.

REGULARITY RESULTS FOR A CLASS OF NON-DIFFERENTIABLE OBSTACLE PROBLEMS

MICHELA ELEUTERI – ANTONIA PASSARELLI DI NAPOLI

Abstract. In this paper we prove the higher differentiability in the scale of Besov spaces of the solutions to a class of obstacle problems of the type

$$\min\left\{\int_\Omega F(x,z,Dz) : z \in \mathcal{K}_\psi(\Omega)\right\}.$$

Here Ω is an open bounded set of $\mathbb{R}^n$, $n \geq 2$, ψ is a fixed function called *obstacle* and $\mathcal{K}_\psi(\Omega)$ is set of admissible functions $z \in W^{1,p}(\Omega)$ such that $z \geq \psi$ a.e. in Ω. We assume that the gradient of the obstacle belongs to a suitable Besov space. The main novelty here is that we are not assuming any differentiability on the partial maps $x \mapsto F(x,z,Dz)$ and $z \mapsto F(x,z,Dz)$, but only their Hölder continuity.

$\boxed{\text{January 19, 2019}}$

1. Introduction

The aim of this paper is the study of the higher differentiability properties of the gradient of the solutions $u \in W^{1,p}(\Omega)$ to variational obstacle problems of the form

$$\min\left\{\int_\Omega F(x,z,Dz) : z \in \mathcal{K}_\psi(\Omega)\right\}. \tag{1.1}$$

where Ω is a bounded open set of $\mathbb{R}^n$, $n \geq 2$. The function $\psi : \Omega \to [-\infty, +\infty)$, called *obstacle*, belongs to the Sobolev class $W^{1,p}(\Omega)$ and the class $\mathcal{K}_\psi(\Omega)$ is defined as follows

$$\mathcal{K}_\psi(\Omega) := \left\{z \in W^{1,p}(\Omega) : z \geq \psi \text{ a.e. in } \Omega\right\}. \tag{1.2}$$

The set $\mathcal{K}_\psi$ is not empty, since it contains the function ψ.
We shall consider integrand F satisfying the following set of conditions:

$$z \mapsto F(x,y,z) \text{ is } \mathcal{C}^2 \tag{F1}$$

$$\tilde{\nu}(\tilde{\mu}^2 + |z|^2)^{p/2} \leq F(x,y,z) \leq \tilde{L}(\tilde{\mu}^2 + |z|^2)^{p/2} \tag{F2}$$

$$\tilde{\nu}_1(\tilde{\mu}^2 + |z|^2)^{\frac{p-2}{2}}|\lambda|^2 \leq \langle F_{zz}(x,y,z)\lambda, \lambda\rangle \leq \tilde{L}_1(\tilde{\mu}^2 + |z|^2)^{\frac{p-2}{2}}|\lambda|^2 \tag{F3}$$

$$|F(x_1,y_1,z) - F(x_2,y_2,z)| \leq \tilde{L}_2\,\omega(|x_1 - x_2| + |y_1 - y_2|)(\tilde{\mu}^2 + |z|^2)^{p/2} \tag{F4}$$

for all $x, x_1, x_2 \in \Omega$, $y, y_1, y_2 \in \mathbb{R}$ and all $z, \lambda \in \mathbb{R}^n$, where $p \geq 2$ and $0 \leq \tilde{\nu} \leq \tilde{L}$, $0 \leq \tilde{\nu}_1 \leq \tilde{L}_1$, $\tilde{L}_2 \geq 1$ are fixed constants, $\tilde{\mu} \in [0,1]$ is a fixed parameter, and $\omega : \mathbb{R}^+ \to (0,1)$ is a continuous non-decreasing function such that for some $\alpha \in (0,1)$

$$\omega(s) \leq s^\alpha. \tag{1.3}$$

2000 *Mathematics Subject Classification.* 26A27; 49J40; 47J20.

Key words and phrases. Non-differentiability, obstacle problems, higher differentiability.

The work of the authors is supported by GNAMPA (Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni) of INdAM (Istituto Nazionale di Alta Matematica), by MIUR through the project FFABR and by the University of Modena and Reggio Emilia through the project FAR2017 "Equazioni differenziali: problemi evolutivi, variazionali ed applicazioni" (coord. Prof. S. Gatti). This reaserch started while A. Passarelli di Napoli was visiting the University of Modena and Reggio Emilia . The hospitality of this Institution is warmly adknowledged.

These assumptions are standard and express that the integrand F is strictly convex with respect to the gradient variable z and it is Hölder continuous with exponent α with respect to the variable (x, y). Note that the parameter $\tilde{\mu} \in [0, 1]$ allows us to consider both the degenerate ($\tilde{\mu} = 0$) and the non-degenerate situation ($\tilde{\mu} > 0$).

The study of the regularity theory for obstacle problems is a classical topic in Partial Differential Equations and Calculus of Variations.

The obstacle problem appeared in the mathematical literature in the works by G. Stampacchia [40] and G. Fichera [20] It is well known that the solutions to the obstacle problem cannot be of class $\mathcal{C}^2$ independently of how regular the obstacle is; this lead to the origin of the concept of weak solution and to the theory of *variational inequalities*, after the fundamental work of J.-L. Lions and G. Stampacchia [32] (for more details we refer to classical monograph [21], [28]); these problems can generally be solved by applying methods of functional analysis, so the question to give conditions to establish that weak solutions are, in many cases, classical ones is of fundamental importance (see [9]).

It is usually observed that the regularity of solutions to the obstacle problems is influenced by the one of obstacle; for example, for linear obstacle problems, obstacle and solutions have the same regularity [6], [10], [28]. This does not apply in the nonlinear setting, hence along the years, there have been intense research activity for the regularity of the obstacle problem in this case. Among the others we quote [33], [11], [12], [22], [3], [14], [15], [16], [4], [17], [8], [18], [2], [5], [34], [35], [7], [36].

In our previous paper ([19]), we analyzed how an extra differentiability (of integer or fractional order) of the gradient of the obstacle transfers to the gradient of the solutions.

More precisely, in case the integrand in (1.1) is independent on the z-variable, i.e.

$$F(x, z, Dz) = \tilde{f}(x, Dz),$$

we found sufficient conditions on the partial map $x \mapsto D_\xi \tilde{f}(x, \xi)$ in order to have that the Besov or Sobolev regularity of the gradient of the obstacle implies a Besov or Sobolev regularity of the gradient of the solutions, with no loss in the order of differentiability.

Recall that the Besov or the Sobolev regularity of a function can be characterized in pointwise term, i.e. assuming $D\psi \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is equivalent to assume that there exists a function $k \in L^p_{\mathrm{loc}}(\Omega)$ such that the following inequality

$$|D\psi(x) - D\psi(y)| \le (k(x) + k(y))|x - y| \tag{1.4}$$

holds, for a.e. $x, y \in \Omega$ (see [26]). Analogously assuming $D\psi \in B^{\kappa}_{p,\infty,\mathrm{loc}}(\Omega)$, for $\kappa \in (0, 1)$ is equivalent to assume that there exists a sequence $(g_k)_k \in \ell^\infty(\mathbb{Z}; L^p(\mathbb{R}^n))$ of measurable, non-negative functions $g_k : \mathbb{R}^n \to \mathbb{R}$, together with a null set $N \subset \mathbb{R}^n$, such that the inequality

$$|D\psi(x) - D\psi(y)| \le \; (g_k(x) + g_k(y))\, |x - y|^\alpha$$

holds whenever $k \in \mathbb{Z}$ and $x, y \in \mathbb{R}^n \setminus N$ are such that $2^{-k} \le |x - y| < 2^{-k+1}$ ([29]). For precise definition and properties of Besov spaces see Subsection 2.1 below.

The main tool in [19] has been the fact that the solutions to obstacle problems of the form

$$\min \left\{ \int_\Omega \tilde{f}(x, Dz) : z \in \mathcal{K}_\psi(\Omega) \right\}$$

solve a suitable variational inequality and that their regularity is strictly connected with that of the solutions to partial differential equation of the form

$$\mathrm{div} D_\xi \tilde{f}(x, Du) = \mathrm{div} D_\xi \tilde{f}(x, D\psi).$$

whose higher differentiability properties have been widely investigated (see for example [1], [13], [23][24], [37], [38], [39]).
Here we continue our study of the higher differentiability properties of the solutions to obstacle problems. The novelty of this paper consists in dealing with functionals depending also on the z-variable and we are not assuming any differentiability for the partial map $z \mapsto F(x, z, Dz)$. Therefore we cannot use the fact that solutions to (1.1) are solutions to a suitable variational inequality.
However, we are able to establish the following

Theorem 1.1. *Let $u \in \mathcal{K}_\psi(\Omega)$ be a solution to the problem* (1.1) *under the assumptions* (F1)-(F4). *If $D\psi \in B^{\kappa}_{p,\infty,\mathrm{loc}}(\Omega)$, then there exists $\sigma = \sigma(\alpha, p, \kappa) > 0$, where α is the exponent appearing in assumption* (1.3)*, such that*

$$V_p(Du) := (\tilde{\mu}^2 + |Du|^2)^{\frac{p-2}{4}} Du \in B^{t}_{2,\infty,\mathrm{loc}}(\Omega) \qquad \text{for all } t \in \left(0, \frac{\kappa\sigma}{2}\right)$$

and the following estimate

$$\int_{\Omega'} |\tau_h V_p(Du)|^2 \, dx \leq C\, |h|^{2t} \left[\int_{\Omega''} (1 + |Du|^p)\, dx + ||D\psi||_{B^{\kappa}_{p,\infty}(\Omega'')} \right]^{\vartheta} \qquad \text{for all } t \in \left(0, \frac{\kappa\sigma}{2}\right)$$

holds true for every open subset $\Omega' \subset \Omega'' \Subset \Omega$, for an exponent $\vartheta = \vartheta(p, \kappa) > 0$ and a constant $C = C(n, \tilde{\nu}, \tilde{L}_1, \tilde{L}_2, |\Omega''|)$.

The main tool in the proof of our main result is a localization technique introduced in [30] (and eventually upgraded in [31]) and a comparison argument between the solutions to the obstacle problem (1.1) with the solution to a "freezed" obstacle problem (see (3.2) below). By the results in [19], we have that the solutions to the "freezed" obstacle problem inherit the Besov regularity of the obstacle. The core of the comparison argument consists in showing that the fractional difference quotients of u and v are close enough, in an integral sense, to have that u belongs to a suitable Besov space.
The higher integrability of the gradient of the solutions, both at the interior and up to boundary, proven in [14] reveals to be the key ingredient to prove that the closeness condition is proportional to the radius of the ball over which we are comparing the solutions.
Next, we choose such a ball with radius proportional to a suitable power of the modulus of the increment to deduce that the difference quotient of Du decays as some positive power of the modulus of the increment. This implies that the gradient of u belongs to some Besov space.
Finally, we employ a boothstrapping argument to get the maximal higher fractional differentiability. Our proof holds true also in case of Sobolev regularity for the gradient of the obstacle, by virtue of the pointwise characterization at (1.4). Indeed, we have the following

Corollary 1.2. *Let $u \in \mathcal{K}_\psi(\Omega)$ be a solution to the problem* (1.1) *under the assumptions* (F1)-(F4). *If $D\psi \in W^{1,p}_{\mathrm{loc}}(\Omega)$, then there exists $\sigma = \sigma(\alpha, p) > 0$, where α is the exponent appearing in assumption* (1.3)*, such that*

$$V_p(Du) := (\tilde{\mu}^2 + |Du|^2)^{\frac{p-2}{4}} Du \in B^{t}_{2,\infty,\mathrm{loc}}(\Omega) \qquad \text{for all } t \in \left(0, \frac{\sigma}{2}\right)$$

and the following estimate

$$\int_{\Omega'} |\tau_h V_p(Du)|^2 \, dx \leq C\, |h|^{2t} \left[\int_{\Omega''} (1 + |Du|^p + |D\psi|^p + |D^2\psi|^p)\, dx \right]^{\vartheta} \qquad \text{for all } t \in \left(0, \frac{\sigma}{2}\right)$$

holds true for every every open subset $\Omega' \subset \Omega'' \Subset \Omega$, for an exponent $\vartheta = \vartheta(p) > 0$ and a constant $C = C(n, \tilde{\nu}, \tilde{L}_1, \tilde{L}_2, |\Omega''|)$.

