A STABILITY RESULT FOR NONLINEAR NEUMANN PROBLEMS UNDER BOUNDARY VARIATIONS

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Abstract. In this paper we study, in dimension two, the stability of the solutions of some nonlinear elliptic equations with Neumann boundary conditions, under perturbations of the domains in the Hausdorff complementary topology. More precisely, for every bounded open subset Ω of $\mathbb{R}^2$, we consider the problem

$$\begin{cases} -\operatorname{div} a(x, \nabla u_\Omega) + b(x, u_\Omega) = 0 & \text{in} \quad \Omega, \\ a(x, \nabla u_\Omega) \cdot \nu = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $a\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ and $b\colon \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ are two Carathéodory functions which satisfy the standard monotonicity and growth conditions of order p, with $1 < p \leq 2$.

Let Ω_n be a uniformly bounded sequence of open sets in $\mathbb{R}^2$, whose complements Ω_n^c have a uniformly bounded number of connected components. We prove that, if $\Omega_n^c \to \Omega^c$ in the Hausdorff metric and $|\Omega_n| \to |\Omega|$, then $u_{\Omega_n} \to u_\Omega$ and $\nabla u_{\Omega_n} \to \nabla u_\Omega$ strongly in L^p. The proof is obtained by showing the Mosco convergence of the Sobolev spaces $W^{1,p}(\Omega_n)$ to the Sobolev space $W^{1,p}(\Omega)$.

Key words: Boundary value problems, nonlinear elliptic equations, capacity, Hausdorff metric, monotone operators, Mosco convergence

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1. Introduction

In this paper we consider nonlinear elliptic equations with Neumann boundary conditions of the form

$$\begin{cases} -\operatorname{div} a(x, \nabla u) + b(x, u) = 0 & \text{in} \quad \Omega, \\ a(x, \nabla u) \cdot \nu = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^2$ and $a\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ and $b\colon \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ are two Carathéodory functions which satisfy suitable monotonicity, coerciveness, and growth conditions (see (2.2)-(2.4) below). Our purpose is to study the continuity of the mapping $\Omega \mapsto u_\Omega$ which associates to every Ω the corresponding solution u_Ω. The notion of convergence we consider on the sets Ω is given by the Hausdorff complementary topology, which is induced by the Hausdorff distance of the complements of the sets Ω (see Section 2.3).

Many examples (see [13], [23], [14], [10], [11]) show that, if we consider a uniformly bounded sequence Ω_h of open subsets of $\mathbb{R}^2$ which converges to an open set Ω in the Hausdorff complementary topology, it may happen that u_{Ω_h} does not converge to u_Ω. Therefore, in this paper we investigate additional conditions on Ω_h which imply that u_{Ω_h} converges to u_Ω for any choice of the functions a and b. These stability results are useful in the proof of the existence of solutions of some shape optimization problems. More recently, similar stability results have been applied in [12] to study some models in fracture mechanics.

In the special case of the linear problems

$$\begin{cases} -\Delta u + u = f & \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.2}$$

the stability with respect to Ω was first studied by Chenais [9] under the assumption that the domains Ω_h satisfy a uniform cone condition, which allows to use extension operators with uniformly bounded norms. This condition excludes a large class of domains, like for instance domains with cracks, for which there is an increasing interest in view of the applications to fracture mechanics.

The stability of (1.2) in nonsmooth domains is studied in [5], [6], [7] under variuos assumptions on Ω_h. In [8] the problem is studied under the hypothesis that Ω_h converges to Ω in the Hausdorff complementary topology, assuming also a uniform bound for the lenghts of the boundaries $\mathcal{H}^1(\partial\Omega_h)$ and for the number of connected components of $\partial\Omega_h$. This result has been recently improved in [2], where the bound on $\mathcal{H}^1(\partial\Omega_h)$ is replaced by the weaker assumption of convergence of the two-dimensional measures of the domains, i.e., $|\Omega_h| \to |\Omega|$, which is also necessary for the stability of (1.2).

In the present paper we study the stability of the nonlinear problems (1.1) with respect to the Hausdorff complementary topology, assuming that $|\Omega_h| \to |\Omega|$ and that the number of connected components of the complements Ω_h^c is uniformly bounded. To obtain this result we reduce the problem to the convergence in the sense of Mosco of the Sobolev spaces $W^{1,p}(\Omega_h)$ to the Sobolev space $W^{1,p}(\Omega)$, where the exponent p is related in the usual way to the growth condition of the functions a and b (see Section 2.6).

The proof of this property for $1 < p \leq 2$ is obtained in two steps. First, under the same assumptions on Ω_h and Ω, we prove the continuity of the map $\Omega \mapsto \nabla u_\Omega$ for the solutions u_Ω of following nonlinear Neumann problems

$$\begin{cases} -\operatorname{div} a(x, \nabla u) = 0 & \text{in} \quad \Omega, \\ a(x, \nabla u) \cdot \nu = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.3}$$

This result is obtained by using the fact that the rotation by $\pi/2$ of the vectorfield $a(x, \nabla u)$ (extended to 0 on the complemenent Ω^c) is the gradient of a function v_Ω which is constant on each connected component of Ω^c (Proposition 3.6). This function plays the role of the conjugate of u_Ω used in [3] and [12] in the linear case. Another important ingredient in the proof is a result on the stability of nonlinear Dirichlet problems proved in [4], which allows to show that, if each function v_{Ω_h} is constant on each connected component of Ω_h^c, then their weak limit is constant on each connected component of Ω^c (Lemmas 3.3 and 3.5).

The second step in the proof of the convergence of the Sobolev spaces $W^{1,p}(\Omega_h)$ is the approximation of locally constant functions in Ω by functions belonging to $W^{1,p}(\Omega_h)$, which relies on a result obtained in [2].

The main difference with respect to the linear case studied in [2] and [3] is that for the nonlinear problems (1.3) we can not use the method of conformal mappings.

The hypothesis $p \leq 2$ is used both in the first and in the second step. In the case $p > 2$ the stability result for (1.1) and (1.3) is not true under our hypotheses, as shown in Remarks 3.7 and 4.6. The general form of the limit problem will be studied in [16].

In the last part of the paper we consider the case of unbounded open sets and the case of mixed boundary value problems, with a Dirichlet condition on a fixed part of the boundary.

Contents

2. Notation and preliminaries

Throughout the paper p and q are real numbers, with $1 < p \leq 2 \leq q < +\infty$ and $p^{-1}+q^{-1} = 1$. The scalar product of two vectors ξ, $\zeta \in \mathbb{R}^2$ is denoted by $\xi \cdot \zeta$, and the norm of ξ by $|\xi|$. For any E, $F \subset \mathbb{R}^2$, $E \triangle F := (E \setminus F) \cup (F \setminus E)$ is the symmetric difference of E and F, and $|E|$ is the Lebesgue (outer) measure of E.

2.1. **Deny-Lions spaces.** Given an open subset Ω of $\mathbb{R}^2$, the *Deny-Lions space* is defined by

$$L^{1,p}(\Omega) := \{u \in L^p_{\text{loc}}(\Omega) : \nabla u \in L^p(\Omega, \mathbb{R}^2)\}.$$

It is well-known that $L^{1,p}(\Omega) = W^{1,p}(\Omega)$ whenever Ω is bounded and has a Lipschitz boundary. It is also known that the set $\{\nabla u : u \in L^{1,p}(\Omega)\}$ is a closed subspace of $L^p(\Omega, \mathbb{R}^2)$. For further properties of the spaces $L^{1,p}(\Omega)$ the reader is referred to [15] and [21].

In many problems it is useful to consider the following equivalence relation in $L^{1,p}(\Omega)$:

$$v_1 \sim v_2 \qquad \text{if and only if} \qquad \nabla v_1 = \nabla v_2 \text{ a.e. in } \Omega. \tag{2.1}$$

The corresponding quotient space is denoted by $L^{1,p}(\Omega)/_\sim$.

2.2. **Capacity.** Let $1 < r < \infty$. For every subset E of $\mathbb{R}^2$, the $(1,r)$-*capacity* of E in $\mathbb{R}^2$, denoted by $C_r(E)$, is defined as the infimum of $\int_{\mathbb{R}^2}(|\nabla u|^r + |u|^r)\,dx$ over the set of all functions $u \in W^{1,r}(\mathbb{R}^2)$ such that $u \geq 1$ a.e. in a neighborhood of E. If $r > 2$, then $C_r(E) > 0$ for every nonempty set E. On the contrary, if $r = 2$ there are nonempty sets E with $C_r(E) = 0$ (for instance, $C_r(\{x\}) = 0$ for every $x \in \mathbb{R}^2$).

We say that a property $\mathcal{P}(x)$ holds C_r-*quasi everywhere* (abbreviated as C_r-*q.e.*) in a set E if it holds for all $x \in E$ except a subset N of E with $C_r(N) = 0$. We recall that the expression *almost everywhere* (abbreviated as *a.e.*) refers, as usual, to the Lebesgue measure.

A function $u\colon E \to \overline{\mathbb{R}}$ is said to be *quasi-continuous* if for every ε there exists $A_\varepsilon \subset \mathbb{R}^2$, with $C_r(A_\varepsilon) < \varepsilon$, such that the restriction of u to $E \setminus A_\varepsilon$ is continuous. If $r > 2$ every quasi-continuous function is continuous, while for $r = 2$ there are quasi-continuous functions that are not continuous. It is well known that, for any open subset Ω of $\mathbb{R}^2$, every function $u \in W^{1,r}(\Omega)$ has a *quasi-continuous representative* $\overline{u}\colon \Omega \to \mathbb{R}$, which satisfies

$$\lim_{\rho\to 0^+} \frac{1}{|B(x,\rho)|}\int_{B(x,\rho)} |u(y) - \overline{u}(x)|\,dy = 0 \quad \text{for q.e. } x \in \Omega,$$

where $B(x,\rho)$ is the open ball with centre x and radius ρ. We recall that if u_h converges strongly to u in $W^{1,r}(\Omega)$, then a subsequence of $\overline{u}_h$ converges to $\overline{u}$ pointwise C_r-q.e. on Ω. For these and other properties on quasi-continuous representatives the reader is referred to [17], [19], [21], [25].

To simplify the notation, we always identify each function $u \in W^{1,r}(\Omega)$ with its quasi-continuous representative $\overline{u}$.

2.3. **Convergence of sets.** We recall here the notion of convergence in the sense of *Kuratowski*. We say that a sequence (C_h) of closed subsets of $\mathbb{R}^2$ converges to a closed set C in the sense of Kuratowski if the following two properties hold:

(K_1) for every $x \in C$, there exists a sequence $x_h \in C_h$ such that $x_h \to x$;

(K_2) if (h_k) is a sequence of indices converging to ∞, (x_k) is a sequence such that $x_k \in C_{h_k}$ for every k, and x_k converges to some $x \in \mathbb{R}^2$, then $x \in C$.

Let us recall also that the *Hausdorff distance* between two nonempty closed subsets C_1 and C_2 of $\mathbb{R}^2$ is defined by

$$d_H(C_1, C_2) := \max\left\{ \sup_{x\in C_1} \operatorname{dist}\big(x, C_2\big)\,,\ \sup_{x\in C_2} \operatorname{dist}\big(x, C_1\big) \right\}.$$

We say that a sequence (C_h) of nonempty closed subsets of $\mathbb{R}^2$ converges to a nonempty closed subset C in the *Hausdorff metric* if $d_H(C_h\,, C)$ converges to 0.

A sequence of subsets of $\mathbb{R}^2$ is said to be *uniformly bounded* if there exists a bounded subset of $\mathbb{R}^2$ which contains all sets of the sequence.

The convergence in the Hausdorff metric implies the convergence in the sense of Kuratowski, while in general the converse is false. However, if (C_h) is a uniformly bounded sequence of nonempty closed sets in $\mathbb{R}^2$, then (C_h) converges to a closed set C in the Hausdorff metric if and only if (C_h) converges to C in the sense of Kuratowski.

We say that a sequence (Ω_h) of open subsets of $\mathbb{R}^2$ converges to an open set Ω in the *Hausdorff complementary topology*, if $d_H(\Omega_h^c\,, \Omega^c)$ converges to 0, where Ω_h^c and Ω^c are the complements of Ω_h and Ω in $\mathbb{R}^2$, It is well-known (see, e.g., [18, Blaschke's Selection Theorem]) that every uniformly bounded sequence of nonempty closed sets is compact with respect to the Hausdorff convergence. This implies that every uniformly bounded sequence of open sets is compact with respect to the Hausdorff complementary topology.

Moreover, a uniformly bounded sequence of open sets (Ω_h) converges to an open set Ω in the Hausdorff complementary topology, if and only if the sequence (Ω_h^c) converges to Ω^c in the sense of Kuratowski.

2.4. **The Neumann problems.** Let $a\colon \mathbb{R}^2\times\mathbb{R}^2 \to \mathbb{R}^2$ and $b\colon \mathbb{R}^2\times\mathbb{R} \to \mathbb{R}$ be two Carathéodory functions that satisfy the following assumptions: there exist $0 < c_1 \le c_2$, $\alpha \in L^q(\mathbb{R}^2)$, and $\beta \in L^1(\mathbb{R}^2)$ such that, for almost every $x \in \mathbb{R}^2$ and for every ξ, $\xi_1, \xi_2 \in \mathbb{R}^2$ with $\xi_1 \neq \xi_2$

$$(a(x,\xi_1) - a(x,\xi_2)) \cdot (\xi_1 - \xi_2) > 0; \tag{2.2}$$

$$|a(x,\xi)| \le \alpha(x) + c_2|\xi|^{p-1}; \tag{2.3}$$

$$a(x,\xi)\cdot\xi \ge -\beta(x) + c_1|\xi|^p. \tag{2.4}$$

We assume that b satisfies the same inequalities for every ξ, $\xi_1, \xi_2 \in \mathbb{R}$.

