CONFORMAL KÄHLER RIGIDITY OF EINSTEIN FOUR-MANIFOLDS

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Abstract. For a compact, connected, oriented Einstein four-manifold, we prove that, if the largest eigenvalue of the self-dual Weyl curvature W^+ is everywhere simple, then, after at worst passing to a double cover, the metric is conformally Kähler with positive scalar curvature; more generally, this result holds for metrics with harmonic self-dual Weyl curvature. For Einstein metrics satisfying a uniform simplicity hypothesis on the largest eigenvalue, we further prove that either $W^+ \equiv 0$, or W^+ nowhere vanishes and the previous conclusion holds. We also obtain extensions to complete Ricci-flat four-manifolds and an optimal pinching theorem for the holomorphic sectional curvature of compact Kähler–Einstein surfaces. The proof combines LeBrun's conformal normalization with the resulting weighted divergence equation and new first-order identities. A zero-capacity argument allows this method to be used across the zero set of W^+ and at infinity in the noncompact case. Finally, K3 surfaces and multicentered Gibbons–Hawking gravitational instantons show that our assumptions are sharp.

1. Introduction

Let (M,h) be an oriented *Einstein* manifold of dimension four, i.e., its Ricci tensor Ric_h is proportional to the metric h: in this case, the curvature information, apart from the value of the scalar curvature, is all encoded in the *self-dual* and *anti-self-dual Weyl tensors*, which can be regarded as linear, symmetric, trace-free endomorphisms

$$W_h^{\pm} : \Lambda^{\pm} \longrightarrow \Lambda^{\pm}$$

of the bundles $\Lambda^{\pm}$, arising as eigenspaces of the Hodge operator acting on the bundle Λ^2 of 2-forms. Throughout the paper, we assume that every manifold we consider is connected.

We focus on the symmetric operator W_h^+ and we denote its eigenvalues by

$$\alpha_h \geq \beta_h \geq \gamma_h, \quad \alpha_h + \beta_h + \gamma_h = 0.$$

The main aim of this paper is to show that a rather weak assumption on the two largest eigenvalues α_h and β_h actually implies strong rigidity for (M,h), connecting the conformal properties of h with the existence of Kähler

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structures on M. More precisely, we show that, after possibly passing to a double cover of M (see Section 2 for a more complete explanation),

$$\alpha_h > \beta_h \quad \text{on } M \quad \Longrightarrow \quad \beta_h = \gamma_h = -\frac{1}{2}\alpha_h \quad \text{on } M.$$

A well-known result due to Derdziński [16, Proposition 5] implies that the conformal metric $g = \alpha_h^{2/3} h$ is *Kähler with positive scalar curvature* and, as a consequence, h has to be a Hermitian Einstein metric with positive scalar curvature.

More precisely, we prove the following.

Theorem 1.1. *Let (M, h) be a compact oriented Einstein four-manifold. Assume that*

$$\alpha_h > \beta_h \quad \text{on } M.$$

Then, after at worst passing to a double cover of M, h is conformal to a Kähler metric with positive scalar curvature. In particular, h is a Hermitian Einstein metric with positive scalar curvature.

This result gains importance when viewed in the context of classifying compact Einstein four-manifolds with positive scalar curvature: indeed, LeBrun [33, 34] proved that, in the Hermitian, simply connected setting, (M, h) is either Kähler–Einstein or isometric to the first del Pezzo surface $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ endowed with the Page metric [47], or to the second del Pezzo surface $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$ endowed with the Chen–LeBrun–Weber metric [14]. In the Kähler–Einstein case, by the classification and existence results of Tian [54] and Odaka–Spotti–Sun [45], the underlying complex surface (M, J) is biholomorphic to $\mathbb{CP}^2$, $\mathbb{CP}^1 \times \mathbb{CP}^1$ or $\mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}$, $3 \leq k \leq 8$, with the blown-up points in general position. As a consequence, Theorem 1.1 provides the classification of all compact, simply connected Einstein four-manifolds (M, h) with positive scalar curvature and simple largest eigenvalue of W_h^+ (see Corollary 1.6 for a more general result).

We point out that, in this context, the proof of Theorem 1.1 only relies on h having *harmonic self-dual Weyl curvature*, i.e.

$$\delta_h W_h^+ \equiv 0,$$

where δ_h denotes the divergence operator: this condition is satisfied by all Einstein four-manifolds [6]. Theorem 1.1 is a direct consequence of the following result.

Theorem 1.2. *Let (M, h) be a compact oriented Riemannian four-manifold with*

$$\delta_h W_h^+ = 0 \quad \text{on } M.$$

Assume that

$$\alpha_h > \beta_h \quad \text{on } M.$$

Then, after at worst passing to a double cover of M, h is conformal to a Kähler metric with positive scalar curvature.

Conversely, if (M, g, J) is a compact Kähler surface, with respect to the complex orientation, and with $s_g > 0$, then $h = s_g^{-2} g$ satisfies $\delta_h W_h^+ = 0$ and $\alpha_h > \beta_h$.

Given a Riemannian four-manifold (M, h), on the set $\{W_h^+ \neq 0\}$ we can define the conformally invariant ratio

$$\rho = \frac{\beta_h}{\alpha_h} :$$

this is a continuous function which satisfies

$$-\frac{1}{2} \leq \rho \leq 1,$$

since W_h^+ is trace-free. One can immediately observe that

$$\begin{aligned} \rho = -\frac{1}{2} \quad &\iff \quad \beta_h = \gamma_h = -\frac{1}{2}\alpha_h, \\ \rho = 1 \quad &\iff \quad \alpha_h = \beta_h, \quad \gamma_h = -2\alpha_h. \end{aligned}$$

A simple computation shows that

$$\frac{\det(W_h^+)}{|W_h^+|^3} = -\frac{\rho(1+\rho)}{2^{3/2}(1+\rho+\rho^2)^{3/2}} :$$

the right-hand side is non-increasing on $[-1/2, 1]$ and attains its minimum at $\rho = 1$, i.e.

$$\frac{\det(W_h^+)}{|W_h^+|^3} = -\frac{1}{3\sqrt{6}} \quad \iff \quad \alpha_h = \beta_h > 0.$$

With this in mind, Theorem 1.2 can be rephrased as

$$\rho < 1 \quad \text{on } M \quad \Longrightarrow \quad \rho = -\frac{1}{2} \quad \text{on } M. \tag{1.1}$$

This result improves upon the previously known results: indeed, in his seminal paper, Derdziński proved the existence of a conformal Kähler metric on $\{W_h^+ \neq 0\}$ under the hypothesis $\#\operatorname{spec}(W_h^+) \leq 2$, i.e. $\rho \in \{-1/2, 1\}$ [16]. More recently, Wu improved this result by assuming that $\det(W_h^+) > 0$, i.e. $\rho < 0$ [56], and LeBrun showed that it is sufficient to assume

$$\frac{\det(W_h^+)}{\left|W_h^+\right|^3} \geq -\frac{5\sqrt{2}}{21\sqrt{21}},$$

i.e. $\rho \leq 1/4$ [36]. By (1.1) and the previous considerations, Theorem 1.2 gives the following consequence.

Corollary 1.3. *Let (M, h) be a compact oriented Riemannian four-manifold with $\delta_h W_h^+ = 0$. Assume that*

$$\det(W_h^+) > -\frac{1}{3\sqrt{6}}|W_h^+|^3 \quad \textit{on } M.$$

Then, after at worst passing to a double cover of M, h is conformal to a Kähler metric with positive scalar curvature.

The proofs build upon LeBrun's conformal method [36]: indeed, we make use of the fact that, up to passing to the double cover of M determined by the eigenspace of W_h^+ associated to α_h, the smooth, real line subbundle $L \subset \Lambda^+$ defined by this eigenspace is trivial and, therefore, we can exploit LeBrun's conformal normalization

$$g = \alpha_h^{2/3} h.$$

Together with the weighted version of the well-known Weitzenböck formula for metrics with harmonic self-dual Weyl curvature discovered by Derdziński, we obtain the crucial integral identity (3.5). The new ingredient is a system of first-order identities for the eigenvalues of W_g^+ on its regular spectral set (Proposition 2.3); these equations imply (3.5) and therefore force an eigenform ω_1, associated to α_g, to be parallel and hence to be the Kähler form of g. We point out that this approach comes from merging the ideas of Derdziński [16] and LeBrun [36]: indeed, we make use of Derdziński's formulas for the derivatives of the eigenvalues of W_h^+, applied to the weighted self-dual Weyl curvature $\alpha_h^{-1/3} W_g^+$.

For Einstein metrics, we may allow W_h^+ to vanish *a priori* by replacing the pointwise simplicity hypothesis with a uniform simplicity assumption on the largest eigenvalue. More precisely, the following results extend Theorem 1.2 and Corollary 1.3:

Theorem 1.4. *Let (M, h) be a compact oriented Einstein four-manifold. Assume that $W_h^+ \not\equiv 0$ and that, for some constant $\rho_0 < 1$,*

$$\beta_h \leq \rho_0 \alpha_h \quad \text{on } M.$$

Then, after at worst passing to a double cover of M, h is conformal to a Kähler metric with positive scalar curvature. In particular, h is a Hermitian Einstein metric with positive scalar curvature.

Since M is compact, obviously, if $\alpha_h > \beta_h$ everywhere, the eigenvalues satisfy the hypothesis of Theorem 1.4. We can rephrase this result in terms of $\det(W_h^+)$ as well:

Corollary 1.5. *Let (M, h) be a compact oriented Einstein four-manifold. Assume that $W_h^+ \not\equiv 0$ and that*

$$\det(W_h^+) \geq -c|W_h^+|^3 \quad \text{on } M$$

for some constant

$$0 \leq c < \frac{1}{3\sqrt{6}}.$$

Then, after at worst passing to a double cover of M, h is conformal to a Kähler metric with positive scalar curvature. In particular, h is a Hermitian Einstein metric with positive scalar curvature.

Combining Theorem 1.4 with the classification of anti-self-dual Einstein four-manifolds with positive scalar curvature [20, 28] and the classification of

Hermitian Einstein four-manifolds cited above, we can classify all compact Einstein manifolds with positive scalar curvature and uniformly simple largest eigenvalue.

Corollary 1.6. *Let (M, h) be a compact simply connected oriented Einstein four-manifold with positive scalar curvature. Assume that*

$$\beta_h \leq \rho_0 \alpha_h \quad \text{on } M$$

for some $\rho_0 < 1$. Then, one of the following holds:

- *(M, h) is, up to homothety and isometry, either $\mathbb{S}^4$ or $\overline{\mathbb{CP}^2}$, endowed with their standard metrics;*
- *h is Kähler–Einstein, and the underlying complex surface is biholomorphic to*
$$\mathbb{CP}^2, \quad \mathbb{CP}^1 \times \mathbb{CP}^1, \quad \text{or} \quad \mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}, \quad 3 \leq k \leq 8,$$
with the blown-up points in general position;
- *h is Hermitian non-Kähler and is, up to homothety and isometry, the Page metric on $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ or the Chen–LeBrun–Weber metric on $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$.*

The idea to prove these results is to apply the same computations used in Theorem 1.2, combined with a zero-capacity argument for the zero set of W_h^+. More precisely, we use the Einstein condition to show that W_h^+ is a non-trivial solution of a system of first-order elliptic PDEs and we exploit a known result due to Bär to control the size of the zero set of W_h^+ [4]: then, we construct a suitable sequence of Lipschitz functions χ_j, which are compactly supported in $\{W_h^+ \neq 0\}$ and allow us to deal with the boundary terms arising in the integration by parts procedure in $\{W_h^+ \neq 0\}$, thanks to which we obtain again the identity (3.5) on this open subset. At this point, it is sufficient to use Derdziński's celebrated result on the zeros of W_h^+ on Einstein four-manifolds to show that, if $W_h^+ \not\equiv 0$, then W_h^+ nowhere vanishes and this concludes the proof [16, Proposition 5 (iv)].

With these results in mind, taking the contrapositive of Theorem 1.4, we can obtain a useful restriction on Einstein metrics of nonpositive scalar curvature.

Corollary 1.7. *Let (M, h) be a compact oriented Einstein four-manifold with nonpositive scalar curvature. If $W_h^+ \not\equiv 0$, then*

$$\sup_{\{W_h^+ \neq 0\}} \frac{\beta_h}{\alpha_h} = 1.$$

We highlight the fact that the spectral hypotheses in Theorems 1.1 and 1.4 are sharp: indeed, we manage to show that some K3 surfaces, endowed with a Calabi–Yau metric, admit points at which, after reversing the hyperkähler orientation, we have

$$\alpha_h = \beta_h > 0;$$

since these spaces do not admit metrics with positive scalar curvature, we obtain that the conclusion of Theorem 1.4 fails.

The same method can be adapted to study compact Kähler–Einstein surfaces (M,g,J) with nonpositive scalar curvature. In this setting, we can analyze the holomorphic sectional curvature H and obtain a sharp classification result, under the same spectral hypothesis as in Theorem 1.4: more precisely, if we denote by $H_{\min}$ and $H_{\max}$ the minimum and the maximum of the holomorphic sectional curvature, respectively, and by H_{av} its average over $\mathbb{CP}^1$, we obtain the following result.

Theorem 1.8. *Let (M,h,J) be a compact Kähler–Einstein surface with nonpositive scalar curvature. Assume that there exists a constant $\theta<2/3$ such that, at every point where $H_{\max}>H_{\min}$,*

$$H_{\mathrm{av}}-H_{\min}\leq\theta(H_{\max}-H_{\min}).$$

Then $W_h^-\equiv 0$. Consequently, either h is flat and M is finitely covered by a flat complex torus, or h has constant negative holomorphic sectional curvature and M is a compact quotient of the complex ball.

