HÖLDER REGULARITY OF THE SOLUTION TO THE LEAST GRADIENT PROBLEM WITH NONHOMOGENEOUS TERM

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ABSTRACT. In this paper, we study the Hölder regularity of the solution u to the following least gradient problem with nonhomogeneous term

$$\inf\left\{\int_\Omega |Du| + \lambda\int_\Omega u \;:\; u \in BV(\Omega),\quad u = f \text{ on } \partial\Omega\right\}, \tag{0.1}$$

in terms of the regularity of the boundary datum f. Inspired by [9, 5], we show that under some assumptions on the mean curvature of Ω and the constant λ, we have the following statements:

$$\begin{cases} f \in C^{0,\alpha}(\partial\Omega) \Rightarrow u \in C^{0,\frac{\alpha}{2}}(\overline{\Omega}), \\ f \in C^{1,\alpha}(\partial\Omega) \Rightarrow u \in C^{0,\frac{1+\alpha}{2}}(\overline{\Omega}). \end{cases} \tag{0.2}$$

1. Introduction

The classical least gradient problem consists in minimizing the total variation of a BV function u subject to the constraint that the trace of u on the boundary of a domain $\Omega \subset \mathbb{R}^N$ equals a given boundary datum $f \in L^1(\partial\Omega)$. Concretely, we minimize

$$\inf\left\{\int_\Omega |Du| \;:\; u \in BV(\Omega),\, u|_{\partial\Omega} = f\right\}. \tag{1.1}$$

Under the assumptions that Ω satisfies a "barrier condition" and f is continuous on $\partial\Omega$, the authors of [9] prove that Problem (1.1) has a unique minimizer u. Moreover, they show that the Hölder regularity of the boundary datum f is inherited by the solution u (see (0.2)), provided that the boundary $\partial\Omega$ has strictly positive mean curvature. In [2, 6], a connection between the least gradient problem (1.1) and the Monge-Kantorovich optimal transport problem was established when $N = 2$ and $f \in BV(\partial\Omega)$. Thanks to this equivalence, the authors in [2] proved Sobolev regularity on the solution u of (1.1) by studying the L^p summability of the transport density between the positive and negative parts of the tangential derivative of f. If $\Omega \subset \mathbb{R}^2$ is uniformly convex, then the solution u belongs to $W^{1,p}(\Omega)$ as soon as $f \in W^{1,p}(\partial\Omega)$ and $p \le 2$. For $p > 2$, $u \in W^{1,p}(\Omega)$ provided that f is in $C^{1,\alpha}(\partial\Omega)$ with $\alpha = 1 - \frac{2}{p}$.

In [1], we consider the following generalization of Problem (1.1) (which corresponds to the case $\psi = 0$):

$$\inf\left\{\int_\Omega |Du| + \int_\Omega \psi\, u \;:\; u \in BV(\Omega),\quad u = f \text{ on } \partial\Omega\right\}, \tag{1.2}$$

where ψ is some given function in $L^N(\Omega)$. We note that a more general version of this problem was already studied in [3], where the density ψ is replaced by a measure $\mu << \mathcal{H}^{N-1}$. But there, the authors proved existence of a solution to a "relaxation" of Problem (1.2), where the Dirichlet boundary condition is removed and replaced by a boundary penalization on u. In fact, proving existence of a

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solution to (1.2) is nontrivial and requires identifying suitable conditions on Ω and ψ. In [1], we prove that Problem (1.2) admits a minimizer provided that Ω satisfies a $\psi-$barrier condition, ψ belongs to $L^p(\Omega)$ with $p > N$, $f \in C(\partial\Omega)$, and the following inequality holds

$$\inf\left\{\frac{\int_\Omega \psi\, u}{\int_\Omega |Du|}\ :\ u \in BV(\Omega),\ \ u = 0\ \text{ on }\ \partial\Omega\right\} > -1. \tag{1.3}$$

If u is a solution to Problem (1.2), then u solves formally the following 1−Laplacian PDE with nonhomogeneous term:

$$\begin{cases} \nabla\cdot[\frac{\nabla u}{|\nabla u|}] = \psi & \text{in }\ \Omega,\\ u = f & \text{on }\ \partial\Omega. \end{cases} \tag{1.4}$$

On the other hand, the uniqueness of solutions to Problem (1.2) presents significant difficulties. In [1], we prove that the solution is unique under a strong restriction on ψ; we need to restrict our analysis to the case $\psi = \lambda$. In this case, (1.3) becomes

$$|\lambda|\ \inf\left\{\frac{-\int_\Omega u}{\int_\Omega |Du|}\ :\ u \in BV(\Omega),\ \ u = 0\ \text{ on }\ \partial\Omega\right\} > -1.$$

Thus,

$$|\lambda| < \inf\left\{\frac{\int_\Omega |Du|}{|\int_\Omega u|}\ :\ u \in BV(\Omega),\ \ u = 0\ \text{ on }\ \partial\Omega\right\} = \inf\left\{\frac{Per(F)}{|F|}\ :\ F \subset \Omega\right\} := \lambda_\Omega.$$

We note that λ_Ω is the Cheeger constant of Ω. The proof of uniqueness is based on the derivation of the following comparison principle (see [1, Proposition 3.7]): suppose that u_1 and u_2 are two solutions to Problem (1.2) with $u_1 = f_1$, $u_2 = f_2$, and $f_1 \le f_2$ on $\partial\Omega$, then $u_1 \le u_2$ a.e. in Ω. Thanks to (1.3), it is easy to see that when the boundary datum f in (1.2) is constant (i.e., $f = c$ on $\partial\Omega$), then $u = c$ is the unique solution to (1.2). In particular, we infer the following estimate:

$$||u||_{L^\infty(\Omega)} \le ||f||_{L^\infty(\partial\Omega)}.$$

If $|\lambda| < \lambda_\Omega$, then Problem (0.1) has a unique solution u. In addition, this solution is continuous on $\overline{\Omega}$ provided that $N \le 7$. Now, assume $f \in C^{0,\alpha}(\partial\Omega)$ (or even more regular). The main question is therefore the following: under what assumptions on the domain Ω and possibly on the constant λ, can one establish Hölder regularity of the solution u to (0.1)? Since the answer to this question is nontrivial, we found it worthwhile to address it in this short paper. In the case when $\lambda = 0$, the approach introduced in [9] (see also [5, 7] for related techniques) to prove Hölder regularity of the solution u of Problem (1.1) consists in constructing in a neighborhood of each boundary point x_0 two barrier functions w^+ and w^-. In this paper, we adapt the arguments developed in [9] to prove Hölder regularity of the solution u of problem (0.1) in the general case when $\lambda \neq 0$. This will be done in two steps: first, we will reduce the interior estimates to boundary estimates by means of the comparison principle, and then we will use the barriers in order to get the required bounds near the boundary. This technique is standard, and has been widely used (see, for instance, [8, 10, 4]).