For every open set $\Omega \subset \mathbb{R}^2$, we consider the following nonlinear Neumann boundary value problems, where ν denotes the outward unit normal to $\partial\Omega$:

$$\begin{cases} -\operatorname{div} a(x,\nabla u) + b(x,u) = 0 & \text{in} \quad \Omega, \\ a(x,\nabla u)\cdot\nu = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.5}$$

and

$$\begin{cases} -\operatorname{div} a(x,\nabla v) = 0 & \text{in} \quad \Omega\,, \\ a(x,\nabla v)\cdot\nu = 0 & \text{on} \quad \partial\Omega\,. \end{cases} \tag{2.6}$$

A function u is a solution of (2.5) if

$$\begin{cases} u \in W^{1,p}(\Omega), \\ \displaystyle\int_\Omega \big[a(x,\nabla u)\cdot\nabla z + b(x,u)z\big]\,dx = 0 \quad \forall z \in W^{1,p}(\Omega). \end{cases} \tag{2.7}$$

while v is a solution of (2.6) if

$$\begin{cases} v \in L^{1,p}(\Omega), \\ \displaystyle\int_\Omega a(x,\nabla v)\cdot\nabla z\;dx = 0 \quad \forall z \in L^{1,p}(\Omega). \end{cases} \tag{2.8}$$

By well-known existence results for nonlinear elliptic equations with strictly monotone operators (see, e.g., Lions [20]), one can easily see that (2.7) has a unique solution in $W^{1,p}(\Omega)$. Similarly one can prove that (2.8) has a solution, and that if v_1 and v_2 are solutions of (2.8), then $\nabla v_1 = \nabla v_2$ a.e. in Ω. Note that problem (2.8) can be formulated in the quotient space $L^{1,p}(\Omega)/_\sim$, where a uniqueness result holds.

2.5. **Stability of Neumann problems.** In order to study the stability of (2.5) and (2.6) with respect to the variations of the open set Ω, we should be able to compare two solutions defined in two different domains. For any subset E of $\mathbb{R}^2$, the characteristic function 1_E of E is defined by $1_E(x) := 1$ for $x \in E$ and $1_E(x) := 0$ for $x \in E^c$. For every $u \in L^{1,p}(\Omega)$, the functions $u1_\Omega$ and $\nabla u 1_\Omega$ are the extensions of the functions u and ∇u which vanish in Ω^c. By means of these extensions, $W^{1,p}(\Omega)$ will be identified with the closed linear subspace X_Ω of $L^p(\mathbb{R}^2)\times L^p(\mathbb{R}^2,\mathbb{R}^2)$ defined by

$$X_\Omega := \{(u1_\Omega, \nabla u 1_\Omega) : u \in W^{1,p}(\Omega)\}, \tag{2.9}$$

while the quotient space $L^{1,p}(\Omega)/_\sim$ will be identified with the closed linear subspace Y_Ω of $L^p(\mathbb{R}^2,\mathbb{R}^2)$ defined by

$$Y_\Omega := \{\nabla u 1_\Omega : u \in L^{1,p}(\Omega)\}. \tag{2.10}$$

Let Ω be an open subset of $\mathbb{R}^2$ and let (Ω_h) be a sequence of open subsets of $\mathbb{R}^2$. Given a pair of Carathéodory functions $a\colon \mathbb{R}^2\times\mathbb{R}^2 \to \mathbb{R}^2$ and $b\colon \mathbb{R}^2\times\mathbb{R}\to\mathbb{R}$ satisfying (2.2)–(2.4), let u be the solution to problem (2.5) in Ω and, for every h, let u_h be the solution to problem (2.5) in Ω_h.

Definition 2.1. *We say that Ω is stable for the Neumann problems (2.5) along the sequence (Ω_h) if for every pair of functions a, b satisfying (2.2)–(2.4) the sequence $(u_h 1_{\Omega_h})$ converges to $u1_\Omega$ strongly in $L^p(\mathbb{R}^2)$ and the sequence $(\nabla u_h 1_{\Omega_h})$ converges to $\nabla u 1_\Omega$ strongly in $L^p(\mathbb{R}^2,\mathbb{R}^2)$.*

Similarly, let v be a solution to problem (2.6) in Ω and, for every h, let v_h be a solution to problem (2.6) in Ω_h.

Definition 2.2. *We say that Ω is stable for the Neumann problems (2.6) along the sequence (Ω_h) if for every function a satisfying (2.2)–(2.4) the sequence $(\nabla v_h 1_{\Omega_h})$ converges to $\nabla v 1_\Omega$ strongly in $L^p(\mathbb{R}^2,\mathbb{R}^2)$.*

2.6. **Mosco convergence.** We shall prove that the notion of stability introduced in the previous definitions is equivalent to a notion of convergence for subspaces of a Banach space introduced by Mosco in [22].

Let Ω_h and Ω be open subsets of $\mathbb{R}^2$, and let X_{Ω_h} and X_Ω be the corresponding subspaces defined by (2.9). We recall that X_{Ω_h} converges to X_Ω in the sense of Mosco (see [22, Definition 1.1]) if the following two properties hold:

(M_1) for every $u \in W^{1,p}(\Omega)$, there exists a sequence $u_h \in W^{1,p}(\Omega_h)$ such that $u_h 1_{\Omega_h}$ converges to $u1_\Omega$ strongly in $L^p(\mathbb{R}^2)$ and $\nabla u_h 1_{\Omega_h}$ converges to $\nabla u 1_\Omega$ strongly in $L^p(\mathbb{R}^2,\mathbb{R}^2)$;

(M_2) if (h_k) is a sequence of indices converging to ∞, (u_k) is a sequence such that $u_k \in W^{1,p}(\Omega_{h_k})$ for every k, and $u_k 1_{\Omega_{h_k}}$ converges weakly in $L^p(\mathbb{R}^2)$ to a function ϕ, while $\nabla u_k 1_{\Omega_{h_k}}$ converges weakly in $L^p(\mathbb{R}^2,\mathbb{R}^2)$ to a function ψ, then there exists $u \in W^{1,p}(\Omega)$ such that $\phi = u1_\Omega$ and $\psi = \nabla u 1_\Omega$ a.e. in $\mathbb{R}^2$.

Analogously, the convergence in the sense of Mosco of the spaces Y_{Ω_h} to Y_Ω defined by (2.10) is obtained by using only the convergence of the extensions of gradients, that is:

(M_1') for every $u \in L^{1,p}(\Omega)$, there exists a sequence $u_h \in L^{1,p}(\Omega_h)$ such that $\nabla u_h 1_{\Omega_h}$ converges strongly to $\nabla u 1_\Omega$ in $L^p(\mathbb{R}^2, \mathbb{R}^2)$;

(M_2') if (h_k) is a sequence of indices converging to ∞, (u_k) is a sequence such that $u_k \in L^{1,p}(\Omega_{h_k})$ for every k, and $\nabla u_k 1_{\Omega_{h_k}}$ converges weakly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$ to a function ψ, then there exists $u \in L^{1,p}(\Omega)$ such that $\psi = \nabla u 1_\Omega$ a.e. in $\mathbb{R}^2$.

Theorem 2.3. *Let Ω_h and Ω be open subsets of $\mathbb{R}^2$, and let X_{Ω_h} and X_Ω be the corresponding subspaces defined by (2.9). Then Ω is stable for the Neumann problems (2.5) along the sequence (Ω_h) if and only if X_{Ω_h} converges to X_Ω in the sense of Mosco.*

Proof. Assume that Ω is stable for the Neumann problems (2.5) along the sequence (Ω_h). We want to prove that X_{Ω_h} converges to X_Ω in the sense of Mosco by using only the stability of the solutions corresponding to functions a and b of the special form

$$a(x,\xi) := a_0(x) + a_1(\xi), \qquad b(x,t) := b_0(x) + b_1(t), \tag{2.11}$$

with

$$a_1(\xi) := |\xi|^{p-2}\xi, \qquad b_1(t) := |t|^{p-2}t, \qquad a_0 \in L^q(\mathbb{R}^2, \mathbb{R}^2), \qquad b_0 \in L^q(\mathbb{R}^2). \tag{2.12}$$

Let us prove (M_1). Given $u \in W^{1,p}(\Omega)$, let $a_0 := -|\nabla u|^{p-2}\nabla u 1_\Omega$ and $b_0 := -|u|^{p-2}u 1_\Omega$. Then u is the solution of (2.7) in Ω with a and b given by (2.11). Let $u_h \in W^{1,p}(\Omega_h)$ be the solution of (2.7) in Ω_h with the same a and b. By Definition 2.1 the sequence $(u_h 1_{\Omega_h})$ converges to $u 1_\Omega$ strongly in $L^p(\mathbb{R}^2)$ and $(\nabla u_h 1_{\Omega_h})$ converges to $\nabla u 1_\Omega$ strongly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$. This proves (M_1).

Let us prove (M_2). Let (h_k) be a sequence of indices converging to ∞ and let (u_k) be a sequence, with $u_k \in W^{1,p}(\Omega_{h_k})$ for every k, such that $(u_k 1_{\Omega_{h_k}})$ converges weakly in $L^p(\mathbb{R}^2)$ to a function ϕ, while $(\nabla u_k 1_{\Omega_{h_k}})$ converges weakly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$ to a function ψ. Let $a_0 := -a_1(\psi) = -|\psi|^{p-2}\psi$ and $b_0 := -b_1(\phi) = -|\phi|^{p-2}\phi$, let a and b be defined by (2.11), and let u^* and $u^*_{h_k}$ be the solutions of problems (2.7) in Ω and Ω_{h_k} respectively. By the stability assumption the sequence $(u^*_{h_k} 1_{\Omega_{h_k}})$ converges to $u^* 1_\Omega$ strongly in $L^p(\mathbb{R}^2)$ and $(\nabla u^*_{h_k} 1_{\Omega_{h_k}})$ converges to $\nabla u^* 1_\Omega$ strongly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$. This implies that $a(x, \nabla u^*_{h_k} 1_{\Omega_{h_k}})$ converges to $a(x, \nabla u^* 1_\Omega)$ strongly in $L^q(\mathbb{R}^2, \mathbb{R}^2)$ and $b(x, u^*_{h_k} 1_{\Omega_{h_k}})$ converges to $b(x, u^* 1_\Omega)$ strongly in $L^q(\mathbb{R}^2)$. Therefore

$$\lim_{k\to\infty} \int_{\Omega_{h_k}} \big[a(x, \nabla u^*_{h_k}) \cdot (\nabla u_{h_k} - \nabla u^*_{h_k}) + b(x, u^*_{h_k})(u_{h_k} - u^*_{h_k})\big]\, dx =$$

$$= \int_{\mathbb{R}^2} \big[a(x, \nabla u^* 1_\Omega) \cdot (\psi - \nabla u^* 1_\Omega) + b(x, u^* 1_\Omega)(\phi - u^* 1_\Omega)\big]\, dx. \tag{2.13}$$

By (2.7) the left hand side of (2.13) is zero. Therefore, using (2.11) and (2.12) we obtain

$$\int_{\mathbb{R}^2} \big[(a_1(\nabla u^* 1_\Omega) - a_1(\psi)) \cdot (\psi - \nabla u^* 1_\Omega) + (b_1(u^* 1_\Omega) - b_1(\phi))(\phi - u^* 1_\Omega)\big]\, dx = 0.$$

Using the strict monotonicity of a_1 and b_1 we obtain that $\psi = \nabla u^* 1_\Omega$ and $\phi = u^* 1_\Omega$ a.e. in $\mathbb{R}^2$.

Conversely, assume now that X_{Ω_h} converges to X_Ω in the sense of Mosco and let us prove the stability. Let a and b be two Carathéodory functions satisfying (2.2)–(2.4) and let u_h and u be the solutions to problems (2.5) in Ω_h and Ω. The weak convergence in $L^p(\mathbb{R}^2) \times L^p(\mathbb{R}^2, \mathbb{R}^2)$

of $(u_h 1_{\Omega_h}, \nabla u_h 1_{\Omega_h})$ to $(u 1_\Omega, \nabla u 1_\Omega)$ is a particular case of [22, Theorem A]. For the reader's convenience, we give here the simple proof.

Using $z := u_h$ as test function in (2.7) for Ω_h, from (2.4) we obtain that $\|u_h\|_{W^{1,p}(\Omega_h)}$ is bounded. Passing to a subsequence, we have that $u_h 1_{\Omega_h}$ converges weakly in $L^p(\mathbb{R}^2)$ to a function ϕ, while $\nabla u_h 1_{\Omega_h}$ converges weakly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$ to a function ψ. By (M_2) there exists $u^* \in W^{1,p}(\Omega)$ such that $\phi = u^* 1_\Omega$ and $\psi = \nabla u^* 1_\Omega$ a.e. in $\mathbb{R}^2$. By monotonicity for every $v \in W^{1,p}(\Omega)$ we have

$$\begin{aligned} &\int_{\mathbb{R}^2} \big[a(x, \nabla v 1_\Omega) \cdot (\nabla v 1_\Omega - \nabla u_h 1_{\Omega_h}) + b(x, v 1_\Omega)(v 1_\Omega - u_h 1_{\Omega_h})\big] dx \geq \\ &\geq \int_{\mathbb{R}^2} \big[a(x, \nabla u_h 1_{\Omega_h}) \cdot (\nabla v 1_\Omega - \nabla u_h 1_{\Omega_h}) + b(x, u_h 1_{\Omega_h})(v 1_\Omega - u_h 1_{\Omega_h})\big] dx. \end{aligned} \tag{2.14}$$

By (M_1) there exists a sequence $v_h \in W^{1,p}(\Omega_h)$ such that $v_h 1_{\Omega_h}$ converges to $v 1_\Omega$ strongly in $L^p(\mathbb{R}^2)$ and $\nabla v_h 1_{\Omega_h}$ converges to $\nabla v 1_\Omega$ strongly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$. As $v_h - u_h \in W^{1,p}(\Omega_h)$, by (2.7) we have

$$\begin{aligned} &\int_{\mathbb{R}^2} \big[a(x, \nabla u_h 1_{\Omega_h}) \cdot (\nabla v 1_\Omega - \nabla u_h 1_{\Omega_h}) + b(x, u_h 1_{\Omega_h})(v 1_\Omega - u_h 1_{\Omega_h})\big] dx = \\ &= \int_{\mathbb{R}^2} \big[a(x, \nabla u_h 1_{\Omega_h}) \cdot (\nabla v 1_\Omega - \nabla v_h 1_{\Omega_h}) + b(x, u_h 1_{\Omega_h})(v 1_\Omega - v_h 1_{\Omega_h})\big] dx. \end{aligned} \tag{2.15}$$

Since $a(x, \nabla u_h 1_{\Omega_h})$ is bounded in $L^q(\mathbb{R}^2, \mathbb{R}^2)$ and $b(x, u_h 1_{\Omega_h})$ is bounded in $L^q(\mathbb{R}^2)$, passing to the limit in (2.14) and (2.15) we obtain

$$\begin{aligned} &\int_\Omega \big[a(x, \nabla v) \cdot (\nabla v - \nabla u^*) + b(x, v)(v - u^*)\big] dx \geq \\ &\geq \lim_{h\to\infty} \int_{\mathbb{R}^2} \big[a(x, \nabla u_h 1_{\Omega_h}) \cdot (\nabla v 1_\Omega - \nabla v_h 1_{\Omega_h}) + b(x, u_h 1_{\Omega_h})(v 1_\Omega - v_h 1_{\Omega_h})\big] dx = 0. \end{aligned} \tag{2.16}$$

Then we take $v = u^* \pm \varepsilon z$ in (2.16), with $z \in W^{1,p}(\Omega)$ and $\varepsilon > 0$. Dividing by ε, and passing to the limit as ε tends to 0, we obtain that u^* satisfies (2.7) in Ω. This proves that $u^* = u$. Therefore $u_h 1_{\Omega_h}$ converges to $u 1_\Omega$ weakly in $L^p(\mathbb{R}^2)$ and $\nabla u_h 1_{\Omega_h}$ converges to $\nabla u 1_\Omega$ weakly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$.

