BOUNDED VARIATION SPACES WITH GENERALIZED ORLICZ GROWTH RELATED TO IMAGE DENOISING

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ABSTRACT. Motivated by the image denoising problem and the undesirable stair-casing effect of the total variation method, we introduce bounded variation spaces with generalized Orlicz growth. Our setup covers earlier variable exponent and double phase models. We study the norm and modular of the new space and derive a formula for the modular in terms of the Lebesgue decomposition of the derivative measure and a location dependent recession function. We also show that the modular can be obtained as the Γ-limit of uniformly convex approximating energies.

1. INTRODUCTION

In PDE-based image processing, a function $u : \Omega \to \mathbb{R}$ represents the gray-scale intensity at each location of an image. Edges of objects correspond to discontinuities of u and make this field challenging for function spaces and the calculus of variations. The space BV of functions of bounded variation has proven to be useful in the field. We refer to the book [3] by Aubert and Kornprobst for an overview. The classical ROF image restoration/denoising model [43] calls for minimizing the energy

$$\inf_{u\in \mathrm{BV}(\Omega)} \int_\Omega |Du| + |u-f|^2\,dx,$$

where $f \in L^2(\Omega)$ is the given, corrupted input image that is to be restored. The *fidelity term* $|u-f|^2$ forces u to be close to f on average, whereas the *regularizing term* $|Du|$ limits the variation of u. This model suffers from a stair-casing effect that leads to piecewise constant minimizers [8, 34]. For a recent overview of autonomous variants of the model we refer to [41]. Recession functions are often used in relaxation including in image processing (see, e.g., [2, 41]) and there is a well-developed theory in the case of functionals of linear growth, e.g. [42, Chapter 11]. However, since we consider the non-autonomous case without assuming linear growth, our recession function depends on x and so acts as an unbounded weight on the singular part of the function. Handling the transition between areas of finite and infinite recession function is the critical innovation of this article.

Image restoration has also been approached with non-autonomous energies that treat different locations differently. The first such model, by Chen, Levine and Rao [9], involves the minimization of

$$\min_{u\in \mathrm{BV}(\Omega)} \int_\Omega \varphi_{clr}(x, |Du|) + |u-f|^2\,dx, \tag{1.1}$$

where the regularizing term has variable exponent growth for small energies and is given by

$$\varphi_{clr}(x,t) := \begin{cases} \frac{1}{p(x)} t^{p(x)}, & \text{when } t\in[0,1],\\ t-1+\frac{1}{p(x)}, & \text{when } t>1.\end{cases}$$

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The variable exponent $p : \Omega \to (1, 2]$ is a function bounded away from 1 (i.e. $p^- := \inf p > 1$) which should be chosen close to 2 in smooth areas of the image and close to 1 near likely edges to avoid stair-casing as well as blurring. Since $\varphi(x, t) \sim t$ as $t \to \infty$, this model can be analyzed in the classical BV-space. Furthermore, using the Lebesgue decomposition of the derivative measure Du, Chen, Levine and Rao define

$$\int_\Omega \varphi_{clr}(x, |Du|)\, dx := \int_\Omega \varphi_{clr}(x, |\nabla^a u|)\, dx + |D^s u|(\Omega),$$

where $\nabla^a u$ is the density of the absolutely continuous part of the derivative. They prove for instance that

$$\int_\Omega \varphi_{clr}(x, |Du|)\, dx = \sup_{w \in C_0^1(\Omega;\mathbb{R}^n), |w| \leqslant 1} \int_\Omega u \operatorname{div} w - \tfrac{1}{p'(x)} |w|^{p'(x)}\, dx \tag{1.2}$$

and use this duality formulation to prove existence and properties of minimizers of (1.1). The reason why we call this a duality formulation and the rationale behind the term $\frac{1}{p'(x)}|w|^{p'(x)}$ will become clear once we introduce a more general framework.

Subsequently, Li, Li and Pi [36] proposed an image restoration model in the variable exponent space $W^{1,p(\cdot)}(\Omega)$ with energy $\varphi_{p(\cdot)}(x, t) := t^{p(x)}$ and $p^- > 1$. The last restriction implies that the problem involves only reflexive Sobolev spaces and that the minimizers are $C^{1,\alpha}$, so theoretically it is ill-suited to the image processing context. Harjulehto, Hästö, Latvala and Toivanen [26, 27] considered the same energy without the restriction $p^- > 1$. In this case, a relaxation procedure shows that the "correct" energy for BV-functions is

$$\int_\Omega \varphi_{p(\cdot)}(x, |Du|)\, dx := \int_\Omega \varphi_{p(\cdot)}(x, |\nabla^a u|)\, dx + |D^s u|(\{p = 1\})$$

provided $|D^s u|(\{p > 1\}) = 0$, analogously to the Chen–Levine–Rao formula (1.2).

More recently, double phase energies have attracted the attention of many in the field of non-standard growth [4, 6, 11, 14, 18, 37, 38]. Most important for image processing is the version $\varphi_{dp}(x, t) := t + a(x)t^2$ with $a \geqslant 0$ and powers 1 and 2. Harjulehto and Hästö [22] considered this energy with the interpretation

$$\int_\Omega \varphi_{dp}(x, |Du|)\, dx := \int_\Omega \varphi_{dp}(x, |\nabla^a u|)\, dx + |D^s u|(\{a = 0\})$$

provided $|D^s u|(\{a > 0\}) = 0$. For instance they showed that it is the Γ-limit as $\varepsilon \to 0^+$ of the uniformly convex approximating energies given by $\varphi_\varepsilon(x, t) := t^{1+\varepsilon} + a(x)t^2$.

The purpose of the present article is to introduce a general model which covers all these cases as well as countless variants like the perturbed variable exponent model and the Orlicz double phase model (see [31, 32] for a list on variants with references). Generalized Orlicz spaces, also known as Musielak–Orlicz spaces, have been widely studied recently (see, e.g., [10, 20, 33, 44, 45]). We consider a generalized Φ-function $\varphi : \Omega \times [0, \infty) \to [0, \infty]$ which may have linear growth at infinity at some points and superlinear growth at others. The dual space in the linear case is L^∞ which can be seen in the restriction $|w| \leqslant 1$ in (1.2). This space lacks several nice properties but it is nevertheless very concrete. However, to deal with the general case we consider the space $L^{\varphi^*}(\Omega)$ given by the conjugate function φ^*. Now the linearity of φ means that φ^* is not doubling; in fact, it is not even finite. Consequently, we can neither use the theory of doubling Φ-functions, nor the concreteness of the space $L^\infty(\Omega)$. Fortunately, the theory of non-doubling variable exponent and generalized Orlicz spaces has been developed in [15, 21] and we know for instance that the maximal operator is bounded irrespective of doubling. Nevertheless, we need new types of approximation estimates that handle the transition between the L^1-, L^p-

and L^∞-regimes without extra constants which can ruin an argument in the non-doubling case. These techniques require subtly stronger assumptions on φ, as the usual (A1) does not suffice (see Example 4.3).

Duality is a commonly used strategy in BV-spaces and image restoration. We use it to define appropriate norms V_φ and modulars $\varrho_{V,\varphi}$ and study their properties in Section 4. To our knowledge, this is the first time that the duality approach has been used to define a modular in a Sobolev-type space. In Section 5, we consider approximation with respect to V_φ and the new space $\mathrm{BV}^\varphi(\Omega)$ which generalizes $\mathrm{BV}(\Omega)$. The main result (Theorem 6.4) provides the formula

$$\varrho_{V,\varphi}(u) = \varrho_\varphi(|\nabla^a u|) + \int_\Omega \varphi'_\infty \, d|D^s u|$$

for the modular in terms of the *recession function* $\varphi'_\infty : \Omega \to [0, \infty]$ defined by

$$\varphi'_\infty(x) := \limsup_{t\to\infty} \frac{\varphi(x,t)}{t}.$$

To see that our recession function depends on x and so acts as an unbounded weight on the singular part of the function consider the example $\varphi(x,t) := t^{p(x)}$ where $\varphi'_\infty = 1$ in the set $\{p = 1\}$ and $\varphi'_\infty = \infty$ elsewhere. This also shows that the continuity of φ does not ensure the continuity of φ'_∞. Furthermore, this makes the non-autonomous case much more difficult than the autonomous case, where the space BV^φ reduces to classical BV- or Sobolev spaces (see Corollary 6.5).

Using this formula we conclude the paper in Section 7 by showing the Γ-convergence of regularized functionals from [16] to $\varrho_{V,\varphi}$. We start with background (Section 2) and auxiliary results (Section 3). A critical tool of independent interest is the Young convolution inequality with asymptotically sharp constants (Corollary 3.4).

2. Background

Notation and terminology. Throughout the paper we always consider a bounded domain $\Omega \subset \mathbb{R}^n$, i.e. an open and connected set. By $p' := \frac{p}{p-1}$ we denote the Hölder conjugate exponent of $p \in [1, \infty]$. The notation $f \lesssim g$ means that there exists a constant $c > 0$ such that $f \leqslant cg$. The notation $f \approx g$ means that $f \lesssim g \lesssim f$ whereas $f \simeq g$ means that $f(t/c) \leqslant g(t) \leqslant f(ct)$ for some constant $c \geqslant 1$. By c we denote a generic constant whose value may change between appearances. A function f is *almost increasing* (more precisely, L-almost increasing) if there exists $L \geqslant 1$ such that $f(s) \leqslant Lf(t)$ for all $s \leqslant t$. *Almost decreasing* is defined analogously. By *increasing* we mean that the inequality holds for $L = 1$ (some call this non-decreasing), similarly for *decreasing*.

Consider a function $\|\cdot\| : X \to [0, \infty]$ on a real vector space X and the following conditions:

(N1) $\|f\| = 0$ implies that $f = 0$.
(N2) $\|af\| = |a|\|f\|$ for all $f \in X$ and $a \in \mathbb{R}$;
(N3) $\|f + g\| \leqslant \|f\| + \|g\|$ for all $f, g \in X$.
(N3′) $\|f + g\| \lesssim \|f\| + \|g\|$ for all $f, g \in X$.

We use the following terminology for $\|\cdot\|$:

	(N1)	(N2)	(N3)	(N3′)
quasi-seminorm		✓		✓
seminorm		✓	✓	
quasinorm	✓	✓		✓
norm	✓	✓	✓	

Generalized Orlicz spaces. We first define types of modulars that generate our spaces. Note that our terminology differs from Musielak [39]. Our justification is the following: a quasi-semimodular generates a quasi-seminorm, a semimodular generates a seminorm, etc.

Definition 2.1. Let X be a real vector space. A function $\varrho : X \to [0,\infty]$ is called a *quasi-semimodular* on X if:

(1) $\varrho(0_X) = 0$;
(2) the function $\lambda \mapsto \varrho(\lambda x)$ is increasing on $[0,\infty)$ for every $x \in X$;
(3) $\varrho(-x) = \varrho(x)$ for every $x \in X$;
(4) there exists $\beta \in (0,1]$ such that $\varrho(\beta(\alpha x + (1-\alpha)y)) \leqslant \alpha\varrho(x) + (1-\alpha)\varrho(y)$ for every $x, y \in X$ and every $\alpha \in [0,1]$.

If (4) holds with $\beta = 1$, then ϱ is a *semimodular*. A (quasi-)semimodular is called a *(quasi)modular* provided $\varrho(x) = 0$ if and only if $x = 0_X$.

Definition 2.2. If ϱ is a quasi-semimodular in X, then the *modular space* $X_\varrho := \{x \in X \mid \|x\|_\varrho < \infty\}$ is defined by the quasi-seminorm

$$\|x\|_\varrho := \inf\Big\{\lambda > 0 \,\Big|\, \varrho\Big(\frac{x}{\lambda}\Big) \leqslant 1\Big\}.$$

The next definitions are from [21]. Our previous works were based on conditions defined for almost every point $x \in \Omega$. In this article we also use singular measures, so the assumptions are adjusted to hold for every point, following [30]. We denote by $L^0(\Omega)$ the set of measurable functions in Ω.

Definition 2.3. We say that $\varphi : \Omega \times [0,\infty) \to [0,\infty]$ is a *weak Φ-function*, and write $\varphi \in \Phi_{\rm w}(\Omega)$, if the following conditions hold for every $x \in \Omega$:

- $\varphi(\cdot, |f|)$ is measurable for every $f \in L^0(\Omega)$.
- $t \mapsto \varphi(x,t)$ is increasing.
- $\varphi(x,0) = \lim\limits_{t\to 0^+} \varphi(x,t) = 0$ and $\lim\limits_{t\to\infty} \varphi(x,t) = \infty$.
- $t \mapsto \frac{\varphi(x,t)}{t}$ is L-almost increasing on $(0,\infty)$ with constant L independent of x.

If $\varphi \in \Phi_{\rm w}(\Omega)$ is additionally convex and left-continuous with respect to t for every $x \in \Omega$, then φ is a *convex Φ-function* and we write $\varphi \in \Phi_{\rm c}(\Omega)$. If φ does not depend on x, then we omit the set and write $\varphi \in \Phi_{\rm w}$ or $\varphi \in \Phi_{\rm c}$.

Since the range of φ is $[0,\infty]$, convexity can be defined as usual by the inequality

$$\varphi(x, \theta t + (1-\theta)s) \leqslant \theta\varphi(x,t) + (1-\theta)\varphi(x,s)$$

including the case $\infty \leqslant \infty$. As we deal with conjugates of linear growth at infinity, it is crucial that we allow extended real-valued Φ-functions. Chlebicka, Gwiazda and colleagues (e.g. [7, 10]) have considered the case of non-doubling N-functions; however, this is not sufficient here since N-functions exclude L^1- and L^∞-spaces which are needed.

