}(p)$.

This leads to a different criterion for minimality.

PROPOSITION 3.20. *Fix two oriented p-planes $P_0, P_1 \in Gr^+(p, H)$ and choose oriented representatives $Y_0, Y_1 \in St(p, H)$ so that $P_0 = [Y_0]_{Gr+}$ and $P_1 = [Y_1]_{Gr+}$. Consider a geodesic γ connecting P_0 to P_1 in Gr^+. A lift $Y : [0, 1] \to St(p, H)$ may be chosen so that*

- *Y is horizontal,* [17]
- *$Y_0 = Y(0)$ and*
- *$Y_1 Q = Y(1)$ with $Q \in \mathrm{SO}(p)$.*

Let

$$S = \dot Y(0)^\top \dot Y(0)\ ,\ M = Y(0)^\top Y(1)\ .$$

Up to permuting the indices, we assume that the eigenvalues λ_i of S are in increasing order.

Then γ is a minimizing geodesic connecting P_0 to P_1 if and only if one of the following cases occurs:

(a) $\det M \geq 0$ *and all the eigenvalues λ_i of S satisfy $\lambda_i \leq \pi^2/4$; or*

(b) $\det M < 0$ *and $\lambda_i \leq \pi^2/4$ for $1 \leq i < p$ whereas $\pi^2/4 < \lambda_p \leq \pi^2$ and moreover*

$$\sqrt{\lambda_i} \leq \pi - \sqrt{\lambda_p}$$

for $i < p$.

PROOF. Let

$$S = \dot Y(0)^\top \dot Y(0)\ ,\ M = Y(0)^\top Y(1)\ ,\ N = \dot Y(0)^\top Y(1)\ .$$

We know from Lemma 3.11 that S, M and N are symmetric and can be diagonalized simultaneously. Let $O \in \mathrm{SO}(p)$ be a matrix that diagonalizes them; also let

$$\lambda_i, \sigma_i, \theta_i,$$

([17]) We recall that ''horizontal with respect to the $\mathrm{SO}(p)$-action'' is equivalent to ''horizontal with respect to the $\mathrm{O}(p)$-action''.

be the eigenvalues of S, M, N respectively; each associated to the eigenvector that is the i-th column of O.

Up to permuting the indices, all in the same way, we assume that the eigenvalues λ_i of S are in increasing order.

By the Lemma we know that

$$\sigma_i = \cos d_i \ , \quad \theta_i = d_i \sin d_i$$

where

$$d_i = \sqrt{\lambda_i} \ .$$

In particular

$$\det M = \prod_i \sigma_i = \prod_i \cos d_i \ . \tag{3.21}$$

So the hypotheses become:

(a) $\det M \geq 0$ and all $d_i \in [0, \pi/2]$; or

(b) $\det M < 0$ and $d_i \leq \pi/2$ for $1 \leq i < p$ whereas $\pi/2 < d_p \leq \pi$ and moreover

$$d_i \leq \pi - d_p$$

for $i < p$.

By the geometric considerations in Section 3.2.1, the intuition is as follows.

Define $Z(t) = Y(t)O$, so $Z_0 = Y_0 O$ and $Z_1 = Y_1 Q O$; let $z_{0,i}$ and $z_{1,i}$ denote their i-th columns; so

$$\langle Z_i(0), Z_i(1) \rangle_H = \sigma_i = \cos(d_i) \tag{3.22}$$

$$\left\langle \dot{Z}_i(0), Z_i(1) \right\rangle_H = \theta_i = d_i \sin(d_i) \ . \tag{3.23}$$

Let $\dot{Z}(0)$ be its initial velocity, then $d_i = \|\dot{Z}_i(0)\|_H$.

Consider the vector spaces K, I, π_i defined in Proposition 3.8 and the decomposition in equation (3.11).

The configuration in the plane π_i is then shown in figure 1.

Since $Z(t)$ is fixed, the decomposition (3.11) is fixed as well. In particular the terminal vector $Z_i(1)$ lies in the two-dimensional plane π_i, and the columns are separated by this decomposition. Therefore, within this fixed decomposition, the only ways in which $Z(1)$ can represent the same unoriented endpoint are

$$Z_i(1) = z_{1,i} \quad \text{or} \quad Z_i(1) = -z_{1,i} \qquad \text{for every } i.$$

It represents the same oriented endpoint P_1 precisely when the number of minus signs is even.

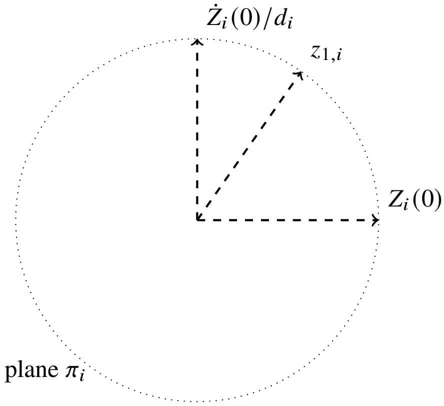

FIGURE 1. π_i plane

Recall that $d_i = \|\dot Z_i(0)\|$; if $d_i \neq 0$ then

$$Z_i(t) = Z_i(0)\cos(td_i) + \frac{\dot Z_i(0)}{d_i}\sin(td_i) \tag{3.24}$$

(as in 3.17); if $d_i = 0$ then $Z_i(t)$ is constant and always in the invariant space I.

Assume first that Y, and hence Z, has minimum length. The length is

$$\sqrt{\sum_{i=1}^{p} d_i^2}.$$

This already tells us that

$$d_i = \|\dot Z_i(0)\| \in [0, \pi].$$

Indeed, if this is not the case,

- if $d_i \geq 2\pi$ we may subtract $2\pi k$ so that $d_i \in [0, 2\pi]$, and rescale $\dot Z_i(0)$ accordingly;
- if $d_i \in [\pi, 2\pi]$, then we reverse the direction of $\dot Z_i(0)$ and rescale it to be of speed $2\pi - d_i$.

All the above single-flip moves do not alter the endpoint $Z(1)$. See Figure 2.

Knowing that $d_i \in [0, \pi]$ and using (3.21) we can say that

- $\det M = 0$ iff there is at least one $d_i = \pi/2$;
- $\det M > 0$ iff there is no $d_i = \pi/2$ and the number of $\pi/2 < d_i \leq \pi$ is even;
- $\det M < 0$ iff there is no $d_i = \pi/2$ and the number of $\pi/2 < d_i \leq \pi$ is odd.

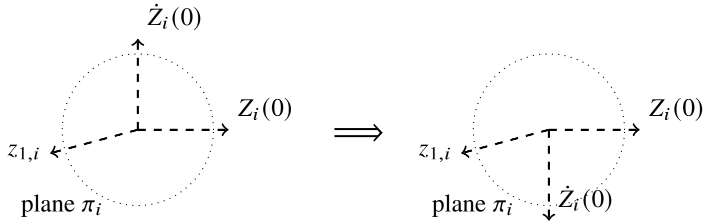

FIGURE 2. A single-flip move shortens a branch longer than π without changing the endpoint.

There is another possible move, which we call a double-flip: replace the endpoint $z_{1,i}$ with $-z_{1,i}$, reverse the initial direction, and replace the length d_i with $\pi - d_i$. See Figure 3. Equivalently, when $d_i \neq 0$, the initial velocity changes as

$$\dot Z_i(0) \mapsto -\frac{\pi - d_i}{d_i} \dot Z_i(0).$$

Note that this move changes the orientation of the frame $Z(1)$, and hence of $Y(1)$. If $d_i = \pi/2$, then this move does not change the length of the geodesic. If $\pi/2 < d_i \leq \pi$ then this move reduces the length of the geodesic.

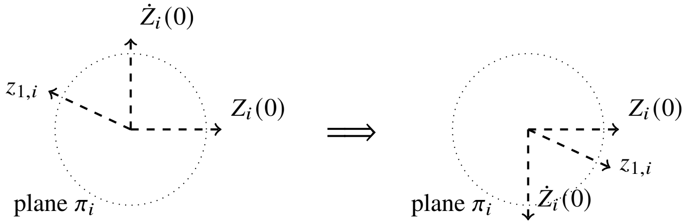

FIGURE 3. A double-flip replaces the endpoint by its opposite and changes the branch length from d_i to $\pi - d_i$.

When we consider the problem in $\mathrm{Gr}(p, H)$, those moves are all permissible, and the conclusion is that Y is a minimal-length geodesic if and only if $0 \leq d_i \leq \pi/2$ for every i: hence the characterization (3.15) for minimal geodesics.

When we consider the problem in $\mathrm{Gr}^+(p, H)$, we reason as follows.

- If there is an even number of i such that $\pi/2 < d_i \leq \pi$, then we can reduce all of them by double-flip, and we obtain the same geodesic as in the $\mathrm{Gr}(p, H)$ case. In this situation $\det M \geq 0$.

- If there is an odd number, and there is at least one $d_i = \pi/2$, we can again double-flip them all. Again we obtain the same geodesic as in the $\mathrm{Gr}(p, H)$ case. Note that $d_i = \pi/2$ implies $\det M = 0$.
- Suppose there is an odd number of indices i such that $\pi/2 < d_i \le \pi$, and no index j such that $d_j = \pi/2$; as we proved above, in this situation $\det M < 0$; we double-flip as much as possible, but one will be left in the range $\pi/2 < d_i \le \pi$: so we choose to skip the one that provides the worst improvement, and this is the one where d_i is minimum. (This will be explained better later on.)

Summarizing, if a geodesic is minimal, then either (a) or (b) must be satisfied; otherwise there are moves that reduce its length while still connecting the same endpoints.

Conversely, if (a) holds, then the geodesic is minimal also in Gr, by Prop. 3.15.

Consider a geodesic $\tilde{Y}$ that satisfies (b). To prove that (b) implies that $\tilde{Y}$ is minimal, we lower bound the length of any admissible geodesic Y, when $\det M < 0$. Let

$$S = \dot{Y}(0)^\top \dot{Y}(0) \ , \ M = Y(0)^\top Y(1) \ ,$$

let λ_i, σ_i be the eigenvalues, as above. Let $d_i = \sqrt{\lambda_i}$. We know that $d_i \in [0, \pi]$ and that $\sigma_i = \cos d_i$. Since $\det M < 0$, there is no $d_i = \pi/2$, and letting

$$J_Y = \{i : \pi/2 < d_i \le \pi\}$$

then $\# J_Y$ is odd (as we proved above).