At every point $p\in M$ such that $H_{\max}>H_{\min}$, the ratio

$$\Theta:=\frac{H_{\mathrm{av}}-H_{\min}}{H_{\max}-H_{\min}}$$

lies in $[1/3,2/3]$: thus, Theorem 1.8 only assumes a uniform gap below the upper bound $2/3$. Siu and Yang obtained the same rigidity theorem under a nonpositive bisectional curvature assumption and imposing $\Theta<\frac{2}{3(1+\sqrt{6/11})}$ [51], while Guan proved the same conclusion under the condition $\Theta\leq 1/2$ without a sign assumption on the bisectional curvature [25]. Our statement requires no sign assumption on the sectional or the bisectional curvature and covers every uniform bound strictly below $2/3$. We point out that Theorem 1.8 is optimal: indeed, recently, compact negatively curved Kähler–Einstein manifolds which are not covered by the complex ball have been constructed *via* gluing and deformation methods [26], which means that the value $2/3$ cannot be included.

The combination of Derdziński's Weitzenböck formula, LeBrun's conformal method and the zero-capacity argument (Proposition 4.1) can be, to some extent, also applied to noncompact Einstein four-manifolds: in particular, in the presence of suitable volume growth and curvature decay assumptions at infinity, we can adapt our line of reasoning to the complete case. It is well known that the study of the eigenvalues of W_h^+ is closely related to the classification of the so-called *gravitational instantons*, i.e. complete, noncompact four-manifolds admitting a Ricci-flat metric such that the L^2-norm of the curvature tensor is finite: indeed, these spaces can be distinguished into three different types, which correspond to different multiplicities of the eigenvalues of W_h^+ (see e.g. the recent survey [39] and the references therein). In our setting, we are able to prove the following general result for noncompact Ricci-flat manifolds:

Theorem 1.9. *Let (M, h) be a complete, oriented, noncompact Ricci-flat four-manifold. Assume that $W_h^+ \not\equiv 0$ and that, for some $\rho_0 < 1$,*

$$\beta_h \leq \rho_0 \alpha_h \quad \text{on } M. \tag{1.2}$$

Suppose moreover that there exist $\nu, q \geq 0$ and $p_0 \in M$ such that

$$\begin{cases} \mathrm{Vol}_h(B_h(p_0, R)) = O(R^\nu) \\ |\,\mathrm{Rm}_h\,| = O(R^{-q}) \quad \text{on } M \setminus B_h(p_0, R) \end{cases} \quad \text{as } R \to \infty, \quad \nu - 2 - \frac{2q}{3} < 0. \tag{1.3}$$

Then, after at worst passing to a double cover of M, h is conformal to a Kähler metric with positive scalar curvature. In particular, h is Ricci-flat, Hermitian and non-Kähler.

As far as gravitational instantons are concerned, we can obtain two rigidity results. The first is a direct consequence of Theorem 1.9: indeed, recalling the celebrated work of Bando, Kasue and Nakajima [3], we show that (M, h) is a Type II gravitational instanton (i.e. Hermitian non-Kähler) having an asymptotically locally Euclidean (ALE) end, under a non-collapsing hypothesis on the volume of geodesic balls (see Corollary 6.1). Furthermore, we also apply the zero-capacity argument to prove a classification result for collapsed gravitational instantons, under the same hypothesis as in Theorem 1.4, obtaining the examples found by Biquard and Gauduchon [7], recently classified as the only collapsed Type II instantons with an asymptotically locally flat (ALF) end by Li [38] (Theorem 6.3). We highlight the fact that, as explained later (see Remark 6.4), this result can be seen as a direct consequence of Theorem 1.9 and a celebrated result due to Cheeger and Tian [12]: however, the proof we included here only relies on the zero-capacity argument and Hölder's inequality. Similar techniques involving the use of the weighted Weitzenböck formula (2.9) (but without the zero-capacity argument) are also exploited in [8].

As for the compact case, the uniform bound $\beta_h \leq \rho_0 \alpha_h$, with $\rho_0 < 1$, is optimal: indeed, we show that there exist *multicentered Gibbons–Hawking gravitational instantons* (M, h) [22, 23] with tetrahedral symmetry which, if we consider the orientation opposite to the one induced by the hyperkähler structure, admit points where $\alpha_h = \beta_h$ and for which the conclusion of Theorem 1.9 does not hold. In this case, we also describe an explicit loss of continuity phenomenon for ρ around a point where $W_h^+ = 0$.

The paper is organized as follows. In Section 2, we provide some preliminaries and describe LeBrun's conformal normalization: furthermore, we prove the weighted Derdziński identities and the crucial first-order identities (2.5). In Section 3, we prove Theorem 1.2 (and, therefore, Theorem 1.1) and Corollary 1.3. Section 4 deals with the zero set of W_h^+: here we prove Theorem 1.4 and its consequences (Corollary 1.5, Corollary 1.7). Section 5 is devoted to the proof of Theorem 1.8. Section 6 treats complete Ricci-flat manifolds and gravitational instantons: here we prove Theorem 1.9 and the classification results for noncollapsed (Corollary 6.1) and collapsed (Theorem 6.3) gravitational instantons. Finally, in Section 7 we construct the compact and noncompact examples for which the largest eigenvalue has multiplicity two, which show the optimality of our results.

Remark 1.10. We point out that, given any oriented Riemannian four-manifold (M, h), reversing the orientation causes the subbundles Λ^+ and Λ^- to swap, which implies that the operators W_h^+ and W_h^- are exchanged. Hence, all the results stated for W_h^+ can also be written in terms of W_h^-: the only important caveat is that, if we choose to work with the anti-self-dual Weyl curvature W_h^-, the conformal metric $g = \alpha_h^{2/3} h$ will be Kähler with respect to the *opposite* orientation (here α_h is the largest eigenvalue of W_h^- with respect to the previous orientation).

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2. LeBrun's conformal change

In this section, we adapt a strategy due to LeBrun [36] based on a specific conformal change of the metric h. First, we recall some well-known facts about four-dimensional Riemannian geometry (see e.g. [6] for more detailed discussions). Let (M, h) be a connected, oriented Riemannian four-manifold: the action of the Hodge operator $* = *_h$ on the bundle of 2-forms Λ^2 induces the conformally invariant splitting $\Lambda^2 = \Lambda_h^+ \oplus \Lambda_h^-$, where, at every point, $\Lambda_h^{\pm}$ is the eigenspace of $*$ associated to the eigenvalue ± 1. We call $\Lambda_h^{\pm}$ the subbundles of *self-dual* and *anti-self-dual* forms, respectively, and we say that a 2-form ω is self-dual (respectively, anti-self-dual) if $*\omega = \omega$ (respectively, if $*\omega = -\omega$): by conformal invariance of the Hodge operator on 2-forms, we know that the subbundles $\Lambda_h^{\pm}$ are conformally invariant, hence, throughout the paper, we will write $\Lambda^{\pm} = \Lambda_{\overline{h}}^{\pm}$ for every $\overline{h} \in [h]$. Since the Weyl tensor W_h preserves the subbundles, this decomposition also induces a splitting of W_h into a self-dual and an anti-self-dual part, denoted respectively by W_h^+ and W_h^-. Both can be regarded as linear, trace-free, symmetric endomorphisms of the subbundles $\Lambda^{\pm}$ and one has $*W_h^{\pm} = \pm W_h^{\pm}$: equivalently, W_h^+ and W_h^- may be viewed as a self-dual and an anti-self-dual $\Lambda^{\pm}$-valued 2-form, respectively.

We say that (M, h) has *harmonic Weyl curvature* if $\delta_h W_h = 0$ on M, where δ_h denotes the divergence operator: since M is four-dimensional, the splitting of W_h allows us to define metrics with harmonic (anti-)self-dual Weyl curvature, i.e. metrics satisfying $\delta_h W_h^{\pm} = 0$ on M. The harmonic Weyl condition enjoys a weighted conformal covariance property (see also [36, 48]):

Lemma 2.1. *Let (M, h) be a Riemannian four-manifold and let $g = f^{-2}h$, for some smooth positive function f on M. Then,*

$$\delta_g(fW_g) = f(\delta_h W_h).$$

In particular, we have that

$$\delta_h W_h = 0 \quad \Longleftrightarrow \quad \delta_g(fW_g) = 0; \tag{2.1}$$

furthermore, (2.1) holds independently for $W_h^{\pm}$.

Proof. First, let $g = e^{2u}h$, $u \in C^\infty(M)$: by the conformal invariance of the $(1,3)$ version of the Weyl tensor, we obtain (see also [11])

$$\delta_g W_g = \delta_h W_h + W_h(\nabla^h u, \cdot, \cdot, \cdot).$$

An easy computation then shows that, for every smooth function f on M,

$$\delta_g(fW_g) = W_h(\nabla^h f, \cdot, \cdot, \cdot) + f[\delta_h W_h + W_h(\nabla^h u, \cdot, \cdot, \cdot)].$$

Now, putting $f = e^{-u}$, we immediately obtain

$$\begin{aligned}\delta_g(fW_g) =& W_h(\nabla^h f, \cdot, \cdot, \cdot) + f\left[\delta_h W_h - \frac{1}{f} W_h(\nabla^h f, \cdot, \cdot, \cdot)\right] \\ =& f\delta_h W_h,\end{aligned}$$

which proves the claims. The same holds for $W_h^{\pm}$ since the Hodge operator commutes with the covariant derivative induced by the Levi-Civita connection. □

Remark 2.2. One can actually prove a more general result: if (M, h) is a Riemannian manifold of dimension $n \geq 4$, then $\delta_g(f^{n-3}W_g) = f^{n-3}(\delta_h W_h)$, where $g = f^{-2}h$.

From now on, we focus on W_h^+: since, at every point $p \in M$, W_h^+ is a symmetric, linear map from $(\Lambda^+)_p$ to itself, we can define its eigenvalues

$$\alpha_h \geq \beta_h \geq \gamma_h.$$

Let $U \subset M$ be an open set on which $\alpha_h > \beta_h$: then, α_h is a smooth positive function on U. We set

$$f = \alpha_h^{-1/3}, \quad g = f^{-2}h = \alpha_h^{2/3}h.$$

By the conformal invariance of $*$ and W_h^+, viewed as a $(1,3)$-tensor, the eigenvalues of $W_g^+ : \Lambda^+ \to \Lambda^+$ are given by

$$\alpha = f^2\alpha_h, \quad \beta = f^2\beta_h, \quad \gamma = f^2\gamma_h.$$

Consequently,

$$f\alpha = 1, \quad \frac{\beta}{\alpha} = \frac{\beta_h}{\alpha_h} = \rho, \quad \gamma = -(1+\rho)\alpha.$$

All the geometric quantities below are computed with respect to g, unless explicitly stated otherwise. We assume that h has harmonic self-dual Weyl curvature,

$$\delta_h W^+_h = 0$$

and we introduce

$$\mathcal{W}^+ = f W^+,$$

whose eigenvalues are

$$\mu_1 = 1, \quad \mu_2 = \rho, \quad \mu_3 = -(1+\rho);$$

in particular, the largest eigenvalue μ_1 is constant. Moreover, by (2.1),

$$\delta \mathcal{W}^+ = 0.$$

Since $\alpha_h > \beta_h$ on U, the eigenspace of W^+_h associated to α_h varies smoothly on U, hence determining a smooth, real line bundle

$$L \subset \Lambda^+|_U$$

over U. As explained in [36, Section 2], L need not to be orientable; however, its sphere bundle

$$\widehat{U} := \{\omega \in L : |\omega|^2 = 2\}$$

is a double cover of U and the pullback of L to $\widehat{U}$ has a global smooth section ω_1 such that

$$|\omega_1|^2 = 2, \quad W^+(\omega_1) = \alpha\omega_1;$$

Therefore, up to passing to this double cover, we can always assume that L is orientable and, hence, trivial; for notational simplicity, whenever we pass to such a cover we use the same symbols for the pulled-back tensors and metrics. The normalization $|\omega_1|^2 = 2$ is chosen since, given any point p, a self-dual form $\eta \in (\Lambda^+)_p$ with $|\eta|^2 = 2$ corresponds to an almost complex structure J, compatible with g and the orientation, on T_pM such that $\eta = g(J\cdot, \cdot)$.

Following Derdziński [16], let M_W denote the regular spectral set of W^+_g, namely the set of points at which the number of distinct eigenvalues is locally constant. Let $M^U_W := M_W \cap U$. Since $\alpha > \beta$ on U, if

$$Z := \{x \in U : \beta(x) = \gamma(x)\},$$

then

$$M^U_W = (U \setminus Z) \cup \operatorname{int}_U Z = U \setminus \partial_U Z.$$

Observe that Z is closed in U and its boundary has empty interior; thus M^U_W is open and dense in U. Locally, we can complete the section ω_1 to an oriented orthogonal frame $\{\omega_1, \omega_2, \omega_3\}$ of Λ^+, which we can express as

$$\omega_1 = e^{12} + e^{34}, \quad \omega_2 = e^{13} - e^{24}, \quad \omega_3 = e^{14} + e^{23}, \tag{2.2}$$

where $e^{ij} = e^i \wedge e^j$ and $\{e^i\}$ is a local oriented g-orthonormal coframe. Near any point at which $\beta \neq \gamma$, the forms ω_2 and ω_3 are chosen to be smooth eigenforms of W^+, i.e.,

$$W^+(\omega_2) = \beta\omega_2, \quad W^+(\omega_3) = \gamma\omega_3;$$

on the other hand, on $\mathrm{int}_U Z$ the restriction of W^+ to $L^\perp$ is a multiple of the identity, so any smooth oriented orthogonal frame of $L^\perp$ is an eigenframe. This observation allows us to perform all the following local computations on M_W^U. Since $|\omega_1|$ is constant, there are locally defined one-forms a and c [16] such that

$$\nabla\omega_1 = a \otimes \omega_2 - c \otimes \omega_3,$$

where

$$a = \sum_{i=1}^{4} a_i e^i, \quad c = \sum_{i=1}^{4} c_i e^i.$$

We note that, on M_W^U, the completion of ω_1 to an orthogonal basis defined by (2.2) also provides the equations for the covariant derivatives of ω_2 and ω_3, which yield the system

$$\begin{cases} \nabla\omega_1 = a \otimes \omega_2 - c \otimes \omega_3, \\ \nabla\omega_2 = -a \otimes \omega_1 + b \otimes \omega_3 \\ \nabla\omega_3 = c \otimes \omega_1 - b \otimes \omega_2, \end{cases} \tag{2.3}$$

where b is a one-form locally defined by $b = \sum_{i=1}^4 b_i e^i$ (see [16, Equation (32)]).

