

MINIMIZERS OF ABSTRACT GENERALIZED ORLICZ-BOUNDED VARIATION ENERGY

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ABSTRACT. A way to measure the lower growth rate of $\varphi : \Omega \times [0, \infty) \rightarrow [0, \infty)$ is to require $t \mapsto \varphi(x, t)t^{-r}$ to be increasing in $(0, \infty)$. If this condition holds with $r = 1$, then

$$\inf_{u \in f + W_0^{1,\varphi}(\Omega)} \int_{\Omega} \varphi(x, |\nabla u|) dx$$

with boundary values $f \in W^{1,\varphi}(\Omega)$ does not necessary have a minimizer. However, if φ is replaced by φ^p , then the growth condition holds with $r = p > 1$ and thus (under some additional conditions) the corresponding energy integral has a minimizer. We show that a sequence (u_p) of such minimizers converges when $p \rightarrow 1^+$ in a suitable BV-type space involving generalized Orlicz growth and obtain the Γ -convergence of functionals with fixed boundary values and of functionals with fidelity terms.

1. INTRODUCTION

Since its introduction in the early 1970's, Γ -convergence has gained more and more importance, being a very flexible instrument and the most natural notion of convergence for variational problems. Therefore in the last decades much literature has been devoted to the description of the asymptotic behaviour of families of minimum problems, usually depending on some parameters, appearing in various contexts.

Γ -convergence is mostly applied to families of integral functionals, among which a key role is played by the p -Dirichlet integral,

$$\int_{\Omega} |\nabla u|^p dx,$$

as a prototype for integral functionals with standard growth. The case $p = 1$ deserves particular attention: questions like semicontinuity and relaxation require the use of functions of bounded variation, $BV(\Omega)$, instead of the space $W^{1,1}(\Omega)$ [9] unless some additional assumptions are made [15]. Recall that $BV(\Omega)$ is the Banach space of all $L^1(\Omega)$ -functions whose first order distributional derivatives are bounded Radon measures.

It is natural to ask if the functionals for $p \rightarrow 1^+$ are connected to the case $p = 1$. However, few results are presented in literature, at least to our knowledge. Juutinen [22] studied the problem for so-called *functions of least gradient* in $BV(\Omega) \cap C(\overline{\Omega})$ with given boundary values and continuous minimizers of the p -Dirichlet energy. More precisely, for suitable domains Ω , the sequence of the unique p -minimizers $u_p \in W_{\text{loc}}^{1,p}(\Omega) \cap C(\overline{\Omega})$ with boundary value $w \in C(\partial\Omega)$ converges uniformly, as $p \rightarrow 1^+$, to a function $u \in BV(\Omega) \cap C(\overline{\Omega})$ that is the unique function of least gradient with boundary value w . Apart from the continuity requirement, these variational problems represent the fundamental minimization problems in their respective spaces BV and $W^{1,p}$.

The concept of Γ -convergence, introduced by De Giorgi and Franzoni [12], has been systematically presented in [3, 10]. Γ -convergence was not used by Juutinen [22], but it seems reasonable

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to use it in this context, as well. A family of functionals $\mathcal{I}_\varepsilon : X \rightarrow \overline{\mathbb{R}}$ is said to Γ -converge (in topology τ) to $\mathcal{I} : X \rightarrow \overline{\mathbb{R}}$ if the following hold for every positive sequence (ε_i) converging to zero:

- (a) $\mathcal{I}(u) \leq \liminf_{i \rightarrow \infty} \mathcal{I}_{\varepsilon_i}(u_i)$ for every $u \in X$ and every $(u_i) \subset X$ τ -converging to u ;
- (b) $\mathcal{I}(u) \geq \limsup_{i \rightarrow \infty} \mathcal{I}_{\varepsilon_i}(u_i)$ for every $u \in X$ and some $(u_i) \subset X$ τ -converging to u .

We will use this definition for $\mathcal{I}_{\varepsilon_i} = E_{1+\varepsilon_i}$; for simplicity we denote $p_i = 1 + \varepsilon_i$ and talk about the Γ -convergence of E_{p_i} as $p_i \rightarrow 1^+$.

In this paper we consider the following functionals with generalized Orlicz growth. This is a very active field recently, boosted by work on the double phase problem by Baroni, Colombo and Mingione, e.g. [4, 8]. The generalized Orlicz (also known as Musielak–Orlicz) case unifies the study of the double phase problem and the variable exponent growth, which has been intensively studied in the last 20 years [13]. We mention as examples the following very recent studies [2, 5, 7, 11, 23, 25, 27, 28, 29]. One motivating factor is applications to image processing initially proposed by Chen, Levine and Rao [6], see also [18, 20]. In this context one is especially interested in the minimization problem with lower growth equal to 1. Here we study such functionals in an abstract and general setting.

To state our results we use the notation from Section 2 and refer to [16] for more background. Let φ be a weak Φ -function and $u_0 \in W^{1,\varphi}(\Omega)$ be the boundary value function. In Section 4 we study the functional $E_p : L^1(\Omega) \rightarrow [0, \infty]$ with fixed boundary values $u_0 \in W^{1,\varphi^r}(\Omega)$, where $r > 1$, for $p \in (1, r)$ defined by

$$(1.1) \quad E_p(u) := \begin{cases} \int_{\Omega} \varphi(x, |\nabla u|)^p dx & \text{when } u - u_0 \in L_0^{1,\varphi^p}(\Omega); \\ +\infty & \text{otherwise.} \end{cases}$$

We define the limit functional $E : L^1(\Omega) \rightarrow [0, \infty]$ by

$$E(u) := \inf \left\{ \liminf_{i \rightarrow \infty} \int_{\Omega} \varphi(x, |\nabla u_i|) dx : u_i - u_0 \in L_0^{1,\varphi}(\Omega) \text{ and } u_i \rightarrow u \text{ in } L^1(\Omega) \right\}.$$

In Section 5 we consider corresponding functionals with a fidelity term. We assume this time that $f \in L^2(\Omega)$. For $p > 1$ we define $F_p : L^2(\Omega) \rightarrow [0, \infty]$ by

$$(1.2) \quad F_p(u) := \begin{cases} \int_{\Omega} \varphi(x, |\nabla u|)^p + |u - f|^2 dx & \text{when } u \in L^{1,\varphi^p}(\Omega) \cap L^2(\Omega); \\ +\infty & \text{otherwise.} \end{cases}$$

Then we define the limit functional $F : L^2(\Omega) \rightarrow [0, \infty]$ by

$$F(u) := \inf \left\{ \liminf_{i \rightarrow \infty} \int_{\Omega} \varphi(x, |\nabla u_i|) + |u_i - f|^2 dx : u_i \in L^{1,\varphi}(\Omega) \cap L^2(\Omega) \text{ and } u_i \rightarrow u \text{ in } L^2(\Omega) \right\}$$

Our main result therefore is the following:

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $\varphi \in \Phi_w(\Omega)$ satisfy (A0) and (Dec).*

- (1) E_p Γ -converges to E in the topology of $L^1(\Omega)$ as $p \rightarrow 1^+$.
- (2) F_p Γ -converges to F in the topology of $L^2(\Omega)$ as $p \rightarrow 1^+$ if $C^\infty(\overline{\Omega})$ is dense in $L^{1,\varphi}(\Omega) \cap L^2(\Omega)$.

Due to the fidelity term in L^2 , it seems natural to consider the latter convergence in $L^2(\Omega)$, although other choices are also possible. From the proofs we see that assumption (A0) is only needed for the case $E(u) = \infty$ or $F(u) = \infty$; if we consider Γ -convergence on the space $X :=$

$\{u \in L^1(\Omega) : E(u) < \infty\}$ or $X := \{u \in L^2(\Omega) : F(u) < \infty\}$, then this assumption can be omitted.

The proof of Theorem 1.3(1) is presented in Section 4 along with results for minimizers of the energy E . The companion section, 5, covers Theorem 1.3(2) and minimizers of the energy F . The results for minimizers of the energies E and F generalize the corresponding ones in [22]. Finally, in Section 6, we give explicit formulas for F in the important special cases $\varphi(x, t) = t + a(x)t^2$ (the double phase case) and $\varphi(x, t) = t^{p(x)}$ (the variable exponent case) based on [18, 19, 20].

We would like to remark that the limit energies E and F turn to be particular cases of the more general modular we introduce and study in Section 3; this new definition is inspired by the work of Miranda [24] and covers classical bounded variation spaces and Sobolev spaces.