2. Preliminary results

In this paper we shall denote by C or c a general positive constant that may vary on different occasions, even within the same line of estimates. Relevant dependencies will be suitably emphasized using parentheses or subscripts. In what follows, $B(x,r) = B_r(x) = \{y \in \mathbb{R}^n : \ |y-x| < r\}$ will denote the ball centered at x of radius r. We shall omit the dependence on the center and on the radius when no confusion arises. For a function $u \in L^1(B)$, the symbol

$$u_B := \fint_B u(x)\,dx = \frac{1}{|B|}\int_B u(x)\,dx$$

will denote the integral mean of the function u over the set B.
It is convenient and usual, when dealing with functionals with p-growth, to introduce the auxiliary function

$$V_p(\xi) := \left(\tilde{\mu}^2 + |\xi|^2\right)^{\frac{p-2}{4}}\xi, \tag{2.1}$$

defined for all $\xi \in \mathbb{R}^n$ and $\tilde{\mu} \geq 0$. For the auxiliary function $V_p(\xi)$, we recall the following estimate (see the proof of [25, Lemma 8.3]):

Lemma 2.1. *Let $1 < p < \infty$. There exists a constant $c = c(n,p) > 0$ such that*

$$c^{-1}\left(\tilde{\mu}^2 + |\xi|^2 + |\eta|^2\right)^{\frac{p-2}{2}} \leq \frac{|V_p(\xi) - V_p(\eta)|^2}{|\xi-\eta|^2} \leq c\left(\tilde{\mu}^2 + |\xi|^2 + |\eta|^2\right)^{\frac{p-2}{2}}$$

for any ξ, $\eta \in \mathbb{R}^n$, $\tilde{\mu} \geq 0$.

2.1. **Besov-Lipschitz spaces.**

Given $h \in \mathbb{R}^n$ and $v : \mathbb{R}^n \to \mathbb{R}$, let us introduce the notation $\tau_h v(x) = v(x+h) - v(x)$. As in [27, Section 2.5.12], given $0 < \alpha < 1$ and $1 \leq p, q < \infty$, we say that v belongs to the Besov space $B^\alpha_{p,q}(\mathbb{R}^n)$ if $v \in L^p(\mathbb{R}^n)$ and

$$\|v\|_{B^\alpha_{p,q}(\mathbb{R}^n)} = \|v\|_{L^p(\mathbb{R}^n)} + [v]_{B^\alpha_{p,q}(\mathbb{R}^n)} < \infty,$$

where

$$[v]_{B^\alpha_{p,q}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\frac{|v(x+h)-v(x)|^p}{|h|^{\alpha p}}dx\right)^{\frac{q}{p}}\frac{dh}{|h|^n}\right)^{\frac{1}{q}} < \infty. \tag{2.2}$$

Equivalently, we could simply say that $v \in L^p(\mathbb{R}^n)$ and $\frac{\tau_h v}{|h|^\alpha} \in L^q\left(\frac{dh}{|h|^n}; L^p(\mathbb{R}^n)\right)$. As usual, if one simply integrates for $h \in B(0,\delta)$ for a fixed $\delta > 0$, then an equivalent norm is obtained, because

$$\left(\int_{\{|h|\geq\delta\}}\left(\int_{\mathbb{R}^n}\frac{|v(x+h)-v(x)|^p}{|h|^{\alpha p}}dx\right)^{\frac{q}{p}}\frac{dh}{|h|^n}\right)^{\frac{1}{q}} \leq c(n,\alpha,p,q,\delta)\,\|v\|_{L^p(\mathbb{R}^n)}.$$

Similarly, we say that $v \in B^\alpha_{p,\infty}(\mathbb{R}^n)$ if $v \in L^p(\mathbb{R}^n)$ and

$$[v]_{B^\alpha_{p,\infty}(\mathbb{R}^n)} = \sup_{h\in\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\frac{|v(x+h)-v(x)|^p}{|h|^{\alpha p}}dx\right)^{\frac{1}{p}} < \infty. \tag{2.3}$$

Again, one can simply take supremum over $|h| \leq \delta$ and obtain an equivalent norm. By construction, $B^\alpha_{p,q}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$. One also has the following version of Sobolev embeddings (a proof can be found at [27, Prop. 7.12]).

Lemma 2.2. *Suppose that $0 < \alpha < 1$.*

(a) *If* $1 < p < \frac{n}{\alpha}$ *and* $1 \le q \le p^*_\alpha := \frac{np}{n-\alpha p}$, *then there is a continuous embedding* $B^\alpha_{p,q}(\mathbb{R}^n) \subset L^{p^*_\alpha}(\mathbb{R}^n)$.

(b) *If* $p = \frac{n}{\alpha}$ *and* $1 \le q \le \infty$, *then there is a continuous embedding* $B^\alpha_{p,q}(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$, *where* BMO *denotes the space of bounded mean oscillations* [25, Chapter 2].

For further needs, we recall the following inclusions ([27, Proposition 7.10 and Formula (7.35)]).

Lemma 2.3. *Suppose that* $0 < \beta < \alpha < 1$.

(a) *If* $1 < p \le +\infty$ *and* $1 \le q \le r \le +\infty$ *then* $B^\alpha_{p,q}(\mathbb{R}^n) \subset B^\alpha_{p,r}(\mathbb{R}^n)$.

(b) *If* $1 < p \le +\infty$ *and* $1 \le q, r \le +\infty$ *then* $B^\alpha_{p,q}(\mathbb{R}^n) \subset B^\beta_{p,r}(\mathbb{R}^n)$.

(c) *If* $1 \le q \le \infty$, *then* $B^\alpha_{\frac{n}{\alpha},q}(\mathbb{R}^n) \subset B^\beta_{\frac{n}{\beta},q}(\mathbb{R}^n)$.

Combining Lemmas 2.2 and 2.3, we get the following Sobolev type imbedding theorem for Besov spaces $B^\alpha_{p,\infty}(\mathbb{R}^n)$ that are excluded from the assumptions of (a) in Lemma 2.2.

Lemma 2.4. *Suppose that* $0 < \alpha < 1$. *There is a continuous embedding* $B^\alpha_{p,\infty}(\mathbb{R}^n) \subset L^{p^*_\beta}(\mathbb{R}^n)$, *for every* $\beta < \alpha$. *Moreover the following local estimate*

$$||F||_{L^{\frac{pn}{n-p\beta}}(B_\rho)} \le c\left([F]_{B^\alpha_{p,\infty}(B_R)} + ||F||_{L^p(B_R)}\right), \tag{2.4}$$

holds for every balls $B_\rho \subset B_R$ *with* $c = c(n, R, \rho, \alpha, \beta)$.

Given a domain $\Omega \subset \mathbb{R}^n$, we say that v belongs to the local Besov space $B^\alpha_{p,q,\mathrm{loc}}$ if $\varphi\, v$ belongs to the global Besov space $B^\alpha_{p,q}(\mathbb{R}^n)$ whenever φ belongs to the class $\mathcal{C}^\infty_c(\Omega)$ of smooth functions with compact support contained in Ω. It is worth noticing that one can prove suitable versions of Lemma 2.2 and of Lemma 2.3, by using local Besov spaces.
The following Lemma is an easy exercise and its proof can be found in [1].

Lemma 2.5. *A function* $v \in L^p_{\mathrm{loc}}(\Omega)$ *belongs to the local Besov space* $B^\alpha_{p,q,\mathrm{loc}}$ *if and only if*

$$\left\|\frac{\tau_h v}{|h|^\alpha}\right\|_{L^q\left(\frac{dh}{|h|^n};L^p(B)\right)} < \infty$$

for any ball $B \subset 2B \Subset \Omega$ *with radius* r_B. *Here the measure* $\frac{dh}{|h|^n}$ *is restricted to the ball* $B(0, r_B)$ *on the* h*-space.*

It is known that Besov-Lipschitz spaces of fractional order $\alpha \in (0, 1)$ can be characterized in pointwise terms. Given a measurable function $v : \mathbb{R}^n \to \mathbb{R}$, a *fractional* α*-Hajłasz gradient for* v is a sequence $(g_k)_k$ of measurable, non-negative functions $g_k : \mathbb{R}^n \to \mathbb{R}$, together with a null set $N \subset \mathbb{R}^n$, such that the inequality

$$|v(x) - v(y)| \le (g_k(x) + g_k(y))\, |x - y|^\alpha$$

holds whenever $k \in \mathbb{Z}$ and $x, y \in \mathbb{R}^n \setminus N$ are such that $2^{-k} \le |x - y| < 2^{-k+1}$. We say that $(g_k) \in \ell^q(\mathbb{Z}; L^p(\mathbb{R}^n))$ if

$$\|(g_k)_k\|_{\ell^q(L^p)} = \left(\sum_{k\in\mathbb{Z}} \|g_k\|^q_{L^p(\mathbb{R}^n)}\right)^{\frac{1}{q}} < \infty.$$

The following result was proved in [29].

Theorem 2.6. *Let $0 < \alpha < 1$, $1 \le p < \infty$ and $1 \le q \le \infty$. Let $v \in L^p(\mathbb{R}^n)$. One has $v \in B^\alpha_{p,q}(\mathbb{R}^n)$ if and only if there exists a fractional α-Hajłasz gradient $(g_k)_k \in \ell^q(\mathbb{Z}; L^p(\mathbb{R}^n))$ for v. Moreover,*

$$\|v\|_{B^\alpha_{p,q}(\mathbb{R}^n)} \simeq \inf \|(g_k)_k\|_{\ell^q(L^p)},$$

where the infimum runs over all possible fractional α-Hajłasz gradients for v.

2.2. **Difference quotient.** We recall some properties of the finite difference operator that will be needed in the sequel.

Definition 2.7. For every vector valued function $F : \mathbb{R}^n \to \mathbb{R}^N$ the finite difference operator is defined by

$$\tau_{s,h} F(x) = F(x + he_s) - F(x)$$

where $h \in \mathbb{R}$, e_s is the unit vector in the x_s direction and $s \in \{1, \dots, n\}$.
The difference quotient is defined for $h \in \mathbb{R} \setminus \{0\}$ as

$$\Delta_{s,h} F(x) = \frac{\tau_{s,h} F(x)}{h}.$$

We start with the description of some elementary properties that can be found, for example, in [25].

