EFFECTIVE BRINKMAN LAW OF THE VISCOELASTOPLASTIC MODELS ON RANDOMLY PERFORATED DOMAINS

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Abstract. We perform the asymptotic analysis of a viscoelastoplastic models with selected smooth and nonsmooth stress-strain relations leading to a Brinkman-like system. The models are posed on randomly perforated domains with critical scaling regime under almost minimal assumptions on the geometry of the perforations.

Keywords: Viscoelastoplastic fluids, Stochastic homogenisation, Brinkman law, Non-smooth potentials, Leray-Hopf solutions, Perforated domains, Incompressible Navier-Stokes
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1. Introduction

The evolutionary equations of viscoelastoplasticity have recently seen abundance of applications in the mathematical theory of geodynamics. They are particularly used to model a variety of convective processes within the lithosphere of the Earth and connatural celestial bodies. Conventionally, upon assuming a large time-scale, the planet's upper mantle part may exhibit a rheological nature featuring a non-Newtonian fluid of viscoelastoplastic type. An overarching description of the concept along with a survey on current research advancements may be found for instance in the works of [37, 31, 29].

In this paper we focus on deriving the effective model of a viscoelastoplastic fluid-like body posed on a domain subject to stochastically varying microgeometry. In concrete mathematical terms, for a Lipschitz regular domain $D \subset \mathbb{R}^3$ and a microstructure scale $\varepsilon > 0$, we consider a family of sets D_ε which are constructed by removing, from the reference domain D, a random set consisting of spherical holes:

$$D_\varepsilon := D \setminus \bigcup_{i \in N_\varepsilon} \overline{B}_{\varepsilon^\alpha \rho_i}(z_\varepsilon^i).$$

Hereby N_ε is a finite set and $\varepsilon^\alpha \ll \varepsilon$ is the scaling order of the holes against the microstructure scale. The radii ρ_i are unbounded random variables and the centres $z_\varepsilon^i \in \mathbb{R}^3$ are probabilistically distributed according to a Poisson point process in $\mathbb{R}^3$. Subsequently, we look into an incompressible fluid under time evolution, whose flow is spatially confined in the perforated domains D_ε. More precisely, for a time interval $(0, T)$ with $T \in (0, +\infty]$, we study the following coupled evolutionary

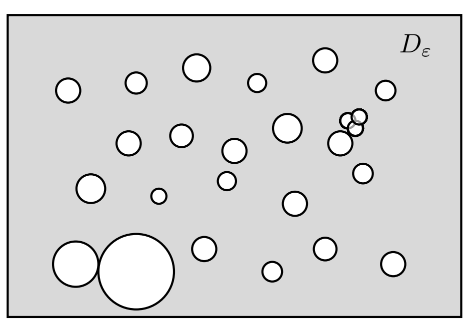

Figure 1. Perforated domains D_ε

system:

$$
\begin{aligned}
\partial_t(\varrho v_\varepsilon)+\operatorname{div}(v_\varepsilon\otimes\varrho v_\varepsilon)-\operatorname{div}(\eta S_\varepsilon+2\nu\nabla^{\mathrm{sym}}v_\varepsilon)+\nabla p_\varepsilon&=f_\varepsilon &&\text{in } D_\varepsilon\times(0,T),\\
\operatorname{div}(v_\varepsilon)&=0 &&\text{in } D_\varepsilon\times(0,T),\\
\overset{\triangledown}{S}_\varepsilon+\partial\mathcal{P}(S_\varepsilon)-\gamma\Delta S_\varepsilon&\ni\eta\nabla^{\mathrm{sym}}v_\varepsilon &&\text{in } D_\varepsilon\times(0,T).
\end{aligned}
\tag{1}
$$

The model exhibits the momentum equation of incompressible Navier-Stokes type along with Maxwell-type stress evolutionary relation. Here the unknowns $v_\varepsilon\colon(0,T)\times D_\varepsilon\to\mathbb{R}^3$ and $S_\varepsilon\colon(0,T)\times D_\varepsilon\to\mathbb{R}^{3\times3}$ are the time-dependent Eulerian velocity and internal stress respectively. The function $p_\varepsilon\colon(0,T)\times D_\varepsilon\to\mathbb{R}$ accounts for the pressure and the density $f_\varepsilon\colon(0,T)\times D_\varepsilon\to\mathbb{R}^3$ denotes the external acting force in D_ε. The rheological quantities of the body are its density $\varrho>0$, viscosity $\nu>0$ and the shear modulus coefficient $\eta>0$ encoding the elastic property. The symmetrised gradient is understood as $\nabla^{\mathrm{sym}}v=(\nabla v+\nabla v^\top)/2$. The diffusive term $\gamma\Delta S$ for $\gamma>0$, besides giving regularising effects, accounts for a source of dissipation and stems from a nonlocal energy storage mechanism. Our choice for the objective derivative of the stress S_ε, is the co-rotational Zaremba-Jaumann derivative with respect to velocity which is customarily employed in the geodynamics literature, see for instance [22, 31] and is given by

$$\overset{\triangledown}{S}:=\partial_t S+\operatorname{div}(S\otimes v)+S\nabla^{\mathrm{skw}}v-\nabla^{\mathrm{skw}}vS$$

wherein $\nabla^{\mathrm{skw}}v=(\nabla v-\nabla v^\top)/2$ is the skew-symmetric gradient. It is a special case of a more general class of Gordon-Schowalter rates. The coupled differential system in (1) is subjected to the no-slip boundary conditions on ∂D_ε: the zero Dirichlet boundary condition on v_ε and zero Neumann boundary condition on S_ε. Physically, the flow is spatially confined in D_ε and it encounters rough boundary at ∂D_ε which means that the confinement is impermeable. In addition, since temporal evolution is considered, we impose appropriate initial conditions for v_ε and S_ε.

The convex potential $\mathcal{P}$ appearing in the Maxwellian part of (1) accounts for viscoplastic effect and contributes further to a source of nonlinear dissipation. The intricacy here is that the functional $\mathcal{P}\colon L^2_{\mathrm{dev}}(D)\to L^2_{\mathrm{dev}}(D)$ might be non-smooth whereby its possibly set-valued subdifferential $\partial\mathcal{P}$ is understood as

$$\partial\mathcal{P}(S):=\big\{\hat S\in L^2_{\mathrm{dev}}(D)^*\colon\mathcal{P}(\Phi)\geq\mathcal{P}(S)+\langle\hat S,\Phi-S\rangle_{L^2_{\mathrm{dev}}(D)},\text{for all }\Phi\in L^2_{\mathrm{dev}}(D)\big\}$$

with $L^2_{\mathrm{dev}}(D):=\{\Phi\in L^2(D;\mathbb{R}^{3\times3})\colon\Phi=\Phi^\top,\operatorname{Tr}(\Phi)=0\}$. In Section 2.3, formula (18) we give a prototypical example of such a non-smooth potential which is frequently used in geodynamics. Consequently the relation (1) is understood as a differential inclusion. In case $\mathcal{P}$ is $C^{1,1}$-regular, which plays a role for instance in polymeric fluid models [2, 8], then $\partial\mathcal{P}$ coincides with the usual differential of $\mathcal{P}$. This then clearly renders the inclusion in (1) an equality where the Maxwellian part would be understood in a weak sense. In contrast, to operate with the notion of solutions encompassing a general dissipation potential, we shall resort to the notion of evolutionary variational solution for non-smooth stress-strain relation recently formulated in [16], see also [34]. Therein the authors prove global temporal existence of a Leray-Hopf type solution with a suitable variational inequality satisfies by the stress, cf. Sections 2.3 and 2.4, where we describe the precise definitions and constructions.

The primary objective of our work is to perform the homogenisation procedure of the system in (1) for a selected class of non-smooth as well as smooth dissipation potentials $\mathcal{P}$. We prove that under *nearly minimal* probabilistic assumptions on the perforations with the *critical* scaling regime $\varepsilon^\alpha\rho_i\simeq\varepsilon^3\rho_i$, the solutions $(v_\varepsilon,S_\varepsilon)$ to (1) converge almost surely as $\varepsilon\to0+$ to a pair (v,S)

which solves the deterministic Brinkmann-type model

$$\begin{aligned} \partial_t(\varrho v) + \operatorname{div}(v\otimes\varrho v) + \mathrm{M}_0 v - \operatorname{div}(\eta S + 2\nu\nabla^{\mathrm{sym}} v) + \nabla p = f \qquad & \text{in } D\times(0,T), \\ \operatorname{div}(v) = 0 \qquad & \text{in } D\times(0,T), \\ \overset{\triangledown}{S} + \partial\mathcal{P}(S) - \gamma\Delta S \ni \eta\nabla^{\mathrm{sym}} v \qquad & \text{in } D\times(0,T). \end{aligned} \tag{2}$$

Here the drag coefficient is given by $\mathrm{M}_0 := 6\pi\nu\lambda\mathbb{E}[\rho]I$ where I is the identity matrix in $\mathbb{R}^{3\times3}$, $\mathbb{E}[\rho]$ is the expectation of the mark ρ and $\lambda > 0$ is the intensity of the associated Poisson process, see Theorem 3.1 and 3.2 below for more details. Loosely speaking, the mesoscopic friction caused by the geometry of the spatial confinement of the flow gives rise to observable velocity drag which is de facto Brinkman-like behaviour.

The outcome of the homogenisation differs considerably when the power law of the perforation scale lies in the *subcritical* regime, that is, when $\varepsilon^\alpha\rho_i \ll \varepsilon^3\rho_i$. It turns out that in such cases, the viscoelastoplastic system is entirely asymptotically stable. Specifically for the scale $\alpha > 3$, in Theorem 6.1 we confirm that a pair of solution $(v_\varepsilon, S_\varepsilon)$ converges almost surely to (v, S) satisfying

$$\begin{aligned} \partial_t(\varrho v) + \operatorname{div}(v\otimes\varrho v) - \operatorname{div}(\eta S + 2\nu\nabla^{\mathrm{sym}} v) + \nabla p = f \qquad & \text{in } D\times(0,T), \\ \operatorname{div}(v) = 0 \qquad & \text{in } D\times(0,T), \\ \overset{\triangledown}{S} + \partial\mathcal{P}(S) - \gamma\Delta S \ni \eta\nabla^{\mathrm{sym}} v \qquad & \text{in } D\times(0,T). \end{aligned} \tag{3}$$

Namely the unchanged system is recovered. In other words, the combined mesoscopic cross-section of the perforations are too meagre to produce a non-negligible forcing against the flow on a large scale.

The matrix M_0 appearing in the momentum equation in (2), commonly known as the *Stokes resistance*, is inherited from the geometric and stochastic properties of the perforations. It comes predominantly from the effect on the elliptic part of the Cauchy-stress in (1) by the imposed Dirichlet boundary, see Section 4.1. The homogenised momentum equation bears resemblance with the behaviour observed in the case of deterministic incompressible Navier-Stokes models [1, 6, 19, 33], to mention a couple of examples among an extensive literature on the subject. As already examined in the aforementioned contributions, proceeded by the foundational work for the rate-independent Dirichlet problem [9], the appearance of the extra forcing term is closely linked to the assumed *critical* scaling of the radii. In actuality, the Stokes resistance is directly proportional to the newtonian capacity of the perforations, i.e. $\mathrm{M}_0 \simeq \operatorname{cap}\big(D\backslash D_\varepsilon;\mathbb{R}^3\big)$. On the other hand, as for the Maxwellian differential inclusion of the stress in (2), the equivalent relation as in (1) persists under limitation exhibiting a Neumann-type stability. Indeed, similarly to results in the static elliptic formulations, e.g. [12, 23, 14], the homogenous Neumann boundary conditions combined with *good* asymptotic behaviour of the perforated domains D_ε are sufficient for the same differential inclusion to be recovered in the limit.

A crucial feature of our analysis is the random nature of the varying domains D_ε upon which the viscoelastoplastic system is posed. Our inspiration draws from the considerations in [26], cf. also [15, 25], which is the stochastic variant of the homogenisation of the static Stokes equation on a domain with critically sized periodic holes [1]. In the former, an *almost minimal* probabilistic assumption on the spherical perforations is proposed. Crucially, contrary to the mentioned periodic setting, the random holes may overlap or form clusters with non-negligible probability. To afford the narrative sufficient context, we give a schematic explanation, while Section 2.1 addresses the detailed descriptions. For that, let us designate $(\Omega, \mathcal{T}, \mathbb{P})$ the ambient probability space. We select a marked point process $(\Theta, \mathcal{R})$ on $\mathbb{R}^3 \times (0,\infty)$ which serves as the instruction for assembling the perforations. Hereby Θ is a Poisson point process for the centres and $\mathcal{R}$ is the associated collection of marks $\mathcal{R}$ of identically independently distributed random variables for the respective radii which will eventually satisfy the *moment condition* $\mathbb{E}[\rho^q] < \infty$ for some $q > 1$. The randomly perforated domains are then defined, for $\omega \in \Omega$, as

$$D_\varepsilon = D_\varepsilon^\omega := D \setminus \bigcup_{x_i \in \Theta(\omega) \cap \frac{1}{\varepsilon} D} \overline{B}_{\varepsilon^\alpha \rho_i}(\varepsilon x_i).$$

where $\alpha \geq 1$, $(x_i, \rho_i) \in (\Theta(\omega), \mathcal{R}(\omega))$ and $\Theta(\omega)$ is a locally finite set. It is worth underlying that the expected number of balls $B_{\varepsilon^\alpha \rho_i}(\varepsilon x_i)$ non-trivially intersecting is strictly positive. Simultaneously, the radii ρ_i might be unbounded. Albeit, as shown in [26, Lemma 2.5] the assumed *moment condition* and the strong law of large numbers respected by the stochastic process, suffices to find a suitable corrector operator with the divergence constraint preserved and thus to reach the almost sure probabilistic counterpart of [1]. This then gives rise to the drag M_0 which depends on the chosen stochasticity. Morally one shows $\mathrm{M}_0 \simeq \sum_{x_i} \mathrm{cap}(B_{\varepsilon^3 \rho_i}(\varepsilon x_i); \mathbb{R}^3) \simeq \mathbb{E}[\mathrm{cap}(B_{\rho_i}(0); \mathbb{R}^3)]$. In the same spirit one sees that the capacitary contribution for $\alpha > 3$ is infinitesimal, i.e. $\lim_{\varepsilon \to 0+} \mathrm{cap}(D \setminus D_\varepsilon^\omega; \mathbb{R}^3) = 0$ for $\mathbb{P}$-a.e $\omega \in \Omega$. These two geometric limiting behaviours serve as a heuristic reasoning behind the distinction of the homogenised momentum equations in (2) and (3). Indeed, subject to appropriate adaptations, such a capacitary relation originating from rate-independent Stokes problem turns out also relevant in deriving the desired limit equation for the velocity equation in our setup.

On another note, owing to the said *moment condition* of the point process $\mathbb{E}[\rho^q] < \infty$, for a generic choice of the power law $\alpha > 1$, we may establish an almost sure Mosco-type asymptotic behaviour of the perforated domains D_ε in the context of Bochner-Sobolev spaces, see Theorem 2.4 below. This is quintessential to prove the stability of the differential inclusion in the two regimes (2) and (3) when passing to $\varepsilon \to 0+$, since it allows to obtain a compactness property of internal stresses and to identify the right Sobolev regularity of the limit map.

To authors awareness the hereby proposed analysis incorporating viscoelastoplasticity with non-smooth stress-strain relation appears to be an unprecedented consideration in the related literature on homogenisation both in deterministic and stochastic realms. The chief novelty of our work is the analysis of the Navier-Stokes coupled system defined on a large class of perforated domains with very complex microgeometry. In comparison with the existing theory, see e.g. [3, 25, 26, 24, 27, 32, 4, 38], in our assumptions the Cauchy stress in the momentum equation is augmented by adding the internal stress which is governed by an evolutionary relation in its own regard. As such our result is twofold. We first generalise the existence of solutions in [16] to less regular domains. Secondly, we derive the homogenisation procedure in probability law for generalised solutions for a collection of non-smooth convex potentials $\mathcal{P}$. Afterwards, we investigate the case of weak solutions for regular potentials of quadratic type drawing inspirations from the celebrated models proposed in [28]. As a byproduct of our analysis, we achieve an existence result for a Brinkman-type viscoelastoplastic model along with energy dissipation inequalities in the two respective stress-strain regimes. To complete the characterisation we verify the stability of the model for a specified scaling range.

2. Setup and preliminary results

Let us give a couple of comments on the notation and conventions present in the paper:

(a) for a measurable set $A \subset \mathbb{R}^3$ the expression $|A|$ means the Lebesgue measure of A;
(b) we denote by $\mathcal{B}(\mathbb{R}^3)$ the σ-algebra of Borel subsets of $\mathbb{R}^3$;
(c) for $A \subset \mathbb{R}^3$ open, V a finite-dimensional real vector space, $p \in [1, +\infty]$ and $u \colon A \to V$, we set $\|u\|_{L^p(A)} := \|u\|_{L^p(A;V)}$ and $\|u\|_{H^1(A)} := \|u\|_{H^1(A;V)}$;
(d) given $s \in [1, d)$ the associated Sobolev exponent is $s^* := ds/d - s$;
(e) the symmetrised gradient of $u \colon A \to \mathbb{R}^3$ is denoted by $\nabla^{\mathrm{sym}} u = (\nabla u + \nabla u^\top)/2$ and the skew-symmetric gradient by $\nabla^{\mathrm{skw}} u = (\nabla u - \nabla u^\top)/2$;

(f) for two vectors $x, y \in \mathbb{R}^3$ their scalar product is written as $x \cdot y$;
(g) for two matrices $M, N \in \mathbb{R}^{3\times 3}$ we denote by $M : N = \mathrm{tr}(M^\top N)$ the Frobenius scalar product and by $|M|$ the Frobenius norm of M;
(h) $\{e_1, e_2, e_3\}$ is the standard euclidean basis in $\mathbb{R}^3$;
(i) the usual tensor product of two vectors $x, y \in \mathbb{R}^3$ is the matrix $x \otimes y \in \mathbb{R}^{3\times 3}$ such that $(x \otimes y)_{ij} = x_i y_j$; if $M \in \mathbb{R}^{3\times 3}$ and $x \in \mathbb{R}^3$ then $M \otimes x \in \mathbb{R}^{3\times 3\times 3}$ is the 3rd-order tensor with $(M \otimes x)_{kij} = M_{ki} x_j$;
(j) for a Banach space X we denote the dual of X by X^* and the dual pairing by $\langle x^*, x\rangle_X$ for $x^* \in X^*$ and $x \in X$;
(k) $c > 0$ denotes a generic constant where relevant dependence is indicated in the subscript;
(l) with sequences indexed by ε, we mean elements of a sequence $\varepsilon_n \to 0$.

2.1. **Geometric and stochastic setting of the domains.** In this section we conceptualise the type of perforated domains within our considerations. Throughout we fix a reference domain $D \subset \mathbb{R}^3$ which is an open, bounded and star-shaped set with respect to the origin and has Lipschitz boundary. Let us also fix a probability space $(\Omega, \mathcal{T}, \mathbb{P})$. For a random variable $X\colon \Omega \to \mathbb{R}$, its expected value with respect to $\mathbb{P}$ is written as

$$\mathbb{E}[X] := \int X(\omega)\,\mathrm{d}\mathbb{P}(\omega).$$

In our analysis we consider marked point processes $(\Theta, \mathcal{R}) \subset \mathbb{R}^3 \times (0,\infty)$ in Ω where $\Theta = (\Theta_n)_{n\in\mathbb{N}}$ is a Poisson point process in $\mathbb{R}^3$ and $\mathcal{R} = (\rho_n)_{x_n\in\Theta}$ is the assigned marked space with ρ_n identically and independently distributed random variables. In what follows, for a detailed description of the notion of marked point processes we refer for instance to the monographs [13, 36]. As such, let us recount that the Campbell's theorem for point processes says that for any measurable function $g\colon \mathbb{R}^3 \times (0,\infty) \to [0,\infty)$ we have

$$\mathbb{E}\Big[\sum_{n\in\mathbb{N}} g(x_n, \rho_n)\Big] = \iint g(x, r)\,\mathrm{d}\mathbb{P}^{\mathcal{R}}_x(r)\,\mathrm{d}m_\Theta(x) \tag{4}$$

where $\mathbb{P}^{\mathcal{R}}_x$ is the probability marginal measure at x and m_Θ is the intensity measure of Θ given by $m_\Theta(B) = \mathbb{E}[n(\Theta \cap B)]$ and $n(\Theta \cap B)$ is the random variable defined by the number of points of Θ in $B \subset \mathbb{R}^3$ at a given realisation in Ω. A further variant of Campbell's theorem also provides an integral representation of point processes with respect to second moments:

$$\mathbb{E}\Big[\sum_{n\in\mathbb{N}} \sum_{\substack{m\in\mathbb{N}\\ x_m\neq x_n}} G(x_n, x_m, \rho_n, \rho_m)\Big] = \iint G(x, y, r, s)\,\mathrm{d}\mathbb{P}^{\mathcal{R}}_{x,y}(r, s)\,\mathrm{d}M_\Theta(x, y). \tag{5}$$

for any measurable function $G\colon (\mathbb{R}^3)^2 \times (0,\infty)^2 \to [0,\infty)$. Here $\mathbb{P}^{\mathcal{R}}_{x,y}$ is the two-point mark distribution and M_Θ is a measure defined as $M_\Theta(A \times B) := m_\Theta(A \cap B) + \mathbb{E}[n(\Theta \cap A)n(\Theta \cap B)]$.

The class of admissible processes shall satisfy properties in close similarity to the ones assumed in the setup of the stochastic Stokes problem in [26].

Definition 2.1 (Admissible process)**.** Given $\alpha > 1$, we say that a marked point process $(\Theta, \mathcal{R})$ with marginals as in (4)-(5) is admissible provided it satisfies the following conditions:

(H0) The measures $\mathbb{P}^{\mathcal{R}}_x$ and $\mathbb{P}^{\mathcal{R}}_{x,y}$ in (4)-(5) are absolutely continuous with respect to the Lebesgue measure: there exist densities f and F such that

$$\mathbb{P}^{\mathcal{R}}_x(\mathcal{I}) = \int_{\mathcal{I}} f(x, r)\,\mathrm{d}r \quad \text{and} \quad \mathbb{P}^{\mathcal{R}}_{x,y}(\mathcal{I}\times\mathcal{J}) = \int_{\mathcal{I}}\int_{\mathcal{J}} F(x, y, r, s)\;\mathrm{d}(r, s) \tag{6}$$

for all $\mathcal{I}, \mathcal{J} \in \mathcal{B}((0,\infty))$.