Throughout this paper, we will assume that $\partial\Omega$ is of class C^2. Let us denote by $\kappa_1, \ldots, \kappa_{N-1}$ the principal curvatures of $\partial\Omega$. Then, we will prove the following main result.

Theorem 1.1. *Assume* $\sum_{i=1}^{N-1} \kappa_i > |\lambda|$. *Then, the solution* u *of Problem* (0.1) *belongs to* $C^{0,\frac{\alpha}{2}}(\overline{\Omega})$ *as soon as* $f \in C^{0,\alpha}(\partial\Omega)$. *Moreover, if* $f \in C^{1,\alpha}(\partial\Omega)$ *then* $u \in C^{0,\frac{1+\alpha}{2}}(\overline{\Omega})$, *for all* $\alpha \in [0,1]$.

2. Key Lemmas

In this section, we present two lemmas that are important for proving the Hölder regularity of the solution u to Problem (0.1). So, these lemmas will be used repeatedly in the next sections. The first lemma shows that if the solution u has a modulus of continuity at boundary points, then the same modulus of continuity holds for u in the interior of the domain. In the sequel, we say that u is a $\lambda-$*least gradient function* in Ω if for every $v \in BV(\Omega)$ such that $v = 0$ on $\partial\Omega$, we have

$$\int_\Omega |Du| + \lambda \int_\Omega u \le \int_\Omega |D(u+v)| + \lambda \int_\Omega u + v.$$

Lemma 2.1. *Let u be a $\lambda-$least gradient function in Ω. Assume that u is continuous on $\overline{\Omega}$ and that there exists a function $\gamma : \mathbb{R}^+ \mapsto \mathbb{R}^+$ such that*

$$|u(x) - u(y)| \le \gamma(|x-y|), \ \textit{for all } x \in \partial\Omega \ \textit{ and } \ y \in \overline{\Omega}. \tag{2.1}$$

Then, we must have

$$|u(x) - u(y)| \le \gamma(|x-y|), \ \textit{for all } x, y \in \overline{\Omega}. \tag{2.2}$$

Proof. Fix two points x_0 and y_0 in Ω. Set $h = x_0 - y_0$. Define $\Omega_h := \Omega \oplus h := \{x + h : x \in \Omega\}$ and $E := \Omega \cap \Omega_h = \{x \in \Omega : x - h \in \Omega\}$. It is clear that E is an open set. We note also that $x_0 \in E$. Moreover, we have $\partial E = [\partial\Omega \cap \overline{\Omega_h}] \cup [\partial\Omega_h \cap \Omega]$. Now, we define the function u_h on $\overline{\Omega_h}$ as follows:

$$u_h(x) = u(x-h) + \gamma(|h|).$$

Then, we claim that both u and u_h are $\lambda-$least gradient functions on E. Let $v \in BV(E)$ be such that $v_{|\partial E} = u^+_{|\partial E}$, where in the sequel $u^+_{|\partial E}$ and $u^-_{|\partial E}$ will denote the inner and outer traces of u on ∂E, respectively. Set

$$\tilde{v} = \begin{cases} v & \text{on } E, \\ u & \text{on } \Omega \backslash E. \end{cases}$$

Hence, it is easy to see that $\tilde{v} = u$ on $\partial\Omega$ (so, $\tilde{v}$ is admissible in (0.1)). Thanks to the minimality of u in Problem (0.1), we infer that

$$\int_\Omega |Du| + \lambda \int_\Omega u \le \int_\Omega |D\tilde{v}| + \lambda \int_\Omega \tilde{v}.$$

Yet,

$$\int_\Omega |Du| + \lambda \int_\Omega u = \int_E |Du| + \lambda \int_E u + \int_{\Omega\backslash\overline{E}} |Du| + \lambda \int_{\Omega\backslash E} u + \int_{\partial E \cap \Omega} |u^+_{|\partial E} - u^-_{|\partial E}|$$

and

$$\int_\Omega |D\tilde{v}| + \lambda \int_\Omega \tilde{v} = \int_E |Dv| + \lambda \int_E v + \int_{\Omega\backslash\overline{E}} |Du| + \lambda \int_{\Omega\backslash E} u + \int_{\partial E \cap \Omega} |v_{|\partial E} - u^-_{|\partial E}|.$$

Thus, we get

$$\int_E |Du| + \lambda \int_E u \le \int_E |Dv| + \lambda \int_E v.$$

Now, let us show that u_h is of $\lambda-$least gradient in Ω_h (so, it will be also of $\lambda-$least gradient in E). Let $v_h \in BV(\Omega_h)$ be such that $v_h = u_h$ on $\partial\Omega_h$. Define $v(x) := v_h(x+h) - \gamma(|h|)$, for all $x \in \Omega$. Hence, it is easy to see that $v = u$ on $\partial\Omega$. Therefore,

$$\int_\Omega |Du| + \lambda \int_\Omega u \le \int_\Omega |Dv| + \lambda \int_\Omega v.$$

But, we have

$$\int_{\Omega_h} |Du_h| + \lambda \int_{\Omega_h} u_h = \int_\Omega |Du| + \lambda \int_\Omega u + \lambda\, \gamma(|h|)\, |\Omega|$$

and
$$\int_{\Omega_h} |Dv_h| + \lambda \int_{\Omega_h} v_h = \int_{\Omega} |Dv| + \lambda \int_{\Omega} v + \lambda\,\gamma(|h|)\,|\Omega|.$$
Hence, we get
$$\int_{\Omega_h} |Du_h| + \lambda \int_{\Omega_h} u_h \leq \int_{\Omega_h} |Dv_h| + \lambda \int_{\Omega_h} v_h.$$
Finally, we claim that $u^+_{|\partial E} \leq u_h{}^+_{|\partial E}$. We recall that $\partial E = \{x \in \partial\Omega \,:\, x-h \in \overline{\Omega}\} \cup \{x \in \Omega \,:\, x-h \in \partial\Omega\}$. First, assume $x \in \partial\Omega$ and $x-h \in \overline{\Omega}$. Thanks to (2.1), we have
$$u(x) \leq u(x-h) + \gamma(|h|) = u_h(x).$$
If $x \in \Omega$ and $x-h \in \partial\Omega$, then by (2.1) we get again the same inequality. Consequently, u and u_h are two $\lambda-$least gradient functions on E with $u \leq u_h$ on ∂E. Hence, by the comparison principle in [1, Proposition 3.7], we infer that $u \leq u_h$ on E. In particular, we get that
$$u(x_0) \leq u_h(x_0) = u(x_0 - h) + \gamma(|h|) = u(y_0) + \gamma(|x_0 - y_0|).$$

Interchanging the points x_0 and y_0, we obtain estimate (2.2). Yet, this concludes the proof. □

In the next lemma, we establish a comparison principle (when $\psi = \lambda$) for sub/supersolutions of (1.4). That is we will show that if a subsolution is below a supersolution on the boundary, so it stays below it everywhere in the interior. This principle will be used to construct two barrier functions w^+ and w^- for the solution u of Problem (0.1) and so, it will allow us to control the behavior of u and obtain Hölder regularity estimates in the next sections.