Proposition 2.8. *Let F and G be two functions such that $F, G \in W^{1,p}(\Omega; \mathbb{R}^N)$, with $p \ge 1$, and let us consider the set*

$$\Omega_{|h|} := \{x \in \Omega : dist(x, \partial\Omega) > |h|\}.$$

Then

- $(d1)$ *$\tau_h F \in W^{1,p}(\Omega; \mathbb{R}^N)$ and*
$$D_i(\tau_h F) = \tau_h(D_i F).$$
- $(d2)$ *If at least one of the functions F or G has support contained in $\Omega_{|h|}$, then*
$$\int_\Omega F\, \tau_h G\, dx = \int_\Omega G\, \tau_{-h} F\, dx.$$
- $(d3)$ *We have*
$$\tau_h(FG)(x) = F(x+h)\tau_h G(x) + G(x)\tau_h F(x).$$

The next result about finite difference operator is a kind of integral version of Lagrange Theorem.

Lemma 2.9. *If $0 < \rho < R$, $|h| < \frac{R-\rho}{2}$, $1 < p < +\infty$, and $F \in L^p(B_R; \mathbb{R}^N)$, $DF \in L^p(B_R; \mathbb{R}^{N\times n})$ then*

$$\int_{B_\rho} |\tau_h F(x)|^p \, dx \le c(n,p)|h|^p \int_{B_R} |DF(x)|^p \, dx.$$

Moreover

$$\int_{B_\rho} |F(x+h)|^p \, dx \le \int_{B_R} |F(x)|^p \, dx.$$

Now, we recall the fundamental Sobolev embedding property.

Lemma 2.10. *Let $F : \mathbb{R}^n \to \mathbb{R}^N$, $F \in L^p(B_R; \mathbb{R}^N)$ with $1 < p < n$. Suppose that there exist $\rho \in (0, R)$ and $M > 0$ such that*

$$\sum_{s=1}^n \int_{B_\rho} |\tau_{s,h} F(x)|^p \, dx \le M^p |h|^p,$$

for every h with $|h| < \frac{R-\rho}{2}$. Then $F \in W^{1,p}(B_\rho) \cap L^{\frac{np}{n-p}}(B_\rho)$. Moreover

$$||DF||_{L^p(B_\rho)} \leq M$$

and

$$||F||_{L^{\frac{np}{n-p}}(B_\rho)} \leq c\left(M + ||F||_{L^p(B_R)}\right),$$

with $c = c(n, N, p)$.

For the proof see, for example, [25, Lemma 8.2].

3. Higher integrability

Throughout this and the next two sections we will denote by u a solution to the obstacle problem (1.1) and fix a ball $B = B(x_0, R) \Subset \Omega$, for a given radius $R > 0$ and $x_0 \in \Omega$. Let us consider the so called "freezed" functional

$$\int_B \tilde{F}(Dz)\, dx := \int_B F(x_0, u_B, Dz)\, dx \tag{3.1}$$

and let $v \in u + W_0^{1,p}(B)$ be the solution to

$$\min\left\{\int_B F(x_0, u_B, Dz)\, dx : z \in \mathcal{K}_\psi(\Omega), \quad z = u \text{ on } \partial B\right\}. \tag{3.2}$$

It is well known that $v \in u + W_0^{1,p}(B)$ is a solution to (3.2) if and only if it satisfies the following variational inequality

$$\int_\Omega \langle D_\xi \tilde{F}(Dv), D(\varphi - v)\rangle\, dx \geq 0 \tag{3.3}$$

for all $\varphi \in W_0^{1,p}(B) \cap \mathcal{K}_\psi(\Omega)$.
Note that the assumptions on F imply that there exist positive constants ν, L, ℓ such that the following p-ellipticity and p-growth conditions are satisfied:

$$\langle D_\xi \tilde{F}(\xi) - D_\xi \tilde{F}(\eta), \xi - \lambda\rangle \geq \nu|\xi - \eta|^2(\tilde{\mu}^2 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} \tag{$\mathcal{A}1$}$$

$$|D_\xi \tilde{F}(\xi) - D_\xi \tilde{F}(\eta)| \leq L\, |\xi - \eta|(\tilde{\mu}^2 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} \tag{$\mathcal{A}2$}$$

$$|D_\xi \tilde{F}(\xi)| \leq \ell\, (\tilde{\mu}^2 + |\xi|^2)^{\frac{p-1}{2}} \tag{$\mathcal{A}3$},$$

for every $\xi, \eta \in \mathbb{R}^n$, $\tilde{\mu} \in [0, 1]$.
We shall need the following two higher integrability results that can be found in [14].

Theorem 3.1. *Let u be a solution to the obstacle problem* (1.1) *where the integrand F satisfies* $(F2)$ *and the function ψ is such that $|D\psi| \in L^{q_0}_{\mathrm{loc}}(\Omega)$, for some $q_0 > p$. Then, there exist an exponent $p < q < q_0$ and a positive constant C depending on $p, \tilde{L}$ such that, for any $B \subset 2B \Subset \Omega$, then*

$$\left(\fint_B |Du(x)|^q dx\right)^{1/q} \leq C\left(\fint_{2B} |Du(x)|^p\, dx\right)^{1/p} + C\left(\fint_{2B} |D\psi(x)|^{q_0}\, dx\right)^{1/q_0}. \tag{3.4}$$

The higher integrability of the minimizer u stated in the previous Theorem allows us to obtain also the following higher integrability up to the boundary result for the solution to the problem (3.2).

Theorem 3.2. *Let $v \in u + W^{1,p}(B)$ be a solution to (3.2) where the integrand F satisfies $(F2)$ and the function ψ is such that $|D\psi| \in L^{q_0}_{\rm loc}(\Omega)$ for some $q_0 > p$. If moreover $u \in W^{1,\bar q}(B)$ for some $p < \bar q < q_0$, then there exist $p < q < \bar q < q_0$ and C depending on $p, \tilde L$ such that $v \in W^{1,q}(B)$ and*

$$\left(\fint_B |Dv(x)|^q\,dx\right)^{1/q} \le C\left[\fint_{2B}(1+|Du(x)|^{\bar q})\,dx\right]^{1/\bar q} + C\left[\fint_{2B}(1+|D\psi(x)|^{\bar q})\,dx\right]^{1/\bar q}. \tag{3.5}$$

4. Higher differentiability for the comparison maps

The higher differentiability of the solution v to (3.2) has been already established in [19] in a more general contest. We give the proof here for the sake of completeness and with the purpose to establish precise estimates on the difference quotient that will be crucial to perform the comparison argument. More precisely our first result reads as follows.

Lemma 4.1. *Let $v \in u + W^{1,p}_0(B)$ be the solution to (3.2) under the assumptions $(\mathcal{A}1) - (\mathcal{A}2)$, where u is the solution of problem (1.1). If*

$$D\psi \in B^{\kappa}_{p,\infty}(B), \tag{4.1}$$

then

$$V_p(Dv) \in B^{\kappa}_{2,\infty,\rm loc}(B)$$

and the following estimate

$$\int_{B_{\frac r2}} |\tau_h V_p(Dv)|^2\,dx \le C\,|h|^{2\kappa}\left([D\psi]^p_{B^{\kappa}_{p,\infty}} + \frac{1}{r^2}\int_{B_{2r}}\left(\tilde\mu^p + |Dv|^p + |D\psi|^p\right)dx\right), \tag{4.2}$$

holds for all balls $B_r \subset B_{2r} \Subset B$, with $C \equiv C(n,p,\nu,L)$.

Proof. Let us fix a ball B_r such that $B_{2r} \Subset B$ and a cut off function $\eta \in C^\infty_0(B_r)$, $\eta \equiv 1$ on $B_{\frac r2}$ such that $|\nabla\eta| \le \frac{c}{r}$. Due to the local nature of our results, there is no loss of generality in supposing $r \le 1$, that we will do from now on. Then, for $|h| < \frac r4$, we consider

$$v_1(x) = \eta^2(x)[(v-\psi)(x+h) - (v-\psi)(x)]. \tag{4.3}$$

It is immediate to check that $v_1 \in W^{1,p}_0(B_r)$. Moreover, for a.e. $x \in \Omega$ and for any $t \in [0,1)$

$$\begin{aligned} & v(x) - \psi(x) + tv_1(x) \\ = \;& v(x) - \psi(x) + t\eta^2(x)[(v-\psi)(x+h) - (v-\psi)(x)] \\ = \;& t\,\eta^2(x)(v-\psi)(x+h) + (1 - t\eta^2(x))(v-\psi)(x) \ge 0, \end{aligned}$$

because $v \in \mathcal{K}_\psi(\Omega)$. Therefore $\varphi := v + tv_1 \in \mathcal{K}_\psi(\Omega)$, for every $t \in (0,1]$, and so it is an admissible test function for the variational inequality (3.3). With this choice of the test function, (3.3) becomes

$$0 \le \int_B \langle D_\xi \tilde F(Dv(x)), D[\eta^2(x)[(v-\psi)(x+h) - (v-\psi)(x)]]\rangle\,dx. \tag{4.4}$$

On the other hand, if we define

$$v_2(x) = \eta^2(x-h)[(v-\psi)(x-h) - (v-\psi)(x)] \tag{4.5}$$

then $v_2 \in W^{1,p}_0(B_r)$ and again we have $\varphi := v + tv_2 \in \mathcal{K}_\psi(\Omega)$, due to the fact that

$$\begin{aligned} & v(x) - \psi(x) + tv_2(x) \\ = \;& v(x) - \psi(x) + t\eta^2(x-h)[(v-\psi)(x-h) - (v-\psi)(x)] \end{aligned}$$

$$= \quad t\,\eta^2(x-h)(v-\psi)(x-h)+(1-t\eta^2(x-h))(v-\psi)(x) \geq 0.$$

Choosing in (3.3) as test function $\varphi = v + tv_2$, where v_2 is defined in (4.5), we get

$$0 \leq \int_B \Big\langle D_\xi \tilde{F}(Dv(x)), D\big[\eta^2(x-h)[(v-\psi)(x-h)-(v-\psi)(x)]\big]\Big\rangle\, dx.$$

By means of a simple change of variable, we obtain

$$0 \leq \int_B \Big\langle D_\xi \tilde{F}(Dv(x+h)), D\big[\eta^2(x)[(v-\psi)(x)-(v-\psi)(x+h)]\big]\Big\rangle\, dx. \tag{4.6}$$