The constraints $\sigma_i = \cos d_i$ and $d_i \in [0, \pi]$ force only one possible choice in each component: $d_i = a_i$ or $d_i = \pi - a_i$, where

$$a_i = \arccos |\sigma_i| = \min\{d_i, \pi - d_i\} \in [0, \pi/2].$$

The numbers a_i depend only on the endpoint data P_0, P_1, through $|\sigma_i|$, and not on the particular admissible geodesic. Indeed $|\sigma_i|$ are the singular values of $Y_0^\top Y_1$, for any choice of oriented representatives Y_0, Y_1 of P_0, P_1. We first define the baseline length squared

$$L_0 = \sum_i a_i^2.$$

Let $J \subseteq \{1, \dots, p\}$. If $\# J$ is odd, then we can build a geodesic connecting P_0 to P_1 using the branch lengths a_i for $i \notin J$ and $\pi - a_i$ for $i \in J$, and then define its squared length by

$$L_J = \sum_{i \notin J} a_i^2 + \sum_{i \in J} (\pi - a_i)^2 = L_0 + \sum_{i \in J} (\pi^2 - 2\pi a_i).$$

This is minimized when $J = \{j\}$, where $a_j = \max_i a_i$. For this minimizing choice, the only long branch is $d_j = \pi - a_j$, and for every $i \ne j$ one has $d_i = a_i \le a_j = \pi - d_j$.

After reordering the $\lambda_i = d_i^2$ in increasing order, the long branch is indexed by p, and this is exactly condition (b). Hence the length of $\tilde{Y}$ realizes the lower bound, and $\tilde{Y}$ is minimal. ∎

COROLLARY 3.21. *Under the same hypotheses as before, let*

$$M = Y(0)^\top Y(1) = U\Sigma V^\top$$

be an SVD decomposition. Assume that the eigenvalues λ_i of S are ordered in increasing order.

Then γ is a minimizing geodesic connecting P_0 to P_1 if and only if one of the following cases occurs:

- $\det\Sigma = 0$ *and all the eigenvalues of S are less than or equal to $\pi^2/4$; or*
- $\det\Sigma > 0$, $\det U = \det V$, *and all the eigenvalues of S are less than or equal to $\pi^2/4$; or*
- $\det\Sigma > 0$ *and* $\det U = -\det V$ *and $\lambda_i \le \pi^2/4$ for $1 \le i < p$, whereas $\pi^2/4 < \lambda_p \le \pi^2$, and moreover*

  $$\sqrt{\lambda_i} \le \pi - \sqrt{\lambda_p}$$

  for $i < p$.

PROOF. Since $M = U\Sigma V^\top$, we have

$$\det M = \det U \ \det\Sigma \ \det V.$$

If $\det\Sigma = 0$, then $\det M = 0$. If $\det\Sigma > 0$, then $\det M > 0$ is equivalent to $\det U = \det V$, while $\det M < 0$ is equivalent to $\det U = -\det V$. The three alternatives are therefore exactly the two alternatives in Proposition 3.20, with the singular case $\det\Sigma = 0$ separated from the non-singular positive-determinant case. ∎

For this reason we use the following modified SVD.

DEFINITION 3.22 (**SVD$^+$**). *In the standard SVD decomposition*

$$B = U\Sigma V^\top ,$$

the entries on the diagonal of Σ are non-negative and ordered in decreasing order, but U and V need not belong to $\mathrm{SO}(p)$.

If $\det U = \det V = -1$, *we change the sign of the last column of both U and V; the diagonal matrix Σ is unchanged.*

If instead $\det U = -\det V$*, up to changing signs of the last columns, assume* $\det U = 1$ *and* $\det V = -1$*; then we change the sign of the last column of* V *and also change the sign of the last diagonal entry of* Σ*.*

We call the resulting decomposition SVD$^+$ *and denote it by*

$$B = \tilde{U}\tilde{\Sigma}\tilde{V}^\top .$$

Note that if the smallest singular value is zero, then the above operation does not change the diagonal matrix. This occurs iff $\det \Sigma = \det \tilde{\Sigma} = 0$.

Proposition 3.23. *Let* $P_0, P_1 \in Gr^+(p, H)$ *and choose oriented representatives* $Y_0, Y_1 \in St(p, H)$ *so that* $P_0 = [Y_0]_{Gr+}, P_1 = [Y_1]_{Gr+}$*. The following construction produces a minimal geodesic from* P_0 *to* P_1*. Let*

$$Y_0^\top Y_1 = \tilde{U}\tilde{\Sigma}\tilde{V}^\top$$

be an SVD$^+$ *decomposition with*

$$\tilde{\Sigma} = \mathrm{diag}(\sigma_1, \ldots, \sigma_p), \quad 1 \geq \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{p-1} \geq |\sigma_p| \geq 0 . \tag{3.25}$$

If some singular values have multiplicity, the adapted bases $\tilde{U}, \tilde{V}$ *are not unique. In the construction below we fix one such choice. Inside a non-zero equal-singular-value block, the rotations of the left and right adapted bases are linked; inside a zero singular block they are independent. See also Appendix B. These rotations may give other adapted bases, and possibly other minimizing geodesics, but the minimality argument depends only on the signed singular values* σ_i*. Set*

$$v_i = \arccos(\sigma_i), \qquad 0 \leq v_i \leq \pi .$$

We call them ***oriented principal angles****. See Figure 4.*

- *If* $\sigma_p \geq 0$*, then the horizontal geodesic constructed from the pairs* $Z(0) = Y_0\tilde{U}$*,* $Z(1) = Y_1\tilde{V}$*, as in Section 3.3, is a minimizing geodesic.*
- *We now consider the case* $\sigma_p < 0$*. We define* $Z_0 = Y_0\tilde{U}$ *and* $Z_1 = Y_1\tilde{V}$*, which represent* P_0 *and* P_1*, respectively, in* $Gr^+(p, H)$*. Let* $z_{0,i}$ *and* $z_{1,i}$ *denote their i-th columns. If* $|\sigma_i| < 1$*, we define*

$$\hat{z}_i = \frac{1}{\sqrt{1 - (\sigma_i)^2}} \left(z_{1,i} - \sigma_i z_{0,i} \right) . \tag{3.26}$$

Note that $\sigma_i = \langle z_{0,i}, z_{1,i} \rangle_H$*, so* $\hat{z}_i$ *is a unit vector parallel to the projection of* $z_{1,i}$ *onto the orthogonal complement of* $z_{0,i}$*; moreover* $\langle \hat{z}_i, z_{1,i} \rangle_H > 0$*. If* $\sigma_i = 1$*, the i-th column is constant, so we set* $\hat{z}_i = 0$ *for convenience.*

If $\sigma_p = -1$*, choose any unit vector* $\hat{z}_p$ *orthogonal to all columns* $z_{0,i}$ *and to all* $\hat{z}_i$ *with* $i < p$*; the resulting geodesic is not unique, but the endpoint and its length are independent of this choice. Note that the above choice is possible since we assumed* $\dim H \geq p+1$. *See Remark C.1 in Appendix C.*

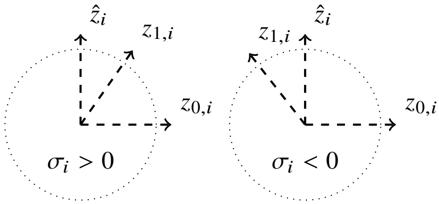

FIGURE 4. Positive and negative signed entries in the SVD⁺ construction.

Since $\sigma_p < 0$*, this gives* $v_p \in (\pi/2, \pi]$*, whereas* $v_i \in [0, \pi/2]$ *for* $i < p$*.*
We then define an initial velocity frame $W = (w_1, \ldots, w_p)$ *by*

$$w_i = v_i \hat{z}_i \ .$$

The resulting geodesic is

$$Z_i(t) = z_{0,i} \cos(tv_i) + \hat{z}_i \sin(tv_i) \tag{3.27}$$

that is

$$Z(t) = Z_0 \cos(tD) + WD^{-1} \sin(tD) \tag{3.28}$$

where $D = \mathrm{diag}(v_1, \ldots, v_p)$*, with the usual limiting interpretation when some* $v_i = 0$*, as defined in Lemma 3.2.*

Moreover,

$$d_{Gr^+}(P_0, P_1) = \left(\sum_{i=1}^{p} v_i^2 \right)^{1/2} .$$

PROOF. The formula above gives a horizontal geodesic. Indeed, in the diagonal coordinates $Z_0 = Y_0 \tilde{U}$ and $Z_1 = Y_1 \tilde{V}$, the columns satisfy

$$\langle z_{0,i}, z_{1,j} \rangle_H = 0 \quad \text{for } i \neq j, \qquad \langle z_{0,i}, z_{1,i} \rangle_H = \sigma_i .$$

Hence the two-dimensional planes generated by $z_{0,i}$ and $z_{1,i}$ are mutually orthogonal after removing the common components, and the construction above is the usual separated Grassmann geodesic in these planes.

At $t = 1$ the geodesic reaches the end point with the correct orientation. When $|\sigma_i| < 1$, $Z_i(1) = z_{1,i}$, because $\cos v_i = \sigma_i$ and $\sin v_i = \sqrt{1-\sigma_i^2}$ so we have

$$z_{1,i} = \sigma_i z_{0,i} + \sqrt{1-\sigma_i^2}\,\hat{z}_i = z_{0,i}\cos v_i + \hat{z}_i \sin v_i .$$

This is just (3.26) rearranged. If $\sigma_i = 1$, then $v_i = 0$, $\hat{z}_i = 0$, and the column is constant. If $\sigma_p = -1$, then $v_p = \pi$, and $Z_p(1) = -z_{0,p} = z_{1,p}$, independently of the admissible choice of $\hat{z}_p$.

It remains to check minimality. If $\sigma_p \geq 0$, then $v_i \leq \pi/2$ for every i, so condition (a) in Proposition 3.20 holds. If $\sigma_p < 0$, then $v_p = \pi - \arccos|\sigma_p|$ is the only long branch. Since the singular values are ordered in decreasing order,

$$\arccos \sigma_i \leq \arccos|\sigma_p| = \pi - v_p \qquad \text{for } i < p.$$

After reordering the eigenvalues of S in increasing order, this is exactly condition (b). Thus the geodesic is minimizing in both cases, and its length is $\left(\sum_i v_i^2\right)^{1/2}$. ∎

Remark 3.24. *Fix two p-planes $\hat{P}_0, \hat{P}_1 \in Gr(p,H)$. Each has two lifts to the oriented Grassmannian, denoted by $P_i, \overline{P}_i \in Gr^+(p,H)$, such that $P_i \neq \overline{P}_i$ but $[P_i]_{Gr} = [\overline{P}_i]_{Gr} = \hat{P}_i$, for $i = 1,2$.*

Let $\hat{\gamma}$ be a minimal-length geodesic between $\hat{P}_0$ and $\hat{P}_1$. Since $Gr^+(p,H) \to Gr(p,H)$ is a two-fold Riemannian covering, $\hat{\gamma}$ has two lifts, and both are minimal geodesics in Gr^+. Depending on the endpoint orientation, they are either

- *one connecting P_0 to P_1 and the other connecting $\overline{P}_0$ to $\overline{P}_1$; or*
- *one connecting P_0 to $\overline{P}_1$ and the other connecting $\overline{P}_0$ to P_1.*

Remark 3.25. *The alternatives (a) and (b) can also be read at the level of oriented representatives.*

Fix $P_0 \in Gr^+(p,H)$ and an oriented representative Y_0.