2. BACKGROUND

Throughout the paper we always consider a **bounded domain** $\Omega \subset \mathbb{R}^n$, i.e. a bounded, open and connected set. By $p' := \frac{p}{p-1}$ we denote the Hölder conjugate exponent of $p \in [1, \infty]$. The notation $f \lesssim g$ means that there exists a constant $c > 0$ such that $f \leq cg$. The notation $f \approx g$ means that $f \lesssim g \lesssim f$. By c we denote a generic constant whose value may change between appearances. A function f is *almost increasing* if there exists $L \geq 1$ such that $f(s) \leq Lf(t)$ for all $s \leq t$ (more precisely, L -almost increasing). *Almost decreasing* is defined analogously. By *increasing* we mean that this inequality holds for $L = 1$ (some call this non-decreasing), similarly for *decreasing*.

Definition 2.1. We say that $\varphi : \Omega \times [0, \infty) \rightarrow [0, \infty]$ is a *weak Φ -function*, and write $\varphi \in \Phi_w(\Omega)$, if the following conditions hold for almost every $x \in \Omega$:

- For every measurable function $f : \Omega \rightarrow \mathbb{R}$ the function $y \mapsto \varphi(y, f(y))$ is measurable.
- The function $t \mapsto \varphi(x, t)$ is non-decreasing.
- $\varphi(x, 0) = \lim_{t \rightarrow 0^+} \varphi(x, t) = 0$ and $\lim_{t \rightarrow \infty} \varphi(x, t) = \infty$.
- The function $t \mapsto \frac{\varphi(x, t)}{t}$ is L -almost increasing on $(0, \infty)$ with L independent of x .

If $\varphi \in \Phi_w(\Omega)$ is additionally convex and left-continuous with respect to t for almost every x , then φ is a *convex Φ -function*, and we write $\varphi \in \Phi_c(\Omega)$.

For $\varphi : \Omega \times [0, \infty) \rightarrow [0, \infty)$ and $p, q > 0$ we define some conditions:

(A0) There exists $\beta \in (0, 1]$ such that $\varphi(x, \beta) \leq 1 \leq \varphi(x, \frac{1}{\beta})$ for a.e. $x \in \Omega$.

(aInc)_p $t \mapsto \frac{\varphi(x, t)}{t^p}$ is L_p -almost increasing in $(0, \infty)$ for some $L_p \geq 1$ and a.e. $x \in \Omega$.

(aDec)_q $t \mapsto \frac{\varphi(x, t)}{t^q}$ is L_q -almost decreasing in $(0, \infty)$ for some $L_q \geq 1$ and a.e. $x \in \Omega$.

We say that (aInc) holds if (aInc)_p holds for some $p > 1$, and similarly for (aDec). We say that φ satisfies (Inc)_p if (aInc)_p holds with $L_p = 1$, similarly for (Dec)_q.

If $\varphi \in \Phi_w(\Omega)$, then $\varphi(\cdot, 1) \approx 1$ implies (A0), and if φ satisfies (aDec), then (A0) and $\varphi(\cdot, 1) \approx 1$ are equivalent. For instance, $\varphi(x, t) = t^p$ always satisfies (A0), since $\varphi(x, 1) \equiv 1$. Note that $\varphi \in \Phi_w(\Omega)$ includes the assumption that φ satisfies (aInc)₁. Finally, (aInc) and (aDec) measure the lower and upper growth rates.

We recall some basic notions of generalized Orlicz spaces [16].

Definition 2.2. Let $\varphi \in \Phi_w(\Omega)$ and $\varrho_\varphi(u) := \int_\Omega \varphi(x, |u|) dx$ for all measurable functions $u \in L^0(\Omega)$. The set

$$L^\varphi(\Omega) := \{u \in L^0(\Omega) : \varrho_\varphi(\lambda u) < \infty \text{ for some } \lambda > 0\}$$

is called a *generalized Orlicz space*. We define a (quasi-)norm on this space by

$$\|u\|_\varphi := \inf \left\{ \lambda > 0 : \varrho_\varphi\left(\frac{u}{\lambda}\right) \leq 1 \right\}.$$

We use the abbreviation $\|v\|_\varphi := \|\|v\|\|_\varphi$ for vector-valued functions.

We will often use the following lower semi-continuity result, for which we require the stronger, convex Φ -function assumption.

Lemma 2.3 (Theorem 2.2.8, [13]). *Let $\varphi \in \Phi_c(\Omega)$. If $u_i \rightharpoonup u$ in $L^\varphi(\Omega)$, then*

$$\int_{\Omega} \varphi(x, |u|) dx \leq \liminf_{i \rightarrow \infty} \int_{\Omega} \varphi(x, |u_i|) dx,$$

i.e. the modular ϱ_φ is weakly (sequentially) lower semicontinuous.

3. ABSTRACT BV-TYPE SPACES

We define a Sobolev-type space with function in L^1 and gradient in L^φ :

$$L^{1,\varphi}(\Omega) := \{u \in W^{1,1}(\Omega) : \|u\|_{L^{1,\varphi}(\Omega)} < \infty\} \quad \text{where} \quad \|u\|_{L^{1,\varphi}(\Omega)} := \|u\|_{L^1(\Omega)} + \|\nabla u\|_{L^\varphi(\Omega)}.$$

Note that $L^{1,\varphi}(\Omega) \subset W^{1,1}(\Omega)$, and $W^{1,\varphi}(\Omega) = L^{1,\varphi}(\Omega) \cap L^\varphi(\Omega)$. In the next definition and the rest of the article we use the $^\circ$ -symbol to indicate only the gradient part of a Sobolev-type norm and the $^-$ -symbol to indicate the lim inf-approximation process.

Definition 3.1. For $\varphi \in \Phi_w(\Omega)$ and $u \in L^1(\Omega)$ and define

$$\|u\|_{1,\varphi}^\circ := \inf \left\{ \liminf_{i \rightarrow \infty} \|\nabla u_i\|_\varphi : u_i \in L^{1,\varphi}(\Omega) \text{ and } u_i \rightarrow u \text{ in } L^1(\Omega) \right\},$$

$$\bar{\varrho}_{1,\varphi}^\circ(u) := \inf \left\{ \liminf_{i \rightarrow \infty} \varrho_\varphi(|\nabla u_i|) : u_i \in L^{1,\varphi}(\Omega) \text{ and } u_i \rightarrow u \text{ in } L^1(\Omega) \right\},$$

$$\bar{\varrho}_{1,\varphi}(u) := \varrho_1(u) + \bar{\varrho}_{1,\varphi}^\circ(u) \quad \text{and} \quad \bar{L}^{1,\varphi}(\Omega) := \{u \in L^1(\Omega) : \bar{\varrho}_{1,\varphi}(\lambda u) < \infty \text{ for some } \lambda > 0\}.$$

By a diagonal argument we find, for every $u \in L^1(\Omega)$, functions $u_i \in L^{1,\varphi}(\Omega)$ with $u_i \rightarrow u$ in $L^1(\Omega)$ and

$$\bar{\varrho}_{1,\varphi}^\circ(u) = \lim_{i \rightarrow \infty} \varrho_\varphi(|\nabla u_i|).$$

We call this a test sequence or a sequence that gives $\bar{\varrho}_{1,\varphi}^\circ$. The same idea works for the norm $\|u\|_{1,\varphi}^\circ$. Let us first motivate our definitions with two examples which show that we cover the spaces $W^{1,\varphi}(\Omega)$ and $BV(\Omega)$.

Example 3.2 (Bounded variation spaces). If $\varphi(x, t) \equiv t$, then $\bar{\varrho}_{1,\varphi}^\circ(u) = V(u, \Omega)$, the total variation of u , for every $u \in BV(\Omega)$. Indeed, by [14, Theorem 5.3] or [1, Theorem 3.9] there exist $u_i \in C^\infty(\Omega)$ such that $u_i \rightarrow u$ in $L^1(\Omega)$ and

$$V(u, \Omega) = \lim_{i \rightarrow \infty} \int_{\Omega} |\nabla u_i| dx.$$

Thus, by the definition of $\bar{\varrho}_{1,\varphi}^\circ(u)$, we obtain that

$$\bar{\varrho}_{1,\varphi}^\circ(u) \leq \lim_{i \rightarrow \infty} \int_{\Omega} |\nabla u_i| dx = V(u, \Omega).$$

On the other hand, let (u_i) give $\bar{\varrho}_{1,\varphi}^\circ(u)$. Since $u_i \rightarrow u$ in $L^1(\Omega)$, we obtain by the lower semicontinuity of the variational measure [14, Theorem 5.2] that

$$V(u, \Omega) \leq \liminf_{i \rightarrow \infty} V(u_i, \Omega) = \liminf_{i \rightarrow \infty} \int_{\Omega} |\nabla u_i| dx = \bar{\varrho}_{1,\varphi}^\circ(u).$$

Example 3.3 (Sobolev spaces). If $\varphi \in \Phi_c(\Omega)$, $C_0^\infty(\Omega)$ is dense in the dual space $(L^\varphi(\Omega))^*$ and $u \in L^{1,\varphi}(\Omega)$, then $\bar{\varrho}_{1,\varphi}^\circ(u) = \varrho_\varphi(|\nabla u|)$. First of all, since $u \in L^{1,\varphi}(\Omega)$, we can choose the constant sequence $u_i = u$ so that $\bar{\varrho}_{1,\varphi}^\circ(u) \leq \varrho_\varphi(|\nabla u|)$ by the definition.

