REGULARITY FOR SCALAR INTEGRALS WITHOUT STRUCTURE CONDITIONS

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Abstract. Integrals of the Calculus of Variations with p,q-growth may have not smooth minimizers, not even bounded, for general p,q exponents. In this paper we consider the scalar case, which contrary to the vector-valued one, allows us not to impose structure conditions on the integrand $f(x,\xi)$ with dependence on the modulus of the gradient i.e. $f(x,\xi)=g(x,|\xi|)$. Without imposing structure conditions, we prove that if $\frac{q}{p}$ is sufficiently close to 1 then every minimizer is locally Lipschitz-continuous.

1. Introduction

The fundamental classical problem of the Calculus of Variations in the scalar case usually is formulated as finding a function u assuming a given value u_0 at the boundary $\partial\Omega$ of an open bounded set $\Omega\subset\mathbb{R}^n$ which minimizes the integral

$$\int_\Omega f(x,Dv)\,dx \tag{1.1}$$

among all functions $v:\Omega\to\mathbb{R}$, assuming the same boundary value u_0 as u. The precise functional space where to look for solutions depends on the growth conditions of $f=f(x,\xi)$ as $\xi\in\mathbb{R}^n$ grows in modulus to $+\infty$. Usually this growth is stated in terms of an inequality of the type

$$f(x,\xi)\geq M_1\,|\xi|^p, \tag{1.2}$$

for a.e. $x\in\Omega$, $\xi\in\mathbb{R}^n$ and for some positive constant M_1. Here $p=1$ is associated to the $BV(\Omega)$ space of functions with bounded variation, while $p>1$ is related to the Sobolev space $W^{1,p}(\Omega)$. Usually the condition $p>1$ and the strict convexity of $f(x,\xi)$ with respect to ξ are sufficient conditions for the existence and uniqueness of minimizers.

A different problem is the regularity of minimizers. A large literature is known about regularity (see for instance [26], [28], [30]) partly based on the nowadays classical well known Hölder continuity result by De Giorgi [16]. To this aim it seems necessary to impose also a

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growth condition from above, to be associated to the growth condition from below in (1.2), of the type

$$f(x,\xi) \leq M_2 \, (1+|\xi|^q) \tag{1.3}$$

for a.e. $x \in \Omega$, $\xi \in \mathbb{R}^n$, for $q \geq p$ and for some positive constant M_2. The so-called "natural growth conditions" appear if $q = p$, while the more general assumption $q > p$ allows us to consider a much larger class of integrals of the Calculus of Variations, such as for example

$$f(\xi) = |\xi|^p \log(1+|\xi|) \tag{1.4}$$

or

$$f(x,\xi) = |\xi|^{p(x)} \qquad \text{or} \qquad f(x,\xi) = (1+|\xi|^2)^{p(x)/2}. \tag{1.5}$$

We recall also the integrands recently considered in [11], [12], [19], [20], see also [2], [3], [4],

$$f(x,\xi) = a(x)|\xi|^p + b(x)|\xi|^q, \tag{1.6}$$

where $a(x), b(x) \geq 0$ and possibly zero on some part of Ω, being at least one of the two coefficients positive at almost every $x \in \Omega$. The above examples (1.4), (1.5), (1.6) enter in the theory presented in this paper. However here we study the more general case with $f = f(x,\xi)$ without a structure, i.e. not necessarily depending on the modulus of ξ of the type $f(x,\xi) = g(x,|\xi|)$.

We assume that $f : \Omega \times \mathbb{R}^n \to [0,+\infty)$ is a convex function with respect to the gradient variable and it is strictly convex only at infinity. More precisely, there exists $M_0 > 0$ such that $f_{\xi\xi}$, $f_{\xi x}$ are Carathéodory functions satisfying

$$\begin{cases} M_1 \, |\xi|^{p-2}|\lambda|^2 \leq \sum_{i,j} f_{\xi_i\xi_j}(x,\xi)\lambda_i\lambda_j, \\ |f_{\xi\xi}(x,\xi)| \leq \, M_2 \, |\xi|^{q-2}, \\ |f_{\xi x}(x,\xi)| \leq \, h(x) \, |\xi|^{\frac{p+q-2}{2}}, \end{cases} \tag{1.7}$$

for a.e. $x \in \Omega$ and for all $\lambda, \xi \in \mathbb{R}^n$, with $|\xi| \geq M_0$ and for positive constants M_1, M_2. Here $1 < p \leq q$ and $h \in L^r(\Omega)$ for some $r > n$.

Model integrands satisfying condition (1.7) are, for instance, the function $f(x,\xi)$ in (1.6) and also

$$f(x,\xi) = |\xi|^p + c(x) \, |\xi|^s + |\xi_n|^q \ , \tag{1.8}$$

ξ_n being the last component (or any other component) of the vector $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$, when

$$s \leq \frac{p+q}{2} \, .$$

For instance, when $s = p$ and $q \geq p$ we are considering energy integrals with integrand of the type (we denote here $a(x) = 1 + c(x)$ a generic positive coefficient)

$$f(x,\xi) = a(x) \, |\xi|^p + |\xi_n|^q \, . \tag{1.9}$$

Note that the cases (1.8) and (1.9) can be handled with Theorem 1.1. On the other hand the example (1.10) below enters in Theorem 1.2

$$f(x,\xi) = |\xi|^p + b(x) \, |\xi|^q \, . \tag{1.10}$$

The main regularity result that we prove here is the following a-priori estimate.

Theorem 1.1. (A-PRIORI ESTIMATE) *Let $u \in W^{1,p}(\Omega)$ be a smooth local minimizer of the integral functional* (1.1) *with exponents p, q fulfilling*

$$\frac{q}{p} < 1 + 2\left(\frac{1}{n} - \frac{1}{r}\right). \tag{1.11}$$

Under the growth assumption (1.7)*, there exist positive constants C, β, γ depending on n, r, p, q, M_0, M_1, M_2 such that, for every $0 < \rho < R \leq \rho + 1$*

$$\|Du\|_{L^\infty(B_\rho;\mathbb{R}^n)} \leq C\left(\frac{\|1+h\|_{L^r(\Omega)}}{R-\rho}\right)^{\beta\gamma}\left(\int_{B_R}\{1+|Du|^p\}\,dx\right)^{\frac{\gamma}{p}}. \tag{1.12}$$

Note that to get regularity of solutions it is natural, and also necessary, to assume that the gap $q-p$ is small or that q/p is close to 1, since the known counterexamples [27], [31], [33].

The $L^\infty-$bound of the gradient is obtained through several steps. The first step of the a-priori estimate is Lemma 2.3 below, where in the right hand side of the a-priori estimate there is the norm of the minimizer u in $W^{1,qm}(\Omega)$ ($m = \frac{r}{r-2}$), and the exponents p, q are related by the condition

$$\frac{q}{p} < 1 + \frac{2n}{n-2}\left(\frac{1}{n} - \frac{1}{r}\right) \tag{1.13}$$

with $n \geq 3$. Note that if $r = +\infty$ in (1.11) and (1.13), we recover the bounds in Theorem 2.1 and Theorem 3.1 of [33]. An interpolation method allows us to obtain (1.12).

The mathematical literature on the regularity under p, q growth is now very large; we refer to [36] for a complete survey on the subject. A new impulse to the subject has been given by the recent articles already cited [11], [12], [19], by [34], [35] for the case of elliptic equations and by [7], [8], [9] for the case of parabolic equations and systems under $p, q-$growth. We observe that here the ellipticity and growth assumptions hold only for large values of the gradient variable, i.e., we consider functionals which are *uniformly convex only at infinity.* In this context see [10], [25], [13] and recently [19], [20] and [15]. The Sobolev dependence on x recently has been considered in [37], [1] and for obstacle problems in [21].

The previous a-priori estimate, more precisely Theorem 2.6, under the assumptions (1.7) where the last condition is replaced by

$$|f_{\xi x}(x,\xi)| \leq h(x)\,|\xi|^{q-1}, \tag{1.14}$$

for a.e. $x \in \Omega$, with $|\xi| \geq M_0$, allows us to obtain the following existence and regularity result.

Theorem 1.2. (EXISTENCE AND REGULARITY) *Assume that f satisfies* (1.7) *and* (1.14) *with $1 < p \leq q$ and*

$$\frac{q}{p} < 1 + \frac{1}{n} - \frac{1}{r}. \tag{1.15}$$

The Dirichlet problem $\min\{F(u) : u \in W_0^{1,p}(\Omega) + u_0\}$*, with F defined in* (1.16) *below and $u_0 \in W^{1,q}(\Omega)$, has at least one locally Lipschitz continuous solution.*

Here we emphasize the definition (i.e. the precise meaning) of the integral $F(u)$ to be minimized; in fact the integral in (1.1) is well defined if $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$, since the growth assumption in (1.3), but a-priori it is not uniquely defined if $u \in W^{1,p}(\Omega) \setminus W^{1,q}_{\mathrm{loc}}(\Omega)$. In this context of $x-$dependence, we cannot a-priori exclude the Lavrentiev phenomenon; however note that in Section 5 we assume a special form of f to rule out this possibility.

For the gap in the Lavrentiev phenomenon we refer to [39], [5] and recently [23], [24] for related results.

For the functional F we adopt the classical definition which refers to the pionieristic research by Serrin [38] (see also [29]), which is related to the $\Gamma-$convergence theory by De Giorgi [17]. Precisely, for all $u \in W^{1,p}(\Omega)$

$$F(u) = \inf\left\{\liminf_k \int_\Omega f(x, Du_k)\, dx,\;\; u_k \in W_0^{1,q}(\Omega) + u_0,\;\; u_k \stackrel{w}{\rightharpoonup} u \text{ in } W^{1,p}(\Omega)\right\}. \tag{1.16}$$

We discuss more in details in Section 3 the definition of F in (1.16), while in Section 2 we give the proof of the a-priori estimate. Finally in Section 4 we give the proof of Theorem 1.2.

2. A-priori estimates

Let us start with two technical lemmas.

Lemma 2.1. *The following inequality holds*

$$(1+t)^\beta \le c_\beta \left(1 + \int_0^t (1+s)^{\beta-2}\, s\, ds\right) \tag{2.1}$$

for every $t \in [0, +\infty)$ and every $\beta \in (0, +\infty)$, where

$$c_\beta = \frac{\beta}{1 - (1 + \beta(\beta-1))^{\frac{1}{1-\beta}}} \tag{2.2}$$

if $\beta \neq 1$, while (by continuity)

$$c_1 = \lim_{\beta \to 1} c_\beta = \frac{e}{e-1}\,. \tag{2.3}$$

Proof. In order to prove the inequality (2.1) we first consider the case $\beta = 1$.