Consider an endpoint $P_1 \in Gr^+(p,H)$ with $[P_0]_{Gr} \neq [P_1]_{Gr}$. Let $\hat{P}_1 = [P_1]_{Gr}$.

The group $\mathrm{O}(p)$ has two connected components, $S^+ = \mathrm{SO}(p)$ and $S^- = \mathrm{O}(p) \setminus \mathrm{SO}(p)$. Hence the fiber of the unoriented endpoint $\hat{P}_1$ splits into two $\mathrm{SO}(p)$-orbits, which we call $\mathcal{O}_+$ and $\mathcal{O}_-$.

- *If P_0, P_1 are oblique, these two orbits are distinguished by the sign of $\det(Y_0^\top Y)$: one has positive sign and the other negative sign. We choose notation so that $Y \in \mathcal{O}_+$ iff $\det(Y_0^\top Y) > 0$, while $Y \in \mathcal{O}_-$ iff $\det(Y_0^\top Y) < 0$.*
  With this convention, geodesics satisfying (a) are the minimizing horizontal Stiefel geodesics from Y_0 to the positive-sign orbit $\mathcal{O}_+$. Geodesics satisfying (b) are the minimizing horizontal Stiefel geodesics from Y_0 to the negative-sign orbit $\mathcal{O}_-$: the

ordinary Grassmann minimizer reaches the wrong oriented lift, so one uses the cheapest possible long branch to reach the required orientation.

- *If P_0, P_1 are semi-perpendicular, then both orbits are at the same distance, and they are reached by minimal horizontal Stiefel geodesics of type (a). In this case* $\det(Y_0^\top Y) = 0$*, at least one principal angle is $\pi/2$; this is a cut-locus phenomenon, discussed in Sec. 4.3.*

3.5.1. Invariant geometric decomposition. For minimal geodesics in the oriented Grassmann manifold, the decomposition in Proposition 3.18 is mostly unchanged.

PROPOSITION 3.26. *Fix two oriented p-planes $P_0, P_1 \in Gr^+(p, H)$. Let $P(t)$ be a minimal geodesic connecting them.*

Let $\hat{P}_0, \hat{P}_1 \in Gr(p, H)$ be given by $[P_i]_{Gr} = \hat{P}_i$ for $i = 0, 1$, and let $\hat{P}(t) = [P(t)]_{Gr}$.

The decomposition $K = I \oplus V_1 \oplus \cdots \oplus V_m$ given by $\hat{P}_0, \hat{P}_1$ in Proposition 3.18 applies as well.

If the geodesic is of type (a), then $\hat{P}(t)$ is the minimal Grassmann geodesic and the motion in each block V_j has speed a_j.

If the geodesic is of type (b), then the oriented constraint changes only the motion inside the block V_m corresponding to the largest ordinary principal angle a_m: one direction in that block is traversed with speed $\pi - a_m$ instead of a_m.

PROOF. Choose oriented representatives and a horizontal lift as in the SVD+ construction above. Let $s_1 \geq \cdots \geq s_p \geq 0$ be the ordinary singular values of $Y_0^\top Y_1$, and let

$$a_i = \arccos s_i \in [0, \pi/2]$$

be the ordinary principal angles. These angles determine the endpoint decomposition of Proposition 3.18, after grouping equal values a_i into the block speeds a_j.

In type (a), the signed SVD+ entries are all non-negative. Hence the oriented geodesic uses the same angles a_i as the ordinary Grassmann minimizer, and the statement follows directly from Proposition 3.18.

In type (b), the SVD+ construction changes the sign of one entry with the smallest ordinary singular value s_p. Equivalently, it changes one ordinary angle $a_m = \arccos s_p$, the largest principal angle, into the long branch $\pi - a_m$. All other directions keep their ordinary angles. Thus only the two-dimensional direction chosen inside the eigenspace for the largest principal angle is changed. If this largest angle has multiplicity, the chosen direction may vary, but it always lies in the same endpoint-defined block V_m. Therefore the decomposition $K = I \oplus V_1 \oplus \cdots \oplus V_m$ is unchanged, and only the motion inside V_m is modified as stated. ∎

3.5.2. Examples.

Example 3.27. *Suppose that $H = \ell^2(\mathbb{N})$ or $H = \mathbb{R}^N$ (with $N > p$). Let $\theta \in [0, \pi]$. Consider the two frames in $St(p, H)$*

$$x_0 = (e_1, e_2, \dots, e_p) \,, \quad x_\theta = (e_1 \cos\theta + e_{p+1} \sin\theta, e_2, \dots, e_p) \,.$$

Then

$$d_{Gr}(x_0, x_\theta) = \min\{\theta, \pi - \theta\} \quad , \quad d_{Gr^+}(x_0, x_\theta) = \theta \,.$$

Example 3.28. *In $H = \mathbb{R}^3$ consider*

$$Y_0 = (e_1, e_3) \,,\; Y_1 = (e_2, -e_3)$$

and let $P_i = [Y_i] \in Gr^+(2, \mathbb{R}^3)$ be the corresponding oriented p-planes; these p-planes are semi-perpendicular. To build the minimal geodesic, we compute

$$M = Y_0^\top Y_1 = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

This is a singular case, since the singular values are 1 *and* 0*. We follow the steps in Definition 3.22. A standard SVD is*

$$M = U \Sigma V^\top, \qquad U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Here $\det U = -1$ *and* $\det V = 1$*. Since the last singular value is zero, the corresponding singular vectors are not uniquely determined. We may therefore change the basis in the zero singular block and choose instead*

$$\tilde U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \tilde\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde V = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then $\tilde U, \tilde V \in$ SO(2), and $M = \tilde U \tilde\Sigma \tilde V^\top$. In this singular example, the signed last entry of $\tilde\Sigma$ is still zero; hence the SVD$^+$ coincides with this SO(2)*-adapted SVD. Hence*

$$Z(0) = Y_0 \tilde U = (e_3, -e_1), \qquad Z(1) = Y_1 \tilde V = (e_3, e_2).$$

The corresponding angles are $v_1 = 0$ and $v_2 = \pi/2$, and the geodesic is

$$Z(t) = \left(e_3,\; -e_1 \cos\left(\frac{\pi t}{2}\right) + e_2 \sin\left(\frac{\pi t}{2}\right)\right).$$

This is a geodesic of type (a). It lies on the cut-locus boundary, so the minimizing geodesic is not unique. Its length is $\pi/2$.

EXAMPLE 3.29. *Let $\pi/4 < \theta < \pi/2$ and let $a = \cos\theta$ (note that $a > 0$). In $H = \mathbb{R}^3$ let $u = e_1\cos\theta + e_2\sin\theta$, and consider*

$$Y_0 = (e_1, e_3)\ ,\ Y_1 = (u, -e_3)$$

and let $P_i = [Y_i] \in Gr^+(2, \mathbb{R}^3)$ be the corresponding oriented p-planes; these p-planes are oblique. To build the minimal geodesic, we compute

$$M = Y_0^\top Y_1 = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix}$$

The matrix M is non-singular, and its singular values are 1 and a. One possible SVD$^+$ decomposition is

$$M = Y_0^\top Y_1 = \tilde{U}\tilde{\Sigma}\tilde{V}^\top$$

with

$$\tilde{U} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\ ,\ \tilde{\Sigma} = \begin{pmatrix} 1 & 0 \\ 0 & -a \end{pmatrix}\ ,\ \tilde{V} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

where $\tilde{U}, \tilde{V} \in \mathrm{SO}(2)$. Hence

$$Z(0) = Y_0\tilde{U} = (e_3, -e_1), \qquad Z(1) = Y_1\tilde{V} = (e_3, u).$$

The corresponding angles are $v_1 = 0$ and $v_2 = \arccos(-a) = \pi - \theta$, and the geodesic is

$$Z(t) = \big(e_3,\ -e_1\cos\left(t(\pi-\theta)\right) + e_2\sin\left(t(\pi-\theta)\right)\big).$$

This is a geodesic of type (b). It does not lie on the cut locus, so the minimizing geodesic is unique. Its length is $\pi - \theta$.

If $\theta = \pi/2$, then the second example would coincide with the first example. If $\pi/2 \le \theta \le \pi$, then the geodesic is of type (a), and its length is $\pi - \theta$. It is interesting to note that an infinitesimal shift of the endpoint can switch the minimizing geodesic from type (a) to type (b).

4. Cut locus and diameter

4.1 – *In the Stiefel manifold*

For the Euclidean metric on $\mathrm{St}(p, \mathbb{R}^n)$, the injectivity radius is [25]

$$\mathrm{inj}\big(\mathrm{St}(p, \mathbb{R}^n)\big) = \pi.$$

Moreover, for $n \geq 2p$, recent sharp distance bounds imply the diameter formula [14, Theorem 6.6]

$$\operatorname{diam}\big(\mathrm{St}(p, \mathbb{R}^n)\big) = \pi\sqrt{p}\ .$$

The maximal distance is attained precisely between antipodal points X and $-X$.

At present, an explicit global description of the cut locus of a point in $\mathrm{St}(p, \mathbb{R}^n)$ for the Euclidean metric does not appear to be known in a form comparable to that available for Grassmannians. The injectivity-radius result shows that geodesics are minimizing and unique below distance π, and that this bound is sharp.

In the Hilbert manifolds used in this paper, the finite-dimensional reduction in Theorem A.1 is the bridge between these finite-dimensional results and the infinite-dimensional setting.

4.2 – *In the Grassmann manifold*

For the Grassmann manifold $\mathrm{Gr}(p, \mathbb{R}^n)$, the situation is much simpler [1, 3, 21].

Proposition 4.1. *Let $P_0 = [Y_0] \in Gr(p, \mathbb{R}^n)$. Then the cut locus of P_0 consists of all P_1 that are semi-perpendicular to it:*

$$\mathrm{Cut}(P_0) = \{[Y_1] \in Gr(p, \mathbb{R}^n) : \det(Y_0^\top Y_1) = 0\}.$$

Equivalently, P_1 belongs to the cut locus of P_0 if and only if at least one principal angle between the two p-planes equals $\pi/2$.