(H1) Θ is stationary: for all $x \in \mathbb{R}^3$ the processes $\Theta = (\Theta_n)_{n\in\mathbb{N}}$ and $\Theta + x = (\Theta_n + x)_{n\in\mathbb{N}}$ possess the same probability distribution. As a consequence the measure m_Θ appearing in (4) admits a homogeneous intensity: there exists $\lambda > 0$ such that

$$m_\Theta(B) = \mathbb{E}\big[n(\Theta \cap B)\big] = \lambda|B|; \tag{7}$$

(H2) Θ has finite intensity rate: for any unit cube $Q \subset \mathbb{R}^3$

$$\mathbb{E}\big[n(\Theta \cap Q)^2\big] \leq \lambda^2$$

where λ is the intensity from (7).

(H3) Strong mixing condition: given $A \in \mathcal{B}(\mathbb{R}^3)$ by $\mathcal{T}(A)$ we denote the smallest σ-algebra such that for any Borel subset $B \subset A$ the random variable $n(\Theta\cap B)$ is $\mathbb{P}$-measurable. In addition, we assume there exist $c_1 > 0$ and $\vartheta > 0$ such that for all $A \in \mathcal{B}(\mathbb{R}^3)$, all $|x| > \mathrm{diam}(A)$ and all random variables X, Y which are $\mathbb{P}$-measurable with respect to $\mathcal{T}(A)$ and $\mathcal{T}(x+A)$ respectively, one has

$$\Big|\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]\Big| \leq \frac{c_1 \mathbb{E}[X^2]^{\frac{1}{2}}\mathbb{E}[Y^2]^{\frac{1}{2}}}{1 + (|x| - \mathrm{diam}(A))^\vartheta}. \tag{8}$$

(H4) In view of the stationarity of Θ in condition (H1) the density f of $\mathbb{P}_x^{\mathcal{R}}$ from (6) satisfies the equality $f(x,\cdot) = h(\cdot)$ for all $x \in \mathbb{R}^3$ whereby $h \in L^1([0,\infty))$ is a nonnegative function. Here we make a requirement that elements in $\mathcal{R}$ satisfy a finite q-moment for some $q > 3/\alpha$:

$$\mathbb{E}[\rho^q] = \int r^q h(r)\,\mathrm{d}r < +\infty$$

for all marks ρ in $\mathcal{R}$.

(H5) The stationarity of Θ in condition (H1) implies the density F in (6) satisfies the relation $F(x, y, \cdot, \cdot) = F(x - y, x - y, \cdot, \cdot)$. We assume here that there exists a function K and constants $c > 0$, $\beta > 2$ such that

$$F(x, y, r, s) = h(r)h(s) + K(|x - y|, r, s), \quad |K(d, r, s)| \leq \frac{c}{(1 + d^q)(1 + r^\beta)(1 + s^\beta)}$$

where h and q are as in (H4).

For the relevance of the above conditions as well as a list of particular examples of marked point processes satisfying the properties (H0)-(H5), we refer to the discussion in [25, Section 2.1].

Given an admissible process $(\Theta, \mathcal{R})$, we shall annotate the randomly perforated domains D_ε associated with $(\Theta, \mathcal{R})$ by

$$D_\varepsilon := D \setminus \overline{H}_\varepsilon \quad \text{where} \quad H_\varepsilon := \bigcup_{x_n\in\mathcal{I}_\varepsilon} B_{\varepsilon^\alpha\rho_n}(\varepsilon x_n) \tag{9}$$

where $\mathcal{I}_\varepsilon$ denotes the index of holes intersecting the reference domain D as $\mathcal{I}_\varepsilon := \{n \in \mathbb{N} \colon \varepsilon x_n \in D\}$.

Henceforth all the randomly perforated domains D_ε associated with the processes $(\Theta, \mathcal{R})$ are going to be assumed admissible.

The star-shaped geometry of D makes, up to possible reordering, the collection of sets $\{D_\varepsilon\}_{\varepsilon>0}$ nested. Let us also recount that for an admissible process $(\Theta, \mathcal{R})$, the random radii (ρ_n) are not necessarily bounded. Nonetheless, the moment condition (H4), see [27, Lemma 3.1] and [24, Lemma 1.1], along with finite intensity guarantee that the expected number of such radii is sufficiently small, so that $\mathbb{P}$-a.e. in Ω:

$$\lim_{\varepsilon\to 0} |D_\varepsilon| = |D|.$$

In addition in view of [25, Lemma 5.1], see also [38, Proposition 6.1] the perforations H_ε generated by an admissible process fulfil a variant of the stong law of large numbers in terms of the capacity:

$$\lim_{\varepsilon\to 0+}\sum_{x_n\in\mathcal{I}_\varepsilon}\varepsilon^3\mathrm{cap}(B_{\rho_n}(0);\mathbb{R}^3)=\lambda\mathbb{E}[\rho]|D|$$

holds $\mathbb{P}$-a.e. in Ω where $\mathrm{cap}\big(\,\cdot\,;\mathbb{R}^3\big)$ denotes the usual newtonian capacity in $\mathbb{R}^3$.

The perforations H_ε generated by the admissible process $(\Theta,\mathcal{R})$ satisfy a representation classifying the overlapping spheres. The forthcoming classification was established for instance in [24, Lemma 1.2] in the case of the power laws $\alpha\in(1,3)$ and in [26, Lemma 3.1, Lemma 5.1] for $\alpha=3$. The argument is in essence a consequence of the Borel-Cantelli lemma along with properties (H1), (H5) of the process $(\Theta,\mathcal{R})$. The claim for the broader range $\alpha>1$ can be then recovered from the aforementioned results by rescaling the radii by the respective order of ε. Therefore for a detailed proof of the statement below we refer to the listed contributions.

Lemma 2.2. There exist $N=N(\alpha,\beta)\in\mathbb{N}, M=M(\alpha,\beta)\in\mathbb{N}$ and $\sigma=\sigma(\alpha,\beta)>0$ along with $\mathcal{I}_1^\varepsilon,\dots,\mathcal{I}_N^\varepsilon\subset\mathcal{I}_\varepsilon$ disjoint sets such that for $\mathbb{P}$-a.e. $\omega\in\Omega$ there exists $\varepsilon_0(\omega)>0$ such that for all $\varepsilon\le\varepsilon_0(\omega)$ it holds

$$H_\varepsilon=\bigcup_{k=1}^N\bigcup_{x_n\in\mathcal{I}_k^\varepsilon}B_{\varepsilon^\alpha\rho_n}(\varepsilon x_n)\quad\text{and}\quad\sup_{x_n\in\mathcal{I}_\varepsilon}\varepsilon^\alpha\rho_n\le\varepsilon^\sigma.$$

Moreover for every $k\in 1,\dots,N$ the elements in $\{B_{2\varepsilon^\alpha\rho_n}(\varepsilon x_n)\colon x_n\in\mathcal{I}_k^\varepsilon\}$ have at most M nontrivial intersections.

2.2. **Function spaces.** Let us recount some underlying function spaces which are going to be utilised throughout the discourse. For an open set $A\subset\mathbb{R}^3$ we define

$$\begin{aligned}
C^\infty_{0,\mathrm{div}}(A)&:=\{u\in C_0^\infty(A;\mathbb{R}^3)\colon \mathrm{div}(u)=0\}\\
H^1_{\mathrm{dev}}(A)&:=\{U\in H^1(A;\mathbb{R}^{3\times3})\colon U=U^\top,\mathrm{Tr}(U)=0\}\\
L^2_{\mathrm{div}}(A)&:=\{u\in L^2(A;\mathbb{R}^3)\colon \mathrm{div}(u)=0, u\cdot\vec{n}=0\quad\text{on }\partial A\}\\
L^2_{\mathrm{dev}}(A)&:=\{U\in L^2(A;\mathbb{R}^{3\times3})\colon U=U^\top,\mathrm{Tr}(U)=0\}.
\end{aligned}$$

We declare $H^1_{0,\mathrm{div}}(A)$ as the closure of $C^\infty_{0,\mathrm{div}}(A)$ in $H^1(A;\mathbb{R}^3)$ with respect to the H^1-norm. In addition, for an interval $I\subset\mathbb{R}$, we define

$$\begin{aligned}
C^\infty_{0,\mathrm{div}}(A\times I)&:=\{u\in C_0^\infty(A\times I;\mathbb{R}^3)\colon \mathrm{div}(u)=0\}\\
C^\infty_{0,\mathrm{dev}}(\overline{A}\times I)&:=\{U\in C_0^\infty(\overline{A}\times I;\mathbb{R}^{3\times3})\colon U=U^\top,\mathrm{Tr}(U)=0\}.
\end{aligned}$$

Given a Banach space X and $a,b\in\mathbb{R}$ with $a<b$, for a X-valued mapping $g\in L^1_{\mathrm{loc}}(a,b;X)$ we define the time regularisation of g by

$$g^\tau(t):=(\varrho_\tau*g)(t):=\int_{\mathbb{R}}\varrho_\tau(t-s)g(s)\,\mathrm{d}s\quad\text{ for }t\in(a+\tau,b-\tau),\tag{10}$$

where $\tau\in(a,b)$ and $\varrho_\tau(s):=(1/\tau)\varrho(s/\tau)$ with $\varrho\in C_0^\infty(\mathbb{R})$, $\mathrm{supp}(\varrho)\subset(-1,1)$ is a standard mollifier.

Let us now present a technical result which elucidates the construction of suitable corrector maps for a general choice of the scaling regimes $\alpha>1$ in the perforations. This will find its implementation in a number of key arguments in this paper further on.

Proposition 2.3. Let $\alpha>1$ and $\varphi\in C^\infty(\mathbb{R}^3;\mathbb{R}^3)\cap W^{1,\infty}(\mathbb{R}^3;\mathbb{R}^3)$ with $\mathrm{div}(\varphi)=0$ in $\mathbb{R}^3$ be given. For $\mathbb{P}$-a.e $\omega\in\Omega$ there exists a sequence $(\varphi_\varepsilon)\subset C^\infty(\mathbb{R}^3;\mathbb{R}^3)$ with the following properties:

i) $\varphi_\varepsilon=0$ in H_ε and $\mathrm{div}(\varphi_\varepsilon)=0$ in $\mathbb{R}^3$;

ii) $\|\varphi_\varepsilon\|_{L^\infty} \leq \|\varphi\|_{L^\infty}$;

iii) for any $p \in [1, +\infty)$ one has:

$$\begin{aligned}
\|\varphi - \varphi_\varepsilon\|_{L^p(D)} &\leq c\|\varphi\|_{L^\infty} |D \backslash D_\varepsilon|^{\frac{1}{p}} \\
\|\nabla\varphi - \nabla\varphi_\varepsilon\|_{L^2(D)} &\leq c\|\varphi\|_{C^1} \left(\sum_{x_n \in \mathcal{I}_\varepsilon} \operatorname{cap}\big(B_{\varepsilon^\alpha \rho_n}(0); \mathbb{R}^3\big) \right)^{\frac{1}{2}}.
\end{aligned} \tag{11}$$

Moreover if $\varphi \in C^\infty_{0,\mathrm{div}}(D)$, then $(\varphi_\varepsilon) \subset C^\infty_{0,\mathrm{div}}(D_\varepsilon)$.

Proof. Consider the families $\{B_{2\varepsilon^\alpha \rho_n}(\varepsilon x_n) \colon x_n \in \cup_{k=1}^N \mathcal{I}_k^\varepsilon\}$ with the partition $\mathcal{I}_1^\varepsilon, \dots, \mathcal{I}_N^\varepsilon$ of $\mathcal{I}_\varepsilon$ decomposing H_ε from Lemma 2.2. Since each family $\{B_{2\varepsilon^\alpha \rho_n}(\varepsilon x_n) \colon x_n \in \mathcal{I}_k^\varepsilon\}$ has at most M non-trivial intersections and both $N, M \in \mathbb{N}$ are independent of $\varepsilon > 0$, after a possible reorganisation, we may write

$$H_\varepsilon = \bigcup_{k=1}^{\tilde{N}} \bigcup_{x_n \in \tilde{\mathcal{I}}_k^\varepsilon} B_{\varepsilon^\alpha \rho_n}(\varepsilon x_n)$$

where $\tilde{N} = \tilde{N}(M, N) \in \mathbb{N}$, with indexing sets $\tilde{\mathcal{I}}_1^\varepsilon, \dots, \tilde{\mathcal{I}}_{\tilde{N}}^\varepsilon \subset \mathcal{I}_\varepsilon$ such that each $\{B_{2\varepsilon^\alpha \rho_n}(\varepsilon x_n) \colon x_n \in \tilde{\mathcal{I}}_k^\varepsilon\}$ is a pairwise disjoint family. We shall construct the sought sequence φ_ε in an iterative modifications of φ across the indices $k = 1, \dots, \tilde{N}$. For such a fixed k define $\varphi_\varepsilon^k \colon \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\varphi_\varepsilon^k := \begin{cases} \varphi_\varepsilon^{k-1} & \text{in } \mathbb{R}^3 \setminus \bigcup_{x_n \in \tilde{\mathcal{I}}_k^\varepsilon} B_{2\varepsilon^\alpha \rho_n}(\varepsilon x_n) \\ 0 & \text{in } B_{\varepsilon^\alpha \rho_n}(\varepsilon x_n) \text{ for all } x_n \in \tilde{\mathcal{I}}_k^\varepsilon \\ \psi_n^k & \text{in } B_{2\varepsilon^\alpha \rho_n}(\varepsilon x_n) \backslash B_{\varepsilon^\alpha \rho_n}(\varepsilon x_n) \text{ for all } x_n \in \tilde{\mathcal{I}}_k^\varepsilon, \end{cases}$$

where each $\psi_n^k \in H^1_{0,\mathrm{div}}(B_{2\varepsilon^\alpha \rho_n}(\varepsilon x_n)) + \varphi_\varepsilon^{k-1}$ is the solution to the following Stokes problem:

$$\begin{aligned}
-\Delta\psi_n^k + \nabla p &= -\Delta\varphi_\varepsilon^{k-1} && \text{in } B_{2\varepsilon^\alpha \rho_n}(\varepsilon x_n) \backslash B_{\varepsilon^\alpha \rho_n}(\varepsilon x_n), \\
\operatorname{div}(\psi_n^k) &= 0 && \text{in } B_{2\varepsilon^\alpha \rho_n}(\varepsilon x_n) \backslash B_{\varepsilon^\alpha \rho_n}(\varepsilon x_n); \\
\psi_n^k &= 0 && \text{in } B_{\varepsilon^\alpha \rho_n}(\varepsilon x_n).
\end{aligned}$$

Here the assumption is $\varphi_\varepsilon^0 := \varphi$. By standard regularity theory of the Stokes problem and the fact that φ is C^∞, one has the inclusion $\varphi_\varepsilon^1 \in C^\infty(\mathbb{R}^3; \mathbb{R}^3)$. Iteratively we obtain $\varphi_\varepsilon^k \in C^\infty(\mathbb{R}^3; \mathbb{R}^3)$ for all $k = 1, \dots, \tilde{N}$. Consequently define the desired sequence by

$$\varphi_\varepsilon := \varphi_\varepsilon^{\tilde{N}}.$$

As an immediate observation, from the very construction φ_ε vanishes in every $B_{\varepsilon^\alpha \rho_n}(\varepsilon x_n)$ and satisfies the divergence-free condition. This concludes part i).

Considering [30, Theorem 5.1, Theorem 6.1] along with a scaling and translation argument, one is led to the estimate $\|\psi_n^1\|_{L^\infty} \leq c\|\varphi\|_{L^\infty}$ and so, by construction we have $\|\varphi_n^1\|_{L^\infty} \leq c\|\varphi\|_{L^\infty}$. Iteratively it follows

$$\|\psi_n^k\|_{L^\infty} \leq c\|\varphi_\varepsilon^{k-1}\|_{L^\infty} \tag{12}$$

and thus, by construction $\|\varphi_\varepsilon^k\|_{L^\infty(D)} \leq c\|\varphi_\varepsilon^{k-1}\|_{L^\infty(D)}$ which yields part ii).

As for part iii), using the fact that $B_{2\varepsilon^\alpha \rho_n}(\varepsilon x_n)$ with $x_n \in \mathcal{I}_k^\varepsilon$ are pairwise disjoint for all k we obtain

$$\begin{aligned}
\|\varphi_\varepsilon^k - \varphi_\varepsilon^{k-1}\|^p_{L^p(D)} &= \sum_{x_n \in \tilde{\mathcal{I}}_k^\varepsilon} \|\varphi_\varepsilon^k - \varphi_\varepsilon^{k-1}\|^p_{L^p(D \cap B_{2\varepsilon^\alpha \rho_n}(\varepsilon x_n))} \leq c\|\varphi_\varepsilon^{k-1}\|^p_{L^\infty(D)} \sum_{x_n \in \tilde{\mathcal{I}}_k^\varepsilon} \big|D \cap B_{2\varepsilon^\alpha \rho_n}(\varepsilon x_n)\big| \\
&\leq c\|\varphi_\varepsilon^{k-1}\|^p_{L^\infty(D)} |D \backslash D_\varepsilon|.
\end{aligned} \tag{13}$$

Moreover in view of (13) we have

$$\|\varphi-\varphi_\varepsilon\|^p_{L^p(D)} \le \tilde N^{p-1}\sum_{k=1}^{\tilde N}\|\varphi^k_\varepsilon-\varphi^{k-1}_\varepsilon\|^p_{L^p(D)} \le c|D\backslash D_\varepsilon|\sum_{k=1}^{\tilde N}\|\varphi^{k-1}_\varepsilon\|^p_{L^\infty(D)}$$

and by part ii), the first estimate in (11) follows. Now we go on to verify the remaining claim. We may invoke [30, Theorem 5.1, Theorem 6.1] again to conclude, after scaling and translations, that for all $k=1,\dots,\tilde N$ there holds

$$\|\nabla\psi^k_n\|^2_{L^2(A_n)} \le c\left(\|\nabla\varphi^{k-1}_\varepsilon\|^2_{L^2(A_n)} + \frac{\|\varphi^{k-1}_\varepsilon\|^2_{L^2(A_n)}}{\varepsilon^{2\alpha}\rho_n^2}\right), \quad A_n := B_{2\varepsilon^\alpha\rho_n}(\varepsilon x_n)\backslash\overline{B}_{\varepsilon^\alpha\rho_n}(\varepsilon x_n). \tag{14}$$

Let $\tilde D\subset\mathbb{R}^3$ be an open set such that $D\subset\tilde D$ and $B_{2\varepsilon^\alpha\rho_n}(\varepsilon x_n)\subset\tilde D$ for all $x_n\in\mathcal{I}_\varepsilon$. This is possible in view of Lemma 2.2. We now claim that for all $k=1,\dots,\tilde N$ the following estimate holds

$$\|\nabla\varphi^k_\varepsilon-\nabla\varphi\|^2_{L^2(\tilde D)} \le c\sum_{j=1}^{k}\sum_{x_n\in\tilde{\mathcal{I}}^\varepsilon_j}\Big(\|\nabla\varphi\|^2_{L^2(B_n)}+\varepsilon^\alpha\rho_n\|\varphi\|^2_{L^\infty}\Big), \quad B_n := B_{2\varepsilon^\alpha\rho_n}(\varepsilon x_n). \tag{15}$$

Indeed, arguing by induction on k, for $k=1$ by construction of φ^1_ε and the bound in (14) and (12) we have

$$\begin{aligned}
\|\nabla\varphi^1_\varepsilon-\nabla\varphi\|^2_{L^2(\tilde D)} &= \sum_{x_n\in\tilde{\mathcal{I}}^\varepsilon_1}\|\nabla\varphi^1_\varepsilon-\nabla\varphi\|^2_{L^2(\tilde D\cap B_n)} \le \sum_{x_n\in\tilde{\mathcal{I}}^\varepsilon_1}\Big(\|\nabla\varphi\|^2_{L^2(\tilde D\cap B_n)}+\|\nabla\psi^1_n\|^2_{L^2(A_n)}\Big)\\
&\le c\sum_{x_n\in\tilde{\mathcal{I}}^\varepsilon_1}\left(\|\nabla\varphi\|^2_{L^2(B_n)}+\frac{\|\varphi\|^2_{L^2(B_n)}}{\varepsilon^{2\alpha}\rho_n^2}\right)\\
&\le c\sum_{x_n\in\tilde{\mathcal{I}}^\varepsilon_1}\Big(\|\nabla\varphi\|^2_{L^2(B_n)}+\varepsilon^\alpha\rho_n\|\varphi\|^2_{L^\infty}\Big).
\end{aligned}$$

Assume, for the inductive step, that the estimate in (15) hold for some k. Analysing the case $k+1$ we have

$$\begin{aligned}
\|\nabla\varphi^{k+1}_\varepsilon-\nabla\varphi\|^2_{L^2(\tilde D)} &\le c\Big(\|\nabla\varphi^{k+1}_\varepsilon-\nabla\varphi^k_\varepsilon\|^2_{L^2(\tilde D)}+\|\nabla\varphi^k_\varepsilon-\nabla\varphi\|^2_{L^2(\tilde D)}\Big)\\
&\le c\|\nabla\varphi^{k+1}_\varepsilon-\nabla\varphi^k_\varepsilon\|^2_{L^2(\tilde D)}+c\sum_{j=1}^{k}\sum_{x_n\in\tilde{\mathcal{I}}^\varepsilon_j}\Big(\|\nabla\varphi\|^2_{L^2(B_n)}+\varepsilon^\alpha\rho_n\|\varphi\|^2_{L^\infty}\Big)
\end{aligned}$$

where in the last step we employed the inductive step. However, by the very construction of φ^k_ε and the bound in (14) it follows

$$\|\nabla\varphi^{k+1}_\varepsilon-\nabla\varphi^k_\varepsilon\|^2_{L^2(\tilde D)} \le c\sum_{x_n\in\tilde{\mathcal{I}}^\varepsilon_{k+1}}\left(\|\nabla\varphi^k_\varepsilon\|^2_{L^2(A_n)}+\frac{\|\varphi^{k-1}_\varepsilon\|^2_{L^2(A_n)}}{\varepsilon^{2\alpha}\rho_n^2}\right)$$

and therefore

$$\|\nabla\varphi^{k+1}_\varepsilon-\nabla\varphi^k_\varepsilon\|^2_{L^2(\tilde D)} \le c\left(\|\nabla\varphi^k_\varepsilon-\nabla\varphi\|^2_{L^2(\tilde D)}+\sum_{x_n\in\tilde{\mathcal{I}}^\varepsilon_{k+1}}\left(\|\nabla\varphi\|^2_{L^2(A_n)}+\frac{\|\varphi^{k-1}_\varepsilon\|^2_{L^2(A_n)}}{\varepsilon^{2\alpha}\rho_n^2}\right)\right).$$

Using the inductive step in the first summand above and part ii) in the second summand we arrive at

$$\|\nabla\varphi_\varepsilon^{k+1} - \nabla\varphi_\varepsilon^{k}\|^2_{L^2(\tilde D)} \leq c\sum_{j=1}^{k+1}\sum_{x_n\in\tilde{\mathcal{I}}_j^\varepsilon}\Big(\|\nabla\varphi\|^2_{L^2(A_n)} + \varepsilon^\alpha\rho_n\|\varphi\|^2_{L^\infty}\Big).$$

This in turn completes the induction argument justifying (15). Setting $k = \tilde N$ therein implies

$$\begin{aligned}\|\nabla\varphi_\varepsilon - \nabla\varphi\|^2_{L^2(D)} \leq \|\nabla\varphi_\varepsilon^{\tilde N} - \nabla\varphi\|^2_{L^2(\tilde D)} &\leq c\sum_{j=1}^{\tilde N}\sum_{x_n\in\tilde{\mathcal{I}}_j^\varepsilon}\Big(\|\nabla\varphi\|^2_{L^2(A_n)} + \varepsilon^\alpha\rho_n\|\varphi\|^2_{L^\infty}\Big)\\ &\leq c\|\varphi\|^2_{C^1}\sum_{x_n\in\mathcal{I}_\varepsilon}\Big(\varepsilon^{3\alpha}\rho_n^3 + \varepsilon^\alpha\rho_n\Big) \leq c\|\varphi\|^2_{C^1}\sum_{x_n\in\mathcal{I}_\varepsilon}\varepsilon^\alpha\rho_n\\ &= c\|\varphi\|^2_{C^1}\sum_{x_n\in\mathcal{I}_\varepsilon}\operatorname{cap}\big(B_{\varepsilon^\alpha\rho_n}(0);\mathbb{R}^3\big)\end{aligned}$$

where in the last estimate we used the bound on the radii $\sup_{x_n\in\mathcal{I}_\varepsilon}\varepsilon^{2\alpha}\rho_n^2 \leq \varepsilon^{2\sigma}$ from Lemma 2.2 in the first summand. This ultimately proving the second estimate in part iii).

