Lipschitz continuity of harmonic maps between $\mathrm{RCD}(K, N)$ spaces and $\mathrm{CAT}(\kappa)$ spaces

Luca Gennaioli *

July 29, 2026

Abstract

We are going to prove that energy minimizing harmonic maps from a domain Ω inside an $\mathrm{RCD}(K, N)$ whose image lies in a small ball inside a $\mathrm{CAT}(\kappa)$ space are locally Lipschitz continuous. This completes the picture concerning regularity of harmonic maps between singular spaces and justifies a variant of the Bochner-Eells-Sampson inequality between singular spaces.

Contents

1 Introduction

This paper is concerned with the regularity property of harmonic maps from $\mathrm{RCD}(K, N)$ spaces to $\mathrm{CAT}(\kappa)$ spaces. The former consists of all $\mathrm{CD}(K, N)$ metric measure spaces where, roughly speaking, one can impose a lower bound on the Ricci curvature K, an upper bound on the dimension N and impose an hilbertian structure. Indeed general $\mathrm{CD}(K, N)$ in general do not enjoy several "Riemannian" theorems, like the splitting theorem for example. This Riemannian structure was added in [Gig15], where the author (see [Gig23a] for a rather complete account on this subject) introduced the notion of *infinitesimal hilbertianity*, singling out the class of $\mathrm{RCD}(K, N)$ spaces. Of course the various contributions to this topic are numerous. We recall here some of the most significant ones: [AGS14a], [AGS14b] (the infinite dimensional case), [AGS15], [AGS12], [AGMR12], [Gig18], [Gig26] and [GKO13].
For what concerns $\mathrm{CAT}(\kappa)$ spaces, they are metric spaces where one can express synthetically the fact of having sectional curvature bounded from above by a value κ and such terminology was used for the first time in [Gro87].
To convince the reader of the fact that this is the natural non-smooth setting where one can hope to obtain "good" regularity properties of harmonic functions (arguably the class of functions which should be the most regular), we recall that if $u : \Omega \subset (M^n, g) \to (N^m, h)$ with $\mathrm{Ric}_{g_M} \geq 0$ (to illustrate the principle we stick to nonnegative Ricci curvature but a general lower bound would be enough) and $\mathrm{Sec}_{g_N} \leq 0$ (again to illustrate the principle we stick to nonpositive sectional curvature) is harmonic, one can write the Bochner-Eells-Sampson formula

$$\Delta\left(\frac{|\,\mathrm{d}u|^2_{\mathsf{HS}}}{2}\right) = |\nabla \,\mathrm{d}u|^2_{\mathsf{HS}} + \mathrm{Ric}_{g_M}(\nabla u, \nabla u) - \sum_{i,j \leq n} \langle u_* \mathcal{R}^N(e_i, e_j)e_i, e_j\rangle, \tag{1.1}$$

*L.Gennaioli@warwick.ac.uk

where $\mathcal{R}^N$ is the Riemann tensor of N and $|\nabla\,\mathrm{d}u|^2_{\mathrm{HS}}$ denotes the square Hilbert-Schmidt norm of the Hessian of u. Using the inequalities on the Ricci and on the sectional curvature we get

$$\Delta\left(\frac{|\,\mathrm{d}u|^2_{\mathsf{HS}}}{2}\right) \geq |\nabla\,\mathrm{d}u|^2_{\mathsf{HS}} \geq 0,$$

and a simple application of Harnack's inequality allows to infer that $|\,\mathrm{d}u|_{\mathrm{HS}}$ is locally bounded, proving local Lipschitz regularity of u.
For what concerns the non-smooth setting, the problem of regularity is tied to the availability of (1.1), which was instead a tool to exploit in the smooth setting. The first contributions appeared for what concerns local Lipschitz regularity of harmonic maps between Alexandrov spaces (i.e. with bounds on the sectional curvature): for real-valued maps we have [Pet03], while the problem was solved in [ZZ18]. For what concerns the case of a smooth domain inside a manifold (M, g_M) with $\mathrm{Ric}_{g_M} \geq K$ and a non-smooth target with sectional curvature bounded above by κ, in [Ser95] was able to achieve the sought regularity because, strongly exploiting the smoothness, he was able to write a PDE of the form

$$\tfrac{|\,\mathrm{d}u|_{\mathsf{HS}}}{\cos(f)}\mathrm{div}\Big(\cos^2(f)\nabla\Big(\tfrac{|\,\mathrm{d}u|_{\mathsf{HS}}}{\cos(f)}\Big)\Big) \geq K|\,\mathrm{d}u|^2_{\mathsf{HS}}, \tag{1.2}$$

where $f(x) := \sqrt{\kappa}\mathsf{d}_{\mathsf{Y}}(u(x), p)$ and $(\mathsf{Y}, \mathsf{d}_{\mathsf{Y}})$ is the target $\mathrm{CAT}(\kappa)$ space and $u : \Omega \subset M \to \mathsf{Y}$ is a minimizing harmonic map with $u(\Omega) \subset B_\rho(o)$, for some $o \in \mathsf{Y}$ and $\rho < \frac{\pi}{2\sqrt{\kappa}}$. Equation (1.2) allows for a Moser iteration and again implies the local Lipschitz regularity. We want to remark that the constraint on the image of u, namely that $u(\Omega) \subset B_\rho(o)$, is not a technicality but it is crucial to obtain such regularity and even continuity. The heuristic reason for which this is important is that the map $y \mapsto \mathsf{d}_{\mathsf{Y}}(o, y)$ cease to be geodesically convex outside $B_\rho(o)$. Indeed if one considers the vortex map $u : B_1(0) \subset \mathbb{R}^7 \to \mathbb{S}^6$ with $u(x) = \frac{x}{|x|}$, such map is an energy minimizing harmonic map (among maps with the same boundary datum) and it is clearly not continuous (see [JK83], [Lin87] and [CG89]). Finally, moving to the RCD setting for what concerns the domain (which in view of the previous discussion appears to be the natural setting), in the works [Gig23b] and [MS26] the authors were able to independently establish the local Lipschitz regularity of energy minimizing harmonic maps, together with a variant of the Bochner-Eells-Sampson formula where one has to replace $|\,\mathrm{d}u|_{\mathrm{HS}}$ with $\mathrm{lip}(u)$, i.e. the local Lipschitz constant of u (see [MS26, Theorem 7.1] for the inequality with an Hessian-type term). However both of the previous work do not pursue the regularity of maps into general $\mathrm{CAT}(\kappa)$ spaces as their strategy is designed to exploit a variant of (1.1) in a crucial way. When $\kappa > 0$ such an inequality cannot immediately be exploited as on the r.h.s. one would have a term of the type $\kappa\|\,\mathrm{d}u\|^4_{\mathrm{HS}}$, which is not clearly an L^1_{loc} function. Moreover, using their proof strategy, to actually be even able to write this term one would already need to require that the map u is locally Lipschitz. Therefore one needs to completely change the strategy and step by step try to upgrade the regularity of the map u and more precisely of its Hopf-Lax regularization. Using this type of regularization was the main idea introduced in [ZZ18], which led the authors in [Gig23b] and [MS26] to adapt this strategy to their setting (in an extremely non-trivial way).
In this paper we are going to exploit ourselves this Hopf-Lax regularization scheme and, building on the recent higher integrability of $|\,\mathrm{d}u|_{\mathrm{HS}}$ for harmonic maps (see Theorem 2.5) established in [GGZZ26]. Another crucial tool will be the Hölder regularity of the map u (see 2.6, established in [GGZZ26, Theorem 3.12]). Finally, exploting all of these results and a further regularization of the Hopf-Lax semigroup, we are going to establish an inequality of the type (1.2) (see (3.4)). The latter will then yield the sought local Lipschitz continuity via a standard argument.
We shall state and prove all of our result in the setting of $\mathrm{CAT}(1)$ spaces, as the case of $\mathrm{CAT}(\kappa)$ spaces with $\kappa > 0$ can be recovered by scaling the distance function on the target space. Our main Theorem is the following.