Step 1 ($\beta = 1$). We compute the integral in the right hand side of (2.1):

$$\int_0^t (1+s)^{-1}\, s\, ds = \int_0^t \left(1 - (1+s)^{-1}\right) ds = t - \log(1+t)$$

and the inequality (2.1) becomes

$$1 + t \le c_1 (1 + t - \log(1+t))$$

which is equivalent to

$$\frac{\log(1+t)}{1+t} \le \frac{c_1 - 1}{c_1}\,.$$

A computation shows that $g(t) =: \frac{\log(1+t)}{1+t}$ is positive for $t \in (0, +\infty)$ and has a maximum at $t = e-1$, thus

$$g(t) =: \frac{\log(1+t)}{1+t} \le g(e-1) = \frac{1}{e};$$

with the position $\frac{c_1-1}{c_1} =: \frac{1}{e}$ we find (2.3).

Step 2 ($\beta \neq 1$). We compute the integral in the right hand side of (2.1) under the condition $\beta \neq 1$ and with the notation $r =: 1 + s$

$$
\begin{aligned}
& \int_0^t (1+s)^{\beta-2} s\, ds = \int_1^{t+1} r^{\beta-2} (r-1)\, dr \\
= & \int_1^{t+1} r^{\beta-1} dr - \int_1^{t+1} r^{\beta-2} dr = \left[\frac{r^\beta}{\beta} \right]_{r=1}^{r=t+1} - \left[\frac{r^{\beta-1}}{\beta-1} \right]_{r=1}^{r=t+1} \\
= & \frac{(t+1)^\beta}{\beta} - \frac{(t+1)^{\beta-1}}{\beta-1} + \frac{1}{\beta(\beta-1)} .
\end{aligned}
$$

The inequality (2.1) takes then the form

$$\frac{1}{c_\beta} (1+t)^\beta \leq 1 + \frac{(t+1)^\beta}{\beta} - \frac{(t+1)^{\beta-1}}{\beta-1} + \frac{1}{\beta(\beta-1)} .$$

We can write it equivalently

$$g(t) \leq 1 + \frac{1}{\beta(\beta-1)} , \tag{2.4}$$

where

$$g(t) =: \frac{(t+1)^{\beta-1}}{\beta-1} - \left(\frac{1}{\beta} - \frac{1}{c_\beta} \right) (t+1)^\beta .$$

We can compute the maximum of $g(t)$ when $t \in [0, +\infty)$. We find that the derivative $g'(t)$ is equal to zero if $t = \frac{\beta}{c_\beta - \beta}$ and, since $c_\beta > \beta$, we obtain

$$
\begin{aligned}
\max \{ g(t) :\ t \in [0, +\infty) \} &= g\left(\frac{\beta}{c_\beta - \beta} \right) \\
&= \left(\frac{c_\beta}{c_\beta - \beta} \right)^{\beta-1} \frac{1}{\beta(\beta-1)} .
\end{aligned}
$$

Therefore the inequality (2.4) holds if we choose c_β to satisfy the condition

$$\left(\frac{c_\beta}{c_\beta - \beta} \right)^{\beta-1} \frac{1}{\beta(\beta-1)} = 1 + \frac{1}{\beta(\beta-1)} .$$

A further computation gives for c_β the explicit expression in (2.2). Note that $c_\beta \to c_1$ as $\beta \to 1$. $\square$

In the sequel we apply the previous lemma to get the a-priori estimates in particular to deal with the left hand side of (2.31), with $\beta = \frac{\gamma}{2} + \frac{p}{2}$, for $\gamma \geq 0$; thus $\beta \geq \frac{p}{2}$. In the next result in fact we consider $\beta \in [\beta_0, +\infty)$ for some fixed $\beta_0 > 0$.

Lemma 2.2. *Let $\beta_0 > 0$. There exist constants c' and c'', depending on β_0 but independent of $\beta \geq \beta_0$ and of $t \geq 0$, such that*

$$(1+t)^\beta \leq c' \frac{\beta^2}{\log(1+\beta)} \left(1 + \int_0^t (1+s)^{\beta-2} s\, ds \right) , \tag{2.5}$$

$$(1+t)^\beta \leq c'' \beta^2 \left(1 + \int_0^t (1+s)^{\beta-2} s\, ds \right) , \tag{2.6}$$

for every $\beta \in [\beta_0, +\infty)$ *and every* $t \in [0, +\infty)$.

Proof. First we show that the constant c_β is bounded independently of $\beta \leq 1$ if $\beta \in [\beta_0, 1]$ (here we assume that $\beta_0 \in (0,1)$, otherwise nothing to be proved at this step). Precisely we show that

$$c_\beta \leq \frac{\beta}{1 - e^{-\beta_0}}, \quad \forall \beta \in [\beta_0, 1]. \tag{2.7}$$

In fact, by the inequality $\log t \leq t - 1$,valid for all $t > 0$, by posing $t = 1 + \beta(\beta - 1)$, if $\beta < 1$ we obtain

$$(1 + \beta(\beta - 1))^{\frac{1}{1-\beta}} = e^{\frac{\log(1+\beta(\beta-1))}{1-\beta}} \leq e^{\frac{\beta(\beta-1)}{1-\beta}} = e^{-\beta}$$

and (2.7) follows if $\beta \in [\beta_0, 1)$, since

$$c_\beta = \frac{\beta}{1 - (1 + \beta(\beta - 1))^{\frac{1}{1-\beta}}} \leq \frac{\beta}{1 - e^{-\beta}} \leq \frac{\beta}{1 - e^{-\beta_0}}.$$

Finally, if $\beta = 1$ then $c_1 = \frac{e}{e-1} < \frac{1}{1-e^{-\beta_0}}$ holds, since it is equivalent to $1 < e^{1-\beta_0}$.

We now consider the case $\beta > 1$. By Taylor's formula we get

$$\begin{aligned} & (1 + \beta(\beta - 1))^{\frac{1}{1-\beta}} = e^{\frac{\log(1+\beta(\beta-1))}{1-\beta}} \\ = \;& 1 + \frac{\log(1 + \beta(\beta - 1))}{1 - \beta} + o\left(\frac{\log(1 + \beta(\beta - 1))}{1 - \beta}\right) \end{aligned}$$

and thus the quantity

$$\frac{c_\beta \log(1 + \beta)}{\beta^2} = \frac{\log(1 + \beta)}{\beta\left[\frac{\log(1+\beta(\beta-1))}{\beta-1} + o\left(\frac{\log(1+\beta(\beta-1))}{1-\beta}\right)\right]}$$

has a finite limit as $\beta \to +\infty$ (equal to $1/2$) and it is a bounded function for $\beta \in [1, +\infty)$, let us say bounded by c'. This proves (2.5). The other inequality (2.6) is a direct consequence of (2.5). □

Let now Ω be an open bounded subset of $\mathbb{R}^n$, for $n \geq 2$ and assume that f satisfies (1.7). We observe that we can transform $f(x, \xi)$ into $f(x, M_0\xi)$, which satisfies the same assumptions for $|\xi| \geq 1$ (with different constants depending on M_0). Then it is sufficient to obtain the a-priori bound and the regularity results for $v = \frac{1}{M_0}u$. Therefore, for clarity of exposition and without loss of generality, we assume $M_0 = 1$. Throughout the paper we will denote by B_ρ and B_R balls of radii, respectively, ρ and R (with $\rho < R$) compactly contained in Ω and with the same center, let us say, $x_0 \in \Omega$.

In this section we assume the following supplementary assumptions on f. Assume that $f \in \mathcal{C}^2(\Omega \times \mathbb{R}^n)$ and there exist two positive constants k, K such that $\forall \xi \in \mathbb{R}^n$, $\forall x \in \Omega$

$$\begin{cases} k\,(1 + |\xi|^2)^{\frac{q-2}{2}} |\lambda|^2 \leq \sum_{i,j} f_{\xi_i \xi_j}(x, \xi) \lambda_i \lambda_j, \\ |f_{\xi\xi}(x, \xi)| \leq K\,(1 + |\xi|^2)^{\frac{q-2}{2}}, \\ |f_{\xi x}(x, \xi)| \leq K\,(1 + |\xi|^2)^{\frac{q-1}{2}}. \end{cases} \tag{2.8}$$

In the next lemma, we obtain an a-priori estimate for the $L^\infty-$norm of the gradient of u which is independent of k and K.

Lemma 2.3. *Let u be a local minimizer of the integral functional* (1.1) *with f satisfying* (1.7) *and* (2.8) *with*

$$\frac{q}{p} < 1 + \frac{2\alpha}{n-2} \qquad \text{with} \quad \alpha = 1 - \frac{n}{r} \tag{2.9}$$

if $n \geq 3$ and $p < q$ if $n = 2$. Then there exists a positive constants C depending only on n, r, p, q, M_1, M_2 (depending also on $|\Omega|$ when $n = 2$) such that

$$\|Du\|_{L^\infty(B_\rho;\mathbb{R}^n)} \leq C \left[\frac{\|1+h\|_{L^r(\Omega)}}{(R-\rho)}\right]^{\theta\tilde{\beta}} \left(\int_{B_R} \{1 + |Du|^{qm}\}\, dx\right)^{\frac{\theta}{qm}}, \tag{2.10}$$

for every $0 < \rho < R \leq \rho + 1$, where

$$\tilde{\beta} := \frac{2^*}{p\frac{2^*}{2} - qm} \qquad \theta := \frac{qm\left(\frac{2^*}{2m} - 1\right)}{p\frac{2^*}{2} - qm} \qquad m := \frac{r}{r-2}. \tag{2.11}$$

Remark 2.4. We observe that

$$1 \leq m := \frac{r}{r-2} < \frac{n}{n-2} = \frac{2^*}{2}, \qquad \text{since } r > n; \tag{2.12}$$

the last inequality holds for $n > 2$, while we set 2^* equal to any fixed real number greater than 2, if $n = 2$. Moreover we also have

$$\frac{1}{2m} - \frac{1}{2^*} = \frac{r-2}{2r} - \frac{n-2}{2n} = \frac{n(r-2) - r(n-2)}{2nr} = \frac{r-n}{nr} = \frac{1}{n} - \frac{1}{r} = \frac{\alpha}{n}, \tag{2.13}$$

therefore (2.9) can be equivalently expressed as

$$\frac{q}{p} < \frac{2^*}{2m} \tag{2.14}$$

because

$$1 + \frac{2\alpha}{n-2} = 1 + \frac{2^*\alpha}{n} \overset{(2.13)}{=} 1 + 2^*\left(\frac{1}{2m} - \frac{1}{2^*}\right) = \frac{2^*}{2m}.$$

Therefore, due to (2.14), in (2.11) we have $\tilde{\beta} > 0$ and $\theta > 1$.