For the opposite inequality we choose $u_i \in L^{1,\varphi}(\Omega)$ with $u_i \rightarrow u$ in $L^1(\Omega)$. Let $g \in C_0^\infty(\Omega; \mathbb{R}^n)$. By the definition of the weak derivative and the L^1 -convergence,

$$\int_{\Omega} \nabla u \cdot g \, dx = - \int_{\Omega} u \operatorname{div} g \, dx = - \int_{\Omega} u_i \operatorname{div} g \, dx = \int_{\Omega} \nabla u_i \cdot g \, dx.$$

Since $C_0^\infty(\Omega)$ is dense in $(L^\varphi(\Omega))^*$ we obtain the claim for any $g \in (L^\varphi(\Omega; \mathbb{R}^n))^*$ (with component-wise approximation), so $\nabla u_i \rightharpoonup \nabla u$ and also $|\nabla u_i| \rightharpoonup |\nabla u|$ in L^φ . Thus it follows from Lemma 2.3 that

$$\varrho_\varphi(|\nabla u|) \leq \liminf \varrho_\varphi(|\nabla u_i|).$$

Taking the infimum over such sequences (u_i) , we obtain the opposite inequality, $\varrho_\varphi(|\nabla u|) \leq \bar{\varrho}_{1,\varphi}^\circ(u)$.

Let us then consider $\bar{\varrho}_{1,\varphi}^\circ$. Note that $u \mapsto \bar{\varrho}_{1,\varphi}^\circ(u)$ is not a semimodular on $L^1(\Omega)$ in the sense of [13, Definition 2.1.1] since $\bar{\varrho}_{1,\varphi}^\circ(c) = 0$ for every constant $c \in \mathbb{R}$. Indeed, as we saw in the example, $\bar{\varrho}_{1,\varphi}^\circ$ is like $\varrho_\varphi(|\nabla u|)$, so it is natural to add the modular of the function to it, otherwise we can only expect it to generate a seminorm, not a norm.

Lemma 3.4. *Let $\varphi \in \Phi_w(\Omega)$. Then $\bar{\varrho}_{1,\varphi}$ satisfies the following properties:*

- (a) $\lambda \mapsto \bar{\varrho}_{1,\varphi}(\lambda u)$ is nondecreasing for every $u \in L^1(\Omega)$;
- (b) $\bar{\varrho}_{1,\varphi}(0) = 0$;
- (c) $\bar{\varrho}_{1,\varphi}(-u) = \bar{\varrho}_{1,\varphi}(u)$;
- (d) there exists a constant $\beta > 0$ such that

$$\bar{\varrho}_{1,\varphi}(\beta(\theta u + (1 - \theta)v)) \leq \theta \bar{\varrho}_{1,\varphi}(u) + (1 - \theta) \bar{\varrho}_{1,\varphi}(v)$$

for every $u, v \in L^1(\Omega)$ and for every $\theta \in (0, 1)$;

- (e1) $\bar{\varrho}_{1,\varphi}(\lambda u) = 0$ for all $\lambda > 0$ implies $u = 0$.

If additionally φ satisfies (aDec), then

- (e2) $\bar{\varrho}_{1,\varphi}(u) = 0$ if and only if $u = 0$.

Moreover, if φ is convex or left-continuous, then so is $\bar{\varrho}_{1,\varphi}$.

Proof. We consider only $\bar{\varrho}_{1,\varphi}^\circ$ for the first four properties, since these properties are known for ϱ_φ and thus follow for the sum $\varrho_\varphi + \bar{\varrho}_{1,\varphi}^\circ$.

(a) Let $0 \leq \lambda_1 \leq \lambda_2$. It is enough to show that $\bar{\varrho}_{1,\varphi}^\circ(\lambda_1 u) \leq \bar{\varrho}_{1,\varphi}^\circ(\lambda_2 u)$. Let (v_i) be a sequence that gives $\bar{\varrho}_{1,\varphi}^\circ(\lambda_2 u)$. Then $\frac{\lambda_1}{\lambda_2} v_i \rightarrow \lambda_1 u$, and thus

$$\bar{\varrho}_{1,\varphi}^\circ(\lambda_1 u) \leq \lim_{i \rightarrow \infty} \varrho_\varphi(v_i) = \bar{\varrho}_{1,\varphi}^\circ(\lambda_2 u).$$

(b) If $u = 0$ a.e., then it can be approximated by the constant sequence (u) , and hence $\bar{\varrho}_{1,\varphi}^\circ(u) = 0$.

(c) This follows directly from the definition of $\bar{\varrho}_{1,\varphi}^\circ$.

(d) Note that by [16, Corollary 2.2.2] there exists $\beta > 0$ such that $\varphi(\beta(\theta u + (1 - \theta)v)) \leq \theta \varphi(u) + (1 - \theta) \varphi(v)$. Let $u, v \in L^1(\Omega)$ and $\theta \in (0, 1)$. Choose (u_i) and (v_i) which give $\bar{\varrho}_{1,\varphi}^\circ(u)$

and $\bar{\varrho}_{1,\varphi}^\circ(v)$. Then $\theta u_i + (1 - \theta)v_i \rightarrow \theta u + (1 - \theta)v$ in $L^1(\Omega)$. Hence we obtain

$$\begin{aligned} \bar{\varrho}_{1,\varphi}^\circ(\beta(\theta u + (1 - \theta)v)) &\leq \liminf_{i \rightarrow \infty} \varrho_\varphi(\beta|\nabla(\theta u_i + (1 - \theta)v_i)|) \\ &\leq \liminf_{i \rightarrow \infty} \int_{\Omega} \varphi(x, \beta\theta|\nabla u_i| + \beta(1 - \theta)|\nabla v_i|) dx \\ &\leq \liminf_{i \rightarrow \infty} \int_{\Omega} \theta\varphi(x, |\nabla u_i|) + (1 - \theta)\varphi(x, |\nabla v_i|) dx \\ &= \theta \lim_{i \rightarrow \infty} \varrho_\varphi(|\nabla u_i|) + (1 - \theta) \lim_{i \rightarrow \infty} \varrho_\varphi(|\nabla v_i|) \\ &= \theta \bar{\varrho}_{1,\varphi}^\circ(u) + (1 - \theta) \bar{\varrho}_{1,\varphi}^\circ(v), \end{aligned}$$

where in the third inequality we used the quasiconvexity of φ .

(e1) If $\bar{\varrho}_{1,\varphi}(\lambda u) = 0$ for all $\lambda > 0$, then $\varrho_\varphi(\lambda u) = 0$ for all $\lambda > 0$. Since ϱ_φ is a semimodular, it follows that $u = 0$ a.e., see [16, Lemma 3.2.2].

(e2) Assume that (aDec) holds. If $\bar{\varrho}_{1,\varphi}(u) = 0$, then $\varrho_\varphi(u) = 0$. Thus $\varphi(x, |u(x)|) = 0$ a.e., and (aDec) implies that $u = 0$ a.e.

If φ is convex, then we may choose $\beta = 1$ in (d), and hence $\bar{\varrho}_{1,\varphi}$ is convex.