If $\operatorname{supp}(\varphi) \subset\subset D$, then investigating the construction of φ_ε^0 and from the radii bound in Lemma 2.2 one observes that

$$\operatorname{dist}(\operatorname{supp}(\varphi_\varepsilon^0), \partial D) \geq \operatorname{dist}(\operatorname{supp}(\varphi), \partial D) - 2\varepsilon^\sigma.$$

Iteratively one obtains $\operatorname{dist}(\operatorname{supp}(\varphi_\varepsilon^k), \partial D) \geq \operatorname{dist}(\operatorname{supp}(\varphi), \partial D) - 2k\varepsilon^\sigma$ and so choosing $\varepsilon > 0$ sufficiently small gives $\operatorname{supp}(\varphi_\varepsilon) \subset\subset D_\varepsilon$. Hence the claim follows. □

We shall now give an essential result concerning the compactness property of sequences of the internal stresses. This property will turn out significant when treating the homogenisation of the Maxwellian inclusion. Since the perforations intersect and form clusters, the domains D_ε may not necessarily be sufficiently regular to recover the right compactness from uniform Sobolev extensions. Thus, to ensure the regularity of the limiting map of the sequence of internal stresses, we shall adhere to a Mosco-type property.

Lemma 2.4. Let $(u_\varepsilon) \subset L^2(D)$ be a sequence such that $u_\varepsilon \in H^1(D_\varepsilon)$ for all $\varepsilon > 0$ and

$$\sup_\varepsilon \|u_\varepsilon\|_{H^1(D_\varepsilon)} < +\infty.$$

Then there exists $u \in H^1(D)$ such that for $\mathbb{P}$-a.e. $\omega \in \Omega$, up to subsequences, $u_\varepsilon\chi_{D_\varepsilon} \rightharpoonup u$ in $L^2(D)$ and $\nabla u_\varepsilon\chi_{D_\varepsilon} \rightharpoonup \nabla u$ in $L^2(D;\mathbb{R}^3)$.

Proof. For all $i \in \{1, 2, 3\}$ we consider a sequence $(w_\varepsilon^i) \subset W^{1,\infty}(D;\mathbb{R}^3)$ such that for $\mathbb{P}$-a.e $\omega \in \Omega$ there exists $\varepsilon_0(\omega) > 0$ such that for all $\varepsilon \leq \varepsilon_0(\omega)$ there holds

- $w_\varepsilon^i = 0$ in H_ε
- $\operatorname{div}(w_\varepsilon^i) = 0$ in D
- $w_\varepsilon^i \to e_i$ in $L^p(D;\mathbb{R}^3)$ for all $p \in [1, +\infty)$.

Indeed, we may apply Lemma 2.3 to $\varphi := e_i$ and set $w_\varepsilon^i := \varphi_\varepsilon|_D$.

Having obtained the auxiliary maps w_ε^i we now turn to the verification of the lemma. From the given uniform bound, up to subsequences, there exist $u \in L^2(D)$ and $\Psi \in L^2(D;\mathbb{R}^3)$ such that $u_\varepsilon\chi_{D_\varepsilon} \rightharpoonup u$ in $L^2(D)$ and $\nabla u_\varepsilon\chi_{D_\varepsilon} \rightharpoonup \Psi$ in $L^2(D;\mathbb{R}^3)$. Let $\varphi \in C_0^\infty(D)$ be arbitrarily chosen. Then writing $\Psi = (\Psi^i)_{i=1}^3$ and using the weak convergence of $\nabla u_\varepsilon\chi_{D_\varepsilon}$ it follows

$$\lim_{\varepsilon\to 0}\int_{D_\varepsilon}\nabla u_\varepsilon\cdot\varphi w_\varepsilon^i\,\mathrm{d}x = \lim_{\varepsilon\to 0}\int_D\nabla u_\varepsilon\cdot\varphi w_\varepsilon^i\,\mathrm{d}x = \int_D\Psi^i\varphi\,\mathrm{d}x.$$

At the same time using the divergence-free condition of w_ε^i, integrating by parts, we have

$$\lim_{\varepsilon\to 0}\int_{D_\varepsilon} \nabla u_\varepsilon\cdot\varphi w_\varepsilon^i\,\mathrm{d}x = -\lim_{\varepsilon\to 0}\int_{D_\varepsilon} u_\varepsilon\nabla\varphi\cdot w_\varepsilon^i\,\mathrm{d}x = -\int_D u\nabla\varphi\cdot e_i\,\mathrm{d}x = -\int_D u\partial_i\varphi\,\mathrm{d}x.$$

In particular, since $i\in\{1,2,3\}$ was arbitrary, $\Psi=\nabla u$ and the proof is complete. □

Proposition 2.5. Let $T>0$ and let $(\Phi_\varepsilon)\subset L^2(0,T;H^1_{\mathrm{dev}}(D_\varepsilon))$ be a sequence such that

$$\sup_\varepsilon\left(\int_0^T \|\Phi_\varepsilon\|^2_{H^1(D_\varepsilon)}\,\mathrm{d}s\right)<+\infty.$$

Then there exists $\Phi\in L^2(0,T;H^1_{\mathrm{dev}}(D))$ such that, up to subsequences, for $\mathbb{P}$-a.e. $\omega\in\Omega$ the convergences $\Phi_\varepsilon\chi_{D_\varepsilon}\rightharpoonup\Phi$ in $L^2(0,T;L^2_{\mathrm{dev}}(D))$ and $\nabla\Phi_\varepsilon\chi_{D_\varepsilon}\rightharpoonup\nabla\Phi$ in $L^2(0,T;L^2(D;\mathbb{R}^{3\times3\times3}))$ hold.

Proof. Firstly, up to subsequences, we infer existence of maps $\Phi\in L^2(0,T;L^2_{\mathrm{dev}}(D))$ as well as $\mathbb{M}\in L^2((0,T)\times D;\mathbb{R}^{3\times3\times3}))$ such that

$$\begin{aligned}\Phi_\varepsilon\chi_{D_\varepsilon}&\rightharpoonup\Phi \quad \text{in } L^2(0,T;L^2_{\mathrm{dev}}(D))\\ \nabla\Phi_\varepsilon\chi_{D_\varepsilon}&\rightharpoonup\mathbb{M} \quad \text{in } L^2((0,T)\times D;\mathbb{R}^{3\times3\times3})).\end{aligned}\tag{16}$$

Let $0<\tau\ll T$ be given. Extend by reflection each Φ_ε at the endpoints of the time interval, without relabelling and consider the time regularisation Φ_ε^τ of the extended Φ_ε as per (10). Then by the definition of Φ_ε^τ and properties of the Bochner-Lebesgue integral, we have $\Phi_\varepsilon^\tau\in C^\infty(0,T;H^1_{\mathrm{dev}}(D_\varepsilon))$ and there exists a constant $c_\tau>0$ independent of $\varepsilon>0$ such that

$$\sup_\varepsilon\|\Phi_\varepsilon^\tau(t)\|^2_{H^1(D_\varepsilon)}<c_\tau$$

for all $t\in(0,T)$. In particular Φ_ε^τ satisfies the hypotheses of Lemma 2.4 pointwisely in time. Also from the preliminarily established weak convergences of $\Phi_\varepsilon\chi_{D_\varepsilon}$ and $\nabla\Phi_\varepsilon\chi_{D_\varepsilon}$ in (16) as well as the construction of the convolution in (10), we infer that $\Phi_\varepsilon^\tau(t)\chi_{D_\varepsilon}\rightharpoonup\Phi^\tau(t)$ in $L^2_{\mathrm{dev}}(D)$ and $\nabla\Phi_\varepsilon^\tau(t)\chi_{D_\varepsilon}\rightharpoonup\mathbb{M}^\tau(t)$ in $L^2(D;\mathbb{R}^{3\times3\times3})$ where Φ^τ and $\mathbb{M}^\tau$ are the corresponding time regularisations of Φ and $\mathbb{M}$ respectively obtained in the same fashion as Φ_ε^τ. We may now apply Lemma 2.4 to $\Phi_\varepsilon^\tau(t)$ entrywise for all $t\in(0,T)$. In particular this means that $\Phi^\tau(t)\in H^1_{\mathrm{dev}}(D)$ and $\nabla\Phi_\varepsilon^\tau(t)\chi_{D_\varepsilon}\rightharpoonup\nabla\Phi^\tau(t)$. By uniqueness of limits the equality $\nabla\Phi^\tau=\mathbb{M}^\tau$ holds pointwisely a.e. in $(0,T)$. Finally passing to the limit $\tau\to0$, the properties of mollifiers in (10) as well as the uniqueness of limits again imply $\mathbb{M}=\nabla\Phi$ in $L^2((0,T)\times D;\mathbb{R}^{3\times3\times3}))$ and this finishes the proof. □

Remark 2.6. Suppose that $A_\delta\subset D$ and $A\subset D$ are open sets such that:

i. ∂A and ∂A_δ for all $\delta>0$ are locally graphs of continuous functions;
ii. $A\subset A_\delta$ for all $\delta>0$;
iii. $\lim_{\delta\to0}\sup_{x\in A_\delta\setminus A}\mathrm{dist}(x,\partial A)=0$.

Then by the Mosco-convergence properties, see for instance [21, Proposition 2.11] and [35, Section 1] the assertions of Lemma 2.4 hold when in the hypotheses $H^1(D_\varepsilon)$ is replaced by $H^1(A_\delta)$ or $H^1_0(A_\delta)$ when and $H^1(D)$ is replaced by $H^1(A)$ or $H^1_0(A)$ respectively with the limit $\delta\to0$. Consequently by applying the same argument, an analogue of the compactness property of Proposition 2.5 can be obtained: let Ψ_δ be in $L^2(0,T;H^1_{\mathrm{dev}}(A_\delta))$ or in $L^2(0,T;H^1_{0,\mathrm{div}}(A_\delta))$ such that

$$\sup_\delta\left(\int_0^T\|\Psi_\delta\|^2_{H^1(A_\delta)}\,\mathrm{d}s\right)<+\infty.$$

Then there exists $\Psi\in L^2(0,T;H^1_{\mathrm{dev}}(A))$ or in $\Psi\in L^2(0,T;H^1_{0,\mathrm{div}}(A))$ respectively such that, up to subsequences, we have $\Psi_\delta\chi_{A_\delta}\rightharpoonup\Psi\chi_A$ in $L^2((0,T)\times D;\mathbb{R}^{3\times3})$ and $\nabla\Psi_\delta\chi_{A_\delta}\rightharpoonup\nabla\Psi\chi_A$ in $L^2((0,T)\times D;\mathbb{R}^{3\times3\times3})$ as $\delta\to0$.

For later purposes let us state an auxiliary lemma, a proof of which can be found for instance in [17, Lemma 2.2].

Lemma 2.7. Let $f \in L^1(0,T)$, $g \in L^\infty(0,T)$ and $g_0 \in \mathbb{R}$. Then the following two statements are equivalent:

i. The inequality
$$-\int_0^T \phi'(s)g(s)\mathrm{d}s + \int_0^T \phi(s)f(s)\mathrm{d}s \le g_0$$
holds for all $\phi \in \tilde{C}^1([0,T)) := \{\phi \in C^1([0,T]) : \phi \ge 0, \phi' \le 0, \phi(0) = 1, \phi(T) = 0\}$;

ii. the inequality
$$g(t) - g_0 + \int_0^t f(s)\mathrm{d}s \le 0$$
holds for a.e. $t \in (0,T)$.

2.3. **The generalised solution system.** Let $T > 0$ be given. We consider a viscoelastoplastic model of a fluid with constant mass density $\varrho > 0$ and viscosity $\nu > 0$ which is regarded mesoscopically on perforated domains D_ε. More precisely at a scale $\varepsilon > 0$ we look into the following family of evolutionary system:

$$\partial_t(\varrho v) + \operatorname{div}(v \otimes \varrho v) - \operatorname{div}(\eta S + 2\nu\nabla^{\mathrm{sym}} v) + \nabla p = f_\varepsilon \qquad \text{in } D_\varepsilon \times (0,T), \tag{17a}$$
$$\operatorname{div}(v) = 0 \qquad \text{in } D_\varepsilon \times (0,T), \tag{17b}$$
$$\overset{\triangledown}{S} + \partial\mathcal{P}(S) - \gamma\Delta S \ni \eta\nabla^{\mathrm{sym}} v \qquad \text{in } D_\varepsilon \times (0,T) \tag{17c}$$

accompanied by prescribed boundary and initial conditions

$$v = 0 \qquad \text{on } \partial D_\varepsilon \times (0,T), \tag{17d}$$
$$\vec{n} \cdot \nabla S = 0 \qquad \text{on } \partial D_\varepsilon \times (0,T), \tag{17e}$$
$$(v,S)|_{t=0} = (v_\varepsilon^0, S_\varepsilon^0) \qquad \text{in } D_\varepsilon. \tag{17f}$$

Hereby $\mathcal{P}$ is the convex dissipation potential and $\eta > 0$ is the elastic coefficient called the shear modulus. The symbol $\partial\mathcal{P}$ denotes the, possibly multi-valued, subdifferential of of $\mathcal{P}$ given by:

$$\partial\mathcal{P}(S) := \left\{\hat{S} \in L^2_{\mathrm{dev}}(D)^* \colon \mathcal{P}(\Phi) \ge \mathcal{P}(S) + \langle \hat{S}, \Phi - S\rangle_{L^2_{\mathrm{dev}}(D)}, \text{for all } \Phi \in L^2_{\mathrm{dev}}(D)\right\}$$

for all $S \in L^2_{\mathrm{dev}}(D)$. Moreover, the internal stress S is transported with respect to velocity according to the Zaremba-Jaumann rate defined as:

$$\overset{\triangledown}{S} := \partial_t S + \operatorname{div}(S \otimes v) + S\nabla^{\mathrm{skw}} v - \nabla^{\mathrm{skw}} v S.$$

The term $\eta S + 2\nu\nabla^{\mathrm{sym}} v + p_\varepsilon$ in (17a) signifies the Cauchy stress tensor. In the model, the condition of incompressibility corresponds to the assumption of $\mathrm{Tr}(S) = 0$ and the pressure p_ε accounting for full spherical part of the Cauchy stress.

In this work we shall consider the dissipation potentials $\mathcal{P} \colon L^2_{\mathrm{dev}}(D) \to [0, +\infty]$ satisfying the following conditions:

(P1) $\mathcal{P}$ is convex and lower semicontinuous in $L^2_{\mathrm{dev}}(D)$;
(P2) $\mathcal{P}(0) = 0$;
(P3) for any $D' \subset D$ measurable and all $S \in L^2_{\mathrm{dev}}(D)$ one has $\mathcal{P}(S\chi_{D'}) \le \mathcal{P}(S)$.

Moreover we extend the domain of definition of $\mathcal{P}$ to $L^2_{\mathrm{dev}}(D_\varepsilon)$ by setting $\mathcal{P}(u) = \mathcal{P}(u\chi_{D_\varepsilon})$ for any $u \in L^2_{\mathrm{dev}}(D_\varepsilon)$. In case the potential $\mathcal{P}$ in $C^{1,1}$-regular, we likewise define $\partial\mathcal{P}(u) = \partial\mathcal{P}(u\chi_{D_\varepsilon})$ for $u \in L^2_{\mathrm{dev}}(D_\varepsilon)$.

The relation (P3) can be thought of as a monotonicity-type property. A prominent example of dissipation potentials in viscoelastoplasticity satisfying conditions (P1)-(P3) which is widely used in geodynamics for models of deformations of tectonic plates is embodied by

$$\mathcal{P}(S) := \int_D \mathfrak{B}(S)\,\mathrm{d}x, \quad \mathfrak{B}(S) := \begin{cases} \dfrac{a|S|^2}{2} & \text{if } |S| \le \sigma_{\text{yield}}; \\ +\infty & \text{otherwise} \end{cases} \tag{18}$$

where $a \in \mathbb{R}$ is a given viscosity coefficient and $\sigma_{\text{yield}} > 0$ is a *yield stress* constant associated to transitions in plastic deformations of tectonic plates.

To account for the possibly multivalued subdifferential $\partial\mathcal{P}$, which stems from the nonsmoothness of $\mathcal{P}$, the concept of solutions to the system (17) shall be understood via a suitable variational inequality. In the forthcoming analysis we shall operate with the general solution framework as per [16, 17, 10, 11] which builds upon the constructions presented in [34, Chapter 10]. Without loss of generality, in what follows, we renormalise the system and assume the fluid to possess density equal to one, i.e. we set $\varrho = 1$.

Definition 2.8 (Generalised solutions)**.** Let $f_\varepsilon \in L^2([0,T]; H^{-1}(D_\varepsilon;\mathbb{R}^3))$ and $\mathcal{P}$ satisfy (P1)-(P2). We call $(v_\varepsilon, S_\varepsilon)$ a generalised solution of the system (17a)-(17f) subject to initial data $v_\varepsilon^0 \in L^2_{\text{div}}(D_\varepsilon)$ and $S_\varepsilon^0 \in L^2_{\text{dev}}(D_\varepsilon)$, provided the two inclusions

$$\begin{aligned} v_\varepsilon &\in L^\infty([0,T]; L^2_{\text{div}}(D_\varepsilon)) \cap L^2([0,T]; H^1_{0,\text{div}}(D_\varepsilon)), \\ S_\varepsilon &\in L^\infty([0,T]; L^2_{\text{dev}}(D_\varepsilon)) \cap L^2([0,T]; H^1_{\text{dev}}(D_\varepsilon)), \end{aligned}$$

hold and, in addition, two conditions are satisfied:

1. *The velocity equation:*

$$\begin{aligned} &-\int_0^T \int_{D_\varepsilon} v_\varepsilon \cdot \partial_t \varphi - (v_\varepsilon \otimes v_\varepsilon) : \nabla\varphi + \nu \nabla v_\varepsilon : \nabla\varphi + \eta S_\varepsilon : \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{D_\varepsilon} v_\varepsilon^0 \cdot \varphi(0)\,\mathrm{d}x + \int_0^T \langle f_\varepsilon, \varphi\rangle_{H^1}\,\mathrm{d}t \end{aligned} \tag{19a}$$

for all $\varphi \in C^\infty_{0,\text{div}}(D_\varepsilon \times [0,T))$,

2. *The stress inequality:*

$$\begin{aligned} &\int_0^{T'} \int_{D_\varepsilon} \partial_t \Phi : (\Phi - S_\varepsilon) - (S_\varepsilon \otimes v_\varepsilon) \,\vdots\, \nabla\Phi + (S_\varepsilon \nabla^{\text{skw}} v_\varepsilon - \nabla^{\text{skw}} v_\varepsilon S_\varepsilon) : \Phi \,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^{T'} \mathcal{P}(\Phi) - \mathcal{P}(S_\varepsilon)\,\mathrm{d}t + \int_0^{T'} \int_{D_\varepsilon} \gamma \nabla S_\varepsilon \,\vdots\, \nabla(\Phi - S_\varepsilon) - \eta \nabla v_\varepsilon : (\Phi - S_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \\ &\ge \frac{1}{2}\|\Phi(T') - S_\varepsilon(T')\|^2_{L^2(D_\varepsilon)} - \frac{1}{2}\|\Phi(0) - S_\varepsilon^0\|^2_{L^2(D_\varepsilon)} \end{aligned} \tag{19b}$$

for all $\Phi \in C^\infty_{0,\text{dev}}(\overline{D}_\varepsilon \times [0,T))$ and for a.e. $T' \in (0,T)$.

The result contained in [16, Proposition 3.3, Theorem 3.4], see also [17, Theorem 3.2], provides an overarching existence theory of weak solutions to the system (17a)-(17f) for Lipschitz regular domains. The subtlety in the current setup is that D_ε might consist of finitely many holes which are tangential and form cusps. Hence the boundary of D_ε is continuous but not necessarily Lipschitz. The idea to achieve existence here is to carefully approximate the domain D_ε by sufficiently regular superdomains where existence holds and then to perform a suitable limit passage and recover solutions in D_ε. In doing so we view the variational inequality (19b) on a more regular collection of test maps as originally phrased in [16]. This is assumed for simplicity and more general test maps can be selected in our framework by resorting to approximation arguments.