We can add (4.4) and (4.6), thus obtaining

$$\begin{aligned} 0 \quad &\leq \quad \int_B \Big\langle D_\xi \tilde{F}(Dv(x)), D\big[\eta^2(x)[(v-\psi)(x+h)-(v-\psi)(x)]\big]\Big\rangle\, dx \\ &\quad + \int_B \Big\langle D_\xi \tilde{F}(Dv(x+h)), D\big[\eta^2(x)[(v-\psi)(x)-(v-\psi)(x+h)]\big]\Big\rangle\, dx \\ &= \quad \int_B \Big\langle D_\xi \tilde{F}(Dv(x)) - D_\xi \tilde{F}(Dv(x+h)), D\big[\eta^2(x)[(v-\psi)(x+h)-(v-\psi)(x)]\big]\Big\rangle\, dx, \end{aligned}$$

which implies

$$\begin{aligned} 0 \quad &\geq \quad \int_B \Big\langle D_\xi \tilde{F}(Dv(x+h)) - D_\xi \tilde{F}(Dv(x)), \eta^2(x) D[(v-\psi)(x+h)-(v-\psi)(x)]\Big\rangle\, dx \\ &\quad + \int_B \Big\langle D_\xi \tilde{F}(Dv(x+h)) - D_\xi \tilde{F}(Dv(x)), 2\,\eta(x)\, D\eta(x)\, [(v-\psi)(x+h)-(v-\psi)(x)]\Big\rangle\, dx. \end{aligned}$$

We can write previous inequality as follows

$$\begin{aligned} 0 \quad &\geq \quad \int_B \big\langle D_\xi \tilde{F}(Dv(x+h)) - D_\xi \tilde{F}(Dv(x)), \eta^2 \tau_h Dv\big\rangle\, dx \\ &\quad - \int_B \big\langle D_\xi \tilde{F}(Dv(x+h)) - D_\xi \tilde{F}(Dv(x)), \eta^2 \tau_h D\psi\big\rangle\, dx \\ &\quad + \int_B \big\langle D_\xi \tilde{F}(Dv(x+h)) - D_\xi \tilde{F}(Dv(x)), 2\eta\, D\eta \tau_h (v-\psi)\big\rangle\, dx \\ &=: \quad I + II + III, \end{aligned}$$

that yields

$$I \leq |II| + |III|. \tag{4.7}$$

The ellipticity assumption expressed by $(\mathcal{A}1)$ implies

$$I \geq \nu \int_B \eta^2 |\tau_h Dv|^2 (\tilde{\mu}^2 + |Dv(x+h)|^2 + |Dv(x)|^2)^{\frac{p-2}{2}}\, dx. \tag{4.8}$$

By virtue of assumption $(\mathcal{A}2)$ and Young's inequality, we get

$$\begin{aligned} |II| \quad &\leq \quad L \int_B \eta^2 |\tau_h Dv| (\tilde{\mu}^2 + |Dv(x+h)|^2 + |Dv(x)|^2)^{\frac{p-2}{2}}\, |\tau_h D\psi| dx \\ &\leq \quad \varepsilon \int_B \eta^2 |\tau_h Dv|^2 (\tilde{\mu}^2 + |Dv(x+h)|^2 + |Dv(x)|^2)^{\frac{p-2}{2}}\, dx \\ &\quad + C_\varepsilon(L) \int_B \eta^2 |\tau_h D\psi|^2 (\tilde{\mu}^2 + |Dv(x+h)|^2 + |Dv(x)|^2)^{\frac{p-2}{2}}\, dx \\ &\leq \quad \varepsilon \int_B \eta^2 |\tau_h Dv|^2 (\tilde{\mu}^2 + |Dv(x+h)|^2 + |Dv(x)|^2)^{\frac{p-2}{2}}\, dx \end{aligned}$$

$$+C_\varepsilon(L,p)\left(\int_{B_r}|\tau_h D\psi|^p\,dx\right)^{\frac{2}{p}}\left(\int_{B_{2r}}(\tilde{\mu}^p+|Dv(x)|^p)\,dx\right)^{\frac{p-2}{p}}, \tag{4.9}$$

where, in the last estimate, we used Hölder's inequality, the properties of η and the second estimate of Lemma 2.9. The assumption $D\psi\in B^{\kappa}_{p,\infty}(B)$, recalling definition (2.3), allows us to estimate the second integral of the right hand side of (4.9), thus obtaining

$$\begin{aligned}|II| &\le \varepsilon\int_B \eta^2|\tau_h Dv|^2(\tilde{\mu}^2+|Dv(x+h)|^2+|Dv(x)|^2)^{\frac{p-2}{2}}\,dx\\ &\quad +C_\varepsilon(L,n,p)|h|^{2\kappa}\Big([D\psi]_{B^{\kappa}_{p,\infty}}\Big)^2\left(\int_{B_{2r}}(\tilde{\mu}^p+|Dv(x)|^p)\,dx\right)^{\frac{p-2}{p}}\\ &\le \varepsilon\int_B \eta^2|\tau_h Dv|^2(\tilde{\mu}^2+|Dv(x+h)|^2+|Dv(x)|^2)^{\frac{p-2}{2}}\,dx\\ &\quad +C_\varepsilon(L,n,p)|h|^{2\kappa}\Big([D\psi]_{B^{\kappa}_{p,\infty}}\Big)^p+C_\varepsilon(L,n,p)|h|^{2\kappa}\int_{B_{2r}}(\tilde{\mu}^p+|Dv(x)|^p)\,dx, \end{aligned}\tag{4.10}$$

where, in the last line, we used Young's inequality. Arguing analogously, we get

$$\begin{aligned}|III| &\le 2L\int_B|\tau_h Dv|\,|D\eta|\eta(\tilde{\mu}^2+|Dv(x+h)|^2+|Dv(x)|^2)^{\frac{p-2}{2}}\,|\tau_h(u-\psi)|\,dx\\ &\le \varepsilon\int_B\eta^2|\tau_h Dv|^2\,(\tilde{\mu}^2+|Dv(x+h)|^2+|Dv(x)|^2)^{\frac{p-2}{2}}\,dx\\ &\quad +C_\varepsilon(L)\int_B|D\eta|^2(\tilde{\mu}^2+|Dv(x+h)|^2+|Dv(x)|^2)^{\frac{p-2}{2}}|\tau_h(v-\psi)|^2\,dx\\ &\le \varepsilon\int_B\eta^2|\tau_h Dv|^2\,(\tilde{\mu}^2+|Dv(x+h)|^2+|Dv(x)|^2)^{\frac{p-2}{2}}\,dx\\ &\quad +\frac{C_\varepsilon(L,p)}{r^2}\left(\int_{B_{2r}}(\tilde{\mu}^p+|Dv(x)|^p)\,dx\right)^{\frac{p-2}{p}}\left(\int_{B_r}|\tau_h(v-\psi)|^p\,dx\right)^{\frac{2}{p}},\end{aligned}$$

by virtue of the fact that $|D\eta|\le\frac{c}{r}$. Since $u-\psi\in W^{1,p}_{\mathrm{loc}}(\Omega)$, we may use the first estimate of Lemma 2.9 to control last integral in the right hand side of previous estimate, obtaining

$$\begin{aligned}|III| &\le \varepsilon\int_B\eta^2|\tau_h Du|^2\,(\tilde{\mu}^2+|Dv(x+h)|^2+|Dv(x)|^2)^{\frac{p-2}{2}}\,dx\\ &\quad +|h|^2\frac{C_\varepsilon(L,n,p)}{r^2}\left(\int_{B_{2r}}(\tilde{\mu}^2+|Dv(x)|)^p\,dx\right)^{\frac{p-2}{p}}\left(\int_{B_{2r}}|D(v-\psi)(x)|^p\,dx\right)^{\frac{2}{p}}\\ &\le \varepsilon\int_B\eta^2|\tau_h Dv|^2\,(\tilde{\mu}^2+|Dv(x+h)|^2+|Dv(x)|^2)^{\frac{p-2}{2}}\,dx\\ &\quad +|h|^2\frac{C_\varepsilon(L,n,p)}{r^2}\int_{B_{2r}}(\tilde{\mu}^p+|Dv(x)|^p)\,dx+|h|^2\frac{C_\varepsilon(L,n,p)}{r^2}\int_{B_{2r}}|D\psi(x)|^p\,dx,\end{aligned}\tag{4.11}$$

where we used also Young's inequality.
Inserting estimates (4.8), (4.10), (4.11) in (4.7), we infer the existence of a constant $C_\varepsilon\equiv C_\varepsilon(\varepsilon,L,n,p)$ such that

$$\begin{aligned}&\nu\int_B\eta^2|\tau_h Dv|^2(\tilde{\mu}^2+|Dv(x+h)|^2+|Dv(x)|^2|)^{\frac{p-2}{2}}\,dx\\ &\le 2\varepsilon\int_B\eta^2|\tau_h Dv|^2(\tilde{\mu}^2+|Dv(x+h)|^2+|Dv(x)|^2|)^{\frac{p-2}{2}}\,dx\end{aligned}$$

$$
\begin{aligned}
&+C_\varepsilon |h|^{2\kappa}\left([D\psi]_{B^\kappa_{p,\infty}}\right)^p + C_\varepsilon |h|^{2\kappa}\int_{B_{2r}} (\tilde{\mu}^p + |Dv(x)|^p)\,dx \\
&+|h|^2\frac{C_\varepsilon}{r^2}\int_{B_{2r}} (\tilde{\mu}^p + |Dv(x)|^p)\,dx + |h|^2\frac{C_\varepsilon}{r^2}\int_{B_{2r}} |D\psi(x)|^p\,dx.
\end{aligned}
$$

Choosing $\varepsilon = \frac{\nu}{4}$, we can reabsorb the first integral in the right hand side by the left hand side thus getting

$$
\begin{aligned}
&\int_B \eta^2 |\tau_h Dv|^2 (\tilde{\mu}^2 + |Dv(x+h)|^2 + |Dv(x)|^2|)^{\frac{p-2}{2}}\,dx \\
\leq\ & C\,|h|^{2\kappa}\left([D\psi]^p_{B^\kappa_{p,\infty}} + \frac{1}{r^2}\int_{B_{2r}} \left(\tilde{\mu}^p + |Dv|^p + |D\psi|^p\right)dx\right),
\end{aligned}
$$

with $C = C(\nu, L, n, p) > 0$. Using Lemma 2.1 in the left hand side of previous estimate and recalling that $\eta \equiv 1$ on $B_{\frac{r}{2}}$, we get

$$
\int_{B_{\frac{r}{2}}} |\tau_h V_p(Dv)|^2\,dx \leq C\,|h|^{2\kappa}\left([D\psi]^p_{B^\kappa_{p,\infty}} + \frac{1}{r^2}\int_{B_{2r}} \left(\tilde{\mu}^p + |Dv|^p + |D\psi|^p\right)dx\right), \tag{4.12}
$$

i.e. the conclusion.

□

Arguing similarly, but assuming that $D\psi \in W^{1,p}_{\mathrm{loc}}(\Omega)$ that, by virtue of Lemma 2.9, implies

$$
\int_{B_\rho} |\tau_h D\psi|^p \leq C \int_{B_R} |D^2\psi|^p,
$$

for every balls $B_\rho \subset B_R \Subset \Omega$, we have the following

Corollary 4.2. *Let $v \in u + W^{1,p}_0(B)$ be the solution to* (3.2) *under the assumptions* $(\mathcal{A}1) - (\mathcal{A}2)$*, where u is the solution of problem* (1.1)*. If*

$$
D\psi \in W^{1,p}(B), \tag{4.13}
$$

then

$$
V_p(Dv) \in W^{1,2}_{\mathrm{loc}}(B)
$$

and the following estimate

$$
\int_{B_{\frac{r}{2}}} |\tau_h V_p(Dv)|^2\,dx \leq C\,|h|^2\left(\int_{B_{2r}} |D^2\psi|^p\,dx + \frac{1}{r^2}\int_{B_{2r}} \left(\tilde{\mu}^p + |Dv|^p + |D\psi|^p\right)dx\right), \tag{4.14}
$$

holds for all balls $B_r \subset B_{2r} \Subset B$, with $C \equiv C(n,p,\nu,L)$.