If exactly one principal angle is equal to $\pi/2$, then P_1 is in the singular part of the cut locus, but not in the conjugate part; if at least two principal angles are equal to $\pi/2$, then P_1 is in the singular part and also in the conjugate part of the cut locus. *The proof is in Appendix C, p. 53.*[18]

See Theorem 9 in [21] or [18]. [19]

If $\theta_1, \ldots, \theta_p \in [0, \pi/2]$ denote the principal angles between P_0 and P_1, then

$$d(P_0, P_1) = \Big(\sum_{i=1}^{p} \theta_i^2\Big)^{1/2}.$$

([18]) We include the proof for completeness, and because the proof in [21] uses different coordinates and notation for $\mathrm{Gr}(p, H)$. Moreover the provided proofs will be reused in Section 4.3.3.

([19]) These references give the cut-locus characterization, but do not separate the singular and conjugate parts using the terminology adopted here.

It follows immediately that

$$\mathrm{inj}\big(\mathrm{Gr}(p, \mathbb{R}^n)\big) = \frac{\pi}{2},$$

For $n \geq 2p$,

$$\mathrm{diam}\big(\mathrm{Gr}(p, \mathbb{R}^n)\big) = \frac{\pi}{2}\sqrt{p}.$$

For $p + 1 \leq n < 2p$,

$$\mathrm{diam}\big(\mathrm{Gr}(p, \mathbb{R}^n)\big) = \frac{\pi}{2}\sqrt{n-p}.$$

The diameter is attained when all principal angles are equal to $\pi/2$, i.e. when the two p-planes are perpendicular.

The above results can be easily proved using the decomposition in Proposition 3.18. Equivalently, along a unit-speed geodesic with velocity eigenvalues v_i^2, the ordinary Grassmann cut time is $\pi/(2\max_i v_i)$; see Proposition 5.1 in [1].

Again, Theorem A.2 is the bridge between these finite-dimensional Grassmannian results and the Hilbert-manifold setting used in this paper.

4.3 – *In the oriented Grassmann manifold*

We now characterize the cut locus in $\mathrm{Gr}^+(p, H)$.

THEOREM 4.2. *Let us fix $P_0 \in Gr^+(p, H)$.*
Fix an endpoint $P_1 \in Gr^+(p, H)$ with $P_1 \neq P_0$.
Then P_1 is in the cut locus of P_0 if and only if at least one of the following conditions is satisfied.

(A) *P_0, P_1 are doubly semi-perpendicular, that is, there are vectors $u_i \neq 0, u_i \in P_0$ for $i = 0, 1$ that are mutually orthogonal and are orthogonal to P_1. Equivalently, the largest ordinary principal angle is $\pi/2$ and has multiplicity at least two.*

(B) *The largest ordinary principal angle between the underlying unoriented p-planes has multiplicity at least two, and, for any oriented representatives Y_0, Y_1, one has* $\det(Y_0^\top Y_1) \leq 0$.

(C) *P_0, P_1 represent the same unoriented p-plane.* [20]

While proving Theorem 4.2 below, we will also show in Remark 4.10 and Proposition 4.11 that, when $p \geq 2$, every point in the cut locus is both singular and conjugate.

([20]) Since $P_0 \neq P_1$, they have opposite orientations.

REMARK 4.3. *If $p = 1$, then $Gr^+(1, H) \sim S^{\dim H - 1}$. The cut locus of a point is its antipodal point, corresponding to condition (C); conditions (A) and (B) do not apply. The injectivity radius and the diameter are both equal to π. The antipodal point is singular. If* $\dim H \geq 3$, *it is also conjugate; in the limiting case* $\dim H = 2$, *the manifold is the circle S^1, and the antipodal point is singular but not conjugate.*

We will divide the proof into two parts, which will be presented in Sections 4.3.2 and 4.3.3.

LEMMA 4.4. *Choose oriented representatives $Y_0, Y_1 \in St(p, H)$ so that $P_0 = [Y_0]_{Gr+}$ and $P_1 = [Y_1]_{Gr+}$, and decompose as before:*

$$M = Y_0^\top Y_1 = \tilde{U}\tilde{\Sigma}\tilde{V}^\top \; ; \tag{4.1}$$

where $\tilde{U}, \tilde{V} \in \mathrm{SO}(p)$, and $\tilde{\Sigma}, \sigma_i$ satisfy the relations in equation (3.25). Define the oriented principal angles $v_i = \arccos(\sigma_i) \in [0, \pi]$.

Then the three conditions in the theorem are equivalent to the following conditions on the oriented principal angles.

(A) *For two distinct indices i, j, $v_i = v_j = \pi/2$.*

(B) *$v_p \geq \pi/2$ and $v_p = \pi - v_{p-1}$, that is, $\sigma_p = -\sigma_{p-1}$.*

(C) *$v_p = \pi$.*

The proof is in Appendix C, p. 54.

REMARK 4.5. *Condition (B) actually overlaps with the others:*

- *if $v_p = v_{p-1} = \pi/2$, then it overlaps with (A);*
- *if $v_p = \pi, v_{p-1} = 0$, then it overlaps with (C); indeed in that case* $\det M < 0$ *and $\sigma_p = -1$ and $\sigma_i = 1$ for $i \leq p - 1$.*

To obtain three mutually exclusive conditions, one may add $\det M < 0$ *and $v_p, v_{p-1} \notin \{0, \pi/2, \pi\}$ to (B).*

THEOREM 4.6. *Assume $p \geq 2$ and* $\dim H \geq p + 2$. *The injectivity radius of $Gr^+(p, H)$ is*

$$\pi/\sqrt{2}\;.$$

If, in addition, $\dim H \geq 2p$, *then the diameter of $Gr^+(p, H)$ is*

$$\max\{\pi, \sqrt{p}\pi/2\}\;.$$

PROOF. We consider the three types of cut locus and, for each type, compute the maximum squared distance, under the condition $\dim H \geq 2p$, and the minimum squared distance, in general, between a point and a point in its cut locus.

(A) The distance squared is

$$\pi^2/2 + \sum_{i=1}^{p-2} v_i^2$$

with $0 \le v_i \le \pi/2$ for $i \le p-2$. Hence the minimum squared distance is $\pi^2/2$, while the maximum is

$$p\pi^2/4 \, .$$

(B) We know that $v_p \in I = [\pi/2, \pi]$. The distance squared is

$$v_p^2 + (\pi - v_p)^2 + \sum_{i=1}^{p-2} v_i^2$$

with $0 \le v_i \le \pi - v_p$ for $i \le p-2$. The minimum squared distance is again

$$\min_{v_p \in I} v_p^2 + (\pi - v_p)^2 = \pi^2/2$$

while the maximum is

$$\max_{v_p \in I} v_p^2 + (p-1)(\pi - v_p)^2 = \max_{v_p \in I} \left(p v_p^2 - 2(p-1)\pi v_p + (p-1)\pi^2 \right) = \max\{\pi^2, p\pi^2/4\}$$

(it is attained at one of the extremes).

(C) The squared distance is always π^2.

Note how the extremes of case (B) coincide with the other cases, as remarked above. Therefore the minimum squared distance from a point to its cut locus is $\min\{\pi^2/2, p\pi^2/4\} = \pi^2/2$, since $p \ge 2$. Under the additional assumption $\dim H \ge 2p$, the maximum squared distance between two points is $\max\{\pi^2, p\pi^2/4\}$. ■

The finite-dimensional cases $\dim H < 2p$ can be handled by passing to orthogonal complements. We use the isomorphism

$$\mathrm{Gr}^+(p, \mathbb{R}^n) \cong \mathrm{Gr}^+(n-p, \mathbb{R}^n)$$

valid for $1 \le p \le n-1$. It associates $P \in \mathrm{Gr}^+(p, \mathbb{R}^n)$ with $P^\perp \in \mathrm{Gr}^+(n-p, \mathbb{R}^n)$, orienting $P^\perp$ so that $P + P^\perp$ is positively oriented. The case $\dim H = p+1$ reduces to the sphere case of Remark 4.3; the remaining cases are covered by the following corollary.

Corollary 4.7. *If $p + 2 \le n \le 2p - 1$, then the diameter of $Gr^+(p, \mathbb{R}^n)$ is*

$$\max\{\pi, \sqrt{n-p}\,\pi/2\} \, .$$

4.3.1. Examples. We adapt Example 3.27 as follows.

EXAMPLE 4.8. *Suppose that $p \geq 2$ and $H = \ell^2(\mathbb{N})$ or $H = \mathbb{R}^N$ (with $N \geq p+2$). Let $\theta \in [\pi/2, \pi]$. Consider the two frames in $St(p, H)$*

$$Y_0 = (e_1, e_2, \ldots, e_p)\,, \quad Y_\theta = \big(e_1 \cos\theta + e_{p+1} \sin\theta, -e_2 \cos\theta + e_{p+2} \sin\theta, e_3, \ldots, e_p\,\big).$$

Let $P_0 = [Y_0]_{Gr+}$ and $P_\theta = [Y_\theta]_{Gr+}$. We have

$$M = Y_0^\top Y_\theta = \mathrm{diag}(\cos\theta, -\cos\theta, 1, \ldots, 1)\,,$$

hence $\det M = 0$ *when $\theta = \pi/2$, whereas* $\det M < 0$ *for $\pi/2 < \theta \leq \pi$. The oriented principal angles are*

$$0, \ldots, 0, \pi - \theta, \theta\,.$$

A minimal geodesic connecting them is

$$Y(t) = \big(e_1 \cos(t\theta) + e_{p+1} \sin(t\theta), e_2 \cos(t(\pi-\theta)) + e_{p+2} \sin(t(\pi-\theta)), e_3, \ldots, e_p\big)$$

for $t \in [0, 1]$. Its length, that is, the distance between the endpoints, is

$$\sqrt{\theta^2 + (\pi-\theta)^2}\,.$$

For every $\theta \in [\pi/2, \pi]$, P_θ is in the cut locus of P_0:

- *If $\theta = \pi/2$, then P_0, P_θ are doubly semi-perpendicular, so we are in case (A).*
- *If $\pi/2 < \theta < \pi$, then P_0, P_θ are in case (B) (in the strict sense explained in Remark 4.5);*
- *If $\theta = \pi$, then we are in case (C): P_0 and P_θ are the two different orientations of the same unoriented p-plane.*

This example shows that case (B) interpolates between the other two cases.

4.3.2. Sufficient. Let us fix $P_0 \in \mathrm{Gr}^+(p, H)$, with $p \geq 2$, and fix an endpoint $P_1 \in \mathrm{Gr}^+(p, H)$ with $P_1 \neq P_0$, throughout this subsubsection.

PROPOSITION 4.9. *Suppose at least one of the conditions (A), (C) is satisfied. Then P_1 is in the cut locus of P_0, and it is both a singular point and a conjugate point to P_0.*

PROOF. As in Proposition 3.3, let

$$Y(t) = Y(0)\cos(tR) + \dot{Y}(0) R^{-1} \sin(tR)$$

be the minimal geodesic connecting $Y(0) = Y_0$ to $Y(1) = Y_1$. Up to using the SVD$^+$, we assume that R is diagonal, with $R_{i,i} = \upsilon_i, \sigma_i = \cos\upsilon_i$.