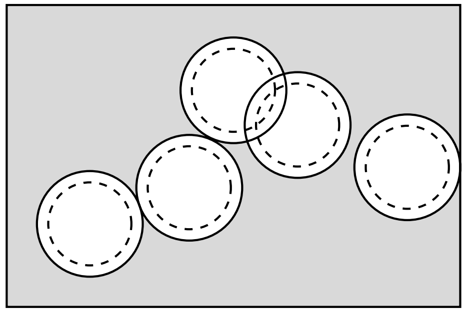

Figure 2. Shrunk balls

Theorem 2.9 (Existence of generalised solutions)**.** Let $v_\varepsilon^0 \in L^2_{\text{div}}(D_\varepsilon)$ and $S_\varepsilon^0 \in L^2_{\text{dev}}(D_\varepsilon)$. Further let $\mathcal{P}$ satisfy (P1)-(P2) and suppose that $f_\varepsilon = f_\varepsilon^1 - \text{div}(f_\varepsilon^2)$ for some $f_\varepsilon^1 \in L^2([0,T); L^2(D_\varepsilon; \mathbb{R}^3))$ and $f_\varepsilon^2 \in L^2([0,T); L^2(D_\varepsilon; \mathbb{R}^{3\times 3}))$. Then for $\mathbb{P}$-a.e $\omega \in \Omega$ there exists a pair $(v_\varepsilon, S_\varepsilon)$ which is a solution to (17a)-(17f) in the sense of Definition 2.8. Moreover $(v_\varepsilon, S_\varepsilon)$ admits the following energy estimates: for a.e. $t \in (0,T)$:

$$\begin{aligned}
&\frac{1}{2}\|v_\varepsilon(t)\|^2_{L^2(D_\varepsilon)} + \int_0^t \nu \|\nabla v_\varepsilon\|^2_{L^2(D_\varepsilon)} \, \mathrm{d}s + \int_0^t \int_{D_\varepsilon} \eta S_\varepsilon : \nabla v_\varepsilon \, \mathrm{d}x \, \mathrm{d}s \leq \frac{1}{2}\|v_\varepsilon^0\|^2_{L^2(D_\varepsilon)} + \int_0^t \langle f_\varepsilon, v_\varepsilon \rangle_{H^1} \, \mathrm{d}s \,, \\
&\frac{1}{2}\|S_\varepsilon(t)\|^2_{L^2(D_\varepsilon)} + \int_0^t \gamma \|\nabla S_\varepsilon\|^2_{L^2(D_\varepsilon)} \, \mathrm{d}s + \int_0^t \mathcal{P}(S_\varepsilon) \, \mathrm{d}s \leq \frac{1}{2}\|S_\varepsilon^0\|^2_{L^2(D_\varepsilon)} + \int_0^t \int_{D_\varepsilon} \eta S_\varepsilon : \nabla v_\varepsilon \, \mathrm{d}x \, \mathrm{d}s \,.
\end{aligned} \tag{20}$$

Proof. Let $\tilde{\mathcal{I}}_\varepsilon \subseteq \mathcal{I}_\varepsilon$ be the collection of tangential balls i.e. $x_n, x_m \in \mathcal{I}_\varepsilon$ such that:

$$|\varepsilon x_n - \varepsilon x_m| = \varepsilon^\alpha(\rho_n + \rho_m) \quad n \neq m \quad \text{or} \quad \text{dist}(\varepsilon x_n, \partial D) = \varepsilon^\alpha \rho_n.$$

Since by (H2) one has $\#\mathcal{I}_\varepsilon = n(\Theta(\omega) \cap \frac{1}{\varepsilon}D) < \infty$ for $\mathbb{P}$-a.e. $\omega \in \Omega$, we may choose $\delta_j > 0$ with $\delta_j \searrow 0$ as $j \to +\infty$ so that for all $j \in \mathbb{N}$ all nontrivial pairwise intersections of balls $B_{(\varepsilon^\alpha - \delta_j)\rho_n}(\varepsilon x_n)$ with $B_{\varepsilon^\alpha \rho_m}(\varepsilon x_m)$ for $x_m \notin \tilde{\mathcal{I}}_\varepsilon$ have nonempty interior. In other words no *cusps* are formed. Let us define a regularisation of D_ε by:

$$D_\varepsilon^j := D \setminus \overline{H}_\varepsilon^j, \quad H_\varepsilon^j := \Big(\bigcup_{x_n \in \mathcal{I}_\varepsilon \setminus \tilde{\mathcal{I}}_\varepsilon} B_{\varepsilon^\alpha \rho_n}(\varepsilon x_n) \Big) \cup \bigcup_{x_n \in \tilde{\mathcal{I}}_\varepsilon} B_{(\varepsilon^\alpha - \delta_j)\rho_n}(\varepsilon x_n)$$

In other words D_ε^j is an enlargement of D_ε by slightly shrinking the tangential perforations, thus removing potential *cusps* in the domain. In particular we have $D_\varepsilon \subset D_\varepsilon^j$ and ∂D_ε^j is (non-uniformly) Lipschitz. Also for the forthcoming reason we record the following set convergences

$$\lim_{j \to +\infty} |D_\varepsilon^j \backslash D_\varepsilon| = 0 \quad \text{and} \quad \lim_{j \to +\infty} \sup_{x \in D_\varepsilon^j \setminus D_\varepsilon} \text{dist}(x, \partial D_\varepsilon) = 0. \tag{21}$$

Now we utilise the initial data $(v^0_{\varepsilon,j}, S^0_{\varepsilon,j})$ such that $v^0_{\varepsilon,j}$ and $S^0_{\varepsilon,j}$ are defined in D_ε^j by extending v_ε^0, S_ε^0 by zero outside D_ε. Moreover, since f_ε is defined as $f_\varepsilon = f_\varepsilon^1 - \text{div}(f_\varepsilon^2)$, $f_{\varepsilon,j}$ is may also be regarded in D_ε^j by extending f_ε^1 and f_ε^2 to zero outside D_ε. Therefore, by applying the existence result of [16, Theorem 3.2] in D_ε^j, we may find a pair of generalised solutions $(v_{\varepsilon,j}, S_{\varepsilon,j})$ in $D_\varepsilon^j \times (0,T)$

satisfying the inclusions $v_{\varepsilon,j} \in L^2(0,T;H^1_{0,\mathrm{div}}(D^j_\varepsilon))$ and $S_{\varepsilon,j} \in L^2(0,T;H^1_{\mathrm{dev}}(D^j_\varepsilon))$. Further

$$\begin{aligned} -\int_0^T \int_{D^j_\varepsilon} v_{\varepsilon,j} \cdot \partial_t \varphi - (v_{\varepsilon,j} \otimes v_{\varepsilon,j}) : \nabla\varphi + \nu \nabla v_{\varepsilon,j} : \nabla \varphi + \eta S_{\varepsilon,j} : \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t \\ = \int_{D^j_\varepsilon} v^0_{\varepsilon,j} \cdot \varphi(0)\,\mathrm{d}x + \int_0^T \langle f_\varepsilon, \varphi\rangle_{H^1}\,\mathrm{d}t \end{aligned} \tag{22a}$$

holds for all $\varphi \in C^\infty_{0,\mathrm{div}}(D^j_\varepsilon \times [0,T))$, as well as,

$$\begin{aligned} &\int_0^t \int_{D^j_\varepsilon} \partial_t \Phi : (\Phi - S_{\varepsilon,j}) - (S_{\varepsilon,j} \otimes v_{\varepsilon,j}) \vdots \nabla\Phi + (S_{\varepsilon,j} \nabla^{\mathrm{skw}} v_{\varepsilon,j} - \nabla^{\mathrm{skw}} v_{\varepsilon,j} S_{\varepsilon,j}) : \Phi \,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^t \mathcal{P}(\Phi) - \mathcal{P}(S_{\varepsilon,j})\,\mathrm{d}t + \int_0^t \int_{D^j_\varepsilon} \gamma \nabla S_{\varepsilon,j} : \nabla(\Phi - S_{\varepsilon,j}) - \eta \nabla v_{\varepsilon,j} : (\Phi - S_{\varepsilon,j})\,\mathrm{d}x\,\mathrm{d}t \\ &\geq \frac{1}{2}\|\Phi(t) - S_{\varepsilon,j}(t)\|^2_{L^2(D^j_\varepsilon)} - \frac{1}{2}\|\Phi(0) - S^0_{\varepsilon,j}\|^2_{L^2(D^j_\varepsilon)} \end{aligned} \tag{22b}$$

holds for all $\Phi \in C^\infty_{0,\mathrm{dev}}\big(\overline{D}^j_\varepsilon \times [0,T)\big)$ and for a.e. $t \in (0,T)$. Moreover, $(v_{\varepsilon,j}, S_{\varepsilon,j})$ satisfies

$$\begin{aligned} \frac{1}{2}\|v_{\varepsilon,j}(t)\|^2_{L^2(D^j_\varepsilon)} + \int_0^t \nu \|\nabla v_{\varepsilon,j}\|^2_{L^2(D^j_\varepsilon)}\,\mathrm{d}s + \int_0^t \int_{D^j_\varepsilon} \eta S_{\varepsilon,j} : \nabla v_{\varepsilon,j}\,\mathrm{d}x\,\mathrm{d}s \\ \leq \frac{1}{2}\big\|v^0_{\varepsilon,j}\big\|^2_{L^2(D^j_\varepsilon)} + \int_0^t \langle f_\varepsilon, v_{\varepsilon,j}\rangle_{H^1}\,\mathrm{d}s\,, \end{aligned} \tag{22c}$$

$$\begin{aligned} \frac{1}{2}\|S_{\varepsilon,j}(t)\|^2_{L^2(D^j_\varepsilon)} + \int_0^t \gamma \|\nabla S_{\varepsilon,j}\|^2_{L^2(D^j_\varepsilon)}\,\mathrm{d}s + \int_0^t \mathcal{P}(S_{\varepsilon,j})\,\mathrm{d}s \\ \leq \frac{1}{2}\big\|S^0_{\varepsilon,j}\big\|^2_{L^2(D^j_\varepsilon)} + \int_0^t \int_{D^j_\varepsilon} \eta S_{\varepsilon,j} : \nabla v_{\varepsilon,j}\,\mathrm{d}x\,\mathrm{d}s\,. \end{aligned} \tag{22d}$$

We extend $v_{\varepsilon,j}$ to D by zero so that $v_{\varepsilon,j} \in H^1_{0,\mathrm{div}}(D)$ a.e. in $(0,T)$. From the pair of estimates above we infer for any $t \in (0,T)$

$$\begin{aligned} \sup_{j\in\mathbb{N}} \sup_{t\in(0,T)} \Big(\|v_{\varepsilon,j}(t)\|^2_{L^2(D)}\Big) + \sup_{j\in\mathbb{N}} \Big(\int_0^t \|v_{\varepsilon,j}\|^2_{H^1(D)}\,\mathrm{d}t\Big) < +\infty \\ \sup_{j\in\mathbb{N}} \sup_{t\in(0,T)} \Big(\|S_{\varepsilon,j}(t)\chi_{D^j_\varepsilon}\|^2_{L^2(D)}\Big) + \sup_{j\in\mathbb{N}} \Big(\int_0^t \|S_{\varepsilon,j}\|^2_{H^1(D^j_\varepsilon)}\,\mathrm{d}t\Big) < +\infty \end{aligned}$$

Therefore, we may find maps $v_\varepsilon \in L^2(0,T;H^1_{0,\mathrm{div}}(D))$, $S_\varepsilon \in L^2(0,T;L^2_{\mathrm{dev}}(D))$ as well as $\mathbb{S}_\varepsilon$ in $L^2(0,T;L^2(D;\mathbb{R}^{3\times3\times3}))$ such that, up to a suitable subsequence, there holds:

$$v_{\varepsilon,j} \overset{*}{\rightharpoonup} v_\varepsilon \text{ in } L^\infty(0,T;L^2_{\mathrm{div}}(D)), \tag{23}$$

$$\nabla v_{\varepsilon,j} \rightharpoonup \nabla v_\varepsilon \text{ in } L^2(0,T;L^2(D;\mathbb{R}^{3\times3})), \tag{24}$$

$$S_{\varepsilon,j}\chi_{D^j_\varepsilon} \overset{*}{\rightharpoonup} S_\varepsilon \chi_{D_\varepsilon} \text{ in } L^\infty(0,T;L^2_{\mathrm{dev}}(D)), \tag{25}$$

$$\nabla S_{\varepsilon,j}\chi_{D^j_\varepsilon} \rightharpoonup \mathbb{S}_\varepsilon \chi_{D_\varepsilon} \text{ in } L^2(0,T;L^2(D;\mathbb{R}^{3\times3\times3})). \tag{26}$$

as $j \to +\infty$. In addition, since $v_{\varepsilon,j}$ is regarded in $H^1_{0,\mathrm{div}}(D_\varepsilon)$ by trivial extenion, with the help of [16, Lemma 4.10, Remark 4.11] the time derivative $\partial_t v_{\varepsilon,j} \in H^1_{0,\mathrm{div}}(D_\varepsilon)^*$ is bounded uniformly in j. By an application of Aubin-Lions Lemma, we may further assert that

$$v_{\varepsilon,j}\chi_{D_\varepsilon} \to v_\varepsilon \chi_{D_\varepsilon} \text{ in } L^2(0,T;L^2_{\mathrm{div}}(D)). \tag{27}$$

as $j \to +\infty$. Using the set convergence (21) and Remark 2.6, since D_ε and D_ε^j both have continuous boundaries, we may conclude that $v_\varepsilon \in L^2(0,T;H^1_{0,\mathrm{div}}(D_\varepsilon))$ and $\mathbb{S}_\varepsilon\chi_{D_\varepsilon} = \nabla S_\varepsilon\chi_{D_\varepsilon}$ with $S_\varepsilon \in L^2(0,T;H^1_{\mathrm{dev}}(D))$.

Now, we fix $\varphi \in C^\infty_{0,\mathrm{div}}(D_\varepsilon\times[0,T))$. In particular $\varphi \in C^\infty_{0,\mathrm{div}}(D_\varepsilon^j\times[0,T))$ since $D_\varepsilon \subset D_\varepsilon^j$. By the convergence of $v_{\varepsilon,j}\chi_{D_\varepsilon^j}$ in $L^\infty(0,T;L^2_{\mathrm{div}}(D))$ from (23), we have

$$\lim_{j\to+\infty}\int_0^T\int_{D_\varepsilon} v_{\varepsilon,j}\cdot\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_{D_\varepsilon} v_\varepsilon\cdot\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t\,.$$

By the convergence of $\nabla v_{\varepsilon,j}\chi_{D_\varepsilon^j}$ in $L^2(0,T;L^2(D))$ from (24), we derive

$$\lim_{j\to+\infty}\int_0^T\int_{D_\varepsilon} \nu\nabla v_{\varepsilon,j}:\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_{D_\varepsilon} \nu\nabla v_\varepsilon:\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t\,.$$

By the convergence of $S_{\varepsilon,j}\chi_{D_\varepsilon}$ in $L^\infty(0,T;L^2(D))$ from (25), we obtain

$$\lim_{j\to+\infty}\int_0^T\int_{D_\varepsilon} S_{\varepsilon,j}:\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t \int_0^T\int_{D_\varepsilon} S_\varepsilon:\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t\,.$$

By the strong convergence of $v_{\varepsilon,j}$ in $L^2(0,T;L^2(D_\varepsilon))$ from (27) and weak convergence of $\nabla v_{\varepsilon,j}$ in $L^2(0,T;L^2(D_\varepsilon))$ from (24), we deduce

$$\begin{aligned}\lim_{j\to+\infty}\int_0^T\int_{D_\varepsilon} v_{\varepsilon,j}\otimes v_{\varepsilon,j}:\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t &= \lim_{j\to+\infty}\int_0^T\int_{D_\varepsilon}(v_{\varepsilon,j}\cdot\nabla v_{\varepsilon,j})\cdot\varphi\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_0^T\int_{D_\varepsilon}(v_\varepsilon\cdot\nabla v_\varepsilon)\cdot\varphi\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_{D_\varepsilon} v_\varepsilon\otimes v_\varepsilon:\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

with the help of integration by parts. Notice that by definition of $v^0_{\varepsilon,j}$ and $f_{\varepsilon,j}$, we have

$$\int_{D_\varepsilon^j} v^0_{\varepsilon,j}\cdot\varphi(0)\,\mathrm{d}x = \int_{D_\varepsilon} v^0_\varepsilon\cdot\varphi(0)\,\mathrm{d}x\,,$$

as well as

$$\int_0^T\langle f_{\varepsilon,j},\varphi\rangle_{H^1(D_\varepsilon^j)}\,\mathrm{d}t = \int_0^T\langle f_\varepsilon,\varphi\rangle_{H^1(D_\varepsilon)}\,\mathrm{d}t\,.$$

This concludes the weak formulation for velocity.

For the partial energy estimate of velocity, one first applies Lemma 2.7 to (22c). Let $\phi \in \tilde{C}^1([0,T))$ be as in Lemma 2.7. Then by the L^2-lower semicontinuity, cf.(23), (24), we have

$$\begin{aligned}-\int_0^T\phi'\int_{D_\varepsilon}|v_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t &\le \liminf_{j\to+\infty} -\int_0^T\phi\int_{D_\varepsilon^j}|v_{\varepsilon,j}|^2\,\mathrm{d}x\,\mathrm{d}t\,,\\ \int_0^T\phi\int_{D_\varepsilon}|\nabla v_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t &\le \liminf_{j\to+\infty}\int_0^T\phi\int_{D_\varepsilon^j}|\nabla v_{\varepsilon,j}|^2\,\mathrm{d}x\,\mathrm{d}t\,,\end{aligned}$$

as well as

$$\lim_{j\to+\infty}\int_0^T\langle f_{\varepsilon,j},v_{\varepsilon,j}\rangle_{H^1(D_\varepsilon^j)}\,\mathrm{d}t = \int_0^T\langle f_\varepsilon,v_\varepsilon\rangle_{H^1(D_\varepsilon)}\,\mathrm{d}t\,.$$

By the convergence of $\nabla S_{\varepsilon,j}\chi_{D^j_\varepsilon}$ in $L^2(0,T;L^2(D;\mathbb{R}^{3\times3\times3}))$ from (26), the strong convergence of $v_{\varepsilon,j}\chi_{D^j_\varepsilon}$ in $L^2(0,T;L^2(D;\mathbb{R}^3))$ from (27) and the Dirichlet boundary conditions on $v_{\varepsilon,j}$ we derive

$$\begin{aligned}\lim_{j\to+\infty}\int_0^T\phi\int_{D^j_\varepsilon}S_{\varepsilon,j}:\nabla v_{\varepsilon,j}\,\mathrm{d}x\,\mathrm{d}t &= \lim_{j\to+\infty}\int_0^T\phi\int_{D^j_\varepsilon}\operatorname{div}(S_{\varepsilon,j}):v_{\varepsilon,j}\,\mathrm{d}x\,\mathrm{d}t\\ &=\int_0^T\phi\int_{D_\varepsilon}\operatorname{div}(S_\varepsilon):v_\varepsilon\,\mathrm{d}x\,\mathrm{d}t=\int_0^T\phi\int_{D_\varepsilon}S_\varepsilon:\nabla v_\varepsilon\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

where in the first equality integration by parts was applied. Therefore, employing Lemma 2.7 yields the desired partial energy estimate. The energy inequality for the stress is shown in the same manner.

As to the stress inequality, let $\Phi\in C^\infty_{0,\mathrm{dev}}\big(\overline{D}_\varepsilon\times[0,T)\big)$ be fixed. First, extend Φ to $C^\infty_{0,\mathrm{dev}}\big(\overline{D}_\varepsilon\times\mathbb{R}\big)$ by the endpoint values outside $[0,T)$. Applying a Whitney type extension entrywise to Φ, see e.g. the proof of [18, Theorem 6.10] there exists $\tilde\Phi\in C^\infty(\mathbb{R}^3\times\mathbb{R};\mathbb{R}^3)$ such that $\tilde\Phi=\Phi$ on $\overline{D}_\varepsilon\times[0,T)$ and

$$\|\tilde\Phi\|_{C^1(\overline{D}^j_\varepsilon\times[0,T))}\le c_\varepsilon\|\Phi\|_{C^1(\overline{D}_\varepsilon\times[0,T))}\tag{28}$$

where the constant $c_\varepsilon>0$ depends on ε but not on j. By construction, restricting to $\overline{D}^j_\varepsilon$, the map $\tilde\Phi$ is admissible for (22b).