5. Comparison

This section is devoted to the proof of a comparison estimate between u solution to the problem (1.1) and v solution to the problem (3.2), both introduced in Section 3.

Remark 5.1. Here we will take advantage from the higher integrability results that hold true, since by the assumption $D\psi \in B^\kappa_{p,\infty,\mathrm{loc}}(\Omega)$, the Sobolev imbedding Theorem of Lemma 2.4 implies that $D\psi \in L^{\frac{np}{n-p\mu}}$, for every $0 < \mu < \kappa$. Therefore both Du and Dv belong to some L^q, with $p < q < \frac{np}{n-p\mu}$.

Lemma 5.2. *Let u be a solution to* (1.1) *and $v \in u + W_0^{1,p}(B)$ be the solution to* (3.2)*, under the assumptions* (F1) – (F4)*. Then*

$$\int_B |V_p(Du) - V_p(Dv)|^2 \leq C\, R^\sigma \int_{2B} (1 + |Du|^q + |D\psi|^q)\, dx.$$

with $\sigma = \min\{\alpha, q-p\}$, where α is the exponent appearing in the assumption $(F4)$ and where q is the minimum of the two higher integrability exponents of Theorems 3.1 *and Theorem* 3.2*.*

Proof. Assumption (F2) and the minimality of v imply

$$\int_B |Dv|^p\, dx \leq C \int_B \tilde{F}(Dv) \leq C \int_B \tilde{F}(Du) \leq C \int_B (\tilde{\mu}^p + |Du|^p)\, dx$$

and so, recalling that $0 \leq \tilde{\mu} \leq 1$,

$$\int_B |Dv|^p\, dx \leq C \int_B (1 + |Du|^p)\, dx, \tag{5.1}$$

while the up-to-the-boundary higher integrability result Theorem 3.2 yields

$$\int_B |Dv|^q\, dx \leq\, C \int_{2B} (1 + |Du|^q + |D\psi|^q)\, dx, \tag{5.2}$$

where $C \equiv C(\tilde{\nu}, \tilde{L}, n, p)$. The inequality

$$\int_B |V_p(Du) - V_p(Dv)|^2\, dx \leq\, C \int_B \tilde{F}(Du) - \tilde{F}(Dv) - \langle \tilde{F}_\xi(Dv), Du - Dv\rangle\, dx$$

follows immediately from Taylor's expansion formula and assumption (F2); moreover, recalling inequality (3.3), i.e.

$$\int_B \langle \tilde{F}_\xi(Dv), Du - Dv\rangle\, dx \geq\, 0,$$

we have

$$\int_B |V_p(Du) - V_p(Dv)|^2\, dx \leq\, C \int_B \tilde{F}(Du) - \tilde{F}(Dv)\, dx.$$

Therefore, we can write

$$\begin{aligned}
\int_B |V_p(Du) - V_p(Dv)|^2\, dx \quad &\leq \quad C \int_B \tilde{F}(Du) - \tilde{F}(Dv)\, dx \\
&= \quad C \int_B [F(x_0, u_B, Du)\, dx - F(x_0, u_B, Dv)]\, dx \\
&= \quad C \int_B [F(x_0, u_B, Du)\, dx - F(x, u_B, Du)]\, dx \\
&\qquad + C \int_B [F(x, u_B, Du)\, dx - F(x, u, Du)]\, dx \\
&\qquad + C \int_B [F(x, u, Du)\, dx - F(x, v, Dv)]\, dx \\
&\qquad + C \int_B [F(x, v, Dv)\, dx - F(x, v_B, Dv)]\, dx \\
&\qquad + C \int_B [F(x, v_B, Dv)\, dx - F(x, u_B, Dv)]\, dx \\
&\qquad + C \int_B [F(x, u_B, Dv)\, dx - F(x_0, u_B, Dv)]\, dx
\end{aligned}$$

$$=: \quad C\,[J_1 + J_2 + J_3 + J_4 + J_5 + J_6].$$

The minimality of u, since $v \in \mathcal{K}_\psi(B)$, entails that

$$J_3 \le\ 0.$$

Now, by assumptions (F4), (1.3) and estimate (5.1), since $R \le 1$, $0 \le \tilde{\mu} \le 1$ and $2 \le p \le q$, we get

$$J_1 + J_6 \le\ C \int_B \omega(|x - x_0|)(1 + |Du|^p + |Dv|^p)\,dx \le\ C\,R^\alpha \int_B (1 + |Du|^p)\,dx,$$

where $C \equiv C(\tilde{\nu}, \tilde{L}_2, n, p)$.
At this point, from (F4), (1.3), Young's and Poincaré inequalities, it follows that

$$\begin{aligned}
J_2 \ &\le\ C \int_B \omega(|u - u_B|)(1 + |Du|^p)\,dx \\
&\le\ C\,R^\sigma \int_B \left[R^{-p-\sigma} \omega(|u - u_B|)^{\frac{p+\sigma}{\sigma}} + (1 + |Du|^p)^{\frac{p+\sigma}{p}} \right] dx \\
&\le\ C\,R^\sigma \int_B [R^{-p-\sigma} |u - u_B|^{p+\sigma} + (1 + |Du|^{p+\sigma})]\,dx \\
&\le\ C\,R^\sigma \int_B (1 + |Du|^q)\,dx,
\end{aligned}$$

where $C \equiv C(\tilde{\nu}, \tilde{L}_2, n, p, \sigma)$ and $\sigma = \min\{\alpha, q - p\}$.
Arguing exactly as before

$$\begin{aligned}
J_4 \ &\le\ C \int_B \omega(|v - v_B|)(1 + |Dv|^p)\,dx \\
&\le\ C\,R^\sigma \int_B [R^{-p-\sigma} |v - v_B|^{p+\sigma} + (1 + |Dv|^{p+\sigma})]\,dx \\
&\le\ C\,R^\sigma \int_B (1 + |Dv|^q)\,dx \\
&\le\ C\,R^\sigma \int_{2B} (1 + |Du|^q + |D\psi|^q)\,dx,
\end{aligned}$$

where in the last inequality we used (5.2). Finally, since $u = v$ on ∂B, we can use Poincaré inequality also for the function $u - v$, thus getting

$$\begin{aligned}
J_5 \ &\le\ C \int_B \omega(|u_B - v_B|)(1 + |Dv|^p)\,dx \\
&\le\ C\,R^\sigma \int_B \left[R^{-p-\sigma} \omega(|u_B - v_B|)^{\frac{p+\sigma}{\sigma}} + (1 + |Dv|^p)^{\frac{p+\sigma}{\sigma}} \right] dx \\
&\le\ C\,R^\sigma \int_B [R^{-p-\sigma}\,|u - v|^{p+\sigma} + (1 + |Dv|^{p+\sigma})]\,dx \\
&\le\ C\,R^\sigma \int_B (1 + |Du|^q + |Dv|^q)\,dx \\
&\le\ C\,R^\sigma \int_{2B} (1 + |Du|^q + |D\psi|^q)\,dx
\end{aligned}$$

where we still used (5.2) in the last inequality. Collecting the above inequalities, the proof is finished. □

6. The main result

More notation. Before proceeding with the proof of the main result, it is convenient to fix some further notation. For a ball $\mathcal{B} \Subset \Omega$ of radius R, we shall denote by $\mathcal{Q}_1 \equiv \mathcal{Q}_1(\mathcal{B})$ and $\mathcal{Q}_2 \equiv \mathcal{Q}_2(\mathcal{B})$ the largest and the smallest cubes, concentric to $\mathcal{B}$ and with sides parallel to the coordinate axes, contained in $\mathcal{B}$ and containing $\mathcal{B}$ respectively. Clearly $|\mathcal{B}| \approx |\mathcal{Q}_1| \approx |\mathcal{Q}_2| \approx R^n$. We refer to $\mathcal{Q}_1(\mathcal{B})$ and $\mathcal{Q}_2(\mathcal{B})$ as the *inner* and the *outer* cubes of $\mathcal{B}$ respectively. We also denote the enlarged ball by $\hat{\mathcal{B}} \equiv 4\mathcal{B}$. Consistent with this notation, we set

$$\mathcal{Q}_1 \equiv \mathcal{Q}_1(\mathcal{B}), \qquad \hat{\mathcal{Q}}_2 \equiv \mathcal{Q}_2(\hat{\mathcal{B}})$$

so that we have the following chain of inclusions

$$\mathcal{Q}_1 \subset \mathcal{B} \Subset 2\mathcal{B} \Subset Q_1(\hat{\mathcal{B}}) \subset \hat{\mathcal{B}} \subset \hat{\mathcal{Q}}_2. \tag{6.1}$$

In the sequel, we shall always take $\mathcal{B}$ such that $\mathcal{Q}_2(\hat{\mathcal{B}}) \Subset \Omega$.
The proof of our main result is achieved with a booth-strapping argument. The first step consists in showing that the differentiability assumption on the gradient of the obstacle transfers in a extra fractional differentiability for the gradient of the minimizer. More precisely we have the following

Theorem 6.1. *Let u be a solution to* (1.1) *under the assumptions* $(F1)$ – $(F4)$*, and assume that*

$$D\psi \in B^{\kappa}_{p,\infty,\mathrm{loc}}(\Omega). \tag{6.2}$$

Then

$$V_p(Du) \in B^{\frac{\kappa\sigma}{2+\sigma}}_{2,\infty}, \text{ locally in } \Omega,$$

with $\sigma := \min\{\alpha, q-p\}$ where α is the exponent appearing in the assumption $(F4)$ *and q is the minimum of the two higher integrability exponents of Theorems 3.1 and Theorem 3.2.*

Proof. Let us fix arbitrary open subsets $\Omega' \Subset \Omega'' \Subset \Omega$ and choose $x_0 \in \Omega'$. Let $\beta \in (0,\kappa)$ be chosen later and consider the ball $\mathcal{B} \equiv B(h) \equiv B(x_0, |h|^\beta)$ with $|h|$ is sufficiently small, depending on the dimension n, the parameter β and the distance between Ω' and the boundary of Ω'' such that, with the notations in (6.1), $\hat{\mathcal{Q}}_2 \Subset \Omega''$.
Let v be the solution to problem (3.2) where this time the ball B is replaced by our choice of the ball $\hat{\mathcal{B}}$.
We estimate the difference quotient for $V_p(Du)$ as follows

$$\begin{aligned}
\int_{\mathcal{B}} |\tau_h V_p(Du)|^2\,dx &= \int_{\mathcal{B}} |V_p(Du(x+h)) - V_p(Du(x))|^2\,dx \\
&\leq C\int_{\mathcal{B}} |V_p(Du(x+h)) - V_p(Dv(x+h))|^2\,dx \\
&\quad +C\int_{\mathcal{B}} |V_p(Dv(x+h)) - V_p(Dv(x))|^2\,dx \\
&\quad +C\int_{\mathcal{B}} |V_p(Dv(x)) - V_p(Du(x))|^2\,dx \\
&\leq C\int_{\mathcal{B}} |V_p(Du) - V_p(Dv)|^2\,dx + C\int_{\mathcal{B}} |\tau_h V_p(Dv)|^2\,dx.
\end{aligned} \tag{6.3}$$