Theorem 1.1. *Let* $(\mathsf{X}, \mathsf{d}_{\mathsf{X}}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *space, and let* $(\mathsf{Y}, \mathsf{d}_{\mathsf{Y}})$ *be a* $\mathrm{CAT}(1)$ *space. Let*

$$u\colon \Omega \subset X \longrightarrow Y$$

be a harmonic map with Ω *bounded open set and* $\mathfrak{m}(\mathsf{X} \setminus \Omega) > 0$ *such that*

$$u(\Omega) \subset B_\rho(o), \qquad \rho < \frac{\pi}{2}.$$

Then u *is locally Lipschitz continuous.*

Once this Theorem is established one can finally and rigorously write a variant of the Bochner-Eells-Sampson formula.

Theorem 1.2. *Let* $(\mathsf{X}, \mathrm{d}_\mathsf{X}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K,N)$ *space, and let* $(\mathsf{Y}, \mathrm{d}_\mathsf{Y})$ *be a* $\mathrm{CAT}(1)$ *space. Let*

$$u : \Omega \subset X \to Y$$

be a harmonic map with Ω *open and* $\mathfrak{m}(\mathsf{X} \setminus \Omega) > 0$ *such that*

$$u(\Omega) \subset B_\rho(o), \qquad \rho < \frac{\pi}{2}.$$

Then the inequality

$$\Delta\left(\frac{\mathrm{lip}^2 u}{2}\right) \geq |\nabla \mathrm{lip} u|^2 + K\mathrm{lip}^2(u) - \frac{|\,\mathrm{d}u|^2_{\mathrm{HS}}\mathrm{lip}^2 u}{n+2} \tag{1.3}$$

holds in the weak sense in Ω*, where* n *is the essential dimension of* $(\mathsf{X}, \mathrm{d}, \mathfrak{m})$.

We are not going to prove the latter Theorem as it was already proved in [GGZZ26, Theorem 3.26], *assuming* the local Lipschitz continuity of u, which was not available at the time. Finally we refer to [GGZZ26, Theorem 3.27] for the sharp Lipschitz bound on u and to [GGZZ26, Corollary 3.28] for what concerns an L^∞-Liouville result for $u : \mathsf{X} \to \mathsf{Y}$ which is globally defined on X, which is an $\mathrm{RCD}(0,N)$ space, with Y which is a $\mathrm{CAT}(\kappa)$ space and $u(\mathsf{X}) \subset B_\rho(o) \subset \mathsf{Y}$, with $\rho < \frac{\pi}{2\sqrt{\kappa}}$: all the previous results are now rigorous, since the local Lipschitz continuity has been established.
The paper is organized as follows: in Section 2 we are going to introduce the main tools and notation for what concerns RCD spaces, CAT spaces and the notion of minimizing harmonic map, along with several useful and known results concerning the topic. In Section 3 instead we are going to prove state and prove the main results building on top of which we are going to establish Theorem 1.1.

2 Preliminaries

In this section $(\mathsf{X}, \mathrm{d}_\mathsf{X}, \mathfrak{m})$ will be denoting an $\mathrm{RCD}(K,N)$ space with $N < +\infty$ and $(\mathsf{Y}, \mathrm{d}_\mathsf{Y})$ will be a $\mathrm{CAT}(1)$ space. We stress again that since the statements for $\mathrm{CAT}(\kappa)$ follow by scaling, we shall stick to the setting of $\mathrm{CAT}(1)$ here and in the following. Moreover we assume the reader to be familiar with the basic theory of RCD and CAT spaces, as we shall only recall the fundamental tools needed for our purposes. We shall introduce the theory developed in [GT21] (after the seminal work [KS93]) and define the Korevaar-Schoen energy, in order to define minimizing harmonic maps.

Let $u \in L^2(\Omega, \mathsf{Y})$ with $\Omega \subseteq \mathsf{X}$ open set. We call the 2-energy density of u at scale r inside Ω the quantity $\mathbf{ks}_{2,r}[u,\Omega] : \mathsf{X} \to \mathbb{R}_+$, defined as

$$\mathbf{ks}_{2,r}[u,\Omega](x) := \begin{cases} \left(\fint_{B_r(x)} \frac{\mathrm{d}^2_\mathsf{Y}(u(x),u(y))}{r^2} \,\mathrm{d}\mathfrak{m}(y) \right)^{\frac{1}{2}} & \text{if } B_r(x) \subset \Omega \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Moreover we introduce the *total energy* of u in Ω as

$$\mathrm{E}_2[u,\Omega] := \liminf_{r \to 0} \int_\Omega \mathbf{ks}_{2,r}[u,\Omega]^2(x) \,\mathrm{d}\mathfrak{m}(x). \tag{2.2}$$

We can now define Sobolev spaces as follows

Definition 2.1 (Korevaar-Schoen space and harmonic maps)**.** We say that a function $u \in L^2(\Omega, \mathsf{Y})$ is in $\mathrm{KS}^{1,2}(\Omega, \mathsf{Y})$ if $\mathrm{E}_2[u, \Omega] < +\infty$. Moreover, given $w \in \mathrm{KS}^{1,2}(\Omega, \mathsf{Y})$, we say that u is *harmonic* in Ω with boundary datum w, if $u \in \arg\min_{v \in \mathrm{KS}^{1,2}_w(\Omega,\mathsf{Y})} \mathrm{E}_2[v, \Omega]$, where

$$\mathrm{KS}^{1,2}_w(\Omega, \mathsf{Y}) := \Big\{ v \in \mathrm{KS}^{1,2}(\Omega, \mathsf{Y}) : \quad \mathrm{d}_{\mathsf{Y}}(v, w) \in W^{1,2}_0(\Omega) \Big\}.$$

The existence theory for minimizers of $\mathrm{E}_2[\cdot, \Omega]$ has been carried out in [Sak23] (see Theorem 1.2 therein) under the condition that the boundary datum has image contained in a sufficiently small ball of the target space.