Remark 2.5. The result obtained is sharp in the sense that if $m = 1$ ($r = +\infty$), then the relation between p and q reduces to the analogous one in Theorem 2.1 of [33], i.e. $\frac{q}{p} < \frac{n}{n-2}$.

Proof. Let $u \in W^{1,q}(\Omega)$ be a local minimizer of (1.1), then u satisfies the Euler first variation

$$\int_\Omega \sum_{i=1}^n f_{\xi_i}(x, Du)\varphi_{x_i}(x)\, dx = 0 \qquad \forall \varphi \in W_0^{1,q}(\Omega).$$

By (2.8), the technique of the difference quotients (see [30], [18], in particular [28], Chapter 8, Sections 8.1 and 8.2) gives

$$u \in W^{1,\infty}_{\mathrm{loc}}(\Omega) \cap W^{2,\min(2,q)}_{\mathrm{loc}}(\Omega) \text{ and } (1 + |Du|^2)^{\frac{q-2}{2}}|D^2u|^2 \in L^1_{\mathrm{loc}}(\Omega). \tag{2.15}$$

Let $\eta \in C_0^1(\Omega)$ and for any fixed $s \in \{1, \ldots, n\}$ define

$$\varphi = \eta^2 u_{x_s}\Phi((|Du| - 1)_+)$$

for $\Phi : [0, +\infty) \to [0, +\infty)$ increasing, locally Lipschitz continuous function, with Φ and Φ' bounded on $[0, +\infty)$, such that $\Phi(0) = \Phi'(0) = 0$ and

$$\Phi'(s)s \leq c_\Phi \, \Phi(s) \tag{2.16}$$

for a suitable constant $c_\Phi \geq 1$. Here $(a)_+$ denotes the positive part of $a \in \mathbb{R}$; in the following we denote $\Phi((|Du| - 1)_+) = \Phi(|Du| - 1)_+$.

We have then

$$\varphi_{x_i} = 2\eta\eta_{x_i} u_{x_s} \Phi(|Du|-1)_+ + \eta^2 u_{x_s x_i} \Phi(|Du|-1)_+ + \eta^2 u_{x_s} \Phi'(|Du|-1)_+ [(|Du|-1)_+]_{x_i}. \tag{2.17}$$

Let $q \geq 2$, by (2.15) we have that $|D^2u|^2 \in L^1_{\text{loc}}(\Omega)$. Otherwise if $1 < q < 2$, we use the fact that $u \in W^{1,\infty}_{\text{loc}}(\Omega)$ to infer that there exists $M = M(\text{supp}\varphi)$ such that

$$|Du(x)| \leq M \quad \text{for a.e. } x \in \text{supp}\varphi.$$

Now since $q - 2 < 0$ we have

$$(1 + M^2)^{\frac{q-2}{2}} |D^2u|^2 \leq (1 + |Du|^2)^{\frac{q-2}{2}} |D^2u|^2,$$

and by (2.15) we again get $|D^2u|^2 \in L^1(\text{supp}\varphi)$. Therefore we can insert φ_{x_i} in the following second variation

$$\int_\Omega \left\{ \sum_{i,j=1}^n f_{\xi_i \xi_j}(x, Du) u_{x_j x_s} \varphi_{x_i} + \sum_{i=1}^n f_{\xi_i x_s}(x, Du) \varphi_{x_i} \right\} dx = 0 \qquad \forall s = 1, \dots, n \tag{2.18}$$

and we obtain

$$\begin{aligned} 0 \;=\; & \sum_s \Bigg[\int_\Omega 2\eta \Phi(|Du| - 1)_+ \sum_{i,j} f_{\xi_i \xi_j}(x, Du) \eta_{x_i} u_{x_s} u_{x_s x_j} \, dx \\ & + \int_\Omega \eta^2 \Phi(|Du| - 1)_+ \sum_{i,j} f_{\xi_i \xi_j}(x, Du) u_{x_s x_i} u_{x_s x_j} \, dx \\ & + \int_\Omega \eta^2 \Phi'(|Du| - 1)_+ \sum_{i,j} f_{\xi_i \xi_j}(x, Du) u_{x_s} u_{x_s x_j} [(|Du| - 1)_+]_{x_i} \, dx \\ & + \int_\Omega 2\eta \Phi(|Du| - 1)_+ \sum_i f_{\xi_i x_s}(x, Du) \eta_{x_i} u_{x_s} \, dx \\ & + \int_\Omega \eta^2 \Phi(|Du| - 1)_+ \sum_i f_{\xi_i x_s}(x, Du) u_{x_s x_i} \, dx \\ & + \int_\Omega \eta^2 \Phi'(|Du| - 1)_+ \sum_i f_{\xi_i x_s}(x, Du) u_{x_s} [(|Du| - 1)_+]_{x_i} \, dx \Bigg] \\ =: \; & \sum_s (I_1^s + I_2^s + I_3^s + I_4^s + I_5^s + I_6^s). \end{aligned} \tag{2.19}$$

In the following, constants will be denoted by C, regardless of their actual value.

Let us start with the estimate of the first integral in (2.19). By the Cauchy-Schwarz inequality, the Young inequality and (1.7), we have

$$\left|\sum_s I_1^s\right| = \left|\int_\Omega 2\eta\Phi(|Du|-1)_+\sum_{i,j,s} f_{\xi_i\xi_j}(x,Du)\eta_{x_i}u_{x_s}u_{x_sx_j}\,dx\right| \tag{2.20}$$

$$\begin{aligned}
&\le \int_\Omega 2\eta\Phi(|Du|-1)_+\left\{\sum_{i,j,s} f_{\xi_i\xi_j}(x,Du)\eta_{x_i}u_{x_s}\eta_{x_j}u_{x_s}\right\}^{\frac12}\left\{\sum_{i,j,s} f_{\xi_i\xi_j}(x,Du)u_{x_sx_i}u_{x_sx_j}\right\}^{\frac12}dx\\
&\le C\int_\Omega |D\eta|^2\Phi(|Du|-1)_+\,|Du|^q\,dx+\frac12\int_\Omega \eta^2\Phi(|Du|-1)_+\sum_{i,j,s} f_{\xi_i\xi_j}(x,Du)u_{x_sx_i}u_{x_sx_j}\,dx.
\end{aligned}$$

Let us consider the third integral in (2.19). First of all we observe that

$$[(|Du|-1)_+]_{x_i}\sum_s u_{x_s}u_{x_sx_j} = [(|Du|-1)_+]_{x_i}|Du|[(|Du|-1)_+]_{x_j}.$$

This entails, using (1.7)

$$\begin{aligned}
\sum_s I_3^s &= \int_\Omega \eta^2\Phi'(|Du|-1)_+\sum_{i,j,s} f_{\xi_i\xi_j}(x,Du)u_{x_s}u_{x_sx_j}[(|Du-1|_+)]_{x_i}\,dx\\
&\ge M_1\int_\Omega \eta^2\Phi'(|Du|-1)_+|Du|^{p-1}|D(|Du|-1)_+|^2\,dx \ge 0
\end{aligned}$$

We now deal with the fourth integral in (2.19). We have

$$\begin{aligned}
\left|\sum_s I_4^s\right| &= \left|\int_\Omega 2\eta\Phi(|Du|-1)_+\sum_{i,s} f_{\xi_ix_s}(x,Du)\eta_{x_i}u_{x_s}\,dx\right|\\
&\overset{(1.7)}{\le} \int_\Omega 2\eta\Phi(|Du|-1)_+h(x)|Du|^{\frac{p+q-2}{2}}\sum_{i,s}|\eta_{x_i}u_{x_s}|\,dx\\
&\le C\int_\Omega(\eta^2+|D\eta|^2)h(x)\Phi(|Du|-1)_+|Du|^q\,dx.
\end{aligned}$$

Consider now the fifth integral in (2.19). We have

$$\begin{aligned}
\left|\sum_s I_5^s\right| &= \left|\int_\Omega \eta^2\Phi(|Du|-1)_+\sum_{i,s} f_{\xi_ix_s}(x,Du)u_{x_sx_j}\,dx\right|\\
&\overset{(1.7)}{\le} \int_\Omega \eta^2\Phi(|Du|-1)_+h(x)|Du|^{\frac{p+q-2}{2}}|D^2u|\,dx\\
&\le \int_\Omega\left[\eta^2\Phi(|Du|-1)_+|Du|^{p-2}|D^2u|^2\right]^{1/2}\left[\eta^2\Phi(|Du|-1)_+|h(x)|^2|Du|^q\right]^{1/2}dx\\
&\le \varepsilon\int_\Omega \eta^2\Phi(|Du|-1)_+|Du|^{p-2}|D^2u|^2\,dx+C_\varepsilon\int_\Omega \eta^2\Phi(|Du|-1)_+|h(x)|^2|Du|^q\,dx,
\end{aligned}$$

where in the last line we used the Young inequality. Finally, for any $0 < \delta < 1$

$$
\begin{aligned}
\left|\sum_s I_6^s\right| &= \left|\int_\Omega \eta^2 \sum_{i,s} f_{\xi_i x_s}(x,Du)u_{x_s}\Phi'(|Du|-1)_+[(|Du|-1)_+]_{x_i}\,dx\right| \\
&\overset{(1.7)}{\leq} \int_\Omega \eta^2\Phi'(|Du|-1)_+\,h(x)|Du|^{\frac{p+q-2}{2}}|Du||D(|Du|-1)_+|\,dx \\
&\leq \int_\Omega \eta^2\Phi'(|Du|-1)_+h(x)|Du|^{\frac{p+q}{2}}\,|D^2u|\,dx \\
&= \int_\Omega \eta^2\Phi'(|Du|-1)_+h(x)[(|Du|-1)_+ +\delta]\,[(|Du|-1)_+ +\delta]^{-1}|Du|^{\frac{p+q}{2}}\,|D^2u|\,dx \\
&\leq \int_\Omega \eta^2\left\{\frac{1}{c_\Phi}\Phi'(|Du|-1)_+[(|Du|-1)_+ +\delta]|Du|^{p-2}|D^2u|^2\right\}^{1/2} \\
&\quad \times\left\{c_\Phi\Phi'(|Du|-1)_+|h(x)|^2|Du|^{q+2}[(|Du|-1)_+ +\delta]^{-1}\right\}^{1/2}\,dx \\
&\leq C_\varepsilon\,c_\Phi\int_\Omega \eta^2\Phi'(|Du|-1)_+|h(x)|^2|Du|^{q+2}[(|Du|-1)_+ +\delta]^{-1}\,dx \\
&\quad +\frac{\varepsilon}{c_\Phi}\int_\Omega \eta^2\Phi'(|Du|-1)_+[(|Du|-1)_+ +\delta]|Du|^{p-2}|D^2u|^2\,dx.
\end{aligned}
$$