Lemma 2.2. *Assume u is a $\lambda-$least gradient function in Ω and, $u \in C(\overline{\Omega})$. Let $w^+,\, w^- \in C^2(\Omega) \cap C(\overline{\Omega})$ be such that*
$$|\nabla w^{\pm}| > 0 \qquad \textit{and} \qquad \nabla \cdot \left[\frac{\nabla w^+}{|\nabla w^+|}\right] < \lambda < \nabla \cdot \left[\frac{\nabla w^-}{|\nabla w^-|}\right] \qquad \textit{in} \quad \Omega.$$
In addition, we assume that $w^- \leq u \leq w^+$ on $\partial\Omega$. Then, we must have $w^- \leq u \leq w^+$ inside Ω.

Proof. Let us prove that $u \leq w^+$ in Ω (the proof that $u \geq w^-$ is completely similar). Assume by contradiction that this is not the case, i.e. there exists $x_0 \in \Omega$ such that $u(x_0) > w^+(x_0)$. Since u and w^+ are continuous on $\overline{\Omega}$, then there will be an $\varepsilon > 0$ such that the set $E := \{x \in \Omega \,:\, u(x) - w^+(x) > \varepsilon\}$ has a positive Lebesgue measure (one can see clearly that E is open and $\overline{E} \subset \Omega$). Moreover, since $\partial E = \{x \in \Omega \,:\, u(x) - w^+(x) = \varepsilon\}$, then by choosing a suitable $\varepsilon > 0$, one can also assume that $\mathcal{L}^N(\partial E) = 0$. In order to arrive to a contradiction, the idea is to find a function $v \in BV(\Omega)$ which is better than u in Problem (0.1) in the sense that
$$\int_{\Omega} |Dv| + \lambda \int_{\Omega} v < \int_{\Omega} |Du| + \lambda \int_{\Omega} u.$$

Yet, this contradicts clearly the minimality of u in Problem (0.1). Let us define the function v as follows:
$$v = \min\{u, w^+ + \varepsilon\} = \begin{cases} w^+ + \varepsilon & \text{on} \quad E, \\ u & \text{on} \quad \Omega \backslash E. \end{cases}$$
We have
$$\int_{\Omega} |Dv| + \lambda \int_{\Omega} v = \int_{E} |Dw^+| + \lambda \int_{E} (w^+ + \varepsilon) + \int_{\Omega \backslash \overline{E}} |Du| + \lambda \int_{\Omega \backslash E} u \,+\, |Dv|(\partial E).$$
But, since u and v are continuous on $\overline{\Omega}$ and ∂E has zero Lebesgue measure, then we have $|Du|(\partial E) = |Dv|(\partial E) = 0$. Hence, we get
$$\int_{\Omega} |Du| + \lambda \int_{\Omega} u - \int_{\Omega} |Dv| - \lambda \int_{\Omega} v = \int_{E} |Du| - \int_{E} |Dw^+| + \lambda \int_{E} (u - w^+ - \varepsilon). \tag{2.3}$$

Yet, by assumption, $\nabla\cdot[\frac{\nabla w^+}{|\nabla w^+|}]<\lambda$. We recall that $w^+\in C^2(\Omega)$, $E\subset\subset\Omega$, and $u-w^+-\varepsilon=0$ on ∂E. Hence, one has the following

$$-\int_E \frac{Dw^+}{|Dw^+|}\cdot D(u-w^+-\varepsilon)<\lambda\int_E(u-w^+-\varepsilon).$$

Thus,

$$\int_E|Dw^+|-\int_E|Du|\le\int_E|Dw^+|-\int_E\frac{Dw^+}{|Dw^+|}\cdot Du<\lambda\int_E(u-w^+-\varepsilon).$$

Recalling (2.3), we get

$$\int_\Omega|Du|+\lambda\int_\Omega u>\int_\Omega|Dv|+\lambda\int_\Omega v.\quad\square$$

3. Hölder regularity

3.1. **The case $f\in C^{0,\alpha}(\partial\Omega)$.** In this subsection, we prove that the solution u of Problem (0.1) is in $C^{0,\frac{\alpha}{2}}(\overline{\Omega})$. Fix $x_0\in\partial\Omega$. For $\delta>0$ small enough and γ, $M>0$ sufficiently large (all these constants are to be determined later), we define

$$v(x)=|x-x_0|^2+\gamma\,d(x)$$

and

$$w^{\pm}(x)=f(x_0)\pm Mv(x)^{\frac{\alpha}{2}},$$

where $d(x):=d(x,\partial\Omega)$ denotes the distance function to the boundary $\partial\Omega$. Let us denote by $\pi(x)$ the projection of x onto $\partial\Omega$ and Σ the set of points x at which $\pi(x)$ is not a singleton. Since $\partial\Omega$ is of class C^2, then the distance function d is C^2 in $\overline{\Omega}\backslash\overline{\Sigma}$ and $\overline{\Sigma}\subset\Omega$. In particular, there exists a $\delta_0>0$ small such that d is C^2 in $\{x\in\Omega\,:\,0\le d(x,\partial\Omega)\le\delta_0\}$. We note also that

$$\nabla d(x)=\frac{x-\pi(x)}{|x-\pi(x)|}=\nu(x),\quad\text{for all }\ x\in\overline{\Omega}\backslash\overline{\Sigma},$$

where $\nu(x)$ is the unit inward normal vector to $\partial\Omega$ at x. Let $\kappa_1,...,\kappa_{N-1}$ be the principal curvatures of $\partial\Omega$ at $\pi(x)$ and $e_1,...,e_{N-1}$ be the corresponding principal directions (so all these vectors e_i, $i=1,...,N-1$, are tangent to $\partial\Omega$ at $\pi(x)$). Then, one has

$$D^2d(x)=-\sum_{i=1}^{N-1}\frac{\kappa_i(\pi(x))}{1-d(x)\kappa_i(\pi(x))}\,e_i\otimes e_i. \tag{3.1}$$