Definition 2.4. Let $\varphi \in \Phi_{\rm w}(\Omega)$ and $\varrho_\varphi(f) := \displaystyle\int_\Omega \varphi(x,|f|)\,dx$ for all $f \in L^0(\Omega)$. The set

$$L^\varphi(\Omega) := (L^0(\Omega))_{\varrho_\varphi} = \big\{f \in L^0(\Omega) \mid \varrho_\varphi(\lambda f) < \infty \quad \text{for some } \lambda > 0\big\}$$

with quasinorm given by $\|f\|_\varphi := \|f\|_{\varrho_\varphi}$ is called a *generalized Orlicz space*. We use the abbreviation $\|v\|_\varphi := \big\||v|\big\|_\varphi$ for vector-valued functions.

We observe that $\|\cdot\|_\varphi$ is a quasinorm in $L^\varphi(\Omega)$ if $\varphi \in \Phi_{\rm w}(\Omega)$, and a norm if $\varphi \in \Phi_{\rm c}(\Omega)$ [21, Lemma 3.2.2]. We define two Sobolev spaces; the space $L^{1,\varphi}$ is sometimes denoted by V^1L^φ, indicating the that first variation ∇u belongs to L^φ. Note that $W^{1,\varphi}(\Omega) = L^{1,\varphi}(\Omega) \cap L^\varphi(\Omega)$.

Definition 2.5. Let $\varphi \in \Phi_{\rm w}(\Omega)$. A function $u \in W^{1,1}(\Omega)$ belongs to the *Sobolev space* $W^{1,\varphi}(\Omega)$ if $|u|, |\nabla u| \in L^\varphi(\Omega)$ and to the *Sobolev space* $L^{1,\varphi}(\Omega)$ if $|\nabla u| \in L^\varphi(\Omega)$. The spaces are equipped with the (quasi)norms

$$\|u\|_{W^{1,\varphi}(\Omega)} := \|u\|_\varphi + \|\nabla u\|_\varphi \quad \text{and} \quad \|u\|_{L^{1,\varphi}(\Omega)} := \|u\|_{L^1(\Omega)} + \|\nabla u\|_{L^\varphi(\Omega)}.$$

When φ in a sub- or superscript is replaced by a real number (e.g., $L^{1,p}$ or ϱ_2), this is an abbreviation for the Φ-function $\varphi(x,t) \equiv t^p$.

3. Auxiliary results

Regularity conditions for harmonic analysis and PDE. We say that $\omega : [0,\infty) \to [0,\infty]$ is a *modulus on continuity* if it is increasing and $\omega(0) = \lim_{t\to 0^+} \omega(t) = 0$. Note that we do not require concavity and allow extended real values.

For $\varphi : \Omega \times [0,\infty) \to [0,\infty)$ and $p, q > 0$ we define some conditions:

(A0) There exists $\beta \in (0,1]$ such that $\varphi(x,\beta) \leqslant 1 \leqslant \varphi(x, \frac{1}{\beta})$ for every $x \in \Omega$.

(A1) For every $K > 0$ there exists $\beta \in (0,1]$ such that, for every $x, y \in \Omega$,

$$\varphi(x,\beta t) \leqslant \varphi(y,t) + 1 \quad \text{when} \quad \varphi(y,t) \in \left[0, \frac{K}{|x-y|^n}\right].$$

(VA1) For every $K > 0$ there exists a modulus of continuity ω such that, for every $x, y \in \Omega$,

$$\varphi(x, \tfrac{t}{1+\omega(|x-y|)}) \leqslant \varphi(y,t) + \omega(|x-y|) \quad \text{when} \quad \varphi(y,t) \in \left[0, \frac{K}{|x-y|^n}\right].$$

(aInc)$_p$ There exists $L_p \geqslant 1$ such that $t \mapsto \frac{\varphi(x,t)}{t^p}$ is L_p-almost increasing in $(0,\infty)$ for every $x \in \Omega$.

(aDec)$_q$ There exists $L_q \geqslant 1$ such that $t \mapsto \frac{\varphi(x,t)}{t^q}$ is L_q-almost decreasing in $(0,\infty)$ for every $x \in \Omega$.

We say that (aInc) holds if (aInc)$_p$ holds for some $p > 1$, and similarly for (aDec).

If $\varphi \in \Phi_{\rm w}(\Omega)$, then $\varphi(\cdot,1) \approx 1$ implies (A0), and if φ satisfies (aDec), then (A0) and $\varphi(\cdot,1) \approx 1$ are equivalent. For instance, $\varphi(x,t) = t^p$ always satisfies (A0), since $\varphi(x,1) \equiv 1$. Assumption (A1) is an almost continuity condition; in the variable exponent case $\varphi(x,t) := t^{p(x)}$ it corresponds to $\log$-Hölder continuity of $\frac{1}{p}$ [21, Proposition 7.1.2]. Finally, (aInc) and (aDec) are quantitative versions of the ∇_2 and Δ_2 conditions and measure lower and upper growth rates.

Note that the definition of (A1) differs slightly from [21, 28], where it is assumed that

$$\varphi(x,\beta t) \leqslant \varphi(y,t) \quad \text{when} \quad \varphi(y,t) \in \left[1, \frac{1}{|B|}\right]$$

and x and y belong to the ball B. If φ satisfies (A0), then this is equivalent to

$$\varphi(x,\beta t) \leqslant \varphi(y,t) + 1 \quad \text{when} \quad \varphi(y,t) \in \left[0, \frac{1}{|B|}\right]$$

and if φ satisfies (aDec), then we can equivalently add in the K, as well, see [29, 32].

The "vanishing (A1)" condition (VA1) is a continuity condition for φ which was introduced to prove maximal regularity of minimizers [31]. In the variable exponent case it corresponds to vanishing $\log$-Hölder continuity. We need the following weaker version of (VA1) where at least one of the points has to belong to the set $\{\varphi'_\infty < \infty\}$ defined using the recession function:

Definition 3.1. We say that $\varphi \in \Phi_{\mathrm{w}}(\Omega)$ satisfies *restricted* (VA1) if it satisfies (A1) and for every $K > 0$ there exists a modulus of continuity ω such that

$$\varphi(x, \tfrac{t}{1+\omega(|x-y|)}) \leqslant \varphi(y,t) + \omega(|x-y|) \quad \text{when} \quad \varphi(y,t) \in \Big[0, \frac{K}{|x-y|^n}\Big]$$

for every $x, y \in \Omega$ with $\varphi'_\infty(x) < \infty$ or $\varphi'_\infty(y) < \infty$.

In [26, Section 3], it was shown that log-Hölder continuity in the variable exponent case was not sufficient for BV-type spaces and a strong log-Hölder continuity condition was introduced. As mentioned above, $\varphi(x,t) = t^{p(x)}$ satisfies (A1) if and only if $\frac{1}{p}$ is log-Hölder continuous. We now prove a corresponding connection between restricted (VA1) and strong log-Hölder continuity. For simplicity, only the case of finite exponents is considered.

Proposition 3.2. *Let $\varphi(x,t) := t^{p(x)}$ be a variable exponent energy with $p : \Omega \to [1,\infty)$. Then restricted (VA1) is equivalent to the* strong log-Hölder continuity *of $\frac{1}{p}$, i.e.* log*-Hölder continuity with*

$$\lim_{x\to y} \big|1 - \tfrac{1}{p(x)}\big| \log \tfrac{1}{|x-y|} = 0$$

uniformly in $y \in \{p = 1\}$.

Proof. The connection between log-Hölder continuity and (A1) was established in [21, Proposition 7.1.2], so it only remains to consider the vanishing log-Hölder continuity around the set $\{p = 1\}$. Suppose that $p(y) = 1$ or, equivalently, $\varphi'_\infty(y) < \infty$. Then $\varphi(y,t) = t^1 = t$. First we assume restricted (VA1) with $K = 1$ and modulus of continuity ω. Choosing $t := |x-y|^{-n} \geqslant 1$ and denoting $r := |x-y|$, we have

$$\big(\tfrac{t}{1+\omega(r)}\big)^{p(x)} = \varphi(x, \tfrac{t}{1+\omega(r)}) \leqslant \varphi(y,t) + \omega(r) \leqslant (1+\omega(r))t.$$

Taking the logarithm of the equivalent inequality $t^{p(x)-1} \leqslant (1+\omega(r))^{p(x)+1}$, we find that

$$\big|1 - \tfrac{1}{p(x)}\big| \log \tfrac{1}{|x-y|} \leqslant \tfrac{1+p(x)}{np(x)} \log(1+\omega(r)) \leqslant \tfrac{2}{n} \log(1+\omega(r)) \to 0$$

as $r \to 0^+$. Thus p is strongly log-Hölder continuous.

Assume conversely that p is strongly log-Hölder continuous so that

$$\omega_p(r) := \sup_{y\in\{p=1\},\, x\in B_r(y)} \big|1 - \tfrac{1}{p(x)}\big| \log \tfrac{1}{|x-y|} \to 0$$

as $r \to 0^+$. To establish the restricted (VA1)-condition when $p(y) = 1$, it is enough that

$$t^{p(x)-1} \leqslant (1+\omega(r))^{p(x)}.$$

The inequality is trivial if $t \in [0,1]$. So let $t > 1$. The left-hand side is increasing in t so the worst case is when $t = \frac{K}{|x-y|^n}$ and we can choose ω based on the estimate

$$t^{1-\frac{1}{p(x)}} - 1 \leqslant \Big(\frac{K}{r^n}\Big)^{\frac{\omega_p(r)}{\log\frac{1}{r}}} - 1 = e^{\frac{\log K + n\log\frac{1}{r}}{\log\frac{1}{r}}\omega_p(r)} - 1 \leqslant e^{(\log K + n)\omega_p(r)} - 1 =: \omega(r)$$

when $r \leqslant \frac{1}{e}$. The strong log-Hölder continuity ensures that this tends to zero when $x \to y$. On the other hand, if $p(x) = 1$ in the (VA1)-condition, then we need

$$\frac{t}{1+\omega(r)} \leqslant t \leqslant t^{p(y)} + \omega(r),$$

which holds since $\sup_{t\geqslant 0}(t - t^{p(y)}) = p(y)^{-p'(y)}(p(y)-1) \leqslant 1 - \frac{1}{p(y)} \leqslant \omega_p(r) \leqslant \omega(r)$ for all small $r > 0$. □

Inequalities with sharp constants. Analogues of Jensen's inequality [21, Theorem 4.3.2] and Young's convolution inequality [21, Lemma 4.4.6] are known in the generalized Orlicz space under the (A1) assumption, but only with constants $\beta \ll 1$. Here we show that the (VA1) assumption lets us choose the constant $\beta \to 1^-$ at the price of restricting to a small ball. The next result is an improvement of [30, Theorem 2.3]. Note that we do not assume (aDec). This makes the proof more difficult but is critical to the application in this article.

Theorem 3.3 (Jensen's inequality)**.** *If $\varphi \in \Phi_c(\Omega)$ satisfies (VA1) and μ is a probability measure in the ball $B = B_r$ with $|B|\,\|\frac{d\mu}{dx}\|_\infty =: m < \infty$, then*

$$\varphi_B^-\left(\frac{1}{1+\omega(r)}\int_{B\cap\Omega}|f|\,d\mu\right) \leqslant \int_{B\cap\Omega}\varphi(x,f)\,d\mu + \omega(r),$$

where ω be the modulus of continuity from (VA1) with $K := m\varrho_\varphi(f) + 2$ and $r > 0$ is so small that $\omega(r) \leqslant \frac{1}{|B|}$.

Proof. We define $t_0 := \int_{B\cap\Omega}|f|\,d\mu$. By [21, Lemma 4.3.1], there exists $\beta > 0$ such that

$$\varphi_B^-\left(\beta\int_{B\cap\Omega}|f|\,d\mu\right) \leqslant \int_{B\cap\Omega}\varphi(x,f)\,d\mu.$$

If $t_0 = \infty$, this implies that the right-hand side of the claim is infinite so there is nothing to prove. Thus we may assume that $t_0 < \infty$.

Denote by φ' the left-continuous function, increasing in s, with

$$\varphi(x,t) = \int_0^t \varphi'(x,s)\,ds.$$

Such a function exists since φ is convex in the second variable. Fix $x_0 \in B$ with

$$\tfrac{1}{1+\omega(r)}\varphi'\big(x_0, \tfrac{1}{1+\omega(r)}t_0\big) \leqslant (\varphi')_B^-\big(\tfrac{1}{1+\omega(r)}t_0\big)$$

and assume $\beta \leqslant \frac{1}{1+\omega(r)}$ is so small that $\varphi(x_0,\beta t_0) \leqslant \frac{K}{|B|}$.

We define $\psi \in \Phi_c$ by

$$\psi(t) := \int_0^t \varphi'(x_0, \min\{s, \beta t_0\})\,ds\,;$$

ψ is convex since ψ' is increasing. Furthermore, $\psi(\beta t) = \varphi(x_0,\beta t)$ if $t \leqslant t_0$. When $t \leqslant t_0$ we consider two cases to show that

$$\psi(\beta t) \leqslant \varphi(x,t) + \omega(r)$$

for $x \in B$: if $\varphi(x,t) \leqslant \frac{K}{|B|}$ this follows from (VA1) and otherwise it follows from $\varphi(x_0,\beta t) \leqslant \frac{K}{|B|} \leqslant \varphi(x,t)$. When $t > t_0$ we estimate

$$\begin{aligned}\psi(\beta t) &= \psi(\beta t_0) + \beta(t-t_0)\varphi'(x_0,\beta t_0) \leqslant \varphi(x,t_0) + \omega(r) + (t-t_0)(\varphi')_B^-(t_0)\\ &\leqslant \varphi(x,t_0) + \omega(r) + (t-t_0)\varphi'(x,t_0) \leqslant \varphi(x,t) + \omega(r),\end{aligned}$$

where we also used the convexity of φ in the last step.