(A) Suppose that $v_i = v_j = \pi/2$, so $\sigma_i = \sigma_j = 0$; let $W = (W_1, \ldots, W_p)$ be defined by

$$W_i = -\dot{Y}_i(0), \quad W_j = -\dot{Y}_j(0), \quad W_h = \dot{Y}_h(0) \text{ for } h \neq i, j,$$

and let

$$\tilde{Y}(t) = Y_0 \cos(tR) + WR^{-1} \sin(tR) \ .$$

Then

$$\tilde{Y}_h(1) = \begin{cases} -Y_h(1) & h = i, j \\ Y_h(1) & h \neq i, j \end{cases}$$

hence $[\tilde{Y}(1)]_{\mathrm{Gr}+} = P_1$. Note that these two geodesics are of type (a).

(C) Suppose that $v_p = \pi$, so $\sigma_p = -1$. Let $W = (W_1, \ldots, W_p)$ be defined by

$$W_p = -\dot{Y}_p(0) \ , \ W_j = \dot{Y}_j(0) \text{ for } j \neq p$$

and let

$$\tilde{Y}(t) = Y_0 \cos(tR) + WR^{-1} \sin(tR) \ .$$

This geodesic connects $\tilde{Y}(0) = Y_0$ to $\tilde{Y}(1) = Y_1$. Note that these two geodesics are of type (b).

■

REMARK 4.10. *The constructions above also show how large the families of minimizing geodesics are in the degenerate cases.*

- *In case (A), let h be the multiplicity of the zero singular value, equivalently the number of indices i such that $v_i = \pi/2$. Then $h \geq 2$. On this zero block the SVD is not unique: after fixing adapted bases for the two endpoint subspaces, one may change their relative position by an element of* $\mathrm{SO}(h)$*, and this generally gives a different minimizing geodesic.* [21] *This proves that the point is singular. Moreover, since $h \geq 2$, the group* $\mathrm{SO}(h)$ *contains non-constant smooth curves. Differentiating the corresponding family of minimizing geodesics gives a non-zero Jacobi field vanishing at the endpoints; hence the endpoint is conjugate to P_0.*

([21]) This is related to the discussion of uniqueness of the SVD in the presence of zero singular values; see Appendix B.

- *In case (C), the endpoints represent the same unoriented p-plane with opposite orientations. One obtains minimizing geodesics by choosing a direction in P_0 to reverse and a normal direction through which it rotates. In particular, even after fixing the normal direction, the choice inside P_0 is parametrized by S^{p-1}. This again gives a non-trivial smooth family of minimizing geodesics with the same endpoints, so the endpoint is singular and conjugate to P_0.*

Proposition 4.11. *Suppose that (B) is satisfied. Then P_1 is in the cut locus of P_0. Let h be the multiplicity of the largest ordinary principal angle. Then $h \geq 2$, and for each unit vector $q \in \mathbb{R}^h$, there is a different geodesic $P^q(t)$ connecting the endpoints.*

Proof. Choose oriented representatives $Y_0, Y_1 \in \mathrm{St}(p, H)$ so that $P_0 = [Y_0]_{\mathrm{Gr}+}$ and $P_1 = [Y_1]_{\mathrm{Gr}+}$, and let $M = Y_0^\top Y_1$ as usual.

Assume that $v_{p-1} \neq \pi/2$; otherwise $v_p = \pi - v_{p-1} = \pi/2$ and we may use the construction in the point (A). Assume also that $v_p \neq \pi$, otherwise the construction for point (C) applies. Then we are in the situation of a type (b) geodesic, and $\det M < 0$.

We first build the type (a) geodesic using the standard SVD. We proceed as in Section 3.3. We decompose

$$M = Y_0^\top Y_1 = U \Sigma V^\top \ .$$

We assume that $\det U = 1, \det V = -1$ up to changing signs of the last columns. Let $a_i = \arccos \Sigma_{i,i}$ be the principal angles, in increasing order. Define $D = \mathrm{diag}(a_1, \ldots, a_p)$, $\hat{Z}_0 = Y_0 U$, and $\hat{Z}_1 = Y_1 V$. We then define an initial velocity frame $\hat{V}$

$$\hat{V}_i = \begin{cases} \frac{1}{\sin a_i} (\hat{Z}_{1,i} - \cos(a_i) \hat{Z}_{0,i}) & if\ a_i > 0 \\ 0 & if\ a_i = 0 \end{cases}$$

Let

$$\hat{Z}(t) = \hat{Z}_0 \cos(tD) + \hat{V} \sin(tD) \ .$$

Then $[\hat{Z}(1)]_{\mathrm{Gr}}$ is the same underlying p-plane as $[P_1]_{\mathrm{Gr}}$, but it has the opposite orientation, since $\det V = -1$. See Figure 5.

Consider the decomposition

$$K = I \oplus V_1 \oplus \cdots \oplus V_m$$

in Proposition 3.18. Let $2h = \dim V_m$ and let $\hat{p} = p - h$ for convenience. Hypothesis (B) says that $a_p = a_{p-1}$, hence $h \geq 2$ by the relation (3.20). [22]

([22]) In (3.20) the role of v_i and a_j is the opposite to the notation used here.

Let $Q \in \mathrm{SO}(p)$ be given by

$$Q = \begin{pmatrix} \mathbb{I}_{\hat{p}} & 0 \\ 0 & Q_r \end{pmatrix}$$

with $Q_r \in \mathrm{SO}(h)$. Then

$$Q\Sigma Q^\top = \Sigma$$

and

$$Y_0^\top Y_1 = UQ\Sigma Q^\top V^\top$$

is another decomposition, as explained in Appendix B. This means that

$$\hat{Z}^Q(t) = \hat{Z}_0 Q \cos(tD) + \hat{V} Q \sin(tD)$$

is again a geodesic.

If we now apply the construction of Section 3.5 (up to a double flip), we obtain

$$\tilde{Z}(t) = \tilde{Z}^Q(t) = \hat{Z}_0 Q \cos(t\tilde{D}) + \hat{V} Q \sin(t\tilde{D})$$

where $v_i = a_i$ for $i < p$ whereas

$$v_p = a_p - \pi \in [-\pi/2, 0] \; ;$$

and $\tilde{D} = \mathrm{diag}(v_1, \ldots, v_p)$. Then

$$\tilde{Z}_i(1) = \hat{Z}_i(1) \text{ for } i \le p-1, \qquad \tilde{Z}_p(1) = -\hat{Z}_p(1),$$

so $[\tilde{Z}(1)]_{\mathrm{Gr}+} = P_1$.

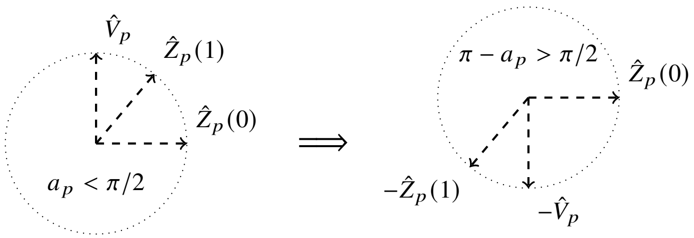

FIGURE 5. The ordinary short branch reaches the opposite oriented lift, while the long branch reaches the required oriented endpoint.

These geodesics are all minimum-length geodesics; indeed, using a single flip, we can write

$$\tilde{Z}(t) = \hat{Z}_0 QF \cos(t\hat{D}) + \hat{V} QF \sin(t\hat{D})$$

where $\hat{v}_i = a_i$ for $i < p$,

$$\hat{v}_p = \pi - a_p \in [\pi/2, \pi] ,$$

$\hat{D} = \mathrm{diag}(\hat{v}_1, \dots, \hat{v}_p)$ and $F = \mathrm{diag}(1, 1, \dots, 1, -1)$. This is a geodesic built by the SVD$^+$ construction.

We now note that $P^q(t) = [\tilde{Z}^Q(t)]_{\mathrm{Gr}+}$ depends only on the last column of Q_r, which we call q: this is the direction in which we perform the long branch. Given q, we can build any Q_r with last column q, and the geodesic $P^q(t)$ will be the same, since different choices rotate $Z_{p-h+1} \dots Z_{p-1}$ over themselves.

At the same time, if $q, \tilde{q} \in \mathbb{R}^h$ are different unit vectors, then $P^q(t)$ and $P^{\tilde{q}}(t)$ are different geodesics. Indeed, their initial tangent spaces in the last block V_m are different: the long branch is taken in different directions q and $\tilde{q}$. Hence the projected geodesics differ for small $t > 0$.

Since the family P^q is non-trivial, the endpoint is singular. Differentiating a non-constant smooth subfamily gives a non-zero Jacobi field vanishing at the endpoints, so the endpoint is also conjugate to P_0. ■

4.3.3. Necessary. We will assume $p \geq 2$ in the following.

The idea of the proof is as follows. Suppose that none of (A), (B), and (C) holds. Then the relevant inequalities are strict, so they remain true after a small reparametrization $t \mapsto rt$, with $r > 1$. Using Lemma 3.6, we will prove that the geodesic remains minimizing slightly past its endpoint; hence the endpoint is not in the cut locus.

In the following $i, j, h, h_j \geq 1$ are integers.

Let us fix $P_0, P_1 \in \mathrm{Gr}^+(p, H)$, with $p \geq 2$ and $P_1 \neq P_0$, throughout this subsubsection. Consider a minimal geodesic $\gamma : [0, 1] \to \mathrm{Gr}^+(p, H)$ connecting the two. Let v_i be its velocities, in increasing order.

We use the criterion in Proposition 3.20. Since $v_i = \sqrt{\lambda_i}$, we know that $v_i \leq \pi$, and that, if $\pi/2 < v_p \leq \pi$, then

$$v_i \leq \pi - v_p \ , i \leq p - 1 \ .$$

If $v_p = \pi$, then $v_i = 0$ for $i \leq p - 1$, so we are in case (C) and then that geodesic is not unique, as shown in Proposition 4.9.

We then consider the case $v_p < \pi$, so we can use Proposition 3.18. Let

$$w_i = \arccos(|\cos v_i|) = \min\{v_i, \pi - v_i\}$$

be the reduced velocities. Then the w_i, up to ordering, are also the principal angles of P_0, P_1. Let $0 < \tau_1 < \dots < \tau_m$ be the distinct positive principal angles of P_0, P_1. Let $I = P_0 \cap P_1, K = P_0 + P_1$, and use the decomposition

$$K = I \oplus V_1 \oplus \dots \oplus V_m \ .$$

We remark that this decomposition depends only on the endpoints.