The next result, which can be found in [GT21, Theorem 3.13], provides a representation formula for the Korevaar-Schoen energy density.

Theorem 2.2. *Let $(\mathsf{X}, \mathrm{d}_{\mathsf{X}}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space and $(\mathsf{Y}, \mathrm{d}_{\mathsf{Y}})$ a complete metric space. Then for every $u \in \mathrm{KS}^{1,2}(\mathsf{X}, \mathsf{Y})$ there exists a function $e_2[u] \in L^2(\mathsf{X})$, called* energy density *of u, such that*

$$\mathbf{ks}_{2,r}[u] \to e_2[u] \quad \mathfrak{m} - \text{a.e. and in } L^2 \text{ as } r \to 0.$$

In particular the $\liminf$ *in* (2.2) *is actually a limit.*

The following is instead an an equivalent way to speak about $e_2[u]$ in terms, based on the Hilbert-Schmidt norm of the differential $|\,\mathrm{d}u|_{\mathrm{HS}}$: we are not going to discuss the meaning of the object $\mathrm{d}u$, referring to [GPS20] for the details. What follows is [GT21, Proposition 6.7].

Theorem 2.3. *Let $(\mathsf{X}, \mathrm{d}_{\mathsf{X}}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space and $\Omega \subset \mathsf{X}$ an open set. Let $(\mathsf{Y}, \mathrm{d}_{\mathsf{Y}})$ be a $\mathrm{CAT}(\kappa)$ space and $u \in \mathrm{KS}^{1,2}(\Omega, \mathsf{Y})$, then for its energy density we have the following representation formula*

$$e_2[u] = (d+2)^{-\frac{1}{2}} |\,\mathrm{d}u|_{\mathrm{HS}}. \tag{2.3}$$

Definition 2.4. Let $(\mathsf{X}, \mathrm{d}_{\mathsf{X}}, \mathfrak{m})$ be a metric measure space and $\Omega \subset \mathsf{X}$ an open and bounded set. Let $\eta : \Omega \to \mathbb{R}$ be locally integrable. We say that a function $f \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is such that $\Delta f \leq \eta$ weakly in Ω if for all $\varphi \in \mathrm{Lip}_c(\Omega)$ with $\varphi \geq 0$ we have

$$-\int_{\mathsf{X}} \nabla f \cdot \nabla \varphi \,\mathrm{d}\mathfrak{m}_{\mathsf{X}} \leq \int_{\mathsf{X}} \varphi \eta \,\mathrm{d}\mathfrak{m}_{\mathsf{X}}.$$

We now introduce some further notation trying to stick to the one of [GGZZ26]. We let $U \Subset \Omega$ be open and consider $r > 0$ and $\hat{x} \in U$ such that $B_{4r}(\hat{x}) \subset U$ and $\|u\|_{C^\alpha(U)}(2r)^\alpha < s < \pi/8$. Finally call $B = B_r(\hat{x})$, $2B := B_{2r}(\hat{x})$, $\widehat{B} = B_{3r/2}(\hat{x})$ and $4B = B_{4r}(\hat{x})$. Let $o_B = u(\hat{x})$ we then have, thanks to the previous conditions, $u(2B) \subset B_s(o_B)$.

For $p > 2$, define the function $F : \mathbb{R} \to \mathbb{R}$ as

$$F(s) := 2\sin\left(\frac{s}{2}\right) + 4\sin^2\left(\frac{s}{2}\right).$$

and $f : \mathsf{X} \times \mathsf{X} \to [-6, 0]$ as

$$f(x,y) = \begin{cases} -F(\mathrm{d}_{\mathsf{Y}}(u(x), u(y))) & \text{if } x, y \in \widehat{B} \\ -6 & \text{otherwise.} \end{cases}$$

Then for all $t > 0$ define the function $f_t : B \to \mathbb{R}$ as

$$f_t(x) := \inf_{y \in \mathsf{X}} \left[\frac{\mathrm{d}^p_{\mathsf{X}}(y, x)}{p\, t^{p-1}} + f(x, y) \right].$$

Now denoting with $y_{t,x}$ a minimizer for $f_t(x)$, and choosing x as a competitor, we get

$$f_t(x) \leq \frac{\mathrm{d}^p_{\mathsf{X}}(x, y_{t,x})}{p t^{p-1}} + f(x, y_{t,x}) \leq 0.$$

This means $\mathsf{d}_{\mathsf{X}}(x, y_{t,x}) \leq 6p\frac{1}{p}t^{\frac{p-1}{p}}$. We now further shrink t_* in a way that $(6p)^{\frac{1}{p}}t_*^{\frac{p-1}{p}} < \operatorname{dist}(\overline{B}, \mathsf{X} \setminus \widehat{B})$ so that

$$f_t(x) := \inf_{y \in \widehat{B}} \left[\frac{\mathsf{d}_{\mathsf{X}}^p(x,y)}{pt^{p-1}} - F(\mathsf{d}_{\mathsf{Y}}(u(x), u(y))) \right] \qquad \forall x \in B$$

Thanks to Theorem 2.6 below we observe that u is locally Hölder continuous inside Ω and thanks to [GGZZ26, Lemma 3.16] so is f_t. Set

$$\gamma_t(x) := -\frac{f_t(x)}{t},$$

$$S_t(x) := \left\{ y \in \mathsf{X} : f_t(x) = \frac{\mathsf{d}_{\mathsf{X}}^p(y,x)}{p\,t^{p-1}} + f(x,y) \right\},$$

and

$$L_t(x) := \min_{y \in S_t(x)} \mathsf{d}_{\mathsf{X}}(y,x), \qquad a_t(x) := \frac{L_t^p(x)}{p\,t^p}, \qquad \frac{D_t(x)}{t} := \gamma_t(x) + a_t(x).$$

We now recall[GGZZ26, Theorem 3.15], which proves higher integrability for gradients of minimizing harmonic maps.