In particular, we have the following:

$$D^2d(x)\nabla d(x)=0\quad\text{and}\quad 1-d(x)\kappa_i(\pi(x))\ge0,\ \text{ for all }\ i\in\{1,...,N-1\}. \tag{3.2}$$

Fix $0<\delta<\delta_0$, we see that v is C^2 in $G:=B(x_0,\delta)\cap\Omega$. But, $v>0$ in G. Hence, w^+ and w^- are C^2 in G. Moreover, we have the following:

Lemma 3.1. *Assume that $\sum_{i=1}^{N-1}\kappa_i>|\lambda|$ and $f\in C^{0,\alpha}(\partial\Omega)$. Then, there exist two large constants γ and M such that $|\nabla w^{\pm}|>0$, $\nabla\cdot[\frac{\nabla w^+}{|\nabla w^+|}]<\lambda$ and $\nabla\cdot[\frac{\nabla w^-}{|\nabla w^-|}]>\lambda$ in G, with $w^-\le u\le w^+$ on ∂G.*

Proof. From the definitions of w and v, we have

$$\nabla w^{\pm}=\pm M\,\frac{\alpha}{2}\,v^{\frac{\alpha}{2}-1}\,\nabla v\qquad\text{and}\qquad\nabla v=2(x-x_0)+\gamma\nabla d(x).$$

Hence,

$$|\nabla v|\ge\gamma-2|x-x_0|>\gamma-2\delta>0\ \text{ and }\ |\nabla w^{\pm}|=M\,\frac{\alpha}{2}\,v^{\frac{\alpha}{2}-1}\,|\nabla v|>0\ \text{ in } G\ \text{ provided that }\ \gamma>2\delta.$$

On the other hand, we clearly have $\partial G = [\partial\Omega \cap \overline{B(x_0,\delta)}] \cup [\partial B(x_0,\delta) \cap \Omega]$. Assume $x \in \partial\Omega \cap \overline{B(x_0,\delta)}$. Since f is $\alpha-$Hölderian on $\partial\Omega$, then we have

$$w^-(x) \leq f(x_0) - Lv(x)^{\frac{\alpha}{2}} \leq f(x_0) - L|x-x_0|^\alpha \leq f(x) \leq f(x_0) + L|x-x_0|^\alpha \leq f(x_0) + Lv(x)^{\frac{\alpha}{2}} \leq w^+(x)$$

as soon as $M \geq L$. Now, assume that $x \in \partial B(x_0,\delta) \cap \Omega$ (so, we have $|x-x_0| = \delta$). Hence, one has the following

$$w^+(x) \geq M\delta^\alpha - ||f||_\infty \geq ||u||_\infty \geq u(x) \geq -||u||_\infty \geq ||f||_\infty - M\delta^\alpha \geq w^-(x)$$

provided that $M\delta^\alpha \geq 2||f||_\infty$. Now, we show that $\nabla \cdot [\frac{\nabla w^+}{|\nabla w^+|}] < \lambda < \nabla \cdot [\frac{\nabla w^-}{|\nabla w^-|}]$. First of all, we clearly have

$$\frac{\nabla w^\pm}{|\nabla w^\pm|} = \pm \frac{\nabla v}{|\nabla v|}.$$

Hence,

$$\nabla \cdot \left[\frac{\nabla w^\pm}{|\nabla w^\pm|}\right] = \pm \nabla \cdot \left[\frac{\nabla v}{|\nabla v|}\right] = \pm \frac{|\nabla v|^2 \Delta v - D^2 v \nabla v \cdot \nabla v}{|\nabla v|^3}. \tag{3.3}$$

Yet,

$$D^2 v = 2I + \gamma D^2 d(x).$$

In particular, we have

$$\Delta v = 2N + \gamma \Delta d(x).$$

On the other hand, one has

$$|\nabla v| = \sqrt{4|x-x_0|^2 + \gamma^2 + 4\gamma \nabla d(x) \cdot (x-x_0)} = \gamma \sqrt{1 + \frac{4\,\nabla d(x) \cdot (x-x_0)}{\gamma} + \frac{4|x-x_0|^2}{\gamma^2}}.$$

Then,

$$|\nabla v|^2 \Delta v = [4|x-x_0|^2 + \gamma^2 + 4\gamma \nabla d(x) \cdot (x-x_0)][2N + \gamma \Delta d(x)]$$
$$= 8N|x-x_0|^2 + 4\gamma|x-x_0|^2 \Delta d(x) + 2N\gamma^2 + \gamma^3 \Delta d(x) + 8N\gamma \nabla d(x) \cdot (x-x_0) + 4\gamma^2 \Delta d(x)\, \nabla d(x) \cdot (x-x_0).$$

While

$$D^2 v \nabla v \cdot \nabla v = [2I + \gamma D^2 d(x)][2(x-x_0) + \gamma \nabla d(x)] \cdot [2(x-x_0) + \gamma \nabla d(x)]$$
$$= [4(x-x_0) + 2\gamma \nabla d(x) + 2\gamma D^2 d(x)(x-x_0) + \gamma^2 D^2 d(x) \nabla d(x)] \cdot [2(x-x_0) + \gamma \nabla d(x)].$$

But, we have $D^2 d(x) \nabla d(x) = 0$ (see (3.2)). Thus, we get

$$D^2 v \nabla v \cdot \nabla v = [4(x-x_0) + 2\gamma \nabla d(x) + 2\gamma D^2 d(x)(x-x_0)] \cdot [2(x-x_0) + \gamma \nabla d(x)]$$
$$= 8|x-x_0|^2 + 8\gamma \nabla d(x) \cdot (x-x_0) + 2\gamma^2 + 4\gamma D^2 d(x)(x-x_0) \cdot (x-x_0).$$