Assume that φ is left-continuous. If $\lambda \in (0, 1)$, then $\bar{\varrho}_{1,\varphi}^\circ(\lambda u) \leq \bar{\varrho}_{1,\varphi}^\circ(u)$ by property (a). We next consider the opposite inequality. Let (λ_i) be a sequence converging to 1 from below. For every i we choose $u_i \in L^{1,\varphi}(\Omega)$ such that $\varrho_\varphi(|\nabla u_i|) \leq \bar{\varrho}_{1,\varphi}^\circ(\lambda_i u) + \frac{1}{i}$ and $\|\lambda_i u - u_i\|_1 < \frac{1}{i}$. Further, $\lambda_i u \rightarrow u$ in $L^1(\Omega)$. Let $\varepsilon > 0$ and choose $i_0 > \frac{2}{\varepsilon}$ such that $\|u - \lambda_i u\|_1 < \frac{\varepsilon}{2}$ for all $i \geq i_0$. Then

$$\|u - u_i\|_1 \leq \|u - \lambda_i u\|_1 + \|\lambda_i u - u_i\|_1 < \varepsilon$$

for all $i \geq i_0$, and thus $u_i \rightarrow u$ in $L^1(\Omega)$. We obtain

$$\bar{\varrho}_{1,\varphi}^\circ(u) \leq \liminf_{i \rightarrow \infty} \varrho_\varphi(|\nabla u_i|) \leq \liminf_{i \rightarrow \infty} (\bar{\varrho}_{1,\varphi}^\circ(\lambda_i u) + \frac{1}{i}) = \liminf_{i \rightarrow \infty} \bar{\varrho}_{1,\varphi}^\circ(\lambda_i u).$$

Thus $\bar{\varrho}_{1,\varphi}^\circ(u) = \lim_{i \rightarrow \infty} \bar{\varrho}_{1,\varphi}^\circ(\lambda_i u)$. □

Remark 3.5. Note that our conditions differ what Musielak [26, Definition 1.1, p. 1] calls a pseudomodular. He requires (b), (c) and

$$\varrho(\theta u + (1 - \theta)v) \leq \varrho(u) + \varrho(v)$$

for every u, v and for every $\theta \in (0, 1)$. However, our condition (d) is more useful when dealing with the Luxemburg norm.

We prove the lower semi-continuity of $\bar{\varrho}_{1,\varphi}^\circ$; note that we can handle Φ_w , since we assume strong rather than weak convergence, in contrast to Lemma 2.3

Lemma 3.6 (Lower semicontinuity of the modular). *Let $\varphi \in \Phi_w(\Omega)$, $u_i \in L^{1,\varphi}(\Omega)$ for $i = 1, 2, \dots$, $u \in L^1(\Omega)$, and $u_i \rightarrow u$ in $L^1(\Omega)$. Then*

$$\bar{\varrho}_{1,\varphi}^\circ(u) \leq \liminf_{i \rightarrow \infty} \bar{\varrho}_{1,\varphi}^\circ(u_i).$$

Moreover, if the limit inferior is finite, then $u \in \bar{L}^{1,\varphi}(\Omega)$.

Proof. By the definition of $\bar{\varrho}_{1,\varphi}^\circ(u_i)$ we can find $u'_i \in L^{1,\varphi}(\Omega)$ with

$$\varrho_\varphi(|\nabla u'_i|) \leq \bar{\varrho}_{1,\varphi}^\circ(u_i) + \frac{1}{i} \quad \text{and} \quad \|u_i - u'_i\|_1 \leq \frac{1}{i}.$$

It follows that $u'_i \rightarrow u$ in $L^1(\Omega)$. By the definition of $\bar{\varrho}_{1,\varphi}^\circ(u)$ we conclude that

$$\bar{\varrho}_{1,\varphi}^\circ(u) \leq \liminf_{i \rightarrow \infty} \varrho_\varphi(|\nabla u'_i|) \leq \liminf_{i \rightarrow \infty} (\bar{\varrho}_{1,\varphi}^\circ(u_i) + \frac{1}{i}) = \liminf_{i \rightarrow \infty} \bar{\varrho}_{1,\varphi}^\circ(u_i). \quad \square$$

We define the Luxemburg (quasi-)norm in the usually way:

$$\|u\|_\varrho := \inf \left\{ \lambda > 0 : \varrho\left(\frac{u}{\lambda}\right) \leq 1 \right\}.$$

If φ is a convex semimodular then $u \mapsto \|u\|_\varrho$ is a norm by [26, Theorem 1.5, p. 2]. As in [17, Lemma 3.2.2], one can check that (a)–(d) from Lemma 3.4 imply that $u \mapsto \|u\|_\varrho$ is a quasi-seminorm.

We next compare $\|\cdot\|_{\bar{\varrho}_{1,\varphi}^\circ}$ induced by the modular via the preceding formula with $\|\cdot\|_{1,\varphi}^\circ$ from Definition 3.1.

Proposition 3.7. *If $\varphi \in \Phi_w(\Omega)$, then $\|\cdot\|_{1,\varphi}^\circ = \|\cdot\|_{\bar{\varrho}_{1,\varphi}^\circ}$.*

Proof. Let us first show that $\|\cdot\|_{1,\varphi}^\circ \geq \|\cdot\|_{\bar{\varrho}_{1,\varphi}^\circ}$. If the left-hand side equals ∞ there is nothing to prove, and thus we may assume without loss of generality that $\|u\|_{1,\varphi}^\circ < \infty$. Choose $u_i \in L^{1,\varphi}(\Omega)$ with $u_i \rightarrow u$ in $L^1(\Omega)$ and $\|\nabla u_i\|_\varphi \rightarrow \|u\|_{1,\varphi}^\circ$. Define $\lambda_i := \|\nabla u_i\|_\varphi + \varepsilon$ and $\lambda := \|u\|_{1,\varphi}^\circ + \varepsilon$ for $\varepsilon > 0$. Then $\|\nabla \frac{u_i}{\lambda_i}\|_\varphi < 1$ and so $\varrho_\varphi(|\nabla \frac{u_i}{\lambda_i}|) \leq 1$. Furthermore,

$$\left\| \frac{u}{\lambda} - \frac{u_i}{\lambda_i} \right\|_1 = \left\| \frac{u - u_i}{\lambda_i} + \left(\frac{1}{\lambda} - \frac{1}{\lambda_i} \right) u \right\|_1 \leq \frac{1}{\lambda_i} \|u - u_i\|_1 + \left(\frac{1}{\lambda} - \frac{1}{\lambda_i} \right) \|u\|_1 \rightarrow 0$$

so that $\frac{u_i}{\lambda_i} \rightarrow \frac{u}{\lambda}$ in $L^1(\Omega)$. Since $(\frac{u_i}{\lambda_i})$ is a valid test sequence for $\bar{\varrho}_{1,\varphi}^\circ$, we conclude that

$$\bar{\varrho}_{1,\varphi}^\circ\left(\frac{u}{\lambda}\right) \leq \liminf_{i \rightarrow \infty} \varrho_\varphi\left(|\nabla \frac{u_i}{\lambda_i}|\right) \leq 1.$$

Hence $\|u\|_{\bar{\varrho}_{1,\varphi}^\circ} \leq \lambda = \|u\|_{1,\varphi}^\circ + \varepsilon$; the inequality follows as $\varepsilon \rightarrow 0^+$.

To prove $\|\cdot\|_{1,\varphi}^\circ \leq \|\cdot\|_{\bar{\varrho}_{1,\varphi}^\circ}$, we may assume that $\|u\|_{\bar{\varrho}_{1,\varphi}^\circ} < \infty$. Since $\bar{\varrho}_{1,\varphi}^\circ\left(\frac{u}{\|u\|_{\bar{\varrho}_{1,\varphi}^\circ} + \varepsilon}\right) < 1$, we can choose $u_i \in L^{1,\varphi}(\Omega)$ with $\varrho_\varphi(|\nabla u_i|) \leq 1$ and $u_i \rightarrow \frac{u}{\|u\|_{\bar{\varrho}_{1,\varphi}^\circ} + \varepsilon}$ in L^1 . Thus

$$\left\| \frac{u}{\|u\|_{\bar{\varrho}_{1,\varphi}^\circ} + \varepsilon} \right\|_{1,\varphi}^\circ \leq \liminf_{i \rightarrow \infty} \|\nabla u_i\|_\varphi \leq 1$$

and so $\|u\|_{1,\varphi}^\circ \leq \|u\|_{\bar{\varrho}_{1,\varphi}^\circ} + \varepsilon$. The claim again follows as $\varepsilon \rightarrow 0^+$. $\square$

4. FUNCTIONALS WITH FIXED BOUNDARY VALUES

Let $\varphi \in \Phi_w(\Omega)$. We say that $u \in L^{1,\varphi}(\Omega)$ belongs to $L_0^{1,\varphi}(\Omega)$ if there exists a $C_0^\infty(\Omega)$ -sequence (ξ_i) such that $\xi_i \rightarrow u$ in $L^{1,\varphi}(\Omega)$. Note that $L_0^{1,\varphi}(\Omega) \subset W_0^{1,1}(\Omega)$.