Theorem 2.5. *Let Ω, Y as above and $u : \Omega \to \mathsf{Y}$ be an harmonic map with $u(\Omega) \subset B_\rho(o)$ and $\rho < \pi?2$. Then there exists an $\varepsilon = \varepsilon(N, K, \operatorname{diam}(\Omega), \rho) > 0$ such that $|\operatorname{d}u|_{\mathrm{HS}} \in L^{2+\varepsilon}_{\mathrm{loc}}(\Omega)$ and*

$$\left(\fint_B |\operatorname{d}u|_{\mathrm{HS}}^{2+\varepsilon} \operatorname{d}\mathfrak{m} \right)^{\frac{2}{2+\varepsilon}} \leqslant C_\varepsilon \fint_{2B} |\operatorname{d}u|_{\mathrm{HS}}^2 \operatorname{d}\mathfrak{m} \tag{2.4}$$

for any ball B with $2B \subset \Omega$, where the constant $C_\varepsilon > 0$ depends only on ε.

We now recall [GGZZ26, Theorem 3.12] which establishes the local Hölder regularity of u.

Theorem 2.6. *Let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ space and $(\mathsf{Y}, \mathsf{d}_{\mathsf{Y}})$ be a $\mathrm{CAT}(1)$ space. Let $u : \Omega \subseteq \mathsf{X} \to \mathsf{Y}$ be a harmonic map such that $\mathrm{Im}(u) \subseteq B_\rho(o)$ with $\rho < \pi/2$. Then u is locally Hölder continuous in Ω.*

For the reader convenience we present [GGZZ26, Lemma 3.19], which is a crucial differential inequality that we are going to exploit later on.

Lemma 2.7. *Let $u : \Omega \to \mathsf{Y}$ be a harmonic map with $\Omega \subset \mathsf{X}$ open set, $(\mathsf{Y}, \mathsf{d}_{\mathsf{Y}})$ which is a $\mathrm{CAT}(1)$ space and $\mathrm{Im}(u) \subseteq B_\rho(o)$ with $o \in \mathsf{Y}$, $\rho < \pi/2$. Let further $f_o(x) := \cos(\mathsf{d}_{\mathsf{Y}}(u(x), o))$, then we have $f_o \in W^{1,2}(\Omega)$ and*

$$\Delta f_o \leq -f_o |\operatorname{d}u|_{\mathrm{HS}}^2 = -f_o(n+2)e_2^2[u], \tag{2.5}$$

weakly in Ω.

We also recall here [GGZZ26, Proposition 3.24], which gives a crucial distributional bound for γ_t. Note that we are dividing by t the equation in [GGZZ26, Proposition 3.24], so that it has the form we need.

Proposition 2.8. *Let $t > 0$ and γ_t, a_t and D_t as above. Then*

$$\Delta \gamma_t \geq pKa_t - c_t(a_t + \gamma_t)|\operatorname{d}u|_{\mathrm{HS}}^2 \quad \text{on } B \tag{2.6}$$

in the weak sense, for all $t < t^*$ and $p \geqslant 2$, where $c_t = 1 + o_t(1)$ uniformly in B.

3 Main results

We stress again that we shall fix $U \Subset \Omega$ and choose a chain of balls with radii such that the conditions at the end of Section 2 are in place.
To prove Theorem 1.1 we need several preliminary results, starting from the following.

Lemma 3.1. *Let $u : \Omega \subset \mathsf{X} \to \mathsf{Y}$ be an harmonic map with $u(\Omega) \subset B_\rho(o)$ and $\rho < \pi/2$ and let $U \Subset \Omega$ be open. There exist $r_0, C > 0$ and $\Lambda \geq 1$ such that, for every $x, y \in U$ with $\mathrm{d}_{\mathsf{X}}(y,x) < r_0$, setting $\ell = \mathrm{d}_{\mathsf{X}}(y,x)$, we have*

$$\mathrm{d}_{\mathsf{Y}}\big(u(y), u(x)\big) \leq C\ell \left(\fint_{B_{\Lambda\ell}(x)} |\,\mathrm{d}u|^2_{\mathrm{HS}} \,\mathrm{d}\mathfrak{m} \right)^{1/2}.$$

Proof. We choose $r_0 > 0$ such that every ball mentioned in the sequel lies inside Ω. For fixed $a \in U$, the map

$$z \longmapsto \mathrm{d}_{\mathsf{Y}}\big(u(z), u(a)\big)$$

is subharmonic; see, for example [GGZZ26, Proposition 3.5]. The Harnack inequality then gives, by [GGZZ26, Theorem 3.7] and Jensen, applied at fixed x and with $\ell = \mathrm{d}_{\mathsf{X}}(x,y) < r_0$,

$$\mathrm{d}^2_{\mathsf{Y}}\big(u(y), u(x)\big) \leq C \fint_{B_\ell(y)} \mathrm{d}^2_{\mathsf{Y}}\big(u(z), u(x)\big) \,\mathrm{d}\mathfrak{m}(z).$$

We now do the same with the other variable, so that

$$\mathrm{d}^2_{\mathsf{Y}}\big(u(y), u(x)\big) \leq C \fint_{B_\ell(x)} \fint_{B_\ell(y)} \mathrm{d}^2_{\mathsf{Y}}\big(u(z), u(w)\big) \,\mathrm{d}\mathfrak{m}(z) \,\mathrm{d}\mathfrak{m}(w).$$

Since

$$B_\ell(x) \cup B_\ell(y) \subset B_{2\ell}(x)$$

and the measure $\mathfrak{m}$ is locally doubling, we get

$$\mathrm{d}^2_{\mathsf{Y}}\big(u(y), u(x)\big) \leq C \fint_{B_{2\ell}(x)} \fint_{B_{2\ell}(x)} \mathrm{d}^2_{\mathsf{Y}}\big(u(z), u(w)\big) \,\mathrm{d}\mathfrak{m}(z) \,\mathrm{d}\mathfrak{m}(w).$$

We now apply the Poincaré inequality (see [GGZZ26, Lemma 3.10]), to get

$$\mathrm{d}^2_{\mathsf{Y}}\big(u(y), u(x)\big) \leq C\ell^2 \fint_{B_{\Lambda\ell}(x)} |\,\mathrm{d}u|^2_{\mathrm{HS}}(z) \,\mathrm{d}\mathfrak{m}(z),$$

proving the lemma. □

We now set

$$\mathcal{M}(|\,\mathrm{d}u|^2_{\mathrm{HS}})(x) := \sup_{0<s<\Lambda r_0} \fint_{B_s(x)} \chi_{2B} |\,\mathrm{d}u|^2_{\mathrm{HS}} \,\mathrm{d}\mathfrak{m},$$

and shrink t_0 so that $B_{\Lambda t_0} \subset 2B$. The previous lemma then reads as

$$\mathrm{d}_{\mathsf{Y}}\big(u(y), u(x)\big) \leq C \,\mathrm{d}_{\mathsf{X}}(y,x) \big(\mathcal{M}(|\,\mathrm{d}u|^2_{\mathrm{HS}})(x)\big)^{1/2}.$$

We now have another crucial lemma.