To derive stress inequality, we start with $\mathcal{P}\in C^{1,1}(L^2_{\mathrm{dev}}(D))$. We first test the inequality in (22b) with the product of $\tilde\Phi$ and then apply Lemma 2.7. In such a case, we have

$$\begin{aligned}&-\int_0^T\phi'\int_{D^j_\varepsilon}\frac12|S_{\varepsilon,j}-\tilde\Phi|^2\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\phi\int_{D^j_\varepsilon}\partial_t\tilde\Phi:(S_{\varepsilon,j}-\tilde\Phi)+(S_{\varepsilon,j}\otimes v_{\varepsilon,j})\,\vdots\,\nabla\tilde\Phi\,\mathrm{d}x\,\mathrm{d}t\\ &-\int_0^T\phi\int_{D^j_\varepsilon}(S_{\varepsilon,j}\nabla^{\mathrm{skw}}v_{\varepsilon,j}-\nabla^{\mathrm{skw}}v_{\varepsilon,j}S_{\varepsilon,j}):\tilde\Phi\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\phi(\mathcal{P}(S_{\varepsilon,j})-\mathcal{P}(\tilde\Phi\chi_{D^j_\varepsilon}))\,\mathrm{d}t\\ &+\int_0^T\phi\int_{D^j_\varepsilon}\gamma\nabla S_{\varepsilon,j}\,\vdots\,\nabla(S_{\varepsilon,j}-\tilde\Phi)-\eta\nabla v_{\varepsilon,j}:(S_{\varepsilon,j}-\tilde\Phi)\,\mathrm{d}x\,\mathrm{d}t\\ &\le\frac12\|\tilde\Phi(0)-S^0_{\varepsilon,j}\|^2_{L^2(D^j_\varepsilon)}\end{aligned}$$

By the weak* convergence of $S_{\varepsilon,j}\chi_{D^j_\varepsilon}$ in $L^\infty(0,T;L^2_{\mathrm{dev}}(D))$ from (25), (28) and (21) we have

$$-\int_0^T\phi'\int_{D_\varepsilon}\frac12|S_\varepsilon-\Phi|^2\,\mathrm{d}x\,\mathrm{d}t\le\liminf_{j\to+\infty}-\int_0^T\phi'\int_{D^j_\varepsilon}\frac12|S_{\varepsilon,j}-\tilde\Phi|^2\,\mathrm{d}x\,\mathrm{d}t$$

as well as

$$\lim_{j\to+\infty}\int_0^T\phi\int_{D^j_\varepsilon}\partial_t\tilde\Phi:(S_{\varepsilon,j}-\tilde\Phi)\,\mathrm{d}x\,\mathrm{d}t=\int_0^T\phi\int_{D_\varepsilon}\partial_t\Phi:(S_\varepsilon-\Phi)\,\mathrm{d}x\,\mathrm{d}t\,.$$

Moreover, using also the weak* convergence of $S_{\varepsilon,j}\chi_{D^j_\varepsilon}$ in $L^\infty(0,T;L^2_{\mathrm{dev}}(D))$ from (25), (21), (28) and the strong convergence of $v_{\varepsilon,j}$ in $L^2(0,T;L^2(D;\mathbb{R}^3))$ from (27), we obtain

$$\lim_{j\to+\infty}\int_0^T\phi\int_{D^j_\varepsilon}S_{\varepsilon,j}\otimes v_{\varepsilon,j}\,\vdots\,\nabla\tilde\Phi\,\mathrm{d}x\,\mathrm{d}t=\int_0^T\phi\int_{D_\varepsilon}S_\varepsilon\otimes v_\varepsilon\,\vdots\,\nabla\Phi\,\mathrm{d}x\,\mathrm{d}t\,.$$

Moreover, by the weak convergence of $\nabla S_{\varepsilon,j}\chi_{D^j_\varepsilon}$ in $L^2(0,T;L^2(D;\mathbb{R}^{3\times3\times3}))$ from (26), the strong convergence of $v_{\varepsilon,j}$ in $L^2(0,T;L^2(D;\mathbb{R}^3))$ from (27) and the Dirichlet condition on $v_{\varepsilon,j}$, with the help of integration by parts, we derive

$$\lim_{j\to+\infty}\int_0^T \phi\int_{D_\varepsilon^j}(S_{\varepsilon,j}\nabla^{\mathrm{skw}}v_{\varepsilon,j}-\nabla^{\mathrm{skw}}v_{\varepsilon,j}S_{\varepsilon,j}):\tilde{\Phi}\,\mathrm{d}x\,\mathrm{d}t=\int_0^T\phi\int_{D_\varepsilon}(S_\varepsilon\nabla^{\mathrm{skw}}v_\varepsilon-\nabla^{\mathrm{skw}}v_\varepsilon S_\varepsilon):\Phi\,\mathrm{d}x\,\mathrm{d}t\,.$$

On the other hand, since $\mathcal{P}$ is convex and lower semicontinuous, we further have

$$\liminf_{j\to+\infty}\int_0^T \phi\mathcal{P}(S_{\varepsilon,j}\chi_{D_\varepsilon^j})\,\mathrm{d}t\geq\int_0^T\phi\mathcal{P}(S_\varepsilon)\,\mathrm{d}t\,,$$

since $\mathcal{P}$ is weakly lower semicontinuous. Thanks to the Lipschitz continuity of $\mathcal{P}$, we obtain

$$\liminf_{j\to+\infty}-\int_0^T \phi\mathcal{P}(\tilde{\Phi}\chi_{D_\varepsilon^j})\,\mathrm{d}t\geq-\int_0^T\phi\mathcal{P}(\Phi)\,\mathrm{d}t\,.$$

Also by the convergence of $\nabla S_{\varepsilon,j}\chi_{D_\varepsilon^j}$ in $L^2(0,T;L^2(D;\mathbb{R}^{3\times3\times3}))$ from (26) in conjunction with weak lower semicontinuity of quadratic forms we deduce

$$\liminf_{j\to+\infty}\int_0^T\phi\int_{D_\varepsilon^j}\gamma\nabla S_{\varepsilon,j}\,\vdots\,\nabla S_{\varepsilon,j}\,\mathrm{d}x\,\mathrm{d}t\geq\int_0^T\phi\int_{D_\varepsilon}\gamma\nabla S_\varepsilon\,\vdots\,\nabla S_\varepsilon\,\mathrm{d}x\,\mathrm{d}t$$
$$\lim_{j\to+\infty}\int_0^T\phi\int_{D_\varepsilon^j}\gamma\nabla S_{\varepsilon,j}\,\vdots\,\nabla\tilde{\Phi}\,\mathrm{d}x\,\mathrm{d}t=\int_0^T\phi\int_{D_\varepsilon}\gamma\nabla S_\varepsilon\,\vdots\,\nabla\Phi\,\mathrm{d}x\,\mathrm{d}t\,.$$

By similar arguments as above, we also have

$$\lim_{j\to+\infty}\int_0^T\phi\int_{D_\varepsilon^j}\eta\nabla v_{\varepsilon,j}:(S_{\varepsilon,j}-\tilde{\Phi})\,\mathrm{d}x\,\mathrm{d}t=\int_0^T\phi\int_{D_\varepsilon}\eta\nabla v_\varepsilon:(S_\varepsilon-\Phi)\,\mathrm{d}x\,\mathrm{d}t\,.$$

Notice that by the definition of $S^0_{\varepsilon,j}$, we have

$$\lim_{j\to+\infty}\frac{1}{2}\|\tilde{\Phi}(0)-S^0_{\varepsilon,j}\|^2_{L^2(D_\varepsilon^j)}=\frac{1}{2}\|\Phi(0)-S^0_\varepsilon\|^2_{L^2(D_\varepsilon)}.$$

Now invoking Lemma 2.7 concludes the desired inequality. For general convex, lower semicontinuous $\mathcal{P}$, we approximate $\mathcal{P}$ by Moreau envelope and follows the proof of [16, Theorem 3.4]. This ultimately finishes the proof. □

Remark 2.10. The velocity v_ε in Theorem 2.9 additionally satisfies bounds on the time derivative. Directly from [16, Lemma 4.10, Remark 4.11] we may infer that the inclusion $\partial_t v_\varepsilon(t)\in H^1_{0,\mathrm{div}}(D_\varepsilon)^*$ holds for a.e $t\in(0,T)$ and

$$\sup_\varepsilon\left(\int_0^T\|\partial_t v_\varepsilon\|_{H^1_{0,\mathrm{div}}(D_\varepsilon)^*}\,\mathrm{d}t\right)<+\infty$$

2.4. **The weak solution system.** In this section we discuss the model in (17) in the case when the potential $\mathcal{P}$ is quadratic. As such, the boost in regularity of ∂P allows one to consider equality in the stress relation (17c) and thus to formulate a notion of Leray-Hopf weak solutions. In more precise terms we consider the potential $\mathcal{P}\colon L^2_{\mathrm{dev}}(D)\to[0,+\infty]$ satisfying (P1)-(P3) and also

(P4) $\mathcal{P}\in C^{1,1}(L^2_{\mathrm{dev}}(D))$ and $\partial\mathcal{P}=\mathcal{L}$ for some linear mapping $\mathcal{L}\colon L^2_{\mathrm{dev}}(D)\to L^2_{\mathrm{dev}}(D)$.

As such the viscoelasoplastic model can be written as

$$\partial_t v+\mathrm{div}(v\otimes v)-\mathrm{div}(\eta S+2\nu\nabla^{\mathrm{sym}}v)+\nabla p_\varepsilon=f_\varepsilon\qquad\text{in } D_\varepsilon\times(0,T),\tag{29a}$$
$$\mathrm{div}(v)=0,\qquad\text{in } D_\varepsilon\times(0,T),\tag{29b}$$
$$\overset{\triangledown}{S}+\mathcal{L}(S)-\gamma\Delta S=\eta\nabla^{\mathrm{sym}}v\qquad\text{in } D_\varepsilon\times(0,T),\tag{29c}$$

subject to the same initial conditions as in (17).

Let us observe that the change in comparison to the consideration in Section 2.3 is the imposed equality in (29). This is consistent since the $C^{1,1}$-regularity assumption renders $\partial\mathcal{P}$ a well-defined map coinciding with the usual definition of the differential of $\mathcal{P}$. We may also generalise the contribution in [16] to give an existence result of weak solution in the current setup:

Theorem 2.11 (Existence of weak solutions)**.** Assume the hypotheses of Theorem 2.9 and, in addition, assume (P4) on $\mathcal{P}$. Then there exists pair $(v_\varepsilon, S_\varepsilon)$ of weak solutions to (29) in the following sense:

1. v_ε satisfies the equation (19a) for all $\varphi \in C^\infty_{0,\mathrm{div}}(D_\varepsilon \times [0,T))$.
2. S_ε satisfies the equation:

$$\int_0^T \int_{D_\varepsilon} \gamma \nabla S_\varepsilon \,\vdots\, \nabla\Phi \,\mathrm{d}x\mathrm{d}t - \int_0^T \int_{D_\varepsilon} S_\varepsilon : \partial_t \Phi \,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_{D_\varepsilon} S_\varepsilon \otimes v_\varepsilon \,\vdots\, \nabla\Phi \,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_0^T \int_{D_\varepsilon} (S_\varepsilon \nabla^{\mathrm{skw}} v_\varepsilon - \nabla^{\mathrm{skw}} v_\varepsilon S_\varepsilon) : \Phi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{D_\varepsilon} \mathcal{L}(S_\varepsilon) : \Phi \,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_{D_\varepsilon} \eta \nabla v_\varepsilon : \Phi \,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_{D_\varepsilon} S^0_\varepsilon : \Phi(0)\,\mathrm{d}x.$$

for all $\Phi \in C^\infty_{0,\mathrm{dev}}(\overline{D}_\varepsilon \times [0,T))$.

Moreover $(v_\varepsilon, S_\varepsilon)$ satisfy, for all $t \in (0,T)$, the first inequality in (20) for the velocity and also the following energy estimate:

$$\begin{aligned}\frac{1}{2}\|S_\varepsilon(t)\|^2_{L^2(D_\varepsilon)} + \int_0^t \int_{D_\varepsilon} \gamma |\nabla S_\varepsilon(t)|^2 \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_{D_\varepsilon} \mathcal{L}(S_\varepsilon) : S_\varepsilon \,\mathrm{d}x\,\mathrm{d}s \\ \le \frac{1}{2}\big\|S^0_\varepsilon\big\|^2_{L^2(D_\varepsilon)} + \int_0^t \int_{D_\varepsilon} \eta S_\varepsilon : \nabla v_\varepsilon \,\mathrm{d}x\,\mathrm{d}s\,.\end{aligned} \tag{30}$$

Proof. The proof of this claim follows verbatim the argument in the proof of Theorem 2.9. □

3. Statement of the main results

Here we formulate the homogenisation result of the viscoelastoplastic systems on perforated domains defined in (17) and (29). Throughout the rest of the paper, we shall focus on the critical case $\alpha = 3$ of the power law for the radii, that is we consider

$$D_\varepsilon = D \setminus \bigcup_{x_n \in \mathcal{I}_\varepsilon} \overline{B}_{\varepsilon^3 \rho_n}(\varepsilon x_n).$$

Let $T > 0$ be given. We say that the initial data $(v^0_\varepsilon, S^0_\varepsilon)$ from Definition 2.8 are *suitable* provided there exist $v^0 \in L^2_{\mathrm{div}}(D)$ and $S^0 \in L^2_{\mathrm{dev}}(D)$ such that for $\mathbb{P}$-a.e. $\omega \in \Omega$:

$$v^0_\varepsilon \chi_{D_\varepsilon} \to v^0 \text{ in } L^2_{\mathrm{div}}(D) \quad \text{and} \quad S^0_\varepsilon \chi_{D_\varepsilon} \to S^0 \text{ in } L^2_{\mathrm{dev}}(D). \tag{31}$$

Moreover, we say that the outer loading $f_\varepsilon \in L^2([0,T]; H^{-1}(D_\varepsilon;\mathbb{R}^3))$ is *suitable* if there exists $f^1_\varepsilon \in L^2([0,T)\times D_\varepsilon;\mathbb{R}^3))$ and $f^2_\varepsilon \in L^2([0,T)\times D_\varepsilon;\mathbb{R}^{3\times 3}))$ such that for a.e. $t \in (0,T)$

$$\langle f_\varepsilon(t), u\rangle_{H^1(D_\varepsilon)} = \int_{D_\varepsilon} f^1_\varepsilon(t)\cdot u \,\mathrm{d}x + \int_{D_\varepsilon} f^2_\varepsilon(t) : \nabla u \,\mathrm{d}x \text{ for all } u \in H^1_0(D_\varepsilon;\mathbb{R}^3), \tag{32a}$$

as well as $f^1 \in L^2([0,T)\times D;\mathbb{R}^3))$ and $f^2 \in L^2([0,T)\times D;\mathbb{R}^{3\times 3}))$ such that

$$f^1_\varepsilon \rightharpoonup f^1 \text{ in } L^2([0,T)\times D;\mathbb{R}^3)) \text{ and } f^2_\varepsilon \to f^2 \text{ in } L^2([0,T)\times D;\mathbb{R}^{3\times 3})). \tag{32b}$$

Thereafter, we define $f \in L^2([0,T]; H^{-1}(D;\mathbb{R}^3))$ by

$$\langle f(t), u\rangle_{H^1(D)} := \int_D f^1(t)\cdot u \,\mathrm{d}x + \int_D f^2(t) : \nabla u \,\mathrm{d}x \quad \text{for all } u \in H^1_0(D;\mathbb{R}^3). \tag{32c}$$

Theorem 3.1 (Main Theorem)**.** Let $(v^0_\varepsilon, S^0_\varepsilon)$ be *suitable* data as in (31) and $f_\varepsilon \in L^2([0,T]; H^{-1}(D_\varepsilon;\mathbb{R}^3))$ be *suitable* outer loading as in (32), and consider $\mathcal{P}$ which satisfies (P1)-(P3). Let $(v_\varepsilon, S_\varepsilon)$ be a pair of generalised solutions of (17) from Theorem 2.9. Then there exist $v \in L^2([0,T]; H^1_{0,\mathrm{div}}(D))$ and $S \in L^2([0,T]; H^1_{\mathrm{dev}}(D))$ such that, up to subsequences, for $\mathbb{P}$-a.e. $\omega \in \Omega$ it holds

$$\begin{aligned} v_\varepsilon &\rightharpoonup v \quad \text{weakly in } L^2([0,T]; H^1_{0,\mathrm{div}}(D)) \\ v_\varepsilon &\overset{*}{\rightharpoonup} v \quad \text{weakly* in } L^\infty([0,T]; L^2(D;\mathbb{R}^3)) \\ S_\varepsilon\chi_{D_\varepsilon} &\overset{*}{\rightharpoonup} S \quad \text{weakly* in } L^\infty([0,T]; L^2_{\mathrm{dev}}(D)) \\ \nabla S_\varepsilon\chi_{D_\varepsilon} &\rightharpoonup \nabla S \quad \text{weakly in } L^2([0,T]; L^2(D;\mathbb{R}^{3\times3\times3})) \end{aligned}$$

and, in addition, (v,S) is a generalised solution to the system:

$$\partial_t v + \mathrm{div}(v\otimes v) + \mathrm{M}_0 v - \mathrm{div}(\eta S + 2\nu\nabla^{\mathrm{sym}} v) + \nabla p = f \qquad \text{in } D\times(0,T), \tag{33a}$$

$$\mathrm{div}(v) = 0, \qquad \text{in } D\times(0,T) \tag{33b}$$

$$\overset{\triangledown}{S} + \partial\mathcal{P}(S) - \gamma\Delta S \ni \eta\nabla^{\mathrm{sym}} v \qquad \text{in } D\times(0,T). \tag{33c}$$

subject to $\vec{n}\cdot\nabla S = 0$ on $\partial D\times(0,T)$, $(v,S)|_{t=0} = (v^0, S^0)$ in D, and where $\mathrm{M}_0 \in \mathbb{R}^{3\times3}$ is the averaged resistance matrix given by

$$\mathrm{M}_0 := 6\pi\nu\lambda\mathbb{E}[\rho] I \tag{34}$$

and $\lambda > 0$ is as in (H0)-(H6). Herein the generalised solution to (33) is understood as the pair (v,S) satisfying the limiting equation

$$\begin{aligned} &-\int_0^T\int_D v\cdot\partial_t\varphi - v\otimes v : \nabla\varphi + \mathrm{M}_0 v\cdot\varphi + \nu\nabla v : \nabla\varphi + \eta S : \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_D v^0\cdot\varphi(0)\,\mathrm{d}x + \int_0^T \langle f, \varphi\rangle_{H^1(D)}\mathrm{d}t \end{aligned} \tag{35}$$

for all $\varphi \in C^\infty_{0,\mathrm{div}}(D\times[0,T))$ and the limiting inequality:

$$\begin{aligned} &\int_0^{T'}\int_D \partial_t\Phi : (\Phi - S) - (S\otimes v) \mathbin{\vdots} \nabla\Phi + (S\nabla^{\mathrm{skw}} v - \nabla^{\mathrm{skw}} v S) : \Phi \,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^{T'} \mathcal{P}(\Phi) - \mathcal{P}(S)\,\mathrm{d}t + \int_0^{T'}\int_D \gamma\nabla S \mathbin{\vdots} \nabla(\Phi - S) - \eta\nabla v : (\Phi - S)\,\mathrm{d}x\,\mathrm{d}t \\ &\geq \frac{1}{2}\|\Phi(T') - S(T')\|^2_{L^2(D)} - \frac{1}{2}\|\Phi(0) - S^0\|^2_{L^2(D)} \end{aligned} \tag{36}$$

for all $\Phi \in C^\infty_{0,\mathrm{dev}}(\overline{D}\times[0,T))$ and for a.e. $T' \in (0,T)$.

Moreover the pair (v,S) satisfies the following partial energy estimate:

$$\frac{1}{2}\|v(t)\|^2_{L^2(D)} + \int_0^t\int_D \nu|\nabla^{\mathrm{sym}} v|^2 + \eta S : \nabla v + \mathrm{M}_0 v\cdot v \,\mathrm{d}x\,\mathrm{d}s \leq \frac{1}{2}\|v_0\|^2_{L^2(D)} + \int_0^t \langle f, v\rangle_{H^1(D)}\mathrm{d}s \tag{37}$$

for a.e. $t \in (0,T)$.

Secondly, we also provide a version of homogenisation in the case of smooth potentials and what follows for systems of weak solutions.

Theorem 3.2 (Convergence of weak solutions)**.** Consider the hypotheses of Theorem 3.1 and, in addition, let $\mathcal{P}$ satisfy (P1)-(P4). Then, up to subsequences, there exist $v \in L^2([0,T); H^1_{0,\mathrm{div}}(D))$ and $S \in L^2([0,T); H^1_{\mathrm{dev}}(D))$ such that for almost $\mathbb{P}$-a.e. $\omega \in \Omega$

$$\begin{aligned} v_\varepsilon &\rightharpoonup v \quad \text{weakly in } L^2([0,T); H^1_{0,\mathrm{div}}(D)) \\ v_\varepsilon &\overset{*}{\rightharpoonup} v \quad \text{weakly* in } L^\infty([0,T); L^2(D;\mathbb{R}^3)) \\ S_\varepsilon \chi_{D_\varepsilon} &\overset{*}{\rightharpoonup} S \quad \text{weakly* in } L^\infty([0,T); L^2_{\mathrm{dev}}(D)) \\ \nabla S_\varepsilon \chi_{D_\varepsilon} &\rightharpoonup \nabla S \quad \text{weakly in } L^2([0,T); L^2(D;\mathbb{R}^{3\times3\times3})) \end{aligned}$$

and, in addition, (v,S) is a weak solution to the system:

$$\partial_t v + \mathrm{div}(v \otimes v) + \mathrm{M}_0 v - \mathrm{div}(\eta S + 2\nu \nabla^{\mathrm{sym}} v) + \nabla p = f \qquad \text{in } D\times(0,T), \tag{38a}$$

$$\mathrm{div}(v) = 0 \qquad \text{in } D\times(0,T), \tag{38b}$$

$$\overset{\triangledown}{S} + \partial\mathcal{P}(S) - \gamma\Delta S = \eta\nabla^{\mathrm{sym}} v \qquad \text{in } D\times(0,T). \tag{38c}$$

subject to $\vec{n}\cdot\nabla S = 0$ on $\partial D\times(0,T)$, $(v,S)|_{t=0} = (v^0, S^0)$ in D. Here M_0 is as in (34) and $\lambda > 0$ is as in (H0)-(H6). Here the weak solution of the velocity in (38) is understood as in (35) and the stress equation is regarded as in Theorem 2.11 part 2 with obvious change of integration domain to D. Moreover, the pair (v,S) satisfies the partial energy estimate as in (37) and also the following partial energy estimate

$$\frac{1}{2}\|S(t)\|^2_{L^2(D)} + \int_0^t \int_D \gamma|\nabla S|^2 \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_D \mathcal{L}(S) : S \,\mathrm{d}x\,\mathrm{d}s \le \frac{1}{2}\|S_0\|^2_{L^2(D)} + \int_0^t \int_D \eta S : \nabla v \,\mathrm{d}x\,\mathrm{d}s$$

for a.e. $t \in (0,T)$.

Remark 3.3. The results in Theorem 3.1 and Theorem 3.2 are stated for any finite time interval $[0,T)$. In the case when time is infinite, i.e. when $T = \infty$, possible blow-up in finite-time of solutions can be ruled out. In this situation an analogous result holds, though locally in time. This means that existence of solutions and their respective convergence is guaranteed only on every compact time interval of $[0,+\infty)$.

4. Proof of Theorem 3.1

In this section we give a proof of the homogenisation result in Theorem 3.1. We divide the narrative into two parts treating the asymptotic behaviour of the velocity equation and the stress inequality separately. Before embarking on the core derivation, we perform an essential preliminary compactness argument.