It follows from Jensen's inequality for ψ that

$$\varphi(x_0,\beta t_0) = \psi\left(\beta\int_{B\cap\Omega}|f|\,d\mu\right) \leqslant \int_{B\cap\Omega}\psi(\beta|f|)\,d\mu \leqslant \int_{B\cap\Omega}\varphi(x,|f|)\,d\mu + \omega(r).$$

This is the claim once we show that we can choose $\beta = \frac{1}{1+\omega(r)}$. Since the integral on the right-hand side can be estimated by $\frac{m}{|B|}\varrho_\varphi(f) = \frac{K-2}{|B|}$ and $\omega(r) \leqslant \frac{1}{|B|}$, the inequality gives $\varphi(x_0,\beta t_0) \leqslant \frac{K-1}{|B|}$. To summarize, we have shown that $\varphi(x_0,\beta t_0) \leqslant \frac{K}{|B|}$ implies $\varphi(x_0,\beta t_0) \leqslant \frac{K-1}{|B|}$.

We next investigate how large we can make β. Consider the set

$$\Theta := \Big\{\theta \in (0,1] \,\Big|\, \varphi\big(x_0, \tfrac{\theta}{1+\omega(r)} t_0\big) \leqslant \tfrac{K}{|B|}\Big\}.$$

Since $\varphi(x_0,t) \to 0$ when $t \to 0^+$, the set is non-empty. If $\theta_k \in \Theta$ with $\theta_k \nearrow \theta_0$, then the left-continuity of $\varphi(x_0,\cdot)$ implies that $\theta_0 \in \Theta$. If $\sup\Theta = 1$, then this means that the previous Jensen inequality holds for $\beta = \frac{1}{1+\omega(r)}$ and the claim is proved. Suppose then that $\theta_0 := \sup\Theta \in (0,1)$. For $\theta_0 < \theta$, this implies that

$$\varphi\big(x_0, \tfrac{\theta_0}{1+\omega(r)} t_0\big) \leqslant \tfrac{K-1}{|B|} < \tfrac{K}{|B|} < \varphi\big(x_0, \tfrac{\theta}{1+\omega(r)} t_0\big).$$

Since $\varphi(x_0,\cdot)$ is convex, such discontinuity is only possible if the right-hand side equals infinity for every $\theta > \theta_0$. If $\varphi(x_0, \frac{1}{1+\omega(r)} t_0) = \infty$, then

$$\infty = \varphi'(x_0, \tfrac{1}{1+\omega(r)} t_0) \leqslant (1+\omega(r))(\varphi')^-_B(\tfrac{1}{1+\omega(r)} t_0)$$

by the choice of x_0. It follows that $\varphi(x,t_0) = \infty$ for every $x \in B$. The set $A := \{x \in B \mid |f(x)| \geqslant t_0\}$ has positive μ-measure since t_0 is the μ-average of $|f|$. Thus also $\int_{B\cap\Omega} \varphi(x,|f|)\,d\mu = \infty$, so the claim holds in the form $\infty \leqslant \infty$ in this case. □

The convolution in the next result should be understood as

$$f * \eta(x) := \int_\Omega f(y)\eta(x-y)\,dy$$

to account for the fact that f and φ are only defined in Ω. Extending φ outside Ω while preserving (VA1) is non-trivial, but luckily that is not needed here.

Corollary 3.4 (Young's convolution inequality)**.** *Let $\varphi \in \Phi_c(\Omega)$ satisfy (VA1) and η be the standard mollifier. Then there exists a modulus of continuity ω such that*

$$\varrho_\varphi\big(\tfrac{1}{1+\omega(\delta)} f * \eta_\delta\big) \leqslant \varrho_\varphi(f) + \omega(\delta)$$

for every $\delta > 0$.

Proof. We may assume that $\varrho_\varphi(f) < \infty$ since otherwise there is nothing to prove. Let ω be the modulus of continuity from (VA1) with $K := m\varrho_\varphi(f) + 2$ and let $r > 0$ be so small that $\omega(r) \leqslant \frac{1}{|B_r|}$. Thus Theorem 3.3 yields

$$\varphi^-_{B_r}\left(\frac{|f * \eta_r(x)|}{1+\omega(r)}\right) \leqslant (\varphi(\cdot,f) * \eta_r)(x) + \omega(r).$$

This yields

$$\varphi^-_{B_r}\left(\frac{|f * \eta_r(x)|}{1+\omega(r)}\right) \leqslant \frac{m}{|B_r|}\int_{B_r\cap\Omega} \varphi(x,f)\,dx + \frac{1}{|B_r|} < \frac{K}{|B_r|}.$$

Thus we obtain by (VA1) that

$$\varphi\left(x, \frac{|f * \eta_r(x)|}{(1+\omega(r))^2}\right) \leqslant \varphi^-_{B_r}\left(\frac{|f * \eta_r(x)|}{1+\omega(r)}\right) + \omega(r) \leqslant (\varphi(\cdot,f) * \eta_r)(x) + 2\omega(r).$$

We integrate this over Ω and use Fubini's Theorem to conclude that

$$\varrho_\varphi\big(\tfrac{1}{(1+\omega(r))^2} f * \eta_r\big) \leqslant \int_\Omega \varphi(x,f) * \eta_r\,dx + 2|\Omega|\omega(r) \leqslant \int_\Omega \varphi(x,f)\,dx + 2|\Omega|\,\omega(r).$$

This gives the claim with the modulus of continuity $\hat\omega(r) := \max\{2\omega(r) + \omega(r)^2, 2|\Omega|\,\omega(r)\}$; when $\omega(r) > \frac{1}{|B_r|}$ we set $\hat\omega(r) := \infty$. □

Associate spaces and conjugate modulars. The associate space is a variant of the dual function space which works better at the end-points $p = 1$ and $p = \infty$. We define the *associate space* $(L^\varphi(\Omega))' \subset L^0(\Omega)$ by the norm

$$\|u\|_{(L^\varphi(\Omega))'} := \sup_{\|v\|_\varphi \leqslant 1} \int_\Omega uv\,dx.$$

According to [21, Theorem 3.4.6], $(L^\varphi(\Omega))' = L^{\varphi^*}(\Omega)$ for $\varphi \in \Phi_{\rm w}(\Omega)$, where

$$\varphi^*(x,t) := \sup_{s\geqslant 0}(st - \varphi(x,s)).$$

The conjugate function φ^* has the following properties:

- If $\varphi \in \Phi_{\rm w}(\Omega)$, then $\varphi^* \in \Phi_{\rm c}(\Omega)$, so φ^* is always convex and left-continuous [21, Lemma 2.4.1].
- For $p, q \in (1,\infty)$, φ satisfies (aInc)$_p$ or (aDec)$_q$ if and only if φ^* satisfies (aDec)$_{p'}$ or (aInc)$_{q'}$, respectively [21, Proposition 2.4.9].
- If $\varphi \in \Phi_{\rm c}(\Omega)$, then $\varphi^*(x, \frac{\varphi(x,t)}{t}) \leqslant \varphi(x,t)$ [21, p. 35] and $\varphi^{**} = \varphi$ [15, Corollary 2.6.3].
- If φ satisfies (A0) or (A1), then so does φ^* [21, Lemmas 3.7.6 and 4.1.7].

Every $\varphi \in \Phi_{\rm c}$ can be represented as

$$\varphi(t) = \int_0^t \varphi'(\tau)\,d\tau.$$

The function $\varphi' : \Omega \times [0,\infty) \to [0,\infty)$ can be the left-continuous left-derivative, the right-continuous right-derivative, or something in between. Additionally, it is known that $\varphi(t) \approx t\varphi'(t)$ if φ satisfies (aDec) [23, Lemma 3.3].

Functions of bounded variation. A function $u \in L^1(\Omega)$ has *bounded variation*, denoted $u \in \mathrm{BV}(\Omega)$, if

$$V(u,\Omega) := \sup\left\{ \int_\Omega u \operatorname{div} w\,dx \,\middle|\, w \in C_0^1(\Omega;\mathbb{R}^n), |w| \leqslant 1 \right\} < \infty.$$

Such functions have weak first derivatives which are Radon measures which we denote Du. By [1, Proposition 3.6], $V(u,\Omega)$ equals the total variation $|Du|(\Omega)$ of the measure Du, defined as

$$|Du|(A) := \sup_{\cup A_i = A} \sum_i |Du(A_i)|$$

where the supremum is taken over finite partitions of A by measurable sets A_i. Furthermore, we use the Lebesgue decomposition

$$Du = D^a u + D^s u,$$

where $D^a u$ is the absolutely continuous part of the derivative and $D^s u$ is the singular part. The density of $D^a u$ is the vector valued function $\nabla^a u$ such that

$$\int_\Omega w \cdot dD^a u = \int_\Omega w \cdot \nabla^a u\,dx$$

for all $w \in C_0^\infty(\Omega;\mathbb{R}^n)$. The space BV has the following compactness-type property [1, Proposition 3.13]: if $\sup_i \big(\|u_i\|_{L^1(\Omega)} + |Du_i|(\Omega)\big) < \infty$, then there exists a subsequence and $u \in \mathrm{BV}(\Omega)$ such that

$$u_{i_j} \to u \text{ in } L^1(\Omega) \quad \text{and} \quad |Du|(\Omega) \leqslant \liminf_{j\to\infty} |Du_{i_j}|(\Omega).$$

We refer to [1] for more information about BV spaces. The next lemma shows that the equality $V(u,\Omega) = |Du|(\Omega)$ holds separately for the singular part.

Lemma 3.5. *If* $u \in \mathrm{BV}(\Omega)$*, then* $|D^s u|(\Omega) = \sup\left\{ \int_\Omega w \cdot dD^s u \,\middle|\, w \in C_0^1(\Omega;\mathbb{R}^n), |w| \leqslant 1 \right\}$.

Proof. By $|Du|(\Omega) = V(u,\Omega)$, the definition of the weak derivative and $Du = D^a u + D^s u$, we see that

$$\begin{aligned}
|Du|(\Omega) &= \sup_{w\in C_0^1(\Omega;\mathbb{R}^n), |w|\leqslant 1} \left(\int_\Omega w\cdot dD^a u + \int_\Omega w \cdot dD^s u \right) \\
&\leqslant \sup_{w\in L^\infty(\Omega;\mathbb{R}^n), |w|\leqslant 1} \int_\Omega w\cdot dD^a u + \sup_{w\in L^\infty(\Omega;\mathbb{R}^n), |w|\leqslant 1} \int_\Omega w\cdot dD^s u \\
&= |D^a u|(\Omega) + |D^s u|(\Omega).
\end{aligned}$$

Since $|D^a u|(\Omega) + |D^s u|(\Omega) = |Du|(\Omega)$ as D^a and D^s are mutually singular, each inequality has to be an equality, and so the claim follows. □

4. Basic properties of dual norms and modulars

In this section we use a duality approach to define a norm and a modular. The "dual norm" V_φ is related to the associate space and Hölder's inequality, whereas the "dual modular" $\varrho_{V,\varphi}$ is related to Young's inequality. Note that V_φ is not the norm generated by $\varrho_{V,\varphi}$; their relationship is explored in Lemma 4.7.

Definition 4.1. Let $\varphi \in \Phi_{\mathrm{w}}(\Omega)$. For $u \in L^1(\Omega)$, we define the "dual norm"

$$V_\varphi(u,\Omega) := V_\varphi(u) := \sup\left\{ \int_\Omega u \operatorname{div} w\,dx \,\middle|\, w \in C_0^1(\Omega;\mathbb{R}^n), \|w\|_{\varphi^*} \leqslant 1 \right\}$$

and the "dual modular"

$$\varrho_{V,\varphi}(u) := \sup\left\{ \int_\Omega u \operatorname{div} w - \varphi^*(x,|w|)\,dx \,\middle|\, w \in C_0^1(\Omega;\mathbb{R}^n) \right\}.$$

We say that $u \in L^\varphi(\Omega)$ belongs to $\mathrm{BV}^\varphi(\Omega)$ if

$$\|u\|_{\mathrm{BV}^\varphi} := \|u\|_\varphi + V_\varphi(u) < \infty.$$

The next example shows that this definition is an extension of the ordinary BV-space, in which the norm and modular coincide. Example 4.3 shows that interesting things can happen in the non-autonomous case, which do not appear at all when φ is independent of x.

Example 4.2. Let $\varphi(x,t) := t$ and consider the corresponding functions V_1 and $\varrho_{V,1}$. Then $\varphi^*(x,t) = \infty\chi_{(1,\infty)}(t)$ so that $\varrho_{\varphi^*}(w) < \infty$ if and only if $w \leqslant 1$ almost everywhere, in which case $\varrho_{\varphi^*}(w) = 0$. Hence $V_1(u) = \varrho_{V,1}(u) = |Du|(\Omega) = V(u,\Omega)$.

Example 4.3. Let $\varphi(x,t) := \frac{1}{p(x)} t^{p(x)}$ for $p : \mathbb{R} \to [1,\infty)$. Then $\varphi^*(x,t) = \frac{1}{p'(x)} t^{p'(x)}$ when $p(x) > 1$ and $\varphi^*(x,t) = \infty\chi_{(1,\infty)}(t)$ when $p(x) = 1$. Consider the Heaviside function $h = \chi_{(0,\infty)}$ so that $Dh = \delta_{\{0\}}$, the Dirac measure. Now

$$\varrho_{V,\varphi}(h) = \sup\big\{ w(0) - \varrho_{\varphi^*}(|w|) \,|\, w \in C_0^1(\Omega) \big\}.$$

Since w is continuous, there exists for every $\varepsilon \in (0, w(0))$ a number $\delta > 0$ such that

$$\varrho_{\varphi^*}(w) \geqslant \int_{-\delta}^{\delta} \frac{1}{p'(x)} (w(0) - \varepsilon)^{p'(x)}.$$

Suppose first that $p(x) := 1 + \frac{c_{\log}}{\log(1/|x|)}$ for small $|x|$. Then $p'(x) = \frac{\log(1/|x|)}{c_{\log}} + 1$ and

$$(w(0) - \varepsilon)^{p'(x)} = |x|^{-\frac{\log(w(0)-\varepsilon)}{c_{\log}}}(w(0) - \varepsilon).$$

Hence the previous integral converges if $\log w(0) < c_{\log}$ and diverges if $\log w(0) > c_{\log}$ (when $\varepsilon \to 0^+$). On the other hand, we can choose w such that $0 \leqslant w \leqslant w(0)\chi_{[-\delta,\delta]}$. From this we see that $\inf_w \varrho_{\varphi^*}(w) = 0$ when $\log w(0) < c_{\log}$. It follows that

$$\varrho_{V,\varphi}(h) = e^{c_{\log}}.$$

In the same way we can show that $\varrho_{V,\varphi}(h) = 1$ if $p(x) := 1 + |x|^\alpha$ for some $\alpha > 0$. This example shows that $\varrho_{V,\varphi}(h)$ depends on the behavior of the exponent in a neighborhood of 0, even though the support of the derivative is only $\{0\}$.