Taking $v := u$ in (2.16) we obtain that

$$\begin{aligned} &\int_{\mathbb{R}^2} (a(x, \nabla u_h 1_{\Omega_h}) - a(x, \nabla u 1_\Omega)) \cdot (\nabla u_h 1_{\Omega_h} - \nabla u 1_\Omega) dx + \\ &+ \int_{\mathbb{R}^2} (b(x, u_h 1_{\Omega_h}) - b(x, u 1_\Omega))(u_h 1_{\Omega_h} - u 1_\Omega) dx \end{aligned}$$

tends to 0 as $h \to \infty$. Using the monotonicity of a and b we conclude that each integral tends to 0. The strong convergence of $(u_h 1_{\Omega_h}, \nabla u_h 1_{\Omega_h})$ is now a consequence of the following lemma. $\square$

Lemma 2.4. *Let (ψ_h) be a sequence in $L^p(\mathbb{R}^2, \mathbb{R}^2)$ converging weakly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$ to a function ψ. Assume that*

$$\lim_{h\to\infty} \int_{\mathbb{R}^2} (a(x, \psi_h) - a(x, \psi)) \cdot (\psi_h - \psi) dx = 0. \tag{2.17}$$

Then ψ_h *converges to* ψ *strongly in* $L^p(\mathbb{R}^2, \mathbb{R}^2)$.

Proof. Various forms of this lemma have been used in the study of Leray-Lions operators (see, e.g., [1, Lemma 5]). For the sake of completeness, we give here the short proof of the present version.

Let $g_h := (a(x, \psi_h) - a(x, \psi)) \cdot (\psi_h - \psi)$. By monotonicity we have $g_h \geq 0$ a.e. in $\mathbb{R}^2$, thus (2.17) implies that g_h converges to 0 strongly in $L^1(\mathbb{R}^2)$. Passing to a subsequence, we may assume that g_h converges to 0 a.e. in $\mathbb{R}^2$. Using the Cauchy inequality, from (2.3) and (2.4) we obtain for every $\varepsilon > 0$

$$\begin{aligned} c_1|\psi_h|^p &\leq \beta + a(x, \psi_h) \cdot \psi_h = \beta + g_h + a(x, \psi_h) \cdot \psi + a(x, \psi) \cdot (\psi_h - \psi) \leq \\ &\leq g_h + \beta + \alpha|\psi| + |a(x, \psi)||\psi| + c_2\big(\frac{\varepsilon^q}{q} + \frac{\varepsilon^p}{p}\big)|\psi_h|^p + c_2\big(\frac{|\psi|^p}{p\varepsilon^p} + \frac{|a(x, \psi)|^q}{q\varepsilon^q}\big). \end{aligned}$$

Choosing ε small enough, we obtain that there exist a constant $c_3 > 0$ and a function $\gamma \in L^1(\mathbb{R}^2)$ such that

$$c_3|\psi_h|^p \leq g_h + \gamma. \tag{2.18}$$

Let us fix a point $x \in \mathbb{R}^2$ where $\gamma(x) < +\infty$ and where $g_h(x)$ tends to 0. By (2.18) the sequence $\psi_h(x)$ is bounded in $\mathbb{R}^2$, thus a subsequence (depending on x) converges to a vector $\xi \in \mathbb{R}^2$. By the definition of $g_h(x)$ and by the continuity of $a(x, \cdot)$ we get $(a(x, \xi) - a(x, \psi(x))) \cdot (\xi - \psi(x)) = 0$, which implies $\xi = \psi(x)$ by (2.2). Therefore the whole sequence $\psi_h(x)$ converges to $\psi(x)$. Since this is true for a.e. $x \in \mathbb{R}^2$, the strong convergence in $L^p(\mathbb{R}^2, \mathbb{R}^2)$ follows from (2.18) by the dominated convergence theorem. □

Remark 2.5. Let us observe that, if (Ω_h) is a uniformly bounded sequence of open subsets of $\mathbb{R}^2$ such that X_{Ω_h} converges to X_Ω in the sense of Mosco, then $|\Omega_h \triangle \Omega|$ converges to 0.

Let $\Sigma \subset \mathbb{R}^2$ be a bounded closed set such that $\Omega_h \subset \Sigma$ for every h. From property (M_1) it follows that $\Omega \subset \Sigma$. Indeed, if $\Omega \setminus \Sigma \neq \emptyset$, let $B \subset \Omega \setminus \Sigma$ be an open ball and let $\varphi \in C_c^\infty(B)$ with $\varphi \neq 0$. By property (M_1), there exists $u_h \in W^{1,p}(\Omega_h)$ such that $u_h 1_{\Omega_h} \to \varphi$ strongly in $L^p(\mathbb{R}^2)$. So,

$$0 < \int_B |\varphi|^p \, dx = \lim_{h\to\infty} \int_B |u_h 1_{\Omega_h}|^p \, dx = 0,$$

which is absurd.

Now, let $u := 1_\Omega$. Since $u \in W^{1,p}(\Omega)$, by property (M_1) there exists $u_h \in W^{1,p}(\Omega_h)$ such that $u_h 1_{\Omega_h} \to 1_\Omega$ strongly in $L^p(\mathbb{R}^2)$. As $|u_h 1_{\Omega_h} - 1_\Omega|^p = 1$ a.e. on $\Omega \setminus \Omega_h$, we have

$$\lim_{h\to\infty} |\Omega \setminus \Omega_h| \;\leq\; \lim_{h\to\infty} \int_{\mathbb{R}^2} |u_h 1_{\Omega_h} - 1_\Omega|^p \, dx \;=\; 0. \tag{2.19}$$

On the other hand, up to a subsequence, 1_{Ω_h} converges weakly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$ to some ϕ. Hence from property (M_2), there exists $v \in W^{1,p}(\Omega)$ such that $\phi = v 1_\Omega$ a.e. in $\mathbb{R}^2$. So we have that

$$\lim_{h\to\infty} |\Omega_h \setminus \Omega| = \lim_{h\to\infty} \int_{\mathbb{R}^2} 1_{\Omega_h} 1_{\Sigma\setminus\Omega} \, dx = \int_{\mathbb{R}^2} v 1_\Omega 1_{\Sigma\setminus\Omega} \, dx = 0,$$

which together with (2.19) gives $|\Omega_h \triangle \Omega| \to 0$.

Note that, if the open sets Ω_h are not uniformly bounded, it is possible that X_{Ω_h} converges to X_Ω in the sense of Mosco while $|\Omega_h \triangle \Omega|$ does not converge to 0. Consider, for example, the sequence of open sets $\Omega_h := B(0,1) \cup ((B(0, h+h^{-1}) \setminus \overline{B(0,h)})$ and $\Omega := B(0,1)$. We have that $|\Omega_h \triangle \Omega| = 2\pi + h^{-2}\pi \to 2\pi$.

Let us verify that X_{Ω_h} converges to X_Ω in the sense of Mosco. For every $u \in W^{1,p}(\Omega)$, property (M_1) is satisfied by the sequence $u_h := u1_\Omega$. For property (M_2), let (h_k) be a sequence of indices converging to ∞; let (u_k) be a sequence, with $u_k \in W^{1,p}(\Omega_{h_k})$ for every k, such that $u_k 1_{\Omega_{h_k}}$ converges weakly in $L^p(\mathbb{R}^2)$ to a function ϕ, while $\nabla u_k 1_{\Omega_{h_k}}$ converges weakly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$ to a function ψ. We set $u := \phi|_\Omega$. As $u_k \rightharpoonup u$ weakly in $L^p(\Omega)$ and $\nabla u_k \rightharpoonup \psi|_\Omega$ weakly in $L^p(\Omega, \mathbb{R}^2)$, we have $u \in W^{1,p}(\Omega)$ and $\nabla u = \psi|_\Omega$ a.e. on Ω. Now, for every ball $D \subset \Omega^c$ we have $D \subset \Omega_h^c$ for h large enough, hence

$$\int_D \phi \, dx \;=\; \lim_{k\to\infty} \int_D u_k 1_{\Omega_{h_k}} dx \;=\; 0.$$

So, $\phi = 0$ a.e. in Ω^c and similarly also $\psi = 0$ a.e. in Ω^c. Hence, $\phi = u1_\Omega$ and $\psi = \nabla u 1_\Omega$ a.e. in $\mathbb{R}^2$.

Note that, in this case, Ω_h converges to Ω in the Hausdorff complementary topology, since $d_H(\Omega_h^c, \Omega^c) = h^{-1} \to 0$. By adding a small strip whose width tends to zero one can obtain an example with connected sets.

The following theorem can be proved as Theorem 2.3.

Theorem 2.6. *Let Ω_h and Ω be open subsets of $\mathbb{R}^2$, and let Y_{Ω_h} and Y_Ω be the corresponding subspaces defined by (2.10). Then Ω is stable for the Neumann problems (2.6) along the sequence (Ω_h) if and only if Y_{Ω_h} converges to Y_Ω in the sense of Mosco.*

Remark 2.7. If (Ω_h) is a uniformly bounded sequence of open subsets of $\mathbb{R}^2$ such that Y_{Ω_h} converges to Y_Ω in the sense of Mosco, then $|\Omega_h \triangle \Omega|$ converges to 0. Let $\Sigma \subset \mathbb{R}^2$ be a bounded set such that $\Omega_h \subset \Sigma$ for every h. Arguing as in Remark 2.5, we get that $\Omega \subset \Sigma$.

Now, let $u(x) := \xi \cdot x$ with $\xi \in \mathbb{R}^2$ and $|\xi| = 1$. Since $u \in L^{1,p}(\Omega)$, by property (M_1') there exists $u_h \in L^{1,p}(\Omega_h)$ such that $\nabla u_h 1_{\Omega_h} \to \nabla u 1_\Omega$ strongly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$. As $|\nabla u_h 1_{\Omega_h} - \nabla u 1_\Omega|^p = 1$ a.e. on $\Omega \setminus \Omega_h$, we have

$$\lim_{h\to\infty} |\Omega \setminus \Omega_h| \;\le\; \lim_{h\to\infty} \int_{\mathbb{R}^2} |\nabla u_h 1_{\Omega_h} - \nabla u 1_\Omega|^p \, dx \;=\; 0. \tag{2.20}$$

On the other hand, we consider the functions $v_h \in L^{1,p}(\Omega_h)$ defined by $v_h(x) := \xi \cdot x$. Up to a subsequence, $\nabla v_h 1_{\Omega_h} = \xi 1_{\Omega_h}$ converges weakly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$ to some function ψ. By property (M_2'), there exists a function $v \in L^{1,p}(\Omega)$ such that $\psi = \nabla v 1_\Omega$ a.e. in $\mathbb{R}^2$. So, it follows that

$$\xi \lim_{h\to\infty} |\Omega_h \setminus \Omega| = \lim_{h\to\infty} \int_{\mathbb{R}^2} \nabla v_h 1_{\Omega_h} 1_{\Sigma\setminus\Omega} \, dx = \int_{\mathbb{R}^2} \nabla v 1_\Omega 1_{\Sigma\setminus\Omega} \, dx \;=\; 0,$$

which together with (2.20) gives $|\Omega \triangle \Omega_h| \to 0$.

Remark 2.8. Theorems 2.3 and 2.6 can be applied in the following easy case. If (Ω_h) is increasing and Ω is the union of the sequence, then it is easy to see that X_{Ω_h} converges to

X_Ω and Y_{Ω_h} converges to Y_Ω in the sense of Mosco. Therefore every open set is stable for the Neumann problems (2.5) and (2.6) along any increasing sequence converging to it.

3. Mosco convergence of Deny-Lions spaces

In this section we study the Mosco convergence of the subspaces Y_Ω introduced in (2.10) and corresponding to the Deny-Lions spaces $L^{1,p}(\Omega)$. By Theorem 2.6, this is equivalent to the stability for the Neumann problems (2.6).

Theorem 3.1. *Let (Ω_h) be a uniformly bounded sequence of open subsets of $\mathbb{R}^2$ that converges to an open set Ω in the Hausdorff complementary topology. Assume that $|\Omega_h|$ converges to $|\Omega|$ and that Ω_h^c has a uniformly bounded number of connected components. Then Ω is stable for the Neumann problems (2.6) along the sequence (Ω_h).*

To prove Theorem 3.1 we use the following lemmas.