We now prove new fundamental first-order identities on the coefficients of ω_i, in the spirit of Derdziński's local computations [16].

Proposition 2.3. *On M_W^U, one has*

$$(1-\rho)\iota_{a^\sharp}\omega_3 + (2+\rho)\iota_{c^\sharp}\omega_2 = 0. \tag{2.4}$$

Equivalently, if

$$k = \frac{1-\rho}{2+\rho},$$

then

$$c_2 = ka_1, \quad c_1 = -ka_2, \quad c_4 = ka_3, \quad c_3 = -ka_4. \tag{2.5}$$

Proof. Fix $p \in M_W^U$ and a neighborhood V on which the eigenframe $\{\omega_1, \omega_2, \omega_3\}$ is smooth. Let

$$A = \frac{1}{2}\sum_{r=1}^{3} \mu_r\, \omega_r \otimes \omega_r$$

be a divergence-free section of $S_0^2\Lambda^+$ which is diagonal in this frame, i.e.

$$A(\omega_r) = \mu_r \omega_r, \quad |\omega_r|^2 = 2.$$

Equivalently,

$$A_{ijkl} = \frac{1}{2}\sum_{r=1}^{3} \mu_r (\omega_r)_{ij}(\omega_r)_{kl}.$$

We compute at a fixed point and choose the tangent frame $e_1, \dots, e_4$ to be geodesic there: we use the fact that, for every $p \in M_W^U$, the operator W^+

can be smoothly diagonalized in an open neighborhood of p. The condition $\delta A = 0$ is
$$0 = (\delta A)_{jkl} = -\nabla_i A_{ijkl},$$
i.e.

(2.6)
$$0 = \sum_{i=1}^{4}\sum_{r=1}^{3}\left[d\mu_r(e_i)(\omega_r)_{ij}(\omega_r)_{kl} + \mu_r(\nabla_i\omega_r)_{ij}(\omega_r)_{kl} + \mu_r(\omega_r)_{ij}(\nabla_i\omega_r)_{kl}\right].$$

First, we compute (2.6) putting $j = l = 2$, $k = 1$, which yields
$$\begin{aligned}0 &= \sum_{i=1}^{4}\sum_{r=1}^{3}\left[d\mu_r(e_i)(\omega_r)_{i2}(\omega_r)_{12} + \mu_r(\nabla_i\omega_r)_{i2}(\omega_r)_{12} + \mu_r(\omega_r)_{i2}(\nabla_i\omega_r)_{12}\right]\\ &=: I + II + III.\end{aligned}$$

By (2.2) and (2.3), a direct computation gives
$$\begin{aligned}I &= \sum_{i,r} d\mu_r(e_i)(\omega_r)_{i2}(\omega_r)_{12} = d\mu_1(e_1);\\ II &= \sum_{i=1}^{4}\mu_1(\nabla_i\omega_1)_{i2} = \sum_{i=1}^{4}\mu_1(a_i\omega_2 - c_i\omega_3)_{i2} = \mu_1(a_4 + c_3);\\ III &= \mu_1(\nabla_1\omega_1)_{12} + \mu_2(\nabla_4\omega_2)_{12} - \mu_3(\nabla_3\omega_3)_{12}\\ &= \mu_2(-a_4\omega_1 + b_4\omega_3)_{12} - \mu_3(c_3\omega_1 - b_3\omega_2)_{12} = -\mu_2 a_4 - \mu_3 c_3;\end{aligned}$$

this implies that
$$d\mu_1(e_1) = -(\mu_1 - \mu_2)a_4 - (\mu_1 - \mu_3)c_3.$$

Performing analogous computations with $(j, k, l) = (1, 1, 2), (3, 1, 2), (4, 1, 2)$, we also obtain
$$\begin{aligned}d\mu_1(e_2) &= -(\mu_1 - \mu_2)a_3 + (\mu_1 - \mu_3)c_4\\ d\mu_1(e_3) &= (\mu_1 - \mu_2)a_2 + (\mu_1 - \mu_3)c_1\\ d\mu_1(e_4) &= (\mu_1 - \mu_2)a_1 - (\mu_1 - \mu_3)c_2,\end{aligned}$$

i.e.
$$\begin{aligned}d\mu_1 =& [-(\mu_1 - \mu_2)a_4 - (\mu_1 - \mu_3)c_3]e^1 + [-(\mu_1 - \mu_2)a_3 + (\mu_1 - \mu_3)c_4]e^2\\ &+ [(\mu_1 - \mu_2)a_2 + (\mu_1 - \mu_3)c_1]e^3 + [(\mu_1 - \mu_2)a_1 - (\mu_1 - \mu_3)c_2]e^4\\ =& (\mu_1 - \mu_2)(-a_4e^1 - a_3e^2 + a_2e^3 + a_1e^4)\\ &+ (\mu_1 - \mu_3)(-c_3e^1 + c_4e^2 + c_1e^3 - c_2e^4).\end{aligned}$$

Now, one has that

(2.7)
$$\begin{aligned}\iota_{a^\sharp}\omega_3 &= -a_4e^1 - a_3e^2 + a_2e^3 + a_1e^4\\ \iota_{c^\sharp}\omega_2 &= -c_3e^1 + c_4e^2 + c_1e^3 - c_2e^4,\end{aligned}$$

which means that

(2.8)
$$d\mu_1 = (\mu_1 - \mu_2)\iota_{a^\sharp}\omega_3 + (\mu_1 - \mu_3)\iota_{c^\sharp}\omega_2;$$

we point out that this equation also appears in [56, Section 2] and it is a rewriting of Derdziński's first-order identity for the first eigenvalue [16].

We now apply (2.8) to $A = \mathcal{W}^+ = fW^+$. By (2.1), A is divergence-free, and its eigenvalues are

$$\mu_1 = 1, \quad \mu_2 = \rho, \quad \mu_3 = -(1+\rho).$$

Thus $d\mu_1 = 0$, and (2.8) becomes

$$0 = (1-\rho)\iota_{a^\sharp}\omega_3 + (2+\rho)\iota_{c^\sharp}\omega_2,$$

which proves (2.4). Substituting (2.7) into (2.4) and comparing coefficients yields (2.5). □

Remark 2.4. The weight f is crucial for removing the terms involving $d\alpha$ from (2.8). Indeed, $f = \alpha_h^{-1/3}$ is precisely the conformal factor for which $\delta(fW^+) = 0$ and $\mu_1 = 1$.

We shall also use the Weitzenböck formula for divergence-free sections of $S^2_0\Lambda^+$ in the weighted conformal form exploited by LeBrun [36, Equations (8)–(9)], derived from the standard identity for metrics with harmonic self-dual Weyl curvature due to Derdziński [16]. Since $\mathcal{W}^+ = fW^+$ is divergence-free, we have

$$\delta^\nabla \mathcal{W}^+ \equiv 0,$$

where we view $\mathcal{W}^+$ as a Λ^+-valued 2-form, $\delta^\nabla = - * d^\nabla *$ and d^∇ is the exterior covariant derivative induced by the Levi-Civita connection on Λ^+: since $\mathcal{W}^+$ is self-dual, we also obtain

$$0 = *(\delta^\nabla \mathcal{W}^+) = - * (*d^\nabla * (\mathcal{W}^+)) = d^\nabla \mathcal{W}^+,$$

which implies that $\Delta_H \mathcal{W}^+ \equiv 0$, where $\Delta_H = d^\nabla\delta^\nabla + \delta^\nabla d^\nabla$ is the Hodge Laplacian. Thus, by exploiting the classical Weitzenböck formula for trace-free, symmetric bundle endomorphisms (see e.g. [10, Proposition 4.2]), we obtain the well-known

Lemma 2.5. *Let (M, g) be an oriented Riemannian four-manifold, and let f be a positive smooth function such that*

$$\delta_g(fW_g^+) = 0 \quad \textit{on } M.$$

Then

$$0 = \nabla^*\nabla(fW_g^+) + \frac{s_g}{2} fW_g^+ - 6fW_g^+ \circ W_g^+ + 2f\left|W_g^+\right|^2 I, \tag{2.9}$$

where I denotes the identity endomorphism of Λ^+.

3. Proof of Theorem 1.2 and Corollary 1.3

Proof of Theorem 1.2. By the discussion at the beginning of Section 2, we know that the subbundle $L \subset \Lambda^+$ defined by the eigenspace of α_h is smooth and, after passing, if necessary, to a double cover of M, we may assume that L is trivial. As before, we set

$$g = \alpha_h^{2/3} h.$$

From now on, unless specified otherwise, all geometric quantities are computed with respect to g; hence we write

$$\Lambda^+ = \Lambda_g^+, \quad \alpha = \alpha_g, \quad W^+ = W_g^+, \quad s = s_g.$$

Furthermore, we recall that, for every $\eta_1, \eta_2 \in \Lambda^+$,

$$\left\langle W^+(\eta_1), \eta_2 \right\rangle = W^+(\eta_1, \eta_2) = \left\langle W^+, \eta_1 \otimes \eta_2 \right\rangle.$$

Choose a globally defined smooth section ω_1 of L such that

$$W^+(\omega_1) = \alpha\omega_1, \quad |\omega_1|^2 = 2;$$

our first step is to derive a pointwise identity, also appearing implicitly in [36] (see also [35]), using the weighted Weitzenböck formula (2.9).

Lemma 3.1. *At every point of M, we have*

$$\begin{aligned} 0 = {} & \frac{1}{2}|\nabla\omega_1|^2 - \frac{2}{\alpha} W^+(\nabla_e\omega_1, \nabla^e\omega_1) \\ & + \frac{3}{2}\left\langle \omega_1, (d+d^*)^2\omega_1 \right\rangle + 2\left(\frac{2|W^+|^2}{\alpha} - 3\alpha \right). \end{aligned} \tag{3.1}$$

Proof. We claim that, at every $p \in M$,

$$\left\langle \nabla^*\nabla(fW^+), \omega_1 \otimes \omega_1 \right\rangle = 2\left\langle \omega_1, \nabla^*\nabla\omega_1 \right\rangle - \frac{2}{\alpha} W^+(\nabla_e\omega_1, \nabla^e\omega_1). \tag{3.2}$$

We observe that, since $|\omega_1|$ is constant and $fW^+(\omega_1) = \omega_1$,

$$\nabla(fW^+(\omega_1, \omega_1)) = \nabla\left(\left\langle fW^+(\omega_1), \omega_1 \right\rangle \right) = \nabla\left\langle f\alpha\omega_1, \omega_1 \right\rangle = \nabla|\omega_1|^2 = 0;$$

now, recalling that $\nabla^*\nabla = -\nabla_e\nabla^e$, we use the standard formula for covariant derivatives, together with the fact that fW^+ and $\nabla(fW^+)$ are self-adjoint, to obtain

$$\begin{aligned} \left\langle \nabla^*\nabla(fW^+), \omega_1 \otimes \omega_1 \right\rangle = {} & 4\left\langle \nabla_e(fW^+)(\omega_1), \nabla^e\omega_1 \right\rangle \\ & + 2\left\langle fW^+(\nabla_e\omega_1), \nabla^e\omega_1 \right\rangle - 2\left\langle \omega_1, \nabla^*\nabla\omega_1 \right\rangle. \end{aligned} \tag{3.3}$$

We use again the fact that $fW^+(\omega_1) = \omega_1$: if we differentiate this equation, we obtain

$$\nabla_e\omega_1 = \nabla_e(fW^+(\omega_1)) = \nabla_e(fW^+)(\omega_1) + fW^+(\nabla_e\omega_1)$$

and, if we contract the previous identity with $\nabla^e\omega_1$, we get

$$\left\langle \nabla_e(fW^+)(\omega_1), \nabla^e\omega_1 \right\rangle = |\nabla\omega_1|^2 - \left\langle fW^+(\nabla_e\omega_1), \nabla^e\omega_1 \right\rangle.$$

Furthermore, by the usual Bochner formula for smooth tensors,

$$0 = \nabla^*\nabla|\omega_1|^2 = 2\left\langle \omega_1, \nabla^*\nabla\omega_1 \right\rangle - 2|\nabla\omega_1|^2 \quad \Longrightarrow \quad |\nabla\omega_1|^2 = \left\langle \omega_1, \nabla^*\nabla\omega_1 \right\rangle.$$