Consequently,

$$\begin{aligned} &|\nabla v|^2 \Delta v - D^2 v \nabla v \cdot \nabla v \\ &= 8(N-1)|x-x_0|^2 + 4\gamma|x-x_0|^2 \Delta d(x) + 2(N-1)\gamma^2 + \gamma^3 \Delta d(x) + 8(N-1)\gamma \nabla d(x) \cdot (x-x_0) \\ &\quad + 4\gamma^2\, \Delta d(x)\, \nabla d(x) \cdot (x-x_0) - 4\gamma D^2 d(x)(x-x_0) \cdot (x-x_0) \\ &= \gamma^3 \Bigg[\Delta d + \frac{8(N-1)|x-x_0|^2}{\gamma^3} + \frac{4|x-x_0|^2 \Delta d + 8(N-1) \nabla d \cdot (x-x_0) - 4D^2 d\,(x-x_0) \cdot (x-x_0)}{\gamma^2} \\ &\quad + \frac{2(N-1) + 4\,\Delta d(x)\, \nabla d(x) \cdot (x-x_0)}{\gamma}\Bigg]. \end{aligned} \tag{3.4}$$

Recalling (3.3), we infer that

$$\nabla \cdot \left[\frac{\nabla w^\pm}{|\nabla w^\pm|}\right]$$
$$= \pm \frac{\Delta d + \frac{8(N-1)|x-x_0|^2}{\gamma^3} + \frac{4|x-x_0|^2 \Delta d + 8(N-1)\nabla d \cdot (x-x_0) - 4D^2 d\,(x-x_0)\cdot(x-x_0)}{\gamma^2} + \frac{2(N-1) + 4\,\Delta d(x)\, \nabla d(x) \cdot (x-x_0)}{\gamma}}{\left(1 + \frac{4\,\nabla d(x) \cdot (x-x_0)}{\gamma} + \frac{4|x-x_0|^2}{\gamma^2}\right)^{\frac{3}{2}}}.$$

Therefore, we have

$$\nabla\cdot\left[\frac{\nabla w^+}{|\nabla w^+|}\right]\le\frac{\Delta d+\frac{C}{\gamma}}{(1-\frac{C}{\gamma})^{\frac32}}\qquad\text{and}\qquad\nabla\cdot\left[\frac{\nabla w^-}{|\nabla w^-|}\right]\ge-\frac{\Delta d+\frac{C}{\gamma}}{(1-\frac{C}{\gamma})^{\frac32}}\tag{3.5}$$

where C is a uniform constant (it does not depend on x_0, δ, γ and, M). From (3.1), it is easy to see that

$$\Delta d(x)=-\sum_{i=1}^{N-1}\frac{\kappa_i(\pi(x))}{1-d(x)\kappa_i(\pi(x))}=-\sum_{i=1}^{N-1}\kappa_i(\pi(x))-\sum_{i=1}^{N-1}\kappa_i(\pi(x))\left[\frac{1}{1-d(x)\kappa_i(\pi(x))}-1\right]$$
$$=-\sum_{i=1}^{N-1}\kappa_i(\pi(x))-d(x)\sum_{i=1}^{N-1}\frac{\kappa_i(\pi(x))^2}{1-d(x)\kappa_i(\pi(x))}.$$

Hence, by (3.2), we get

$$\Delta d(x)\le-\sum_{i=1}^{N-1}\kappa_i(\pi(x)).$$

Recalling (3.5), we infer that

$$\nabla\cdot\left[\frac{\nabla w^+}{|\nabla w^+|}\right]\le\frac{-\sum_{i=1}^{N-1}\kappa_i+\frac{C}{\gamma}}{(1-\frac{C}{\gamma})^{\frac32}}<\lambda\qquad\text{and}\qquad\nabla\cdot\left[\frac{\nabla w^-}{|\nabla w^-|}\right]\ge\frac{\sum_{i=1}^{N-1}\kappa_i-\frac{C}{\gamma}}{(1-\frac{C}{\gamma})^{\frac32}}>\lambda\tag{3.6}$$

as soon as γ is sufficiently large. □

Corollary 3.2. *Assume $\sum\limits_{i=1}^{N-1}\kappa_i>|\lambda|$ and $f\in C^{0,\alpha}(\partial\Omega)$. Then, the solution u of Problem (0.1) belongs to $C^{0,\frac{\alpha}{2}}(\overline{\Omega})$.*

Proof. From Lemma 3.1, we know that there exist two large constants γ and M (which do not depend on x_0) such that the functions $w^+=f(x_0)+M[|x-x_0|^2+\gamma d(x)]^{\frac{\alpha}{2}}$ and $w^-=f(x_0)-M[|x-x_0|^2+\gamma d(x)]^{\frac{\alpha}{2}}$ satisfy the following:

$$|\nabla w^{\pm}|>0,\quad\nabla\cdot\left[\frac{\nabla w^+}{|\nabla w^+|}\right]<\lambda<\nabla\cdot\left[\frac{\nabla w^-}{|\nabla w^-|}\right]\ \text{ in }\ B(x_0,\delta)\cap\Omega,\ \text{ and }\ w^-\le u\le w^+\ \text{ on }\ \partial[B(x_0,\delta)\cap\Omega].$$

Thanks to Lemma 2.2, this yields that

$$f(x_0)-C|x-x_0|^{\frac{\alpha}{2}}\le w^-(x)\le u(x)\le w^+(x)\le f(x_0)+C|x-x_0|^{\frac{\alpha}{2}}\ \text{ in }\ B(x_0,\delta)\cap\Omega,$$

where C is a uniform constant that does not depend on x_0. But, $u(x_0)=f(x_0)$. Hence, we get the following estimate:

$$|u(x)-u(x_0)|\le C|x-x_0|^{\frac{\alpha}{2}},\ \text{ for all }\ x\in B(x_0,\delta)\cap\Omega.$$

Now, assume $x\in\Omega$ and $|x-x_0|\ge\delta$. Suppose also that $C\ge\frac{2||u||_{L^\infty(\Omega)}}{\delta^{\frac{\alpha}{2}}}$. Then, we clearly have

$$|u(x)-u(x_0)|\le 2||u||_{L^\infty(\Omega)}\le C\delta^{\frac{\alpha}{2}}\le C|x-x_0|^{\frac{\alpha}{2}}.$$

Consequently, we get that

$$|u(x)-u(x_0)|\le C|x-x_0|^{\frac{\alpha}{2}},\quad\text{for all }\ x\in\Omega.$$

Yet, this estimate is true for all $x_0\in\partial\Omega$ and the constant C does not depend on the point x_0. Recalling Lemma 2.1, we infer that

$$|u(x)-u(y)|\le C|x-y|^{\frac{\alpha}{2}},\quad\text{for all }\ x,\,y\in\Omega.\quad\square$$