Lemma 3.2. *Let (Ω_h) be a uniformly bounded sequence of open subsets of $\mathbb{R}^2$ which converges to an open set Ω in the Hausdorff complementary topology. Assume that $|\Omega_h| \to |\Omega|$. Then $1_{\Omega_h} \to 1_\Omega$ in measure, i.e., $|\Omega_h \triangle \Omega| \to 0$. Moreover, if $\varphi_h \rightharpoonup \varphi$ weakly in $L^r(\mathbb{R}^2)$ for some $1 < r < +\infty$, and $\varphi_h = 0$ a.e. in Ω_h^c, then $\varphi = 0$ a.e. in Ω^c.*

Proof. From the convergence of Ω_h to Ω in the Hausdorff complementary topology we have that $1_{\Omega\setminus\Omega_h} \to 0$ pointwise, hence $|\Omega \setminus \Omega_h| \to 0$ by the dominated convergence theorem. Since $|\Omega_h| - |\Omega| = |\Omega_h \setminus \Omega| - |\Omega \setminus \Omega_h|$, and the left hand side tends to 0 by hypothesis, we conclude that $|\Omega_h \setminus \Omega| \to 0$ too.

Now, let $\psi \in L^\infty(\mathbb{R}^2)$. As $1_{\Omega_h} \to 1_\Omega$ strongly in $L^s(\mathbb{R}^2)$ for every $1 \le s < +\infty$, we have

$$\int_{\Omega^c} \varphi\psi \, dx = \lim_{h\to\infty} \int_{\Omega^c} \varphi_h \psi \, dx = \lim_{h\to\infty} \int_{\Omega^c} 1_{\Omega_h} \varphi_h \psi \, dx = \int_{\Omega^c} 1_\Omega \varphi\psi \, dx = 0,$$

which implies $\varphi = 0$ a.e. in Ω^c. □

Lemma 3.3. *Let (Ω_h) be a uniformly bounded sequence of open subsets of $\mathbb{R}^2$, converging to an open set Ω in the Hausdorff complementary topology. Assume that the sets Ω_h^c have a uniformly bounded number of connected components. Let (v_h) be a sequence in $W^{1,q}(\mathbb{R}^2)$ converging weakly in $W^{1,q}(\mathbb{R}^2)$ to a function v and with $v_h = 0$ C_q-q.e. on Ω_h^c. Then $v = 0$ C_q-q.e. on Ω^c.*

Proof. Let Δ_q be the q-Laplacian, defined by $\Delta_q u := \operatorname{div}(|\nabla u|^{q-2}\nabla u)$. Let w_h and w be the solutions of the problems

$$\begin{aligned} &w_h \in W^{1,q}(\mathbb{R}^2), \qquad \Delta_q w_h = \Delta_q v \quad \text{in} \quad \Omega_h, \qquad w_h = 0 \quad C_q\text{-q.e. in } \Omega_h^c, \\ &w \in W^{1,q}(\mathbb{R}^2), \qquad \Delta_q w = \Delta_q v \quad \text{in} \quad \Omega, \qquad w = 0 \quad C_q\text{-q.e. in } \Omega^c. \end{aligned} \tag{3.1}$$

Using a result on the stability of Dirichlet problems proved by Bucur and Trebeschi in [4] (see also Šverák [24] for the case $q = 2$), we obtain that w_h converges to w strongly in $W^{1,q}(\mathbb{R}^2)$. Taking $v_h - w_h$ as test function in (3.1), which is possible since $v_h - w_h \in W_0^{1,q}(\Omega_h)$ (see, e.g., [19, Theorem 4.5]), we obtain

$$\langle \Delta_q w_h, v_h - w_h \rangle = \langle \Delta_q v, v_h - w_h \rangle, \tag{3.2}$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between $W^{-1,p}(\mathbb{R}^2)$ and $W^{1,q}(\mathbb{R}^2)$. Passing to the limit in (3.2) we obtain

$$\langle \Delta_q w, v - w \rangle = \langle \Delta_q v, v - w \rangle,$$

which implies $v = w$ by the strict monotonicity of $-\Delta_q$. Since, by definition, $w = 0$ C_q-q.e. in Ω^c, we conclude that $v = 0$ C_q-q.e. in Ω^c. □

Lemma 3.4. *Let $v \in W^{1,q}(\mathbb{R}^2)$ and let C_1 and C_2 be two connected closed subsets of $\mathbb{R}^2$ with $C_1 \cap C_2 \neq \emptyset$. If v is constant C_q-q.e. in C_1 and in C_2, then v is constant C_q-q.e. in $C_1 \cup C_2$.*

Proof. For $q = 2$ we the reader is referred to Proposition 2.5 in [12], while for $q > 2$ the result follows from the Sobolev embedding theorem, which yields the continuity of v. □

Lemma 3.5. *Let (Ω_h) be a uniformly bounded sequence of open subsets of $\mathbb{R}^2$ which converges to an open set Ω in the Hausdorff complementary topology, and let (v_h) be a sequence in $W^{1,q}(\mathbb{R}^2)$, which converges to a function v weakly in $W^{1,q}(\mathbb{R}^2)$. Assume that Ω_h^c has a uniformly bounded number of connected components and that every function v_h is constant C_q-q.e. in each connected component of Ω_h^c. Then v is constant C_q-q.e. in each connected component of Ω^c.*

Proof. Let $C_h^1, \dots, C_h^{n_h}$ be the connected components of Ω_h^c. Passing to a subsequence, we can assume that n_h does not depend on h, and that the sets C_h^i converge in the Hausdorff metric to some connected sets C^i as $h \to \infty$. Let us prove that v is constant q.e. in each C^i.

This is trivial if C^i contains only a single point. If C^i has more than one point, there exists $r > 0$ such that $\operatorname{diam}(C_h^i) > 2r$ for h large enough. Let us prove that the constant values c_h^i taken by v_h on C_h^i are bounded uniformly with respect to h. To this aim let us consider a point $x_h \in C_h^i$. Since $\operatorname{diam}(C_h^i) > 2r$, we have $C_h^i \setminus B(x_h, r) \neq \emptyset$, and by connectedness

$$C_h^i \cap \partial B(x_h, \rho) \neq \emptyset \qquad \text{for every } 0 < \rho < r. \tag{3.3}$$

As $v_h = c_h^i$ C_q-q.e. on C_h^i, by using polar coordinates we deduce from (3.3) the Poincaré inequality

$$\int_{B(x_h,r)} |v_h - c_h^i|^q \, dx \leq M r^q \int_{B(x_h,r)} |\nabla v_h|^q \, dx,$$

where the constant M is independent of h, i, and r. Since v_h is bounded in $W^{1,q}(\mathbb{R}^2)$, it follows that c_h^i is bounded, and so it converges (up to a subsequence) to some constant c^i.

To prove that $v = c^i$ C_q-q.e. on C^i, we fix two open balls B_1 and B_2 with $B_1 \subset\subset B_2$, and a cut-off function $\varphi \in C_c^\infty(B_2)$ with $\varphi = 1$ in B_1. Then we have that $\varphi(v_h - c_h^i) = 0$ C_q-q.e. on $(B_2 \setminus C_h^i)^c$. By Lemma 3.3 we get $\varphi(v - c^i) = 0$ C_q-q.e. in $(B_2 \setminus C^i)^c$, hence $v = c^i$ C_q-q.e on $B_1 \cap C^i$. As B_1 is arbitrary, we obtain $v = c^i$ C_q-q.e. on C^i. If $C^i \cap C^j \neq \emptyset$, by Lemma 3.4 we have that v is constant C_q-q.e. on $C^i \cup C^j$. As Ω^c is the union of the sets C^i, we conclude that v is constant C_q-q.e. on each connected component of Ω^c. □

Lemma 3.6. *Let Ω be a bounded open subset of $\mathbb{R}^2$ and let u be a solution of problem (2.6). Let R be the rotation on $\mathbb{R}^2$ defined by $R(y_1, y_2) := (-y_2, y_1)$. Then there exists a unique function $v \in W^{1,q}(\mathbb{R}^2)$ such that $\nabla v = Ra(x, \nabla u) 1_\Omega$ a.e. in $\mathbb{R}^2$. Moreover v is constant C_q-q.e. on each connected component of Ω^c.*

Proof. We consider the vector field $\Phi \in L^q(\mathbb{R}^2, \mathbb{R}^2)$ defined by $\Phi := a(x, \nabla u)1_\Omega$. By (2.8) we have $\operatorname{div}\Phi = 0$ in $\mathcal{D}'(\mathbb{R}^2)$, hence $\operatorname{rot}(R\Phi) = 0$ in $\mathcal{D}'(\mathbb{R}^2)$. Since Ω is bounded, there exists a potential $v \in W^{1,q}(\mathbb{R}^2)$ such that $\nabla v = R\Phi$ a.e. in $\mathbb{R}^2$ and $v = 0$ a.e. in the interior of the unbounded connected component of Ω^c.

Given a connected component C of Ω^c, it remains to prove that v is constant C_q-q.e. on C. For every $\varepsilon > 0$ let $C_\varepsilon := \{x \in \mathbb{R}^2 : \operatorname{dist}(x, C) < \varepsilon\}$, and let u_ε be a solution of problem (2.6) in $\Omega_\varepsilon := \Omega \setminus \overline{C}_\varepsilon$. Let v_ε be the unique function in $W^{1,q}(\mathbb{R}^2)$ such that $\nabla v_\varepsilon = Ra(x, \nabla u_\varepsilon)1_{\Omega_\varepsilon}$ a.e. in $\mathbb{R}^2$. By Remark 2.8, ∇u_ε converges to ∇u strongly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$ and so v_ε converges to v strongly in $W^{1,q}(\mathbb{R}^2)$. By construction $\nabla v_\varepsilon = 0$ in C_ε. As C_ε is a connected open set containing C, we have that v_ε is constant C_q-q.e. on C. Since a subsequence of v_ε converges to v C_q-q.e. on $\mathbb{R}^2$, we conclude that v is constant C_q-q.e. on C. □

Proof of Theorem 3.1. Let $a \colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ be a Carathéodory function satisfying (2.2)–(2.4) and let u_h and u be solutions to problems (2.6) in Ω_h and Ω. Taking u_h as test function in (2.8) in Ω_h and using (2.4) we obtain that $\nabla u_h 1_{\Omega_h}$ is bounded in $L^p(\mathbb{R}^2, \mathbb{R}^2)$. By (2.3) we obtain also that $a(x, \nabla u_h)1_{\Omega_h}$ is bounded in $L^q(\mathbb{R}^2)$. Passing to a subsequence, we may assume that $\nabla u_h 1_{\Omega_h} \rightharpoonup \Psi$ weakly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$ and $a(x, \nabla u_h)1_{\Omega_h} \rightharpoonup \Phi$ weakly in $L^q(\mathbb{R}^2, \mathbb{R}^2)$. By (2.8) we have $\operatorname{div}(a(x, \nabla u_h)1_{\Omega_h}) = 0$ in $\mathcal{D}'(\mathbb{R}^2)$, hence $\operatorname{div}\Phi = 0$ in $\mathcal{D}'(\mathbb{R}^2)$.

If $\Omega' \subset\subset \Omega$, by the Hausdorff complementary convergence we have $\Omega' \subset\subset \Omega_h$ for h large enough. Since the set of gradients of functions of $L^{1,p}(\Omega')$ is closed in $L^p(\Omega', \mathbb{R}^2)$, the vectorfield Ψ is the gradient of a function of $L^{1,p}(\Omega')$. As Ω' is arbitrary, we can construct $u^* \in L^{1,p}(\Omega)$ such that $\Psi = \nabla u^*$ a.e. in Ω. On the other hand, by Lemma 3.2 we have $\Psi = 0$ a.e. in Ω^c, hence $\Psi = \nabla u^* 1_\Omega$ a.e. in $\mathbb{R}^2$.

Let us prove that $\Phi = a(x, \nabla u^*)1_\Omega$ a.e. in $\mathbb{R}^2$. By Lemma 3.2 it is enough to prove the equality in every open ball $B \subset\subset \Omega$. Note that by the Hausdorff complementary convergence we have $B \subset\subset \Omega_h$ for h large enough. By adding suitable constants, we may assume that the mean values of u_h and u^* on B are zero. Thus the Poincaré inequality and the Rellich theorem imply that u_h converges to u^* strongly in $L^p(B)$.

Let $z \in W^{1,p}(B)$ and let $\varphi \in C_c^\infty(B)$ with $\varphi \geq 0$. For h large enough we have $B \subset\subset \Omega_h$, thus by monotonicity we have

$$\int_B (a(x, \nabla z) - a(x, \nabla u_h)) \cdot (\nabla z - \nabla u_h)\varphi dx \geq 0. \tag{3.4}$$

By (2.8) we have also

$$\int_B a(x, \nabla u_h) \cdot \nabla((z - u_h)\,\varphi) dx = 0,$$

which, together with (3.4), gives

$$\int_B a(x, \nabla z) \cdot \nabla((z - u_h)\varphi) dx - \int_B (a(x, \nabla z) - a(x, \nabla u_h)) \cdot \nabla\varphi\,(z - u_h) dx \geq 0. \tag{3.5}$$

We can pass to the limit in each term of (3.5) and we get

$$\int_B a(x, \nabla z) \cdot \nabla((z - u^*)\varphi) dx - \int_B (a(x, \nabla z) - \Phi) \cdot \nabla\varphi\,(z - u^*) dx \geq 0. \tag{3.6}$$

As $\operatorname{div} \Phi = 0$ in $\mathcal{D}'(B)$, we have

$$\int_B \Phi \cdot \nabla((z-u^*)\varphi)dx = 0. \tag{3.7}$$

From (3.6) and (3.7) we obtain

$$\int_B (a(x,\nabla z) - \Phi) \cdot (\nabla z - \nabla u^*)\, \varphi dx \geq 0.$$

As φ is arbitrary, we get $(a(x,\nabla z) - \Phi) \cdot (\nabla z - \nabla u^*) \geq 0$ a.e. in B. In particular, taking $z(x) := u^*(x) \pm \varepsilon\xi \cdot x$, with $\xi \in \mathbb{R}^2$ and $\varepsilon > 0$, we obtain $\pm(a(x,\nabla u^* \pm \varepsilon\xi) - \Phi) \cdot \xi \geq 0$ a.e. in B. As ε tends to zero we get $(a(x,\nabla u^*) - \Phi) \cdot \xi = 0$ a.e. in B, which implies that $a(x,\nabla u^*) = \Phi$ a.e. in B by the arbitrariness of ξ.