Substituting these identities into (3.3) and using $f\alpha = 1$ proves the claim. Contracting (2.9) with $\omega_1 \otimes \omega_1$ and using again $f\alpha = 1$, $|\omega_1|^2 = 2$, and (3.2), we obtain

$$\begin{aligned} 0 &= 2\langle \omega_1, \nabla^*\nabla\omega_1\rangle - \frac{2}{\alpha} W^+(\nabla_e\omega_1, \nabla^e\omega_1) + s - 12\alpha + \frac{4}{\alpha}\left|W^+\right|^2 \\ &= \frac{1}{2}|\nabla\omega_1|^2 + \frac{3}{2}\langle \omega_1, \nabla^*\nabla\omega_1\rangle - \frac{2}{\alpha} W^+(\nabla_e\omega_1, \nabla^e\omega_1) + s - 12\alpha + \frac{4}{\alpha}\left|W^+\right|^2. \end{aligned}$$

Finally, by the well-known Weitzenböck formula for self-dual two-forms (see e.g. [36, Equation (7)])

$$(d + d^*)^2\omega_1 = \nabla^*\nabla\omega_1 - 2W^+(\omega_1) + \frac{s}{3}\omega_1,$$

substitution gives (3.1). □

We now integrate (3.1) over M: first, we point out that, since $*\omega_1 = \omega_1$ and $d^*\omega_1 = - * d * \omega_1$, using the properties of the Hodge operator we obtain

$$\int_M \left\langle \omega_1, (d + d^*)^2\omega_1 \right\rangle d\mu_g = 2\int_M |d\omega_1|^2 \, d\mu_g. \tag{3.4}$$

Then, setting again

$$\rho = \frac{\beta}{\alpha} = \frac{\beta_h}{\alpha_h}$$

and using (3.1) and (3.4), together with the fact that $\left|W^+\right|^2 = 2\alpha^2(1+\rho+\rho^2)$, we get the equality

$$\int_M \left[\mathcal{P} + 8\alpha\left(\rho + \frac{1}{2}\right)^2\right] d\mu_g = 0, \tag{3.5}$$

where

$$\mathcal{P} := 3|d\omega_1|^2 + \frac{1}{2}|\nabla\omega_1|^2 - \frac{2}{\alpha}W^+(\nabla_e\omega_1, \nabla^e\omega_1). \tag{3.6}$$

The crucial new point is that the weighted Derdziński relation in Proposition 2.3 allows us to evaluate the indefinite gradient term $\mathcal{P}$ exactly, rather than estimate it. Let $M_W \subset M$ be the regular spectral set for W^+, equivalently for $\mathcal{W}^+ = fW^+$. Since $\alpha > \beta$ and $\alpha > 0$ by hypothesis, the discussion in Section 2 implies that M_W is an open dense subset of M and that Proposition 2.3 holds locally on the whole M_W.

Indeed, we can complete ω_1 to the oriented orthogonal frame $\{\omega_1, \omega_2, \omega_3\}$ of Λ^+ defined in (2.2) locally on M_W: the covariant derivatives of the forms ω_i are then given by (2.3), which immediately implies that

$$|\nabla\omega_1|^2 = 2\left(|a|^2 + |c|^2\right)$$

on M_W. A direct computation gives

$$\begin{aligned} d\omega_1 &= a \wedge \omega_2 - c \wedge \omega_3 \\ &= -(a_2 + c_1)e^{123} + (-a_1 + c_2)e^{124} + (a_4 + c_3)e^{134} + (a_3 - c_4)e^{234}, \end{aligned}$$

where $e^{ijk} = e^i \wedge e^j \wedge e^k$, which implies that

$$|d\omega_1|^2 = (a_1 - c_2)^2 + (a_2 + c_1)^2 + (a_3 - c_4)^2 + (a_4 + c_3)^2.$$

Finally, using the first equation in (2.3) again, we can compute

$$W^+(\nabla_e \omega_1, \nabla^e \omega_1) = 2\left(\beta\,|a|^2 + \gamma\,|c|^2\right);$$

by (2.5), locally on M_W we have

$$\begin{cases} |d\omega_1|^2 = (1-k)^2\,|a|^2\,, \\ |\nabla\omega_1|^2 = 2(1+k^2)\,|a|^2\,, \\ W^+(\nabla_e\omega_1, \nabla^e\omega_1) = 2\alpha\left(\rho - (1+\rho)k^2\right)|a|^2\,, \end{cases} \tag{3.7}$$

where

$$k = \frac{1-\rho}{2+\rho}.$$

Now, we can express $\mathcal{P}$ using (3.7) as

$$\begin{aligned} \mathcal{P} &= [3(1-k)^2 + (1+k^2) - 4(\rho - (1+\rho)k^2)]\,|a|^2 \\ &= [4(1-\rho) - 6k + 4(2+\rho)k^2]\,|a|^2 \\ &= \frac{6(1-\rho)}{2+\rho}\,|a|^2\,. \end{aligned}$$

Moreover, (3.7) gives

$$|\nabla\omega_1|^2 = 2(1+k^2)\,|a|^2 = \frac{2(2\rho^2+2\rho+5)}{(2+\rho)^2}\,|a|^2\,,$$

and hence

$$\mathcal{P} = \mathcal{Q}_\rho\,|\nabla\omega_1|^2\,, \quad \mathcal{Q}_\rho := \frac{3(1-\rho)(2+\rho)}{2\rho^2+2\rho+5}. \tag{3.8}$$

This identity holds on M_W.

By (3.6), $\mathcal{P}$ is smooth on M, since ω_1 is a globally defined smooth eigenform. Moreover, ρ is continuous on M, so $\mathcal{Q}_\rho\,|\nabla\omega_1|^2$ is globally defined and continuous. These two functions coincide on the open dense set M_W; therefore,

$$\mathcal{P} = \mathcal{Q}_\rho\,|\nabla\omega_1|^2 \quad \text{on } M.$$

Consequently, (3.5) becomes

$$\int_M \left[\frac{3(1-\rho)(2+\rho)}{2\rho^2+2\rho+5}\,|\nabla\omega_1|^2 + 8\alpha\left(\rho+\frac{1}{2}\right)^2\right] d\mu_g = 0. \tag{3.9}$$

Since $-\frac{1}{2} \le \rho < 1$, both terms in the integrand are nonnegative and hence vanish identically. Since $\alpha > 0$ and $\rho < 1$ on M, we conclude that

$$\nabla\omega_1 = 0, \quad \rho = -\frac{1}{2} \quad \text{on } M,$$

which means that g is a Kähler metric on M [16, Proposition 5], whose Kähler form is ω_1 and whose self-dual Weyl operator W^+ has eigenvalues given by (see [16, Proposition 2])

$$\alpha = \frac{s}{6}, \quad \beta = \gamma = -\frac{s}{12}. \tag{3.10}$$

Furthermore, since $\alpha > 0$ on M, by (3.10) we obtain that g is a Kähler metric with positive scalar curvature on M.

Conversely, let (M, g, J) be a Kähler surface with $s_g > 0$ and set $h = s_g^{-2} g$. By [16, Proposition 2], $s_g^{-1} W_g^+$ is parallel, and hence $\delta_g(s_g^{-1} W_g^+) = 0$. Then, (2.1) implies $\delta_h W_h^+ = 0$, while (3.10) gives $\alpha_h > 0$ and $\alpha_h > \beta_h$. This concludes the proof of Theorem 1.2. □

Proof of Corollary 1.3. The strict inequality rules out zeros of W_h^+. Hence W_h^+ is nowhere zero and $\alpha_h > 0$, $-1/2 \leq \rho \leq 1$, and $\gamma_h = -(1+\rho)\alpha_h$. Hence, we immediately obtain

$$|W_h^+|^2 = 2\alpha_h^2(1 + \rho + \rho^2), \quad \det(W_h^+) = -\rho(1+\rho)\alpha_h^3,$$

which imply that

$$\frac{\det(W_h^+)}{|W_h^+|^3} = \mathcal{F}(\rho),$$

where

$$\mathcal{F}(\rho) := -\frac{\rho(1+\rho)}{2^{3/2}(1+\rho+\rho^2)^{3/2}}.$$

A direct computation gives

$$\mathcal{F}'(\rho) = \frac{(\rho-1)(\rho+2)\left(\rho+\frac{1}{2}\right)}{2^{3/2}(1+\rho+\rho^2)^{5/2}} \leq 0.$$

Thus, $\mathcal{F}$ is non-increasing on $[-1/2, 1]$, and its minimum is attained at $\rho = 1$, where

$$\mathcal{F}(1) = -\frac{1}{3\sqrt{6}}.$$

Since $\mathcal{F}(\rho) > \mathcal{F}(1)$ by assumption, we have $\rho < 1$ on M, and the conclusion follows from Theorem 1.2. □

4. Compact Einstein manifolds

The main aim of this section is to show how the method used to prove Theorem 1.2 can be adapted to Einstein four-manifolds, without the non-vanishing assumption on W_h^+. We prove that, if W_h^+ is not identically zero and its eigenvalues satisfy

$$\beta_h \leq \rho_0 \alpha_h \quad \text{on } M, \quad \rho_0 < 1, \tag{4.1}$$

then $W_h^+ \neq 0$ at every point of M. The idea is to work on the set

$$X := M \setminus N, \quad N := \{W_h^+ = 0\}$$

and, by constructing suitable cutoff functions satisfying a zero-capacity condition, to prove that $g = \alpha_h^{2/3} h$ is Kähler on X. We then use Derdziński's rigidity result for the zero set of W_h^+ on Einstein four-manifolds [16, Proposition 5(iv)].

We first work under the weaker assumption $\delta_h W_h^+ = 0$: by (4.1), on X we have that $\alpha_h > \beta_h$ and, therefore, $\alpha_h > 0$, which means that, using again Proposition 2.3, we can do the same algebraic computations we performed at the end of the previous section in order to obtain that (3.1) and (3.8) hold on $X \cap M_W$ and, therefore, on X (recall that $X \cap M_W$ is an open dense subset of X). Since X is not a closed set, we have to modify our argument in order to use integration and obtain an analogue of (3.9). We begin with a general result for manifolds with harmonic self-dual Weyl curvature.

Proposition 4.1. *Let (M, h) be an oriented four-manifold, not necessarily complete, such that $\delta_h W_h^+ = 0$ and let*

$$N = \{W_h^+ = 0\}, \quad X = M \setminus N.$$

Assume that $W_h^+ \not\equiv 0$ and that, on X,

$$\beta_h \leq \rho_0 \alpha_h, \quad \rho_0 < 1.$$

Let $g = \alpha_h^{2/3} h$. Suppose that there exist cutoff functions $\chi_j \in \mathrm{Lip}_c(X)$, $0 \leq \chi_j \leq 1$, such that $\chi_j \to 1$ locally uniformly on X and

$$\int_X |d\chi_j|_g^2 \, d\mu_g \longrightarrow 0. \tag{4.2}$$

Then, after at worst passing to a double cover of X, there exists a smooth eigenform ω_1, normalized by $|\omega_1|_g^2 = 2$, such that

$$\nabla^g \omega_1 = 0, \quad \beta_h = \gamma_h = -\frac{1}{2}\alpha_h \quad \text{on } X$$

and g is Kähler with positive scalar curvature on X.

Proof. Since $\delta_h W_h^+ = 0$, by (2.1) we have $\delta_g(\alpha_h^{-1/3} W_g^+) = 0$: hence, since ρ is continuous on X, (3.1) and (3.8) hold on X. Let $L \subset \Lambda^+|_X$ be the line subbundle associated with the eigenspace of α_h. If L is not orientable, we pass to its standard double cover; the capacity condition is unchanged up to the degree of the cover. Hence we may assume that L is orientable and choose a smooth eigenform ω_1 associated to α_h normalized by $|\omega_1|_g^2 = 2$. If the line subbundle L associated to the eigenspace of α_h is not orientable, We recall that, since ω_1 is self-dual,

$$d^*(\chi_j^2 \omega_1) = - * d(\chi_j^2 \omega_1) = \chi_j^2 d^* \omega_1 - 2\chi_j * (d\chi_j \wedge \omega_1)$$

almost everywhere on X. Hence, since $|d^*\omega_1| = |d\omega_1|$ and $|\omega_1|^2 = 2$, multiplying (3.1) by χ_j^2 and integrating by parts we obtain

$$\begin{aligned} \int_X \chi_j^2 \left\langle \omega_1, (d+d^*)^2\omega_1 \right\rangle \, d\mu_g =& 2\int_X \chi_j^2 |d\omega_1|^2 \, d\mu_g \\ &+ 4\int_X \chi_j \left\langle d\chi_j \wedge \omega_1, d\omega_1 \right\rangle \, d\mu_g. \end{aligned} \tag{4.3}$$

Using again $|\omega_1|^2 = 2$ and $|d\omega_1| \le C\,|\nabla\omega_1|$, the Cauchy–Schwarz and Young inequalities give

$$\begin{aligned} \left| \int_X \chi_j \left\langle d\chi_j \wedge \omega_1, d\omega_1 \right\rangle \, d\mu_g \right| &\le C \int_X |\chi_j|\,|d\chi_j|\,|\nabla\omega_1| \, d\mu_g \\ &\le \varepsilon \int_X \chi_j^2\,|\nabla\omega_1|^2 \, d\mu_g + C_\varepsilon \int_X |d\chi_j|^2 \, d\mu_g \end{aligned} \tag{4.4}$$

for every $\varepsilon > 0$ and some $C_\varepsilon > 0$.