Lemma 3.2. *There exist $t_0 > 0$ and an appropriate choice of $p > 3$ such that*

$$\sup_{t \in (0,t_0)} \|\gamma_t\|_{L^2(B)} < +\infty.$$

Proof. Let $t_0 > 0$ and $t \in (0, t_0)$, and choose $y_{t,x} \in S_t(x)$ so that

$$L_t(x) = \mathrm{d}_{\mathsf{X}}(x, y_{t,x}).$$

By continuity, we can decrease t_0 so that

$$y_{t,x} \in B_{r_0}(x),$$

where r_0 is the one of Lemma 3.1. We can therefore apply Lemma 3.1, together with the fact that $F(s) \le Cs$, to obtain

$$D_t(x) = F\left(\mathrm{d}_{\mathsf{Y}}\big(u(x), u(y_{t,x})\big)\right) \le C L_t(x)\big(\mathcal{M}(|\,\mathrm{d}u|^2_{\mathrm{HS}})(x)\big)^{1/2}.$$

Consequently,

$$\gamma_t(x) = \frac{D_t(x)}{t} - a_t(x) \le C\frac{L_t(x)}{t}\big(\mathcal{M}(|\,\mathrm{d}u|^2_{\mathrm{HS}})(x)\big)^{1/2} - \frac{L_t^p(x)}{p\,t^p}.$$

For

$$A = C\big(\mathcal{M}(|\,\mathrm{d}u|^2_{\mathrm{HS}})(x)\big)^{1/2},$$

we apply Young's inequality, where

$$q^{-1} + p^{-1} = 1,$$

to get

$$A\frac{L_t(x)}{t} - \frac{L_t^p(x)}{p\,t^p} \le \frac{A^q}{q}.$$

Hence

$$0 \le \gamma_t \le C\big(\mathcal{M}(|\,\mathrm{d}u|^2_{\mathrm{HS}})(x)\big)^{q/2}.$$

Choosing p large enough so that

$$|\,\mathrm{d}u|^2_{\mathrm{HS}} \in L^q, \qquad q > 1,$$

which is possible thanks to Theorem 2.5, we obtain

$$\int_B \gamma_t^2 \,\mathrm{d}\mathfrak{m} \le C\int_B \big(\mathcal{M}(|\,\mathrm{d}u|^2_{\mathrm{HS}})\big)^q \,\mathrm{d}\mathfrak{m} \le C\int_{2B} |\,\mathrm{d}u|^{2q}_{\mathrm{HS}} \,\mathrm{d}\mathfrak{m},$$

which proves the claim. Here we also used the standard maximal-function inequality. □

The idea is now to integrate γ_t in order to obtain important estimates on its integrated version, which will propagate the regularity.

Lemma 3.3 (Time averaging)**.** *Let*

$$0 < T < \delta < t_0,$$

with t_0 as in the previous lemma, and define

$$G_T := T\int_T^\delta \frac{\gamma_t}{t^2}\,dt, \qquad A_T := T\int_T^\delta \frac{a_t}{t^2}\,dt.$$

Then

$$G_T, A_T \in W^{1,2}_{\mathrm{loc}}(B)$$

and

$$(p-1)A_T = 2G_T + \frac{T}{\delta}\gamma_\delta - \gamma_T \qquad \mathfrak{m}\text{−a.e. in } B.$$

Moreover we have

$$\Delta G_T \ge -pK^- A_T - c_\delta(A_T + G_T)|\,\mathrm{d}u|^2_{\mathrm{HS}}, \tag{3.1}$$

weakly in B, where $c_\delta = \sup_{t\in(0,\delta)} c_t$.

Proof. First let us introduce

$$\widehat{L}_t(x) := \max_{y \in S_t(x)} \mathrm{d}_{\mathsf{X}}(y,x).$$

and then we observe that the map

$$t \longmapsto f_t(x)$$

is locally Lipschitz on $(0, t_0)$ and

$$\partial_t^- f_t(x) = -\frac{p-1}{p\, t^p} L_t^p(x), \qquad \partial_t^+ f_t(x) = -\frac{p-1}{p\, t^p} \widehat{L}_t^p(x).$$

thanks to the usual differentiability properties of the Hopf-Lax semigroup (see [ACM+21, Theorem 3.3] and the monograph [AGS08] for example). At every t for which $f_t(x)$ is differentiable, we therefore get

$$\partial_t f_t(x) = -(p-1) a_t(x).$$

Using the relation between f_t and γ_t, we obtain

$$(p-1) a_t = \gamma_t + t\, \partial_t \gamma_t$$

for $\mathscr{L}^1 \otimes m$-a.e. (t,x).

Now fix $T > 0$. Up to choosing $\delta > 0$ small, there exists a relatively compact set $W \Subset \widehat{B}$ containing every minimizer y associated with $x \in B$ and $t \in [T, \delta]$. For these minimizers, set

$$q_{t,y}(x) := \frac{\mathrm{d}_{\mathsf{X}}^p(x,y)}{p\, t^{p-1}} - F\left(\mathrm{d}_{\mathsf{Y}}\big(u(y), u(x)\big)\right).$$

First observe that since u is continuous and X is separable we can actually write $f_t(x) = \inf_{j \in \mathbb{N}} q_{t,y_j}(x)$, where $(y_j)_j$ is a *countable* and dense set of $\overline{\widehat{B}}$. Then we have

$$|Dq_{t,y_j}| \le C_{T,\delta}\big(1 + |\,\mathrm{d}u|_{\mathrm{HS}}\big),$$

uniformly in $j \in \mathbb{N}$ and $t \in [T, \delta]$. Since these functions all have the same bound on the weak upper gradient, taking a countable infimum allows that bound to go through the infimization and we get

$$|Df_t| \le C_{T,\delta}\big(1 + |\,\mathrm{d}u|_{\mathrm{HS}}\big),$$

and hence

$$f_t \in W^{1,2}(B).$$

Thanks to this, G_T and DG_T are well-defined as Bochner integrals, and

$$DG_T = T \int_T^\delta \frac{D\gamma_t}{t^2}\, dt \in L^2(B).$$

Integrating by parts, we get

$$\begin{aligned}
(p-1) A_T &= T \int_T^\delta \left(\frac{\gamma_t}{t^2} + \frac{\partial_t \gamma_t}{t} \right) dt \\
&= 2T \int_T^\delta \frac{\gamma_t}{t^2}\, dt + \frac{T \gamma_\delta}{\delta} - \gamma_T.
\end{aligned}$$

Therefore,

$$(p-1) A_T = 2G_T + \frac{T}{\delta}\gamma_\delta - \gamma_T,$$

which also implies that

$$A_T \in W^{1,2}(B).$$

Finally, to prove (3.1), let

$$\varphi \in \mathrm{Lip}_c(B), \qquad \varphi \geq 0,$$

and integrate (2.6) against the measure $Tt^{-2}\,\mathrm{d}t$ to obtain

$$-\int_B \langle DG_T, D\varphi\rangle \,\mathrm{d}\mathfrak{m} \geq -pK\int_B A_T\varphi \,\mathrm{d}\mathfrak{m} - c_\delta \int_B (A_T + G_T)|\,\mathrm{d}u|^2_{\mathrm{HS}}\varphi \,\mathrm{d}\mathfrak{m}.$$

□

Remark. Observe that, for $t \in [T, \delta]$, a_t and γ_t are uniformly bounded, so all the terms in the previous lemma are well-defined.