Remark 4.4. If $\varrho_{\varphi^*}(|w|) = \infty$, then $\int_\Omega u \operatorname{div} w - \varphi^*(x, |w|)\, dx = -\infty$ since $\int_\Omega u \operatorname{div} w\, dx$ is finite as $u \in L^1(\Omega)$ and $w \in C^1_0(\Omega; \mathbb{R}^n)$. Testing with $w \equiv 0$, we see that the supremum in $\varrho_{V,\varphi}$ is always non-negative. Therefore test-functions w with $\varrho_{\varphi^*}(|w|) = \infty$ can be omitted and we obtain the alternative, equivalent formulation

$$\varrho_{V,\varphi}(u) = \sup\left\{ \int_\Omega u \operatorname{div} w - \varphi^*(x, |w|)\, dx \,\middle|\, w \in C^1_0(\Omega; \mathbb{R}^n),\, \varrho_{\varphi^*}(|w|) < \infty \right\}.$$

Note that $\varrho_{\varphi^*}(|w|) < \infty$ does not follow from $w \in C^1_0(\Omega; \mathbb{R}^n)$ as φ^* does not satisfy (aDec).

Remark 4.5. In our definition we use test-functions from $C^1_0(\Omega; \mathbb{R}^n)$. This corresponds to the definition of the usual BV-space. An alternative in duality formulations (e.g. [2, 9]) is $C^1(\Omega; \mathbb{R}^n)$, which means that the boundary values of the function u also influence the norm and modular. The restriction $|w| \leqslant 1$ carries over nicely to the boundary and leads to a boundary term in $L^1(\partial\Omega)$. This is not the case with $\varrho_{\varphi^*}(|w|) < \infty$. It seems that an additional boundary term for w of fractional Sobolev space-type is needed in $\varrho_{V,\varphi}$ if we want to obtain appropriate boundary values in the generalized Orlicz case. This remains a problem for future research.

Let $w \in C^1_0(\Omega; \mathbb{R}^n)$ with $|w| \leqslant 1$. Since Ω is bounded and (A0) for φ implies (A0) for φ^*, we find that $w \in L^{\varphi^*}(\Omega)$ and $\|w\|_{\varphi^*} \leqslant c\,\|w\|_\infty \leqslant c$, see Corollary 3.7.10 in [21]. By the definition of V_φ,

$$\int_\Omega u \operatorname{div} w\, dx = \|w\|_{\varphi^*} \int_\Omega u \operatorname{div} \tfrac{w}{\|w\|_{\varphi^*}}\, dx \leqslant \|w\|_{\varphi^*} V_\varphi(u)$$

and taking supremum over all such w, we obtain that $V(u, \Omega) \leqslant cV_\varphi(u)$. Since Ω is bounded and φ satisfies (A0), we have $L^\varphi(\Omega) \hookrightarrow L^1(\Omega)$ by [21, Corollary 3.7.9]. Thus $\mathrm{BV}^\varphi(\Omega) \hookrightarrow \mathrm{BV}(\Omega)$ provided φ satisfies (A0).

Lemma 4.6. *If $\varphi \in \Phi_{\mathrm{w}}(\Omega)$, then V_φ is a seminorm and $\|\cdot\|_{\mathrm{BV}^\varphi}$ is a quasinorm in $\mathrm{BV}^\varphi(\Omega)$. Moreover, if $\varphi \in \Phi_{\mathrm{c}}(\Omega)$, then $\|\cdot\|_{\mathrm{BV}^\varphi}$ is a norm.*

Proof. The homogeneity property $V_\varphi(au) = |a|V_\varphi(u)$ is clear. Let us show that V_φ satisfies the triangle inequality. If $u, v \in L^1(\Omega)$, then

$$\int_\Omega (u + v) \operatorname{div} w\, dx = \int_\Omega u \operatorname{div} w\, dx + \int_\Omega v \operatorname{div} w\, dx \leqslant V_\varphi(u) + V_\varphi(v)$$

for $w \in C^1_0(\Omega; \mathbb{R}^n)$ with $\|w\|_{\varphi^*} \leqslant 1$. By taking the supremum over such w we have

$$V_\varphi(u + v) \leqslant V_\varphi(u) + V_\varphi(v).$$

Note that $\|\cdot\|_\varphi$ is a quasinorm if $\varphi \in \Phi_{\rm w}(\Omega)$, and a norm if $\varphi \in \Phi_{\rm c}(\Omega)$. Combining these two results, we obtain the (quasi)triangle inequality for the sum that is $\|\cdot\|_{\mathrm{BV}^\varphi(\Omega)}$. These properties also imply that $\mathrm{BV}^\varphi(\Omega)$ is a vector space. □

In [24] we showed that $\varrho_{V,\varphi}$ defines a seminorm and so with the Luxenburg method (Definition 2.2) we obtain a norm $\|\cdot\|_{\varrho_{V,\varphi}}$. We next show that this seminorm is comparable to V_φ.

Lemma 4.7. *If $\varphi \in \Phi_{\rm w}(\Omega)$ and $u \in \mathrm{BV}^\varphi(\Omega)$, then*

$$\|u\|_{\varrho_{V,\varphi}} \leqslant V_\varphi(u) \leqslant 2\|u\|_{\varrho_{V,\varphi}}.$$

Proof. If $V_\varphi(u) = 0$, then $\varrho_{V,\varphi}(\frac{u}{\lambda}) = 0$ for every $\lambda > 0$ so that $\|u\|_{\varrho_{V,\varphi}} = 0$. The case $V_\varphi(u) = \infty$ is excluded by the assumption $u \in \mathrm{BV}^\varphi(\Omega)$. Since the claim is homogeneous, the case $V_\varphi(u) \in (0,\infty)$ reduces to $V_\varphi(u) = 1$. By the definition of V_φ, it then follows that

$$\int_\Omega u \,\mathrm{div}\, w\, dx \leqslant V_\varphi(u)\|w\|_{\varphi^*} = \|w\|_{\varphi^*} \leqslant 1 + \varrho_{\varphi^*}(|w|);$$

the last step is a general property of the Luxemburg norm, see [15, Corollary 2.1.15]. Thus

$$\varrho_{V,\varphi}(u) \leqslant \sup_{w\in C^1_0(\Omega;\mathbb{R}^n),\varrho_{\varphi^*}(|w|)<\infty} \big(1 + \varrho_{\varphi^*}(|w|) - \varrho_{\varphi^*}(|w|)\big) = 1,$$

and so $\|u\|_{\varrho_{V,\varphi}} \leqslant 1$. This concludes the proof of the first inequality.

We next establish the opposite inequality $2\|u\|_{\varrho_{V,\varphi}} \geqslant 1$, which is equivalent to $\varrho_{V,\varphi}(\frac{2u}{\lambda}) \geqslant 1$ for every $\lambda < 1$. Since $\varrho_{\varphi^*}(|w|) \leqslant 1$ when $\|w\|_{\varphi^*} \leqslant 1$, we conclude that

$$\begin{aligned}\varrho_{V,\varphi}(\tfrac{2u}{\lambda}) &\geqslant \sup\left\{ \int_\Omega \tfrac{2u}{\lambda}\,\mathrm{div}\, w - \varphi^*(x,|w|)\, dx \,\middle|\, w \in C^1_0(\Omega;\mathbb{R}^n), \|w\|_{\varphi^*} \leqslant 1\right\}\\ &\geqslant \tfrac{2}{\lambda}V_\varphi(u) - 1 > 1.\end{aligned}$$ □

The following result is the counterpart of Theorem 5.2 in [17], see also Theorem 1.9 in [19].

Lemma 4.8 (Weak lower semicontinuity)**.** *Let $\varphi \in \Phi_{\rm w}(\Omega)$, $u, u_k \in L^1(\Omega)$ with $u_k \rightharpoonup u$ in $L^1(\Omega)$. Then*

$$V_\varphi(u) \leqslant \liminf_{k\to\infty} V_\varphi(u_k) \qquad \text{and} \qquad \varrho_{V,\varphi}(u) \leqslant \liminf_{k\to\infty} \varrho_{V,\varphi}(u_k).$$

Proof. If $w \in C^1_0(\Omega;\mathbb{R}^n)$, then $\mathrm{div}\, w \in L^\infty(\Omega)$ and weak convergence in $L^1(\Omega)$ with $\|w\|_{\varphi^*} \leqslant 1$ give

$$\int_\Omega u\,\mathrm{div}\, w\, dx = \lim_{k\to\infty}\int_\Omega u_k \,\mathrm{div}\, w\, dx \leqslant \liminf_{k\to\infty} V_\varphi(u_k).$$

The first inequality of the claim follows by taking the supremum over all such w. Subtracting $\varrho_{\varphi^*}(|w|) < \infty$ from both sides of the equality similarly gives the second inequality. □

5. Approximation properties of the dual norm

In this section we prove a compactness-type property of BV^φ and estimate the V_φ-norm of $W^{1,1}_{\rm loc}$-functions by $\|\nabla u\|_\varphi$. We first connect the norm V_φ with the associate space $(L^{\varphi^*})'$-norm of the gradient. One crucial difference between these norms is that in the associate space norm we test with functions in L^{φ^*} whereas in V_φ the test functions are smooth. Thus some approximation is needed, but we cannot use density in $L^{\varphi^*}(\Omega)$ since φ^* is not, in general, doubling. We start with a property of lower semicontinuous functions. Although the result is known, we did not find a reference for these exact properties of the approximating functions, so we provide a proof for completeness.

Lemma 5.1. *Let $f : \Omega \to [0, \infty]$ be lower semicontinuous. Then there exist functions $w_i \in C_0^1(\Omega)$ with $0 \leqslant w_i \leqslant f$ and $w_i \to f$.*

Proof. We first define

$$f_i := \sum_{k=1}^{\infty} 2^{-i} \chi_{\{f > 2^{-i}k\}}.$$

If $f(x) \in (2^{-i}k, 2^{-i}(k+1)]$, then $f_i(x) = 2^{-i}k$. Hence $0 \leqslant f_i \leqslant f$ and $f_i \nearrow f$. Thus it suffices to approximate f_i and use a diagonal argument. Since $\{f > 2^{-i}k\}$ is open, we can find non-negative functions $w_j^{k,i} \in C_0^1(\{f > 2^{-i}k\})$ with $w_j^{k,i} \nearrow \chi_{\{f>2^{-i}k\}}$ as $j \to \infty$. Set

$$w_j^i := \sum_{k=1}^{j} 2^{-i} w_j^{k,i}.$$

Since each sum is finite, $w_j^i \in C_0^1(\Omega)$. Furthermore, $w_j^i \nearrow f_i$ as $j \to \infty$. □

In Theorem 5.2(1) we assume that $C_0^1(\Omega; \mathbb{R}^n)$ is dense in $L^{\varphi^*}(\Omega; \mathbb{R}^n)$. If φ^* satisfies (A0) and (aDec), then this holds by [21, Theorem 3.7.15]. Furthermore, φ^* satisfies these conditions if and only if φ satisfies (A0) and (aInc). In other words, this is exactly the opposite of the linear growth case that we are interested in. However, in this case we can give an exact formula for the variation V_φ in terms of the norm of the associate space.

The case when φ^* does not satisfy (aDec) is more interesting and involves the technical difficulties that we expect with BV-type spaces. Now we need to approximate not the test function but the function itself so the regularity of φ matters.

Theorem 5.2. *Let $\varphi \in \Phi_{\mathrm{w}}(\Omega)$ and $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$. Then $V_\varphi(u) \leqslant \|\nabla u\|_{(L^{\varphi^*}(\Omega))'}$.*

(1) If $C_0^1(\Omega; \mathbb{R}^n)$ is dense in $L^{\varphi^*}(\Omega; \mathbb{R}^n)$, then $V_\varphi(u) = \|\nabla u\|_{(L^{\varphi^*}(\Omega))'}$.
(2) If φ satisfies (A0), (A1) and (aDec), then $V_\varphi(u) \approx \|\nabla u\|_\varphi$.

Proof. Since $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$, it follows from the definition of V_φ and integration by parts that

$$V_\varphi(u) = \sup\left\{ \int_\Omega \nabla u \cdot w \, dx \,\middle|\, w \in C_0^1(\Omega; \mathbb{R}^n), \|w\|_{\varphi^*} \leqslant 1 \right\}. \tag{5.3}$$

The definition of the associate space norm implies that

$$\int_\Omega \nabla u \cdot w \, dx \leqslant \int_\Omega |\nabla u| \, |w| \, dx \leqslant \|\nabla u\|_{(L^{\varphi^*}(\Omega))'} \|w\|_{L^{\varphi^*}(\Omega)}.$$

Taking the supremum over $w \in C_0^1(\Omega; \mathbb{R}^n)$ with $\|w\|_{L^{\varphi^*}(\Omega)} \leqslant 1$, we conclude that $V_\varphi(u) \leqslant \|\nabla u\|_{(L^{\varphi^*}(\Omega))'}$.