Lemma 4.1. *Let $\varphi \in \Phi_w(\Omega)$ satisfy (aInc) and (aDec). Then $L^{1,\varphi}(\Omega)$ and $L_0^{1,\varphi}(\Omega)$ are reflexive.*

Proof. Let $(u_i) \subset L^{1,\varphi}(\Omega)$ be a bounded sequence. Since $L^{1,\varphi}(\Omega) \subset \text{BV}(\Omega)$, a subsequence, denoted again (u_i) , converges in $L^1(\Omega)$ to some function $u \in \text{BV}(\Omega)$. On the other hand, $(\partial_{x_k} u_i)$ is a bounded sequence in $L^\varphi(\Omega)$, which is a reflexive space [16, Theorem 3.6.6]. Hence we can find $v_1, \dots, v_n \in L^\varphi(\Omega)$ such that $\partial_{x_k} u_i \rightharpoonup v_k$ in $L^\varphi(\Omega)$, up to a subsequence. So we need only prove that $\nabla u = (v_1, \dots, v_n)$. Let $k \in \{1, \dots, n\}$ and let $h \in C_0^\infty(\Omega)$ be a test function. We obtain by the previous convergences that

$$\int_\Omega u \partial_{x_k} h \, dx = \lim_{i \rightarrow \infty} \int_\Omega u_i \partial_{x_k} h \, dx = - \lim_{i \rightarrow \infty} \int_\Omega h \partial_{x_k} u_i \, dx = - \int_\Omega h v_k \, dx,$$

and thus $v_k = \partial_{x_k} u$.

The usual diagonal argument shows that $L_0^{1,\varphi}(\Omega)$ is a closed subspace of $L^{1,\varphi}(\Omega)$. Hence it is also reflexive. $\square$

We will study φ^p -energies when $p \geq 1$ is a constant. For that purpose we note some properties of φ^p when $\varphi \in \Phi_w(\Omega)$.

Lemma 4.2. *Let $p \geq 1$ and $\varphi \in \Phi_w(\Omega)$. Then $\varphi^p \in \Phi_w(\Omega)$ satisfies (aInc)_p and, moreover,*

- (1) if φ satisfies **(aDec)_q**, then φ^p satisfies **(aDec)_{qp}**;
- (2) if φ is left-continuous, then so is φ^p ;
- (3) if φ satisfies **(A0)**, then so does φ^p .

Proof. Since $t \mapsto t^p$ is continuous and increasing in $[0, \infty)$, the first three requirements of Definition 2.1 hold for φ^p . Since φ satisfies **(aInc)₁** and $t \mapsto t^p$ is increasing, we obtain by $\frac{\varphi^p(x, t)}{t^p} = \left(\frac{\varphi(x, t)}{t}\right)^p$ that φ^p satisfies **(aInc)_p**.

- (1) Assume that φ satisfies **(aDec)_q**. Then by $\frac{\varphi^p(x, t)}{t^{pq}} = \left(\frac{\varphi(x, t)}{t^q}\right)^p$ we see that φ^p satisfies **(aDec)_{pq}**.
- (2) This claim follows since $t \mapsto t^p$ is continuous.
- (3) It follows from **(A0)** that $\varphi(x, \beta)^p \leq 1 \leq \varphi(x, \frac{1}{\beta})^p$, which is **(A0)** of φ^p . $\square$

4.1. Γ -convergence.

Assume that $u_0 \in L^{1, \varphi^r}(\Omega) \cap L^1(\Omega)$ for some $r > 1$. For $p \in (1, r)$ we consider $E_p : L^1(\Omega) \rightarrow [0, \infty]$ and $E : L^1(\Omega) \rightarrow [0, \infty]$ from the introduction, see (1.1). Note that E is a variant of $\bar{g}_{1, \varphi}^\circ$ where we have fixed the boundary values.

As we mentioned in the introduction, we want to show that E_{p_i} Γ -converges to E in the topology of $L^1(\Omega)$ as $p_i \rightarrow 1^+$. We do so by first considering the Γ -liminf inequality, and then finding the recovery sequence in Lemma 4.4. This establishes the two properties of Γ -convergence and completes the proof of Theorem 1.3(1).

Lemma 4.3. *Let $\varphi \in \Phi_w(\Omega)$. Assume that $u_0 \in L^{1, \varphi^r}(\Omega)$ for some $r > 1$. Let $u \in L^1(\Omega)$, $u_i \rightarrow u$ in $L^1(\Omega)$ and $p_i \rightarrow 1^+$. Then*

$$E(u) \leq \liminf_{i \rightarrow \infty} E_{p_i}(u_i).$$

Proof. If $K := \liminf_{i \rightarrow \infty} E_{p_i}(u_i) = \infty$, then there is nothing to prove, so we assume that $K < \infty$. We restrict our attention to a subsequence with $\lim_{i \rightarrow \infty} E_{p_i}(u_i) = K$ and $u_i - u_0 \in L_0^{1, \varphi^{p_i}}(\Omega)$. Since $u_i \rightarrow u$ in $L^1(\Omega)$, we obtain by the definition of $E(u)$ that

$$E(u) \leq \liminf_{i \rightarrow \infty} \int_{\Omega} \varphi(x, |\nabla u_i|) dx.$$

From Young's inequality we conclude that $ab \leq a^p + (p-1)\left(\frac{b}{p}\right)^{p'}$. Using this with $a = \varphi(x, |\nabla u_i|)$, $b = 1$ and $p = p_i$, we continue the previous estimate by

$$E(u) \leq \liminf_{i \rightarrow \infty} \left(\int_{\Omega} \varphi^{p_i}(x, |\nabla u_i|) dx + |\Omega|(p_i - 1)p_i^{-p'_i} \right) = \liminf_{i \rightarrow \infty} \int_{\Omega} \varphi^{p_i}(x, |\nabla u_i|) dx,$$

where the equality holds since $(p_i - 1)p_i^{-p'_i} \rightarrow 0 \cdot \frac{1}{e} = 0$ as $p_i \rightarrow 1^+$. $\square$

We now turn our attention to the Γ -limsup property. For this part we need to assume **(Dec)**. Recall that **(aDec)** implies **(Dec)** if φ is convex [17, Lemma 2.2.6].

Lemma 4.4. *Assume that $\varphi \in \Phi_w(\Omega)$ satisfies **(A0)** and **(Dec)**. Let $u \in L^1(\Omega)$ and $u_0 \in L^{1, \varphi^r}(\Omega)$ for some $r > 1$. For every $p_i \rightarrow 1^+$ there exist $u_i \in L^{1, \varphi}(\Omega)$ such that $u_i - u_0 \in L_0^{1, \varphi}(\Omega)$, $u_i \rightarrow u$ in $L^1(\Omega)$ and*

$$E(u) \geq \limsup_{i \rightarrow \infty} E_{p_i}(u_i).$$

Proof. If $E(u) = \infty$, then any approximating sequence in $u_0 + C_0^\infty(\Omega)$ works [16, Corollary 3.7.10], so we assume that $E(u) < \infty$. By the definition of $E(u)$, there exists a sequence (v_k) from $u_0 + L_0^{1, \varphi}(\Omega)$ such that $v_k \rightarrow u$ in $L^1(\Omega)$ and

$$E(u) = \lim_{k \rightarrow \infty} \int_{\Omega} \varphi(x, |\nabla v_k|) dx.$$

Since $v_k - u_0 \in L_0^{1,\varphi}(\Omega)$, there exists a sequence (ζ_j^k) from $C_0^\infty(\Omega)$ with $u_0 + \zeta_j^k \rightarrow v_k$ in $L^{1,\varphi}(\Omega)$. Since φ satisfies (Dec) this yields by Lemma 3.1.6 of [16] that

$$\int_{\Omega} \varphi(x, |\nabla v_k|) dx = \lim_{j \rightarrow \infty} \int_{\Omega} \varphi(x, |\nabla(u_0 + \zeta_j^k)|) dx.$$

Thus by the usual diagonal argument, we can choose $\xi_j := \zeta_{k_j}^j$ such that $u_0 + \xi_j \rightarrow u$ in $L^1(\Omega)$ and

$$E(u) = \lim_{j \rightarrow \infty} \int_{\Omega} \varphi(x, |\nabla(u_0 + \xi_j)|) dx.$$

By Hölder's inequality with exponent $q > 1$ we obtain

$$\int_{\Omega} \varphi(x, |\nabla(u_0 + \xi_j)|)^{\frac{1}{q} + \frac{r}{q'}} dx \leq \left(\int_{\Omega} \varphi(x, |\nabla(u_0 + \xi_j)|) dx \right)^{\frac{1}{q}} \left(\int_{\Omega} \varphi(x, |\nabla(u_0 + \xi_j)|)^r dx \right)^{\frac{1}{q'}}.$$

The second integral is finite since $u_0 + \xi_j \in W^{\varphi^r}(\Omega)$. Thus we may choose $\overline{q}_j \in (1, 1 + \frac{1}{j})$ close enough to 1 that the second factor on the right hand side is at most $1 + \frac{1}{j}$ for all $q \in [1, \overline{q}_j]$. Since $p_i \rightarrow 1^+$ there exists $i_j \geq j$ such that $q_i := \frac{r-1}{r-p_i} \leq \overline{q}_j$ for all $i \geq i_j$. Note that $\frac{1}{q_i} + \frac{r}{q_i'} = p_i$. We now choose $u_i := u_0 + \xi_j$ for all $i \in [\max\{i_1, \dots, i_j\}, i_{j+1})$. Then

$$\int_{\Omega} \varphi(x, |\nabla u_i|)^{p_i} dx \leq \left(1 + \frac{1}{j(i)} \right) \left(\int_{\Omega} \varphi(x, |\nabla u_i|) dx \right)^{\frac{1}{q_i}},$$

where $j(i) \rightarrow \infty$ as $i \rightarrow \infty$ since $\overline{q}_j > 1$ so only finitely many values of i can have $q_i > \overline{q}_j$.