Proposition 3.4. *There exist $\delta > 0$ and constants $\lambda < 2$ and $\mu \geq 0$ such that, for every $T \in (0, \delta)$, we can construct*

$$\overline{G}_T \geq 0, \qquad \overline{G}_T \in W^{1,2}(B),$$

satisfying

$$\Delta \overline{G}_T \geq -\mu \overline{G}_T - \lambda|\,\mathrm{d}u|^2_{\mathrm{HS}}\overline{G}_T \tag{3.2}$$

weakly in B. Moreover,

$$\sup_{T\in(0,\delta)} \|\overline{G}_T\|_{L^2(B)} < \infty, \qquad \gamma_T \leq 2\overline{G}_T \quad \mathfrak{m}\text{−a.e.} \tag{3.3}$$

Finally, if

$$f_{o_B} = \cos\big(\mathrm{d}_{\mathsf{Y}}(u, o_B)\big), \qquad \psi := f_{o_B} - \frac{1}{2}, \qquad Z_T := \frac{\overline{G}_T}{\psi},$$

with $o_B \in B$, then

$$\int_B \psi^2 \langle DZ_T, D\varphi\rangle \,\mathrm{d}\mathfrak{m} \leq \mu \int_B \psi^2 Z_T \varphi \,\mathrm{d}\mathfrak{m} \tag{3.4}$$

for every $\varphi \in \mathrm{Lip}_c(B)$ with $\varphi \geq 0$.

Proof. We first observe that Proposition 2.8 is in place and we choose $p > 3$ and $\eta > 0$ sufficiently small so that

$$\lambda := (1+\eta)\left(1 + \frac{2}{p-1}\right) < 2.$$

We can then choose δ so that

$$c_t \leq 1 + \eta \qquad \text{for } t \in (0, \delta).$$

Set

$$M_\delta := \|\gamma_\delta\|_{L^\infty(B)} \leq \frac{6}{\delta}, \qquad \overline{G}_T := G_T + \frac{T}{2\delta}M_\delta.$$

Since $A_T \geq 0$, the identity for A_T in the previous lemma gives

$$A_T \leq \frac{2}{p-1}\overline{G}_T, \qquad \gamma_T \leq 2\overline{G}_T.$$

Indeed,

$$(p-1)A_T \leq 2G_T + \frac{T}{\delta}M_\delta = 2\overline{G}_T,$$

whereas

$$\gamma_T = 2G_T + \frac{T}{\delta}\gamma_\delta - (p-1)A_T \leq 2\overline{G}_T.$$

Moreover we have

$$\|G_T\|_{L^2(B)} \leq T\int_T^\delta \frac{\gamma_t}{t^2}\,\mathrm{d}t \leq C_0\left(1 - \frac{T}{\delta}\right) \leq C_0,$$

which proves $\sup_{T\in(0,\delta)} \|\overline{G}_T\|_{L^2(B)} < \infty$. We now apply the previous lemma together with this latter inequality to get

$$\begin{aligned}\Delta\overline{G}_T &\geq -pK^- A_T - (1+\eta)(A_T + G_T)|\,\mathrm{d}u|^2_{\mathrm{HS}} \\ &\geq -\underbrace{\frac{2pK^-}{p-1}}_{\mu}\,\overline{G}_T - \underbrace{(1+\eta)\left(1+\frac{2}{p-1}\right)}_{\lambda}|\,\mathrm{d}u|^2_{\mathrm{HS}}\overline{G}_T,\end{aligned}$$

which proves (3.2).
We now remove the energy density from (3.2) to obtain a better differential inequality.
Since we assumed

$$u(\widehat{B}) \subset B_s(o_B), \qquad s < \frac{\pi}{8},$$

we get

$$\frac{3}{4} \leq f_{o_B} \leq 1, \qquad \frac{1}{4} \leq \psi \leq \frac{1}{2}.$$

Moreover,

$$f_{o_B} \geq 2\psi.$$

Now, by Lemma 2.7 (here we are also using that u minimizes the energy in every ball among maps with the same boundary values),

$$\Delta f_{o_B} \leq -f_{o_B}|\,\mathrm{d}u|^2_{\mathrm{HS}}.$$

Since $\lambda < 2$ and $f_{o_B} \geq 2\psi$, this becomes

$$\Delta\psi \leq -\lambda|\,\mathrm{d}u|^2_{\mathrm{HS}}\psi,$$

since we clearly have $\Delta\psi = \Delta f_{o_B}$.

Let now

$$\varphi \in \mathrm{Lip}_c(B), \qquad \varphi \geq 0.$$

Testing (3.2) against $\varphi\psi$ (by approximation) we obtain

$$\int_B \left\langle D\overline{G}_T, D(\varphi\psi)\right\rangle \,\mathrm{d}\mathfrak{m} \leq \int_B (\mu + \lambda|\,\mathrm{d}u|^2_{\mathrm{HS}})\overline{G}_T\psi\varphi\,\mathrm{d}\mathfrak{m}. \tag{3.5}$$

Now we test the inequality

$$\Delta\psi \leq -\lambda|\,\mathrm{d}u|^2_{\mathrm{HS}}\psi$$

against $\overline{G}_T\varphi$. We obtain

$$\int_B \left\langle D\psi, D(\overline{G}_T\varphi)\right\rangle \,\mathrm{d}\mathfrak{m} \geq \lambda\int_B |\,\mathrm{d}u|^2_{\mathrm{HS}}\psi\overline{G}_T\varphi\,\mathrm{d}\mathfrak{m}. \tag{3.6}$$

Subtracting (3.6) from (3.5) gives

$$\int_B \left\langle \psi D\overline{G}_T - \overline{G}_T D\psi, D\varphi\right\rangle \,\mathrm{d}\mathfrak{m} \leq \mu\int_B \psi\overline{G}_T\varphi\,\mathrm{d}\mathfrak{m}.$$