Since $\Omega = \{x : |Du(x)| \geq 2\} \cup \{x : |Du(x)| < 2\}$ and $(|Du|-1)_+ \geq 1$ in $\{x : |Du(x)| \geq 2\}$, we also have

$$
(|Du|-1)_+ +\delta \leq\, 2(|Du|-1)_+ \tag{2.21}
$$

as long as we have chosen $\delta < 1$. Therefore, using (2.16), we can estimate the last integral as

$$
\begin{aligned}
&\int_\Omega \eta^2\Phi'(|Du|-1)_+[(|Du|-1)_+ +\delta]|Du|^{p-2}|D^2u|^2\,dx \\
=\ &\int_{|Du|\geq 2}\eta^2\Phi'(|Du|-1)_+[(|Du|-1)_+ +\delta]|Du|^{p-2}|D^2u|^2\,dx \\
&+\int_{1<|Du|<2}\eta^2\Phi'(|Du|-1)_+[(|Du|-1)_+ +\delta]|Du|^{p-2}|D^2u|^2\,dx \\
\overset{(2.21)}{\leq}\ & 2\int_{|Du|\geq 2}\eta^2\Phi'(|Du|-1)_+(|Du|-1)_+|Du|^{p-2}|D^2u|^2\,dx \\
&+\int_{1<|Du|<2}\eta^2\Phi'(|Du|-1)_+(|Du|-1)_+|Du|^{p-2}|D^2u|^2\,dx \\
&+\delta\int_{1<|Du|<2}\eta^2\Phi'(|Du|-1)_+|Du|^{p-2}|D^2u|^2\,dx \\
\overset{(2.16)}{\leq}\ & 2\,c_\Phi\int_\Omega \eta^2\Phi(|Du|-1)_+|Du|^{p-2}|D^2u|^2\,dx +\,\delta\int_{1<|Du|<2}\eta^2\Phi'(|Du|-1)_+|Du|^{p-2}|D^2u|^2\,dx.
\end{aligned}
$$

Therefore we finally have

$$\begin{aligned}\left|\sum_s I_6^s\right| \;\le\;& C_\varepsilon\, c_\Phi \int_\Omega \eta^2 \Phi'(|Du|-1)_+ |h(x)|^2 |Du|^{q+2}[(|Du|-1)_+ + \delta]^{-1}\, dx\\ &+2\,\varepsilon \int_\Omega \eta^2 \Phi(|Du|-1)_+ |Du|^{p-2}|D^2u|^2\, dx\\ &+\delta\,\varepsilon \int_{1<|Du|<2} \eta^2 \Phi'(|Du|-1)_+ |Du|^{p-2}|D^2u|^2\, dx.\end{aligned}$$

Now, for ε sufficiently small and putting together all the previous estimates, we deduce that there exists a constant C depending on n, r, p, q, M_1 such that

$$\begin{aligned}&\int_\Omega \eta^2 \Phi(|Du|-1)_+ |Du|^{p-2}|D^2u|^2\, dx\\ \le\;& C\, c_\Phi \int_\Omega (\eta^2+|D\eta|^2)(1+h(x))^2 |Du|^q\\ &\times \left[\Phi(|Du|-1)_+ + \Phi'(|Du|-1)_+ |Du|^2 [(|Du|-1)_+ +\delta]^{-1}\right]\, dx\\ &+\delta \int_{1<|Du|<2} \eta^2 \Phi'(|Du|-1)_+ |Du|^{p-2}|D^2u|^2\, dx.\end{aligned} \tag{2.22}$$

Let now set

$$\Phi(s) := (1+s)^{\gamma-2} s^2 \qquad \gamma \ge 0; \tag{2.23}$$

with

$$\Phi'(s) = (\gamma s + 2)s(1+s)^{\gamma-3}. \tag{2.24}$$

It is easy to check that Φ satisfies (2.16) with $c_\Phi = 2(1+\gamma)$.
We now approximate this function Φ by a sequence of functions Φ_h, each of them being equal to Φ in the interval $[0,h]$, and then extended to $[h,+\infty)$ with the constant value $\Phi(h)$. Since Φ_h and Φ_h' converge monotonically to Φ and Φ', by inserting Φ_h in (2.22), it is possible to pass to the limit as $h \to +\infty$ by the Monotone Convergence Theorem.

Therefore, for every $0<\delta<1$, since

$$\frac{(|Du|-1)_+}{(|Du|-1)_+ + \delta} \le 1 \qquad \forall\, \delta > 0$$

and $\Phi'(t-1)_+ \le C(\gamma)$ when $1<t<2$, we obtain

$$\begin{aligned}&\int_\Omega \eta^2 (1+(|Du|-1)_+)^{\gamma-2}(|Du|-1)_+^2 |Du|^{p-2}|D^2u|^2\, dx\\ \le\;& C\,(1+\gamma)^2 \int_\Omega (\eta^2+|D\eta|^2)(1+h(x))^2(1+(|Du|-1)_+)^{\gamma+q}\, dx\\ &+\delta\, C(\gamma) \int_{1<|Du|<2} \eta^2 |Du|^{p-2}|D^2u|^2\, dx.\end{aligned} \tag{2.25}$$

Using formula (3.51) of Lemma 3.3 in [19], namely the fact that $|Du|^{p-2} \le C(p)\,(1+|Du|^2)^{\frac{p-2}{2}}$ when $|Du|>1$, we have

$$\int_{1<|Du|<2} \eta^2 |Du|^{p-2}|D^2u|^2\,dx \le C \int_{1<|Du|<2} \eta^2 (1+|Du|^2)^{\frac{p-2}{2}}|D^2u|^2\,dx < +\infty$$

by (2.15), and for $\delta \to 0$ and the last term in the previous inequality vanishes.

Since $h \in L^r(\Omega)$, by the Hölder inequality, since $\frac{1}{m}+\frac{2}{r}=1$, by (2.25) we have

$$\begin{aligned} &\int_\Omega \eta^2 (1+(|Du|-1)_+)^{\gamma-2}(|Du|-1)_+^2|Du|^{p-2}|D((|Du|-1)_+)|^2\,dx \\ \le\ & C\,(1+\gamma)^2\,\|1+h\|^2_{L^r(\Omega)} \left[\int_\Omega (\eta^2+|D\eta|^2)^m (1+(|Du|-1)_+)^{(\gamma+q)m}\,dx\right]^{\frac{1}{m}}. \end{aligned} \tag{2.26}$$

Let us introduce

$$G(t) = 1 + \int_0^t (1+s)^{\frac{\gamma}{2}+\frac{p}{2}-2}\, s\,ds \tag{2.27}$$

and we obtain

$$[G(t)]^2 \le 4\,(1+t)^{\gamma+p} \le 4\,(1+t)^{\gamma+q}, \tag{2.28}$$

where we used the fact that $p \le q$. On the other hand

$$G'(t) = (1+t)^{\frac{\gamma}{2}+\frac{p}{2}-2}t \tag{2.29}$$

which in turn allows us to give the following estimate for the gradient of the function $w = \eta G((|Du|-1)_+)$

$$\begin{aligned} &\int_\Omega |D(\eta G((|Du|-1)_+))|^2\,dx \\ \le\ & 2\int_\Omega |D\eta|^2 |G((|Du|-1)_+)|^2\,dx + 2\int_\Omega \eta^2 [G_t((|Du|-1)_+)]^2[D((|Du|-1)_+)]^2\,dx \\ \overset{(2.26),(2.28),(2.29)}{\le}\ & C\,(1+\gamma)^2\,\|1+h\|^2_{L^r(\Omega)} \left[\int_\Omega (\eta^2+|D\eta|^2)^m [1+(|Du|-1)_+]^{(\gamma+q)m}\,dx\right]^{\frac{1}{m}}. \end{aligned} \tag{2.30}$$

By Sobolev's inequality there exists a constant C (depending also on $|\Omega|$ when $n=2$) such that

$$\left\{\int_\Omega [\eta G((|Du|-1)_+)]^{2^*}\,dx\right\}^{\frac{2}{2^*}} \le C\int_\Omega |D(\eta G((|Du|-1)_+))|^2\,dx$$

and by the previous inequality we get (for a different constant)

$$\begin{aligned} &\left\{\int_\Omega [\eta G((|Du|-1)_+)]^{2^*}\,dx\right\}^{\frac{2}{2^*}} \\ \le\ & C\,(1+\gamma)^2\,\|1+h\|^2_{L^r(\Omega)} \left[\int_\Omega (\eta^2+|D\eta|^2)^m [1+(|Du|-1)_+]^{(\gamma+q)m}\,dx\right]^{\frac{1}{m}}. \end{aligned} \tag{2.31}$$

We take into account the definition of $G(t)$ in (2.27) and we use Lemma 2.2, and in particular formula (2.6) with $\beta = \frac{\gamma+p}{2}$. Being $\gamma \geq 0$, we have $\beta \geq \beta_0 := p/2 > 0$ and

$$(1+t)^{\frac{\gamma+p}{2}} \leq c'' \left(\frac{\gamma+p}{2}\right)^2 \left(1 + \int_0^t (1+s)^{\frac{\gamma+p}{2}-2} s\, ds\right), \tag{2.32}$$

for every $\gamma \geq 0$ and every $t \in [0, +\infty)$. In terms of $G(t) = 1 + \int_0^t (1+s)^{\frac{\gamma}{2}+\frac{p}{2}-2} s\, ds$ equivalently

$$(1+t)^{\frac{\gamma+p}{2}} \leq c'' \left(\frac{\gamma+p}{2}\right)^2 G(t), \qquad \forall\, \gamma \geq 0, \ \ \forall\, t \geq 0.$$

Therefore, if $t := (|Du|-1)_+$,

$$\left(1 + (|Du|-1)_+\right)^{\frac{\gamma+p}{2}2^*} \leq (c'')^{2^*} \left(\frac{\gamma+p}{2}\right)^{2\cdot 2^*} \left[G\left((|Du|-1)_+\right)\right]^{2^*}, \quad \forall\, \gamma \geq 0,$$

and by (2.31) we finally get

$$\begin{aligned} &\left\{\int_\Omega \eta^{2^*} \left[1 + (|Du|-1)_+\right]^{\frac{\gamma+p}{2}2^*} dx\right\}^{\frac{2}{2^*}} \\ \leq\ & (c'')^2 \left(\frac{\gamma+p}{2}\right)^4 \left\{\int_\Omega [\eta G((|Du|-1)_+)]^{2^*} dx\right\}^{\frac{2}{2^*}} \\ \leq\ & C\, (\gamma+1)^6\, \|1+h\|^2_{L^r(\Omega)} \left[\int_\Omega (\eta^2 + |D\eta|^2)^m [1 + (|Du|-1)_+]^{(\gamma+q)m} dx\right]^{\frac{1}{m}} \end{aligned}$$

with a new constant C and for every $\gamma \geq 0$.