Estimate (4.12) applied over the ball $\mathcal{B}$ implies

$$\int_{\mathcal{B}} |\tau_h V_p(Dv)|^2\,dx \leq C\,|h|^{2\kappa}\left[D\psi\right]^p_{B^{\kappa}_{p,\infty}(\hat{\mathcal{B}})} + C\,|h|^{2\kappa-2\beta}\int_{\hat{\mathcal{B}}}\left(\tilde{\mu}^p + |Dv|^p + |D\psi|^p\right)dx, \tag{6.4}$$

recalling that the radius of the ball $\mathcal{B}$ is proportional to $|h|^\beta$. Now Lemma 5.2, applied over the ball $\hat{\mathcal{B}}$, implies that

$$\int_{\hat{\mathcal{B}}} |V_p(Du) - V_p(Dv)|^2\, dx \le C\, |h|^{\beta\sigma} \int_{\hat{\mathcal{Q}}_2} (1 + |Du|^q + |D\psi|^q)\, dx, \tag{6.5}$$

where we used that the radius of $\hat{\mathcal{B}}$ is proportional to $|h|^\beta$. Inserting (6.4) and (6.5) in (6.3) we get

$$\begin{aligned}
\int_{\mathcal{Q}_1} |\tau_h V_p(Du)|^2\, dx \quad \le \quad & C\, |h|^{\beta\sigma} \int_{\hat{\mathcal{Q}}_2} (1 + |Du|^q + |D\psi|^q)\, dx + C\, |h|^{2\kappa} \left[D\psi\right]^p_{B^\kappa_{p,\infty}(\hat{\mathcal{Q}}_2)} \\
& + C\, |h|^{2\kappa - 2\beta} \int_{\hat{\mathcal{Q}}_2} \left(\tilde{\mu}^p + |Dv|^p + |D\psi|^p\right) dx \\
\le \quad & C\, |h|^{\beta\sigma} \int_{\hat{\mathcal{Q}}_2} (1 + |Du|^q + |D\psi|^q)\, dx + C\, |h|^{2\kappa} \left[D\psi\right]^p_{B^\kappa_{p,\infty}(\hat{\mathcal{Q}}_2)} \\
& + C\, |h|^{2\kappa - 2\beta} \int_{\hat{\mathcal{Q}}_2} \left(1 + |Dv|^p\right) dx + C|h|^{2\kappa - 2\beta + p\mu} \left(\int_{\hat{\mathcal{Q}}_2} |D\psi|^{\frac{np}{n-\mu p}}\, dx\right)^{\frac{n-\mu p}{n}} \\
\le \quad & C\, |h|^{\beta\sigma} \int_{\hat{\mathcal{Q}}_2} (1 + |Du|^q + |D\psi|^q)\, dx + C\, |h|^{2\kappa}\, ||D\psi||^p_{B^\kappa_{p,\infty}(\hat{\mathcal{Q}}_2)} \\
& + C\, |h|^{2\kappa - 2\beta} \int_{\hat{\mathcal{Q}}_2} \left(1 + |Dv|^p\right) dx,
\end{aligned}$$

where we used Sobolev embedding Theorem at Lemma 2.2 with $0 < \mu < \kappa$, that $p \ge 2$ and that $|h| < 1$. Using (5.1) in the last integral of previous estimate, we get

$$\begin{aligned}
\int_{\mathcal{Q}_1} |\tau_h V_p(Du)|^2\, dx \le\; & C\, |h|^{\beta\sigma} \int_{\hat{\mathcal{Q}}_2} (1 + |Du|^q + |D\psi|^q)\, dx \\
& + C\, |h|^{2\kappa}\, ||D\psi||^p_{B^\kappa_{p,\infty}(\hat{\mathcal{Q}}_2)} + C\, |h|^{2\kappa - 2\beta} \int_{\hat{\mathcal{Q}}_2} \left(1 + |Du|^p\right) dx,
\end{aligned} \tag{6.6}$$

Now we choose β in order to minimize the right hand side of the previous estimate. One can easily check that the best possible estimate is given by the choice

$$\beta = \frac{2\kappa}{2 + \sigma} \in (0, \kappa).$$

With such a choice estimate (6.6) becomes

$$\int_{\mathcal{Q}_1} |\tau_h V_p(Du)|^2\, dx \le C\, |h|^{\frac{2\kappa\sigma}{2+\sigma}} \left[\int_{\hat{\mathcal{Q}}_2} (1 + |Du|^q + |D\psi|^q)\, dx + ||D\psi||^p_{B^\kappa_{p,\infty}(\hat{\mathcal{Q}}_2)}\right]. \tag{6.7}$$

At this point, arguing as in Lemma 4.5 in [31] a covering argument allows us to replace the cubes $\mathcal{Q}_1$ and $\hat{\mathcal{Q}}_2$ with the fixed open subsets Ω' and Ω'', respectively. Indeed for each $h \in \mathbb{R}$ sufficiently small we can find balls $\mathcal{B}_1 \equiv \mathcal{B}_1(x_1, |h|^\beta), \dots, \mathcal{B}_K \equiv \mathcal{B}_K(x_K, |h|^\beta)$, being $K \equiv K(h) \in \mathbb{N}$, such that the corresponding inner cubes $\mathcal{Q}_1(\mathcal{B}_1), \dots, \mathcal{Q}_1(\mathcal{B}_K)$ are disjoint and satisfy

$$\left|\Omega' \setminus \bigcup_{k=1}^{K} \mathcal{Q}_1(\mathcal{B}_k)\right| = 0.$$

By assumption we have that $\mathcal{Q}_2(\hat{\mathcal{B}}_k) \subset \Omega''$, for every $k \le K$, and each of the dilated outer cubes $\mathcal{Q}_2(\hat{\mathcal{B}}_k)$ intersects at most $(16\sqrt{n})$ of the other dilated cubes $\mathcal{Q}_2(\hat{\mathcal{B}}_j)$, with $j \ne k$. Hence, after summing up (6.7) over the inner cubes $\mathcal{Q}_1 \in \{\mathcal{Q}_1(\mathcal{B}_1), \dots, \mathcal{Q}_1(\mathcal{B}_K)\}$, and enlarging the constant by a fixed factor only depending on n (in particular independent of h), we arrive at

$$\int_{\Omega'} |\tau_h V_p(Du)|^2 \, dx \le C\, |h|^{\frac{2\kappa\sigma}{2+\sigma}} \left[\int_{\Omega''} (1 + |Du|^q + |D\psi|^q) \, dx + ||D\psi||^p_{B^\kappa_{p,\infty}(\Omega'')} \right]. \tag{6.8}$$

Since the right hand side is finite by our assumptions, we conclude that $V_p(Du) \in B^{\frac{\kappa\sigma}{2+\sigma}}_{2,\infty}(\Omega')$. The result follows by the arbitrariness of Ω'. □

The Besov regularity of Du established in previous Theorem allows us to start the booth strapping argument, since the right hand side of (6.8) depends on the integrability of Du. More precisely, we have

Proof of Theorem 1.1. For $t \in (0, \kappa)$, let us consider the function

$$A(t) = \frac{\kappa\sigma}{2\left(1 - \frac{t}{\kappa}\right) + \sigma}$$

One can easily check that $A(t)$ is increasing and that

$$t < A(t) < \frac{\kappa\sigma}{2} \qquad \forall t \in \left(0, \frac{\kappa\sigma}{2}\right)$$
$$A\left(\frac{\kappa\sigma}{2}\right) = \frac{\kappa\sigma}{2}.$$

Therefore the sequence

$$\vartheta_0 = \frac{\kappa\sigma}{2+\sigma} \qquad \vartheta_j = A(\vartheta_{j-1})$$

is increasing and

$$\vartheta_j \nearrow \frac{\kappa\sigma}{2}. \tag{6.9}$$

Now consider the sequence

$$\gamma_0 = \vartheta_0 \qquad \gamma_j = \frac{\vartheta_j + A(\gamma_{j-1})}{2}$$

From (6.9), it follows that

$$\lim_j \gamma_j = \frac{\kappa\sigma}{2}. \tag{6.10}$$

Arguing by induction, we shall prove that

$$V_p(Du) \in B^{\gamma_j}_{2,\infty,\mathrm{loc}}(\Omega) \ \text{ for every } j \in \mathbb{N}.$$

The case $j = 0$ follows from our choice of $\gamma_0 = \frac{\kappa\sigma}{\sigma+2}$ and Theorem 6.1. Now, let us prove the implication

$$V_p(Du) \in B^{\gamma_{j-1}}_{2,\infty,\mathrm{loc}}(\Omega) \implies V_p(Du) \in B^{\gamma_j}_{2,\infty,\mathrm{loc}}(\Omega). \tag{6.11}$$

By virtue of Lemma 2.4, the assumption $V_p(Du) \in B^{\gamma_{j-1}}_{2,\infty,\mathrm{loc}}(\Omega)$ implies $V_p(Du) \in L^{\frac{2n}{n-2\lambda}}(\hat{\mathcal{Q}}_2)$, for every $\lambda < \gamma_{j-1}$ and so $Du \in L^{\frac{np}{n-2\lambda}}(\hat{\mathcal{Q}}_2)$. Therefore, we can use Hölder's inequality in estimate (6.6) thus getting

$$\begin{aligned} \int_{\mathcal{Q}_1} |\tau_h V_p(Du)|^2 \, dx \le \ & |h|^{\beta\sigma} \int_{\hat{\mathcal{Q}}_2} (1 + |Du|^q + |D\psi|^q) \, dx \\ & + C\, |h|^{2\kappa}\, ||D\psi||^p_{B^\kappa_{p,\infty}(\hat{\mathcal{Q}}_2)} + C\, |h|^{2\kappa - 2\beta}\, |\hat{\mathcal{Q}}_2|^{\frac{2\lambda}{n}} \left(\int_{\hat{\mathcal{Q}}_2} \left(1 + |Du|^{\frac{np}{n-2\lambda}}\right) dx \right)^{\frac{n-2\lambda}{n}} \\ \le \ \ & C\, |h|^{\beta\sigma} \int_{\hat{\mathcal{Q}}_2} (1 + |Du|^q + |D\psi|^q) \, dx + C\, |h|^{2\kappa}\, ||D\psi||^p_{B^\kappa_{p,\infty}(\hat{\mathcal{Q}}_2)} \end{aligned}$$

$$+C\,|h|^{2\kappa-2\beta+2\beta\lambda}\left(\int_{\hat{\mathcal{Q}}_2}\left(1+|Du|^{\frac{np}{n-2\lambda}}\right)dx\right)^{\frac{n-2\lambda}{n}}, \tag{6.12}$$

where we used the fact that the radius of the ball $\hat{\mathcal{B}}$ is $|h|^{\beta}$ and so also $|\hat{\mathcal{Q}}_1|\approx|\hat{\mathcal{Q}}_2|\approx|h|^{\beta n}$. Therefore, choosing β in order to maximize the right hand side of (6.12), i.e.