Since $\psi \geq \frac{1}{4}$, setting $Z_T := \overline{G}_T/\psi \in W^{1,2}_{\mathrm{loc}}(B)$, the chain rule gives

$$\psi D\overline{G}_T - \overline{G}_T D\psi = \psi^2 D\left(\frac{\overline{G}_T}{\psi}\right) = \psi^2 DZ_T,$$

therefore proving (3.4). □

Lemma 3.5. *Let*

$$Z \geq 0 \quad \text{with } Z \in W^{1,2}_{\mathrm{loc}}(B)$$

and suppose that, for every $\varphi \in \mathrm{Lip}_c(B)$ *with* $\varphi \geq 0$,

$$\int_B \theta \langle DZ, D\varphi \rangle \, \mathrm{d}\mathfrak{m} \leq \mu \int_B \theta Z \varphi \, \mathrm{d}\mathfrak{m}, \tag{3.7}$$

for some $\mu \geq 0$. *Assume that* $\theta \in L^\infty(B)$, *with*

$$0 < \theta_0 \leq \theta \leq \theta_1 < +\infty \quad \mathfrak{m} - \text{a.e. in } B.$$

Then for all $B_R(x_0)$ *such that* $B_{2R}(x_0) \Subset B$ *we have*

$$\operatorname*{ess\,sup}_{B_R(x_0)} Z \leq C \left(\fint_{B_{2R}(x_0)} Z^2 \, \mathrm{d}\mathfrak{m} \right)^{1/2}, \tag{3.8}$$

where

$$C = C(\theta_0, \theta_1, K, R, |\mu|) > 0$$

is equibounded as $R \to 0$.

Proof. Let $\beta \geq 2$ and let η be a Lipschitz cutoff function. Using $\eta^2 Z^{\beta-1}$ as a test function in (3.7) (via approximation), we obtain

$$\int_{B_{2R}(x_0)} \eta^2 \left| D\left(Z^{\beta/2}\right)\right|^2 \, \mathrm{d}\mathfrak{m} \leq C \frac{\theta_1}{\theta_0} \beta^2 \int_{B_{2R}(x_0)} (|D\eta|^2 + \mu\eta^2) Z^\beta \, \mathrm{d}\mathfrak{m}.$$

Indeed, using the chain rule, we have

$$\begin{aligned}
&\int_{B_{2R}(x_0)} \eta^2 \theta \left\langle DZ, D\left(Z^{\beta-1}\right)\right\rangle \, \mathrm{d}\mathfrak{m} \\
&\quad + \int_{B_{2R}(x_0)} \theta Z^{\beta-1} \left\langle DZ, D(\eta^2)\right\rangle \, \mathrm{d}\mathfrak{m} \\
&\leq \mu \int_{B_{2R}(x_0)} \theta Z^\beta \eta^2 \, \mathrm{d}\mathfrak{m}.
\end{aligned}$$

Now

$$D\left(\eta^2 Z^{\beta-1}\right) = 2\eta Z^{\beta-1} D\eta + (\beta-1)\eta^2 Z^{\beta-2} DZ,$$

so that

$$\begin{aligned}
(\beta-1) \int_{B_{2R}(x_0)} \theta \eta^2 Z^{\beta-2} |DZ|^2 \, \mathrm{d}\mathfrak{m} \leq 2 &\int_{B_{2R}(x_0)} \theta \eta Z^{\beta-1} |DZ| \, |D\eta| \, \mathrm{d}\mathfrak{m} \\
&+ \mu \int_{B_{2R}(x_0)} \theta \eta^2 Z^\beta \, \mathrm{d}\mathfrak{m}.
\end{aligned}$$

Using Young's inequality and absorbing terms gives

$$\begin{aligned}
\int_{B_{2R}(x_0)} \theta \eta^2 Z^{\beta-2} |DZ|^2 \, \mathrm{d}\mathfrak{m} \leq &\frac{C}{(\beta-1)^2} \int_{B_{2R}(x_0)} \theta Z^\beta |D\eta|^2 \, \mathrm{d}\mathfrak{m} \\
&+ \frac{C\mu}{\beta-1} \int_{B_{2R}(x_0)} \theta \eta^2 Z^\beta \, \mathrm{d}\mathfrak{m}.
\end{aligned}$$

Since

$$\left| D\left(Z^{\beta/2}\right)\right|^2 = \frac{\beta^2}{4} Z^{\beta-2} |DZ|^2,$$

we conclude with the previous estimate, using also

$$\theta_0 \leq \theta \leq \theta_1.$$

Finally (3.8) follows from the local Sobolev inequality on $\mathrm{RCD}(K, N)$ spaces, along with the classical De Giorgi-Nash-Moser iteration. $\square$

Proposition 3.6. *For every $V \Subset B$, there exist $\delta > 0$ and $C_V > 0$ such that*

$$\sup_{0<T<\delta} \|\gamma_T\|_{L^\infty(V)} \le C_V.$$

Proof. Cover V with finitely many balls $B_R(x_i)$ such that

$$B_{2R}(x_i) \Subset B.$$

Apply the previous lemma to Z_T with $\theta = \psi^2$, so that

$$\theta_0 := \frac{1}{16} \le \psi^2 \le \frac{1}{4} =: \theta_1.$$

Now

$$Z_T^2 = \frac{\overline{G}_T^2}{\psi^2} \le 16\overline{G}_T^2.$$

The L^2-norms of $\overline{G}_T$ are uniformly bounded thanks to (3.3), i.e. $\sup_{T\in(0,\delta)} \left\|\overline{G}_T\right\|_{L^2(B)} < \infty$. We can then apply (3.8) and infer

$$Z_T \in L^\infty\big(B_R(x_i)\big)$$

with a bound independent of T and depending on $\|Z_T\|_{L^2(B)}$. Since

$$\overline{G}_T = \psi Z_T \qquad \text{and} \qquad \gamma_T \le 2\overline{G}_T,$$

we obtain the claim, exploting also the Hölder continuity of u (which allows to use a supremum instead of an essential supremum). $\square$

We are finally ready to prove the main Theorem.