Under assumption (1), we next show the opposite inequality, $\|\nabla u\|_{(L^{\varphi^*}(\Omega))'} \leqslant V_\varphi(u)$. Let $w \in L^{\varphi^*}(\Omega; \mathbb{R}^n)$ with $\|w\|_{\varphi^*} = 1$ and let (w_j) be a sequence from $C_0^1(\Omega; \mathbb{R}^n)$ with $w_j \to w$ in $L^{\varphi^*}(\Omega; \mathbb{R}^n)$ and pointwise a.e. Since also $w_j / \|w_j\|_{\varphi^*} \to w$ in $L^{\varphi^*}(\Omega; \mathbb{R}^n)$, we may assume that $\|w_j\|_{\varphi^*} = 1$. By Fatou's Lemma,

$$\liminf_{j \to \infty} \int_\Omega \nabla u \cdot w_j \, dx \geqslant \int_\Omega \nabla u \cdot w \, dx,$$

so it follows from (5.3) that

$$V_\varphi(u) \geqslant \sup\left\{ \int_\Omega \nabla u \cdot w \, dx \,\middle|\, w \in L^{\varphi^*}(\Omega; \mathbb{R}^n), \|w\|_{\varphi^*} \leqslant 1 \right\}.$$

Let $h \in L^{\varphi^*}(\Omega)$. We set $w := \frac{\nabla u}{|\nabla u|} h$ if $|\nabla u| \neq 0$ and 0 otherwise. This gives

$$V_\varphi(u) \geqslant \sup \bigg\{ \int_\Omega |\nabla u|\, h \, dx \,\bigg|\, h \in L^{\varphi^*}(\Omega), \|h\|_{\varphi^*} \leqslant 1 \bigg\} = \|\nabla u\|_{(L^{\varphi^*}(\Omega))'}.$$

Hence $V_\varphi(u) = \|\nabla u\|_{(L^{\varphi^*}(\Omega))'}$ and the proof of (1) is complete.

Then we prove (2). Fix $h \in C^1_0(\Omega)$. Since $w_{\varepsilon,\delta} := (\frac{\nabla u}{|\nabla u|+\varepsilon}) * \eta_\delta$ is bounded and converges to $\frac{\nabla u}{|\nabla u|+\varepsilon}$ in L^1 and a.e. as $\delta \to 0^+$, it follows by L^1-convergence and dominated convergence with majorant $|\nabla u|\,|h|$ that

$$\lim_{\varepsilon \to 0^+} \lim_{\delta \to 0^+} \int_\Omega \nabla u \cdot (w_{\varepsilon,\delta} h) \, dx = \lim_{\varepsilon \to 0^+} \int_\Omega \frac{|\nabla u|^2}{|\nabla u| + \varepsilon} h \, dx = \int_\Omega |\nabla u|\, h \, dx.$$

Since $w_{\varepsilon,\delta} h \in C^1_0(\Omega; \mathbb{R}^n)$, this and (5.3) imply that

$$V_\varphi(u) \geqslant \sup \bigg\{ \int_\Omega |\nabla u|\, h \, dx \,\bigg|\, h \in C^1_0(\Omega), \|h\|_{\varphi^*} \leqslant 1 \bigg\}.$$

Let $g \in L^{\varphi^*}(\Omega)$ with $\|g\|_{\varphi^*} \leqslant 1$. Since φ^* satisfies (A0), (A1) and (aInc), the Hardy–Littlewood maximal operator M is bounded in $L^{\varphi^*}(\Omega)$, with some constant $c_M > 0$ [28]. The function $\tilde g := \frac{Mg}{c_M} \in L^{\varphi^*}(\Omega)$ is lower semi-continuous, $\|\tilde g\|_{\varphi^*} \leqslant 1$ and $\tilde g \geqslant \frac{g}{c_M}$. By Lemma 5.1, we can find $h_i \in C^1_0(\Omega)$ with $h_i \to \tilde g$ and $0 \leqslant h_i \leqslant \tilde g$. By dominated convergence, with $|\nabla u|\, \tilde g$ as a majorant, we find that

$$V_\varphi(u) \geqslant \lim_{i \to \infty} \int_\Omega |\nabla u|\, h_i \, dx = \int_\Omega |\nabla u|\, \tilde g \, dx \geqslant \frac{1}{c_M} \int_\Omega |\nabla u|\, g \, dx.$$

Since g is arbitrary, this implies that

$$V_\varphi(u) \geqslant \frac{1}{c_M} \sup \bigg\{ \int_\Omega |\nabla u|\, g \, dx \,\bigg|\, g \in L^{\varphi^*}(\Omega), \|g\|_{\varphi^*} \leqslant 1 \bigg\} = \frac{\|\nabla u\|_{(L^{\varphi^*}(\Omega))'}}{c_M}.$$

By Theorem 3.4.6 and Proposition 2.4.5 of [21], $\|\nabla u\|_{(L^{\varphi^*}(\Omega))'} \approx \|\nabla u\|_\varphi$, so the proof of (2) is complete. □

The next lemma is the counterpart of [17, Theorem 5.3] and [19, Theorem 1.17], albeit with an extra constant c_φ. The extra constant is expected, since we assume only (A1), cf. Example 4.3 and Proposition 7.1.

Lemma 5.4 (Approximation by smooth functions)**.** *Assume that $\varphi \in \Phi_{\rm w}(\Omega)$ satisfies (A0), (A1) and (aDec). Then there exists $c_\varphi \geqslant 1$ such that for every $u \in L^\varphi(\Omega)$ we can find $u_k \in C^\infty(\Omega)$ with*

$$u_k \to u \text{ in } L^\varphi(\Omega) \quad \text{and} \quad V_\varphi(u) \leqslant \lim_{k \to \infty} V_\varphi(u_k) \leqslant c_\varphi V_\varphi(u).$$

If additionally $u \in W^{1,p}(\Omega)$ or $u \in L^p(\Omega)$, $p \in [1, \infty)$, then the sequence can be chosen with $u_k \to u$ in $W^{1,p}(\Omega)$ or $L^p(\Omega)$ as well.

Proof. For $k \in \mathbb{Z}_+$, we define

$$U_k := \bigg\{ x \in \Omega \,\bigg|\, \operatorname{dist}(x, \partial\Omega) > \frac{1}{m+k} \bigg\},$$

where $m > 0$ is chosen such that U_1 is non-empty. Set $V_1 := U_2$ and $V_k := U_{k+1} \setminus \overline{U_{k-1}}$ for $k \geqslant 2$. Let (ξ_k) be a partition of unity subordinate to (V_k), i.e. $\xi_k \in C^\infty_0(V_k)$, $0 \leqslant \xi_k \leqslant 1$ and $\sum_{k=1}^\infty \xi_k = 1$ for all $x \in \Omega$.

Let $\varepsilon > 0$ and let η be the standard mollifier. Choose $\varepsilon_k \in (0, \varepsilon)$ so small that $\operatorname{supp}(\eta_{\varepsilon_k} * (u\xi_k)) \subset V_k$,

$$\|\eta_{\varepsilon_k} * (u\xi_k) - u\xi_k\|_\varphi \leqslant \frac{\varepsilon}{2^k} \qquad \text{and} \qquad \|\eta_{\varepsilon_k} * (u\nabla\xi_k) - u\nabla\xi_k\|_\varphi \leqslant \frac{\varepsilon}{2^k}; \tag{5.5}$$

the last conditions are possible by (A0), (A1) and (aDec) since $u \in L^\varphi(\Omega)$ and $\xi_k, |\nabla\xi_k| \in L^\infty(\Omega)$ [21, Theorem 4.4.7] . Let us define

$$u_\varepsilon := \sum_{k=1}^\infty \eta_{\varepsilon_k} * (u\xi_k).$$

In a neighborhood of each point there are at most three non-zero terms in the sum, hence $u_\varepsilon \in C^\infty(\Omega)$.

Since $\|\cdot\|_\varphi$ is equivalent to a norm, it satisfies a countable quasitriangle inequality [21, Corollary 3.2.5]. Using $u = \sum_{k=1}^\infty \xi_k u$ and (5.5) with this inequality, we find that

$$\|u_\varepsilon - u\|_\varphi \leqslant \Big\| \sum_{k=1}^\infty (\eta_{\varepsilon_k} * (u\xi_k) - \xi_k u) \Big\|_\varphi \lesssim \sum_{k=1}^\infty \|\eta_{\varepsilon_k} * (u\xi_k) - u\xi_k\|_\varphi \leqslant \sum_{k=1}^\infty \frac{\varepsilon}{2^k} = \varepsilon.$$

Thus $u_\varepsilon \to u$ in $L^\varphi(\Omega)$ and so Lemma 4.8 yields

$$V_\varphi(u) \leqslant \liminf_{\varepsilon\to 0^+} V_\varphi(u_\varepsilon).$$

If we assume $u \in L^p(\Omega)$ or $|\nabla u| \in L^p(\Omega)$, then we can add to (5.5) also the requirement

$$\|\eta_{\varepsilon_k} * (u\xi_k) - u\xi_k\|_p \leqslant \frac{\varepsilon}{2^k} \quad \text{or} \quad \|\eta_{\varepsilon_k} * (\nabla u\,\xi_k) - \nabla u\,\xi_k\|_p \leqslant \frac{\varepsilon}{2^k}$$

and estimate in the same way $\|u_\varepsilon - u\|_p \leqslant \varepsilon$ or $\|\nabla u_\varepsilon - \nabla u\|_p \leqslant \varepsilon$. Thus also $u_\varepsilon \to u$ in $L^p(\Omega)$ or $W^{1,p}(\Omega)$, as claimed.

Fix $w \in C^1_0(\Omega; \mathbb{R}^n)$ with $\|w\|_{\varphi^*} \leqslant 1$. Since w has a compact support in Ω, only finitely many of the terms $(\eta_{\varepsilon_k} * (u\xi_k)) \operatorname{div} w$ are non-zero. Thus the sums below are really finite and can be interchanged with integrals and derivatives. Using the definition of u_ε, Fubini's Theorem in the convolution, the product rule and $\sum_{k=1}^\infty \nabla\xi_k = \nabla \sum_{k=1}^\infty \xi_k = \nabla 1 = 0$, we conclude that

$$\begin{aligned}
\int_\Omega u_\varepsilon \operatorname{div} w\,dx &= \sum_{k=1}^\infty \int_\Omega (\eta_{\varepsilon_k} * (u\xi_k)) \operatorname{div} w\,dx = \sum_{k=1}^\infty \int_\Omega (u\xi_k) \operatorname{div}(\eta_{\varepsilon_k} * w)\,dx \\
&= \sum_{k=1}^\infty \int_\Omega u \operatorname{div}(\xi_k(\eta_{\varepsilon_k} * w))\,dx - \sum_{k=1}^\infty \int_\Omega u\nabla\xi_k \cdot (\eta_{\varepsilon_k} * w)\,dx \\
&= \underbrace{\sum_{k=1}^\infty \int_\Omega u \operatorname{div}(\xi_k(\eta_{\varepsilon_k} * w))\,dx}_{=:I} - \underbrace{\sum_{k=1}^\infty \int_\Omega w \cdot (\eta_{\varepsilon_k} * (u\nabla\xi_k) - u\nabla\xi_k)\,dx}_{=:II}.
\end{aligned}$$

For II we obtain by Hölder's inequality and (5.5) that

$$|II| \lesssim \sum_{k=1}^\infty \|w\|_{\varphi^*} \big\|\eta_{\varepsilon_k} * (u\nabla\xi_k) - u\nabla\xi_k\big\|_\varphi \leqslant \sum_{k=1}^\infty \frac{\varepsilon}{2^k} = \varepsilon.$$

As $\sum_{k=1}^{\infty}\xi_k(\eta_{\varepsilon_k}*w)\in C_0^1(\Omega;\mathbb{R}^n)$ is a viable test function (up to a constant), we obtain that

$$|I|=\bigg|\int_\Omega u\,\mathrm{div}\,\Big(\sum_{k=1}^{\infty}\xi_k\,\eta_{\varepsilon_k}*w\Big)\,dx\bigg|\leqslant V_\varphi(u)\bigg\|\sum_{k=1}^{\infty}\xi_k\,\eta_{\varepsilon_k}*w\bigg\|_{\varphi^*}$$
$$\lesssim V_\varphi(u)\|Mw\|_{\varphi^*}.$$

Since φ^* satisfies (A0), (A1) and (aInc), maximal operator M is bounded in $L^{\varphi^*}(\Omega)$ [28]. So the estimates for I and II give

$$\bigg|\int_\Omega u_\varepsilon\,\mathrm{div}\,w\,dx\bigg|\lesssim V_\varphi(u)+\varepsilon.$$

Hence $V_\varphi(u_\varepsilon)\lesssim V_\varphi(u)+\varepsilon\to V_\varphi(u)$ as $\varepsilon\to 0^+$. By choosing a subsequence we ensure that $\lim_k V_\varphi(u_k)$ exists. □

A bounded domain $\Omega\subset\mathbb{R}^n$ is a *John domain* if there exist constants $0<\alpha\leqslant\beta<\infty$ and a point $x_0\in\Omega$ such that each point $x\in\Omega$ can be joined to x_0 by a rectifiable curve $\gamma:[0,\ell_\gamma]\to\Omega$ parametrized by arc length with $\gamma(0)=x$, $\gamma(\ell_\gamma)=x_0$, $\ell_\gamma\leqslant\beta$, and

$$t\leqslant\tfrac{\beta}{\alpha}\,\mathrm{dist}\,\big(\gamma(t),\partial\Omega\big)\quad\text{for all}\quad t\in[0,\ell_\gamma].$$

Examples of John domains include convex domains and domains with Lipschitz boundary, but also some domains with fractal boundaries such as the von Koch snowflake. The next compactness-type result is the counterpart of Theorem 5.5 in [17], see also Theorem 1.19 in [19].