3.2. **The case $f \in C^{1,\alpha}(\partial\Omega)$.** To obtain $\alpha-$Hölder regularity of the solution u to Problem (0.1) for $\alpha \in (\frac{1}{2}, 1]$, we need to strengthen the regularity assumption on the boundary datum f. In this subsection, we therefore assume that

$$f \in C^{1,\alpha}(\partial\Omega).$$

So, we can see f as the restriction of a $C^{1,\alpha}-$function (we still call it f) to the boundary $\partial\Omega$. Then, there exists a constant L such that

$$|f(x) - f(x_0) - \nabla f(x_0) \cdot (x - x_0)| \leq L|x - x_0|^{1+\alpha}, \text{ for all } x_0, x \in \partial\Omega.$$

For $x_0 \in \partial\Omega$, we set

$$W^{\pm}(x) = f(x_0) + \nabla f(x_0) \cdot (x - x_0) \pm Mv(x)^{\frac{1+\alpha}{2}}, \text{ for all } x \in G := B(x_0, \delta) \cap \Omega,$$

where we recall that $v(x) = |x - x_0|^2 + \gamma\, d(x)$. Then, we have

$$\nabla W^{\pm}(x) = \nabla f(x_0) \pm M\left(\frac{1+\alpha}{2}\right)v(x)^{\frac{\alpha-1}{2}}\nabla v(x)$$

and so,

$$|\nabla W^{\pm}(x)| = \left|\nabla f(x_0) \pm M\left(\frac{1+\alpha}{2}\right)v(x)^{\frac{\alpha-1}{2}}\nabla v(x)\right| \geq M\left(\frac{1+\alpha}{2}\right)v(x)^{\frac{\alpha-1}{2}}|\nabla v(x)| - |\nabla f(x_0)|$$

$$\geq M\left(\frac{1+\alpha}{2}\right)v(x)^{\frac{\alpha-1}{2}}|2(x - x_0) + \gamma\nabla d(x)| - |\nabla f(x_0)|$$

$$\geq M\left(\frac{1+\alpha}{2}\right)(\delta^2 + \gamma\delta)^{\frac{\alpha-1}{2}}[\gamma - 2\delta] - ||\nabla f||_{L^\infty(\partial\Omega)} > 0$$

as soon as γ and M are large enough. Let us show that $W^-(x) \leq u(x) \leq W^+(x)$ on ∂G. Fix $x \in \partial G \cap \partial\Omega$. Then, we have

$$f(x_0) + \nabla f(x_0) \cdot (x - x_0)| - L|x - x_0|^{1+\alpha} \leq u(x) = f(x) \leq f(x_0) + \nabla f(x_0) \cdot (x - x_0)| + L|x - x_0|^{1+\alpha}.$$

Yet,

$$L|x - x_0|^{1+\alpha} \leq M|x - x_0|^{1+\alpha} \leq Mv(x)^{\frac{1+\alpha}{2}}$$

as soon as $M \geq L$. Hence, $W^-(x) \leq u(x) \leq W^+(x)$. Now, assume that $x \in \partial G \cap \partial B(x_0, \delta)$ (so, $|x - x_0| = \delta$). Then, one has

$$u(x) - f(x_0) - \nabla f(x_0) \cdot (x - x_0) \leq ||u||_\infty + ||f||_\infty + ||\nabla f||_\infty\delta \leq M\delta^{1+\alpha} \leq Mv(x)^{\frac{1+\alpha}{2}}$$

provided that

$$M \geq \frac{2||f||_\infty + ||\nabla f||_\infty\delta}{\delta^{1+\alpha}}.$$

Moreover, we have

$$u(x) - f(x_0) - \nabla f(x_0) \cdot (x - x_0) \geq -2||f||_\infty - ||\nabla f||_\infty\delta \geq -M\delta^{1+\alpha} \geq -Mv(x)^{\frac{1+\alpha}{2}}.$$

Therefore, we conclude that $W^-(x) \leq u(x) \leq W^+(x)$, for all $x \in \partial G$. Finally, we claim that the following holds

$$\nabla \cdot \left[\frac{\nabla W^-}{|\nabla W^-|}\right] < \lambda < \nabla \cdot \left[\frac{\nabla W^+}{|\nabla W^+|}\right] \quad \text{ in } G.$$

Indeed, we have

$$\frac{\nabla W^{\pm}(x)}{|\nabla W^{\pm}(x)|} = \pm\frac{\nabla v(x) \pm \frac{2}{M(1+\alpha)}v(x)^{\frac{1-\alpha}{2}}\nabla f(x_0)}{|\nabla v(x) \pm \frac{2}{M(1+\alpha)}v(x)^{\frac{1-\alpha}{2}}\nabla f(x_0)|}.$$

This yields that

$$\nabla\cdot\left[\frac{\nabla W^{\pm}(x)}{|\nabla W^{\pm}(x)|}\right]=\pm\nabla\cdot\left[\frac{\nabla v(x)\pm\frac{2}{M(1+\alpha)}v(x)^{\frac{1-\alpha}{2}}\nabla f(x_0)}{|\nabla v(x)\pm\frac{2}{M(1+\alpha)}v(x)^{\frac{1-\alpha}{2}}\nabla f(x_0)|}\right]$$
$$=\pm\frac{E_1-E_2}{|\nabla v(x)\pm\frac{2}{M(1+\alpha)}v(x)^{\frac{1-\alpha}{2}}\nabla f(x_0)|^3}$$

where

$$E_1=\left[\Delta v\pm\frac{1-\alpha}{M(1+\alpha)}v^{\frac{-\alpha-1}{2}}\nabla v\cdot\nabla f(x_0)\right]\left|\nabla v(x)\pm\frac{2}{M(1+\alpha)}v^{\frac{1-\alpha}{2}}\nabla f(x_0)\right|^2$$

and

$$E_2=\left[D^2v\pm\frac{1-\alpha}{M(1+\alpha)}v^{\frac{-\alpha-1}{2}}\nabla f(x_0)\otimes\nabla v\right]\left[\nabla v\pm\frac{2}{M(1+\alpha)}v^{\frac{1-\alpha}{2}}\nabla f(x_0)\right]\cdot\left[\nabla v\pm\frac{2}{M(1+\alpha)}v^{\frac{1-\alpha}{2}}\nabla f(x_0)\right].$$

Yet,

$$|\nabla v\pm\frac{2}{M(1+\alpha)}v^{\frac{1-\alpha}{2}}\nabla f(x_0)|^2=|\nabla v|^2+\frac{4}{M^2(1+\alpha)^2}v^{1-\alpha}|\nabla f(x_0)|^2\pm\frac{4}{M(1+\alpha)}v^{\frac{1-\alpha}{2}}[\nabla v\cdot\nabla f(x_0)].$$