Next we combine all the above estimates, and obtain

$$\begin{aligned} E(u) &= \lim_{i \rightarrow \infty} \int_{\Omega} \varphi(x, |\nabla u_i|) dx \geq \limsup_{i \rightarrow \infty} \left(1 + \frac{1}{j(i)} \right)^{-q_i} \left(\int_{\Omega} \varphi(x, |\nabla u_i|)^{p_i} dx \right)^{q_i} \\ &= \limsup_{i \rightarrow \infty} \int_{\Omega} \varphi(x, |\nabla u_i|)^{p_i} dx = \limsup_{i \rightarrow \infty} E_{p_i}(u_i), \end{aligned}$$

where the last equality follows since $u_i - u_0 = \xi_j \in L_0^{1,\varphi}(\Omega)$. □

4.2. A sequence of minimizers.

We continue by establishing some results about sequences of minimizers of the functionals E_p from (1.1). These results are in the spirit of Juutinen [22].

Lemma 4.5. *Let $\varphi \in \Phi_c(\Omega)$ satisfy (A0) and (aDec), $p > 1$ and $u_0 \in L^{1,\varphi^p}(\Omega)$. Then there exists a unique function $u_p \in L^{1,\varphi^p}(\Omega)$ which satisfies $u_p - u_0 \in L_0^{1,\varphi^p}(\Omega)$ and*

$$E_p(u_p) = \inf_{u \in L^1(\Omega)} E_p(u).$$

Proof. Let (u_j) be a minimizing sequence of E_p . Since

$$\inf_{u \in L^1(\Omega)} E_p(u) \leq E_p(u_0) = \int_{\Omega} \varphi(x, |\nabla u_0|)^p dx < \infty,$$

we may assume that $(|\nabla u_j|)$ is bounded in $L^{\varphi^p}(\Omega)$. By the Poincaré inequality in $W_0^{1,1}(\Omega)$, we obtain that $(u_j - u_0)$ is bounded in $L^1(\Omega)$:

$$\|u_j - u_0\|_1 \lesssim \|\nabla(u_j - u_0)\|_1 \lesssim \|\nabla u_j\|_{\varphi^p} + \|\nabla u_0\|_{\varphi^p},$$

where the embedding in the last inequality holds since φ^p satisfies (A0) by Lemma 4.2. Since φ^p satisfies (aInc) and (aDec), $L_0^{1,\varphi^p}(\Omega)$ is reflexive, by Lemma 4.1. Thus there exists a subsequence $(u_{j_k} - u_0)$ and $u \in L_0^{1,\varphi}(\Omega)$ such that $\nabla(u_{j_k} - u_0) \rightharpoonup \nabla u$ in $L^{\varphi^p}(\Omega; \mathbb{R}^n)$. Set $u_p := u + u_0$. Since $\varphi \in \Phi_c(\Omega)$, Lemma 2.3 implies that

$$E_p(u_p) \leq \liminf_{k \rightarrow \infty} E_p(u_{j_k}).$$

Thus u_p is a minimizer of E_p .

Since φ is convex and $t \mapsto t^p$ strictly convex, we obtain that $t \mapsto \varphi(x, t)^p$ is strictly convex. The usual argument yields uniqueness, namely, if u and v are distinct minimizers, then we obtain a contradiction from $E_p(\frac{u+v}{2}) < \frac{1}{2}(E_p(u) + E_p(v))$. $\square$

Take a sequence (p_j) such that $p_j \rightarrow 1^+$ and a sequence of minimizers (u_j) , where u_j is the minimizer of E_{p_j} . Let us also assume that the φ^{p_j} -modular of u_0 is finite for every j . Then the limit function exists in $\bar{L}^{1,\varphi}(\Omega)$.

Lemma 4.6. *Let $p_j \rightarrow 1^+$ and $\varphi \in \Phi_w(\Omega)$. Assume that $\varrho_{\varphi^{p_j}}(u_0) \leq K$ for every j . Let (u_j) be a sequence of minimizers of E_{p_j} with boundary value u_0 . Then there exist $u \in \bar{L}^{1,\varphi}(\Omega)$ and a subsequence (u_{j_i}) converging to u in $L^1(\Omega)$ with*

$$E(u) \leq \liminf_{i \rightarrow \infty} E_{p_{j_i}}(u_{j_i}).$$

Proof. Since u_j is a minimizer, we see as in the previous lemma that $E_{p_j}(u_j) \leq K$. By $L^{\varphi^{p_1}}(\Omega) \hookrightarrow L^\varphi(\Omega)$ and $\|u\|_\varphi \leq \varrho_\varphi(u) + 1$ we conclude that $\|\nabla u_j\|_\varphi$ is bounded. Since $u_j - u_0 \in W_0^{1,1}(\Omega)$, we obtain as in the previous lemma by the Poincaré inequality that $\|u_j - u_0\|_1 \lesssim 1 + \|\nabla u_0\|_{\varphi^{p_1}}$. Thus $(u_j - u_0)$ is bounded in $W_0^{1,1}(\Omega)$. Since $W_0^{1,1}(\Omega) \hookrightarrow L^1(\Omega)$, there exists a subsequence (u_{j_i}) converging to u in $L^1(\Omega)$. The claim follows from the first property of Γ -convergence by Theorem 1.3(1). Finally, Lemma 3.6 yields that $u \in \bar{L}^{1,\varphi}(\Omega)$. $\square$

Proposition 4.7. *Let $\varphi \in \Phi_w(\mathbb{R}^n)$ and $\varrho_{\varphi^{p_j}}(u_0) \leq K$ for every j . The function $u \in \bar{L}^{1,\varphi}(\Omega)$ from Lemma 4.6 minimizes the E -energy, i.e.*

$$E(u) = \inf_{v \in L^1(\Omega)} E(v).$$

Proof. Assume by contradiction that there exists $v \in L^1(\Omega)$ such that $E(v) < E(u)$. Let $p_i \rightarrow 1^+$. By the second property of Γ -convergence from Theorem 1.3(1), there exists a sequence (v_i) such that $v_i - u_0 \in L_0^{1,\varphi}(\Omega)$, $v_i \rightarrow v$ in $L^1(\Omega)$ and

$$E(v) \geq \limsup_{i \rightarrow \infty} E_{p_i}(v_i).$$

By the assumption and Lemma 4.6, we have

$$\limsup_{i \rightarrow \infty} E_{p_i}(v_i) \leq E(v) < E(u) \leq \liminf_{j \rightarrow \infty} E_{p_{j_i}}(u_{j_i}).$$

Let $\varepsilon := \liminf_{j \rightarrow \infty} E_{p_{j_i}}(u_{j_i}) - \limsup_{i \rightarrow \infty} E_{p_i}(v_i)$, and choose M so large that

$$E_{p_i}(v_i) < \limsup_{i \rightarrow \infty} E_{p_i}(v_i) + \frac{\varepsilon}{2} \quad \text{and} \quad E_{p_{i_j}}(v_{i_j}) > \liminf_{j \rightarrow \infty} E_{p_{i_j}}(u_{i_j}) - \frac{\varepsilon}{2}$$

as $i, j \geq M$. Thus for all $i, j \geq M$ we have

$$E_{p_i}(v_i) < E_{p_{i_j}}(u_{i_j}).$$

Then we just choose $j \geq M$ with $i_j \geq M$ and obtain

$$E_{p_{i_j}}(v_{i_j}) < E_{p_{i_j}}(u_{i_j}).$$

But this contradicts that u_{i_j} minimizes the $E_{p_{i_j}}$ -energy with the boundary values u_0 , so the assumption $E(v) < E(u)$ was wrong and the claim is proved. $\square$

5. FUNCTIONALS WITH THE FIDELITY TERM

Assume then that $f \in L^2(\Omega)$. For $p > 1$ we defined $F_p : L^2(\Omega) \rightarrow [0, \infty]$ and $F : L^2(\Omega) \rightarrow [0, \infty]$ in the introduction, see (1.2). Note that F is another a variant of $\bar{\varrho}_{1,\varphi}^\circ$, this time including a “fidelity term” in L^2 . Since this implies that $u \in L^2(\Omega)$, it seems natural to consider the convergence in $L^2(\Omega)$, although convergence in $L^1(\Omega)$ is also a possibility.