Combining (3.1), (3.8), and (4.3), we obtain

$$0 = \int_X \chi_j^2 \left[\mathcal{Q}_\rho |\nabla\omega_1|^2 + 8\alpha \left(\rho + \frac{1}{2}\right)^2 \right] d\mu_g + 6\int_X \chi_j \left\langle d\chi_j \wedge \omega_1, d\omega_1 \right\rangle d\mu_g, \tag{4.5}$$

where $\mathcal{Q}_\rho$ is defined in (3.8): observe that, by the pinching assumption, since $\mathcal{Q}_\rho$ is continuous and positive on $[-1/2, \rho_0]$, there exists a constant $q_0 > 0$, depending only on ρ_0, such that

$$\mathcal{Q}_\rho \ge q_0 > 0 \quad \text{on } X. \tag{4.6}$$

Set

$$I_j := \int_X \chi_j^2 \left[\mathcal{Q}_\rho |\nabla\omega_1|^2 + 8\alpha \left(\rho + \frac{1}{2}\right)^2 \right] d\mu_g.$$

Using (4.4), (4.5) and (4.6), we get

$$I_j \le 6\varepsilon \int_X \chi_j^2 |\nabla\omega_1|^2 \, d\mu_g + 6C_\varepsilon \int_X |d\chi_j|^2 \, d\mu_g \le \frac{6\varepsilon}{q_0} I_j + 6C_\varepsilon \int_X |d\chi_j|^2 \, d\mu_g.$$

Now, taking $\varepsilon > 0$ small enough such that $6\varepsilon/q_0 < 1$, we obtain

$$0 \le I_j \le C \int_X |d\chi_j|^2 \, d\mu_g,$$

for some $C > 0$. By (4.2), $I_j \to 0$ and, since $\chi_j \to 1$ locally uniformly on X and the integrand of I_j is nonnegative, Fatou's lemma yields

$$0 = \int_X \left[\mathcal{Q}_\rho |\nabla\omega_1|^2 + 8\alpha \left(\rho + \frac{1}{2}\right)^2 \right] d\mu_g. \tag{4.7}$$

As in (3.9), both terms in the integrand vanish identically. Hence

$$\nabla^g \omega_1 = 0, \quad \beta_h = \gamma_h = -\frac{1}{2}\alpha_h \quad \text{on } X.$$

Thus g is Kähler on X with $s_g > 0$, which concludes the proof. □

Now, we are ready to prove our main result on compact Einstein four-manifolds.

Proof of Theorem 1.4. We need to construct cutoff functions that satisfy the capacity hypothesis in Proposition 4.1 for the zero set of W_h^+.

If $N = \varnothing$, Proposition 4.1 applies with $\chi_j \equiv 1$ and shows that, after possibly passing to a double cover of M, $g = \alpha_h^{2/3} h$ is Kähler with positive scalar curvature.

Henceforth suppose that $N \neq \varnothing$. The construction of the cutoff functions relies on the fact that, under our hypotheses, the zero set of W_h^+ cannot be too large. Since (M, h) is Einstein, W_h^+ satisfies the second Bianchi identity and it is divergence-free: in particular, since we are assuming that $W_h^+ \not\equiv 0$, W_h^+ is a non-trivial solution of the first-order elliptic system (see e.g. [49])

$$\mathcal{D} W_h^+ = 0,$$

where $\mathcal{D} := d^\nabla + \delta^\nabla$ is the Hodge–Dirac operator on Λ^+-valued 2-forms (see also the discussion before Lemma 2.5). Thus, we can apply a result by Bär [4, Corollary 1] and conclude that N is countably 2-rectifiable and has finite $\mathcal{H}_h^2$-measure: in particular, N has codimension at least 2.

We now exploit the following consequence of the definition of Hausdorff measure. By compactness of N, for every $\delta > 0$, there are finitely many geodesic balls $B_h(p_i, r_i)$, $p_i \in N$, and $0 < r_i < \delta$ such that

$$N \subset \bigcup_i B_h(p_i, r_i), \quad \sum_i r_i^2 \leq C_N, \tag{4.8}$$

where C_N is independent of δ. Indeed, finite Hausdorff measure implies uniformly bounded Hausdorff δ-content for every $\delta > 0$; see, for instance, [19, Section 2.1]. Since W_h^+ is smooth and vanishes on N, α_h is a Lipschitz function in a neighborhood of N: hence, for x sufficiently close to N,

$$\alpha_h(x) \leq C\, d_h(x, N)$$

for some uniform constant $C > 0$. In particular, choosing δ sufficiently small, on each ball $B_h(p_i, 2r_i)$ we have

$$\alpha_h \leq C r_i. \tag{4.9}$$

For every Lipschitz function φ, one has

$$|d\varphi|_g^2 \, d\mu_g = \alpha_h^{2/3} |d\varphi|_h^2 \, d\mu_h \tag{4.10}$$

almost everywhere on X. We want to construct suitable cutoff functions, adapted to the chosen cover of N, in order to apply Proposition 4.1. We choose $0 < \delta < \mathrm{inj}_h(M)/2$ and, for every i, we can define $\theta_i \in C^\infty(M)$ such that

$$0 \leq \theta_i \leq 1, \quad \theta_i = 0 \quad \text{on } B_h(p_i, r_i), \quad \theta_i = 1 \quad \text{on } M \setminus B_h(p_i, 2r_i),$$

with

$$|d\theta_i|_h^2 \leq C r_i^{-2},$$

where C is a constant independent of δ. Since the functions θ_i are smooth, the function

$$\chi_\delta := \min_i \theta_i$$

is Lipschitz on M. It vanishes on an open neighborhood of N, so its restriction belongs to $\mathrm{Lip}_c(X)$. Moreover,

$$0 \le \chi_\delta \le 1, \quad \chi_\delta = 0 \quad \text{on } \bigcup_i B_h(p_i, r_i), \quad \chi_\delta = 1 \quad \text{on } M \setminus \bigcup_i B_h(p_i, 2r_i),$$

and

$$|d\chi_\delta|_h^2 \le C \sum_i r_i^{-2} \mathbf{1}_{B_h(p_i,2r_i)}$$

almost everywhere on X. Let $K \Subset X$. Since $\bigcup_i B_h(p_i, 2r_i) \subset \{x : d_h(x, N) < 2\delta\}$, the set K is disjoint from this union for all sufficiently small δ. Therefore $\chi_\delta \to 1$ uniformly on K. Finally, we need to verify that

$$\int_X |d\chi_\delta|_g^2 \, d\mu_g \longrightarrow 0 \quad \text{as } \delta \downarrow 0. \tag{4.11}$$

After possibly decreasing δ, we use the standard volume bound for small geodesic balls

$$\mathrm{Vol}_h \, B_h(p_i, 2r_i) \le C r_i^4;$$

using (4.8), (4.9) and (4.10), we obtain

$$\begin{aligned} \int_X |d\chi_\delta|_g^2 \, d\mu_g &= \int_X \alpha_h^{2/3} |d\chi_\delta|_h^2 \, d\mu_h \\ &\le C \sum_i r_i^{2/3} r_i^{-2} \, \mathrm{Vol}_h \, B_h(p_i, 2r_i) \le C \sum_i r_i^{8/3} \\ &\le C\delta^{2/3} \sum_i r_i^2 \le C\delta^{2/3}, \end{aligned}$$

which proves (4.11). Therefore, the cutoffs χ_δ satisfy the capacity condition (4.2) and Proposition 4.1 applies. Thus, after at worst passing to a double cover of X,

$$\nabla^g \omega_1 = 0, \quad \beta_h = \gamma_h = -\frac{1}{2}\alpha_h \quad \text{on } X$$

and $g = \alpha_h^{2/3} h$ is a Kähler metric on X. Since $W_h^+ \neq 0$ on X, the eigenvalues of W_g^+ are

$$\alpha = \frac{s_g}{6}, \quad \beta = \gamma = -\frac{s_g}{12}.$$

Since $\alpha > 0$ on X, it follows that $s_g > 0$ on X. Now, since $W_h^+ = 0$ on N and $\beta_h = \gamma_h$ on X, we have that $\#\mathrm{spec}(W_h^+)$, i.e. the number of distinct eigenvalues of W_h^+, is at most 2 on M. Therefore, by [16, Proposition 5(iv)] and the fact that $W_h^+ \not\equiv 0$, we conclude that W_h^+ is nowhere zero, contrary to $N \neq \varnothing$. Hence $N = \varnothing$, and, after at worst passing to a double cover of

M, g is Kähler with positive scalar curvature on M. Since $h = \alpha_h^{-2/3} g$, the conformal transformation law for the scalar curvature gives

$$\mathrm{Vol}_h(M) s_h = \int_M s_h \, d\mu_h = \int_M \left(\frac{2}{3} \alpha_h^{-8/3} |\nabla \alpha_h|_g^2 + s_g \alpha_h^{-2/3} \right) d\mu_g > 0.$$

Since h is Einstein, s_h is constant, and hence positive: thus h is a Hermitian Einstein metric with positive scalar curvature, which concludes the proof. □

Proof of Corollary 1.5. We follow the notation in the proof of Corollary 1.3. By assumption and the continuity and strict monotonicity of $\mathcal{F}$ on the relevant interval, there is a unique $\rho_0 < 1$ such that

$$\mathcal{F}(\rho) \geq -c = \mathcal{F}(\rho_0)$$

at every point where $W_h^+ \neq 0$, where $c < \frac{1}{3\sqrt{6}}$. Hence, by the monotonicity of $\mathcal{F}(\rho)$,

$$\rho \leq \rho_0,$$

i.e.

$$\beta_h \leq \rho_0 \alpha_h$$

on the set $\{W_h^+ \neq 0\}$. Since, on the zero set of W_h^+, the same inequality is automatically true, the conclusion follows from Theorem 1.4. □

Proof of Corollary 1.7. Suppose that

$$\sup_{\{W_h^+ \neq 0\}} \frac{\beta_h}{\alpha_h} = \rho_0 < 1.$$

Then $\beta_h \leq \rho_0 \alpha_h$ on $\{W_h^+ \neq 0\}$, and the same inequality holds trivially on the zero set. Since $W_h^+ \not\equiv 0$, Theorem 1.4 implies that h has positive scalar curvature, contradicting the hypothesis. □

5. Kähler–Einstein surfaces

We are now ready to prove Theorem 1.8. Let (M, h, J) be a Kähler–Einstein surface with the orientation induced by J. At each $p \in M$, denote by H the holomorphic sectional curvature, defined by

$$H(v) = h(R(v, Jv)Jv, v) \tag{5.1}$$

for every unit vector $v \in T_pM$. We denote its pointwise maximum and minimum by $H_{\max}$ and $H_{\min}$, respectively, and its Fubini–Study average over $\mathbb{CP}^1$ by H_{av}. Following the notation of Siu and Yang [51], when $H_{\max} > H_{\min}$, we define the pinching ratio

$$\Theta = \frac{H_{\mathrm{av}} - H_{\min}}{H_{\max} - H_{\min}}.$$

It is well known that, with respect to the complex orientation, the tensor W_h^- of a Kähler–Einstein surface is precisely the *Bochner tensor* B; see, for instance, [9, 53, 55]. As in the previous sections, let

$$\alpha_- \geq \beta_- \geq \gamma_-$$

denote the eigenvalues of W_h^-, and define

$$\rho_- := \frac{\beta_-}{\alpha_-}$$

at every point where $W_h^- \neq 0$. Let ω be the Kähler form, normalized by $|\omega|_h^2 = 2$. For every unit vector $v \in T_pM$, set

$$\sigma_v := v^\flat \wedge (Jv)^\flat, \quad \varphi_v := \sqrt{2}\left(\sigma_v - \frac{1}{2}\omega\right) \in \Lambda_p^-.$$

Then $|\varphi_v|_h = 1$, and the curvature-operator decomposition of the Kähler–Einstein metric h gives

$$H(v) = \langle \mathcal{R}_h(\sigma_v), \sigma_v \rangle_h = \frac{s_h}{6} + \frac{1}{2}\langle W_h^-(\varphi_v), \varphi_v \rangle_h.$$

As v varies over the unit tangent sphere, φ_v ranges over the unit sphere in Λ_p^-. Consequently,

$$H_{\max} = \frac{s_h}{6} + \frac{\alpha_-}{2}, \quad H_{\min} = \frac{s_h}{6} + \frac{\gamma_-}{2}.$$

Furthermore, a well-known result due to Berger [5] implies that

$$H_{\mathrm{av}} = \frac{s_h}{6}.$$

Putting everything together, we can rewrite the pinching ratio Θ as

$$\Theta = \frac{-\gamma_-}{\alpha_- - \gamma_-} = \frac{1+\rho_-}{2+\rho_-}. \tag{5.2}$$

Since $-1/2 \leq \rho_- \leq 1$ and

$$\frac{d\Theta}{d\rho_-} = \frac{1}{(2+\rho_-)^2} > 0,$$

we have

$$\frac{1}{3} \leq \Theta \leq \frac{2}{3}.$$

Moreover,

$$\Theta = \frac{2}{3} \quad \Longleftrightarrow \quad \rho_- = 1 \quad \Longleftrightarrow \quad \alpha_- = \beta_-.$$

Proof of Theorem 1.8. Assume first that $W_h^- \not\equiv 0$: since $\Theta \geq 1/3$, we immediately see that $1/3 \leq \theta < 2/3$. Then, using the pinching hypothesis $\Theta \leq \theta$ together with (5.2), we obtain the uniform spectral gap

$$\frac{\beta_-}{\alpha_-} \leq \rho_0 < 1$$

on $\{W_h^- \neq 0\}$, where $\rho_0 = (2\theta - 1)/(1-\theta)$. After reversing the orientation, W_h^- becomes W_h^+. The uniform bound above then contradicts Corollary 1.7 unless W_h^- vanishes identically in the original complex orientation. Hence $W_h^- \equiv 0$. If h is Ricci-flat, it is therefore flat, and the Bieberbach theorem implies that M is finitely covered by a flat complex torus. We may thus assume that $s_h < 0$. For a Kähler–Einstein surface, $W_h^- = 0$ is equivalent to the vanishing of the Bochner tensor. A constant-scalar-curvature Kähler metric with vanishing Bochner tensor has constant holomorphic sectional curvature [43]. Consequently, the universal cover is the complex hyperbolic plane [30, Chapter IX], and M is a compact quotient of the complex ball. □

6. Noncompact Ricci-flat manifolds

In this section, we show how to extend our results to noncompact Ricci-flat manifolds: in particular, under suitable volume growth and curvature decay assumptions, we show how to adapt the argument of Proposition 4.1.