Let us prove now that

$$\int_\Omega a(x,\nabla u^*) \cdot \nabla z\, dx = 0 \qquad \forall z \in L^{1,p}(\Omega). \tag{3.8}$$

By Lemma 3.6 for every h there exists $v_h \in W^{1,q}(\mathbb{R}^2)$ such that $\nabla v_h = Ra(x,\nabla u_h)1_{\Omega_h}$ a.e. in $\mathbb{R}^2$. Moreover v_h is constant C_q-q.e. on each connected component of Ω_h^c. As $a(x,\nabla u_h)1_{\Omega_h}$ converges to $a(x,\nabla u^*)1_\Omega$ weakly in $L^q(\mathbb{R}^2,\mathbb{R}^2)$, there exists a function $v \in W^{1,q}(\mathbb{R}^2)$ such that $v_h \rightharpoonup v$ weakly in $W^{1,q}(\mathbb{R}^2)$ and $\nabla v = Ra(x,\nabla u^*)1_\Omega$ a.e. in $\mathbb{R}^2$. So, we have to prove that

$$\int_\Omega R\nabla v \cdot \nabla z\, dx = 0 \qquad \forall z \in L^{1,p}(\Omega).$$

From the Lemma 3.5 it follows that v is constant C_q-q.e. on the connected components of Ω^c. By [19, Theorem 4.5] we can approximate v strongly in $W^{1,q}(\mathbb{R}^2)$ by a sequence of functions $v_h \in C_c^\infty(\mathbb{R}^2)$ that are constant in suitable neighborhoods U_h^i of each connected component C^i of Ω^c. Let $z \in L^{1,p}(\Omega)$ and $z_h \in W_0^{1,p}(\Omega)$ such that $z_h = z$ in $\Omega \setminus \bigcup_i U_h^i$. Then, we have

$$\int_\Omega R\nabla v_h \cdot \nabla z\, dx = \int_\Omega R\nabla v_h \cdot \nabla z_h\, dx = 0, \tag{3.9}$$

where the last equality follows from the fact that the vector field $R\nabla v_h$ is divergence free. Then, passing to the limit in (3.9) for $h \to \infty$, we get

$$\int_\Omega a(x,\nabla u^*) \cdot \nabla z\, dx = -\int_\Omega R\nabla v \cdot \nabla z dx = 0.$$

So u^* is a solution of (2.8) in Ω, hence $\nabla u^* = \nabla u$ a.e. in Ω by uniqueness of the gradients. This implies that $\nabla u_h 1_{\Omega_h}$ converges to $\nabla u 1_\Omega$ weakly in $L^p(\mathbb{R}^2,\mathbb{R}^2)$ and $a(x,\nabla u_h)1_{\Omega_h}$ converges to $a(x,\nabla u)1_\Omega$ weakly in $L^q(\mathbb{R}^2,\mathbb{R}^2)$. Since $|\Omega_h \triangle \Omega|$ tends to 0 by Lemma 3.2, from the identity $a(x,\nabla u_h 1_{\Omega_h}) = a(x,\nabla u_h)1_{\Omega_h} + a(x,0)1_{\Omega_h^c}$ we conclude also that $a(x,\nabla u_h 1_{\Omega_h})$ converges to $a(x,\nabla u 1_\Omega)$ weakly in $L^q(\mathbb{R}^2,\mathbb{R}^2)$.

To prove the strong convergence, we consider the integral

$$I_h := \int_{\mathbb{R}^2} (a(x,\nabla u_h 1_{\Omega_h}) - a(x,\nabla u 1_\Omega)) \cdot (\nabla u_h 1_{\Omega_h} - \nabla u 1_\Omega) dx.$$

Since by (2.8)

$$\int_{\Omega_h} a(x,\nabla u_h) \cdot \nabla u_h\, dx = 0 \qquad \text{and} \qquad \int_\Omega a(x,\nabla u) \cdot \nabla u\, dx = 0,$$

we have

$$I_h = -\int_\Omega a(x, \nabla u_h 1_{\Omega_h}) \cdot \nabla u \, dx - \int_{\Omega_h} a(x, \nabla u 1_\Omega) \cdot \nabla u_h 1_{\Omega_h} \, dx.$$

Therefore

$$\lim_{h\to\infty} I_h = -2 \int_\Omega a(x, \nabla u) \cdot \nabla u \, dx = 0, \tag{3.10}$$

where the last equality can be deduced from (2.8). The strong convergence in $L^p(\mathbb{R}^2, \mathbb{R}^2)$ of $\nabla u_h 1_{\Omega_h}$ to $\nabla u 1_\Omega$ follows now from (3.10) and from Lemma 2.4. □

Remark 3.7. As we said in the introduction, in the case $p > 2$ the stability result for problem (2.6) is not true under our hypotheses. Indeed, let us consider

$$S := [1, 3] \times \{0\}, \quad S_h := ([1, 2 - 1/h] \cup [2 + 1/h, 3]) \times \{0\},$$

$$\Omega := B(0, 3) \setminus (\overline{B(0, 1)} \cup S) \quad \text{and} \quad \Omega_h := B(0, 3) \setminus (\overline{B(0, 1)} \cup S_h).$$

Let $\varphi \in C_c^\infty(0, \infty)$ be such that $\varphi(\rho) = \rho^{-p}$ for $1 \le \rho \le 3$. We set

$$a(x, \xi) := |\xi|^{p-2}\xi - \varphi(|x|)Rx,$$

where R is the rotation by $\pi/2$ defined by $Rx := (-x_2, x_1)$. Let u_h and u be solutions of problems (2.6) in Ω and Ω_h, with $\int_{\Omega_h} u_h \, dx = \int_\Omega u \, dx = 0$. For every $x \in \Omega$, let $0 < \theta(x) < 2\pi$ be the angle between x and the positive x_1-axis. As $\nabla\theta(x) = Rx/|x|^2$, we have that $u = \theta - \pi$ in Ω.

If the open set Ω were stable for problem (2.6) along the sequence (Ω_h), then ∇u_h would converge strongly to ∇u in $L^p(\Omega, \mathbb{R}^2)$. By the Poincaré inequality we would have that u_h converges strongly to u in $W^{1,p}(\Omega)$.

For every $v \in W^{1,p}(\Omega)$, let v^+ and v^- be the upper and lower traces of v on S, defined by

$$v^+(x) := \lim_{\substack{y\to x\\ y_2>0}} v(y) \quad \text{and} \quad v^-(x) := \lim_{\substack{y\to x\\ y_2<0}} v(y). \tag{3.11}$$

From the convergence of u_h to u in $W^{1,p}(\Omega)$, we obtain that $u_h^+ \to u^+$ and $u_h^- \to u^-$ uniformly on S (recall that $p > 2$ here). Since $u_h^+(2, 0) = u_h^-(2, 0)$ by the continuity of u_h, we obtain $u^+(2, 0) = u^-(2, 0)$, which contradicts the fact that $u^+(2, 0) = -\pi$ and $u^-(2, 0) = \pi$, being $u = \theta - \pi$.

4. Mosco convergence of Sobolev spaces

In this section we study the convergence in the sense of Mosco of the subspaces X_Ω introduced in (2.9) and corresponding to the Sobolev spaces $W^{1,p}(\Omega)$. The convergence of X_{Ω_h} to X_Ω will be obtained from the convergence of Y_{Ω_h} to Y_Ω and from the following approximation theorem for functions which are locally constant on the limit open set Ω.

Theorem 4.1. *Let (Ω_h) be a uniformly bounded sequence of open subsets of $\mathbb{R}^2$ which converges to an open set Ω in the Hausdorff complementary topology. Assume that $|\Omega_h|$ converges to $|\Omega|$ and that Ω_h^c is connected for every h. Then for every $u \in W^{1,p}(\Omega)$ with $\nabla u = 0$ a.e. in Ω, there exists a sequence $u_h \in W^{1,p}(\Omega_h)$ such that $u_h 1_{\Omega_h}$ converges to $u 1_\Omega$ strongly in $L^p(\mathbb{R}^2)$ and $\nabla u_h 1_{\Omega_h}$ converges to 0 strongly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$.*

The proof of this theorem is postponed. We are now in a position to state the main result of the paper.

Theorem 4.2. *Let (Ω_h) be a uniformly bounded sequence of open subsets of $\mathbb{R}^2$ which converges to an open set Ω in the Hausdorff complementary topology, with $|\Omega_h|$ converging to $|\Omega|$. Assume that Ω_h^c has a uniformly bounded number of connected components. Then X_{Ω_h} converges to X_Ω in the sense of Mosco.*

To prove Theorem 4.2 we need the following localization lemma.

Lemma 4.3. *Let (Ω_h) be a uniformly bounded sequence of open subsets of $\mathbb{R}^2$, and let Ω be a bounded open subset of $\mathbb{R}^2$. Assume that for every $x \in \mathbb{R}^2$ there exists $\varepsilon > 0$ such that the sequence $X_{B(x,\varepsilon)\cap\Omega_h}$ converges to $X_{B_\varepsilon(x)\cap\Omega}$ in the sense of Mosco. Then X_{Ω_h} converges to X_Ω in the sense of Mosco.*

Proof. Condition (M_2) is easy, and condition (M_1) can be obtained by using a partition of unity. $\square$

Proof of Theorem 4.2. Step 1. Assume that Ω_h^c is connected for every h. Let us prove (M_2). Let (h_k) be a sequence of indices converging to ∞, (u_k) be a sequence, with $u_k \in W^{1,p}(\Omega_{h_k})$ for every k, such that $u_k 1_{\Omega_{h_k}}$ converges weakly in $L^p(\mathbb{R}^2)$ to a function ϕ, while $\nabla u_k 1_{\Omega_{h_k}}$ converges weakly in $L^p(\mathbb{R}^2,\mathbb{R}^2)$ to a function ψ. From Lemma 3.2 it follows that ϕ and ψ vanish a.e. in Ω^c.

Let $\Omega' \subset\subset \Omega$ be an open set. By the Hausdorff complementary convergence we have $\Omega' \subset\subset \Omega_h$ for h large enough. So, $u_k|_{\Omega'}$ converges weakly to $\phi|_{\Omega'}$ in $L^p(\Omega')$ and $\nabla u_k|_{\Omega'}$ converges weakly to $\psi|_{\Omega'}$ in $L^p(\Omega',\mathbb{R}^2)$. Hence $\phi|_{\Omega'} \in W^{1,p}(\Omega')$ and $\psi|_{\Omega'} = \nabla\phi|_{\Omega'}$ in Ω'. From the arbitrariness of Ω', it follows that the function $u := \phi|_\Omega$ belongs to $W^{1,p}(\Omega)$, $\phi = u1_\Omega$ and $\psi = \nabla u 1_\Omega$ a.e. in $\mathbb{R}^2$.

Now let us prove (M_1). Let $u \in W^{1,p}(\Omega)$. We write $\Omega := \bigcup_{i=1}^m \Omega_i$, where $1 \le m \le \infty$ and (Ω_i) is the family of connected components of Ω. Since the set of functions u satisfying (M_1) is a closed linear subspace of $W^{1,p}(\Omega)$, by a density argument it is enough to prove (M_1) when u belongs to $L^\infty(\Omega)$ and vanishes on all connected components of Ω except one. By renumbering the sequence (Ω_i), we may assume that u vanishes on Ω_i for every $i \ge 2$.

From Theorem 3.1 on the convergence of Y_{Ω_h} to Y_Ω in the sense of Mosco, there exists a sequence $z_h \in L^{1,p}(\Omega_h)$ such that $\nabla z_h 1_{\Omega_h}$ converges strongly to $\nabla u 1_\Omega$ in $L^p(\mathbb{R}^2,\mathbb{R}^2)$. Let us fix a nonempty open set $A_0 \subset\subset \Omega_1$. We can assume that $\int_{A_0} z_h dx = \int_{A_0} u dx$. For every smooth connected open set A, with $A_0 \subset\subset A \subset\subset \Omega_1$, by the Poincaré inequality we have that $z_h|_A \to u|_A$ strongly in $W^{1,p}(A)$.

We consider now $w_h := (-\|u\|_\infty) \vee z_h \wedge \|u\|_\infty$. We have that $w_h|_A \to u|_A$ strongly in $W^{1,p}(A)$ for every open set $A \subset\subset \Omega_1$. Moreover, for every open set $E \subset\subset \Omega$ the function $w_h|_E$ belongs to $W^{1,p}(E)$ for h large enough. As $\|w_h\|_\infty \le \|u\|_\infty$ and $|\nabla w_h|_E| \le |\nabla z_h|_E|$, the sequence $(w_h|_E)$ is bounded in $W^{1,p}(E)$. By the Rellich theorem, there exists $w \in W^{1,p}(\Omega)$ such that $w_h|_E$ converges to $w|_E$ strongly in $L^p(E)$ for every open set $E \subset\subset \Omega$ with smooth boundary. As $w_h|_A \to u|_A$ strongly in $W^{1,p}(A)$ for every open set $A \subset\subset \Omega_1$, we have that $w = u$ a.e. in Ω_1.

For every open set $E \subset\subset \Omega \setminus \Omega_1$, since $|\nabla w_h| \leq |\nabla z_h|$ and $\nabla z_h|_E \to \nabla u|_E = 0$ strongly in $L^p(E, \mathbb{R}^2)$, we have $\nabla w_h|_E \to 0$ strongly in $L^p(E, \mathbb{R}^2)$. Therefore we get $\nabla w = 0 = \nabla u$ a.e. in $\Omega \setminus \Omega_1$, which together with the result obtained in Ω_1 implies that $\nabla w_h|_E$ converges to $\nabla u|_E$ strongly in $L^p(E, \mathbb{R}^2)$ for every $E \subset\subset \Omega$. In particular, we obtain that the function $u - w$ is locally constant in Ω.