5.1. Γ -convergence.

We want to show that F_{p_i} Γ -converges to F in the topology of $L^2(\Omega)$ as $p_i \rightarrow 1^+$, i.e. to prove Theorem 1.3(2). We do so by first considering the Γ -liminf inequality, and then finding the recovery sequence in Lemma 5.2.

Lemma 5.1. *Let $u \in L^2(\Omega)$ and assume that (u_i) is a sequence of L^2 -functions converging to u in $L^2(\Omega)$ and that $p_i \rightarrow 1^+$. Then*

$$F(u) \leq \liminf_{i \rightarrow \infty} F_{p_i}(u_i).$$

Proof. If $K := \liminf_{i \rightarrow \infty} F_{p_i}(u_i) = \infty$, then there is nothing to prove, so we assume that $K < \infty$. We restrict our attention to a subsequence with $\lim_{i \rightarrow \infty} F_{p_i}(u_i) = K$ and $u_i \in L^{1,\varphi^{p_i}}(\Omega) \cap L^2(\Omega)$. Since $u_i \rightarrow u$ in $L^2(\Omega)$, we obtain by the definition of $F(u)$ that

$$F(u) \leq \liminf_{i \rightarrow \infty} \left(\int_{\Omega} \varphi(x, |\nabla u_i|) + |u_i - f|^2 dx \right).$$

From Young’s inequality we conclude that $ab \leq a^p + (p-1)(\frac{b}{p})^{p'}$. Using this with $a = \varphi(x, |\nabla u_i|)$, $b = 1$ and $p = p_i$, we continue the previous estimate by

$$\begin{aligned} F(u) &\leq \liminf_{i \rightarrow \infty} \left(\int_{\Omega} \varphi^{p_i}(x, |\nabla u_i|) dx + |\Omega|(p_i - 1)p_i^{-p'_i} + \int_{\Omega} |u_i - f|^2 dx \right) \\ &= \liminf_{i \rightarrow \infty} \left(\int_{\Omega} \varphi^{p_i}(x, |\nabla u_i|) dx + \int_{\Omega} |u_i - f|^2 dx \right), \end{aligned}$$

where the equality holds since $(p_i - 1)p_i^{-p'_i} \rightarrow 0 \cdot \frac{1}{e} = 0$ as $p_i \rightarrow 1^+$. $\square$

We now turn our attention to the Γ -limsup property. Note that in contrast to Lemma 4.4 we now have an assumption regarding density of smooth functions. The difference comes from the fact that $L_0^{1,\varphi}$ is defined as the completion of C_0^∞ -functions, so smooth functions are automatically dense in that setting. We refer to [21, Corollary 4.4] for a sufficient condition for the density of $C^\infty(\bar{\Omega})$ in $W^{1,\varphi}(\Omega)$ where Ω is an (ε, ∞) -domain.

Lemma 5.2. *Assume that $\varphi \in \Phi_w(\Omega)$ satisfies (A0) and (Dec), and that $C^\infty(\bar{\Omega})$ is dense in $L^{1,\varphi}(\Omega) \cap L^2(\Omega)$. For every $u \in L^2(\Omega)$ and $p_i \rightarrow 1^+$ there exist $u_i \in L^{1,\varphi}(\Omega) \cap L^2(\Omega)$ such that $u_i \rightarrow u$ in $L^2(\Omega)$ and*

$$F(u) \geq \limsup_{i \rightarrow \infty} F_{p_i}(u_i).$$

Proof. If $F(u) = \infty$, any approximating sequence in $C_0^\infty(\Omega)$ works [16, Corollary 3.7.10], so we assume that $F(u) < \infty$. Thus there exists a sequence (v_i) from $L^{1,\varphi}(\Omega)$ such that $v_i \rightarrow u$ in $L^2(\Omega)$ and

$$F(u) = \lim_{i \rightarrow \infty} \int_{\Omega} \varphi(x, |\nabla v_i|) + |v_i - f|^2 dx.$$

Using the density assumption, we can find $\xi_j^i \in C^\infty(\bar{\Omega})$ with $\xi_j^i \rightarrow v_i$ in $L^{1,\varphi}(\Omega) \cap L^2(\Omega)$. Then by a diagonal argument and [16, Lemma 3.1.6] we find $u_i := \xi_{j_i}^i \in C^\infty(\bar{\Omega})$ with $u_i \rightarrow u$ in $L^2(\Omega)$ and

$$F(u) = \lim_{i \rightarrow \infty} \int_{\Omega} \varphi(x, |\nabla u_i|) + |u_i - f|^2 dx.$$

The proof is concluded in the same way as Lemma 4.4. $\square$

5.2. A sequence of minimizers.

Lemma 5.3. *Let $\varphi \in \Phi_c(\Omega)$ satisfy (aDec), $p > 1$ and $f \in L^2(\Omega)$. Then there exists a unique function $u_p \in L^{1,\varphi^p}(\Omega) \cap L^2(\Omega)$ which satisfies*

$$F_p(u_p) = \inf_{u \in L^2(\Omega)} F_p(u).$$

Proof. Let (u_j) be a minimizing sequence of F_p . Since

$$\inf_{u \in L^{1,\varphi^p}(\Omega) \cap L^2(\Omega)} F_p(u) \leq F_p(0) = \int_{\Omega} \varphi(x, 0)^p + |0 - f|^2 dx < \infty,$$

we may assume that $(|\nabla u_j|)$ is bounded in $L^{\varphi^p}(\Omega)$ and (u_j) is bounded in $L^2(\Omega)$. Note that φ^p satisfies (aInc) and (aDec) by Lemma 4.2. Thus $L^2(\Omega)$ and $L^{1,\varphi}(\Omega)$ are reflexive by Lemma 4.1, and hence there exists a subsequence (u_{j_k}) and $u \in L^{1,\varphi^p}(\Omega) \cap L^2(\Omega)$ such that $\nabla u_{j_k} \rightharpoonup \nabla u$ in $L^{\varphi^p}(\Omega; \mathbb{R}^n)$ and $u_{j_k} \rightharpoonup u$ in $L^2(\Omega)$. Thus by weak lower semi-continuity (Lemma 2.3) we have

$$F_p(u) \leq \liminf_{k \rightarrow \infty} F_p(u_{j_k}).$$

Thus u is a minimizer. Uniqueness is proved as in Lemma 4.5. $\square$

Let us take a sequence (p_j) such that $p_j \rightarrow 1^+$ and a sequence of minimizers (u_j) , where u_j is the minimizer of F_{p_j} . Then the limit function exists in $L^{1,\varphi}(\Omega) \cap L^2(\Omega)$.

Lemma 5.4. *Let $\varphi \in \Phi_c(\Omega)$. Assume that $f \in L^2(\Omega)$. Let $p_j \rightarrow 1^+$ and let (u_j) be a sequence of minimizers of F_{p_j} . Then there exist $u \in \bar{L}^{1,\varphi}(\Omega) \cap L^2(\Omega)$ and a subsequence (u_{j_i}) converging to u in $L^2(\Omega)$ with*

$$F(u) \leq \liminf_{i \rightarrow \infty} F_{p_{j_i}}(u_{j_i}).$$

Proof. We take a subsequence that gives $\liminf_{i \rightarrow \infty} F_{p_{j_i}}(u_{j_i})$, and denote this subsequence by (u_j) . Since u_j is a minimizer of F_{p_j} ,

$$F_{p_j}(u_j) \leq \int_{\Omega} \varphi(x, 0)^{p_j} + |0 - f|^2 dx = \int_{\Omega} |f|^2 dx < \infty.$$

Thus by $L^{\varphi^p}(\Omega) \hookrightarrow L^{\varphi}(\Omega)$ we have that $\|\nabla u_j\|_{\varphi} + \|u_j - f\|_2$ is bounded, and by $f \in L^2(\Omega)$ we obtain that $\|u_j\|_2$ is bounded. Since $L^2(\Omega)$ is reflexive, there exists a subsequence, denoted again by (u_j) , and $u \in L^2(\Omega)$ such that $u_j \rightharpoonup u$ in $L^2(\Omega)$.