To simplify the exposition, we set $a:=\nabla f(x_0)$. Hence, we have

$$E_1=\left[\Delta v\pm\frac{1-\alpha}{M(1+\alpha)}v^{\frac{-\alpha-1}{2}}\nabla v\cdot a\right]\left[|\nabla v|^2+\frac{4}{M^2(1+\alpha)^2}v^{1-\alpha}|a|^2\pm\frac{4}{M(1+\alpha)}v^{\frac{1-\alpha}{2}}[\nabla v\cdot a]\right]$$
$$=\Delta v|\nabla v|^2+\frac{4}{M^2(1+\alpha)^2}v^{1-\alpha}\Delta v|a|^2\pm\frac{4}{M(1+\alpha)}v^{\frac{1-\alpha}{2}}\Delta v[\nabla v\cdot a]$$
$$\pm\frac{1-\alpha}{M(1+\alpha)}v^{\frac{-\alpha-1}{2}}[\nabla v\cdot a]|\nabla v|^2\pm\frac{4(1-\alpha)}{M^3(1+\alpha)^3}v^{\frac{1-3\alpha}{2}}|a|^2[\nabla v\cdot a]+\frac{4(1-\alpha)}{M^2(1+\alpha)^2}v^{-\alpha}[\nabla v\cdot a]^2$$

while

$$E_2=D^2v\nabla v\cdot\nabla v\pm\frac{4}{M(1+\alpha)}v^{\frac{1-\alpha}{2}}[D^2v\nabla v\cdot a]+\frac{4}{M^2(1+\alpha)^2}v^{1-\alpha}[D^2v\,a\cdot a]$$
$$\pm\frac{1-\alpha}{M(1+\alpha)}v^{\frac{-\alpha-1}{2}}|\nabla v|^2[\nabla v\cdot a]+\frac{2(1-\alpha)}{M^2(1+\alpha)^2}v^{-\alpha}|\nabla v|^2|a|^2+\frac{2(1-\alpha)}{M^2(1+\alpha)^2}v^{-\alpha}[\nabla v\cdot a]^2$$
$$\pm\frac{4(1-\alpha)}{M^3(1+\alpha)^3}v^{\frac{1-3\alpha}{2}}[\nabla v\cdot a]|a|^2.$$

Hence,

$$E_1-E_2=\Delta v|\nabla v|^2-D^2v\nabla v\cdot\nabla v+\frac{4}{M^2(1+\alpha)^2}v^{1-\alpha}[\Delta v|a|^2-D^2va\cdot a]$$
$$\pm\frac{4}{M(1+\alpha)}v^{\frac{1-\alpha}{2}}[\Delta v[\nabla v\cdot a]-D^2v\nabla v\cdot a]+\frac{2(1-\alpha)}{M^2(1+\alpha)^2}v^{-\alpha}[[\nabla v\cdot a]^2-|\nabla v|^2|a|^2].$$

Yet, we recall that

$$D^2v=2I+\gamma D^2d(x)\quad\text{and so,}\quad\Delta v=2N+\gamma\Delta d(x).$$

Moreover, using (3.4), one has

$$|\nabla v|^2\Delta v-D^2v\nabla v\cdot\nabla v$$
$$=\gamma^3\left[\Delta d(x)+\frac{8(N-1)|x-x_0|^2}{\gamma^3}+\frac{4|x-x_0|^2\Delta d+8(N-1)\nabla d(x)\cdot(x-x_0)-4D^2d(x)\,(x-x_0)\cdot(x-x_0)}{\gamma^2}\right.$$
$$\left.+\ \frac{2(N-1)+4\nabla d(x)\cdot(x-x_0)\,\Delta d(x)}{\gamma}\right],$$

$$\Delta v|a|^2-D^2v\,a\cdot a=[2N+\gamma\Delta d(x)]|a|^2-[2I+\gamma D^2d(x)]a\cdot a=2(N-1)|a|^2+\gamma[\Delta d(x)|a|^2-D^2d(x)a\cdot a]$$
$$=\gamma\left[\Delta d(x)|a|^2-D^2d(x)a\cdot a+\frac{2(N-1)|a|^2}{\gamma}\right],$$

and

$$\begin{aligned}\Delta v[\nabla v\cdot a]-D^2v\nabla v\cdot a&=[2N+\gamma\Delta d(x)][\nabla v\cdot a]-[2I+\gamma D^2d(x)]\nabla v\cdot a\\&=2(N-1)[\nabla v\cdot a]+\gamma(\Delta d(x)[\nabla v\cdot a]-D^2d(x)\nabla v\cdot a)\\&=4(N-1)\,[x-x_0]\cdot a+2\gamma(N-1)[\nabla d(x)\cdot a]+2\gamma\Delta d(x)[x-x_0]\cdot a+\gamma^2\Delta d(x)[\nabla d(x)\cdot a]\\&\quad-2\gamma D^2d(x)\,[x-x_0]\cdot a-\gamma^2D^2d(x)\nabla d(x)\cdot a\\&=\gamma^2\Bigg[\Delta d(x)[\nabla d(x)\cdot a]-D^2d(x)\nabla d(x)\cdot a+\frac{2(N-1)[\nabla d(x)\cdot a]+2\Delta d(x)[x-x_0]\cdot a-2D^2d(x)\,[x-x_0]\cdot a}{\gamma}\\&\qquad+\quad\frac{4(N-1)\,[x-x_0]\cdot a}{\gamma^2}\Bigg].\end{aligned}$$

Consequently,

$$\begin{aligned}E_1-E_2&=\gamma^3[\Delta d+o(1)]+\frac{4}{M^2(1+\alpha)^2}v^{1-\alpha}\,\gamma\Big[\Delta d(x)|a|^2-D^2d(x)a\cdot a+o(1)\Big]\\&\pm\frac{4}{M(1+\alpha)}v^{\frac{1-\alpha}{2}}\gamma^2[\Delta d(x)[\nabla d(x)\cdot a]-D^2d(x)\nabla d(x)\cdot a+o(1)]+\frac{2(1-\alpha)}{M^2(1+\alpha)^2}v^{-\alpha}[[\nabla v\cdot a]^2-|\nabla v|^2|a|^2]\\&=\gamma^3\Big[\Delta d(x)+o(1)\Big]+\frac{2(1-\alpha)}{M^2(1+\alpha)^2}v^{-\alpha}[[\nabla v\cdot a]^2-|\nabla v|^2|a|^2],\end{aligned}$$

where $|o(1)|\le\frac{C}{\gamma}\to 0$ when $\gamma\to\infty$ and C is a uniform constant (does not depend on x_0, δ, M, or γ). On the other hand, we have

$$\begin{aligned}\left|\nabla v(x)\pm\frac{2}{M(1+\alpha)}v(x)^{\frac{1-\alpha}{2}}a\right|^3&=\left|2(x-x_0)+\gamma\nabla d(x)\pm\frac{2}{M(1+\alpha)}v(x)^{\frac{1-\alpha}{2}}a\right|^3\\&=\gamma^3|\nabla d(x)+o(1)|^3.\end{aligned}$$