We claim that $\nabla w_h 1_{\Omega_h}$ converges to $\nabla w 1_\Omega$ strongly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$. Indeed for every $E \subset\subset \Omega$ we have $\|\nabla w_h 1_{\Omega_h} - \nabla u 1_\Omega\|_{L^p(\mathbb{R}^2,\mathbb{R}^2)} \leq \|\nabla w_h 1_E - \nabla u 1_E\|_{L^p(E,\mathbb{R}^2)} + \|\nabla z_h 1_{\Omega_h} 1_{E^c}\|_{L^p(\mathbb{R}^2,\mathbb{R}^2)} + \|\nabla u 1_{\Omega\setminus E}\|_{L^p(\mathbb{R}^2,\mathbb{R}^2)}$. Hence

$$\limsup_{h\to\infty} \|\nabla w_h 1_{\Omega_h} - \nabla u 1_\Omega\|_{L^p(\mathbb{R}^2,\mathbb{R}^2)} \leq 2\|\nabla u 1_{\Omega\setminus E}\|_{L^p(\mathbb{R}^2,\mathbb{R}^2)},$$

and by letting $E \nearrow \Omega$ we prove the claim. In a similar way, we obtain also that $w_h 1_{\Omega_h}$ converges to $w 1_\Omega$ strongly in $L^p(\mathbb{R}^2)$.

Since $u - w$ is locally constant in Ω, from Theorem 4.1 there exists $v_h \in W^{1,p}(\Omega_h)$ such that $v_h 1_{\Omega_h} \to (u - w)1_\Omega$ strongly in $L^p(\mathbb{R}^2)$ and $\nabla v_h 1_{\Omega_h} \to 0$ strongly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$. Therefore $w_h + v_h \in W^{1,p}(\Omega_h)$, $(w_h + v_h)1_{\Omega_h} \to u 1_\Omega$ strongly in $L^p(\mathbb{R}^2)$, and $\nabla(w_h + v_h)1_{\Omega_h} \to \nabla u 1_\Omega$ strongly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$, which give property (M_1).

Step 2. We now remove the hypothesis that Ω_h^c is connected. Let $C_h^1, \dots, C_h^{n_h}$ be the connected components of Ω_h^c. Passing to a subsequence we can assume that n_h does not depend on h and that the sets C_h^i converge in the Hausdorff metric to some connected sets C^i as $h \to \infty$. Let $C^{i_1}, \dots, C^{i_d}$ be those C^i having at least two points. We set

$$\Omega^* := \Big(\bigcup_{j=1}^d C^{i_j}\Big)^c \quad \text{and} \quad \Omega_h^* := \Big(\bigcup_{j=1}^d C_h^{i_j}\Big)^c.$$

We have that $\Omega \subset \Omega^*$, $\Omega_h \subset \Omega_h^*$, and, by construction, Ω_h^* converges in the Hausdorff complementary topology to Ω^* and $|\Omega_h^*| \to |\Omega^*|$ (because $|C_h^i| \to 0$ if $i \neq i_1, \dots, i_d$). There exists some $\eta > 0$ such that $\operatorname{diam} C^{i_j} > \eta$ for every j, hence $\operatorname{diam} C_h^{i_j} > \eta$ for h large enough. Let us observe that, for every $x \in \mathbb{R}^2$, the sequence $B(x, \eta/2) \cap \Omega_h^*$ converges to $B(x, \eta/2) \cap \Omega^*$ in the Hausdorff complementary topology and also $|B(x, \eta/2) \cap \Omega_h^*| \to |B(x, \eta/2) \cap \Omega^*|$ (by Lemma 3.2). As $\operatorname{diam} C_h^{i_j} > \eta$, it is easy to see that $(B(x, \eta/2) \cap \Omega_h^*)^c$ is connected for h large enough. So, from Step 1 we obtain the Mosco convergence of $X_{B(x,\eta/2)\cap\Omega_h^*}$ to $X_{B(x,\eta/2)\cap\Omega^*}$. Now, using Lemma 4.3 we get the Mosco convergence of $X_{\Omega_h^*}$ to X_{Ω^*}.

As $\Omega_h^* \setminus \Omega_h$ is the union of a uniformly bounded number of sets with diameter tending to 0, using the fact that $1 < p \leq 2$, we deduce that $C_p(\Omega_h^* \setminus \Omega_h) \to 0$. Let us show that this implies that X_{Ω_h} converges to X_Ω in the sense of Mosco.

For property (M_1), let $u \in W^{1,p}(\Omega)$. Since the set $\Omega^* \setminus \Omega$ is finite, we have that $C_p(\Omega^* \setminus \Omega) = 0$. Hence, $u \in W^{1,p}(\Omega^*)$. So, there exists $u_h^* \in W^{1,p}(\Omega_h^*)$ such that $u_h^* 1_{\Omega_h^*}$ converges to $u 1_{\Omega^*}$ strongly in $L^p(\mathbb{R}^2)$ and $\nabla u_h^* 1_{\Omega_h^*}$ converges to $\nabla u 1_{\Omega^*}$ strongly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$. Setting $u_h = u_h^*|_{\Omega_h}$, we obtain that $u_h 1_{\Omega_h}$ converges to $u 1_\Omega$ strongly in $L^p(\mathbb{R}^2)$ and $\nabla u_h 1_{\Omega_h}$ converges to $\nabla u 1_\Omega$ strongly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$, and so property (M_1) holds.

Let us prove property (M_2). Let (h_k) be a sequence of indices converging to ∞, (u_k) be a sequence, with $u_k \in W^{1,p}(\Omega_{h_k})$ for every k, such that $u_k 1_{\Omega_{h_k}}$ converges weakly in $L^p(\mathbb{R}^2)$ to a function ϕ, while $\nabla u_k 1_{\Omega_{h_k}}$ converges weakly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$ to a function ψ. From Lemma 3.2 it follows that ϕ and ψ vanish a.e. in Ω^c.

As $C_p(\Omega^*_{h_k} \setminus \Omega_{h_k}) \to 0$, there exists a sequence $\varphi_k \in W^{1,p}(\mathbb{R}^2)$ converging strongly to 0 in $W^{1,p}(\mathbb{R}^2)$ such that $\varphi_k = 1$ a.e. in a neighborhood of $\Omega^*_{h_k} \setminus \Omega_{h_k}$.

We set

$$u^*_k := u_k(1 - \varphi_k).$$

Note that $u^*_k \in W^{1,p}(\Omega^*_{h_k})$ and that $u^*_k 1_{\Omega^*_{h_k}}$ converges weakly in $L^p(\mathbb{R}^2)$ to ϕ, while $\nabla u^*_k 1_{\Omega^*_{h_k}}$ converges weakly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$ to ψ. So, from the Mosco convergence of $X_{\Omega^*_h}$ to X_{Ω^*}, it follows that there exists $u^* \in W^{1,p}(\Omega^*)$ such that $\phi = u^* 1_{\Omega^*}$ and $\psi = \nabla u^* 1_{\Omega^*}$ a.e. in $\mathbb{R}^2$. By setting $u = u^*|_\Omega$, we get that $\phi = u 1_\Omega$ and $\psi = \nabla u 1_\Omega$ a.e. in $\mathbb{R}^2$ and the proof of (M_2) is complete. ☐

The rest of this section is devoted to the proof of Theorem 4.1. To this aim we will need some preliminary results.

Lemma 4.4. *Let (Ω_h) be a uniformly bounded sequence of open subsets of $\mathbb{R}^2$, which converges to an open set Ω in the Hausdorff complementary topology, with $|\Omega_h|$ converging to $|\Omega|$. Assume that $\Omega_h = \Omega^1_h \cup \Omega^2_h$ with Ω^i_h open and $\Omega^1_h \cap \Omega^2_h = \emptyset$. Assume also that (Ω^i_h) converges to an open set Ω^i, $i = 1, 2$, in the Hausdorff complementary topology. Then*

(i) $\Omega^1 \cap \Omega^2 = \emptyset$,
(ii) $\Omega^1 \cup \Omega^2 = \Omega$,
(iii) $|\Omega^i| = \lim_h |\Omega^i_h|$, $i = 1, 2$.

In particular, if Ω^0_h is union of connected components of Ω_h and converges to an open set Ω^0 in the Hausdorff complementary topology, then Ω^0 is union of connected components of Ω and $|\Omega^0_h|$ converges to $|\Omega^0|$.

Proof. (i) and (ii) are easy consequences of the convergence in the Hausdorff complementary topology, while (iii) follows from Lemma 3.2. As Ω^0_h is union of connected components of Ω_h, then the set $\Omega'_h := \Omega_h \setminus \Omega^0_h$ is open in the relative topology of Ω_h. Up to a subsequence, Ω'_h converges to an open set Ω' in the Hausdorff complementary topology. From (i) and (ii) we have that $\Omega^0 \cap \Omega' = \emptyset$, and $\Omega^0 \cup \Omega' = \Omega$; hence Ω^0 is union of connected components of Ω. The last assertion follows from (iii). ☐

The following lemma, proved by Bucur and Varchon in [2], will also be used in the proof of Theorem 4.1.

Lemma 4.5. *Let Ω be a bounded open set in $\mathbb{R}^2$ and let a and b be two points in two different connected components Ω_a and Ω_b of Ω, whose distance from Ω^c is greater than 10δ for some $\delta > 0$. Let U be an open subset of $\mathbb{R}^2$ such that U^c is connected and $d_H(U^c, \Omega^c) < \delta$. Then there exists $x \in \Omega^c$ such that the closed square $Q(x, 9\delta)$, with centre x and side length 9δ, intersects any curve contained in U and joining the points a and b.*

Proof of Theorem 4.1. By a density argument, it is sufficient to prove that for every connected component Ω^0 of Ω there exists a sequence $u_h \in W^{1,p}(\Omega_h)$ such that $u_h 1_{\Omega_h}$ converges strongly to 1_{Ω^0} in $L^p(\mathbb{R}^2)$ and $\nabla u_h 1_{\Omega_h}$ converges strongly to 0 in $L^p(\mathbb{R}^2, \mathbb{R}^2)$. Let $a^0 \in \Omega^0$ and let Ω^0_h

be the connected component of Ω_h which contains a^0 (which is defined for h large enough). Up to a subsequence, Ω_h^0 converges in the Hausdorff complementary topology to some open set E. From Lemma 4.4 it follows that

$$E = \bigcup_{i=0}^{m} \Omega^i,$$

where $0 \le m \le \infty$ and (Ω^i) is a family of connected components of Ω (including Ω^0), and

$$|\Omega_h^0| \to |E|. \tag{4.1}$$

Let $0 < \varepsilon < |\Omega^0|$ be fixed. There exist a finite integer $n_\varepsilon \ge 1$ and an open set Ω_ε such that

$$E = \bigcup_{i=0}^{n_\varepsilon} \Omega^i \cup \Omega_\varepsilon, \tag{4.2}$$

where $|\Omega_\varepsilon| \le \varepsilon$ and $\Omega^i \cap \Omega_\varepsilon = \emptyset$ for every $i \le n_\varepsilon$.

We fix now a point a^i in each set Ω^i. Let $\delta > 0$ be such that $\operatorname{dist}(a^i, \Omega^c) > 10\delta$ for every $i \le n_\varepsilon$. From Lemma 4.5, for h big enough there exist some points $(x_h^{\delta,i})_{i=1}^{n_\varepsilon}$, uniformly bounded in Ω^c, such that for every $i \le n_\varepsilon$ the square $Q(x_h^{\delta,i}, 9\delta)$ intersects any curve contained in Ω_h^0 and joining the points a^0 and a^i. Up to a subsequence, we have that $x_h^{\delta,i} \to x^{\delta,i}$ as $h \to \infty$, for some $x^{\delta,i} \in \Omega^c$. Once again up to a subsequence, we have that $x^{\delta,i} \to x^i$ as $\delta \to 0$, for some $x^i \in \Omega^c$. Let

$$K^{\delta,\varepsilon} := \bigcup_{i=1}^{n_\varepsilon} \overline{B(x^i, 10\delta)}$$

and, for $i = 0, \dots, n_\varepsilon$, let $\Omega_h^{\delta,\varepsilon,i}$ be the connected component of $\Omega_h^0 \setminus K^{\delta,\varepsilon}$ containing a^i. As $K^{\delta,\varepsilon} \supset Q(x_h^{\delta,i}, 9\delta)$ for δ small enough and h large enough, we have

$$\Omega_h^{\delta,\varepsilon,0} \neq \Omega_h^{\delta,\varepsilon,i} \qquad \text{for } i \neq 0. \tag{4.3}$$

Let $\varphi^{\delta,\varepsilon}$ be the C_p-capacitary potential of $K^{\delta,\varepsilon}$, i.e., the solution of the minimum problem

$$\min \left\{ \int_{\mathbb{R}^2} [|\nabla \varphi|^p + |\varphi|^p] \, dx : \varphi \in W^{1,p}(\mathbb{R}^2),\ \varphi = 1 \ C_p\text{-a.e. on } K^{\delta,\varepsilon} \right\}.$$

We set

$$u_h^{\varepsilon,\delta} := \begin{cases} 1 & \text{in } \Omega_h^{\delta,\varepsilon,0}, \\ \varphi^{\delta,\varepsilon} & \text{in } \Omega_h \setminus \Omega_h^{\delta,\varepsilon,0}. \end{cases} \tag{4.4}$$

As $\Omega_h \cap \partial\Omega_h^{\delta,\varepsilon,0} \subset K^{\delta,\varepsilon}$, we have that $u_h^{\varepsilon,\delta} \in W^{1,p}(\Omega_h)$. We observe that

$$\| u_h^{\delta,\varepsilon} 1_{\Omega_h} - 1_{\Omega_h^{\delta,\varepsilon,0}} \|_{L^p(\mathbb{R}^2)} + \| \nabla u_h^{\delta,\varepsilon} 1_{\Omega_h} \|_{L^p(\mathbb{R}^2,\mathbb{R}^2)} \ \le\ 2 C_p(K^{\delta,\varepsilon})^{\frac{1}{p}}.$$

As $C_p(K^{\delta,\varepsilon}) \le n_\varepsilon C_p(B(0, 10\delta))$ and $p \le 2$, we conclude that

$$\limsup_{\delta \to 0} \limsup_{h \to \infty} \big[\| u_h^{\delta,\varepsilon,0} 1_{\Omega_h} - 1_{\Omega_h^{\delta,\varepsilon}} \|_{L^p(\mathbb{R}^2)} + \| \nabla u_h^{\delta,\varepsilon} 1_{\Omega_h} \|_{L^p(\mathbb{R}^2,\mathbb{R}^2)} \big] \ =\ 0. \tag{4.5}$$

We claim that

$$\limsup_{\delta \to 0} \limsup_{h \to \infty} \| u_h^{\varepsilon,\delta,0} 1_{\Omega_h} - 1_{\Omega^0} \|_{L^p(\mathbb{R}^2)}^p \le \varepsilon, \tag{4.6}$$

and

$$\limsup_{\delta\to 0}\limsup_{h\to\infty}\|\nabla u_h^{\varepsilon,\delta}1_{\Omega_h}\|_{L^p(\mathbb{R}^2,\mathbb{R}^2)}=0, \tag{4.7}$$

from which the proof of the theorem is achieved by the arbitrariness of ε.