We want to understand whether an extension of γ is still a minimizing geodesic. Let $\theta(t) = \gamma(rt)$, with $r > 1$. Also let

$$\tilde{M} = \theta(0)^\top \theta(1).$$

Then, by Lemma 3.6, the principal velocities of θ are $\tilde{v}_i = rv_i$. For $r > 1$ sufficiently close to 1, we still have $\tilde{v}_i < \pi$, so θ admits the same geometric decomposition as γ.

Let $h_j = \dim(P_0 \cap V_j) = \frac{1}{2} \dim V_j$. We know that in each space V_j, the geodesic has the form

$$\gamma(t) \cap V_j = \operatorname{span}\left\{\cos(ta_{1,j})e_1 + \sin(ta_{1,j})e_{h+1}, \dots, \cos(ta_{h,j})e_h + \sin(ta_{h,j})e_{2h}\right\}$$

where $h = h_j$ and, for each $i \leq h$, $a_{i,j}$ is either τ_j or $\pi - \tau_j$.

At the same time, these velocities are those of the original geodesic, that is, as multisets,

$$\left\{a_{i,j} : i \leq h_j, j \leq m\right\} = \{v_i : v_i > 0\}$$

but we know that $0 \leq v_i \leq \pi/2$ for $i \leq p-1$: hence all those velocities are short, $a_{i,j} = \tau_j$, except possibly one long branch $a_{i,m} = \pi - \tau_m = v_p$, which happens inside V_m.

In particular, in all the principal spaces V_j with $j < m$, we have $\tau_j < \pi/2$, so the geodesic is uniquely determined from the endpoints by the same proof as in Proposition 4.1. It remains only to study what happens inside $\gamma(t) \cap V_m$.

We now consider all possible cases.

- If $\pi/2 < v_p < \pi$, then we split into two subcases:
  - If $\dim V_m = 2$, then there are no degrees of freedom: $\gamma(t) \cap V_m$ is an oriented line that has to travel minimally to another oriented line, with an angle less than π, hence there is only one geodesic; moreover, since case (B) does not hold, we have $v_i < \pi - v_p$ for $i < p$. This strict inequality persists for all $r > 1$ close enough to 1, giving

    $$\tilde{v}_i < \pi - \tilde{v}_p \quad \text{for } i < p, \qquad \pi/2 < \tilde{v}_p < \pi.$$

    Consequently, by equation (3.21), $\tilde{M}$ has negative determinant, and θ satisfies the long-branch case of Proposition 3.20; it is minimizing and extends γ past $t = 1$.
  - If $\dim V_m \geq 4$, this is (B), and there are multiple geodesics by Proposition 4.11.
- If $v_p = \pi/2$, then we split into two subcases:

  - If $\dim V_m = 2$, then there are no degrees of freedom, so there is only one geodesic; since case (A) does not hold, $v_i < \pi/2$ for $i < p$. For $r > 1$ close enough to 1, this gives
$$\tilde{v}_i < \pi - \tilde{v}_p \quad \text{for } i < p, \qquad \pi/2 < \tilde{v}_p < \pi.$$
Again, by equation (3.21), $\tilde{M}$ has negative determinant, so θ is minimizing by the long-branch case of Proposition 3.20; it extends γ past $t = 1$.
  - If $\dim V_m \geq 4$, this is (A), and there are multiple geodesics by Proposition 4.9.
- If $v_p < \pi/2$, then we have uniqueness by the argument in Proposition 4.1. Choosing $r > 1$ sufficiently close to 1, all reparametrized velocities satisfy $\tilde{v}_i < \pi/2$, so θ is minimizing by the short-branch case of Proposition 3.20.

Therefore, if none of the three conditions occurs, every minimizing geodesic from P_0 to P_1 extends as a minimizing geodesic past P_1. Hence P_1 is not in the cut locus.

A. From infinite to finite dimensions

We recall Theorem 3 from [7] for the reader's convenience.

THEOREM A.1. *Let H be a Hilbert space with $\dim H \geq 2p$. Consider a $2p$-dimensional Hilbert space W and an isometric linear embedding $i : W \to H$. Then i induces an isometric embedding*
$$i_* : St(p, W) \to St(p, H), \qquad (x_1, \ldots, x_p) \mapsto (ix_1, \ldots, ix_p) .$$

(1) *$i_*\big(St(p, W)\big)$ is totally geodesic in $St(p, H)$.*

(2) *Let d_W be the distance in $St(p, W)$ and similarly d_H in $St(p, H)$. Then*
$$d_W(x, y) = d_H\big(i_*(x), i_*(y)\big) . \tag{A.1}$$

(3) *Let $x, y \in St(p, W)$, and let γ be a minimal geodesic connecting x to y in $St(p, W)$. Then $i_* \circ \gamma$ is a minimal geodesic connecting $i_*(x)$ to $i_*(y)$ in $St(p, H)$.*

(4) *The diameter of $St(p, H)$ is equal to the diameter of $St(p, \mathbb{R}^{2p})$.*

(5) *Any two points $x, y \in St(p, H)$ can be connected by a minimal geodesic γ. Any minimal geodesic lies in $St(p, U)$, where U is a $2p$-dimensional subspace of H depending on γ.*

(6) *Let $x, y \in St(p, H)$. Then y is in the cut locus of x if and only if there is a $2p$-dimensional subspace W of H and $\tilde{x}, \tilde{y} \in St(p, W)$ such that $x = i_*(\tilde{x})$, $y = i_*(\tilde{y})$, and $\tilde{y}$ is in the cut locus of $\tilde{x}$ in $St(p, W)$.*

Note that point *(5)* in the above theorem implies that minimal geodesics can be numerically computed using a finite-dimensional algorithm; see Section 3.3.4 in [19].

The corresponding statement for Grassmannians is Theorem 6 in [7].

THEOREM A.2. *Theorem A.1 remains valid when Stiefel manifolds are replaced by Grassmannians. In particular, for any two points $x, y \in Gr(p, H)$, there is a minimal geodesic γ connecting x to y. The same holds for the oriented Grassmannian $Gr^+(p, H)$ of oriented p-planes.*

B. Uniqueness of SVD

Let $B \in \mathbb{R}^{p\times p}$, and let

$$B = U\Sigma V^\top, \qquad U, V \in \mathrm{O}(p), \qquad \Sigma = \mathrm{diag}(\sigma_1, \dots, \sigma_p),$$

where

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0.$$

For each singular value σ, define

$$I_\sigma := \{\, i \in \{1, \dots, p\} : \sigma_i = \sigma \,\},$$

and

$$U_\sigma := \mathrm{span}\{\, U_i : i \in I_\sigma \,\}, \qquad V_\sigma := \mathrm{span}\{\, V_i : i \in I_\sigma \,\},$$

where U_i and V_i denote the i-th columns of U and V, respectively.

Then the singular values σ_i, counted with multiplicity, are uniquely determined by B. Moreover, for every $\sigma > 0$, the subspaces U_σ and V_σ are uniquely determined by B, and

$$\dim U_\sigma = \dim V_\sigma = \#I_\sigma.$$

The individual columns U_i, V_i with $i \in I_\sigma$ are not uniquely determined; they may be changed by a common orthogonal transformation on the block I_σ.

For $\sigma = 0$, the corresponding subspaces U_0 and V_0 are

$$U_0 = \ker B^\top, \qquad V_0 = \ker B,$$

and are also uniquely determined, although the relation between bases in U_0 and V_0 is independent.

Equivalently, if

$$B = \widetilde{U}\Sigma\widetilde{V}^\top$$

is another SVD with the same ordered diagonal matrix Σ, then for each $\sigma > 0$,

$$\operatorname{span}\{U_i : i \in I_\sigma\} = \operatorname{span}\{\widetilde{U}_i : i \in I_\sigma\},$$

and

$$\operatorname{span}\{V_i : i \in I_\sigma\} = \operatorname{span}\{\widetilde{V}_i : i \in I_\sigma\}.$$

C. Proofs

REMARK C.1. *We justify a comment used in Prop. 3.23. If H is infinite-dimensional, the orthogonal complement of the finite-dimensional forbidden span is automatically non-trivial. Suppose instead that* $\dim H < \infty$. *Let*

$$q = \dim \operatorname{span}\big(\{z_{0,i} : i \le p\} \cup \{\hat{z}_j : j < p\}\big).$$

Let $I = P_0 \cap P_1$ and

$$k = \dim I = \#\{i : v_i = 0\} = \#\{i : \sigma_i = 1\} = \#\{i : \hat{z}_i = 0\}.$$

Then $q = 2p - 1 - k$. If $\sigma_p = -1$, we can choose the unit vector $\hat{z}_p$ iff $q \le \dim H - 1$, that is

$$2p - 1 - k \le \dim H - 1$$

and this is equivalent to $k \ge 2p - \dim H$, that follows from $k = \dim I = \dim(P_0 \cap P_1)$ and $\dim(P_0 + P_1) = 2p - k \le \dim H$.

Proof of 2.4. We recall the simple proof that, if $p + 1 \le \dim H$, then $\operatorname{St}(p, H)$, $\operatorname{Gr}(p, H)$, and $\operatorname{Gr}^+(p, H)$ are connected.

Suppose $p + 1 \le \dim H$. Let $Y_0, Y_1 \in \operatorname{St}(p, H)$. Let W_0 be the finite-dimensional subspace spanned by the columns of Y_0 and Y_1. Choose a finite-dimensional subspace $W \subset H$ such that

$$W_0 \subset W, \qquad \dim W = n \ge p + 1.$$

Then $Y_0, Y_1 \in \operatorname{St}(p, W)$. After identifying W with $\mathbb{R}^n$, we may regard Y_0 and Y_1 as $n \times p$ matrices with orthonormal columns. Since $n > p$, each Y_i can be completed to a matrix $M_i \in \operatorname{SO}(n)$, $i = 0, 1$. Define

$$Q = M_1 M_0^\top \in \operatorname{SO}(n).$$

Since every element of $\operatorname{SO}(n)$ is the exponential of a skew-symmetric matrix, there exists A such that

$$Q = e^A.$$

Therefore

$$\gamma(t) = e^{tA} Y_0, \qquad t \in [0,1],$$

is a continuous path in $\mathrm{St}(p, W) \subseteq \mathrm{St}(p, H)$ connecting Y_0 to Y_1. Hence $\mathrm{St}(p, H)$ is connected.

In finite dimension, when $p < n$, one may also use the homogeneous-space description

$$\mathrm{St}(p, \mathbb{R}^n) \cong \mathrm{SO}(n)/\mathrm{SO}(n-p)$$

which immediately implies connectedness, since $\mathrm{SO}(n)$ is connected.