Proof of Theorem 1.9. Again, we set

$$N = \{W_h^+ = 0\}, \quad X = M \setminus N.$$

As we observed in Section 4, the condition (1.2) implies that $\alpha_h > 0$ and that it is simple on X: hence, we need to verify the capacity condition in Proposition 4.1 for $g = \alpha_h^{2/3} h$. The restriction of the zero set of W_h^+ to each ball $\overline{B_h(p_0, 2R)}$ has finite $\mathcal{H}_h^2$-measure by Bär's Theorem [4]: hence, we can apply the same argument used in the proof of Theorem 1.4 in order to construct cutoff functions $\eta_{\delta,R}$, $0 \le \eta_{\delta,R} \le 1$, which vanish near $N \cap \overline{B_h(p_0, 2R)}$, are equal to 1 away from the zero set and satisfy

$$\int_X |d\eta_{\delta,R}|_g^2 \, d\mu_g \le C_R \delta^{2/3}, \tag{6.1}$$

where C_R is a constant independent of δ. Moreover, for fixed R, $\eta_{\delta,R} \to 1$ locally uniformly on $B_h(p_0, 2R) \cap X$ as $\delta \to 0$. Let ψ_R be a Lipschitz radial cutoff function such that $\psi_R = 1$ on $B_h(p_0, R)$, $\psi_R = 0$ on $M \setminus B_h(p_0, 2R)$ and $|d\psi_R|_h \le CR^{-1}$ almost everywhere. Since h is Ricci-flat, $\mathrm{Rm}_h = W_h$, which implies that $\alpha_h \le C|\mathrm{Rm}_h|$: hence, we obtain

$$\begin{aligned} \int_X |d\psi_R|_g^2 \, d\mu_g &= \int_{B_h(p_0,2R)\setminus B_h(p_0,R)} \alpha_h^{2/3} |d\psi_R|_h^2 \, d\mu_h \\ &\le CR^{-2-2q/3} \operatorname{Vol}_h(B_h(p_0, 2R)) \\ &\le CR^{\nu-2-2q/3} = o(1) \end{aligned} \tag{6.2}$$

as $R \to \infty$. We now define $\chi_{\delta,R} := \eta_{\delta,R}\psi_R$: by (6.1) and (6.2), we obtain

$$\int_X |d\chi_{\delta,R}|_g^2 \, d\mu_g \le C_R \delta^{2/3} + o_R(1),$$

where $o_R(1)$ denotes a quantity that goes to 0 as $R \to +\infty$. We choose a sequence of radii R_j such that $R_j \to +\infty$ as $j \to +\infty$. Furthermore, we can find a sequence $\{\delta_j\}$ such that $\delta_j > 0$ and

$$\delta_j < \frac{1}{R_j}, \quad C_{R_j} \delta_j^{2/3} < \frac{1}{R_j}$$

for every j. Setting $\chi_j := \chi_{\delta_j, R_j} \in \mathrm{Lip}_c(X)$, we immediately see that $\chi_j \to 1$ locally uniformly in X and

$$\int_X |d\chi_j|_g^2 \, d\mu_g \longrightarrow 0.$$

Therefore, Proposition 4.1 applies, and we can conclude that $g = \alpha_h^{2/3} h$ is a Kähler metric with positive scalar curvature on X. Thus, as in the proof of Theorem 1.4, we can use our assumption and [16, Proposition 5 (iv)] to deduce that $N = \varnothing$: hence, after passing, if necessary, to the double cover, g is a Kähler metric with positive scalar curvature on M. This implies that h is a Ricci-flat Hermitian metric; however, h is not a Kähler metric, since otherwise W_h^+ would vanish identically [16, Proposition 2], and this concludes the proof. □

Now, we want to adapt these results to the case of gravitational instantons: following [39], a gravitational instanton is a complete, connected, oriented, noncompact Ricci-flat four-manifold with quadratic curvature decay. Equivalently,

$$\int_M |\operatorname{Rm}_h|^2 \, d\mu_h < \infty;$$

see [39, Remark 2.3], where the equivalence is deduced using the ε-regularity theorem of Cheeger and Tian [12]. In our setting, we recall the three types of gravitational instantons, which can be distinguished by the spectral properties of W_h^+. Namely, we say that an oriented gravitational instanton is of

- *Type I* if $W_h^+ = 0$ (in this case, (M, h) is locally hyperkähler);
- *Type II* if W_h^+ has two distinct eigenvalues at every point;
- *Type III* if W_h^+ has three distinct eigenvalues generically.

Every nonflat gravitational instanton has exactly one end: indeed, if it had at least two ends, the Cheeger–Gromoll splitting theorem would give an isometric product $\mathbb{R} \times N^3$, with N compact, and Ricci-flatness would then force N (and, therefore, M) to be flat.

If (M, h) is globally hyperkähler and simply connected, the asymptotic classification theorem of Sun and Zhang [52] implies that its end is of ALE, ALF, ALG, ALH, ALG*, or ALH* type (see [39, Section 4] and [7, 13, 22, 23, 44]); the ALE cases were classified by Kronheimer [31, 32]. Under a noncollapsing volume assumption, Theorem 1.9 gives the following characterization of the Type II case.

Corollary 6.1. *Let (M, h) be an oriented gravitational instanton with Euclidean volume growth, i.e. such that, for some $v > 0$ and some $p_0 \in M$,*

$$\mathrm{Vol}_h(B_h(p_0, R)) \geq vR^4 \quad \textit{for all } R \gg 1.$$

Assume that $W_h^+ \not\equiv 0$ and that, for some $\rho_0 < 1$,

$$\beta_h \leq \rho_0 \alpha_h \quad \textit{on } M.$$

Then M has one ALE end, and, after at worst passing to a double cover of M, h is conformal to a Kähler metric with positive scalar curvature. In particular, h is a Hermitian Ricci-flat and non-Kähler gravitational instanton of Type II.

Proof. By the classical Bishop–Gromov volume estimate, we obtain that (1.3) holds with $\nu = 4$. Since h is Ricci-flat and $|\operatorname{Rm}_h| \in L^2$, the non-collapsing volume assumption allows us to apply the theorem of Bando, Kasue and Nakajima, which implies that the end has an ALE structure of order 4 [3]; in particular $|\operatorname{Rm}_h| = O(R^{-6})$ on $M \setminus B_h(p_0, R)$. Thus (1.3) holds, and the conclusion follows from Theorem 1.9: the fact that (M, h) is of Type II is an immediate consequence of (3.10). $\square$

Remark 6.2. In the noncollapsed ALE case, Corollary 6.1 can be combined with known classification results. If $W_h^+ = 0$, then, up to a quotient, (M, h) is one of Kronheimer's Type I gravitational instantons [31, 32]. If $W_h^+ \not\equiv 0$ and the group at infinity satisfies $\Gamma \subset \mathrm{SU}(2)$ in the chosen orientation, Li's classification [37, 38] implies that, up to homothety and isometry, h is the Eguchi–Hanson metric on $T^*\mathbb{S}^2$ with the opposite orientation [18].

Li [37, 38] proved that collapsed Hermitian non-Kähler gravitational instantons are toric and of ALF or AF type. Combined with the classification of Biquard and Gauduchon [7], this gives the following list (see [39, Section 5] for a complete description):

- Kerr metrics on $\mathbb{S}^2 \times \mathbb{R}^2$, with respect to both orientations [21];
- Chen–Teo metrics on $(\mathbb{S}^2 \times \mathbb{R}^2)\#\overline{\mathbb{CP}^2}$, where $\overline{\mathbb{CP}^2}$ is the complex projective space with the opposite orientation [15];
- Taub-bolt and anti-Taub-bolt metrics [46];
- Taub–NUT metric on $\mathbb{R}^4$ with the opposite orientation, called anti-Taub–NUT metric.

We recall that, with respect to the standard orientation of $\mathbb{R}^4$, the Taub–NUT metric is hyperkähler and hence is a Type I gravitational instanton. We now prove the following result.

Theorem 6.3. *Let (M, h) be an oriented gravitational instanton with at most cubic volume growth, i.e.*

$$\operatorname{Vol}_h(B_h(p_0, R)) = O(R^3).$$

Assume that $W_h^+ \not\equiv 0$ and that, for some $\rho_0 < 1$,

$$\beta_h \le \rho_0 \alpha_h \quad \text{on } M.$$

Then, after at worst passing to a double cover of M, h is conformal to a Kähler metric with positive scalar curvature. In particular, h is a Hermitian Ricci-flat and non-Kähler gravitational instanton of Type II and belongs to one of the following families:

- *a Kerr metric, with either orientation;*
- *a Chen–Teo metric, with its Type II orientation;*

- *the Taub–bolt metric, with either orientation (Taub–bolt or anti–Taub–bolt);*
- *the reversed Taub–NUT metric.*

Proof. First, we observe that the construction of the functions $\eta_{\delta,R}$ described in the proof of Theorem 1.9 applies without change in this setting: hence, we just need to verify the capacity condition at infinity, in order to apply Proposition 4.1. Let ψ_R be the usual radial cutoff function supported in $B_h(p_0, 2R)$, equal to 1 on $B_h(p_0, R)$, and with $|d\psi_R|_h \leq CR^{-1}$. On $A_R = B_h(p_0, 2R) \setminus B_h(p_0, R)$, we can apply Hölder's inequality and use the fact that $\alpha_h \leq C|\operatorname{Rm}_h|$ in order to obtain

$$\begin{aligned}\int_X |d\psi_R|_g^2 \, d\mu_g &= \int_{A_R} \alpha_h^{2/3} |d\psi_R|_h^2 \, d\mu_h \\ &\leq CR^{-2} \left(\int_{A_R} |\operatorname{Rm}_h|^2 \, d\mu_h \right)^{1/3} \operatorname{Vol}_h(A_R)^{2/3} = o_R(1),\end{aligned}$$

where the last equality comes from the fact that $|\mathrm{Rm}_h| \in L^2$ and the volume assumption. As we did in the proof of Theorem 1.9, we can similarly construct cutoff functions χ_j which satisfy the hypotheses of Proposition 4.1: therefore, after at worst passing to a double cover of X, we conclude that g is a Kähler metric with positive scalar curvature on X and, using again [16, Proposition 5(iv)], that $X = M$ as in Theorem 1.9. The classification now follows from the aforementioned classification result of conformally Kähler gravitational instantons due to Li [38, 39]. □

Remark 6.4. The proof of Theorem 6.3 can be deduced directly from Theorem 1.9 combined with the curvature decay proved in [12]. Moreover, the proof uses only the bound $\operatorname{Vol}_h(B_h(p_0, R)) \leq CR^3$ and therefore also yields a rigidity statement for non-ALF collapsed gravitational instantons. Indeed, Li's classification [38] shows that every Type II gravitational instanton is, up to scaling, of ALE, ALF, or AF type. Consequently, if (M, h) is of ALG, ALG*, ALH, or ALH* type and satisfies (1.2), then it must be of Type I and hence locally hyperkähler.

Remark 6.5. The same capacity proof can also be adapted to isolated quotient-orbifold singularities. In local uniformizing charts the lifted metric is smooth, so isolated singular points have zero capacity for $\alpha_h^{2/3} h$; combining these local cutoffs with the annular cutoff at infinity gives the corresponding orbifold variants. We do not state these extensions here.

7. Sharpness of the spectral condition

In this section, we provide examples of Ricci-flat four-manifolds (M, h) such that, at a point p, $\alpha_h = \beta_h \neq 0$: this shows that the uniform hypothesis on the eigenvalues in Theorems 1.4 and 1.9 is optimal.

7.1. **Compact case: K3 surfaces.** First, we recall that a hyperkähler manifold (M, h) is a Riemannian manifold endowed with three Kähler structures I, J and K, which satisfy the quaternionic relations:

$$I^2 = J^2 = K^2 = IJK = -1.$$

In dimension four, according to Kodaira's classification of complex surfaces, the only examples of compact hyperkähler manifolds are flat tori and K3 surfaces. It is well known that every K3 surface admits a Kähler metric: the resolution of the Calabi conjecture by Yau ensures that, given any Kähler metric $\hat{h}$ on M with Kähler form $\hat{\omega}$, there exists a unique Kähler Ricci-flat metric h with Kähler form $\omega \in [\hat{\omega}]$ [57]. Furthermore, by [16, Proposition 2], we know that all these metrics are also anti-self-dual with respect to the orientation induced by the hyperkähler structure (see also [27]), which means that the curvature operator is precisely W_h^-. Since the metrics constructed by Yau are not explicit, in general, no explicit formula for the curvature tensor is known. Every K3 surface is spin, and

$$\hat{A}(M) = -\frac{\tau(M)}{8} = 2$$

with respect to the hyperkähler orientation. The Lichnerowicz formula therefore implies that a K3 surface does not admit a metric of positive scalar curvature [40]. Thus, after reversing the hyperkähler orientation, a nonflat Ricci-flat K3 metric cannot satisfy the conclusions of Theorems 1.2 and 1.4.

We show that, for certain K3 surfaces,

$$\sup_{\{W_h^+ \neq 0\}} \rho = 1$$

and that there is a point p with $W_h^+(p) \neq 0$ and $\rho(p) = 1$.