By the Banach–Saks Theorem, $\frac{1}{k} \sum_{j=1}^k u_j \rightarrow u$ in $L^2(\Omega)$. We then use the definition of F and the convexity of φ and $t \mapsto t^2$ and obtain that

$$\begin{aligned} F(u) &\leq \liminf_{k \rightarrow \infty} F\left(\frac{1}{k} \sum_{j=1}^k u_j\right) = \liminf_{k \rightarrow \infty} \int_{\Omega} \varphi\left(x, \frac{1}{k} \sum_{j=1}^k |\nabla u_j|\right) + \left|\frac{1}{k} \sum_{j=1}^k u_j - f\right|^2 dx \\ &\leq \liminf_{k \rightarrow \infty} \frac{1}{k} \sum_{j=1}^k \int_{\Omega} \varphi(x, |\nabla u_j|) + |u_j - f|^2 dx \end{aligned}$$

Arguing as in Lemma 5.1, we continue:

$$F(u) \leq \liminf_{k \rightarrow \infty} \frac{1}{k} \sum_{j=1}^k \int_{\Omega} \varphi(x, |\nabla u_j|)^{p_j} + |u_j - f|^2 + (p_j - 1)p_j^{-p'_j} dx = \liminf_{k \rightarrow \infty} \frac{1}{k} \sum_{j=1}^k F_{p_j}(u_j).$$

It is well known that if a sequence (z_j) converges to z , so does $(\frac{1}{k} \sum_{j=1}^k z_j)$. Thus we obtain $F(u) \leq \lim_{j \rightarrow \infty} F_{p_j}(u_j)$. Then Lemma 3.6 yields that $u \in \bar{L}^{1,\varphi}(\Omega)$. $\square$

The next result is proved like Proposition 4.7, with the lemmas from this section replacing their counterparts from the previous section.

Proposition 5.5. *Let $\varphi \in \Phi_c(\Omega)$ and $C^\infty(\overline{\Omega})$ be dense in $L^{1,\varphi}(\Omega) \cap L^2(\Omega)$. Assume that $f \in L^2(\Omega)$. The function $u \in L^{1,\varphi}(\Omega) \cap L^2(\Omega)$ from Lemma 5.4 minimizes the F -energy, i.e.*

$$F(u) = \inf_{v \in L^2(\Omega)} F(v).$$

6. DOUBLE PHASE AND VARIABLE EXPONENT CASES

We considered the Γ -convergence of the double phase functional

$$I_\varepsilon(u) := \int_{\Omega} |\nabla u|^{1+\varepsilon} + a(x)|\nabla u|^2 + |u - f|^2 dx$$

in [18]. Specifically, we proved the following, where $\nabla_a u$ is the absolutely continuous part of the BV-gradient and $V(u, \Omega)$ is the total variation of u . (We refer to [1] for more information about BV-spaces.)

Theorem 6.1 (Theorem 4.1, [18]). *Suppose that Ω is an open rectangular cuboid, $a \in C^{0,1}(\overline{\Omega})$, and assume that $a > 0$ $\mathcal{H}^{n-1}$ -a.e. on the boundary $\partial\Omega$. Then I_ε Γ -converges to $I : \text{BV}(\Omega) \rightarrow [0, \infty]$,*

$$I(u) := V(u, \Omega) + \int_{\Omega} a(x)|\nabla_a u|^2 + |u - f|^2 dx,$$

in $L^1(\Omega)$ -topology.

By the uniqueness of the Γ -limit, we obtain the following explicit formula for F :

Corollary 6.2. *Suppose that Ω is an open rectangular cuboid, $a \in C^{0,1}(\overline{\Omega})$, and assume that $a > 0$ $\mathcal{H}^{n-1}$ -a.e. on the boundary $\partial\Omega$. Then*

$$F(u) = V(u, \Omega) + \int_{\Omega} a(x)|\nabla_a u|^2 + |u - f|^2 dx,$$

when F is defined with the double phase function $\varphi(x, t) := t + a(x)t^2$.

Proof. Let $u \in L^2(\Omega)$ and $p_i \rightarrow 1^+$. We use the elementary estimate $(s+t)^p - s^p \geq t^p \geq t - (p-1)$. By the second property of Γ -convergence (Theorem 1.3(2)), there exist $u_i \in L^{1,\varphi}(\Omega) \cap L^2(\Omega)$ and $p_i \in (1, 2)$ with $p_i \rightarrow 1^+$ such that $u_i \rightarrow u$ in $L^2(\Omega)$ and

$$\begin{aligned} F(u) &\geq \limsup_{i \rightarrow \infty} \int_{\Omega} (|\nabla u_i| + a(x)|\nabla u_i|^2)^{p_i} + |u_i - f|^2 dx \\ &\geq \limsup_{i \rightarrow \infty} \int_{\Omega} |\nabla u_i|^{p_i} + a(x)|\nabla u_i|^2 + |u_i - f|^2 - (p_i - 1) dx \\ &\geq V(u, \Omega) + \int_{\Omega} a(x)|\nabla_a u|^2 + |u - f|^2 dx, \end{aligned}$$

where the last step is the Γ -convergence of I_ε (Theorem 6.1).

For the opposite inequality we obtain from the second property of Γ -convergence (Theorem 6.1) functions $(u_i) \subset L^{1,\varphi}(\Omega)$ with $u_i \rightarrow u$ in $L^1(\Omega)$ and

$$\begin{aligned} I(u) &\geq \limsup_{i \rightarrow \infty} \int_{\Omega} |\nabla u_i|^{p_i} + a(x)|\nabla u_i|^2 + |u_i - f|^2 dx \\ &\geq \limsup_{i \rightarrow \infty} \int_{\Omega} |\nabla u_i| + a(x)|\nabla u_i|^2 + |u_i - f|^2 - (p_i - 1) dx \geq F(u), \end{aligned}$$

where we used the definition of F in the last step. $\square$

In the case of bounded functions we were able to prove Γ -convergence under a weaker condition on a , namely $\alpha > \frac{1}{2}$ instead of $\alpha = 1$.

Theorem 6.3 (Theorem 4.2, [18]). *Suppose that Ω is a bounded Lipschitz domain, $a \in C^{0,\alpha}(\overline{\Omega})$ for some $\alpha > \frac{1}{2}$, and assume that $a > 0$ H^{n-1} -a.e. on the boundary $\partial\Omega$. Then $I_\varepsilon|_{L^\infty}$ Γ -converges to $I|_{L^\infty}$ in $L^1(\Omega)$ -topology.*

This gives another corollary. The proof is similar to the previous corollary, so we skip it.

Corollary 6.4. *Suppose that Ω is a bounded Lipschitz domain, $a \in C^{0,\alpha}(\overline{\Omega})$ for some $\alpha > \frac{1}{2}$, and assume that $a > 0$ $\mathcal{H}^{n-1}$ -a.e. on the boundary $\partial\Omega$. For bounded u ,*

$$F(u) = V(u, \Omega) + \int_{\Omega} a(x) |\nabla_a u|^2 + |u - f|^2 dx,$$

when F is defined with the double phase function $\varphi(x, t) := t + a(x)t^2$.

Next we consider the variable exponent case $\varphi(x, t) := t^{p(x)}$, where $p : \Omega \rightarrow [1, \infty)$. We denote $Y := \{x \in \Omega : p = 1\}$ and define $p_\delta(x) := \max\{\delta, p(x)\}$. In [19, Theorem 1.5] we proved Γ -convergence of the p_δ -energy.

Theorem 6.5. *Let Ω be an open rectangular cuboid and let p be strongly log-Hölder continuous, i.e.*

$$|p(x) - p(z)| \log \left(e + \frac{1}{|x - z|} \right) \leq c \quad \text{and} \quad \lim_{x \rightarrow y} |p(x) - 1| \log \frac{1}{|x - y|} = 0$$

for every $x, z \in \Omega$ and $y \in Y$. Then

$$D_\varepsilon(u) := \int_{\Omega} |\nabla u|^{p_{1+\varepsilon}(x)} + |u - f|^2 dx$$

Γ -converges in $L^1(\Omega)$ -topology to

$$D(u) := V(u, Y) + \int_{\Omega \setminus Y} |\nabla u|^{p(x)} + |u - f|^2 dx.$$

Again, the uniqueness of the Γ -limit allows us to obtain a concrete form of E . The proof is again similar to Lemma 6.2 and thus skipped.

Corollary 6.6. *Suppose that Ω is a rectangular cuboid and p is strongly log-Hölder continuous. Then*

$$F(u) = V(u, Y) + \int_{\Omega \setminus Y} |\nabla u|^{p(x)} + |u - f|^2 dx,$$

where F is defined with the variable exponent function $\varphi(x, t) := t^{p(x)}$.

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