$$\beta=\frac{2\kappa}{\sigma+2(1-\lambda)}$$

we have

$$\int_{\mathcal{Q}_1}|\tau_h V_p(Du)|^2\,dx\le C\,|h|^{\frac{2\kappa\sigma}{\sigma+2(1-\lambda)}}\left[\int_{\hat{\mathcal{Q}}_2}(1+|Du|^q+|D\psi|^q)\,dx+||D\psi||^p_{B^{\kappa}_{p,\infty}(\hat{\mathcal{Q}}_2)}\right]$$

and so, again through a covering argument, we can conclude that

$$V_p(Du)\in B_{2,\infty}^{\frac{\kappa\sigma}{\sigma+2(1-\lambda)}}(\hat{\mathcal{Q}}_2)=B^{A(\lambda)}_{2,\infty}(\hat{\mathcal{Q}}_2).$$

So we proved the implication

$$V_p(Du)\in B^{\gamma_{j-1}}_{2,\infty,\mathrm{loc}}(\Omega)\implies V_p(Du)\in B^{A(\lambda)}_{2,\infty,\mathrm{loc}}(\Omega), \tag{6.13}$$

for every $\lambda<\gamma_{j-1}$. On the other hand, since $A(t)$ is increasing and

$$\vartheta_j<\gamma_j<\frac{\kappa\sigma}{2},$$

we have that

$$\vartheta_j<A(\gamma_{j-1}).$$

Then the definition of γ_j implies that

$$\gamma_j<A(\gamma_{j-1})<A(\lambda).$$

Therefore (6.11) follows by (6.13) and (b) of Lemma 2.3. Moreover, we have the estimate

$$\int_{\mathcal{Q}_1}|\tau_h V_p(Du)|^2\,dx\le C\,|h|^{2t}\left[\int_{\hat{\mathcal{Q}}_2}(1+|Du|^q+|D\psi|^q)\,dx+||D\psi||^p_{B^{\kappa}_{p,\infty}(\hat{\mathcal{Q}}_2)}\right]\qquad\text{for all } t\in\left(0,\frac{\kappa\sigma}{2}\right)$$

Dividing previous estimate by $|\hat{\mathcal{Q}}_2|$ we obtain

$$\fint_{\mathcal{Q}_1}|\tau_h V_p(Du)|^2\,dx\le C\,|h|^{2t}\left[\fint_{\hat{\mathcal{Q}}_2}(1+|Du|^q+|D\psi|^q)\,dx+\frac{1}{|\hat{\mathcal{Q}}_2|}||D\psi||^p_{B^{\kappa}_{p,\infty}(\hat{\mathcal{Q}}_2)}\right]\qquad\text{for all } t\in\left(0,\frac{\kappa\sigma}{2}\right).$$

The higher integrability estimate (3.4) implies that

$$\fint_{\hat{\mathcal{Q}}_2}|Du|^q\,dx\le C\left(\fint_{2\hat{\mathcal{Q}}_2}|Du|^p\,dx\right)^{\frac{q}{p}}+C\left(\fint_{2\hat{\mathcal{Q}}_2}|D\psi|^{p^*_\mu}\,dx\right)^{\frac{q}{p^*_\mu}},$$

with $0<\mu<\kappa$, and so, by the Sobolev imbedding Theorem at Lemma 2.4 and Remark 5.1,

$$\begin{aligned}\fint_{\hat{\mathcal{Q}}_2}\left(|Du|^q+|D\psi|^q\right)dx &\le C\left(\fint_{2\hat{\mathcal{Q}}_2}|Du|^p\,dx\right)^{\frac{q}{p}}+C\left(\fint_{2\hat{\mathcal{Q}}_2}|D\psi|^{p^*_\mu}\,dx\right)^{\frac{q}{p^*_\mu}}\\ &\le C\left(\fint_{2\hat{\mathcal{Q}}_2}|Du|^p\,dx\right)^{\frac{q}{p}}+C\left(\fint_{4\hat{\mathcal{Q}}_2}|D\psi|^p\,dx\right)^{\frac{q}{p}}+C\left(\frac{1}{|\hat{\mathcal{Q}}_2|}[D\psi]_{B^{\kappa}_{p,\infty}(4\hat{\mathcal{Q}}_2)}\right)^{\frac{q}{p}}\\ &\le C\left(\fint_{2\hat{\mathcal{Q}}_2}|Du|^p\,dx\right)^{\frac{q}{p}}+C\left(\frac{1}{|\hat{\mathcal{Q}}_2|}||D\psi||_{B^{\kappa}_{p,\infty}(4\hat{\mathcal{Q}}_2)}\right)^{\frac{q}{p}}\end{aligned}$$

Hence, we obtain

$$\fint_{\mathcal{Q}_1} |\tau_h V_p(Du)|^2\,dx \le C\,|h|^{2t}\left(1+\fint_{4\hat{\mathcal{Q}}_2}|Du|^p\,dx+\frac{1}{|\hat{\mathcal{Q}}_2|}||D\psi||_{B^{\kappa}_{p,\infty}(4\hat{\mathcal{Q}}_2)}\right)^{\frac{q}{p}},$$

i.e. the conclusion, again by a covering argument. □

We are now in position to give the

Proof of Corollary 1.2. Again as in the proof of Theorem 1.1, let us fix arbitrary open subsets $\Omega' \Subset \Omega'' \Subset \Omega$ and choose $x_0 \in \Omega'$. According to the notations introduced in Section 6, consider the ball $\mathcal{B} \equiv B(h) \equiv B(x_0, |h|^\beta)$ with $|h|$ is sufficiently small, depending on the dimension n, the parameter $\beta \in (0,1)$ (to be chosen later), and the distance between Ω' and the boundary of Ω'' such that, with the notations in (6.1), $\hat{\mathcal{Q}}_2 \Subset \Omega''$.
Let v be the solution to problem (3.2) with the ball B is replaced by our choice of the ball $\hat{\mathcal{B}}$. Estimate (4.14) of Corollary 4.2 applied over the ball $\mathcal{B}$ implies

$$\int_{\mathcal{B}} |\tau_h V_p(Dv)|^2\,dx \quad\le\quad C\,|h|^2\int_{\hat{\mathcal{B}}}|D^2\psi|^p\,dx + C\,|h|^{2-2\beta}\int_{\hat{\mathcal{B}}}\left(\tilde{\mu}^p+|Dv|^p+|D\psi|^p\right)dx, \quad (6.14)$$

while Lemma 5.2, applied over the ball $\hat{\mathcal{B}}$, yields

$$\int_{\hat{\mathcal{B}}} |V_p(Du)-V_p(Dv)|^2\,dx \le C\,|h|^{\beta\sigma}\int_{\hat{\mathcal{Q}}_2}(1+|Du|^q+|D\psi|^q)\,dx, \quad (6.15)$$

with $\sigma := \min\{\alpha, q-p\}$ where α is the exponent appearing in the assumption $(F4)$ and where this time $q < p^* = \frac{np}{n-p}$, being q the minimum of the two higher integrability exponents of Theorems 3.1 and Theorem 3.2. Thus we get

$$\begin{aligned}
\int_{\mathcal{Q}_1} |\tau_h V_p(Du)|^2\,dx \quad&\le\quad C\,|h|^{\beta\sigma}\int_{\hat{\mathcal{Q}}_2}(1+|Du|^q+|D\psi|^q)\,dx + C\,|h|^2\int_{\hat{\mathcal{Q}}_2}|D^2\psi|^p\,dx\\
&\quad +C\,|h|^{2-2\beta}\int_{\hat{\mathcal{Q}}_2}\left(\mu^p+|Dv|^p+|D\psi|^p\right)dx\\
&\le\quad C\,|h|^{\beta\sigma}\int_{\hat{\mathcal{Q}}_2}(1+|Du|^q+|D\psi|^q)\,dx + C\,|h|^2\int_{\hat{\mathcal{Q}}_2}|D^2\psi|^p\,dx\\
&\quad +C\,|h|^{2-2\beta}\int_{\hat{\mathcal{Q}}_2}\left(1+|Dv|^p\right)dx + C|h|^{2-2\beta+p\beta}\left(\int_{\hat{\mathcal{Q}}_2}|D\psi|^{\frac{np}{n-p}}\,dx\right)^{\frac{n-p}{n}}\\
&\le\quad C\,|h|^{\beta\sigma}\int_{\hat{\mathcal{Q}}_2}(1+|Du|^q+|D\psi|^q)\,dx + C\,|h|^2\int_{\hat{\mathcal{Q}}_2}\left(|D\psi|^p+|D^2\psi|^p\right)dx\\
&\quad +C\,|h|^{2-2\beta}\int_{\hat{\mathcal{Q}}_2}\left(1+|Du|^p\right)dx,
\end{aligned}$$

where we used Sobolev imbedding Theorem, the fact that $p \ge 2$, the fact that $|h| < 1$ and, in the last integral (5.1). Choosing β in order to minimize the right hand side of the previous estimate we get

$$\int_{\mathcal{Q}_1} |\tau_h V_p(Du)|^2\,dx \le C\,|h|^{\frac{2\sigma}{2+\sigma}}\left[\int_{\hat{\mathcal{Q}}_2}(1+|Du|^q)\,dx + \int_{\hat{\mathcal{Q}}_2}\left(|D\psi|^p+|D^2\psi|^p\right)dx\right].$$

By a convering argument we are able to prove that

$$V_p(Du) \in B^{\frac{\sigma}{2+\sigma}}_{2,\infty}, \text{ locally in } \Omega,$$

At this point we proceed arguing in a similar way as in the proof of Theorem 1.1: by induction it is possible to prove that

$$V_p(Du) \in B^{\gamma_j}_{2,\infty,\mathrm{loc}}(\Omega) \quad \text{for every } j \in \mathbb{N},$$

where this time

$$\vartheta_0 = \frac{\sigma}{2+\sigma} \qquad \vartheta_j = A(\vartheta_{j-1}) \qquad A(t) = \frac{\sigma}{2(1-t)+\sigma}.$$