Theorem 5.6. *Let $\Omega\subset\mathbb{R}^n$ be a bounded John domain and $\varphi\in\Phi_{\mathrm{w}}(\Omega)$ satisfy (A0), (A1) and (aDec). If (u_k) is a sequence in $\mathrm{BV}^\varphi(\Omega)$ with $\sup_k\|u_k\|_{\mathrm{BV}^\varphi}<\infty$, then there exists a subsequence (u_{k_j}) and $u\in\mathrm{BV}^\varphi(\Omega)$ such that $u_{k_j}\to u$ in $L^\varphi(\Omega)$ and $\|u\|_{\mathrm{BV}^\varphi}\leqslant\liminf\|u_{k_j}\|_{\mathrm{BV}^\varphi}$.*

Proof. By Lemma 5.4, we may choose functions $v_k\in C^\infty(\Omega)\cap\mathrm{BV}^\varphi(\Omega)$ such that

$$\|u_k-v_k\|_\varphi<\tfrac{1}{k}\qquad\text{and}\qquad\sup_k V_\varphi(v_k)<\infty.$$

Theorem 5.2(2) for $v_k\in C^\infty(\Omega)$ yields that $\sup_k\|\nabla v_k\|_\varphi<\infty$, so the sequence is bounded in $W^{1,\varphi}(\Omega)$. Since Ω is a John domain, the compact embedding $W^{1,\varphi}(\Omega)\hookrightarrow\hookrightarrow L^\varphi(\Omega)$ holds [21, 24], and thus (v_k) has a subsequence (v_{k_j}) converging to some u in $L^\varphi(\Omega)$. Therefore $\|u_{k_j}-v_{k_j}\|_\varphi<\frac{1}{k_j}$ implies that also $u_{k_j}\to u$ in $L^\varphi(\Omega)$ and, by Lemma 4.8, $u\in\mathrm{BV}^\varphi(\Omega)$. □

6. Explicit expression for the dual modular

In this section we derive a formula for the "dual modular" $\varrho_{V,\varphi}$ from Definition 4.1 in terms of ϱ_φ of the derivative's absolutely continuous part and the singular part with weight given by the recession function

$$\varphi'_\infty(x)=\limsup_{t\to\infty}\frac{\varphi(x,t)}{t}.$$

Throughout this section, we assume that $\varphi\in\Phi_{\mathrm{c}}(\Omega)$. Then $t\mapsto\frac{\varphi(\cdot,t)}{t}$ is increasing and the limit superior is a limit. Moreover, if the derivative of φ with respect to t exists, then it is increasing by convexity, so $\lim_{t\to\infty}\varphi'(\cdot,t)$ exists and equals φ'_∞ by l'Hôpital's rule. The following lemma illustrates the significance of φ'_∞.

In [26, Section 3 and Example A.1] it was shown that $\log$-Hölder continuity is not sufficient when working in $\mathrm{BV}^{p(\cdot)}$. Similarly, the (A1) condition (corresponding to $\log$-Hölder continuity) is not sufficient in the next results in view of Example 4.3. Instead, we use the restricted (VA1) which corresponds to strong $\log$-Hölder continuity (Proposition 3.2). Note that here we need the inequality at every point, since we will use the estimate with the singular measure $D^s u$.

Lemma 6.1. *Let $\varphi \in \Phi_c(\Omega)$ satisfy restricted (VA1). If $w \in C(\Omega)$ with $\varrho_{\varphi^*}(w) < \infty$, then $|w| \leqslant \varphi'_\infty$.*

Proof. We assume that $w \geqslant 0$ to simplify notation. Suppose to the contrary that $w(x_0) > \varphi'_\infty(x_0)$ for some point $x_0 \in \Omega$. Since φ is convex, $t \mapsto \frac{\varphi(x_0,t)}{t}$ is increasing and $\frac{\varphi(x_0,t)}{t} \leqslant \varphi'_\infty(x_0)$ for every $t > 0$. Now $\varphi'_\infty(x_0) < \infty$ and $\varphi(x_0,t) \leqslant \varphi'_\infty(x_0)t$ give $\varphi^*(x_0,s) \geqslant \infty\chi_{(\varphi'_\infty(x_0),\infty)}(s)$ and $\varphi^*(x_0,w(x_0)) = \infty$. From this and $\varrho_{\varphi^*}(w) < \infty$ it follows that $w \leqslant \varphi'_\infty$ almost everywhere. However, φ'_∞ need not be continuous, so we cannot directly conclude that the inequality holds everywhere.

Let ω be from (VA1) for $K := 1$. Choose $r_0 > 0$ and $\beta := \frac{1}{1+\omega(r_0)}$ such that $\varphi'_\infty(x_0) < \beta^3 w(x_0) < \beta^2 w(x)$ for every $x \in B(x_0, r_0)$. Note that $\varphi^*(x_0, \beta^3 w(x_0)) = \infty$ and $\varphi(x_0,t) \leqslant \varphi'_\infty(x_0)t \leqslant \beta^2 w(x)t$ for all $t \geqslant 0$. Since $\varphi(x_0,\cdot)$ is finite and convex, it is continuous and we can find t_x with $\varphi(x_0,t_x) = |x-x_0|^{-n}$. By restricted (VA1),

$$\varphi(x,\beta t_x) \leqslant \varphi(x_0,t_x) + \omega(r_0) = \varphi(x_0,t_x) + \tfrac{1}{\beta} - 1 \leqslant \beta^2 w(x)t_x + \tfrac{1}{\beta}.$$

By the definition of φ^* and the previous inequalities, we obtain that

$$\varphi^*(x,w(x)) \geqslant \beta t_x w(x) - \varphi(x,\beta t_x) \geqslant \beta(1-\beta)w(x)t_x - \tfrac{1}{\beta} \geqslant \tfrac{1-\beta}{\beta}\varphi(x_0,t_x) - \tfrac{1}{\beta}.$$

Since $\varphi(x_0,t_x) = |x-x_0|^{-n}$, we conclude that

$$\int_\Omega \varphi^*(x,w)\,dy \gtrsim \int_{B(x_0,r_0)} |x-x_0|^{-n}\,dy - c = \infty.$$

This contradicts the assumption $\varrho_{\varphi^*}(w) < \infty$ and thus the counter-assumption $w(x_0) > \varphi'_\infty(x_0)$ was incorrect and the claim is proved. ☐

We define

$$T^\varphi := \big\{ w \in C^1_0(\Omega;\mathbb{R}^n) \,\big|\, \varrho_{\varphi^*}(|w|) < \infty \big\}.$$

Then the usual test function space of BV is T^1 since $\varrho_\infty(|w|) < \infty$ if and only if $|w| \leqslant 1$ a.e. In the next propositions we first consider the singular and absolutely continuous parts of the derivative separately. Then we combine them to handle the whole function in Theorem 6.4.

Proposition 6.2. *Let $\varphi \in \Phi_c(\Omega)$ satisfy (A0), (aDec) and restricted (VA1). If $u \in \mathrm{BV}(\Omega)$, then*

$$\sup_{w\in T^\varphi} \int_\Omega w \cdot dD^s u = \int_\Omega \varphi'_\infty \, d|D^s u|.$$

Proof. By the definition of the total variation of a measure and Lemma 6.1,

$$\sup_{w\in T^\varphi} \int_\Omega w\cdot dD^s u \leqslant \sup_{w\in T^\varphi} \int_\Omega |w|\,d|D^s u| \leqslant \int_\Omega \varphi'_\infty\,d|D^s u|.$$

For the opposite inequality, we define $h_k : \Omega \to [0,\infty]$ by

$$h_k(x) := \lim_{r\to 0^+} \inf_{y\in B(x,r)} \frac{\varphi(y,k)}{k}.$$

Then h_k is lower semicontinuous with $h_k \leqslant \frac{\varphi(\cdot,k)}{k} \leqslant \varphi'_\infty$. From the first inequality it follows that $\varphi^*(\cdot,h_k) \leqslant \varphi(\cdot,k)$ so $\varrho_{\varphi^*}(h_k) \leqslant \varrho_\varphi(k) < \infty$ since φ satisfies (A0) and (aDec) and Ω is bounded. Let us show that $h_k \to \varphi'_\infty$. If $\varphi'_\infty(x) = \infty$, then since $\varphi^+(k) < \infty$ we can use (A1) in all sufficiently small balls to conclude that

$$h_k(x) = \lim_{r\to 0^+} \inf_{y\in B(x,r)} \frac{\varphi(y,k)}{k} \geqslant \frac{\varphi(x,\beta k) - 1}{k} \to \beta\varphi'_\infty(x) = \infty$$

as $k \to \infty$. If $\varphi'_\infty(x) < \infty$, then we use the same inequality but now with $\beta := \frac{1}{1+\omega(r)}$ from the restricted (VA1) condition; we obtain the desired convergence as $\beta \to 1^-$.

Note that h_k is increasing in k since φ is convex. It follows by monotone convergence that

$$\int_\Omega \varphi'_\infty \, d|D^s u| = \lim_{k\to\infty} \int_\Omega h_k \, d|D^s u|.$$

Let $\varepsilon > 0$ and assume $\int_\Omega \varphi'_\infty \, d|D^s u| < \infty$. We can find $h = h_k$ and $K > 0$ such that

$$\int_\Omega \varphi'_\infty \, d|D^s u| \leqslant \int_\Omega h \, d|D^s u| + \varepsilon \leqslant \sum_{j=1}^{K^2} \int_\Omega \tfrac{1}{K} \chi_{\{h > \frac{j}{K}\}} \, d|D^s u| + 2\varepsilon = \sum_{j=1}^{K^2} \tfrac{1}{K} |D^s u|\big(\{h > \tfrac{j}{K}\}\big) + 2\varepsilon.$$

Since h is lower semicontinuous, $\{h > \frac{j}{K}\}$ is open, and hence by Lemma 3.5 we can choose $w_j \in C^1_0(\{h > \frac{j}{K}\}; \mathbb{R}^n)$ with $|w_j| \leqslant 1$ such that

$$\int_\Omega \varphi'_\infty \, d|D^s u| \leqslant \sum_{j=1}^{K^2} \frac{1}{K} \int_{\{h > \frac{j}{K}\}} w_j \cdot dD^s u + 3\varepsilon = \int_\Omega \underbrace{\Bigg(\sum_{j=1}^{K^2} \tfrac{1}{K} w_j \Bigg)}_{=: w_\varepsilon} \cdot \, dD^s u + 3\varepsilon.$$

Note that $w_\varepsilon \in C^1_0(\Omega; \mathbb{R}^n)$ and

$$|w_\varepsilon| \leqslant \sum_{j=1}^{K^2} \tfrac{1}{K} |w_j| \leqslant \sum_{j=1}^{K^2} \tfrac{1}{K} \chi_{\{h > \frac{j}{K}\}} \leqslant h$$

so that $\varrho_{\varphi^*}(|w_\varepsilon|) < \infty$. Therefore $w_\varepsilon \in T^\varphi$ and

$$\int_\Omega \varphi'_\infty \, d|D^s u| \leqslant \int_\Omega w_\varepsilon \cdot dD^s u + 3\varepsilon \leqslant \sup_{w \in T^\varphi} \int_\Omega w \cdot dD^s u + 3\varepsilon.$$

The upper bound follows from this as $\varepsilon \to 0^+$. If $\int_\Omega \varphi'_\infty \, d|D^s u| = \infty$, then a similar argument gives $\frac{1}{3\varepsilon} \leqslant \sup_{w\in T^\varphi} \int_\Omega w \cdot dD^s u$ and the claim again follows. □

In the next result we assume that φ is continuous in both variables. Removing this somewhat unusual requirement is an open problem. Similar to the case of V_φ in Theorem 5.2(2), the approximation is made much more difficult by the fact that φ^* is not doubling.

Proposition 6.3. *Let $\varphi \in \Phi_c(\Omega) \cap C(\Omega \times [0,\infty))$ satisfy (A0) and (aDec). If $u \in \mathrm{BV}(\Omega)$, then*

$$\sup_{w \in T^\varphi} \int_\Omega \nabla^a u \cdot w - \varphi^*(x, |w|) \, dx = \varrho_\varphi(|\nabla^a u|).$$

Proof. The upper bound follows directly from Young's inequality:

$$\sup_{w \in T^\varphi} \int_\Omega \nabla^a u \cdot w - \varphi^*(x, |w|) \, dx \leqslant \int_\Omega \varphi(x, |\nabla^a u|) \, dx.$$

For the lower bound we make several reductions. Choose $g_i \in C(\Omega; \mathbb{R}^n) \cap L^\varphi(\Omega; \mathbb{R}^n)$ with $g_i \to \nabla^a u$ pointwise a.e and in $L^1(\Omega; \mathbb{R}^n)$. Then Fatou's Lemma and L^1-convergence yield

$$\int_\Omega \varphi(x, |\nabla^a u|) \, dx \leqslant \liminf_{i\to\infty} \int_\Omega \varphi(x, |g_i|) \, dx \quad \text{and} \quad \lim_{i\to\infty} \int_\Omega g_i \cdot w \, dx = \int_\Omega \nabla^a u \cdot w \, dx$$

when $w \in T^\varphi$. Thus it suffices to show that

$$\int_\Omega \varphi(x, |g|) \, dx \leqslant \sup_{w \in T^\varphi} \int_\Omega g \cdot w - \varphi^*(x, |w|) \, dx$$

for $g \in C(\Omega;\mathbb{R}^n) \cap L^\varphi(\Omega;\mathbb{R}^n)$. Furthermore, replacing w by $\frac{g}{\varepsilon+|g|}|w|$ and letting $\varepsilon \to 0^+$, we see that $g \cdot \frac{g}{\varepsilon+|g|}|w| \to |g|\,|w|$ pointwise. Thus by monotone convergence the vector-values of g and w are irrelevant and we need only show that

$$\int_\Omega \varphi(x,|g|)\,dx \leqslant \sup_{w\in C_0^1(\Omega)} \int_\Omega |gw| - \varphi^*(x,|w|)\,dx$$

for $g \in C(\Omega)\cap L^\varphi(\Omega)$. We also exclude test-functions with $\varrho_{\varphi^*}(w)=\infty$ by Remark 4.4.