To this end, given a pair of generalised solutions $(v_\varepsilon, S_\varepsilon)$ to (17), let us extend, without relabelling, v_ε to $L^2([0,T); H^1_{0,\mathrm{div}}(D))$ by zero outside D_ε. Combining the energy estimates (20) in Theorem 2.9 with Poincaré inequality, Young's inequality as well as the boundedness of f_ε in $L^2([0,T); H^{-1}(D;\mathbb{R}^3))$ give us:

$$\sup_\varepsilon \sup_{t\in(0,T)} \Big(\|v_\varepsilon(t)\|^2_{L^2(D)}\Big) + \sup_\varepsilon \Big(\int_0^t \|v_\varepsilon\|^2_{H^1(D)} \,\mathrm{d}t\Big) < +\infty$$

$$\sup_\varepsilon \sup_{t\in(0,T)} \Big(\|S_\varepsilon(t)\chi_{D_\varepsilon}\|^2_{L^2(D)}\Big) + \sup_\varepsilon \Big(\int_0^t \|S_\varepsilon\|^2_{H^1(D_\varepsilon)} \,\mathrm{d}t\Big) < +\infty$$

for a.e. $t \in (0,T)$. The above estimates give uniform bounds on (v_ε) in $L^2(0,T; H^1_{0,\mathrm{div}}(D))$ and (S_ε) in $L^2(0,T; H^1_{\mathrm{dev}}(D_\varepsilon))$; thus applying Proposition 2.5 to the sequence S_ε yields existence of

$v \in L^2([0,T]; H^1_{0,\mathrm{div}}(D))$ and $S \in L^2([0,T]; H^1_{\mathrm{dev}}(D))$ such that

$$\begin{aligned} v_\varepsilon &\rightharpoonup v \quad \text{weakly in } L^2([0,T]; H^1_{0,\mathrm{div}}(D)), \\ v_\varepsilon &\overset{*}{\rightharpoonup} v \quad \text{weakly* in } L^\infty([0,T]; L^2_{\mathrm{div}}(D)), \\ S_\varepsilon \chi_{D_\varepsilon} &\overset{*}{\rightharpoonup} S \quad \text{weakly* in } L^\infty([0,T]; L^2_{\mathrm{dev}}(D)), \\ \nabla S_\varepsilon \chi_{D_\varepsilon} &\rightharpoonup \nabla S \quad \text{weakly in } L^2([0,T]; L^2(D; \mathbb{R}^{3\times3\times3})). \end{aligned} \tag{39}$$

Moreover in light of Remark 2.10, an application of the Aubin-Lions lemma promotes the convergence $v_\varepsilon \to v$ strongly in $L^2([0,T]; L^2(D; \mathbb{R}^3))$.

4.1. **The limiting equation of the velocity.** Here we examine the limit of the velocity fields v_ε. After what was stated in the introduction, we set $\rho = 1$ for simplicity. As a key component of the proof, we shall employ a construction of the so called corrector maps tailored to the random setting. The lemma below is a reformulation of the result in [26, Lemma 2.5].

Lemma 4.1. For $\mathbb{P}$-a.e. ω in Ω there exist $\varepsilon_0(\omega) > 0$ and a family of operators $\mathcal{T}_\varepsilon \colon C^\infty_{0,\mathrm{div}}(D) \to H^1_{0,\mathrm{div}}(D)$ such that for all $\varepsilon \le \varepsilon_0(\omega)$ and any $\tilde v \in C^\infty_{0,\mathrm{div}}(D)$:

(i) $\mathcal{T}_\varepsilon \tilde v = 0$ in $D \backslash D_\varepsilon$;
(ii) $\mathcal{T}_\varepsilon \tilde v \rightharpoonup \tilde v$ in $H^1_0(D; \mathbb{R}^3)$ and $\mathcal{T}_\varepsilon \tilde v \to \tilde v$ in $L^p(D; \mathbb{R}^3)$ for all $p \in [1, \infty)$;
(iii) for any $(u_\varepsilon) \subset H^1_0(D_\varepsilon; \mathbb{R}^3)$ with $\mathrm{div}(u_\varepsilon) = 0$ and $u_\varepsilon \rightharpoonup u$ in $H^1_0(D; \mathbb{R}^3)$ there holds

$$\lim_{\varepsilon \to 0} \int_{D_\varepsilon} \nu \nabla \mathcal{T}_\varepsilon \tilde v : \nabla u_\varepsilon \, \mathrm{d}x = \int_D \nu \nabla \tilde v : \nabla u \, \mathrm{d}x + \int_D \mathrm{M}_0 \tilde v \cdot u \, \mathrm{d}x, \tag{40}$$

where M_0 is as in (34).

Step 1: Preliminary regularisation. Let $0 < \tau \ll T$. Extending v_ε, S_ε for $t < 0$ by v^0_ε and S^0_ε respectively, we may define the regularised quantities $v^\tau_\varepsilon(t) := \varrho_\tau * v_\varepsilon(t)$, $f^\tau_\varepsilon(t) := \varrho_\tau * f_\varepsilon(t)$, $S^\tau_\varepsilon(t) := \varrho_\tau * S_\varepsilon(t)$ and $(v_\varepsilon \otimes v_\varepsilon)^\tau(t) := \varrho_\tau * (v_\varepsilon \otimes v_\varepsilon)(t)$ as per (10). In view of the fact that $(v_\varepsilon, S_\varepsilon)$ is a generalised solution to (17), by applying appropriate test maps, the regularised pair $(v^\tau_\varepsilon, S^\tau_\varepsilon)$ satisfies the following equation for any $t \in (\tau, T - \tau)$:

$$\begin{aligned} &\int_{D_\varepsilon} \partial_t v^\tau_\varepsilon(t) \cdot w \, \mathrm{d}x + \int_{D_\varepsilon} \nu \nabla v^\tau_\varepsilon(t) : \nabla w \, \mathrm{d}x + \int_{D_\varepsilon} \eta S^\tau_\varepsilon(t) : \nabla w \, \mathrm{d}x \\ &= \int_{D_\varepsilon} f^{1,\tau}_\varepsilon \cdot w \, \mathrm{d}x + \int_{D_\varepsilon} f^{2,\tau}_\varepsilon : \nabla w \, \mathrm{d}x + \int_{D_\varepsilon} (v_\varepsilon \otimes v_\varepsilon)^\tau(t) : \nabla w \, \mathrm{d}x \end{aligned} \tag{41}$$

for all $w \in H^1_{0,\mathrm{div}}(D_\varepsilon)$.

Step 2: pointwise Stokes equation. Let $t \in (\tau, T - \tau)$ be fixed. From (41) and uniqueness of solutions to the stationary Stokes problem, it directly follows that $v^\tau_\varepsilon(t) \in H^1_{0,\mathrm{div}}(D_\varepsilon)$ is the unique weak solution in $H^1_{0,\mathrm{div}}(D_\varepsilon)$ to

$$\begin{aligned} -\nu \Delta v^\tau_\varepsilon(t) + \nabla p_\varepsilon &= F^\tau_\varepsilon(t) && \text{in } D_\varepsilon \\ \mathrm{div}(v^\tau_\varepsilon(t)) &= 0 && \text{in } D_\varepsilon \\ v^\tau_\varepsilon(t) &= 0 && \text{on } \partial D_\varepsilon \end{aligned} \tag{42}$$

where $F^\tau_\varepsilon(t) \in H^1_{0,\mathrm{div}}(D)^*$ is given by

$$\begin{aligned} \langle F^\tau_\varepsilon(t), w \rangle_{H^1_{0,\mathrm{div}}} := &\int_{D_\varepsilon} f^{1,\tau}_\varepsilon \cdot w \, \mathrm{d}x + \int_{D_\varepsilon} f^{2,\tau}_\varepsilon : \nabla w \, \mathrm{d}x - \int_{D_\varepsilon} (v_\varepsilon \otimes v_\varepsilon)^\tau(t) : \nabla w \, \mathrm{d}x \\ &- \int_{D_\varepsilon} \partial_t v^\tau_\varepsilon(t) \cdot w \, \mathrm{d}x - \int_{D_\varepsilon} \eta \, \mathrm{div}(S^\tau_\varepsilon(t)) \cdot w \, \mathrm{d}x \end{aligned}$$

for any $w \in H^1_{0,\mathrm{div}}(D)$.

Let us notice that in view of (39) and (10) we have $v_\varepsilon^\tau(t)$ and $S_\varepsilon^\tau(t)$ uniformly bounded in $H^1(D;\mathbb{R}^3)$ and $H^1(D_\varepsilon;\mathbb{R}^{3\times3})$ respectively. Up to subsequences, from the bound above and the convergences in (39), we may thus suppose that $v_\varepsilon^\tau(t) \rightharpoonup v^\tau(t)$ in $H^1_0(D;\mathbb{R}^3)$, $S_\varepsilon^\tau(t)\chi_{D_\varepsilon} \rightharpoonup S^\tau(t)$ in $L^2_{\mathrm{dev}}(D)$ and $\nabla S_\varepsilon^\tau(t)\chi_{D_\varepsilon} \rightharpoonup \nabla S^\tau(t)$ in $L^2(D;\mathbb{R}^{3\times3\times3})$. In fact, the two convergences hold uniformly in time. On the account of the compact Sobolev embedding, up to further extraction of subsequences, the convergences $v_\varepsilon^\tau(t) \to v^\tau(t)$ in $L^{2^*}(D;\mathbb{R}^3)$ follow. Also with the aid of the Sobolev embedding theorem, followed by a standard L^2-L^{2^*} interpolation we may also assume that $(v_\varepsilon \otimes v_\varepsilon)^\tau(t) \to (v\otimes v)^\tau(t)$ strongly in $L^2(D;\mathbb{R}^{3\times3})$ uniformly in time. Altogether, in conjunction with the fact that $f_\varepsilon^{1,\tau} \rightharpoonup f^{1,\tau}$ in $L^2(D;\mathbb{R}^3)$ as well as $f_\varepsilon^{2,\tau} \to f^{2,\tau}$ in $L^2(D;\mathbb{R}^{3\times3})$, for any $w \in H^1_{0,\mathrm{div}}(D)$ and any $(w_\varepsilon) \subset H^1_{0,\mathrm{div}}(D)$ such that $w_\varepsilon \rightharpoonup w$ in $H^1_0(D;\mathbb{R}^3)$ there holds

$$\lim_{\varepsilon\to0}\langle F_\varepsilon^\tau(t), w_\varepsilon\rangle_{H^1_{0,\mathrm{div}}} = \langle F^\tau_{\mathrm{hom}}(t), w\rangle_{H^1_{0,\mathrm{div}}} \tag{43}$$

and in particular $F_\varepsilon^\tau(t) \to F^\tau_{\mathrm{hom}}(t)$ in $H^1_{0,\mathrm{div}}(D)^*$, where

$$\begin{aligned}\langle F^\tau_{\mathrm{hom}}(t), w\rangle_{H^1_{0,\mathrm{div}}} := &\int_D f^{1,\tau}\cdot w\,\mathrm{d}x + \int_D f^{2,\tau} : \nabla w\,\mathrm{d}x - \int_D (v\otimes v)^\tau(t) : \nabla w\,\mathrm{d}x\\ &- \int_D \partial_t v^\tau(t)\cdot w\,\mathrm{d}x - \int_D \eta\,\mathrm{div}(S^\tau(t))\cdot w\,\mathrm{d}x.\end{aligned}$$

Having arrived at the static problem, we may now test the equation (42) with suitable set of correctors. For any $\tilde v \in C^\infty_{0,\mathrm{div}}(D)$ the map $\mathcal{T}_\varepsilon\tilde v$ from Lemma 4.1 is admissible for the weak formulation of (42). In particular testing it with $w = \mathcal{T}_\varepsilon \tilde v$, it holds

$$\int_{D_\varepsilon} \nu\nabla\mathcal{T}_\varepsilon\tilde v : \nabla v_\varepsilon^\tau(t)\,\mathrm{d}x = \langle F_\varepsilon^\tau(t), \mathcal{T}_\varepsilon\tilde v\rangle_{H^1_{0,\mathrm{div}}}.$$

Consequently, passing to $\varepsilon\to0$ and using the properties (ii)-(iii) of $\mathcal{T}_\varepsilon$ in Lemma 4.1 together with the convergences $v_\varepsilon^\tau(t) \rightharpoonup v^\tau(t)$ in $H^1_0(D;\mathbb{R}^3)$ and the one in (43) implies the equality

$$\int_D \nu\nabla\tilde v : \nabla v^\tau(t)\,\mathrm{d}x + \int_D \mathrm{M}_0 v^\tau(t) : \tilde v\,\mathrm{d}x = \langle F^\tau_{\mathrm{hom}}(t), \tilde v\rangle_{H^1_{0,\mathrm{div}}}.$$

Thus, by the arbitrariness of $\tilde v$, the mapping $v^\tau(t)$ is the weak solution in $H^1_{0,\mathrm{div}}(D)$ to

$$\begin{aligned} -\nu\Delta v^\tau(t) + \mathrm{M}_0 v^\tau(t) + \nabla p &= F^\tau_{\mathrm{hom}}(t) && \text{in } D\\ \mathrm{div}(v^\tau(t)) &= 0 && \text{in } D\\ v^\tau(t) &= 0 && \text{on } \partial D.\end{aligned} \tag{44}$$

Step 3: Passage to $\tau\to0$. Recall from the preliminary compactness the convergence $v_\varepsilon \to v$ in $L^2([0,T);L^2(D;\mathbb{R}^3))$ strongly. Since also $v_\varepsilon\otimes v_\varepsilon$ is bounded in $L^2(D;\mathbb{R}^{3\times3})$ uniformly in time, up to subsequences, $v_\varepsilon\otimes v_\varepsilon$ converges weakly in $L^{\frac43}([0,T);L^2(D;\mathbb{R}^{3\times3}))$. Next since v^τ solves (44) pointwisely in $(\tau, T-\tau)$, we may use the properties of mollifiers ϱ_τ and integration by parts to let $\tau\to0$ in (44) and deduce that v satisfies

$$\begin{aligned} -\int_0^T\int_D v\cdot\partial_t\varphi - \mathbb{V} : \nabla\varphi + \mathrm{M}_0 v\cdot\varphi + \nu\nabla v : \nabla\varphi + \eta S : \nabla\varphi\,\mathrm{d}x\,\mathrm{d}t&\\ = \int_D v^0\cdot\varphi(0)\,\mathrm{d}x + \int_0^T\int_D f^1\cdot\varphi\,\mathrm{d}x\mathrm{d}t + \int_0^T\int_D f^2 : \nabla\varphi\,\mathrm{d}x\mathrm{d}t&\end{aligned} \tag{45}$$

for all admissible test maps φ of the form $\varphi(x,t) = \psi(t)w(x)$. By density, see e.g. [20, Lemma 2.3], the above holds for all $\varphi \in C^\infty_{0,\mathrm{div}}(D\times[0,T))$. Here $\mathbb{V} \in L^{\frac43}([0,T);L^2(D;\mathbb{R}^{3\times3}))$ is such that $(v\otimes v)^\tau \rightharpoonup \mathbb{V}$ weakly in $L^{\frac43}([0,T);L^2(D;\mathbb{R}^{3\times3}))$. To conclude the claim it suffices to verify that $\mathbb{V} = v \otimes v$. Herein, by the properties of mollifiers, it is sufficient to show the convergence $v_\varepsilon \otimes v_\varepsilon \rightharpoonup \mathbb{V}$. Writing $v_\varepsilon = (v_\varepsilon^k; \mathbb{R}^3)_{k=1}$ and $\varphi = (\varphi^k; \mathbb{R}^3)_{k=1}$, integration by parts gives

$$\begin{aligned}\int_0^T \int_{D_\varepsilon} v_\varepsilon \otimes v_\varepsilon : \nabla\varphi \, \mathrm{d}x &= -\int_0^T \int_{D_\varepsilon} \operatorname{div}(v_\varepsilon \otimes v_\varepsilon) \cdot \varphi \, \mathrm{d}x \\ &= -\sum_{j,k=1}^3 \int_0^T \int_{D_\varepsilon} v_\varepsilon^j (\partial_j v_\varepsilon^k) \varphi^k \, \mathrm{d}x.\end{aligned}$$

Subsequently utilising in addition the convergences $\nabla v_\varepsilon \rightharpoonup \nabla v$ in $L^2((0,T); L^2(D; \mathbb{R}^{3\times 3}))$ and $v_\varepsilon \to v$ strongly in $L^2([0,T); L^2(D; \mathbb{R}^3))$ leads, after integrating by parts again, to

$$\lim_{\varepsilon\to 0} \int_0^T \int_{D_\varepsilon} v_\varepsilon \otimes v_\varepsilon : \nabla\varphi \, \mathrm{d}x \, \mathrm{d}t = \int_0^T \int_D v \otimes v : \nabla\varphi \, \mathrm{d}x \, \mathrm{d}t$$

and thus (35) is proved.

Step 4: Partial energy inequality. Here we verify (37). Let $\delta > 0$ be fixed. Let $\varphi_\delta \in C^\infty([0,T); C^\infty_{0,\mathrm{div}}(D))$ such that for a.e. $t \in (0,T)$ there holds $\|\varphi_\delta(t) - v^\tau(t)\|_{H^1(D)} < \delta$, where v_ε^τ and v^τ denotes the time regularization as defined above. We similarly define $\mathcal{T}_\varepsilon \varphi_\delta$ where $\mathcal{T}_\varepsilon$ is from Lemma 4.1. Noticing that, by Lemma 4.1 part ii), $\mathcal{T}_\varepsilon\varphi_\delta(t) \rightharpoonup \varphi_\delta(t)$ in $H_0^1(D; \mathbb{R}^3)$ and $\mathcal{T}_\varepsilon\varphi_\delta(t) \to \varphi_\delta(t)$ in $L^p(D; \mathbb{R}^3)$ for all $p \in [1, \infty)$ and, in addition, Lemma 3.1 part iii) implies

$$\begin{aligned}\lim_{\varepsilon\to 0} \int_{D_\varepsilon} \nu \nabla \mathcal{T}_\varepsilon\varphi_\delta : \nabla v_\varepsilon^\tau \, \mathrm{d}x &= \int_D \nu\nabla v^\tau : \nabla\varphi_\delta \, \mathrm{d}x + \int_D \mathrm{M}_0 v^\tau \cdot \varphi_\delta \, \mathrm{d}x \\ \lim_{\varepsilon\to 0} \int_{D_\varepsilon} \nu \nabla \mathcal{T}_\varepsilon\varphi_\delta : \nabla \mathcal{T}_\varepsilon\varphi_\delta \, \mathrm{d}x &= \int_D \nu\nabla \varphi_\delta : \nabla\varphi_\delta \, \mathrm{d}x + \int_D \mathrm{M}_0 \varphi_\delta \cdot \varphi_\delta \, \mathrm{d}x,\end{aligned}$$

pointwisely a.e. in time. Moreover using the estimate

$$2\int_{D_\varepsilon} \nabla\mathcal{T}_\varepsilon\varphi_\delta : \nabla v_\varepsilon^\tau \, \mathrm{d}x \le \int_{D_\varepsilon} |\nabla v_\varepsilon^\tau|^2 \, \mathrm{d}x + \int_{D_\varepsilon} \nabla\mathcal{T}_\varepsilon\varphi_\delta : \nabla\mathcal{T}_\varepsilon\varphi_\delta \, \mathrm{d}x,$$

from the convergences above, it follows

$$\liminf_{\varepsilon\to 0} \int_{D_\varepsilon} \nu|\nabla v_\varepsilon^\tau|^2 \, \mathrm{d}x \ge 2\int_D \nu\nabla v^\tau : \nabla\varphi_\delta \, \mathrm{d}x + 2\int_D \mathrm{M}_0 v^\tau \cdot \varphi_\delta \, \mathrm{d}x - \int_D \nu\nabla\varphi_\delta : \nabla\varphi_\delta \, \mathrm{d}x - \int_D \mathrm{M}_0\varphi_\delta \cdot \varphi_\delta \, \mathrm{d}x,$$

pointwisely a.e. in time. Setting $\delta \to 0$, from the assumption on φ_δ, implies

$$\liminf_{\varepsilon\to 0} \int_{D_\varepsilon} \nu|\nabla v_\varepsilon^\tau|^2 \, \mathrm{d}x \ge \int_D \nu|\nabla v^\tau|^2 \, \mathrm{d}x + \int_D \mathrm{M}_0 v^\tau \cdot v^\tau \, \mathrm{d}x,$$

pointwisely a.e. in time. By applying Fatou's lemma it follows

$$\liminf_{\varepsilon\to 0} \int_0^t \int_{D_\varepsilon} \nu|\nabla v_\varepsilon^\tau|^2 \, \mathrm{d}x \, \mathrm{d}s \ge \int_0^t \int_D \nu|\nabla v^\tau|^2 \, \mathrm{d}x \, \mathrm{d}s + \int_0^t \int_D \mathrm{M}_0 v^\tau \cdot v^\tau \, \mathrm{d}x \, \mathrm{d}s.$$

Finally, by the definition of time-regulisation along with Young's convolution inequality, we have

$$\int_0^t \int_{D_\varepsilon} |\nabla v_\varepsilon|^2 \, \mathrm{d}x \, \mathrm{d}\tau \ge \int_0^t \int_{D_\varepsilon} |\nabla v_\varepsilon^\tau|^2 \, \mathrm{d}x \, \mathrm{d}\tau$$

for all $\varepsilon > 0$ and $\tau > 0$ sufficiently small. Therefore, we may let $\varepsilon \to 0$ followed by $\tau \to 0$ and obtain

$$\liminf_{\varepsilon\to 0} \int_0^t \int_{D_\varepsilon} \nu|\nabla v_\varepsilon|^2 \, \mathrm{d}x \, \mathrm{d}s \ge \int_0^t \int_D \nu|\nabla v|^2 \, \mathrm{d}x \, \mathrm{d}s + \int_0^t \int_D \mathrm{M}_0 v \cdot v \, \mathrm{d}x \, \mathrm{d}s. \tag{46}$$

On the other hand $v_\varepsilon \to v$ in $L^2([0,t);L^2(D;\mathbb{R}^3))$, $\nabla v_\varepsilon \rightharpoonup \nabla v$ in $L^2([0,t);L^2(D;\mathbb{R}^3))$, $f^1_\varepsilon \rightharpoonup f^1$ in $L^2([0,t);L^2(D;\mathbb{R}^3))$ and $f^2_\varepsilon \to f^2_\varepsilon$ in $L^2([0,t);L^2(D;\mathbb{R}^{3\times 3}))$ give

$$\lim_{\varepsilon\to 0}\left(\int_0^t\int_{D_\varepsilon} f^1_\varepsilon\cdot v_\varepsilon \mathrm{d}x\,\mathrm{d}\tau + \int_0^t\int_{D_\varepsilon} f^2_\varepsilon : \nabla v_\varepsilon \mathrm{d}x\,\mathrm{d}\tau\right) = \int_0^t\int_D f^1\cdot v\,\mathrm{d}x\,\mathrm{d}\tau + \int_D f^2 : \nabla v\,\mathrm{d}x\,\mathrm{d}\tau\,. \tag{47}$$

Thus combining (46) and (47) with the bounds (20) and the suitability of the initial data v^0_ε lead to (37). □