Proof of Theorem 1.1. Fix $\hat{x} \in V \Subset B$ satisfying the nested ball assumptions introduced at the end of Section 2 together with $u(\widehat{B}) \subset B_s(u(\hat{x}))$, with $s < \pi/8$. Then, by Proposition 3.6,

$$\gamma_t(x) \le C_V \qquad \text{for every } x \in V, \quad t \in (0,\delta),$$

for some $\delta > 0$. Let

$$r = \mathsf{d}_\mathsf{X}(x,y), \qquad x, y \in V.$$

Assume first that $r \in (0,\delta)$. Then

$$\begin{aligned} f_r(x) &\le \frac{r^p}{p\,r^{p-1}} - F\left(\mathsf{d}_\mathsf{Y}\big(u(y),u(x)\big)\right) \\ &= \frac{r}{p} - F\left(\mathsf{d}_\mathsf{Y}\big(u(y),u(x)\big)\right). \end{aligned}$$

Since

$$f_r = -r\gamma_r,$$

it follows that

$$c\,\mathsf{d}_\mathsf{Y}\big(u(y),u(x)\big) \le F\left(\mathsf{d}_\mathsf{Y}\big(u(y),u(x)\big)\right) \le r\left(\frac{1}{p} + \gamma_r(x)\right) \le \left(\frac{1}{p} + C_V\right) r.$$

If instead $r \ge \delta$, then $u(V) \subset B_s\big(u(\hat{x})\big)$ gives

$$\mathsf{d}_\mathsf{Y}\big(u(y),u(x)\big) \le 2s \le \frac{2s}{\delta}\,\mathsf{d}_\mathsf{X}(x,y).$$

Since x_0 was arbitrary, u is locally Lipschitz in Ω. $\square$

Acknowledgements. This work was supported by UK Research and Innovation (UKRI) under the Horizon Europe funding guarantee [grant number EP/Z000297/1]. The author wishes to thank Nicola Gigli for helpful comments on a preliminary draft of this work.

References

[ACM+21] Afiny Akdemir, Andrew Colinet, Robert McCann, Fabio Cavalletti, and Flavia Santarcangelo. Independence of synthetic curvature dimension conditions on transport distance exponent. *Trans. Amer. Math. Soc.*, 374(8):5877–5923, 2021.

[AGMR12] Luigi Ambrosio, Nicola Gigli, Andrea Mondino, and Tapio Rajala. Riemannian Ricci curvature lower bounds in metric measure spaces with σ-finite measure. *Trans. Amer. Math. Soc.*, 367(7):4661–4701, 2012.

[AGS08] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré. *Gradient flows in metric spaces and in the space of probability measures*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, second edition, 2008.

[AGS12] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré. Heat flow and calculus on metric measure spaces with Ricci curvature bounded below—the compact case. *Boll. Unione Mat. Ital. (9)*, 5(3):575–629, 2012.

[AGS14a] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré. Calculus and heat flow in metric measure spaces and applications to spaces with Ricci bounds from below. *Invent. Math.*, 195(2):289–391, 2014.

[AGS14b] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré. Metric measure spaces with Riemannian Ricci curvature bounded from below. *Duke Math. J.*, 163(7):1405–1490, 2014.

[AGS15] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré. Bakry-Émery curvature-dimension condition and Riemannian Ricci curvature bounds. *The Annals of Probability*, 43(1):339–404, 2015.

[CG89] Jean-Michel Coron and Robert Gulliver. Minimizing p-harmonic maps into spheres. *J. Reine Angew. Math.*, 401:82–100, 1989.

[GGZZ26] Luca Gennaioli, Nicola Gigli, Hui-Chun Zhang, and Xi-Ping Zhu. Comments on the regularity of harmonic maps between singular spaces. *J. Geom. Anal.*, 36(6):Paper No. 208, 34, 2026.

[Gig15] Nicola Gigli. On the differential structure of metric measure spaces and applications. *Mem. Amer. Math. Soc.*, 236(1113):vi+91, 2015.

[Gig18] Nicola Gigli. Nonsmooth differential geometry—an approach tailored for spaces with Ricci curvature bounded from below. *Mem. Amer. Math. Soc.*, 251(1196):v+161, 2018.

[Gig23a] Nicola Gigli. De giorgi and Gromov working together, 2023.

[Gig23b] Nicola Gigli. On the regularity of harmonic maps from $\mathrm{RCD}(K,N)$ to $\mathrm{CAT}(0)$ spaces and related results. *Ars Inven. Anal.*, pages Paper No. 5, 55, 2023.

[Gig26] Nicola Gigli. The splitting theorem in non-smooth context. *Mem. Amer. Math. Soc.*, 317(1609):x+113, 2026. with prologue by Luigi Ambrosio.

[GKO13] Nicola Gigli, Kazumasa Kuwada, and Shin-ichi Ohta. Heat flow on Alexandrov spaces. *Communications on Pure and Applied Mathematics*, 66(3):307–331, 2013.

[GPS20] Nicola Gigli, Enrico Pasqualetto, and Elefterios Soultanis. Differential of metric valued Sobolev maps. *J. Funct. Anal.*, 278(6):108403, 2020.

[Gro87] M. Gromov. Hyperbolic groups. In *Essays in group theory*, volume 8 of *Math. Sci. Res. Inst. Publ.*, pages 75–263. Springer, New York, 1987.

[GT21] Nicola Gigli and Alexander Tyulenev. Korevaar-Schoen's energy on strongly rectifiable spaces. *Calc. Var. Partial Differential Equations*, 60(6):Paper No. 235, 54, 2021.

[JK83] Willi Jäger and Helmut Kaul. Rotationally symmetric harmonic maps from a ball into a sphere and the regularity problem for weak solutions of elliptic systems. *J. Reine Angew. Math.*, 343:146–161, 1983.

[KS93] Nicholas J. Korevaar and Richard M. Schoen. Sobolev spaces and harmonic maps for metric space targets. *Comm. Anal. Geom.*, 1(3-4):561–659, 1993.

[Lin87] Fang-Hua Lin. A remark on the map $x/|x|$. *C. R. Acad. Sci. Paris Sér. I Math.*, 305(12):529–531, 1987.

[MS26] Andrea Mondino and Daniele Semola. Lipschitz continuity and Bochner-Eells-Sampson inequality for harmonic maps from $\mathrm{RCD}(K,N)$ spaces to $\mathrm{CAT}(0)$ spaces. *Amer. J. Math.*, 148(2):505–568, 2026.

[Pet03] Anton Petrunin. Harmonic functions on Alexandrov spaces and their applications. *Electron. Res. Announc. Amer. Math. Soc.*, 9:135–141, 2003.

[Sak23] Yohei Sakurai. Dirichlet problem for harmonic maps from strongly rectifiable spaces into regular balls in $\mathrm{CAT}(1)$ spaces. *Ann. Global Anal. Geom.*, 64(3):Paper No. 19, 18, 2023.

[Ser95] Tomasz Serbinowski. *Harmonic maps into metric spaces with curvature bounded above*. ProQuest LLC, Ann Arbor, MI, 1995. Thesis (Ph.D.)–The University of Utah.

[ZZ18] Hui-Chun Zhang and Xi-Ping Zhu. Lipschitz continuity of harmonic maps between Alexandrov spaces. *Invent. Math.*, 211(3):863–934, 2018.