As usual we consider a test function η equal to 1 in a ball B_ρ, with $\operatorname{supp}\eta \subset B_R$ and such that $|D\eta| \leq \frac{2}{(R-\rho)}$. We get

$$\begin{aligned} &\left[\int_{B_\rho} [1 + (|Du|-1)_+]^{[(\gamma+p)m]\frac{2^*}{2m}} dx\right]^{\frac{2m}{2^*}} \\ \leq\ & C_0\, \|1+h\|^{2m}_{L^r(\Omega)} \frac{(\gamma+q)^{6m}}{(R-\rho)^{2m}} \int_{B_R} [1 + (|Du|-1)_+]^{(\gamma+q)m} dx, \end{aligned} \tag{2.33}$$

where the constant C_0 only depends on n, r, p, q, M_1, M_2 but is independent of γ.

Fixed $0 < \rho_0 < R_0 \leq \rho_0 + 1$, we define the decreasing sequence of radii $\{\rho_k\}_{k\geq 1}$

$$\rho_k = \rho_0 + \frac{R_0 - \rho_0}{2^k} \qquad \forall k \geq 1.$$

We define recursively a sequence α_k in the following way

$$\alpha_1 := 0 \qquad \alpha_{k+1} := (\alpha_k + pm)\frac{2^*}{2m} - qm. \tag{2.34}$$

We have the following representation formula for α_k which can be easily proved by induction

$$\alpha_k = \left(p\frac{2^*}{2} - qm\right) \frac{\left[\left(\frac{2^*}{2m}\right)^{k-1} - 1\right]}{\frac{2^*}{2m} - 1} \tag{2.35}$$

We rewrite (2.33) with $R=\rho_k$, $\rho=\rho_{k+1}$, $\gamma=\frac{\alpha_k}{m}$ and we observe that

$$R-\rho:=\rho_k-\rho_{k+1}=\frac{R_0-\rho_0}{2^{k+1}}.$$

Set, for all $k\geq 1$

$$A_k:=\left(\int_{B_{\rho_k}}[1+(|Du|-1)_+]^{\alpha_k+qm}\,dx\right)^{1/(\alpha_k+qm)}$$

$$C_k:=C_0\,\|1+h\|^{2m}_{L^r(\Omega)}\,\left(\frac{(\alpha_k+qm)^3 2^{k+1}}{R_0-\rho_0}\right)^{2m};$$

we obtain for every $k\geq 1$

$$A_{k+1}\leq\,C_k^{1/(\alpha_k+pm)}\,A_k^{(\alpha_k+qm)/(\alpha_k+pm)}. \tag{2.36}$$

Let θ be defined by

$$\theta:=\prod_{i=1}^{\infty}\frac{\alpha_i+qm}{\alpha_i+pm}. \tag{2.37}$$

We show that θ is finite and is given by

$$\theta=\frac{qm\left(\frac{2^*}{2m}-1\right)}{p\frac{2^*}{2}-qm}.$$

Indeed by (2.37), the recursive definition of α_k, i.e. (2.34) and the representation formula for α_k, namely (2.35), we have

$$\begin{aligned}\theta_k\ &:=\ \prod_{i=1}^{k}\frac{\alpha_i+qm}{\alpha_i+pm}\overset{(2.34)}{=}\frac{qm}{\alpha_k+pm}\left(\frac{2^*}{2m}\right)^{k-1}\overset{(2.35)}{=}\frac{qm\left(\frac{2^*}{2m}\right)^{k-1}}{\frac{\left(p\frac{2^*}{2}-qm\right)\left[\left(\frac{2^*}{2m}\right)^{k-1}-1\right]}{\frac{2^*}{2m}-1}+pm}\\ &=\ \frac{qm\left(\frac{2^*}{2m}\right)^{k-1}\left(\frac{2^*}{2m}-1\right)}{pm\left(\frac{2^*}{2m}-1\right)+\left(p\frac{2^*}{2}-qm\right)\left[\left(\frac{2^*}{2m}\right)^{k-1}-1\right]}\end{aligned}$$

which yields (2.37) once we pass to the limit as $k\to\infty$ as long as $\frac{2^*}{2m}>1$ in view of (2.12). Note that θ makes sense due to (2.12) and the bound (2.14) in Remark 2.4.

Iterating (2.36) we deduce

$$A_{k+1}\leq\tilde{C}\left(\left[\frac{\|1+h\|_{L^r(\Omega)}}{(R_0-\rho_0)}\right]^{\tilde{\beta}}A_1\right)^{\theta_k}\qquad k\geq 1, \tag{2.38}$$

where

$$\tilde{C}:={C_0}^{\tilde{\beta}\theta}\,\exp\left[\theta\sum_{i=1}^{\infty}\frac{\log[(\alpha_i+qm)^{6m}2^{2m(i+1)}]}{\alpha_i+pm}\right]<+\infty$$

which is finite because the series is convergent (α_i from the representation formula (2.35) grows exponentially) and

$$\sum_{i=1}^{\infty}\frac{2m}{\alpha_i+pm}\overset{(2.35)}{=}\sum_{i=1}^{\infty}\frac{2m}{\left(p\frac{2^*}{2}-qm\right)\frac{\left[\left(\frac{2^*}{2}\right)^{i-1}-1\right]}{\frac{2^*}{2m}-1}+pm}\leq\sum_{i=1}^{\infty}\frac{2m}{\left(p\frac{2^*}{2}-qm\right)\frac{\left(\frac{2^*}{2m}\right)^{i-1}}{\frac{2^*}{2m}-1}}$$

$$\leq\ \frac{2m\left(\frac{2^*}{2m}-1\right)}{p\frac{2^*}{2}-qm}\sum_{i=0}^{\infty}\left(\frac{2m}{2^*}\right)^i=\frac{2m\left(\frac{2^*}{2m}-1\right)}{p\frac{2^*}{2}-qm}\frac{1}{\left(1-\frac{2m}{2^*}\right)}=\frac{2^*}{p\frac{2^*}{2}-qm}=:\tilde{\beta},$$

where in the first inequality we used the fact that

$$\left(p\frac{2^*}{2}-qm\right)\frac{\left[\left(\frac{2^*}{2m}\right)^{i-1}-1\right]}{\frac{2^*}{2m}-1}+pm\geq\left(p\frac{2^*}{2}-qm\right)\frac{\left(\frac{2^*}{2m}\right)^{i-1}}{\frac{2^*}{2m}-1}\Leftrightarrow-\frac{p\frac{2^*}{2}-qm}{\frac{2^*}{2m}-1}+pm\geq 0\Leftrightarrow q\geq p.$$

By letting $k\to+\infty$ in (2.38), we have (2.10). Therefore, the proof of Lemma 2.3 is complete. □

The a-priori estimate (1.12) in Theorem 1.1 follows by the classical interpolation inequality

$$\|v\|_{L^s(B_\rho)}\leq\|v\|_{L^p(B_\rho)}^{\frac{p}{s}}\|v\|_{L^\infty(B_\rho)}^{1-\frac{p}{s}},\tag{2.39}$$

for any $s\geq p$, which permits to estimate the essential supremum of the gradient of the local minimizer in terms of its L^p-norm.

Proof of Theorem 1.1. Let us set

$$V(x):=1+(|Du|(x)-1)_+\tag{2.40}$$

then estimate (2.10) becomes

$$\sup_{x\in B_\rho}|V(x)|\leq\ C\left(\left[\frac{\|1+h\|_{L^r(\Omega)}}{R-\rho}\right]^{\tilde{\beta}}\|V\|_{L^{qm}(B_R)}\right)^{\theta}\tag{2.41}$$

for every ρ,R such that $0<\rho<R\leq\rho+1$ and where $C=C(n,r,p,q,M_1,M_2)$.

In the following we denote

$$s:=qm.$$

At this point, (2.41) and (2.39) give

$$\|V\|_{L^s(B_\rho)}\leq\ C^{1-\frac{p}{s}}\|V\|_{L^p(B_\rho)}^{\frac{p}{s}}\left(\left[\frac{\|1+h\|_{L^r(\Omega)}}{R-\rho}\right]^{\tilde{\beta}}\|V\|_{L^s(B_R)}\right)^{\theta\left(1-\frac{p}{qm}\right)}.\tag{2.42}$$

We observe that

$$\tau:=\theta\left(1-\frac{p}{qm}\right)<1,\tag{2.43}$$

because

$$\theta\left(1-\frac{p}{qm}\right)<1\Leftrightarrow\frac{q\frac{2^*}{2}-qm-\frac{p2^*}{2m}+p}{p\frac{2^*}{2}-qm}<1\Leftrightarrow q<p\left(1-\frac{2}{2^*}+\frac{1}{m}\right)\overset{(2.13)}{=}p\left(1+\frac{2\alpha}{n}\right).$$

For $0<\rho<R$ and for every $k\geq 0$, let us define $\rho_k:=R-(R-\rho)2^{-k}$. By inserting in (2.42) $\rho=\rho_k$ and $R=\rho_{k+1}$, (so that $R-\rho=(R-\rho)2^{-(k+1)}$) we have, for every $k\geq 0$

$$\|V\|_{L^s(B_{\rho_k})}\leq C^{1-\frac{p}{s}}\|V\|^{\frac{p}{s}}_{L^p(B_{\rho_k})}\left(2^{\tilde\beta(k+1)}\left[\frac{\|1+h\|_{L^r(\Omega)}}{(R-\rho)}\right]^{\tilde\beta}\|V\|_{L^s(B_{\rho_{k+1}})}\right)^{\tau}. \tag{2.44}$$

By iteration of (2.44), we deduce for $k\geq 0$

$$\begin{aligned}\|V\|_{L^s(B_{\rho_0})} &\leq \left(C^{1-\frac{p}{s}}\left[\frac{\|1+h\|_{L^r(\Omega)}}{(R-\rho)}\right]^{\tilde\beta\tau}\|V\|^{\frac{p}{s}}_{L^p(B_{\rho_k})}\right)^{\sum_{i=0}^{k}\tau^i}\\ &\quad\times 2^{\tilde\beta\sum_{i=0}^{k+1}i\tau^i}(\|V\|_{L^s(B_{\rho_{k+1}})})^{\tau^{k+1}}. \end{aligned}\tag{2.45}$$