We want to look at holomorphic isometries of K3 surfaces: Lye showed that, if a hyperkähler manifold admits a holomorphic isometry of order at least 3 that fixes a subset of positive dimension, then the holomorphic sectional curvatures of I, J and K provide some information on the full Riemann tensor of the manifold [42].

In this setting, given a point $p \in M$, we define the holomorphic sectional curvatures H_{II}, H_{JJ}, H_{KK} and the mixed-type curvatures H_{IJ}, H_{IK}, H_{JK} at p as in (5.1): for instance, for every $v \in T_pM$ of unit norm,

$$H_{II}(v) = h(R(v, Iv)Iv, v), \quad H_{IJ}(v) = h(R(v, Iv)Jv, v)$$

and so on. It is not hard to show that $H_{IJ} = H_{JI}$, $H_{IK} = H_{KI}$ and $H_{JK} = H_{KJ}$; furthermore, since h is Ricci-flat, we have

$$H_{II} + H_{JJ} + H_{KK} = 0.$$

Following [42, Theorem 4.6], if (M, h) admits a holomorphic isometry of order at least 3 which fixes a subset C of M with positive dimension, then

$$H_{IJ}(v) = H_{IK}(v) = H_{JK}(v) = 0, \quad H_{JJ}(v) = H_{KK}(v) = -\frac{1}{2}H_{II}(v) \tag{7.1}$$

for every unit tangent vector $v \in T_pC$, $p \in C$: in particular, this means that all six curvature quantities are uniquely determined by H_{II} along C. Assume that $\dim C = 2$, i.e. C is a surface inside (M, h): since C is fixed by an isometry of h, it is a (possibly non-connected) totally geodesic smooth submanifold with respect to the metric $\iota^*(h)$ induced by h (see e.g. [29]) and, by [42, Corollary 4.8], for every $v \in T_pC$ of unit norm,

$$H_{II}(w) = \Big[(\alpha^2+\beta^2)^2 + (\mu^2+\nu^2)^2 - 4(\alpha^2+\beta^2)(\mu^2+\nu^2)\Big]\,\kappa(p), \tag{7.2}$$

where

$$w = \alpha v + \beta I v + \mu J v + \nu K v, \quad \alpha^2+\beta^2+\mu^2+\nu^2 = 1,$$

and κ is the Gaussian curvature of $(C, \iota^* h)$. Now, let us choose a local orthonormal frame $\{e_i\}$ on an open chart of (M, h) such that e_1 is tangent to C and

$$\begin{aligned} I(e_1) &= e_2, \quad I(e_3) = e_4;\\ J(e_1) &= e_3, \quad J(e_2) = -e_4;\\ K(e_1) &= e_4, \quad K(e_2) = e_3. \end{aligned}$$

Setting $R_{ijkl} = h(R(e_i, e_j)e_l, e_k)$, we can exploit (7.2) with $v = e_1$ and $(\alpha, \beta, \mu, \nu) = (1, 0, 0, 0)$ to obtain, on C,

$$\begin{aligned} R_{1212} &= H_{II}(e_1) = \kappa,\\ R_{1313} &= H_{JJ}(e_1) = -\frac{1}{2}H_{II}(e_1) = -\frac{1}{2}\kappa = R_{1414}; \end{aligned}$$

since h is Ricci-flat and anti-self-dual, we have

$$\begin{aligned} R_{3434} &= -R_{1234} = R_{1212} = \kappa,\\ R_{4242} &= -R_{1342} = R_{1313} = R_{1414} = -R_{1423} = R_{2323} = -\frac{1}{2}\kappa. \end{aligned}$$

Furthermore, by (7.1) and anti-self-duality, we obtain

$$H_{IJ}(e_1) = H_{IK}(e_1) = H_{JK}(e_1) = 0 \Longrightarrow R_{ijik} = 0$$

whenever i, j, k are pairwise distinct. We now reverse the orientation on M: since h is Ricci-flat and self-dual in this orientation, its full curvature operator is W_h^+. A direct computation from the preceding curvature identities gives

$$W_h^+ = \begin{pmatrix} 2\kappa & 0 & 0\\ 0 & -\kappa & 0\\ 0 & 0 & -\kappa \end{pmatrix}. \tag{7.3}$$

If we assume that M is compact, then C is a compact surface with genus g: if $g > 1$, the Gauss–Bonnet Theorem implies that there exist points of C where $\kappa < 0$, which means that $\alpha_h = \beta_h$ at those points.

We observe that, in their classification of non-symplectic holomorphic automorphisms of order 3, Artebani and Sarti [2] proved the existence of K3 surfaces satisfying our hypotheses and such that the fixed point set of the automorphism is a surface of genus $g > 1$, which proves our claim.

Furthermore, if we consider the hyperkähler orientation (i.e. the one for which h is anti-self-dual), by (5.2), (7.3) and the previous discussion, we obtain that there exist points of C at which $\Theta = 2/3$.

Here we also provide an explicit example of an algebraic K3 surface where $\sup_{\{W_h^+\neq 0\}}\rho = 1$. We consider a K3 surface M as a branched double cover of $\mathbb{CP}^2$, defined in the weighted projective space $\mathbb{P}(1,1,1,3)$: namely, using homogeneous coordinates $[x:y:z:w]$, we choose the K3 surface described by the equation

$$w^2 = x^6 + y^6 + z^6,$$

which is branched over the sextic curve $x^6+y^6+z^6=0$ in $\mathbb{CP}^2$. The surface M is a smooth complex submanifold of $\mathbb{P}(1,1,1,3)$. Indeed, the first-order partial derivatives of the defining polynomial do not have a common zero outside the origin of $\mathbb{C}^4$ and the unique singular point $[0:0:0:1]$ of $\mathbb{P}(1,1,1,3)$ does not belong to M.

The weighted projective space $\mathbb{P}(1,1,1,3)$ admits an orbifold Kähler form (see e.g. [50]), whose restriction to M is a smooth Kähler form $\widehat{\omega}$. We define

$$\phi([x:y:z:w]) = [e^{2\pi i/3}x : y : z : w] :$$

it is clear that ϕ is a holomorphic automorphism of M and that $\phi^3 = \mathrm{Id}$. We set

$$\widehat{\omega}_{\mathrm{inv}} := \frac{1}{3}\left(\widehat{\omega} + \phi^*\widehat{\omega} + (\phi^2)^*\widehat{\omega}\right).$$

By construction, $\widehat{\omega}_{\mathrm{inv}}$ is a ϕ-invariant Kähler form on M and, therefore, by Yau's theorem, its Kähler class contains a unique Ricci-flat Kähler form ω [57]. Since ϕ preserves the cohomology class $[\widehat{\omega}_{\mathrm{inv}}]$ and $\omega \in [\widehat{\omega}_{\mathrm{inv}}]$, the form $\phi^*\omega$ lies in the same class and is also Ricci-flat. Consequently, by uniqueness in Yau's Theorem, $\phi^*\omega = \omega$, so ϕ is an isometry of the Ricci-flat Kähler metric h associated with ω.

Now, we look for the fixed point set $\mathrm{Fix}(\phi)$ of ϕ, i.e. for the points $[x:y:z:w]$ such that

$$(e^{\frac{2\pi i}{3}}x, y, z, w) = (\lambda x, \lambda y, \lambda z, \lambda^3 w),$$

for some $\lambda \in \mathbb{C}\setminus\{0\}$. If $\lambda = 1$, a point $p = [x:y:z:w]$ lies in $\mathrm{Fix}(\phi)$ if and only if $x = 0$, equivalently, if and only if $p \in C$, where C is the complex curve defined by the equation $w^2 = y^6 + z^6$. On the other hand, if $\lambda \neq 1$, we immediately get that $\lambda = e^{\frac{2\pi i}{3}}$ and that $y = z = 0$: using the equation which defines M, we obtain the isolated points

$$q_1 = [1:0:0:1], \quad q_2 = [1:0:0:-1],$$

i.e. $\mathrm{Fix}(\phi) = C \cup \{q_1, q_2\}$.

Hence, the fixed point set of ϕ contains a submanifold of (M,h) of dimension 2, which has to be totally geodesic. In order to compute the genus of C, we observe that C can be seen as the double cover of the complex line $\{x = 0\}$, which is a Riemann sphere $\mathbb{CP}^1$, branched over the six roots of the polynomial $P(y,z) = y^6 + z^6$: by the classical Riemann–Hurwitz Theorem, we have

$$2g_C - 2 = d(2g_{\mathbb{CP}^1} - 2) + B = -2d + B,$$

where d is the degree of the cover $C \longrightarrow \mathbb{CP}^1$ and B is the total ramification. Since $d = 2$ and $B = 6$, we have $g_C = 2$. Thus C is a compact totally geodesic surface in (M,h). Since $\chi(C) < 0$, the Gauss–Bonnet theorem gives a point $p \in C$ with $\kappa(p) < 0$. With respect to the hyperkähler orientation, the preceding calculation gives

$$\mathrm{spec}(W_h^-(p)) = \{2\kappa(p), -\kappa(p), -\kappa(p)\}.$$

After reversing the orientation, this becomes the spectrum of $W_h^+(p)$; hence $\alpha_h(p) = \beta_h(p) > 0$ and $\rho(p) = 1$.

We also obtain examples showing that, without a uniform gap, one cannot conclude that W_h^+ is either identically zero or nowhere zero. Certain K3 surfaces with Calabi–Yau metrics admit a holomorphic isometry of order at least 3 fixing an elliptic curve C (see e.g. [2] and the Kummer surface studied in [42]). Since C is totally geodesic and $\chi(C) = 0$, the Gauss–Bonnet theorem implies that its Gaussian curvature vanishes somewhere. After reversing orientation, (7.3) therefore gives a point p with $W_h^+(p) = 0$. On the other hand, $W_h^+ \not\equiv 0$, since a K3 surface admits no flat metric.

7.2. **Noncompact case: the Gibbons–Hawking ansatz.** In the late 1970s, Gibbons and Hawking developed an ansatz for constructing gravitational instantons [22, 23]. Let $U \subset \mathbb{R}^3$ be an open set with $H^2(U;\mathbb{R}) = 0$, endowed with the Euclidean metric h_{Eucl}, and let V be a positive harmonic function on U. Since $*_{\mathbb{R}^3} dV$ is closed, there exists $\omega \in \Omega^1(U)$ such that

$$d\omega = *_{\mathbb{R}^3} dV. \tag{7.4}$$

Equivalently, ω satisfies the Bogomolny monopole equation

$$\nabla^{\mathrm{Eucl}} \times \omega = \nabla^{\mathrm{Eucl}} V.$$

On $M = U \times \mathbb{S}^1$, we can define a Riemannian metric h which can be written as

$$h = V^{-1}(d\tau + \omega)^2 + V h_{\mathrm{Eucl}}, \tag{7.5}$$

where τ is a coordinate along $\mathbb{S}^1$. One has that h is hyperkähler (see e.g. [39, Section 4.1]). Hence, with the orientation induced by the hyperkähler structure, h is a Ricci-flat, anti-self-dual metric on M.

For simplicity, write $x^1 := x$, $x^2 := y$, and $x^3 := z$. We reverse the orientation on M and choose the local orthonormal coframe given by

$$e^0 := V^{-1/2}(d\tau + \omega), \quad e^i = V^{1/2} dx^i,\ i = 1,2,3.$$

We recall that $\{dx^i\}$ is a local orthonormal coframe on $\mathbb{R}^3$, with respect to which the Hodge operator on $\mathbb{R}^3$ can be locally expressed as

$$*_{\mathbb{R}^3}(\eta) = \frac{1}{2}\eta_k \varepsilon_{ijk} dx^i \wedge dx^j,$$

for every 1-form $\eta = \eta_i dx^i$, where ε_{ijk} is the Levi-Civita symbol.

We can find the associated Levi-Civita connection forms $\{\omega^i_j\}$ by means of Cartan's first structure equations

$$de^\alpha = -\omega^\alpha_\beta \wedge e^\beta :$$

by (7.4), a straightforward computation yields (see also [41, Lemma 2.2])

$$\begin{aligned}\omega^0_i &= -\frac{1}{2}V^{-3/2}\left(\partial_i V e^0 + \varepsilon_{ijk}\partial_j V e^k\right)\\ \omega^i_j &= \frac{1}{2}V^{-3/2}\left(\partial_j V e^i - \partial_i V e^j - \varepsilon_{ijk}\partial_k V e^0\right),\end{aligned} \tag{7.6}$$

where $\partial_i V = \frac{\partial}{\partial x^i}V$. It is immediate to observe that

$$\omega^i_j = \varepsilon_{ijk}\omega^0_k, \tag{7.7}$$

for every i, j. Now, we compute the curvature forms using Cartan's second structure equations

$$\frac{1}{2}R^\alpha_{\beta\mu\nu}e^\mu \wedge e^\nu = \Omega^\alpha_\beta = d\omega^\alpha_\beta + \omega^\alpha_\gamma \wedge \omega^\gamma_\beta :$$

by (7.6), (7.7) and the basic properties of the Levi-Civita symbol, it is immediate to show that

$$\Omega^i_j = \varepsilon_{ijk}\Omega^0_k.$$

Using the harmonicity of V and the properties of $*_{\mathbb{R}^3}$, we obtain

$$\Omega^0_i = E_{ij}\left(e^0 \wedge e^j + \frac{1}{2}\varepsilon_{jkl}e^k \wedge e^l\right),$$

where

$$E_{ij} := \frac{1}{2}\left(V^{-2}\partial_i\partial_j V + V^{-3}\left|\nabla V\right|^2\delta_{ij} - 3V^{-3}\partial_i V\partial_j V\right). \tag{7.8}$$

By our choice of orientation, (M, h) is now self-dual and, since h is Ricci-flat, the Riemann curvature operator coincides with W^+_h: choosing the standard orthogonal basis $\{\eta_1, \eta_2, \eta_3\}$ of Λ^+, where

$$\eta_j := e^0 \wedge e^j + \frac{1}{2}\varepsilon_{jkl}e^k \wedge e^l,$$

we obtain

$$\Omega^0_i = E_{ij}\eta_j, \quad \Omega^i_j = \varepsilon_{ijk}E_{kl}\eta_l.$$

Hence, by the definition of the self-dual Weyl operator in Λ^+_h, we can immediately compute

$$W^+_h(\eta_i) = \Omega^0_i + \frac{1}{2}\varepsilon_{ijk}\Omega^j_k = \Omega^0_i + \frac{1}{2}\varepsilon_{ijk}\varepsilon_{jkl}\Omega^0_l = 2\Omega^0_i = 2E_{ij}\eta_j,$$

i.e., viewing W^+_h as a matrix,

$$W^+_h = \frac{1}{V^3}\left[V\,\mathrm{Hess}_{\mathbb{R}^3}(V) + \left|\nabla V\right|^2 h_{\mathrm{Eucl}} - 3dV \otimes dV\right]. \tag{7.9}$$

We now recall the multicentered Gibbons–Hawking construction. Fix distinct points $p_1, \dots, p_N \in \mathbb{R}^3$ and set

$$U = \mathbb{R}^3 \setminus \{p_1, \dots, p_N\}.$$

We can find a smooth, positive harmonic function V in U defined as

$$V(x) := \varepsilon + \sum_{\alpha=1}^{N} \frac{m_\alpha}{|x - p_\alpha|}, \tag{7.10}$$

where $\varepsilon \geq 0$ and m_α is the mass (or charge) at the point p_α.