Hence,

$$\nabla\cdot\left[\frac{\nabla W^{\pm}(x)}{|\nabla W^{\pm}(x)|}\right]=\pm\frac{\Delta d(x)+o(1)+\frac{2(1-\alpha)}{M^2(1+\alpha)^2}\gamma^{-3}\,v^{-\alpha}[[\nabla v\cdot a]^2-|\nabla v|^2|a|^2]}{|\nabla d(x)+o(1)|^3}.$$

We note that the term $\frac{2(1-\alpha)}{M^2(1+\alpha)^2}v^{-\alpha}[[\nabla v\cdot a]^2-|\nabla v|^2|a|^2]$ is not bounded near to x_0 because of $v^{-\alpha}$. However, it has a good sign (negative). Thanks to our assumption that $\sum_{i=1}^{N-1}\kappa_i>|\lambda|$, we infer that for γ large enough,

$$\nabla\cdot\left[\frac{\nabla W^{+}(x)}{|\nabla W^{+}(x)|}\right]<\lambda<\nabla\cdot\left[\frac{\nabla W^{-}(x)}{|\nabla W^{-}(x)|}\right].$$

From Lemma 2.2, we then have

$$f(x_0)-C|x-x_0|^{\frac{1+\alpha}{2}}\le W^-(x)\le u(x)\le W^+(x)\le f(x_0)+C|x-x_0|^{\frac{1+\alpha}{2}}\ \text{ in }\ B(x_0,\delta)\cap\Omega.$$

Hence,

$$|u(x)-u(x_0)|\le C|x-x_0|^{\frac{1+\alpha}{2}},\ \text{ for all }\ x\in B(x_0,\delta)\cap\Omega.$$

The rest of the proof follows exactly as in Corollary 3.2. Finally, we conclude that u belongs to $C^{0,\frac{1+\alpha}{2}}(\overline{\Omega})$.

4. Example

We conclude this paper by showing in 2D that we cannot expect in general $\beta-$Hölder regularity on the solution u of Problem (0.1) for $\beta > \frac{\alpha}{2}$ if the boundary datum f is not better than $\alpha-$Hölder. Assume $\lambda > 0$. From [1], we recall that Problem (0.1) has a unique solution u provided that λ is less than the Cheeger constant $\lambda_\Omega := \inf\{\frac{Per(E)}{|E|} : E \subset \Omega\}$ and, Ω satisfies a $\lambda-$barrier condition. Assume $\Omega = B(0,R) \subset \mathbb{R}^2$ and $\lambda < \frac{1}{R}$. Thanks to [1, Proposition 3.1], we know that the boundary of any superlevel set $E_s = \{u \geq s\}$ is smooth inside Ω with curvature $-\lambda$. Hence, any level set $\partial\{u \geq s\}$ is an arc of a circle with radius $\frac{1}{\lambda}$. Moreover, thanks to [1, Lemma 3.3], $\partial E_s \cap \partial\Omega \subset f^{-1}(s)$. Fix $\alpha \in (0,1]$. Then, we define the boundary datum f on ∂B_1 as follows:

$$f(x_1,x_2) = \begin{cases} |x_1|^\alpha & \text{if} \quad x_2 \geq 0, \\ R^\alpha & \text{if} \quad x_2 < 0. \end{cases}$$

Clearly, $f \in C^{0,\alpha}(\partial\Omega)$. Moreover, we have $f(-x_1,x_2) = f(x_1,x_2)$. Fix a point $x := (x_1,x_2) \in \partial B_R$ with $x_2 \geq 0$ and, set $s_x := f(x)$. Then, we know that $\partial E_{s_x} \cap \partial\Omega = \{(x_1,x_2),(-x_1,x_2)\}$. But, we recall that ∂E_{s_x} is an arc of a circle with center $(0,a_x)$, $a_x > 0$ since the curvature of $\partial E_{s_x} \cap \Omega$ is $-\lambda < 0$, and radius $\frac{1}{\lambda}$. Yet, one has

$$|(0,a_x) - (x_1,x_2)| = \frac{1}{\lambda}.$$

Hence,

$$a_x = x_2 + \sqrt{x_2^2 + \frac{1}{\lambda^2} - R^2}.$$

In particular, the map $x_2 \in [0,R] \mapsto a_x$ is smooth and $a_x \geq \sqrt{\frac{1}{\lambda^2} - R^2}$. Assume $(y_1,y_2) \in \partial E_{s_x} \cap \Omega$. Then, we have

$$u(y_1,y_2) = f(x_1,x_2).$$

Let us compute $u(0,y_2)$ and show that u is not better than $\frac{\alpha}{2}-$Hölderian. From the definition of f, we have

$$u(0,y_2) = |x_1|^\alpha = (R^2 - x_2^2)^{\frac{\alpha}{2}} = (R+x_2)^{\frac{\alpha}{2}}(R-x_2)^{\frac{\alpha}{2}}.$$

Assume that u is $\beta-$Hölderian on $\overline{\Omega}$ for some $\beta > \frac{\alpha}{2}$; the aim is to arrive to a contradiction. Since $u(0,R) = 0$, then we get

$$R^{\frac{\alpha}{2}}|R-x_2|^{\frac{\alpha}{2}} \leq u(0,y_2) \leq M|R-y_2|^\beta.$$

Hence,

$$R^{\frac{\alpha}{2}}\left[\frac{R-x_2}{R-y_2}\right]^{\frac{\alpha}{2}} \leq M|R-y_2|^{\beta-\frac{\alpha}{s}} \to 0 \quad \text{when} \quad y_2 \to R. \tag{4.1}$$

However,

$$|a_x - y_2| = |(0,a_x) - (x_1,x_2)|.$$

Then, one has

$$y_2^2 - 2a_x y_2 = R^2 - 2a_x x_2.$$

This yields that

$$\left[\frac{R-x_2}{R-y_2}\right] = \frac{-(R+y_2)+2a_x}{2a_x} \to \frac{1}{1+\lambda R} \qquad \text{when} \quad y_2 \to R.$$

But, this contradicts (4.1).

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