By (2.43), the series appearing in (2.45) are convergent. Since

$$\|V\|_{L^s(B_{\rho_k})}\leq\|V\|_{L^s(B_R)},$$

we can pass to the limit as $k\to+\infty$ and we obtain for every $0<\rho<R$ with a constant $C=C(n,r,p,q,M_1,M_2)$ independent of k

$$\|V\|_{L^s(B_\rho)}\leq C\left(\left[\frac{\|1+h\|_{L^r(\Omega)}}{(R-\rho)}\right]^{\tilde\beta\tau}\|V\|^{\frac{p}{s}}_{L^p(B_R)}\right)^{\frac{1}{1-\tau}}. \tag{2.46}$$

Combining (2.41) and (2.46), by setting $\rho'=\frac{(R+\rho)}{2}$ we have

$$\begin{aligned}\|V\|_{L^\infty(B_\rho)} &\leq C\left(\left[\frac{\|1+h\|_{L^r(\Omega)}}{(\rho'-\rho)}\right]^{\tilde\beta}\|V\|_{L^s(B_{\rho'})}\right)^{\theta}\\ &\leq C\left(\left[\frac{\|1+h\|_{L^r(\Omega)}}{(\rho'-\rho)}\right]^{\tilde\beta(1-\tau)}\left[\frac{1+\|h\|_{L^r(\Omega)}}{(R-\rho')}\right]^{\tilde\beta\tau}\|V\|^{\frac{p}{s}}_{L^p(B_R)}\right)^{\frac{\theta}{1-\tau}};\end{aligned}$$

now, since

$$(\rho'-\rho)=(R-\rho')=\frac{R-\rho}{2},$$

we get

$$\|Du\|_{L^\infty(B_\rho;\mathbb{R}^n)}\leq C\left(\left[\frac{\|1+h\|_{L^r(\Omega)}}{(R-\rho)}\right]^{\beta}\left(\int_{B_R}\{1+|Du|^p\}\,dx\right)^{1/p}\right)^{\gamma},$$

where

$$\beta:=\tilde\beta\frac{qm}{p}=\frac{\frac{2^*q}{p}m}{p\frac{2^*}{2}-qm},\qquad \gamma:=\frac{\theta p}{qm\left(1-\theta\left(1-\frac{p}{qm}\right)\right)},\qquad \theta:=\frac{qm\left(\frac{2^*}{2m}-1\right)}{p\frac{2^*}{2}-qm},$$

so (1.12) follows.

□

Let now f satisfy (1.7) and (1.14). Under these assumptions on f, we obtain the following result.

Theorem 2.6. *Let $u \in W^{1,p}(\Omega)$ be a local minimizer of the integral functional* (1.1)*. Assume that $f = f(x,\xi)$ in* (1.1) *satisfies* (1.7), (1.14) *and* (2.8)*, with*

$$\frac{q}{p} < 1 + \frac{1}{n} - \frac{1}{r}. \tag{2.47}$$

Then there exist positive constants $C, \hat{\beta}, \hat{\gamma}$ depending on $n, r, p, q, M_1, M_2, \rho, R$ such that, for every $0 < \rho < R \leq \rho + 1$

$$\|Du\|_{L^\infty(B_\rho;\mathbb{R}^n)} \leq C \, \left[\|1+h\|_{L^r(\Omega)}\right]^{\hat{\beta}\,\hat{\gamma}} \left(\int_{B_R} \{1 + f(x,Du)\}\, dx\right)^{\frac{\hat{\gamma}}{p}}. \tag{2.48}$$

Proof. Let $t := 2q - p$. Then (1.14) can be written explicitly in the form

$$|f_{\xi x}(x,\xi)| \leq \, h(x)\, |\xi|^{\frac{t+p-2}{2}}, \qquad |\xi| \geq \, 1.$$

Moreover (2.47) in terms of p and t is equivalent to

$$\frac{t}{p} < 1 + \frac{2\alpha}{n} \qquad \text{with } \ \alpha = 1 - \frac{n}{r}.$$

Thus all the assumptions of Theorem 1.1 are satisfied with q replaced by t. In particular the second inequality in (1.7) holds with q replaced by t since $t \geq q$. Then the conclusion of Theorem 1.1 holds with $q = t$ which corresponds to (2.48) with

$$\hat{\beta} := \frac{\frac{2^*}{p}(2q-p)m}{p\frac{2^*}{2} - (2q-p)m} \qquad\qquad \hat{\gamma} := \frac{\theta p}{(2q-p)m(1-\theta) + p\theta},$$

since $f(x,\xi) \geq \, C|\xi|^p$ for every $|\xi| \geq 1$. $\square$

3. Extension of the integral energy

Let $f : \Omega \times \mathbb{R}^n \to [0, +\infty)$ be a continuous function, convex in ξ such that

$$|\xi|^p \leq f(x,\xi) \leq C(1 + |\xi|^q) \qquad \text{for a.e. } x \in \Omega, \text{ for all } \xi \in \mathbb{R}^n. \tag{3.1}$$

For $u_0 \in W^{1,q}(\Omega)$, we define the extension to $W^{1,p}(\Omega)$ of the integral functional $\int_\Omega f(x,Du)\, dx$, i.e.

$$F(u) = \inf\left\{\liminf_k \int_\Omega f(x,Du_k)\, dx, \;\; u_k \in W_0^{1,q}(\Omega) + u_0, \;\; u_k \xrightarrow{w} u \text{ in } W^{1,p}(\Omega)\right\} \tag{3.2}$$

with

$$\int_\Omega f(x,Du_0)\, dx < +\infty.$$

It is easy to check that

$$F(u) = \int_\Omega f(x,Du)\, dx \qquad \text{for } u \in W_0^{1,q}(\Omega) + u_0.$$

In fact, for $u_k = u$ for all k

$$F(u) \leq \int_\Omega f(x,Du)\, dx.$$

On the other hand, by the semicontinuity of $\int_\Omega f(x,Du)\, dx$ with respect to the weak topology of $W^{1,p}$, the inverse inequality also holds.

Lemma 3.1. *For each $v \in W_0^{1,p}(\Omega) + u_0$, there exists a sequence $v_k \in W_0^{1,q}(\Omega) + u_0$ such that $v_k \rightharpoonup v$ weakly in $W^{1,p}(\Omega)$ and*

$$F(v) = \lim_{k \to +\infty} \int_\Omega f(x, Dv_k)\, dx.$$

Proof. The proof follows similarly as in [6]. We give the sketch of the proof.
Let $v \in W_0^{1,p}(\Omega) + u_0$ such that $F(v) < \infty$. Then, for all k, there exists $v_h^{(k)} \in W_0^{1,q}(\Omega) + u_0$ such that $v_h^{(k)} \overset{w}{\rightharpoonup} v$, as $h \to +\infty$, weakly in $W^{1,p}(\Omega)$ and

$$F(v) \le \lim_{h \to +\infty} \int_\Omega f(x, Dv_h^{(k)})\, dx \le F(v) + \frac{1}{k}.$$

Moreover by the weak convergence of $v_h^{(k)}$ in $W^{1,p}(\Omega)$ we get

$$\lim_{h \to +\infty} \|v_h^{(k)} - v\|_{L^p(\Omega)} = 0$$

and for h sufficiently large

$$\int_\Omega |Dv_h^{(k)}|^p\, dx \le \int_\Omega f(x, Dv_h^{(k)})\, dx \le F(v) + 1.$$

Then

$$\forall k \ \ \exists h_k : \ \forall h \ge h_k \ \ \|v_h^{(k)} - v\|_{L^p(\Omega)} < \frac{1}{k}$$

and for $h = h_k$, by denoting $w_k = v_{h_k}^{(k)}$, we have

$$\|w_k - v\|_{L^p(\Omega)} < \frac{1}{k} \qquad \text{and} \qquad \int_\Omega |Dw_k|^p\, dx \le C,$$

then $w_k \overset{w}{\rightharpoonup} v$, as $k \to +\infty$, in the weak topology of $W^{1,p}(\Omega)$ and

$$F(v) \le \int_\Omega f(x, Dw_k)\, dx \le F(v) + \frac{1}{k}$$

i.e.

$$\lim_{k \to +\infty} \int_\Omega f(x, Dw_k)\, dx = F(v).$$

□

4. Existence and regularity

First of all we prove an approximation theorem for f through a suitable sequence of regular functions.

Proposition 4.1. *Let f be satisfying the growth conditions (3.1), $f_{\xi\xi}$ and $f_{\xi x}$ Carathéodory functions, satisfying (1.7) and (1.14) with $M_0 = 1$ and f strictly convex at infinity. Then there exists a sequence of $\mathcal{C}^2-$functions $f^{\ell k} : \Omega \times \mathbb{R}^n \to [0, +\infty)$, $f^{\ell k}$ convex in the last variable and strictly convex at infinity, such that $f^{\ell k}$ converges to f as $\ell \to \infty$ and $k \to \infty$ for a.e. $x \in \Omega$, for all $\xi \in \mathbb{R}^n$ and uniformly in $\Omega_0 \times K$ where $\Omega_0 \subset\subset \Omega$ and K being a compact set of $\mathbb{R}^n$. Moreover:*

- *there exists $\tilde{C}$, independently of k, ℓ such that*

$$|\xi|^p \le f^{\ell k}(x, \xi) \le \tilde{C}(1 + |\xi|^q) \qquad \text{for a.e. } x \in \Omega, \text{ for all } \xi \in \mathbb{R}^n, \tag{4.1}$$

- *there exists $\tilde{M}_1 > 0$ such that for $|\xi| > 2$ and a.e. $x \in \Omega$*

$$\tilde{M}_1 |\xi|^{p-2}|\lambda|^2 \le \sum_{i,j} f^{\ell k}_{\xi_i \xi_j}(x,\xi)\lambda_i\lambda_j \qquad \lambda \in \mathbb{R}^n, \tag{4.2}$$

- *there exists $c(k) > 0$ such that for all $(x,\xi) \in \Omega \times \mathbb{R}^n$ and $\lambda \in \mathbb{R}^n$*

$$c(k)(1+|\xi|^2)^{\frac{q-2}{2}}|\lambda|^2 \le \sum_{i,j} f^{\ell k}_{\xi_i \xi_j}(x,\xi)\lambda_i\lambda_j, \tag{4.3}$$

- *there exists $\tilde{M}_2 > 0$ such that for $|\xi| > 2$ and a.e. $x \in \Omega$*

$$|f^{\ell k}_{\xi\xi}(x,\xi)| \le \tilde{M}|\xi|^{q-2} \tag{4.4}$$

- *there exists $C(k)$ such that for a.e. $x \in \Omega$ and $\xi \in \mathbb{R}^n$*

$$|f^{\ell k}_{\xi\xi}(x,\xi)| \le C(k)(1+|\xi|^2)^{\frac{q-2}{2}} \tag{4.5}$$

- *there exists a constant $C > 0$ such that for a.e. $x \in \Omega$ and $|\xi| > 2$*

$$|f^{\ell k}_{\xi x}(x,\xi)| \le C\, h_\ell(x)|\xi|^{q-1} \tag{4.6}$$

where $h_\ell \in \mathcal{C}^\infty(\Omega)$ is the regularized function of h which converges to h in $L^r(\Omega)$
- *for $\Omega_0 \subset\subset \Omega$, there exists a constant C such that for $x \in \Omega_0$ and $\xi \in \mathbb{R}^n$*

$$|f^{\ell k}_{\xi x}(x,\xi)| \le C(k,\ell,\Omega_0)(1+|\xi|^2)^{\frac{q-1}{2}}. \tag{4.7}$$

Proof. We argue as in the proof of Theorem 2.7 (Step 3) of [25] and Lemma 4.3 of [19]. For the sake of completeness, we give a sketch of the arguments of the proof.