Let φ' be the left-derivative of φ with respect to the second variable. Then φ' is left-continuous and $\varphi(x,s) \geqslant \varphi(x,s_0)+\varphi'(x,s_0)(s-s_0)$ by convexity. We would like to choose $w := \varphi'(x,|g|)$ in the previous supremum. However, this function is not in general smooth and we cannot use regular approximation by smooth functions since φ^* is not doubling. Instead we define

$$\psi_\varepsilon(x,t) := \int_{-\infty}^{\infty} \varphi(x,\max\{\tau,0\})\zeta_\varepsilon(t-\tau)\,d\tau = \varphi *_t \zeta_\varepsilon(x,t),$$

where ζ_ε is a mollifier in $\mathbb{R}$ with support in $[0,\varepsilon]$. Since φ and φ' are increasing in the second variable and left-continuous, we see that $\psi_\varepsilon \nearrow \varphi$ and $\psi_\varepsilon' \nearrow \varphi'$ as $\varepsilon \to 0^+$. Furthermore, $\psi_\varepsilon' = \varphi *_t (\zeta_\varepsilon')$ is continuous in x since φ is and it continuous in t as a convolution with a smooth function. Let $v_i \in C_0(\Omega)$ with $0 \leqslant v_i \leqslant 1$ and $v_i \nearrow 1$. By uniform continuity in $\operatorname{supp} v_i$, we can choose $\delta = \delta_{\varepsilon,i} > 0$ such that $\psi_\varepsilon'(x,|g(x)|\,v_i(x)) - \varepsilon \leqslant \psi_\varepsilon'(y,|g(y)|\,v_i(y))$ for all $x \in B(y,\delta)$ and $y \in \Omega$. Then

$$w_{\varepsilon,i} := \max\{\psi_\varepsilon'(\cdot,|g|\,v_i)-\varepsilon,0\} *_x \eta_\delta \leqslant \psi_\varepsilon'(\cdot,|g|) \leqslant \varphi'(\cdot,|g|).$$

Now $w_{\varepsilon,i} \to \varphi'(\cdot,|g|)$, so we conclude by Fatou's Lemma that

$$\int_\Omega |g|\varphi'(x,|g|)\,dx \leqslant \liminf_{i\to\infty,\varepsilon\to 0} \int_\Omega |g w_{\varepsilon,i}|\,dx.$$

Since φ satisfies (A0) and (aDec), we see that

$$\varphi^*(x,|w_{\varepsilon,i}|) \leqslant \varphi^*(x,\varphi'(x,|g|)) \leqslant |g|\varphi'(x,|g|) \lesssim \varphi(x,|g|).$$

As $g \in L^\varphi(\Omega)$ and φ satisfies (aDec), $\varrho_\varphi(g) < \infty$. Thus dominated convergence with majorant $c\varphi(\cdot,g)$ yields

$$\int_\Omega \varphi^*(x,\varphi'(x,|g|))\,dx = \lim_{i\to\infty,\varepsilon\to 0} \int_\Omega \varphi^*(x,|w_{\varepsilon,i}|)\,dx.$$

Since w_ε is a valid test-function and $\varphi'(\cdot,|g|) < \infty$ a.e., this together with "Young's equality" [40, Lemma 1.7.3(i)] implies that

$$\begin{aligned}\sup_{w\in C_0^1(\Omega)} \int_\Omega |gw| - \varphi^*(x,|w|)\,dx &\geqslant \liminf_{i\to\infty,\varepsilon\to 0} \int_\Omega |g|\,|w_{\varepsilon,i}| - \varphi^*(x,|w_{\varepsilon,i}|)\,dx \\ &\geqslant \int_\Omega |g|\varphi'(x,|g|) - \varphi^*(x,\varphi'(x,|g|))\,dx = \int_\Omega \varphi(x,|g|)\,dx.\end{aligned}$$

This completes the proof of the lower bound. □

We next derive a simple, closed form expression for $\varrho_{V,\varphi}$. This is the main result of the paper. Note that the right-hand side expression was also obtained recently in the one-dimensional case for a modular based on the Riesz variation assuming the (VA1) condition [30].

Theorem 6.4. *Let $\varphi \in \Phi_c(\Omega)\cap C(\Omega\times[0,\infty))$ satisfy (A0), (aDec) and restricted (VA1). If $u \in \mathrm{BV}(\Omega)$, then*

$$\varrho_{V,\varphi}(u) = \varrho_\varphi(|\nabla^a u|) + \int_\Omega \varphi'_\infty\,d|D^s u|.$$

Proof. Since $Du = D^a u + D^s u$, integration by parts implies that

$$-\int_\Omega u \operatorname{div} w\, dx = \int_\Omega w \cdot dDu = \int_\Omega \nabla^a u \cdot w\, dx + \int_\Omega w \cdot dD^s u$$

for $w \in T^\varphi$. Hence the claim follows from Propositions 6.2 and 6.3 once we prove that

$$\begin{aligned} &\sup_{w\in T^\varphi} \left[\int_\Omega \nabla^a u \cdot w - \varphi^*(x, |w|)\, dx + \int_\Omega w \cdot dD^s u \right] \\ &\qquad = \sup_{w\in T^\varphi} \int_\Omega \nabla^a u \cdot w - \varphi^*(x,|w|)\, dx + \sup_{w\in T^\varphi} \int_\Omega w \cdot dD^s u. \end{aligned}$$

The inequality "$\leqslant$" is clear, so we focus on the opposite one.

We assume first that $\varrho_\varphi(|\nabla^a u|) + \int_\Omega \varphi'_\infty\, d|D^s u| < \infty$ and fix $\varepsilon > 0$. By the definition of supremum we can choose $w_1, w_2 \in T^\varphi$ such that

$$\sup_{w\in T^\varphi} \int_\Omega \nabla^a u \cdot w - \varphi^*(x,|w|)\, dx \leqslant \int_\Omega \nabla^a u \cdot w_1 - \varphi^*(x,|w_1|)\, dx + \varepsilon < \infty$$

and

$$\sup_{w\in T^\varphi} \int_\Omega w \cdot dD^s u \leqslant \int_\Omega w_2 \cdot dD^s u + \varepsilon < \infty.$$

Since $u \in \mathrm{BV}(\Omega)$ and $w_i \in T^\varphi$, we have $|\nabla^a u|\,|w_i| \in L^1(\Omega)$ and $\varrho_{\varphi^*}(|w_i|) < \infty$. Thus, by the absolute continuity of the integral, we find $\delta > 0$ such that

$$\left| \int_{\Omega\setminus\Omega_1} \nabla^a u \cdot w_i - \varphi^*(x,|w_i|)\, dx \right| \leqslant \int_{\Omega\setminus\Omega_1} |\nabla^a u|\,|w_i| + \varphi^*(x,|w_i|)\, dx \leqslant \varepsilon$$

for $i \in \{1,2\}$ and any $\Omega_1 \subset \Omega$ with $|\Omega\setminus\Omega_1| < \delta$ and

$$\left| \int_{\Omega\setminus\Omega_2} w_2 \cdot dD^s u \right| \leqslant \int_{\Omega\setminus\Omega_2} \varphi'_\infty\, d|D^s u| \leqslant \varepsilon$$

for any $\Omega_2 \subset \Omega$ with $|D^s u|(\Omega\setminus\Omega_2) < \delta$.

Since $\operatorname{supp} D^s u$ has Lebesgue measure zero, we can find a finite collection of open rectangles $Q_i \subset \Omega$ with $|D^s u|(\bigcup Q_i) > |D^s u|(\Omega) - \delta$ and $|\bigcup 2Q_i| < \delta$. Then we choose $\theta \in C^1_0(\Omega)$ with $0 \leqslant \theta \leqslant 1$, $\theta = 1$ in $\Omega_2 := \bigcup Q_i$ and $\theta = 0$ in $\Omega_1 := \Omega \setminus \bigcup 2Q_i$. Let $w_\varepsilon := \theta w_2 + (1-\theta) w_1 \in C^1_0(\Omega;\mathbb{R}^n)$. Since w_ε is a pointwise convex combination,

$$\varphi^*(\cdot, |w_\varepsilon|) \leqslant \varphi^*(\cdot, \max\{|w_2|, |w_1|\}) \leqslant \varphi^*(\cdot,|w_2|) + \varphi^*(\cdot,|w_1|).$$

This yields that $\varrho_{\varphi^*}(|w_\varepsilon|) \leqslant \varrho_{\varphi^*}(|w_2|) + \varrho_{\varphi^*}(|w_1|) < \infty$, and so $w_\varepsilon \in T^\varphi$. By Lemma 6.1, $|w_\varepsilon| \leqslant \varphi'_\infty$. We obtain that

$$\begin{aligned} \varrho_{V,\varphi}(u) &\geqslant \int_\Omega \nabla^a u \cdot w_\varepsilon - \varphi^*(x,|w_\varepsilon|)\, dx + \int_\Omega w_\varepsilon \cdot dD^s u \\ &\geqslant \int_{\Omega_1} \nabla^a u \cdot w_1 - \varphi^*(x,|w_1|)\, dx + \int_{\Omega_2} w_2 \cdot dD^s u - c_\theta \\ &\geqslant \int_\Omega \nabla^a u \cdot w_1 - \varphi^*(x,|w_1|)\, dx + \int_\Omega w_2 \cdot dD^s u - 5\varepsilon, \end{aligned}$$

where

$$c_\theta := \int_{\Omega\setminus\Omega_1} |\nabla^a u|\,|w_\varepsilon| + \varphi^*(x,|w_\varepsilon|)\, dx + \int_{\Omega\setminus\Omega_2} |w_\varepsilon|\, d|D^s u| \leqslant 3\varepsilon$$

by the absolute integrability assumptions. By the choice of w_1 and w_2,

$$\varrho_{V,\varphi}(u) \geqslant \sup_{w\in T^\varphi} \int_\Omega \nabla^a u\cdot w - \varphi^*(x,|w|)\,dx + \sup_{w\in T^\varphi}\int_\Omega w\cdot dD^s u - 7\varepsilon.$$

The lower bound follows as $\varepsilon \to 0^+$. This concludes the proof in the case $\varrho_\varphi(|\nabla^a u|) + \int_\Omega \varphi'_\infty\, d|D^s u| < \infty$.

When $\varrho_\varphi(|\nabla^a u|) = \infty$ and $\int_\Omega \varphi'_\infty\, d|D^s u| < \infty$, we estimate

$$\begin{aligned}
&\sup_{w\in T^\varphi}\bigg[\int_\Omega \nabla^a u\cdot w - \varphi^*(x,|w|)\,dx + \int_\Omega w\cdot dD^s u\bigg]\\
&\qquad\geqslant \sup_{w\in T^\varphi}\int_\Omega \nabla^a u\cdot w - \varphi^*(x,|w|)\,dx - \sup_{w\in T^\varphi}\int_\Omega w\cdot dD^s u\\
&\qquad= \varrho_\varphi(|\nabla^a u|) - \int_\Omega \varphi'_\infty\, d|D^s u| = \infty.
\end{aligned}$$

Only the case $\int_\Omega \varphi'_\infty\, d|D^s u| = \infty$ remains. By the proof of Proposition 6.2, there exists $w_\varepsilon \in C^1_0(\Omega;\mathbb{R}^n)$ with

$$\int_\Omega w_\varepsilon\cdot dD^s u > \frac{1}{\varepsilon}$$

and $|w_\varepsilon| \leqslant \frac{\varphi(\cdot,k)}{k}$ for some $k = k_\varepsilon$. For any $\theta : \Omega \to [0,1]$, we find that

$$\nabla^a u\cdot(\theta w_\varepsilon) - \varphi^*(\cdot,|\theta w_\varepsilon|) \geqslant \nabla^a u\cdot(\theta w_\varepsilon) - \varphi^*\Big(\cdot, \frac{\varphi(\cdot,k)}{k}\Big) \geqslant -\Big(\frac{\varphi^+(k)}{k}\,|\nabla^a u| + \varphi^+(k)\Big).$$

Since the function on the right-hand side is integrable, we can choose $\delta_k > 0$ such that its integral over any measurable A with $|A| < \delta_k$ is at least -1. Furthermore, since $\operatorname{supp} D^s u$ has measure zero, we can choose $\theta \in C^\infty_0(\Omega)$ as before to have support with Lebesgue measure at most δ_k and satisfy

$$\int_\Omega \theta w_\varepsilon\cdot dD^s u > \frac{1}{2}\int_\Omega w_\varepsilon\cdot dD^s u > \frac{1}{2\varepsilon}.$$

Then

$$\begin{aligned}
&\sup_{w\in T^\varphi}\bigg[\int_\Omega \nabla^a u\cdot w - \varphi^*(x,|w|)\,dx + \int_\Omega w\cdot dD^s u\bigg]\\
&\qquad\geqslant \int_\Omega \theta w_\varepsilon\cdot dD^s u + \int_\Omega \nabla^a u\cdot(\theta w_\varepsilon) - \varphi^*(x,|\theta w_\varepsilon|)\,dx \geqslant \frac{1}{2\varepsilon} - 1.
\end{aligned}$$

When $\varepsilon \to \infty$, the claim follows in this case also. □

As a special case we obtain the following result in Orlicz spaces. Now the recession function is just a constant, either finite or infinite. As can be seen, we do not obtain any new spaces in this case, only the classical BV-space or the regular Sobolev space.

Corollary 6.5. *Let $\varphi \in \Phi_c$ be independent of x and satisfy (aDec). If $u \in \mathrm{BV}(\Omega)$, then $\varrho_{V,\varphi}(u) = \varrho_\varphi(|\nabla^a u|) + \varphi'_\infty\,|D^s u|(\Omega)$ and so*

(1) $\mathrm{BV}^\varphi(\Omega) = \mathrm{BV}(\Omega)$ if $\varphi'_\infty < \infty$;
(2) $\mathrm{BV}^\varphi(\Omega) = W^{1,\varphi}(\Omega)$ if $\varphi'_\infty = \infty$.

7. Precise approximation and Γ-convergence

We can now prove a precise approximation lemma for the modular using the formula for $\varrho_{V,\varphi}$ from the previous section. In contrast to Lemma 5.4 which provides only approximate equality of the limit we here obtain that the limit exactly equals $\varrho_{V,\varphi}(u)$, under appropriately stronger assumptions on φ. This is critical for Γ-convergence. A similar argument should also work for V_φ with the same assumptions.