By virtue of Lemmas 2.2 and 2.3, the assumption $V_p(Du) \in B^{\gamma_{j-1}}_{2,\infty,\mathrm{loc}}(\Omega)$ implies $V_p(Du) \in L^{\frac{2n}{n-2\lambda}}(\hat{\mathcal{Q}}_2)$, for every $\lambda < \gamma_{j-1}$ and so $Du \in L^{\frac{np}{n-2\lambda}}(\mathcal{B})$. Therefore, we can use Hölder's inequality to obtain

$$\begin{aligned}\int_{\mathcal{Q}_1} |\tau_h V_p(Du)|^2\,dx \leq\; & C\,|h|^{\beta\sigma} \int_{\hat{\mathcal{Q}}_2} (1+|Du|^q+|D\psi|^q)\,dx \\ & + C\,|h|^2 \int_{\hat{\mathcal{Q}}_2} \left(|D\psi|^p + |D^2\psi|^p\right) dx + C\,|h|^{2-2\beta} |\hat{\mathcal{Q}}_2|^{\frac{2\lambda}{n}} \left(\int_{\hat{\mathcal{Q}}_2} \left(1+|Du|^{\frac{np}{n-2\lambda}}\right) dx\right)^{\frac{n-2\lambda}{n}} \\ \leq\;& C\,|h|^{\beta\sigma} \int_{\hat{\mathcal{Q}}_2} (1+|Du|^q+|D\psi|^q)\,dx + C\,|h|^2 \int_{\hat{\mathcal{Q}}_2} \left(|D\psi|^p + |D^2\psi|^p\right) dx \\ & + C\,|h|^{2-2\beta+2\beta\lambda} \left(\int_{\hat{\mathcal{Q}}_2} \left(1+|Du|^{\frac{np}{n-2\lambda}}\right) dx\right)^{\frac{n-2\lambda}{n}},\end{aligned}$$

where we used the fact that the radius of the ball $\hat{\mathcal{B}}$ is $|h|^\beta$. Therefore, choosing β in order to maximize the right hand side of the previous estimate we have

$$\int_{\mathcal{Q}_1} |\tau_h V_p(Du)|^2\,dx \leq C\,|h|^{\frac{2\sigma}{\sigma+2(1-\lambda)}} \left[\int_{\hat{\mathcal{Q}}_2} (1+|Du|^q)\,dx + \int_{\hat{\mathcal{Q}}_2} \left(|D\psi|^p + |D^2\psi|^p\right) dx\right]$$

and so, again through a covering argument, we can conclude that

$$V_p(Du) \in B^{\frac{\sigma}{\sigma+2(1-\lambda)}}_{2,\infty}(\hat{\mathcal{Q}}_2) = B^{A(\lambda)}_{2,\infty}(\hat{\mathcal{Q}}_2).$$

So we proved the implication

$$V_p(Du) \in B^{\gamma_{j-1}}_{2,\infty,\mathrm{loc}}(\Omega) \implies V_p(Du) \in B^{A(\lambda)}_{2,\infty,\mathrm{loc}}(\Omega),$$

for every $\lambda < \gamma_{j-1}$; this, together with (b) of Lemma 2.3 and the fact that, by our assumptions

$$\gamma_j < A(\gamma_{j-1}) < A(\kappa),$$

yields the desired conclusion, together with the estimate

$$\int_{\mathcal{Q}_1} |\tau_h V_p(Du)|^2\,dx \leq C\,|h|^{2t} \left[\int_{\hat{\mathcal{Q}}_2} (1+|Du|^q+|D\psi|^p+|D^2\psi|^p)\,dx\right] \qquad \text{for all } t \in \left(0, \frac{\sigma}{2}\right).$$

Dividing previous estimate by $|\hat{\mathcal{B}}|$ we obtain

$$\fint_{\mathcal{Q}_1} |\tau_h V_p(Du)|^2\,dx \leq C\,|h|^{2t} \left[\fint_{\hat{\mathcal{Q}}_2} (1+|Du|^q+|D\psi|^p+|D^2\psi|^p\right] \qquad \text{for all } t \in \left(0, \frac{\sigma}{2}\right).$$

The higher integrability estimate (3.4) implies that

$$\fint_{\mathcal{Q}_1} |Du|^q\,dx \leq C \left(\fint_{\hat{\mathcal{Q}}_2} |Du|^p\,dx\right)^{\frac{q}{p}} + C \left(\fint_{\hat{\mathcal{Q}}_2} |D\psi|^{p^*}\,dx\right)^{\frac{q}{p^*}},$$

so, by the Sobolev imbedding Theorem, we finally obtain

$$\fint_{\mathcal{Q}_1} |\tau_h V_p(Du)|^2\,dx \leq C\,|h|^{2t} \left(1 + \fint_{4\hat{\mathcal{Q}}_2} |Du|^p\,dx + |D\psi|^p + |D^2\psi|^p\right)^{\frac{q}{p}},$$

which gives the desired conclusion.

□

References

[1] A.L. Baisón, A. Clop, R. Giova, J. Orobitg, A. Passarelli di Napoli: *Fractional differentiability for solutions of nonlinear elliptic equations,* Potential Anal., **46**, (3), (2017), 403-430.

[2] P. Baroni: *Lorentz estimates for obstacle parabolic problems*, Nonlinear Anal., **96**, (2014), 167-188.

[3] M. Bildhauer, M. Fuchs, G. Mingione: *A priori gradient bounds and local $\mathcal{C}^{1,\alpha}$-estimates for (double) obstacle problems under non-standard growth conditions*, Z. Anal. Anwendungen, **20**, (4), (2001), 959-985.

[4] V. Bögelein, F. Duzaar, G. Mingione: *Degenerate problems with irregular obstacles*, J. Reine Angew. Math., **650**, (2011), 107-160.

[5] V. Bögelein, T. Lukkari, C. Scheven: *The obstacle problem for the porous medium equation*, Math. Ann. **363**, (1) (2015), 455-499.

[6] H. Brézis, D. Kinderlehrer: *The smoothness of solutions to nonlinear variational inequalities*, Indiana Univ. Math. J., **23** (1973-1974), 831-844.

[7] S.S. Byun, Y. Cho, J. Ok: *Global gradient estimates for nonlinear obstacle problem with non standard growth,* Forum Math., **28**, (4), (2016), 729-747.

[8] S.S. Byun, Y. Cho, L. Wang: *Calderón-Zygmund theory for nonlinear elliptic problems with irregular obstacles*, J. Funct. Anal., **263**, (10), (2012), 3117-3143.

[9] L. Caffarelli: *The regularity of elliptic and parabolic free boundaries*, Bull. Amer. Math. Soc., **82**, (1976), 616-618.

[10] L.A. Caffarelli, D. Kinderlehrer: *Potential methods in variational inequalities*, J. Anal. Math., **37** (1980), 285-295.

[11] H. Choe: *A regularity theory for a general class of quasilinear elliptic partial differential equations and obstacle problems*, Arch. Rational Mech. Anal., **114**, (4), (1991), 383-394.

[12] H. Choe, J.-L. Lewis: *On the obstacle problem for quasilinear elliptic equations of p Laplacian type*, SIAM J. Math. Anal., **22**, (3), (1991), 623-638.

[13] A. Clop, R. Giova, A. Passarelli di Napoli: *Besov regularity for solutions of p-harmonic equations,* Adv. in Nonlinear Anal., (to appear).

[14] M. Eleuteri: *Regularity results for a class of obstacle problems*, Appl. Math., **52**, (2), (2007), 137-169.

[15] M. Eleuteri, J. Habermann: *Regularity results for a class of obstacle problems under non standard growth conditions*, J. Math. Anal. Appl., **344**, (2), (2008), 1120-1142.

[16] M. Eleuteri, J. Habermann: *Calderón-Zygmund type estimates for a class of obstacle problems with $p(x)$ growth*, J. Math. Anal. Appl., **372**, (1), (2010), 140-161.

[17] M. Eleuteri, J. Habermann: *A Hölder continuity result for a class of obstacle problems under non standard growth conditions*, Math. Nachr., **284**, No. 11-12, (2011), 1404-1434.

[18] M. Eleuteri, P. Harjulehto, T. Lukkari: *Global regularity and stability of solutions to obstacle problems with nonstandard growth*, Rev. Mat. Complut., **26**, No. 1, (2013), 147-181.

[19] M. Eleuteri, A. Passarelli di Napoli: *Higher differentiability for solutions to a class of obstacle problems*, Calc. Var. , **57**, 115 (2018).

[20] G. Fichera: *Problemi elastostatici con vincoli unilaterali: il problema di Signorini con ambigue condizioni al contorno,* Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Nat. Sez. Ia, **7**, (8), (1963-1964), 91-140.

[21] A. Friedman: *Variational principles and free boundary problems*, Wiley, New York, 1982.

[22] M. Fuchs, G. Mingione: *Full $\mathcal{C}^{1,\alpha}-$regularity for free and constrained local minimizers of elliptic variational integrals with nearly linear growth*, Manuscripta Math., **102**, (2000), 227-250.

[23] R. Giova: *Higher differentiability for n-harmonic systems with Sobolev coefficients*, J. Diff. Equ., **259**, (11), (2015), 5667-5687.

[24] R. Giova, A. Passarelli di Napoli: *Regularity results for a priori bounded minimizers of non-autonomous functionals with discontinuous coefficients*, Adv. Calc. Var., (to appear). DOI: https://doi.org/10.1515/acv-2016-0059

[25] E. Giusti: *Direct methods in the calculus of variations.* World scientific publishing Co. (2003).

[26] P. Hajłasz: *Sobolev spaces on an arbitrary metric space*, Potential Anal., **5**, (4), (1996), 403-415.

[27] D. Haroske: *Envelopes and Sharp Embeddings of Function Spaces*, Chapman and Hall CRC (2006).

[28] D. Kinderlehrer, G. Stampacchia: *An introduction to variational inequalities and their applications*, Academic Press, NY (1980).

[29] P. Koskela, D. Yang, Y. Zhou: *Pointwise characterizations of Besov and Triebel- Lizorkin spaces and quasiconformal mappings*, Advances in Mathematics **226** (2011), n.4, 3579-3621.
[30] J. Kristensen, G. Mingione: *The singular set of minima of integral functionals*, Archive Rational Mech. Anal., **180**, (2006), 331-398.
[31] J. Kristensen, G. Mingione: *Boundary regularity in variational problems*, Archive Rational Mech. Anal., **198**, (2010), 369-455.
[32] J.L. Lions, G. Stampacchia: *Variational inequalities*, Comm. Pure Appl. Math., **20**, (1967), 493-519.
[33] J.H. Michael, W.P. Ziemer: *Interior regularity for solutions to obstacle problems*, Nonlinear Anal., **10**, (12), (1986), 1427-1448.
[34] J. Ok: *Regularity results for a class of obstacle problems with nonstandard growth*, J. Math. Anal. Appl., **444**, (2), (2016), 957-979.
[35] J. Ok: *Calderón-Zygmund estimates for a class of obstacle problems with nonstandard growth*, NoDEA, Nonlinear Diff. Equ. Appl., **23**, (4), (2016).
[36] J. Ok: *Gradient continuity for nonlinear obstacle problems*, Mediterr. J. Math., **14**, (1), (2017).
[37] A. Passarelli di Napoli: *Higher differentiability of minimizers of variational integrals with Sobolev coefficients*, Adv. Calc. Var., **7**, (1), (2014), 59-89.
[38] A. Passarelli di Napoli: *Higher differentiability of solutions of elliptic systems with Sobolev coefficients: the case $p = n = 2$*, Pot. Anal., **41**, (3), (2014), 715-735.
[39] A. Passarelli di Napoli: *Regularity results for non-autonomous variational integrals with discontinuous coefficients*, Atti Accad. Naz. Lincei, Rend. Lincei Mat. Appl., **26**, (2015), (4), 475-496.
[40] G. Stampacchia: *Formes bilineaires coercivitives sur les ensembles convexes,* C.R. Ac. Sci. Paris, **258**, (1964), 4413-4416.

Email address: `michela.eleuteri@unimore.it`
Email address: `antonia.passarellidinapoli@unina.it`

Dipartimento di Scienze Fisiche, Informatiche e Matematiche, Università degli Studi di Modena E Reggio Emilia, via Campi 213/b, 41125 Modena, Italy

Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università degli Studi di Napoli "Federico II", Via Cintia, 80126, Napoli (Italy)