Let B be the unit ball of $\mathbb{R}^n$ centered in the origin and consider a positive decreasing sequence $\varepsilon_\ell \to 0$. We introduce

$$f^\ell(x,\xi) = \int_{B\times B} \rho(y)\rho(\eta) f(x+\varepsilon_\ell y, \xi+\varepsilon_\ell \eta)\, d\eta\, dy$$

where ρ is a positive symmetric mollifier, and

$$f^{\ell k}(x,\xi) = f^\ell(x,\xi) + \frac{1}{k}(1+|\xi|^2)^{\frac{q}{2}}. \tag{4.8}$$

It is easy to check that the sequence $f^{\ell k}$ satisfies the conditions (4.1), (4.2), (4.3), (4.4), (4.5), (4.7). Let us verify (4.6). For $|\xi| > 2$ we have

$$|f^{\ell k}_{\xi x}(x,\xi)| \le \int_{B\times B} \rho(y)\rho(\eta)|\xi+\varepsilon_\ell\eta|^{q-1} h(x+\varepsilon_\ell y) dy d\eta \le \; C\, h_\ell(x)|\xi|^{q-1}$$

where

$$h_\ell(x) = \int_B \rho(y) h(x+\varepsilon_\ell y)\, dy$$

h_ℓ is a smooth function and it converges to h in $L^r(\Omega)$. Moreover

$$|f^{\ell k}_{\xi x}(x,\xi)| \le \; C(k,\Omega_0)[\|1+h_\ell\|_{L^\infty(\Omega_0)}](1+|\xi|^2)^{\frac{q-1}{2}}.$$

This concludes the proof. □

Proof of Theorem 1.2. For $u_0 \in W^{1,q}(\Omega)$, let us consider the variational problems

$$\inf\left\{\int_\Omega f^{\ell k}(x, Dv)\,dx,\ \ v\in W_0^{1,q}(\Omega)+u_0\right\}, \tag{4.9}$$

where $f^{\ell k}$ are defined in (4.8). By semicontinuity arguments, there exists $v^{\ell k}\in u_0+W_0^{1,q}(\Omega)$ solution to (4.9). By the growth conditions and the minimality of $v^{\ell k}$, we get

$$\begin{aligned}\int_\Omega |Dv^{\ell k}|^p\,dx &\le \int_\Omega f^{\ell k}(x, Dv^{\ell k})\,dx\\ &\le \int_\Omega f^{\ell k}(x, Du_0)\,dx\\ &= \int_\Omega f^{\ell}(x, Du_0)\,dx+\frac{1}{k}\int_\Omega (1+|Du_0|^2)^{\frac{q}{2}}\,dx.\end{aligned}$$

Moreover the properties of the convolutions imply that

$$f^\ell(x, Du_0)\overset{\ell\to\infty}{\to} f(x, Du_0)\qquad \text{a.e. in }\Omega$$

and since

$$\int_\Omega f^\ell(x,Du_0)\,dx\le C\int_\Omega (1+|Du_0|^2)^{\frac{q}{2}}\,dx,$$

by the Lebesgue Dominated Convergence Theorem we deduce therefore

$$\begin{aligned}\lim_{\ell\to\infty}\int_\Omega |Dv^{\ell k}|^p\,dx &\le \lim_{\ell\to\infty}\int_\Omega f^{\ell}(x, Du_0)\,dx+\frac{1}{k}\int_\Omega (1+|Du_0|^2)^{\frac{q}{2}}\,dx\\ &= \int_\Omega f(x, Du_0)\,dx+\frac{1}{k}\int_\Omega (1+|Du_0|^2)^{\frac{q}{2}}\,dx.\end{aligned}$$

By Proposition 4.1, $f^{\ell k}$ satisfy (1.7), (1.14) and (2.8), so we can apply the a-priori estimate (2.48) to $v^{\ell k}$ and obtain, by standard covering arguments $\forall\,\Omega'\subset\subset\Omega$

$$\|Dv^{\ell k}\|_{L^\infty(\Omega';\mathbb{R}^n)}\le C(\Omega')\left[\|1+h_\ell\|_{L^r(\Omega)}\right]^{\hat\beta\hat\gamma}\left[\int_\Omega (1+f^{\ell k}(x,Dv^{\ell k}))\,dx\right]^{\frac{\hat\gamma}{p}}.$$

Since $\|1+h_\ell\|_{L^r(\Omega)}=\|(1+h)_\ell\|_{L^r(\Omega)}\le\|1+h\|_{L^r(\Omega)}$, we obtain

$$\begin{aligned}\|Dv^{\ell k}\|_{L^\infty(\Omega';\mathbb{R}^n)} &\le C(\Omega')\left[\|1+h\|_{L^r(\Omega)}\right]^{\hat\beta\hat\gamma}\left[\int_\Omega (1+f^{\ell k}(x,Dv^{\ell k}))\,dx\right]^{\frac{\hat\gamma}{p}}\\ &\le C(\Omega')\left[\|1+h\|_{L^r(\Omega)}\right]^{\hat\beta\hat\gamma}\left[\int_\Omega 1+f^{\ell}(x,Du_0)+\frac{1}{k}(1+|Du_0|^2)^{\frac{q}{2}}dx\right]^{\frac{\hat\gamma}{p}},\end{aligned}$$

where $C,\hat\gamma,\hat\beta$ depend on n,r,p,q,M_1,M_2,ρ,R but are independent of ℓ,k. Therefore we conclude that

$$\begin{aligned}&v^{\ell k}\overset{\ell\to\infty}{\to} v^k\qquad \text{weakly in } W_0^{1,p}(\Omega)+u_0\\ &v^{\ell k}\overset{\ell\to\infty}{\to} v^k\qquad \text{weakly star in } W^{1,\infty}_{\mathrm{loc}}(\Omega)\end{aligned}$$

and by the previous estimates

$$\|Dv^k\|_{L^p(\Omega;\mathbb{R}^n)} \le \liminf_{\ell\to\infty} \|Dv^{\ell k}\|_{L^p(\Omega;\mathbb{R}^n)} \le \int_\Omega f(x,Du_0)\,dx + \int_\Omega (1+|Du_0|^2)^{\frac{q}{2}}\,dx$$

and

$$\begin{aligned}\|Dv^k\|_{L^\infty(\Omega';\mathbb{R}^n)} &\le \liminf_{\ell\to\infty} \|Dv^{\ell k}\|_{L^\infty(\Omega';\mathbb{R}^n)}\\ &\le C(\Omega')\left[\|1+h\|_{L^r(\Omega)}\right]^{\hat\beta\hat\gamma}\left[\int_\Omega 1+f(x,Du_0)\,dx + \int_\Omega (1+|Du_0|^2)^{\frac{q}{2}}\,dx\right]^{\frac{\hat\gamma}{p}}.\end{aligned}$$

Thus we can deduce that there exists, up to subsequences, $\bar u \in u_0 + W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} v^k &\to \bar u \qquad \text{weakly in } W_0^{1,p}(\Omega)+u_0\\ v^k &\to \bar u \qquad \text{weakly star in } W^{1,\infty}_{\mathrm{loc}}(\Omega).\end{aligned}$$

Now, for any fixed $k\in\mathbb{N}$, using the uniform convergence of f^ℓ to f in $\Omega_0\times K$ (for any K compact subset of $\mathbb{R}^n$) and the minimality of $v^{\ell k}$, we get, for all $v \in W_0^{1,q}(\Omega)+u_0$

$$\begin{aligned}\int_{\Omega_0} f(x,Dv^k)\,dx &\le \liminf_{\ell\to\infty}\int_{\Omega_0} f(x,Dv^{\ell k})\,dx\\ &= \liminf_{\ell\to\infty}\int_{\Omega_0} f^\ell(x,Dv^{\ell k})\,dx\\ &\le \liminf_{\ell\to\infty}\int_{\Omega_0} f^\ell(x,Dv^{\ell k})\,dx + \frac{1}{k}\int_\Omega (1+|Dv^{\ell k}|^2)^{\frac{q}{2}}\,dx\\ &\le \liminf_{\ell\to\infty}\int_{\Omega} f^\ell(x,Dv^{\ell k})\,dx + \frac{1}{k}\int_\Omega (1+|Dv^{\ell k}|^2)^{\frac{q}{2}}\,dx\\ &\le \liminf_{\ell\to\infty}\int_{\Omega} f^\ell(x,Dv)\,dx + \frac{1}{k}\int_\Omega (1+|Dv|^2)^{\frac{q}{2}}\,dx.\end{aligned}$$

Then, for $\Omega_0\to\Omega$

$$\int_\Omega f(x,Dv^k)\,dx \le \int_\Omega f(x,Dv)\,dx + \frac{1}{k}\int_\Omega (1+|Dv|^2)^{\frac{q}{2}}\,dx.$$

By the definition (3.2), we have

$$F(\bar u) \le \liminf_{k\to\infty}\int_\Omega f(x,Dv^k)\,dx \le \int_\Omega f(x,Dv)\,dx \qquad \forall v\in W_0^{1,q}(\Omega)+u_0. \tag{4.10}$$

Let $w\in W_0^{1,p}(\Omega)+u_0$. By Lemma 3.1, there exists $v_k\in W_0^{1,q}(\Omega)+u_0$ such that $v_k \rightharpoonup w$ weakly in $W^{1,p}(\Omega)$ and

$$\lim_{k\to\infty}\int_\Omega f(x,Dv_k)\,dx = F(w).$$

By (4.10)

$$F(\bar u)\le \int_\Omega f(x,Dv_k)\,dx,$$

we can conclude that

$$F(\bar{u}) \leq \lim_{k\to\infty} \int_\Omega f(x, Dv_k)\, dx = F(w) \qquad \forall w \in W_0^{1,p}(\Omega) + u_0,$$

then $\bar{u} \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ is a solution to the problem $\min\{F(u) : u \in W_0^{1,p}(\Omega) + u_0\}$.