It is easy to see that (4.7) follows from (4.5), while (4.6) is a consequence of (4.5) and of the following inequality

$$\limsup_{\delta\to 0}\limsup_{h\to\infty}|\Omega_h^{\delta,\varepsilon,0}\triangle\Omega^0|\le\varepsilon. \tag{4.8}$$

So, let us prove (4.8). For every $i=0,\dots n_\varepsilon$, up to a subsequence, $\Omega_h^{\delta,\varepsilon,i}$ converges in the Hausdorff complementary topology, when $h\to\infty$, to some open set $\Omega^{\delta,\varepsilon,i}\subset E$. We observe that $\Omega_h^0\setminus K^{\delta,\varepsilon}$ converges to $E\setminus K^{\delta,\varepsilon}$ in the Hausdorff complementary topology when $h\to\infty$. Let $E^{\delta,\varepsilon,i}$ be the connected component of $E\setminus K^{\delta,\varepsilon}$ which contains a^i. It is easy to see that

$$E^{\delta,\varepsilon,i}\subset\Omega^{\delta,\varepsilon,i}. \tag{4.9}$$

Note that, as $\delta\searrow 0$, $K^{\delta,\varepsilon}$ converges decreasingly to the set $\{x_1,\dots,x_{n_\varepsilon}\}$, $E^{\delta,\varepsilon,i}$ converges increasingly to Ω^i, and $\Omega^{\delta,\varepsilon,i}$ converges increasingly to some open set $\Omega^{\varepsilon,i}\subset E$. From (4.9), it follows that $\Omega^i\subset\Omega^{\varepsilon,i}$. From (4.1) and from Lemma 3.2 it follows that

$$|\Omega_h^0\setminus K^{\delta,\varepsilon}|\to|E\setminus K^{\delta,\varepsilon}|. \tag{4.10}$$

By Lemma 4.4 applied to $\Omega_h^1:=\Omega_h^{\delta,\varepsilon,0}$ and $\Omega_h^2:=(\Omega_h^0\setminus K^{\delta,\varepsilon})\setminus\Omega_h^{\delta,\varepsilon,0}$, we have that $\Omega^{\delta,\varepsilon,0}\cap\Omega^{\delta,\varepsilon,i}=\emptyset$ for every $i\neq 0$, from which it follows that $\Omega^{\varepsilon,0}\cap\Omega^{\varepsilon,i}=\emptyset$ and hence $\Omega^{\varepsilon,0}\cap\Omega^i=\emptyset$ for every $1\le i\le n_\varepsilon$. Therefore, there exists an open set Ω'_ε, contained in the set Ω_ε introduced in (4.2), such that

$$\Omega^{\varepsilon,0}=\Omega^0\cup\Omega'_\varepsilon.$$

From (4.10) and from Lemmas 4.4 and 3.2, it follows that

$$|\Omega_h^{\delta,\varepsilon,0}\triangle\Omega^{\delta,\varepsilon,0}|\to 0. \tag{4.11}$$

As $\Omega^{\delta,\varepsilon,0}\subset\Omega^{\varepsilon,0}=\Omega^0\cup\Omega'_\varepsilon$, it follows that

$$|\Omega_h^{\delta,\varepsilon,0}\setminus\Omega^0|\le|\Omega_h^{\delta,\varepsilon,0}\setminus\Omega^{\delta,\varepsilon,0}|+|\Omega^{\delta,\varepsilon,0}\setminus\Omega^0|\le|\Omega_h^{\delta,\varepsilon,0}\setminus\Omega^{\delta,\varepsilon,0}|+|\Omega'_\varepsilon|, \tag{4.12}$$

and

$$|\Omega^0\setminus\Omega_h^{\delta,\varepsilon,0}|\le|\Omega^0\setminus\Omega^{\delta,\varepsilon,0}|+|\Omega^{\delta,\varepsilon,0}\setminus\Omega_h^{\delta,\varepsilon,0}|\le|\Omega^{\varepsilon,0}\setminus\Omega^{\delta,\varepsilon,0}|+|\Omega^{\delta,\varepsilon,0}\setminus\Omega_h^{\delta,\varepsilon,0}|. \tag{4.13}$$

As $\Omega^{\delta,\varepsilon,0}$ converges increasingly to $\Omega^{\varepsilon,0}$ as $\delta\to 0^+$, we have $|\Omega^{\varepsilon,0}\setminus\Omega^{\delta,\varepsilon,0}|\to 0$ as $\delta\to 0$. By (4.11), passing to the limit in (4.12) and (4.13) first as $h\to\infty$ and then as $\delta\to 0^+$ we obtain

$$\limsup_{\delta\to 0}\limsup_{h\to\infty}|\Omega_h^{\delta,\varepsilon,0}\setminus\Omega^0|\le|\Omega_\varepsilon|+\limsup_{h\to\infty}|\Omega_h^{\delta,\varepsilon,0}\setminus\Omega^{\delta,\varepsilon,0}|\le\varepsilon$$

and

$$\limsup_{\delta\to 0}\limsup_{h\to\infty}|\Omega^0\setminus\Omega_h^{\delta,\varepsilon,0}|=0,$$

which give inequality (4.8). □

Remark 4.6. In the case $p > 2$ the stability result for problem (1.1) is not true under our hypotheses. Indeed, let us consider

$$S := [-1,1] \times \{0\}, \quad S_h := ([-1,-1/h] \cup [1/h,1]) \times \{0\},$$
$$\Omega := (-1,1)^2 \setminus S, \quad \text{and} \quad \Omega_h := (-1,1)^2 \setminus S_h.$$

We set

$$a(x,\xi) := |\xi|^{p-2}\xi \quad \text{and} \quad b(x,\eta) := |\eta|^{p-2}\eta - x_2,$$

where $x = (x_1, x_2)$. Let u_h and u be solutions of problems (1.1) in Ω_h and Ω respectively. By the symmetry of Ω, the solution u will depend only on x_2. Therefore, for every $x = (x_1,x_2) \in \Omega$

$$u(x) = \begin{cases} w(x_2) & \text{if } x_2 \in (0,1), \\ -w(-x_2) & \text{if } x_2 \in (-1,0) \end{cases}$$

where w is the solution of the one-dimensional problem

$$\begin{cases} -(|w'|^{p-2}w')' + |w|^{p-2}w = t & \text{in } (0,1), \\ w'(0) = w'(1) = 0, \end{cases} \tag{4.14}$$

which turns out to be of class $C^1([0,1])$. For every $v \in W^{1,p}(\Omega)$, let v^+ and v^- be the upper and lower traces of v on S, defined as in (3.11).

If the open set Ω were stable for problem (1.1) along the sequence (Ω_h), then u_h would converge strongly to u in $W^{1,p}(\Omega)$. Hence we would have that $u_h^+ \to u^+$ and $u_h^- \to u^-$ uniformly on S (recall that $p > 2$ here). Since $u_h^+(0,0) = u_h^-(0,0)$ by the continuity of u_h, we would obtain $u^+(0,0) = u^-(0,0)$, which implies that $w(0) = 0$. Let us prove that this is false. Indeed, by the maximum principle we have that $w(t) \geq 0$ for every $t \in [0,1]$. Since $w'(0) = 0$ and $p > 2$, we have that $w^{p-1}(t) - t < 0$ in a small neighborhood I of 0 in $[0,1]$. So, from equation (4.14) the function $|w'|^{p-2}w'$ is decreasing in I and hence $w'(t) < 0$ for every $t \in I$. If $w(0)$ were equal to 0, we would obtain $w(t) < 0$ for every $t \in I$, which contradicts the fact that $w(t) \geq 0$ for every $t \in [0,1]$. This proves that $w(0) > 0$, and hence Ω is not stable for problem (1.1) along the sequence (Ω_h) for $p > 2$.

5. The case of unbounded domains

We now extend the results of the previous sections to the case of unbounded domains.

Theorem 5.1. *Let (Ω_h) be a sequence of open subsets of $\mathbb{R}^2$ such that (Ω_h^c) converges to Ω^c in the sense of Kuratowski for some open subset Ω. Assume that, for every $R > 0$, $|\Omega_h \cap B(0,R)|$ converges to $|\Omega \cap B(0,R)|$ and that the number of connected components of $(\Omega_h \cap B(0,R))^c$ is uniformly bounded with respect to h. Then the sequence of subspaces X_{Ω_h} (resp. Y_{Ω_h}) converges to X_Ω (resp. Y_Ω).*

Proof. We prove only the Mosco convergence of X_{Ω_h} to X_Ω, since the convergence of Y_{Ω_h} to Y_Ω can be proved in the same way. First of all note that, from the convergence of Ω_h^c to Ω^c in the sense of Kuratowski, it follows that the sequence $\Omega_h \cap B(0,R)$ converges to $\Omega \cap B(0,R)$ in the Hausdorff complementary topology. Moreover, by the assumptions of the theorem, we can apply

Theorem 4.2 to the sequence $\Omega_h \cap B(0,R)$. So, we get that $X_{\Omega_h \cap B(0,R)}$ converges to $X_{\Omega \cap B(0,R)}$ in the sense of Mosco.

Now let us prove (M_2) for X_{Ω_h} and X_Ω. Let (h_k) be a sequence of indices converging to ∞, (u_k) be a sequence, with $u_k \in W^{1,p}(\Omega_{h_k})$ for every k, such that $u_k 1_{\Omega_{h_k}}$ converges weakly in $L^p(\mathbb{R}^2)$ to a function ϕ, while $\nabla u_k 1_{\Omega_{h_k}}$ converges weakly in $L^p(\mathbb{R}^2,\mathbb{R}^2)$ to a function ψ. It follows that $u_k 1_{\Omega_{h_k}\cap B(0,R)}$ converges to $\phi 1_{B(0,R)}$ weakly in $L^p(\mathbb{R}^2)$, while $\nabla u_k 1_{\Omega_{h_k}\cap B(0,R)}$ converges to $\psi 1_{B(0,R)}$ weakly in $L^p(\mathbb{R}^2,\mathbb{R}^2)$. So, by property (M_2) relative to the Mosco convergence of $X_{\Omega_h\cap B(0,R)}$ to $X_{\Omega\cap B(0,R)}$, there exists a function $u_R \in W^{1,p}(\Omega \cap B(0,R))$ such that $\phi 1_{B(0,R)} = u_R 1_{\Omega\cap B(0,R)}$ and $\psi 1_{B(0,R)} = \nabla u_R 1_{\Omega \cap B(0,R)}$ a.e. in $\mathbb{R}^2$. Since R is arbitrary, it is easy to construct $u \in W^{1,p}(\Omega)$ such that $\phi = u 1_\Omega$ and $\psi = \nabla u 1_\Omega$ a.e. in $\mathbb{R}^2$.

Let us prove property (M_1). Let $u \in W^{1,p}(\Omega)$ and let $\varepsilon > 0$. There exists $R_\varepsilon > 0$ such that

$$\int_{\Omega \setminus B(0,R_\varepsilon)} [|u|^p + |\nabla u|^p]\, dx \leq \varepsilon.$$

By property (M_1) relative to the Mosco convergence of $X_{\Omega_h \cap B(0,R_\varepsilon+1)}$ to $X_{\Omega\cap B(0,R_\varepsilon+1)}$, there exists a sequence $w_h^\varepsilon \in W^{1,p}(\Omega_h \cap B(0,R_\varepsilon+1))$ such that $w_h^\varepsilon 1_{\Omega_h\cap B(0,R_\varepsilon+1)}$ converges strongly to $u 1_{\Omega\cap B(0,R_\varepsilon+1)}$ in $L^p(\mathbb{R}^2)$ and $\nabla w_h^\varepsilon 1_{\Omega_h \cap B(0,R_\varepsilon+1)}$ converges strongly to $\nabla u 1_{\Omega\cap B(0,R_\varepsilon+1)}$ in $L^p(\mathbb{R}^2,\mathbb{R}^2)$. Let $\varphi_\varepsilon \in C^1_c(B(0,R_\varepsilon+1))$ such that $0 \leq \varphi_\varepsilon \leq 1$, $\varphi_\varepsilon = 1$ in $B(0,R_\varepsilon)$, and $\|\nabla \varphi_\varepsilon\|_\infty \leq C$. Now we set $u_h^\varepsilon := \varphi_\varepsilon w_h^\varepsilon$. By construction $u_h^\varepsilon \in W^{1,p}(\Omega_h)$, $u_h^\varepsilon 1_{\Omega_h} \to \varphi_\varepsilon u 1_\Omega$ strongly in $L^p(\mathbb{R}^2)$, and $\nabla u_h^\varepsilon 1_{\Omega_h} \to \varphi_\varepsilon \nabla u 1_\Omega + u \nabla\varphi_\varepsilon 1_\Omega$ strongly in $L^p(\mathbb{R}^2,\mathbb{R}^2)$. On the other hand

$$\limsup_{h\to\infty} \int_{\mathbb{R}^2} |u_h^\varepsilon 1_{\Omega_h} - u 1_\Omega|^p + |\nabla u_h^\varepsilon 1_{\Omega_h} - \nabla u 1_\Omega|^p\, dx \;\leq\; 2^{p-1}(C^p+1)\varepsilon,$$

$\varphi_\varepsilon u 1_\Omega \to u 1_\Omega$ strongly in $L^p(\mathbb{R}^2)$, and $\varphi_\varepsilon \nabla u 1_\Omega + u\nabla \varphi_\varepsilon 1_\Omega \to \nabla u 1_\Omega$ strongly in $L^p(\mathbb{R}^2,\mathbb{R}^2)$. Therefore, we can construct a sequence $u_h \in W^{1,p}(\Omega_h)$ which satisfies (M_1) by a standar argument on double sequences. □

6. Problems with Dirichlet boundary conditions

In this section we study the Mosco convergence of Sobolev and Deny-Lions spaces with prescribed Dirichlet conditions on part of the boundary.