4.2. **The limiting evolutionary inequality of the stress.** In this subsection, we examine the limit of stress tensor S_ε. To this end, first recall that

$$v_\varepsilon \rightharpoonup v \text{ in } L^2([0,T);H^1_{0,\mathrm{div}}(D)), \tag{48a}$$

$$v_\varepsilon \overset{*}{\rightharpoonup} v \text{ in } L^\infty([0,T);L^2_{\mathrm{div}}(D)), \tag{48b}$$

$$v_\varepsilon \to v \text{ in } L^2([0,T);L^2_{\mathrm{div}}(D)), \tag{48c}$$

$$S_\varepsilon\chi_{D_\varepsilon} \overset{*}{\rightharpoonup} S \text{ in } L^\infty([0,T);L^2_{\mathrm{dev}}(D)), \tag{48d}$$

$$\nabla S_\varepsilon\chi_{D_\varepsilon} \rightharpoonup \nabla S \text{ in } L^2([0,T);L^2(D;\mathbb{R}^{3\times3\times3})). \tag{48e}$$

Let $\Phi\in C^\infty_{0,\mathrm{dev}}(\overline{D}\times[0,T))$ be a fixed test function. Notice that the restriction Φ from D to D_ε guarantees that Φ is an admissible test function in (19b). Then, we apply Lemma 2.7 to (19b) so that for all $\phi\in\tilde{C}^1([0,T))$ one has

$$\begin{aligned}
&-\int_0^T\int_{D_\varepsilon}\frac{1}{2}\phi'|S_\varepsilon-\Phi|^2\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_{D_\varepsilon}\phi\,\partial_t\Phi:(S_\varepsilon-\Phi)\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_0^T\phi\,(\mathcal{P}(S_\varepsilon\chi_{D_\varepsilon})-\mathcal{P}(\Phi\chi_{D_\varepsilon}))\,\mathrm{d}t+\int_0^T\int_{D_\varepsilon}\phi\,S_\varepsilon\otimes v_\varepsilon \,\vdots\, \nabla\Phi\,\mathrm{d}x\,\mathrm{d}t\\
&-\int_0^T\int_{D_\varepsilon}\phi\left(S_\varepsilon\nabla^{\mathrm{skw}}v_\varepsilon-\nabla^{\mathrm{skw}}v_\varepsilon S_\varepsilon\right):\Phi\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_0^T\int_{D_\varepsilon}\phi\,\gamma\nabla S_\varepsilon \,\vdots\, \nabla(S_\varepsilon-\Phi)\,\mathrm{d}x\,\mathrm{d}t-\eta\int_0^T\int_{D_\varepsilon}\phi\,\nabla^{\mathrm{sym}}v_\varepsilon:(S_\varepsilon-\Phi)\,\mathrm{d}x\,\mathrm{d}t\\
\le&\frac{1}{2}\|S_{0,\varepsilon}-\Phi(0)\|^2_{L^2(D_\varepsilon)}.
\end{aligned}$$

Since $\phi'\le 0$, with the help of convergence in (48d) and the weak lower semicontinuity of the L^2-norm, we obtain

$$\liminf_{\varepsilon\to0}\left(-\int_0^T\int_D\frac{1}{2}\phi'\chi_{D_\varepsilon}|S_\varepsilon-\Phi|^2\,\mathrm{d}x\,\mathrm{d}t\right)\ge-\int_0^T\int_D\frac{1}{2}\phi'|S-\Phi|^2\,\mathrm{d}x\,\mathrm{d}t\,.$$

Thanks to (48d), we also get

$$\lim_{\varepsilon\to0}\int_0^T\int_D\phi\,\partial_t\Phi:\chi_{D_\varepsilon}(S_\varepsilon-\Phi)\,\mathrm{d}x\,\mathrm{d}t=\int_0^T\int_D\phi\,\partial_t\Phi:(S-\Phi)\,\mathrm{d}x\,\mathrm{d}t\,.$$

Now let us fix such function ϕ and consider the auxiliary functional $\mathcal{F}\colon L^2(0,T;L^2_{\mathrm{dev}}(D))\to[0,+\infty]$ defined as

$$\mathcal{F}(\tilde{S}) := \int_0^T\phi\mathcal{P}(\tilde{S})\mathrm{d}t.$$

For any $\tilde S_\varepsilon, \tilde S \in L^2(0,T;L^2_{\mathrm{dev}}(D))$ such that $\tilde S_\varepsilon \to \tilde S$ in $L^2(0,T;L^2_{\mathrm{dev}}(D))$ by Fatou's lemma, we have

$$\liminf_{\varepsilon\to 0} \mathcal{F}(\tilde S_\varepsilon) \geq \int_0^T \phi \liminf_{\varepsilon\to 0} \mathcal{P}(\tilde S_\varepsilon)\,\mathrm{d}t \geq \int_0^T \phi\mathcal{P}(\tilde S)\,\mathrm{d}t = \mathcal{F}(\tilde S).$$

In particular $\mathcal{F}$ is lower semicontinuous on $L^2(0,T;L^2_{\mathrm{dev}}(D))$. In addition, convexity of $\mathcal{P}$ implies convexity of $\mathcal{F}$ and thus $\mathcal{F}$ is weakly lower semicontinuous on $L^2(0,T;L^2_{\mathrm{dev}}(D))$. Consequently, in view of the weak convergence in (48d), the above implies

$$\liminf_{\varepsilon\to 0} \int_0^T \phi\mathcal{P}(S_\varepsilon\chi_{D_\varepsilon})\,\mathrm{d}t = \liminf_{\varepsilon\to 0} \mathcal{F}(S_\varepsilon\chi_{D_\varepsilon}) \geq \int_0^T \phi\mathcal{P}(S)\,\mathrm{d}t\,.$$

Moreover, thanks to (P3), we have

$$-\int_0^T \phi\mathcal{P}(\Phi\chi_{D_\varepsilon})\,\mathrm{d}t \geq -\int_0^T \phi\mathcal{P}(\Phi)\,\mathrm{d}t\,.$$

On the other hand by the convergence in (48e), and weak lower semicontinuity of quadratic forms we derive

$$\liminf_{\varepsilon\to 0} \int_0^T \int_{D_\varepsilon} \phi\,\gamma\nabla S_\varepsilon \,\vdots\, \nabla S_\varepsilon\,\mathrm{d}x\,\mathrm{d}t \geq \int_0^T \int_D \phi\,\gamma\nabla S \,\vdots\, \nabla S\,\mathrm{d}x\,\mathrm{d}t\,,$$

as well as

$$\lim_{\varepsilon\to 0} \int_0^T \int_D \phi\,\gamma\chi_{D_\varepsilon}\nabla S_\varepsilon \,\vdots\, \nabla\Phi\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_D \phi\,\gamma\nabla S \,\vdots\, \nabla\Phi\,\mathrm{d}x\,\mathrm{d}t\,.$$

Let us recall that due to the divergence-free condition of v_ε, with the convention $S_\varepsilon = (S_\varepsilon^{ij})_{i,j=1}^3$, $\Phi = (\Phi^{ij})_{i,j=1}^3$ and $v_\varepsilon = (v_\varepsilon^\ell)_{\ell=1}^3$, we may write

$$\begin{aligned}\int_0^T \int_{D_\varepsilon} \phi\, S_\varepsilon \otimes v_\varepsilon \,\vdots\, \nabla\Phi\,\mathrm{d}x &= -\int_0^T \int_{D_\varepsilon} \phi\,\mathrm{div}(S_\varepsilon\otimes v_\varepsilon) : \Phi\,\mathrm{d}x \\ &= -\sum_{j,k,\ell=1}^3 \int_0^T \int_{D_\varepsilon} \phi\, v_\varepsilon^j(\partial_j S_\varepsilon^{k\ell})\Phi^{k\ell}\,\mathrm{d}x.\end{aligned}$$

Consequently the convergences $v_\varepsilon \to v$ in $L^2((0,T);L^2(D;\mathbb{R}^3))$ and $\nabla S_\varepsilon\chi_{D_\varepsilon} \rightharpoonup \nabla S$ in $L^2((0,T);L^2(D;\mathbb{R}^{3\times3\times3}))$ result in

$$\lim_{\varepsilon\to 0} \int_0^T \int_{D_\varepsilon} \phi\, S_\varepsilon\otimes v_\varepsilon \,\vdots\, \nabla\Phi\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_D \phi\, S\otimes v \,\vdots\, \nabla\Phi\,\mathrm{d}x\,\mathrm{d}t\,.$$

In the same vein, by standard interpolation and the fact that $\nabla\Phi$ belongs to $L^\infty([0,T)\times D;\mathbb{R}^{3\times3\times3})$, we get that $\Phi\otimes v_\varepsilon$ converges strongly in $L^2((0,T);L^2(D;\mathbb{R}^{3\times3\times3}))$, $v_\varepsilon\cdot\nabla\Phi$ converges strongly in $L^1((0,T);L^2(D;\mathbb{R}^{3\times3}))$ and thus integrating by parts and taking the respective limit yields

$$\lim_{\varepsilon\to 0} \int_0^T \int_{D_\varepsilon} \phi\left(S_\varepsilon\nabla^{\mathrm{skw}}v_\varepsilon - \nabla^{\mathrm{skw}}v_\varepsilon S_\varepsilon\right) : \Phi\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_D \phi\left(S\nabla^{\mathrm{skw}}v - \nabla^{\mathrm{skw}}vS\right) : \Phi\,\mathrm{d}x\,\mathrm{d}t\,.$$

Moreover, the weak convergence $\nabla v_\varepsilon \to \nabla v$ in $L^2(0,T;L^2(D;\mathbb{R}^{3\times3}))$ gives

$$\lim_{\varepsilon\to 0} \int_0^T \int_{D_\varepsilon} \eta\phi\nabla^{\mathrm{sym}}v_\varepsilon : \Phi\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_{D_\varepsilon} \eta\phi\nabla^{\mathrm{sym}}v : \Phi\,\mathrm{d}x\,\mathrm{d}t\,.$$

With the additional help of integration by parts and weak convergence of $S_\varepsilon \chi_{D_\varepsilon}$ in $L^2(0,T;L^2_{\mathrm{dev}}(D))$ yields

$$\lim_{\varepsilon\to 0}\int_0^T\int_{D_\varepsilon}\eta\phi\nabla^{\mathrm{sym}}v_e : S_\varepsilon \,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_D \eta\phi\nabla^{\mathrm{sym}}v : S\,\mathrm{d}x\,\mathrm{d}t\,.$$

Ultimately, by the *suitability* of initial data in (31), we have

$$\lim_{\varepsilon\to 0}\frac{1}{2}\int_D \chi_{D_\varepsilon}|S_{0,\varepsilon}-\Phi|^2\,\mathrm{d}x\,\mathrm{d}t = \frac{1}{2}\int_D |S_0-\Phi|^2\,\mathrm{d}x\,.$$

Altogether, we arrive at

$$\begin{aligned}
&-\int_0^T\int_D \frac{1}{2}\phi'|S-\Phi|^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_D \phi\,\partial_t\Phi : (S-\Phi)\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_0^T (\mathcal{P}(S)-\mathcal{P}(\Phi))\,\mathrm{d}t + \int_0^T\int_D \phi\, S\otimes v \,\vdots\, \nabla\Phi\,\mathrm{d}x\,\mathrm{d}t\\
&-\int_0^T\int_D \phi\left(S\nabla^{\mathrm{skw}}v(t) - \nabla^{\mathrm{skw}}vS\right) : \Phi\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_0^T\int_D \phi\,\gamma\nabla S \,\vdots\, \nabla(S-\Phi)\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_D \phi\,\eta\nabla^{\mathrm{sym}}v : (S-\Phi)\,\mathrm{d}x\,\mathrm{d}t\\
\le&\frac{1}{2}\|S_0-\Phi(0)\|^2_{L^2(D)}.
\end{aligned}$$

Since $\phi\in\tilde{C}^1([0,T))$ with $\phi\ge 0$ was arbitrary, by virtue of Lemma 2.7 we have that

$$\begin{aligned}
&\frac{1}{2}\|S(T')-\Phi(T')\|^2_{L^2} + \int_0^{T'}\int_D \partial_t\Phi : (S-\Phi) - S\otimes v \,\vdots\, \nabla\Phi - (S\nabla^{\mathrm{skw}}v - \nabla^{\mathrm{skw}}vS) : \Phi\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_0^{T'} \mathcal{P}(S)-\mathcal{P}(\Phi)\,\mathrm{d}t + \int_0^{T'}\int_D \gamma\nabla S \,\vdots\, \nabla(S-\Phi) - \eta\nabla^{\mathrm{sym}}v : (S-\Phi)\,\mathrm{d}x\,\mathrm{d}t\\
\le&\frac{1}{2}\|S_0-\Phi(0)\|^2_{L^2}
\end{aligned}\tag{49}$$

for a.e. $T'\in(0,T)$ and this ultimately proves (36) completing the proof of Theorem 3.1. □

5. Proof of Theorem 3.2

We may again identify v_ε in $L^2([0,T);H^1_{0,\mathrm{div}}(D))$ by extending by zero outside D_ε. Also from Theorem 2.11, cf. also (20) we arrive at the bounds

$$\begin{aligned}
&\sup_\varepsilon \sup_{t\in(0,T)}\Big(\|v_\varepsilon(t)\|^2_{L^2(D)}\Big) + \sup_\varepsilon\Big(\int_0^t \|v_\varepsilon\|^2_{H^1(D)}\,\mathrm{d}t\Big) < +\infty,\\
&\sup_\varepsilon \sup_{t\in(0,T)}\Big(\|S_\varepsilon(t)\chi_{D_\varepsilon}\|^2_{L^2(D)}\Big) + \sup_\varepsilon\Big(\int_0^t \|S_\varepsilon\|^2_{H^1(D_\varepsilon)}\,\mathrm{d}t\Big) < +\infty
\end{aligned}$$

for a.e. $t\in(0,T)$. Subsequently, arguing as at the beginning of Section 2.3 we obtain existence of $v\in L^2([0,T);H^1_{0,\mathrm{div}}(D))$ and $S\in L^2([0,T);H^1_{\mathrm{dev}}(D))$ such that

$$\begin{aligned}
v_\varepsilon &\rightharpoonup v \quad \text{weakly in } L^2([0,T);H^1_{0,\mathrm{div}}(D))\\
v_\varepsilon &\overset{*}{\rightharpoonup} v \quad \text{weakly* in } L^\infty([0,T);L^2_{\mathrm{div}}(D))\\
S_\varepsilon\chi_{D_\varepsilon} &\overset{*}{\rightharpoonup} S \quad \text{weakly* in } L^\infty([0,T);L^2_{\mathrm{dev}}(D))\\
\nabla S_\varepsilon\chi_{D_\varepsilon} &\rightharpoonup \nabla S \quad \text{weakly in } L^2([0,T);L^2(D;\mathbb{R}^{3\times3\times3})).
\end{aligned}\tag{50}$$

5.1. **The limiting equation of the velocity.** Hereby, the homogenised equation is exactly the same one as in the statement of Theorem 3.1. Thus, in view of the weak convergence of v_ε to v and S_ε to S, we may reproduce the argumentation of Section 4.1 to obtain the desired equation for v.

5.2. **The limiting equation of the stress.** Let $\Phi \in C^\infty_{0,\mathrm{dev}}(\overline{D}\times[0,T))$ be a test function. Thanks to (50), we have

$$\lim_{\varepsilon\to 0}\int_0^T\int_D \partial_t\Phi : S_\varepsilon\chi_{D_\varepsilon}\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_D \partial_t\Phi : S\,\mathrm{d}x\,\mathrm{d}t\,,$$

and

$$\lim_{\varepsilon\to 0}\int_0^T\int_D \chi_{D_\varepsilon}\nabla S_\varepsilon \vdots \nabla\Phi\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_D \nabla S \vdots \nabla\Phi\,\mathrm{d}x\,\mathrm{d}t\,,$$

as well as

$$\lim_{\varepsilon\to 0}\int_0^T\int_{D_\varepsilon} \eta\nabla v_\varepsilon : \Phi\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_D \eta\nabla v : \Phi\,\mathrm{d}x\,\mathrm{d}t\,.$$

Consequently, using the convergences $v_\varepsilon \to v$ in $L^2((0,T);L^2(D;\mathbb{R}^3))$ and $\nabla S_\varepsilon\chi_{D_\varepsilon} \rightharpoonup \nabla S$ in $L^2((0,T);L^2(D;\mathbb{R}^{3\times3\times3}))$ result in

$$\lim_{\varepsilon\to 0}\int_0^T\int_{D_\varepsilon} S_\varepsilon\otimes v_\varepsilon \vdots \nabla\Phi\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_D S\otimes v \vdots \nabla\Phi\,\mathrm{d}x\,\mathrm{d}t\,.$$

In the same vein, with the help of integration by parts, we derive

$$\lim_{\varepsilon\to 0}\int_0^T\int_{D_\varepsilon} \big(S_\varepsilon\nabla^{\mathrm{skw}}v_\varepsilon - \nabla^{\mathrm{skw}}v_\varepsilon S_\varepsilon\big) : \Phi\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_D \big(S\nabla^{\mathrm{skw}}v - \nabla^{\mathrm{skw}}vS\big) : \Phi\,\mathrm{d}x\,\mathrm{d}t\,.$$

In addition, by the convergence of initial data (31), we also have

$$\lim_{\varepsilon\to 0}\int_D S^0_\varepsilon\chi_{D_\varepsilon} : \Phi(0)\,\mathrm{d}x = \int_D S_0 : \Phi(0)\,\mathrm{d}x.$$

Since the mapping $\mathcal{L}$ is linear, the weak convergence of $S_\varepsilon\chi_{D_\varepsilon} \rightharpoonup S$ in $L^\infty([0,T);L^2_{\mathrm{dev}}(D))$ gives $\mathcal{L}(S_\varepsilon\chi_{D_\varepsilon}) \rightharpoonup \mathcal{L}(S)$ in $L^\infty([0,T);L^2_{\mathrm{dev}}(D))$. Hence, we derive

$$\lim_{\varepsilon\to 0}\int_0^T\int_{D_\varepsilon} \mathcal{L}(S_\varepsilon\chi_{D_\varepsilon}) : \Phi\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_D \mathcal{L}(S) : \Phi\,\mathrm{d}x\,\mathrm{d}t\,.$$

Therefore, we conclude

$$\begin{aligned}
&\int_0^T\int_D \gamma\nabla S \vdots \nabla\Phi\,\mathrm{d}x\mathrm{d}t - \int_0^T\int_D S : \partial_t\Phi\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_D S\otimes v \vdots \nabla\Phi\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_0^T\int_D (S\nabla^{\mathrm{skw}}v - \nabla^{\mathrm{skw}}vS) : \Phi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_D \mathcal{L}(S) : \Phi\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_D \eta\nabla v : \Phi\,\mathrm{d}x\,\mathrm{d}t\\
&= \int_D S_0 : \Phi(0)\,\mathrm{d}x.
\end{aligned}$$

For the partial energy-dissipative estimate, one mimics the same idea as in Section 4.2 except for one term

$$\int_0^t\int_{D_\varepsilon} \mathcal{L}(S_\varepsilon) : S_\varepsilon\,\mathrm{d}x\,\mathrm{d}s\,.$$

For this term, we first claim that

$$\mathcal{P}(\Phi) = \frac{1}{2}\int_D \mathcal{L}(\Phi) : \Phi\,\mathrm{d}x \;\text{ for all } \Phi \in L^2_{\mathrm{dev}}(D). \tag{51}$$

Indeed, for fixed $\Phi \in L^2_{\mathrm{dev}}(D)$, we define

$$\begin{aligned} g : [0,1] &\to [0,\infty), \\ t &\mapsto \mathcal{P}(t\Phi). \end{aligned}$$

Since $\mathcal{P} \in C^{1,1}(L^2_{\mathrm{dev}}(D))$, we have

$$g'(t) = t \int_D \mathcal{L}(\Phi) : \Phi \,\mathrm{d}x \,.$$

Integrating in time on $(0,1)$ yields

$$\mathcal{P}(\Phi) = \mathcal{P}(\Phi) - \mathcal{P}(0) = g(1) - g(0) = \int_0^1 t \int_\Omega \mathcal{L}(\Phi) : \Phi \,\mathrm{d}x \,\mathrm{d}t = \frac{1}{2} \int_\Omega \mathcal{L}(\Phi) : \Phi \,\mathrm{d}x \,.$$

This proves the claim. Moreover, since $\mathcal{L}$ is linear, one can see that $\mathcal{P}$ is quadratic. Recall also that $\mathcal{P}$ is convex. Hence, we have

$$\frac{1}{2} \int_\Omega \mathcal{L}(\Phi) : \Phi \,\mathrm{d}x \geq 0 \text{ for all } \Phi \in L^2_{\mathrm{dev}}(D),$$

see [5, Proposition 3.71] for details. This implies

$$\int_0^T \int_\Omega \mathcal{L}(\Phi) : \Phi \,\mathrm{d}x \,\mathrm{d}s \geq 0 \text{ for all } \Phi \in L^2(0,T; L^2_{\mathrm{dev}}(D)).$$

Therefore, the functional

$$\Phi \mapsto \int_0^T \int_\Omega \mathcal{L}(\Phi) : \Phi \,\mathrm{d}x \,\mathrm{d}s$$

is convex and quadratic, and therefore weakly lower semicontinuous. This concludes

$$\int_0^t \int_D \mathcal{L}(S) : S \,\mathrm{d}x \,\mathrm{d}s \leq \liminf_{\varepsilon \to 0} \int_0^t \int_{D_\varepsilon} \mathcal{L}(S_\varepsilon) : S_\varepsilon \,\mathrm{d}x \,\mathrm{d}s \,.$$

This finishes our proof.