The product $U \times \mathbb{S}^1$ is replaced by a principal $U(1)$-bundle over U. Assume that all masses are equal to $m > 0$ and normalize τ to have period $4\pi m$. The Chern number over a small sphere surrounding each p_α is then ± 1, so the restriction of the bundle is the Hopf fibration $\mathbb{S}^3 \to \mathbb{S}^2$. After adding one point over each center, where the circle collapses, the Gibbons–Hawking metric extends to a smooth complete hyperkähler metric; see [24, Section 2.3], [41, Example 2.4], and [39, Section 4.1].

Since the bundle is generally nontrivial, the expression $d\tau + \omega$ in (7.5) must be globally replaced by a connection one-form Θ with curvature $\pi^*(*_{\mathbb{R}^3} dV)$. However, locally, one can choose τ and ω so that (7.5) and all the preceding computations remain valid. If $\varepsilon > 0$, (M, h) becomes an ALF manifold (such as the multi-Taub-NUT space), while if $\varepsilon = 0$, we obtain an ALE manifold. In both cases (M, h) is a gravitational instanton: as an example, if $\varepsilon = 0$ and $N = 2$, we can recover the Eguchi–Hanson metric on $T^*\mathbb{S}^2$ [18] (see e.g. [17, 41] for other examples). We highlight the fact that Anderson, Kronheimer and LeBrun generalized this construction to the case of countably many centers p_α [1].

From now on, we set $N = 4$ and assume that $p_1, \dots, p_4$ are vertices of a regular tetrahedron in $\mathbb{R}^3$: namely, for a constant $R > 0$, we define

$$p_1 := \frac{R}{\sqrt{3}}(1,1,1), \qquad p_2 := \frac{R}{\sqrt{3}}(1,-1,-1),$$
$$p_3 := \frac{R}{\sqrt{3}}(-1,1,-1), \qquad p_4 := \frac{R}{\sqrt{3}}(-1,-1,1),$$

and we assume that $m_\alpha = 1$ for every α, for the sake of simplicity. One can easily observe that $|p_\alpha| = R$ for every α and that, by definition of the p_α's, the following identities hold:

$$\sum_{\alpha=1}^{4} p_\alpha = 0, \qquad \sum_{\alpha=1}^{4} p_\alpha \otimes p_\alpha = \frac{4R^2}{3} I. \tag{7.11}$$

First, we observe that there exists at least one point $p \in M$ such that $\left|W_h^+\right|(p) = 0$: indeed, we consider the barycenter $b = (0,0,0)$ of the tetrahedron and we can compute

$$\nabla\left(\frac{1}{|p_\alpha - x|}\right)\Big|_{x=0} = \frac{p_\alpha}{R^3}, \quad \mathrm{Hess}_{\mathbb{R}^3}\left(\frac{1}{|p_\alpha - x|}\right)\Big|_{x=0} = \frac{3p_\alpha \otimes p_\alpha - R^2 I}{R^5}.$$

By (7.10) and (7.11), the previous identities imply

$$\nabla V(0) = \mathrm{Hess}_{\mathbb{R}^3} V(0) = 0, \tag{7.12}$$

which immediately shows that $W_h^+ = 0$ for every point $p \in \pi^{-1}(b)$, by (7.9).

Now, let

$$r := \{t\nu : t \in \mathbb{R}\}, \quad \nu := \frac{1}{\sqrt{3}}(1,1,1)$$

be the line through the barycenter b and p_1, and let $x = t\nu$ with $t < R$. Let $C_3 = \{e, \phi, \phi^2\}$ be the cyclic group generated by ϕ, the rotation of angle $2\pi/3$ around the axis r: it is clear that r is exactly the fixed locus of every isometry in C_3.

We choose a small, C_3-invariant open ball $B \subset U$ centered at b such that $\pi^{-1}(B) \cong B \times \mathbb{S}^1$ and $p_\alpha \notin B$ for every α. Since V is C_3-invariant and the action preserves the Euclidean orientation, $*_{\mathbb{R}^3} dV$ is also C_3-invariant. It is important to observe that (7.4) does not admit a unique solution ω: by defining

$$\omega_{\mathrm{inv}} := \frac{1}{3}\left(\omega + \phi^*\omega + (\phi^2)^*\omega\right),$$

we obtain $d\omega_{\mathrm{inv}} = *_{\mathbb{R}^3} dV$ and that ω_{inv} is C_3-invariant. Renaming this new form ω, we can extend ϕ to a local isometry $\widetilde{\phi}$ of the total space of the $U(1)$-bundle over B, i.e. we can define

$$\widetilde{\phi}(x, \tau) = (\phi(x), \tau), \tag{7.13}$$

which obviously generates a cyclic group of local isometries of the total space, which we call again C_3.

Now, we fix $p \in \pi^{-1}(r \cap B)$: since $\widetilde{\phi}(p) = p$, its differential induces an isometry Φ of $T_p^* M$. Labeling the coordinates cyclically, its action is

$$\begin{aligned} &\Phi(e^0) = e^0, && \Phi(e^1) = e^2, && \Phi(e^2) = e^3, && \Phi(e^3) = e^1, \\ &\Phi^2(e^0) = e^0, && \Phi^2(e^1) = e^3, && \Phi^2(e^2) = e^1, && \Phi^2(e^3) = e^2 : \end{aligned}$$

we point out that the form e^0 is fixed by all isometries since both V and ω are C_3-invariant. Therefore, the natural action on Λ_p^+ induced by Φ satisfies

$$\begin{aligned} &\Phi(\eta_1) = \eta_2, && \Phi(\eta_2) = \eta_3, && \Phi(\eta_3) = \eta_1, \\ &\Phi^2(\eta_1) = \eta_3, && \Phi^2(\eta_2) = \eta_1, && \Phi^2(\eta_3) = \eta_2 : \end{aligned}$$

hence, $C_3' = \{\mathrm{Id}, \Phi, \Phi^2\}$ acts by orientation-preserving isometries on $(\Lambda_h^+)_p$.

We can visualize Φ and Φ^2 using their representative matrices with respect to the basis $\{\eta_1, \eta_2, \eta_3\}$ of $(\Lambda^+)_p$ as

$$\Phi \longleftrightarrow \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Phi^2 \longleftrightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Now, since the elements of C_3' are isometries, the self-dual Weyl operator W_h^+ commutes with all of them: since W_h^+ is symmetric and trace-free, it is easy to see that, for some $\lambda \in \mathbb{R}$,

$$W_h^+ = \begin{pmatrix} 0 & \lambda & \lambda \\ \lambda & 0 & \lambda \\ \lambda & \lambda & 0 \end{pmatrix} \tag{7.14}$$

at p. Its characteristic polynomial is

$$\det(W_h^+ - s\,\mathrm{Id}) = -(s-2\lambda)(s+\lambda)^2.$$

Thus, its eigenvalues are $2\lambda, -\lambda, -\lambda$. In particular, at every point of $\pi^{-1}(r \cap B)$ where W_h^+ is nonzero, it has exactly two distinct eigenvalues. Our aim is to show that the ratio $\rho = \beta_h/\alpha_h$ does not admit a continuous extension on $\pi^{-1}(b)$. In order to do so, we exploit the conclusions drawn *via* the properties of the C_3-symmetry and consider the expansion of V around b: indeed, we will show that W_h^+ does not admit other zeros on the fibers of points on the axis $t\nu$ close to b, and therefore, we prove that, for $t \to 0$, the limit of $\rho(t) := \rho(t\nu)$ does not exist. First, we can easily compute the following

$$\nabla^3\left(\frac{1}{|p_\alpha - x|}\right)\Big|_{x=0} = \frac{3\left(5p_\alpha \otimes p_\alpha \otimes p_\alpha - 3R^2 h_{\mathrm{Eucl}} \odot p_\alpha^\flat\right)}{R^7},$$

where ∇^3 denotes the third covariant derivative with respect to h_{Eucl} and $h_{\mathrm{Eucl}} \odot p_\alpha^\flat$ denotes the symmetric part of $h_{\mathrm{Eucl}} \otimes p_\alpha^\flat$; then, by definition of the p_α's, we can compute, for every $x = (x_1, x_2, x_3) \in \mathbb{R}^3$,

$$\begin{aligned} \sum_{\alpha=1}^4 \langle p_\alpha, x\rangle^3 =& \frac{R^3}{3\sqrt{3}}\Big[(x_1+x_2+x_3)^3 + (x_1-x_2-x_3)^3 \\ &+ (-x_1+x_2-x_3)^3 + (-x_1-x_2+x_3)^3\Big] = \frac{8R^3}{\sqrt{3}}x_1x_2x_3. \end{aligned} \tag{7.15}$$

We consider the following expansion of $1/|p_\alpha - x|$ centered at b:

$$\begin{aligned} \frac{1}{|p_\alpha - x|} =& \frac{1}{R} + \frac{\langle p_\alpha, x\rangle}{R^3} + \frac{3\langle p_\alpha, x\rangle^2 - R^2|x|^2}{2R^5} \\ &+ \frac{5\langle p_\alpha, x\rangle^3 - 3R^2|x|^2\langle p_\alpha, x\rangle}{2R^7} + O(|x|^4). \end{aligned}$$

By (7.10), (7.11) and (7.15), since $V(0) = \varepsilon + 4/R$, we obtain the following

$$V(x) = \varepsilon + \frac{4}{R} + \frac{20}{\sqrt{3}R^4}x_1x_2x_3 + O(|x|^4)$$

around b. By (7.9), it is clear that, for every $p \in \pi^{-1}(b')$, where b' is close to b,

$$W_h^+ = \frac{20}{\sqrt{3}R^4V(0)^2}\begin{pmatrix} 0 & x_3 & x_2 \\ x_3 & 0 & x_1 \\ x_2 & x_1 & 0 \end{pmatrix} + O(|x|^2). \tag{7.16}$$

Indeed, the product of the first derivatives of V is $O(|x|^4)$, so the leading term in (7.9) becomes $V^{-2}\,\mathrm{Hess}_{\mathbb{R}^3}(V)$.

Now, we restrict ourselves to $x = t\nu$ for some t, which means that

$$\lambda(t) = \frac{20}{3R^4V(0)^2}t + O(t^2)$$

in (7.14). Putting $c = 20/(3R^4V(0)^2)$, the eigenvalues of W_h^+ are

$$2ct + O(t^2), \quad -ct + O(t^2), \quad -ct + O(t^2):$$

by our previous discussion about the action of C_3', the two equal eigenvalues have exactly the same error term for every t. Consequently,

$$\rho(t) = \begin{cases} 1, & t < 0, \\ -\dfrac{1}{2}, & t > 0, \end{cases} \quad \text{for } 0 < |t| \ll 1.$$

In particular,

$$\lim_{t \to 0^+} \rho(t) = -\frac{1}{2}, \quad \lim_{t \to 0^-} \rho(t) = 1,$$

hence there is a degeneration phenomenon for ρ on $\pi^{-1}(b)$: this means that there exists no constant $\rho_0 < 1$ such that $\beta_h \leq \rho_0 \alpha_h$ in M.

This shows that the conclusion of Theorem 1.9 can fail if the spectral hypothesis is omitted. The metric h is a nonflat gravitational instanton with an ALE or ALF end and satisfies the volume-growth and curvature-decay assumptions with $(\nu, q) = (4, 4)$ in the ALE case and $(\nu, q) = (3, 3)$ in the ALF case. Nevertheless, a Kähler metric of positive scalar curvature satisfies $\det(W_g^+) > 0$ everywhere [16]. By conformal invariance, a Kähler representative of this type in $[h]$ would imply $\det(W_h^+) > 0$, contradicting the vanishing of W_h^+ in $\pi^{-1}(b)$.

Remark 7.1. The examples above show that the spectral hypotheses in Theorems 1.4 and 1.9 are sharp. We have not produced a compact example that exhibits the same loss of continuity of ρ near a zero of W_h^+ as in the Gibbons–Hawking example. Despite the lack of explicit formulas for Calabi–Yau metrics on K3 surfaces, they may provide such examples.

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