Now consider the natural projection maps

$$\pi \colon \mathrm{St}(p, H) \longrightarrow \mathrm{Gr}(p, H),$$

and

$$\pi^+ \colon \mathrm{St}(p, H) \longrightarrow \mathrm{Gr}^+(p, H),$$

Both maps are continuous and surjective. Since the continuous image of a connected space is connected, it follows that $\mathrm{Gr}(p, H)$ and $\mathrm{Gr}^+(p, H)$ are connected.

Notice that the natural forgetful map goes in the opposite direction,

$$\mathrm{Gr}^+(p, H) \longrightarrow \mathrm{Gr}(p, H),$$

because an unoriented p-plane has no canonical choice of orientation. Thus one should not use a map $\mathrm{Gr}(p, H) \to \mathrm{Gr}^+(p, H)$. Instead, both Grassmannians are obtained as continuous images of the Stiefel manifold. ■

Proof of 2.7. Let $M = \mathrm{St}(p, H)$. For $V \in T_P M$ we identify $T_V T_P M = T_P M$. The following are equivalent.

- $Q = \gamma(r)$ is a conjugate point along the geodesic with positive multiplicity;
- $Z \in T_P M$ is non-zero and belongs to the kernel of $d \exp_P$ at rV;
- there is a Jacobi field $\eta : [0, r] \to TM$ that projects on γ and satisfies

$$\eta(0) = 0 \ , \ D_\alpha \eta(0) = Z \ , \ \eta(r) = 0 \ ;$$

- setting

  $$Y(t, s) = \exp_P\big(t(V + sZ)\big)$$

  for $t \in [0, r], s \in [-1, 1]$ we have

  $$\left.\frac{\partial}{\partial s}\right|_{s=0} Y = \eta \ .$$

This follows from standard results, see [10] Section 1.12.

In particular, let E be the span of the columns of P, V, Z. Then every geodesic $Y(\cdot, s)$ is contained in $\mathrm{St}(p, E)$, and hence the Jacobi field is tangent to $\mathrm{St}(p, E)$ along $Y(\cdot, 0)$.

This can also be shown analytically, which may be useful for numerical computations. We consider the closed form formula in Proposition 3.1, here rewritten as

$$(Y(t), \dot{Y}) = (P, V + sZ) \exp\big(t\mathcal{M}(s)\big) \exp\big(-t\mathcal{N}(s)\big) \tag{C.1}$$

with

$$\mathcal{M}(s) = \begin{pmatrix} A_0 & -S_0 \\ \mathbb{I} & A_0 \end{pmatrix} \quad , \quad \mathcal{N}(s) = \begin{pmatrix} A_0 & 0 \\ 0 & A_0 \end{pmatrix}$$

and

$$\begin{aligned} A_0 =& P^\top (V + sZ) = P^\top V + sP^\top Z \ , \\ S_0 =& (V + sZ)^\top (V + sZ) = V^\top V + sV^\top Z + sZ^\top V + s^2 Z^\top Z \ . \end{aligned}$$

Let

$$F(t) = \frac{\partial}{\partial s}\Big|_{s=0} \exp\big(t\mathcal{M}(s)\big) \ , \quad G(t) = \frac{\partial}{\partial s}\Big|_{s=0} \exp\big(-t\mathcal{N}(s)\big) \ ;$$

An analytic formula can be written using the explicit formula for the Fréchet derivative

$$D(\exp X)(X; Z) = \int_0^1 \exp\,(tX) Z \exp\,((1-t)X)\, \mathrm{d}t,$$

which can be found in [16] [23]. Differentiating it, we obtain

$$\begin{aligned} \frac{\partial}{\partial s}\Big|_{s=0} \big(Y(t,s), \dot{Y}(t,s)\big) =& (0, Z) \exp\big(t\mathcal{M}(0)\big) \exp\big(-t\mathcal{N}(0)\big) \\ & + (P, V) F(t) \exp\big(-t\mathcal{N}(0)\big) + (P, V) \exp\big(t\mathcal{M}(0)\big) G(t) \end{aligned}$$

Since the above are right matrix multiplications, each column in $\eta(t)$ is contained in the span E of the columns of P, V, Z.

The converse is obvious, since $\mathrm{St}(p, E)$ is totally geodesic in $\mathrm{St}(p, H)$.

In the Grassmannian and the oriented Grassmannian, we first use horizontal Stiefel lifts of the geodesics.

∎

([23]) See also Appendix B in [19].

Proof of 2.10. Let $A = Y_0^\top Y_1$. If $x, y \in \mathbb{R}^p$, then

$$\langle Y_0 x, Y_1 y\rangle_H = x^\top A y.$$

Hence P_0 and P_1 are perpendicular if and only if $x^\top A y = 0$ for all x, y, which is equivalent to $A = 0$.

Moreover, a non-zero vector $Y_0 x \in P_0$ is orthogonal to every vector in P_1 if and only if $x^\top A y = 0$ for every y, that is, if and only if $x^\top A = 0$. Such a non-zero x exists if and only if A is singular. Equivalently, there exists a non-zero vector $Y_1 y \in P_1$ orthogonal to P_0 if and only if $Ay = 0$, again if and only if A is singular. Therefore P_0 and P_1 are semi-perpendicular if and only if $\det(Y_0^\top Y_1) = 0$, and they are oblique if and only if $\det(Y_0^\top Y_1) \neq 0$.

Finally, replacing Y_i by another Stiefel representative $Y_i Q_i$, with $Q_i \in \mathrm{O}(p)$, changes A to $Q_0^\top A Q_1$. This preserves the properties $A = 0$, $\det A = 0$, and $\det A \neq 0$, so the conditions are independent of the representatives. ∎

Proof of 3.1. We recall the argument in Section 2.2.2 of [5]. The tangent space at Y is

$$T_Y \mathrm{St}(p, \mathbb{R}^n) = \{Z : Y^\top Z + Z^\top Y = 0\}.$$

In particular, along any differentiable path in the Stiefel manifold, $A(t) = Y^\top(t)\dot{Y}(t)$ is skew-symmetric.

With the Euclidean metric induced from $\mathbb{R}^{n\times p}$, the normal space at Y is

$$N_Y \mathrm{St}(p, \mathbb{R}^n) = \{YB : B \in \mathbb{R}^{p\times p},\ B^\top = B\}.$$

A constant-speed curve is therefore a geodesic exactly when its acceleration is normal to the Stiefel manifold. Since $S(t) = \dot{Y}^\top(t)\dot{Y}(t)$ is symmetric, the equation

$$\ddot{Y} = -YS$$

has normal acceleration and gives (3.1).

We now record the conserved quantities used in the closed formula. Substituting (3.1), we obtain

$$\dot{A} = \dot{Y}^\top \dot{Y} + Y^\top \ddot{Y} = S - Y^\top Y S = 0,$$

hence $A(t) = A_0$. Similarly,

$$\dot{S} = \ddot{Y}^\top \dot{Y} + \dot{Y}^\top \ddot{Y} = -SA_0 + A_0 S,$$

so

$$S(t) = e^{A_0 t} S_0 e^{-A_0 t}.$$

Taking the trace shows that the speed $\mathbf{tr}(\dot{Y}^\top \dot{Y}) = \mathbf{tr} S(t)$ is constant.

It remains to verify the closed form. Set $E(t) = e^{A_0 t}$, $W(t) = Y(t)E(t)$, and $V(t) = \dot{Y}(t)E(t)$. Since $S(t)E(t) = E(t)S_0$, the equation (3.1) is equivalent to the constant-coefficient first-order system

$$(W, V)' = (W, V) \begin{pmatrix} A_0 & -S_0 \\ \mathbb{I} & A_0 \end{pmatrix}.$$

Solving this system with initial data $W(0) = Y(0)$, $V(0) = \dot{Y}(0)$ gives exactly (3.2). ■

Proof of 3.2. Indeed

$$\frac{\partial}{\partial t} \cos(tB) = \sum_{k=1}^{\infty} \frac{(-1)^k A^k t^{2k-1}}{(2k-1)!} = -\sum_{k=0}^{\infty} \frac{(-1)^k A^{k+1} t^{2k+1}}{(2k+1)!} = -B\sin(tB) ,$$

$$\frac{\partial}{\partial t} B\sin(tB) = \sum_{k=0}^{\infty} \frac{(-1)^k A^{k+1} t^{2k}}{(2k)!} = A\cos(tB) ,$$

and similarly for the third one. ■

Proof of 3.3. Put $Y_0 = Y(0)$ and $V_0 = \dot{Y}(0)$. Since $R^2 = S$ and every analytic function of R commutes with R, the displayed formula gives

$$Y(t) = Y_0 \cos(tR) + V_0 R^{-1} \sin(tR)$$

and hence

$$\dot{Y}(t) = -Y_0 R \sin(tR) + V_0 \cos(tR).$$

Differentiating once more,

$$\ddot{Y}(t) = -Y_0 R^2 \cos(tR) - V_0 R \sin(tR) = -Y(t)R^2 = -Y(t)S.$$

Thus Y satisfies $\ddot{Y} + YS = 0$, which is (3.1) in the horizontal case, where $S = \dot{Y}^\top \dot{Y}$ is constant. At $t = 0$ the formula gives $Y(0) = Y_0$ and $\dot{Y}(0) = V_0$, so it is the required solution.

Note also that differentiating (3.8) we obtain the second set of columns in (3.7), namely

$$\dot{Y}(t) = -Y(0)R\sin(tR) + \dot{Y}(0)\cos(tR) \quad ; \tag{C.2}$$

hence (3.8) and (3.7) are equivalent. ■

Proof of 3.11. We know that $Y(t)$ satisfies all properties in Proposition 3.3. By multiplying equation (3.8) on the left by $Y(0)^\top$ and by $\dot{Y}(0)^\top$, respectively, we obtain that

$$Y(0)^\top Y(1) = \cos(R) \,, \tag{C.3}$$

$$\dot{Y}(0)^\top Y(1) = R\sin(R) \,. \tag{C.4}$$

In particular, $Y(0)^\top Y(1)$ and $\dot{Y}(0)^\top Y(1)$ are symmetric. Again, by the Spectral Theorem, let $O \in \mathrm{SO}(p)$ be an orthogonal matrix such that $D = O^\top RO$ is diagonal with non-negative entries (and then $D^2 = O^\top SO$), so

$$O^\top Y(0)^\top Y(1)O = \cos(D) \tag{C.5}$$

$$O^\top \dot{Y}(0)^\top Y(1)O = D\sin(D) \tag{C.6}$$

but then $O^\top Y(0)^\top Y(1)O$ and $O^\top \dot{Y}(0)^\top Y(1)O$ are diagonal. ■

Proof of 3.12. We note that in the previous proof $|\sigma_i|$ are also the singular values of $Y_0^\top Y_1$, and similarly $|\theta_i|$ are the singular values of $\dot{Y}(0)^\top Y_1$, because right multiplication by $Q \in \mathrm{O}(p)$ does not change singular values. ■