Note that we assume (VA1) for φ^*, not only its restricted version. This is used for Young's convolution inequality. In [21, Lemma 4.1.7] it was shown that (A1) of φ and φ^* are equivalent provided (A0) holds; the corresponding statement is not known for (VA1).

Proposition 7.1 (Modular approximation by smooth functions)**.** *Let $\varphi \in \Phi_c(\Omega)\cap C(\Omega\times[0,\infty))$ satisfy (A0), (aDec) and restricted (VA1) and assume that φ^* satisfies (VA1). For every $u \in L^\varphi(\Omega)$ there exist $u_k \in C^\infty(\Omega)$ such that*

$$u_k \to u \text{ in } L^\varphi(\Omega) \quad \text{and} \quad \varrho_{V,\varphi}(u) = \lim_{k\to\infty} \varrho_\varphi(|\nabla u_k|).$$

If additionally $u \in L^2(\Omega)$, then the sequence can be chosen with $u_k \to u$ in $L^2(\Omega)$ as well.

Proof. Since the case $\varrho_{V,\varphi}(u) = \infty$ is trivial, we may assume that $\varrho_{V,\varphi}(u) < \infty$. Let $\varepsilon \in (0,1)$. We define ξ_k, η_{ε_k}, and u_ε as in the proof of Lemma 5.4 so that $V_\varphi(u_\varepsilon) \lesssim V_\varphi(u)$. It follows from (aDec)$_q$ that

$$\min\{\|u\|_{\varrho_{V,\varphi}}, \|u\|_{\varrho_{V,\varphi}}^q\} \lesssim \varrho_{V,\varphi}(u) \lesssim \max\{\|u\|_{\varrho_{V,\varphi}}, \|u\|_{\varrho_{V,\varphi}}^q\}.$$

Thus Lemma 4.7 and $V_\varphi(u_\varepsilon) \lesssim V_\varphi(u)$ imply that $\varrho_\varphi(|\nabla u_\varepsilon|) \lesssim \varrho_{V,\varphi}(u)^q + 1$. From Theorem 6.4 we see that U_1 can be chosen so large (by choosing m large in Lemma 5.4) that $\varrho_{V\setminus\overline{U_1},\varphi}(u) < \varepsilon$. Then $V_\varphi(u, \Omega\setminus\overline{U_1}) \lesssim \varepsilon^{1/q}$.

Since $u_\varepsilon \in C^\infty(\Omega)$, $\nabla^a u_\varepsilon = \nabla u_\varepsilon$. By the proof of Proposition 6.3 with $|\nabla u_\varepsilon|$ as g, there exists $w_\varepsilon \in C^1_0(\Omega;\mathbb{R}^n)$ with $\varphi^*(x,|w_\varepsilon|) \lesssim \varphi(x,|\nabla u_\varepsilon|)$ and

$$\int_\Omega \varphi(x,|\nabla u_\varepsilon|)\,dx \leqslant (1+\varepsilon)\int_\Omega \nabla u_\varepsilon\cdot w_\varepsilon - \varphi^*(x,|w_\varepsilon|)\,dx.$$

By (aDec) of φ, Theorem 6.4 and the estimates above,

$$\varrho_{\varphi^*}(|w_\varepsilon|) \lesssim \varrho_\varphi(|\nabla u_\varepsilon|) \lesssim \varrho_{V,\varphi}(u)^q + 1.$$

Thus $\|w_\varepsilon\|_{\varphi^*} \leqslant c$. As in Lemma 5.4, we have

$$\int_\Omega \nabla u_\varepsilon\cdot w_\varepsilon\,dx = \underbrace{\sum_{k=1}^\infty \int_\Omega u\operatorname{div}(\xi_k(\eta_{\varepsilon_k} * w_\varepsilon))\,dx}_{=:I} - \underbrace{\sum_{k=1}^\infty \int_\Omega w_\varepsilon\cdot(\eta_{\varepsilon_k} * (u\nabla\xi_k) - (u\nabla\xi_k))\,dx}_{=:II}$$

and the inequality $|II| \leqslant c\varepsilon$ again follows.

We divide the term I into two parts. Let ω be from Corollary 3.4. Using the definition of $\varrho_{V,\varphi}$ to the first part of I, and and estimating the second part of I as in Lemma 5.4 but now with a test-function supported in $\Omega\setminus\overline{U_1}$, we find that

$$\begin{aligned}
|I| &= \left|\int_\Omega u\operatorname{div}(\xi_1(\eta_{\varepsilon_1} * w_\varepsilon))\,dx + \int_\Omega u\operatorname{div}\left(\sum_{k=2}^\infty \xi_k(\eta_{\varepsilon_k} * w_\varepsilon)\right)dx\right| \\
&\leqslant \varrho_{V,\varphi}\big((1+\omega(\varepsilon_1))u\big) + \varrho_{\varphi^*}\left(\frac{\xi_1|\eta_{\varepsilon_1} * w_\varepsilon|}{1+\omega(\varepsilon_1)}\right) + cV_\varphi(u,\Omega\setminus\overline{U_1}) \\
&\leqslant \varrho_{V,\varphi}\big((1+\omega(\varepsilon_1))u\big) + \varrho_{\varphi^*}\left(\frac{\eta_{\varepsilon_1} * |w_\varepsilon|}{1+\omega(\varepsilon_1)}\right) + cV_\varphi(u,\Omega\setminus\overline{U_1})
\end{aligned}$$

By Young's convolution inequality (Corollary 3.4),

$$\varrho_{\varphi^*}\Big(\frac{\eta_{\varepsilon_1} * |w_\varepsilon|}{1+\omega(\varepsilon_1)}\Big) - \varrho_{\varphi^*}(|w_\varepsilon|) \leqslant \varrho_{\varphi^*}(|w_\varepsilon|) + \omega(\varepsilon_1) - \varrho_{\varphi^*}(|w_\varepsilon|) \leqslant \omega(\varepsilon_1) \to 0$$

as $\varepsilon_1 \to 0^+$. Combining the estimates, we obtain that

$$\begin{aligned}\int_\Omega \varphi(x, |\nabla u_\varepsilon|)\,dx &\leqslant (1+\varepsilon)\int_\Omega \nabla u_\varepsilon \cdot w_\varepsilon - \varphi^*(x,|w_\varepsilon|)\,dx \\ &\leqslant (1+\varepsilon)(|I| - \varrho_{\varphi^*}(|w_\varepsilon|)) + c\varepsilon \\ &\leqslant (1+\varepsilon)\varrho_{V,\varphi}\big((1+\omega(\varepsilon_1))u\big) + c(|\Omega|\,\omega(\varepsilon_1) + \varepsilon^{1/q}).\end{aligned}$$

By [21, Lemma 2.2.6], there exists a constant q_2 depending on q such that $\varrho_{V,\varphi}\big((1+\omega(\varepsilon_1))u\big) \leqslant (1+\omega(\varepsilon_1))^{q_2}\varrho_{V,\varphi}(u)$. As $\varepsilon, \varepsilon_1 \to 0^+$, we obtain that $\limsup_{\varepsilon\to 0^+} \varrho_\varphi(|\nabla u_\varepsilon|) \leqslant \varrho_{V,\varphi}(u)$. The opposite inequality follows from Lemma 4.8 as in Lemma 5.4. □

In [16], we introduced abstract BV^φ-type spaces by a limit procedure. We use here the version with a fidelity term which is most relevant for image processing. For $p > 1$ and for a given $f \in L^2(\Omega)$, we defined functionals $F_p : L^2(\Omega) \to [0,\infty]$ by

$$F_p(u) := \begin{cases} \displaystyle\int_\Omega \varphi(x,|\nabla u|)^p + |u-f|^2\,dx & \text{when } u \in L^{1,\varphi^p}(\Omega); \\ \infty & \text{otherwise.}\end{cases}$$

and the limit functional $F : L^2(\Omega) \to [0,\infty]$ by

$$F(u) := \inf\left\{ \liminf_{i\to\infty} \int_\Omega \varphi(x,|\nabla u_k|) + |u_k - f|^2\,dx \,\middle|\, u_k \in L^{1,\varphi}(\Omega)\cap L^2(\Omega) \text{ and } u_k \to u \text{ in } L^2(\Omega)\right\}$$

Note that the energy in F_p satisfies (aInc) and (aDec) so it can be studied in a reflexive space and it is easy to prove existence of minimizers among other things [25].

We compare F with the corresponding version of $\varrho_{V,\varphi}$ including the fidelity term, namely

$$\varrho^f_{V,\varphi}(u) := \varrho_{V,\varphi}(u) + \varrho_2(u-f) = \sup\left\{ \int_\Omega u \operatorname{div} w - \varphi^*(x,|w|) + |u-f|^2\,dx \,\middle|\, w \in C^1_0(\Omega;\mathbb{R}^n)\right\}.$$

Proposition 7.2. *Let $\varphi \in \Phi_c(\Omega) \cap C(\Omega \times [0,\infty))$ satisfy (A0), (aDec) and restricted (VA1) and assume that φ^* satisfies (VA1). Then $\varrho^f_{V,\varphi}(u) \leqslant F(u)$ for all $u \in L^2(\Omega)$ and $\varrho^f_{V,\varphi}(u) = F(u)$ for all $u \in L^2(\Omega) \cap L^\varphi(\Omega)$.*

Proof. Let us prove first that $\varrho^f_{V,\varphi}(u) \leqslant F(u)$. We may assume $F(u) < \infty$ and consider functions $u_k \in L^{1,\varphi}(\Omega) \cap L^2(\Omega)$ realizing the infimum from F with $u_k \to u$ in $L^2(\Omega)$. Weak lower semicontinuity of $\varrho_{V,\varphi}$ (Lemma 4.8) and in L^2 gives

$$\varrho^f_{V,\varphi}(u) \leqslant \liminf_{i\to\infty} \varrho^f_{V,\varphi}(u_k).$$

By Young's inequality, $\varrho^f_{V,\varphi}(u_k) \leqslant \varrho_\varphi(|\nabla u_k|) + \varrho_2(u_k - f)$ so that

$$\varrho^f_{V,\varphi}(u) \leqslant \liminf_{i\to\infty} \big(\varrho_\varphi(|\nabla u_k|) + \varrho_2(u_k - f)\big) = F(u).$$

Thus the inequality is proved.

For the opposite inequality, $F(u) \leqslant \varrho^f_{V,\varphi}(u)$, we may assume that $\varrho^f_{V,\varphi}(u) < \infty$. By Proposition 7.1, there exist $u_k \in C^\infty(\Omega)$ such that $u_k \to u$ in $L^\varphi(\Omega) \cap L^2(\Omega)$ and

$$\varrho_{V,\varphi}(u) = \lim_{i\to\infty} \varrho_\varphi(|\nabla u_k|).$$

Since $\varrho_\varphi(|\nabla u_k|) < \infty$ and $u_k \in L^1(\Omega)$, we see that $u_k \in L^{1,\varphi}(\Omega)$, and so, by the definition of F, using the fact that the limit of the sum is the sum of the limits, we obtain that

$$F(u) \leqslant \liminf_{i\to\infty} \big(\varrho_\varphi(|\nabla u_k|) + \varrho_2(u_k - f)\big) = \varrho^f_{V,\varphi}(u). \qquad \square$$

The concept of Γ-convergence was introduced by De Giorgi and Franzoni [13], see also [5, 12]. A family of functionals $F_p : L^2(\Omega) \to [0,\infty]$ is said to *Γ-converge* to $F : L^2(\Omega) \to [0,\infty]$ in $L^2(\Omega)$ if the following hold for every sequence (p_k) converging to one from above:

(a) $F(u) \leqslant \liminf_{i\to\infty} F_{p_k}(u_i)$ for every $u \in L^2(\Omega)$ and every sequence with $u_i \to u$ in $L^2(\Omega)$;

(b) $F(u) \geqslant \limsup_{i\to\infty} F_{p_k}(u_i)$ for every $u \in L^2(\Omega)$ and some sequence with $u_i \to u$ in $L^2(\Omega)$.

We conclude by showing the Γ-convergence in the situation most relevant to image processing: convex planar domains. This allows us to simplify the assumptions.

Corollary 7.3. *Let $\Omega \subset \mathbb{R}^2$ be convex and let $\varphi \in \Phi_c(\Omega)$ satisfy (A0), (aDec)$_2$ and (VA1) and assume that φ^* satisfies (VA1). Then F_p Γ-converges to $\varrho^f_{V,\varphi}$ in $L^2(\Omega)$.*

Proof. To establish the necessary conditions we use some results from references without defining here all the terms. The references can be consulted if necessary. By [35, Corollary 4.6], $C^\infty(\overline{\Omega})$ is dense in $W^{1,\varphi}(\Omega)$ if Ω is an (ε,δ)-domain and φ satisfies (A0), (A1) and (A2). We note that (A2) holds since Ω is bounded [21, Lemma 4.2.3] and Ω is an (ε,δ)-domain since it is convex.

Since φ satisfies (aDec)$_2$, $L^2(\Omega) \subset L^\varphi(\Omega)$ and thus $L^{1,\varphi}(\Omega) \cap L^2(\Omega) \hookrightarrow W^{1,\varphi}(\Omega)$. Since the dimension is 2 we also have $W^{1,\varphi}(\Omega) \hookrightarrow W^{1,1}(\Omega) \hookrightarrow L^2(\Omega)$. Thus $C^\infty(\overline{\Omega})$ is dense in $L^{1,\varphi}(\Omega) \cap L^2(\Omega)$ with respect to the norm $u \mapsto \|u\|_2 + \|\nabla u\|_\varphi$. By density, [16, Theorem 1.3(2)] yields that F_p Γ-converges to F in $L^2(\Omega)$. Since φ satisfies (VA1), it belongs to $C(\Omega \times [0,\infty))$. Thus Proposition 7.2 gives $F = \varrho^f_{V,\varphi}$ in $L^2(\Omega) = L^2(\Omega) \cap L^\varphi(\Omega)$. $\square$

Statements

The authors have no competing interests, financial or otherwise. No data was used in this research, obviously.

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