Let $A \subset \mathbb{R}^2$ be a bounded open set with Lipschitz boundary ∂A, and let $\partial_D A$ be a relatively open subset of ∂A with a finite number of connected components. For every compact set $K \subset \overline{A}$, for every $g \in W^{1,p}(A)$, and for every pair of function a and b satisfying the properties (2.2)–(2.4), we consider the solutions u and v of the mixed problems

$$\begin{cases} -\operatorname{div} a(x,\nabla u) + b(x,u) = 0 & \text{in} \quad A\setminus K, \\ u = g & \text{on} \quad \partial_D A \setminus K, \\ a(x,\nabla u)\cdot \nu = 0 & \text{on} \quad \partial(A\setminus K)\setminus(\partial_D A\setminus K), \end{cases} \tag{6.1}$$

and

$$
\begin{cases} -\operatorname{div} a(x,\nabla v) = 0 & \text{in} \quad A\setminus K, \\ v = g & \text{on} \quad \partial_D A \setminus K, \\ a(x,\nabla v)\cdot \nu = 0 & \text{on} \quad \partial(A\setminus K)\setminus(\partial_D A\setminus K). \end{cases} \tag{6.2}
$$

Let (K_h) be a sequence of compact subsets of $\overline{A}$, let (g_h) be a sequence in $W^{1,p}(A)$, and let (u_h) be the sequence of the solutions of problems (6.1) corresponding to K_h and g_h.

Definition 6.1. *We say that the pair (K,g) is stable for the mixed problems (6.1) along the sequence (K_h, g_h) if for every pair of functions a, b satisfying (2.2)–(2.4) the sequence $(u_h 1_{K_h^c})$ converges to $u1_{K^c}$ strongly in $L^p(A)$ and the sequence $(\nabla u_h 1_{K_h^c})$ converges to $\nabla u 1_{K^c}$ strongly in $L^p(A,\mathbb{R}^2)$.*

The stability for problems (6.2) is defined in a similar way by using only the convergence of the gradients (as in Definition 2.2).

The stabilty for problems (6.2) has been recently studied in [12] in the case $a(x,\xi)=\xi$. In this section we will study the stability in the general case by using again the notion of Mosco convergence.

We set

$$
W^{1,p}_g(A\setminus K, \partial_D A\setminus K) := \{u \in W^{1,p}(A\setminus K) : u = g \text{ on } \partial_D A \setminus K\},
$$

and

$$
L^{1,p}_g(A\setminus K, \partial_D A\setminus K) := \{u \in L^{1,p}(A\setminus K) : u = g \text{ on } \partial_D A \setminus K\},
$$

where the equality $u=g$ on $\partial_D A\setminus K$ is intended in the usual sense of traces.

As in Section 2.5 the space $W^{1,p}_g(A\setminus K,\partial_D A\setminus K)$ will be identified with the closed linear subspace $X^g_K(A)$ of $L^p(A)\times L^p(A,\mathbb{R}^2)$ defined by

$$
X^g_K(A) := \{(u1_{K^c}, \nabla u 1_{K^c}) : u \in W^{1,p}_g(A\setminus K, \partial_D A\setminus K)\}. \tag{6.3}
$$

For problem (6.2), we consider in $L^{1,p}_g(A\setminus K,\partial_D A\setminus K)$ the equivalence relation $\sim$ defined in (2.1). Note that in this case $v_1\sim v_2$ if and only if $v_1=v_2$ a.e. in those connected components of $A\setminus K$ whose boundary intersects $\partial_D A\setminus K$ and $\nabla v_1 = \nabla v_2$ a.e. in the other connected components of $A\setminus K$. The corresponding quotient space, denoted by $L^{1,p}_g(A\setminus K,\partial_D A\setminus K)/_\sim$, will be identified with the closed linear subspace $Y^g_K(A)$ of $L^p(A,\mathbb{R}^2)$ defined by

$$
Y^g_K(A) := \{\nabla u 1_{K^c} : u \in L^{1,p}_g(A\setminus K, \partial_D A\setminus K)\}. \tag{6.4}
$$

Let K_h, K be compact subsets of $\overline{A}$ and let g_h, $g\in W^{1,p}(A)$. Let $X^{g_h}_{K_h}(A)$ and $X^g_K(A)$ be the corresponding subspaces defined by (6.3). We recall that $X^{g_h}_{K_h}(A)$ converges to $X^g_K(A)$ in the sense of Mosco if the following two properties hold:

(M_1'') for every $u\in W^{1,p}_g(A\setminus K,\partial_D A\setminus K)$, there exists a sequence $u_h\in W^{1,p}_{g_h}(A\setminus K_h,\partial_D A\setminus K_h)$ such that $u_h 1_{K_h^c}$ converges strongly to $u1_{K^c}$ in $L^p(A)$ and $\nabla u_h 1_{K_h^c}$ converges strongly to $\nabla u 1_{K^c}$ in $L^p(A,\mathbb{R}^2)$;

(M_2'') if (h_k) is a sequence of indices converging to ∞, (u_k) is a sequence, with $u_k \in W^{1,p}_{g_{h_k}}(A \setminus K_{h_k}, \partial_D A \setminus K_{h_k})$ for every k, such that $u_k 1_{K^c_{h_k}}$ converges weakly in $L^p(A)$ to a function ϕ, while $\nabla u_k 1_{K^c_{h_k}}$ converges weakly in $L^p(A, \mathbb{R}^2)$ to a function ψ, then there exists $u \in W^{1,p}_g(A \setminus K, \partial_D A \setminus K)$ such that $\phi = u 1_{K^c}$ and $\psi = \nabla u 1_{K^c}$ a.e. in A.

Analogously, the convergence of $Y^{g_h}_{K_h}(A)$ to $Y^g_K(A)$ in the sense of Mosco can be characterized by using only the convergence of the extensions of the gradients.

Remark 6.2. As in Section 2.6 we can prove that the Mosco convergence of $X^{g_h}_{K_h}(A)$ to $X^g_K(A)$ (resp. of $Y^{g_h}_{K_h}(A)$ to $Y^g_K(A)$) is equivalent to the stability of (K, g) for the mixed problems (6.1) (resp. (6.2) along the sequence (K_h, g_h)).

The following theorem is the main result of this section.

Theorem 6.3. *Let A be a bounded open subset of $\mathbb{R}^2$ with Lipschitz boundary ∂A and let $\partial_D A$ be a relatively open subset of ∂A with a finite number of connected components. Let (g_h) be a sequence in $W^{1,p}(A)$ converging strongly to a function g in $W^{1,p}(A)$, and let (K_h) be a sequence of compact subsets of $\overline{A}$ converging to a set K in the Hausdorff metric. Assume that $|K_h|$ converges to $|K|$ and that the sets K_h have a uniformly bounded number of connected components. Then $X^{g_h}_{K_h}(A)$ converges to $X^g_K(A)$ (resp. $Y^{g_h}_{K_h}(A)$ converges to $Y^g_K(A)$) in the sense of Mosco.*

Proof. The main idea of this proof is due to Chambolle. Let us first prove the Mosco convergence of $X^{g_h}_{K_h}(A)$ to $X^g_K(A)$. Let Σ be an open ball in $\mathbb{R}^2$ such that $\overline{A} \subset \Sigma$. Let $\tilde{g}_h$, $\tilde{g} \in W^{1,p}(\Sigma)$ be extensions of g_h and g to Σ such that $\tilde{g}_h$ converges to $\tilde{g}$ strongly in $W^{1,p}(\Sigma)$. We set

$$\Omega_h := \Sigma \setminus (K_h \cup (\partial A \setminus \partial_D A)) \quad \text{and} \quad \Omega := \Sigma \setminus (K \cup (\partial A \setminus \partial_D A)).$$

Note that Ω_h and Ω satisfy the assumptions of Theorem 4.2. Let us prove property $(M2'')$. Let (h_k) be a sequence of indices that tends to ∞, and $u_k \in W^{1,p}_{g_{h_k}}(A \setminus K_{h_k}, \partial_D A \setminus K_{h_k})$ such that $u_k 1_{K^c_{h_k}}$ converges weakly to ϕ in $L^p(A)$ and $\nabla u_k 1_{K^c_{h_k}}$ converges weakly to ψ in $L^p(A, \mathbb{R}^2)$. Let $\tilde{u}_k$ be the extension of u_k defined by

$$\tilde{u}_k := \begin{cases} u_k 1_{K^c_{h_k}} & \text{in } A, \\ \tilde{g}_{h_k} & \text{in } \Sigma \setminus A, \end{cases}$$

and let $\tilde{\phi}$ and $\tilde{\psi}$ be defined by

$$\tilde{\phi} := \begin{cases} \phi & \text{in } A, \\ \tilde{g} & \text{in } \Sigma \setminus A, \end{cases} \qquad \text{and} \qquad \tilde{\psi} := \begin{cases} \psi & \text{in } A, \\ \nabla \tilde{g} & \text{in } \Sigma \setminus A. \end{cases}$$

As $u_k = g_{h_k}$ on $\partial_D A \setminus K_{h_k}$, we have $\tilde{u}_k \in W^{1,p}(\Omega_{h_k})$. Since $\tilde{u}_k 1_{\Omega_{h_k}}$ converges to $\tilde{\phi} 1_\Sigma$ weakly in $L^p(\mathbb{R}^2)$ and $\nabla \tilde{u}_k 1_{\Omega_{h_k}}$ converges to $\tilde{\psi} 1_\Sigma$ weakly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$, by Theorem 4.2 we conclude that there exists $\tilde{u} \in W^{1,p}(\Omega)$ such that $\tilde{\phi} 1_\Sigma = \tilde{u} 1_\Omega$ and $\tilde{\psi} 1_\Sigma = \nabla \tilde{u} 1_\Omega$. Let u be the restriction of $\tilde{u}$ to $A \setminus K$. Then $u \in W^{1,p}(A \setminus K)$ and we have that $\phi = u 1_{K^c}$ and $\psi = \nabla u 1_{K^c}$ a.e. in A. As $\tilde{u} \in W^{1,p}(\Omega)$, the traces of $\tilde{u}$ on both sides of $\partial_D A \setminus K$ coincide. Since $\tilde{u} = \tilde{g}$ a.e. in $\Sigma \setminus A$, we conclude that $u = \tilde{g} = g$ in the sense of traces on $\partial_D A \setminus K$. Therefore $u \in W^{1,p}_g(A \setminus K, \partial_D A \setminus K)$.

Now we prove property (M_1''). Let $u \in W^{1,p}_g(A \setminus K, \partial_D A \setminus K)$. Let $\tilde{u}$ be the extension of u defined by

$$\tilde{u} = \begin{cases} u 1_{K^c} & \text{a.e. in } A, \\ \tilde{g} & \text{a.e. in } \Sigma \setminus A. \end{cases}$$

As $u = g$ on $\partial_D A \setminus K$, we have that $\tilde{u} \in W^{1,p}(\Omega)$. By Theorem 4.2 there exists a sequence $\tilde{u}_h \in W^{1,p}(\Omega_h)$ such that $\tilde{u}_h 1_{\Omega_h}$ converges to $\tilde{u} 1_\Omega$ strongly in $L^p(\mathbb{R}^2)$ and $\nabla \tilde{u}_h 1_{\Omega_h}$ converges to $\nabla \tilde{u} 1_\Omega$ strongly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$. We consider the function

$$\phi_h := (\tilde{u}_h - \tilde{g}_h)|_{\Sigma \setminus \overline{A}}.$$

By construction, $\phi_h \to 0$ strongly in $W^{1,p}(\Sigma \setminus \overline{A})$. Therefore there exists a sequence $v_h \in W^{1,p}(\Sigma)$, converging to 0 strongly in $W^{1,p}(\Sigma)$, such that $v_h|_{\Sigma \setminus \overline{A}} = \phi_h$ a.e. in $\Sigma \setminus \overline{A}$. We set

$$u_h := (\tilde{u}_h - v_h)|_{A \setminus K_h}.$$

By construction, we have that $u_h \in W^{1,p}(A \setminus K_h)$ and $u_h = g_h$ in the sense of traces on $\partial_D A \setminus K_h$. Moreover, we have that $u_h 1_{K_h^c}$ converges to $u 1_{K^c}$ strongly in $L^p(A)$ and $\nabla u_h 1_{K_h^c}$ converges to $\nabla u 1_{K^c}$ strongly in $L^p(A, \mathbb{R}^2)$.

Now let us prove that $Y^{g_h}_{K_h}(A)$ converges to $Y^g_K(A)$ in the sense of Mosco. Property (M_2'') is obtained arguing as in [12, Lemma 4.1]. So, let us prove (M_1''). Let $u \in L^{1,p}_g(A \setminus K, \partial_D A \setminus K)$. We set for every $k \in \mathbb{N}$

$$u^k := (g - k) \vee u \wedge (g + k).$$

Then $u^k \in W^{1,p}_g(A \setminus K, \partial_D A \setminus K)$ and $\nabla u^k 1_{K^c} \to \nabla u 1_{K^c}$ strongly in $L^p(A, \mathbb{R}^2)$. From property (M_1'') proved above for the Mosco convergence of $X^{g_h}_{K_h}(A)$ to $X^g_K(A)$, for every k there exists $u^k_h \in W^{1,p}_{g_h}(A \setminus K_h, \partial_D A \setminus K_h)$ such that $\nabla u^k_h 1_{K_h^c} \to \nabla u^k 1_{K^c}$ in $L^p(A, \mathbb{R}^2)$. Hence, by a standard argument on double sequences, we obtain a sequence of indices k_h converging to ∞ such that, setting $u_h := u^{k_h}_h$, we get $\nabla u_h 1_{K_h^c} \to \nabla u 1_{K^c}$ in $L^p(A, \mathbb{R}^2)$. □

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