Proof of 3.17. Let $I = P \cap Q$ and $k = \dim I$. Let $b_j = \cos\tau_j$. Let us trace through the definition 2.9. The first step is to consider the set

$$Z_0 = \{(u,v) \in P \times Q : \|u\| = 1 \,,\ \|v\| = 1 \,,\ \langle u,v\rangle = 1\}$$

but the conditions imply that $u = v$, so

$$Z_0 = \{(u,u) : u \in I,\ \|u\| = 1\}$$

and in particular $I \neq \{0\}$ iff 0 is a principal angle. Let us set $P_0 = Q_0 = I$. Recursively, for $j = 1, \ldots, m$, define

$$\begin{aligned}
\tilde{b}_j &:= \max_{\substack{u\in P,\ \|u\|=1\\ u\perp P_{j-1}}} \ \max_{\substack{v\in Q,\ \|v\|=1\\ v\perp Q_{j-1}}} \langle u,v\rangle, \\
Z_j &:= \Big\{(u,v) \in P\times Q : \langle u,v\rangle = \tilde{b}_j \,,\ \|u\| = 1 \,,\ \|v\| = 1, \\
&\qquad\quad u \perp P_{j-1} \,,\ v \perp Q_{j-1}\Big\} \\
P_j &:= P_{j-1} \oplus \mathrm{span}\{\pi_P Z_j\} \\
Q_j &:= Q_{j-1} \oplus \mathrm{span}\{\pi_Q Z_j\}
\end{aligned}$$

where π_P, π_Q are the canonical projections from $P \times Q$. Note that Z_j is the "argmax" of the first line. Then we define

$$V_j = \mathrm{span}\big\{\pi_P Z_j \cup \pi_Q Z_j\big\} \,.$$

By induction on j, the max–max construction gives $\tilde b_j = b_j$, and these spaces satisfy the stated properties.

Note that the vectors u_i, v_i defined in 2.9, when grouped by ''same angle τ_j'' are a basis for V_j:

$$V_j = \operatorname{span}\{u_i, v_i : \theta_i = \tau_j\} \ .$$

Another proof may be obtained by studying the SVD in Definition 2.8. In an SVD-adapted pair of orthonormal bases, V_j is the span of the principal vector pairs whose singular value is $\cos\tau_j$. The uniqueness criterion for *''uniqueness of the SVD''*, see Appendix B, then proves that this decomposition is unique. ■

Proof of 3.18. We use Lemma 3.11. Up to lifting and possibly rotating Y_1 by $Q \in \mathrm{O}(p)$, this geodesic γ is represented by a horizontal geodesic $Y : [0, 1] \to \mathrm{St}(p, H)$ with $Y_0 = Y(0)$ and $Y_1 Q = Y(1)$. Let $W = \dot Y(0)$.

Then the matrices

$$M = Y(0)^\top Y(1) = Y_0^\top Y_1 Q \quad \text{and} \quad N = \dot Y(0)^\top Y(1) = W^\top Y_1 Q$$

are symmetric and, together with $S = \dot Y(0)^\top \dot Y(0)$, admit a common orthonormal eigenbasis.

By (3.12)

$$\sigma = \cos\sqrt{\lambda}\ , \quad \theta = \sqrt{\lambda}\sin\sqrt{\lambda}\ . \tag{C.7}$$

since $0 < \sqrt{\lambda} < \pi$ for the positive eigenvalues under discussion, different λ produce different pairs (σ, θ) and vice versa.

Thus the corresponding common eigenspaces of M, N, S are tied together, and the eigenspaces determine the endpoint-dependent summands V_j. Precisely,

- V_j is the span of $Y(0)z, \dot Y(0)z$, where z is an eigenvector associated to λ with $\sqrt{\lambda} = \tau_j$ or $\sqrt{\lambda} = \pi - \tau_j$; equivalently
- V_j is the span of $Y(0)z, Y(1)z$, where z is an eigenvector associated to λ with $\sqrt{\lambda} = \tau_j$ or $\sqrt{\lambda} = \pi - \tau_j$.

These two facts are equivalent. Let $a = \sqrt{\lambda}$ for convenience; since $Rz = az$, we have

$$\cos(tR)z = \cos(ta)z\ , \ \sin(tR)z = \sin(ta)z$$

hence

$$Y(t)z = Y(0)\cos(tR)z + \dot Y(0)R^{-1}\sin(tR)z = Y(0)z\cos(ta) + \frac{1}{a}\dot Y(0)z\sin(ta)$$

so

$$Y(1)z = Y_0 z\cos(a) + \frac{1}{a}\dot{Y}(0)z\sin(a)$$

but also

$$\dot{Y}(0)z = \frac{a}{\sin(a)}\big(Y(1)z - Y_0 z\cos(a)\big)\ .$$

■

Proof of 4.1. If a principal angle is $\pi/2$, then an easy construction shows that P_1 is in the cut locus. Indeed if

$$Y(t) = Y(0)\cos(tR) + \dot{Y}(0)R^{-1}\sin(tR)$$

is a minimal geodesic, with R diagonal, and $R_{p,p} = \pi/2$, then let $W = (W_1, \dots, W_p)$ be defined by

$$W_p = -\dot{Y}_p(0), \quad W_h = \dot{Y}_h(0) \text{ for } h \neq p,$$

so

$$\tilde{Y}(t) = Y_0\cos(tR) + WR^{-1}\sin(tR)$$

is a different minimal geodesic. At $t = 1$, its p-th column is the negative of the p-th column of $Y(1)$, while all other columns are unchanged; hence $\tilde{Y}(1)$ represents the same point $P_1 \in \mathrm{Gr}(p, \mathbb{R}^n)$.

This construction is similar to the one for point (A) in Proposition 4.9; when at least two such angles occur, the zero block also gives a non-trivial family of minimizing geodesics, hence a conjugate point by Proposition 2.3.

Suppose instead that exactly one angle $\pi/2$ occurs. By Proposition 3.17, equal principal angles, equivalently equal singular values, belong to the same orthogonal block. The Jacobi equation along the geodesic, written in principal coordinates, splits according to this orthogonal block decomposition. On every block with principal angle $< \pi/2$, the geodesic is still before the first conjugate time, so no non-zero endpoint-vanishing Jacobi component can occur there. The block associated with the single angle $\pi/2$ is two-dimensional, hence its Grassmann factor is $\mathrm{Gr}(1, \mathbb{R}^2)$; in this one-dimensional factor there is no non-zero Jacobi field vanishing at both endpoints. Hence P_1 is not a conjugate point.

Conversely, if all principal angles are $< \pi/2$, we prove that the minimal geodesic is unique, and its endpoint is not in the cut locus. We use Proposition 3.18 to decompose

$$K = I \oplus V_1 \oplus \cdots \oplus V_m \tag{C.8}$$

into principal spaces. On I, any minimizing geodesic is constant. On each V_j, the common principal angle is $a_j \in (0, \pi/2)$.

Let F_0 be an orthonormal basis of $P_0 \cap V_j$, and let F_1 be the corresponding principal basis of $P_1 \cap V_j$. Then

$$E_1 = \frac{F_1 - \cos(a_j)F_0}{\sin(a_j)}$$

is uniquely determined up to the same right orthogonal change of basis as F_0, F_1. Hence the subspace curve

$$\operatorname{span}\{\cos(ta_j)F_0 + \sin(ta_j)E_1\}$$

is independent of the chosen adapted bases. Thus each principal block, and hence the whole minimizing geodesic, is unique. Moreover, if the geodesic is extended by a little bit, as in Section 4.3.3, the principal velocities are still less than $\pi/2$, so the extension is still minimizing. ■

Proof of 4.4. Let $s_1 \geq \cdots \geq s_p \geq 0$ be the ordinary singular values of $M = Y_0^\top Y_1$. The ordinary principal angles are $a_i = \arccos s_i$. By the construction of the SVD$^+$,

$$\sigma_i = s_i \quad \text{for } i < p, \qquad |\sigma_p| = s_p,$$

and the sign of σ_p is the sign of $\det M$, with the convention that $\sigma_p = 0$ when $\det M = 0$.

For (A), the existence of two orthogonal non-zero vectors in P_0 that are orthogonal to P_1 is equivalent to $\dim \ker M^\top \geq 2$. This is equivalent to M having at least two zero singular values, hence to at least two ordinary principal angles equal to $\pi/2$. Since zero singular values remain zero in the SVD$^+$, this is equivalent to $v_i = v_j = \pi/2$ for two distinct indices.

For (B), the largest ordinary principal angle has multiplicity at least two if and only if the smallest ordinary singular value has multiplicity at least two, that is, $s_{p-1} = s_p$. Together with $\det M \leq 0$, this is equivalent to

$$\sigma_p = -\sigma_{p-1}.$$

Indeed, if $\det M < 0$, then $\sigma_p = -s_p$ and $\sigma_{p-1} = s_{p-1}$; if $\det M = 0$, then $s_p = 0$, and the equality says $s_{p-1} = 0$. Since $v_i = \arccos \sigma_i$, the equality $\sigma_p = -\sigma_{p-1}$ is equivalent to $v_p = \pi - v_{p-1}$, and $\det M \leq 0$ gives $v_p \geq \pi/2$.

Finally, (C) holds if and only if the underlying unoriented p-planes are equal and the orientations are opposite. This is equivalent to all ordinary singular values being equal to 1, and to $\det M < 0$. In the SVD$^+$, this gives $\sigma_i = 1$ for $i < p$ and $\sigma_p = -1$, hence $v_p = \pi$. Conversely, if $v_p = \pi$, then $\sigma_p = -1$; by (3.25), all ordinary singular values are equal to 1, and the sign is negative, so the oriented p-planes represent the same unoriented p-plane with opposite orientations. ■

D. Statements and Declarations

D.1 – *Tool and computational resource disclosure*

During the preparation of this manuscript, the author used automated tools, mostly *OpenAI Codex* based on *GPT-5.5*, for editorial support, LaTeX maintenance, bibliography organization, English phrasing, and interactive checking of notation and proofs. The mathematical arguments and proofs are the author's responsibility; the assistant's role was akin to that of a reviewer, checking the prose and proofreading the presentation.

In very few instances the assistant, while checking the material, suggested standard tools or proof strategies. The most prominent case was the use of Minkowski's integral inequality in Section 3.3. This result is not fundamental to the main results of the paper[24], but it provides an alternative proof of a well-known result, and was therefore included.

The author takes full responsibility for having verified any material proposed by the assistant before including it in the manuscript.

No proof assistant, computer algebra system, or numerical computation was used as evidence for the theorems.

This disclosure is included in the spirit of the *Leiden Declaration on Artificial Intelligence and Mathematics* [12].

D.2 – *Competing Interests*

The author has no competing interests to declare.

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(24) Section 3.3 may be deleted without affecting the rest of the paper.

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Contents

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