6. The case of stability in the subcritical scaling

In the final section we discuss the asymptotic behaviour of (17) when the power law of the perforations in H_ε are in the range $\alpha > 3$. As surmised in the introduction the main difference to the critical scaling is that, here the Brinkman effect is negligible, i.e. $\mathrm{M}_0 = 0$. This occurs due to the fact that the capacity of H_ε in D is infinitesimal for $\alpha > 3$. Indeed recalling the strong law of large numbers in Section 2.1, for $\mathbb{P}$-a.e. $\omega \in \Omega$ there holds

$$\sup_\varepsilon \Big(\sum_{x_n \in \mathcal{I}_\varepsilon} \varepsilon^3 \mathrm{cap}(B_{\rho_n}(0); \mathbb{R}^3) \Big) < +\infty.$$

Therefore by the subadditivity of the newtonian capacity the above bound implies

$$\lim_{\varepsilon \to 0+} \mathrm{cap}(H_\varepsilon; \mathbb{R}^3) \leq \lim_{\varepsilon \to 0+} \sum_{x_n \in \mathcal{I}_\varepsilon} \mathrm{cap}(B_{\varepsilon^\alpha \rho_n}(\varepsilon x_n); \mathbb{R}^3) = \lim_{\varepsilon \to 0+} \varepsilon^{\alpha - 3} \sum_{x_n \in \mathcal{I}_\varepsilon} \varepsilon^3 \mathrm{cap}(B_{\rho_n}(0); \mathbb{R}^3) = 0 \tag{52}$$

Theorem 6.1. Let $\alpha > 3$. Consider $(v^0_\varepsilon, S^0_\varepsilon)$ a pair of *suitable* data and $f_\varepsilon \in L^2(0,T; H^{-1}(D_\varepsilon; \mathbb{R}^3))$ be *suitable* outer loading both defined in an analogous way to (31), (32). Likewise we take $\mathcal{P}$ satisfying (P1)-(P3). Suppose that $(v_\varepsilon, S_\varepsilon)$ is a pair of generalised solutions of (17) from Theorem

2.9. Then there exist $v \in L^2([0,T]; H^1_{0,\mathrm{div}}(D))$ and $S \in L^2([0,T]; H^1_{\mathrm{dev}}(D))$ such that, up to subsequences, for $\mathbb{P}$-a.e. $\omega \in \Omega$ one has

$$\begin{aligned} v_\varepsilon &\rightharpoonup v \quad \text{weakly in } L^2([0,T]; H^1_{0,\mathrm{div}}(D)) \\ v_\varepsilon &\overset{*}{\rightharpoonup} v \quad \text{weakly* in } L^\infty([0,T]; L^2(D;\mathbb{R}^3)) \\ S_\varepsilon \chi_{D_\varepsilon} &\overset{*}{\rightharpoonup} S \quad \text{weakly* in } L^\infty([0,T]; L^2_{\mathrm{dev}}(D)) \\ \nabla S_\varepsilon \chi_{D_\varepsilon} &\rightharpoonup \nabla S \quad \text{weakly in } L^2([0,T]; L^2(D;\mathbb{R}^{3\times3\times3})) \end{aligned}$$

and, (v,S) is a generalised solution to the system:

$$\partial_t v + \mathrm{div}(v \otimes v) - \mathrm{div}(\eta S + 2\nu \nabla^{\mathrm{sym}} v) + \nabla p = f \qquad \text{in } D\times(0,T), \tag{53a}$$

$$\mathrm{div}(v) = 0, \qquad \text{in } D\times(0,T) \tag{53b}$$

$$\overset{\triangledown}{S} + \partial\mathcal{P}(S) - \gamma\Delta S \ni \eta\nabla^{\mathrm{sym}} v \qquad \text{in } D\times(0,T). \tag{53c}$$

subject to $\vec{n}\cdot\nabla S = 0$ on $\partial D\times(0,T)$, $(v,S)|_{t=0} = (v^0, S^0)$ in D. Herein the generalised solution to (53) is understood as the pair (v,S) satisfying (19a) and (19b) with the obvious change of spatial integration of D_ε by D.

Moreover the pair (v,S) satisfies the following partial energy estimate:

$$\frac{1}{2}\|v(t)\|^2_{L^2(D)} + \int_0^t \int_D \nu|\nabla^{\mathrm{sym}} v|^2 + \eta S : \nabla v \,\mathrm{d}x\,\mathrm{d}s \le \frac{1}{2}\|v_0\|^2_{L^2(D)} + \int_0^t \langle f, v\rangle_{H^1(D)}\mathrm{d}s \tag{54}$$

for a.e. $t \in (0,T)$.

6.1. **Proof of Theorem 6.1.** In the final section we prove the homogenisation in the subcritical case of Theorem 3.1. We again divide the justification into two separate arguments for the velocity and the stress. Similarly as in Section 2.3, extending v_ε to D by zero, together with the energy bounds (20) in Theorem 2.9 yield:

$$\sup_\varepsilon \sup_{t\in(0,T)} \Big(\|v_\varepsilon(t)\|^2_{L^2(D)} \Big) + \sup_\varepsilon \Big(\int_0^t \|v_\varepsilon\|^2_{H^1(D)}\,\mathrm{d}t \Big) < +\infty$$

$$\sup_\varepsilon \sup_{t\in(0,T)} \Big(\|S_\varepsilon(t)\chi_{D_\varepsilon}\|^2_{L^2(D)} \Big) + \sup_\varepsilon \Big(\int_0^t \|S_\varepsilon\|^2_{H^1(D_\varepsilon)}\,\mathrm{d}t \Big) < +\infty$$

for a.e. $t \in (0,T)$. From this along with an application of Proposition 2.5 we obtain the following compactness: there exist $v \in L^2(0,T; H^1_{0,\mathrm{div}}(D))$ and $S \in L^2(0,T; H^1_{\mathrm{dev}}(D))$ such that, up to subsequences:

$$\begin{aligned} v_\varepsilon &\rightharpoonup v \quad \text{weakly in } L^2(0,T; H^1_{0,\mathrm{div}}(D)), \\ v_\varepsilon &\overset{*}{\rightharpoonup} v \quad \text{weakly* in } L^\infty(0,T; L^2_{\mathrm{div}}(D)), \\ S_\varepsilon \chi_{D_\varepsilon} &\overset{*}{\rightharpoonup} S \quad \text{weakly* in } L^\infty(0,T; L^2_{\mathrm{dev}}(D)), \\ \nabla S_\varepsilon \chi_{D_\varepsilon} &\rightharpoonup \nabla S \quad \text{weakly in } L^2(0,T; L^2(D;\mathbb{R}^{3\times3\times3})). \end{aligned} \tag{55}$$

6.1.1. *Stability of the momentum equation.* We derive the homogenised equation for the limiting velocity of v_ε. Without loss of generality, we again set $\rho = 1$. The proof is divided into several steps. As in the proof of Theorem 3.1, we use time regularisation and pass to the limit in the corresponding pointwise Stokes equation.

Step 1: Time regularisation. We fix a parameter $0 < \tau \ll T$. Arguing as in Step 1 and Step 2 of Section 4.1, after mollifying the momentum equation, we obtain that, for every $t \in (\tau, T-\tau)$, the regularised field $v_\varepsilon^\tau(t)$ is the unique weak solution of the system:

$$-\nu\Delta v_\varepsilon^\tau(t) + \nabla p_\varepsilon = F_\varepsilon^\tau(t) \quad \text{in } D_\varepsilon \tag{56}$$

in $H^1_{0,\mathrm{div}}(D_\varepsilon)$, where $F_\varepsilon^\tau(t) \in H^1_{0,\mathrm{div}}(D_\varepsilon)^*$ is defined by

$$\begin{aligned}\langle F_\varepsilon^\tau(t), w\rangle_{H^1_{0,\mathrm{div}}} &:= \langle f_\varepsilon^\tau, w\rangle_{H^1} - \int_{D_\varepsilon} (v_\varepsilon \otimes v_\varepsilon)^\tau(t) : \nabla w \,\mathrm{d}x \\ &\quad - \int_{D_\varepsilon} \partial_t v_\varepsilon^\tau(t)\cdot w\,\mathrm{d}x - \int_{D_\varepsilon} \eta \operatorname{div}(S_\varepsilon^\tau(t))\cdot w\,\mathrm{d}x\end{aligned}$$

for every $w \in H^1_{0,\mathrm{div}}(D_\varepsilon)$.

Step 2: Passage to $\varepsilon \to 0$. We fix an arbitrary $t \in (\tau, T-\tau)$. From (55) and the definition of mollification, we infer that $v_\varepsilon^\tau(t)$ and $S_\varepsilon^\tau(t)$ are uniformly bounded in $H^1(D;\mathbb{R}^3)$ and $H^1(D_\varepsilon;\mathbb{R}^{3\times 3})$, respectively. By repeating the arguments in the proof of Theorem 3.1, where the same functional F_ε^τ is considered, we obtain

$$\lim_{\varepsilon\to 0}\langle F_\varepsilon^\tau(t), w_\varepsilon\rangle_{H^1_{0,\mathrm{div}}} = \langle F^\tau_{\mathrm{hom}}(t), w\rangle_{H^1_{0,\mathrm{div}}} \tag{57}$$

for every $w_\varepsilon \rightharpoonup w$ in $H^1_{0,\mathrm{div}}(D)$, where

$$\begin{aligned}\langle F^\tau_{\mathrm{hom}}(t), w\rangle_{H^1_{0,\mathrm{div}}} :=& \langle f^\tau, w\rangle_{H^1} - \int_D (v\otimes v)^\tau(t) : \nabla w\,\mathrm{d}x \\ & - \int_D \partial_t v^\tau(t)\cdot w\,\mathrm{d}x - \int_D \eta\operatorname{div}(S^\tau(t))\cdot w\,\mathrm{d}x.\end{aligned}$$

Let $\tilde v \in C^\infty_{0,\mathrm{div}}(D)$ be fixed. Applying Proposition 2.3 to $\tilde v$, we find a sequence $(\tilde v_\varepsilon) \subset C^\infty_{0,\mathrm{div}}(D_\varepsilon)$ such that

$$\|\nabla\tilde v - \nabla \tilde v_\varepsilon\|_{L^2(D)} \le c\|\tilde v\|_{C^1}\Big(\varepsilon^{\alpha-3}\sum_{x_n\in\mathcal{I}_\varepsilon}\varepsilon^3 \mathrm{cap}(B_{\rho_n}(0);\mathbb{R}^3)\Big)^{\frac{1}{2}}. \tag{58}$$

Combining (58) with the vanishing newtonian capacity of H_ε in (52), we obtain the convergence $\tilde v_\varepsilon \to \tilde v$ in $H^1_{0,\mathrm{div}}(D)$. Since $\tilde v_\varepsilon$ is admissible in the weak formulation of (56), we may use it as a test function and obtain

$$\int_{D_\varepsilon} \nu\nabla\tilde v_\varepsilon : \nabla v_\varepsilon^\tau(t)\,\mathrm{d}x = \langle F_\varepsilon^\tau(t), \tilde v_\varepsilon\rangle_{H^1_{0,\mathrm{div}}}.$$

Passing to the limit $\varepsilon \to 0$, the strong convergence $\tilde v_\varepsilon \to \tilde v$ in $H^1(D;\mathbb{R}^3)$ together with (57) yields

$$\int_D \nu\nabla\tilde v : \nabla v^\tau(t)\,\mathrm{d}x = \langle F^\tau_{\mathrm{hom}}(t), \tilde v\rangle_{H^1_{0,\mathrm{div}}}.$$

Hence $v^\tau(t)$ is the weak solution in $H^1_{0,\mathrm{div}}(D)$ of

$$-\nu\Delta v^\tau(t) + \nabla p = F^\tau_{\mathrm{hom}}(t) \quad \text{in } D. \tag{59}$$

Step 3: Passage to $\tau \to 0$. In view of Remark 2.10, the Aubin-Lions lemma gives the strong convergence $v_\varepsilon \to v$ in $L^2(0,T;L^2(D;\mathbb{R}^3))$. Moreover, by interpolation, $v_\varepsilon \otimes v_\varepsilon$ is uniformly bounded in $L^{\frac{4}{3}}(0,T;L^2(D;\mathbb{R}^{3\times 3}))$. Hence, up to subsequences, $v_\varepsilon\otimes v_\varepsilon$ converges weakly in $L^{\frac{4}{3}}(0,T;L^2(D;\mathbb{R}^{3\times 3}))$. The rest of the proof follows the same argument as Step 3 of Section 4.1 which ultimately yields

$$\begin{aligned}&-\int_0^T\int_D v\cdot\partial_t\varphi - v\otimes v : \nabla\varphi + \nu\nabla v : \nabla\varphi + \eta S : \nabla\varphi\,\mathrm{d}x\,\mathrm{d}t \\ &\quad= \int_D v^0\cdot\varphi(0)\,\mathrm{d}x + \int_0^T\int_D \langle f,\varphi\rangle_{H^1}\,\mathrm{d}x\,\mathrm{d}t\end{aligned} \tag{60}$$

for all admissible $\varphi \in C^\infty_{0,\mathrm{div}}(D\times[0,T))$.

Step 4: Partial energy inequality. We verify (54). From the convergences above, together with the weak lower semicontinuity of norms, it follows that

$$\liminf_{\varepsilon\to 0}\int_0^t\int_{D_\varepsilon}\nu|\nabla v_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}s \geq \int_0^t\int_D \nu|\nabla v|^2\,\mathrm{d}x\,\mathrm{d}s. \tag{61}$$

On the other hand, the convergences $v_\varepsilon \to v$ in $L^2(0,t;L^2(D;\mathbb{R}^3))$, $\nabla v_\varepsilon \rightharpoonup \nabla v$ in $L^2(0,t;L^2(D;\mathbb{R}^3))$, and $f_\varepsilon \rightharpoonup f$ imply that

$$\lim_{\varepsilon\to 0}\int_0^t \langle f_\varepsilon, v_\varepsilon\rangle_{H^1}\,\mathrm{d}x\,\mathrm{d}s = \int_0^t\langle f, v\rangle_{H^1}\,\mathrm{d}x\mathrm{d}s. \tag{62}$$

Combining (61) and (62) with the bounds in (20) and the suitability of the initial data v^0_ε, we obtain (54). $\square$

6.1.2. *Stability of the evolutionary inequality of the stress.* In this subsection, we examine the limit of stress tensor S_ε. To this end, first recall that

$$v_\varepsilon \rightharpoonup v \text{ in } L^2(0,T;H^1_{0,\mathrm{div}}(D)), \tag{63a}$$

$$v_\varepsilon \overset{*}{\rightharpoonup} v \text{ in } L^\infty(0,T;L^2_{\mathrm{div}}(D)), \tag{63b}$$

$$v_\varepsilon \to v \text{ in } L^2(0,T;L^2_{\mathrm{div}}(D)), \tag{63c}$$

$$S_\varepsilon\chi_{D_\varepsilon} \overset{*}{\rightharpoonup} S \text{ in } L^\infty(0,T;L^2_{\mathrm{dev}}(D)), \tag{63d}$$

$$\nabla S_\varepsilon\chi_{D_\varepsilon} \rightharpoonup \nabla S \text{ in } L^2(0,T;L^2(D;\mathbb{R}^{3\times 3\times 3})). \tag{63e}$$

Repeating the calculation as in Section 4.2, we obtain (36) completing the proof of Theorem 6.1. $\square$

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References

[1] G. Allaire: Homogenization of the Navier-Stokes equations in open sets perforated with tiny holes. I. Abstract framework, a volume distribution of holes, *Arch. Ration. Mech. Anal.* 113(3), 209–259 (1990).

[2] M. Bathory, M. Bulicek, J. Malek: Large data existence theory for three- dimensional unsteady flows of rate-type viscoelastic fluids with stress diffusion, *Adv. Nonlin. Analysis*, 10:1, 501–521 (2021).

[3] P. Bella, F. Lemming, R. Marziani and F. Oschmann: Brinkman's law as Γ-limit of compressible low Mach Navier-Stokes equations and application to randomly perforated domains. *arXiv preprint: 2505.11213.*, https://arxiv.org/abs/2505.11213 (2025).

[4] P. Bella, F. Oschmann: Inverse of divergence and homogenization of compressible Navier-Stokes equations in randomly perforated domains. *Arch. Ration. Mech. Anal.*, 247, No. 2, Paper No. 14, 29 pp. (2023).

[5] J. F. Bonnans, A. Shapiro, *Perturbation Analysis of Optimization Problems*, Springer Series in Operations Research, Springer, New York, (2000).

[6] A. Brillard: Asymptotic analysis of incompressible and viscous fluid flow through porous media. Brinkman's law via epi-convergence methods, *Ann. Fac. Sci. Univ. Toulouse Sci. Math.* 8(2), 225–252 (1986–1987).

[7] D. Bucur, E. Feireisl, S.Necasova: Boundary Behavior of Viscous Fluids: Influence of Wall Roughness and Friction-driven Boundary Conditions. *Arch. Rational Mech. Anal.* 197, 117–138 (2010).

[8] M. Bulicek, J. Malek, V. Prusa, and E. Süli; PDE analysis of a class of thermo- dynamically compatible viscoelastic rate-type fluids with stress-diffusion, *Mathematical analysis in fluid mechanics—selected recent results, Contemp. Math.*, vol. 710, *Amer. Math. Soc.*, pp. 25–51 (2018).

[9] D. Cioranescu, F. Murat: A strange term coming from nowhere, *Topics in the Mathematical Modelling of Composite Materials*, in: *Progress in Nonlinear Differential Equations and Their Applications*, vol.31, pp.45–93 (1997).

[10] F. Cheng, R. Lasarzik and M. Thomas: Analysis of a Cahn–Hilliard model for viscoelastoplastic two-phase flows. *Nonlinear Anal., Real World Appl.* Vol. 93 (2027).
[11] F. Cheng, R. Lasarzik and M. Thomas: Energy-variational solutions for geodynamical two-phase flows – From logarithmic to double-obstacle potentials by variational convergence. Accepted in *Commun. Appl. Math. Comput. Sci. arXiv preprint: 2601.08625*, https://arxiv.org/abs/2601.08625 (2026).
[12] D. Chenais: On the existence of a solution in a domain identification problem. *J. Math. Anal. Appl.*, 52, 189–219 (1975).
[13] D. J. Daley, D. Vere-Jones: *An introduction to the theory of point processes.* Vol.II: General theory and structures. Probability and its applications, Springer-Verlag New York (2008).
[14] G. Dal Maso, F. Ebobisse, M. Ponsiglione: A stability result for nonlinear Neumann problems under boundary variations. *J. Math. Pures Appl. (9)* 82, No. 5, 503–532 (2003).
[15] L. Desvillettes, F. Golse, V. Ricci: The mean-field limit for solid particles in a Navier-Stokes flow, *J. Stat. Phys.* 131(5), 941–967 (2008).
[16] T. Eiter, K. Hopf, A. Mielke: Leray–Hopf solutions to a viscoelastoplastic fluid model with nonsmooth stress-strain relation. *Nonlinear Anal., Real World Appl.* Vol. 65 (2022).
[17] T. Eiter, K. Hopf, R. Lasarzik: Weak-strong uniqueness and energy-variational solutions for a class of viscoelastoplastic fluid models. *Adv. Nonlinear Anal.* 12 (1) (2022).
[18] L. C. Evans, R. F. Gariepy: *Measure Theory and Fine Properties of Functions*, Revised Edition (1st ed.). Chapman and Hall/CRC (2015).
[19] E. Feireisl, Y. Namlyeyeva, S. Necasova: Homogenization of the evolutionary Navier–Stokes system. *Manuscripta Math.*, 149, 251–274 (2016).
[20] G. P. Galdi, *An introduction to the Navier-Stokes initial-boundary value problem*, Fundamental directions in mathematical fluid mechanics, Adv. Math. Fluid Mech., Birkhäuser, Basel pp. 1-70 (2000).
[21] M. Fornoni, L. Rondi: Mosco convergence of Sobolev spaces and Sobolev inequalities for nonsmooth domains, *Calc. Var.*, 62, 15 (2023).
[22] T. Gerya, D. A. Yuen: Robust characteristics method for modelling multiphase visco-elasto-plastic thermo-mechanical problems. *Phys. Earth Plan. Inter.* 163:1-4, 83–105 (2007).
[23] A. Giacomini: A Stability Result for Neumann Problems in Dimension $N \geq 3$, *Journal of Convex Analysis*, Vol. 11, No. 1, 41–58 (2004).
[24] A. Giunti: Derivation of Darcy's law in randomly perforated domains. *Calc. Var. Partial Differ. Equ.*, 60, No. 5, Paper No. 172, 30 pp. (2021).
[25] A. Giunti, R. Höfer, J. Velázquez: Homogenization for the Poisson equation in randomly perforated domains under minimal assumptions on the size of the holes. *Comm. Partial Differential Equations*, 43(9), 1377-1412 (2018).
[26] A. Giunti, R. Höfer: Homogenization for the Stokes equations in randomly perforated domains under almost minimal assumptions on the size of the holes. *Annales de l'Institut Henri Poincaré C Analyse Non linéaire*, 36, 7 (2019).
[27] A. Giunti, R. Höfer: Convergence of the pressure in the homogenization of the Stokes equations in randomly perforated domains. *Journal of Differential Equations*, Vol. 320, 399-418 (2022).
[28] P. L. Lions, N. Masmoudi: Global solutions for some Oldroyd models of non-Newtonian flows, *Chin. Ann. Math.*, Ser. B 21:2, 131–146 (2000).
[29] J. Málek, V. Průša, T. Skřivan, E. Süli: Thermodynamics of viscoelastic rate-type fluids with stress diffusion, *Physics of Fluids*, 30 (2): 023101 (2018).
[30] P. Maremonti, R. Russo, G. Starita: On the Stokes Equations: The Boundary Value Problem. *Advances in Fluid Dynamics, Quad. Mat.*, Vol.4, Dept. Math., Seconda Univ. Napoli, Caserta, pp.69–140 (1999).
[31] L. Moresi, F. Dufour, H. B. Mühlhaus: Mantle convection modeling with viscoelastic/brittle lithosphere: Numerical methodology and plate tectonic modelling. *Pure Appl. Geophys.* 159, 2335–2356 (2002).
[32] F. Oschmann: Homogenization of the full compressible Navier-Stokes-Fourier system in randomly perforated domains. *J. Math. Fluid Mech.*, 24, No. 2, Paper No. 45, 20 pp. (2022).
[33] J. Pan: Homogenization of non-homogeneous incompressible Navier-Stokes system in critically perforated domains, *Journal of Mathematical Fluid Mechanics*, 27(2), 1-16 (2025).
[34] T. Roubicek: *Nonlinear partial differential equations with applications*, second ed., International Series of Numerical Mathematics, Vol. 153, Birkhäuser/Springer Basel AG, Basel (2013).
[35] G. Savare, G. Schimperna: Domain perturbations and estimates for the solutions of second order elliptic equations *J. Math. Pures Appl.*, 81, pp. 1071–111 (2002).
[36] R. Schneider, W. Weil: *Stochastic and Integral Geometry.* Probability and its applications, Springer-Verlag New York (2008).

[37] G. Schubert, D.L. Turcotte, P. Olson: *Mantle Convection in the Earth and Planets*, Cambridge University Press, Cambridge (2001).

[38] L. Scardia, K. Zemas, C. Zeppieri: Homogenisation of nonlinear Dirichlet problems in randomly perforated domains under minimal assumptions on the size of perforations. *Probab. Theory Relat.*, Fields 192, No. 1-2, 471-544 (2025).

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