□

5. Regularity of local minimizers in a special case

Let us consider now the case of a special form of integrand

$$f(x, \xi) = \sum_{i=1}^{N} a_i(x) g_i(\xi), \tag{5.1}$$

with $a_i(x) > 0$ a.e. in Ω, $a_i \in W^{1,r}(\Omega)$, $r > n$, $g_i : \mathbb{R}^n \to [0, +\infty)$ convex in ξ and strictly convex for ξ such that $|\xi| \geq M_0$. The following regularity result holds.

Theorem 5.1. *Assume that $f = f(x, \xi)$ as in* (5.1) *satisfies the assumptions of Theorem 2.6. Then every local minimizer $u \in W^{1,p}(\Omega)$ of the integral functional*

$$\int_\Omega f(x, Dv)\, dx = \int_\Omega \sum_{i=1}^{N} a_i(x) g_i(Du)\, dx \tag{5.2}$$

is locally Lipschitz continuous in Ω.

Proof. Let $u \in W^{1,p}(\Omega)$ be a local minimizer of (5.2). For a suitable φ_σ mollifier, consider $u_\sigma = u * \varphi_\sigma \in W^{1,q}_{\mathrm{loc}}(\Omega)$. Consider the sequence of problems in $B_R \subset\subset \Omega$

$$\inf \left\{ \int_{B_R} f^{\ell k}(x, Dv)\, dx, \;\; v \in W_0^{1,q}(B_R) + u_\sigma \right\}, \tag{5.3}$$

where $f^{\ell k}$ are defined in Proposition 4.1.

For fixed σ, ℓ, k, problem (5.3) has a unique solution $v_\sigma^{\ell k} \in W_0^{1,q}(B_R) + u_\sigma$. By proceeding as in the previous theorem, we have that for each fixed σ, by the minimality of $v_\sigma^{\ell k}$

$$\begin{array}{ll} v_\sigma^{\ell k} \overset{\ell\to\infty}{\rightharpoonup} v_\sigma^k & \text{weakly in } W_0^{1,p}(B_R) + u_\sigma \\ v_\sigma^{\ell k} \overset{\ell\to\infty}{\rightharpoonup} v_\sigma^k & \text{weakly star in } W^{1,\infty}_{\mathrm{loc}}(B_R). \end{array}$$

We also have

$$\begin{array}{ll} v_\sigma^{k} \overset{k\to\infty}{\rightharpoonup} v_\sigma & \text{weakly in } W_0^{1,p}(B_R) + u_\sigma \\ v_\sigma^{k} \overset{k\to\infty}{\rightharpoonup} v_\sigma & \text{weakly star in } W^{1,\infty}_{\mathrm{loc}}(B_R) \end{array}$$

and

$$\begin{aligned} \|Dv_\sigma\|_{L^\infty(B_\rho;\mathbb{R}^n)} &\leq C \liminf_{k\to\infty} \left[1 + \int_{B_R} f(x, Du_\sigma)\, dx + \frac{1}{k} \int_{B_R} (1 + |Du_\sigma|^2)^{\frac{q}{2}}\, dx \right]^{\frac{\hat{\gamma}}{p}} \\ &= C \liminf_{k\to\infty} \left[1 + \int_{B_R} f(x, Du_\sigma)\, dx \right]^{\frac{\hat{\gamma}}{p}} \end{aligned} \tag{5.4}$$

for any $0 < \rho < R$ and where C independent of k, σ. For fixed k, by proceeding as in the previous theorem we have

$$\int_{B_R} f(x, Dv_\sigma^k)\,dx \le \int_{B_R} f(x, Du_\sigma)\,dx + \frac{1}{k}\int_{B_R}(1+|Du_\sigma|^2)^{\frac{q}{2}}\,dx$$

then, by the semicontinuity

$$\begin{aligned}\int_{B_R} f(x, Dv_\sigma)\,dx &\le \liminf_{k\to\infty}\int_{B_R} f(x, Du_\sigma)\,dx + \frac{1}{k}\int_{B_R}(1+|Du_\sigma|^2)^{\frac{q}{2}}\,dx \\ &\le \int_{B_R} f(x, Du_\sigma)\,dx. \qquad (5.5)\end{aligned}$$

Now we claim that, by the particular form of f, we may deduce

$$\liminf_{\sigma\to 0}\int_{B_R} f(x, Du_\sigma)\,dx \le \int_{B_R} f(x, Du)\,dx. \qquad (5.6)$$

Since g_i is convex, for $i = 1, \dots, N$, Jensen's inequality (applied to each g_i) yields

$$\begin{aligned}&\int_{B_R} a_i(x) g_i(Du_\sigma)\,dx \\ =\ &\int_{B_R} a_i(x) g_i\left(\int_{B_\sigma} Du(y)\varphi_\sigma(x-y)\,dy\right)dx \\ \le\ &\int_{B_R} a_i(x)\int_{B_\sigma} g_i(Du(y))\varphi_\sigma(x-y)\,dy\,dx \\ =\ &\int_{B_R}\int_{B_\sigma} a_i(x)\varphi_\sigma(x-y)\,dy\,g_i(Du(y))\,dx \\ \le\ &\int_{B_{R+\sigma}} (a_i)_\sigma(y) g_i(Du(y))\,dy,\end{aligned}$$

Then

$$\sum_{i=1}^N \int_{B_R} a_i(x) g_i(Du_\sigma)\,dx \le \sum_{i=1}^N \int_{B_{R+\sigma}} (a_i)_\sigma(x) g_i(Du)\,dx$$

so that passing to the limit as $\sigma \to 0$

$$\liminf_{\sigma\to 0}\sum_{i=1}^N \int_{B_R} a_i(x) g_i(Du_\sigma)\,dx \le \sum_{i=1}^N \int_{B_R} a_i(x) g_i(Du)\,dx$$

because $(a_i)_\sigma \to a_i$ in $L^\infty(B_R)$, $g_i(Du) \in L^1(B_R)$, the Dominated Convergence Theorem may be applied, and (5.6) holds.

By collecting (5.5) and (5.6)

$$\liminf_{\sigma\to 0}\int_{B_R} f(x, Dv_\sigma)\,dx \le \int_{B_R} f(x, Du)\,dx. \qquad (5.7)$$

On the other hand, the growth assumption on f yields, since u is a local minimizer of (5.2)

$$\liminf_{\sigma\to 0}\int_{B_R}|Dv_\sigma|^p\,dx \le \liminf_{\sigma\to 0}\int_{B_R} f(x, Dv_\sigma)\,dx \overset{(5.7)}{\le} \int_{B_R} f(x, Du)\,dx < +\infty.$$

Thus there exists $\bar{v} \in u + W_0^{1,p}(B_R)$ such that, up to a subsequence

$$v_\sigma \rightharpoonup \bar{v} \text{ weakly in } W^{1,p}(B_R).$$

By the semicontinuity of the functional, using (5.5) and (5.7)

$$\int_{B_R} f(x, D\bar{v})\, dx \leq \liminf_{\sigma \to 0} \int_{B_R} f(x, Dv_\sigma)\, dx \leq \int_{B_R} f(x, Du)\, dx. \tag{5.8}$$

Moreover, since (5.4) holds, Dv_σ converges to $D\bar{v}$ as $\sigma \to 0$ in the weak star topology of L^∞ and there exists a constant C such that, for any $0 < \rho < R$

$$\|D\bar{v}\|_{L^\infty(B_\rho;\mathbb{R}^n)} \leq C \left[1 + \int_{B_R} f(x, Du)\, dx\right]^{\frac{\hat{\gamma}}{p}}.$$

Consider the problem in $B_R \subset\subset \Omega$

$$\inf \left\{ \int_{B_R} f(x, Dv)\, dx : \; v \in W_0^{1,p}(B_R) + u \right\}. \tag{5.9}$$

Then (5.8) implies that $\bar{v}$ and u are solutions to (5.9) and $\bar{v} \in W^{1,\infty}_{\mathrm{loc}}(B_R)$.

In the present case the functional is not strictly convex, then we proceed as in Theorem 2.1 in [19] (see also [25]) and we have that $u \in W^{1,\infty}_{\mathrm{loc}}(B_R)$. Indeed, set

$$E_0 := \left\{ x \in B_R : \left| \frac{Du(x) + D\bar{v}(x)}{2} \right| > M_0 \right\} \quad \text{and} \quad w := \frac{u + \bar{v}}{2}.$$

If E_0 has positive measure, then from the convexity of $f(x, \cdot)$ we have

$$\int_{B_R \setminus E_0} f(x, Dw)\, dx \leq \frac{1}{2} \int_{B_R \setminus E_0} f(x, Du)\, dx + \frac{1}{2} \int_{B_R \setminus E_0} f(x, D\bar{v})\, dx. \tag{5.10}$$

Now, by the strict convexity of $f(x, \xi)$ for ξ such that $|\xi| \geq M_0$ and applying two times the following inequality

$$f(x, \eta) > f(x, \xi) + \langle f_\xi(x, \xi), \eta - \xi \rangle, \qquad \text{for } \xi \text{ such that } |\xi| \geq M_0$$

first with $\xi = Dw$ and $\eta = Du$, then for $\xi = Dw$ and $\eta = D\bar{v}$, finally by adding up the two inequalities obtained, we have

$$\int_{B_R \cap E_0} f(x, Dw)\, dx < \frac{1}{2} \int_{B_R \cap E_0} f(x, Du)\, dx + \frac{1}{2} \int_{B_R \cap E_0} f(x, Dv)\, dx. \tag{5.11}$$

Adding (5.10) and (5.11), we get a contradiction with the minimality of u and $\bar{v}$. Therefore the set E_0 has zero measure, which implies that

$$\sup_{B_\rho} |Du(x)| \leq \sup_{B_\rho} |Du(x) + D\bar{v}(x)| + \sup_{B_\rho} |D\bar{v}(x)| \leq 2M_0 + \sup_{B_\rho} |D\bar{v}(x)|$$

and this yields the thesis. □

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